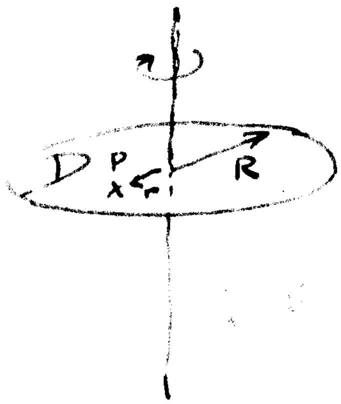
Question 3 *(20 points)* Compute the moment of inertia I of a homogeneous disc of mass M, radius R, about the perpendicular axis through its center.



mass density mass/area

$$\rho = \frac{M}{\pi R^2}$$

$$I = \int_D dA\, \rho(P)\, r^2$$

$$= \frac{M}{\pi R^2} \int_0^R r\,dr \int_0^{2\pi} d\theta \; r^2$$

$$= \frac{M}{\pi R^2} \left[\frac{1}{4} r^4 \right]_0^R \cdot 2\pi$$

$$\underline{I = \frac{1}{2} M R^2}$$

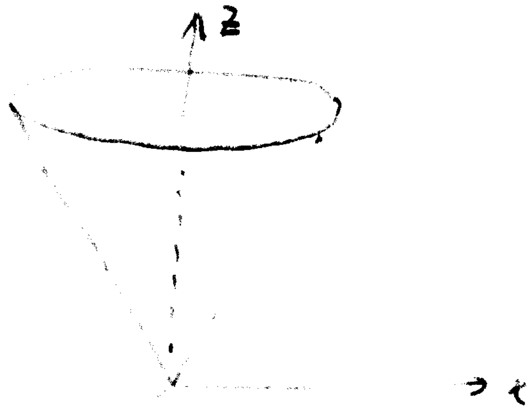
(Working space for Question 1)

iv) $f(r,\theta) = r$

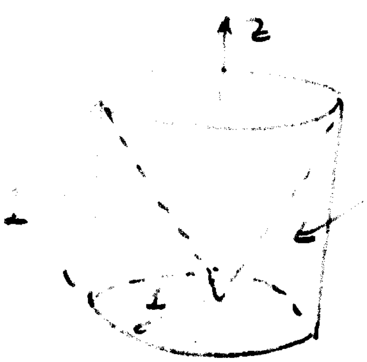
v) $\int_D dA\, f(P) = \int_0^1 r\,dr \int_0^{2\pi} d\theta\, r$

$= 2\pi \left[\frac{1}{3} r^3\right]_0^1 = \frac{2\pi}{3}$

vi) f cone



vii)



- $\int_D dA\, f(P)$ is the volume of this cylinder with the cone removed.

Question 2 *(50 points)* Let the function f be the same as in Question 1 but extend its domain to the plane $\mathbb{R}^2$. Now answer the following questions:

5 (i) Let (x, y) be Cartesian coordinates for $\mathbb{R}^2$. Write down a formula for $f(x, y)$.

5 (ii) Calculate $\frac{\partial f}{\partial x}$. Does $\lim_{(x,y)\to(0,0)} \partial_x f(x, y)$ exist?

10 (iii) Compute the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)\,.$$

Prove that your proposed limit is correct.

5 (iv) Let (r, θ) be polar coordinates for $\mathbb{R}^2$. Write down a formula for $f(r, \theta)$.

10 (v) Let D be a unit disc centered at $\mathcal{O}$. Compute

$$\int_D dA\, f(P).$$

5 (vi) Let Σ be the surface

$$z = f(x, y)\,.$$

Sketch the surface Σ.

5 (vii) Copy out your sketch from part (vi). Indicate an interpretation of the integral in part (v) on this second sketch. (An English sentence may help too!)

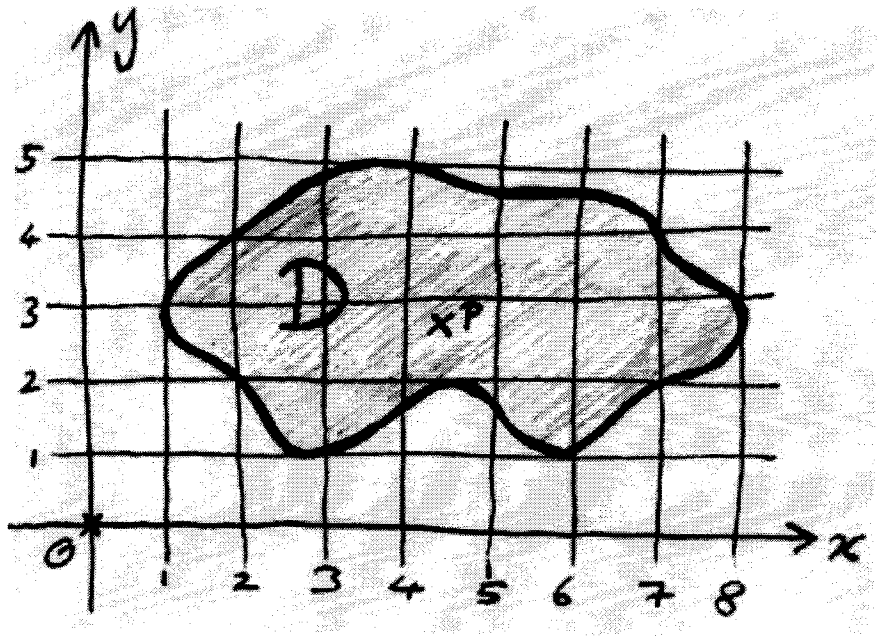
5 (viii) Let R be the solid region given in Cartesian coordinates (x, y, z) by

$$0 \leq z \leq f(x, y)\,, \qquad x^2 + y^2 \leq 1\,.$$

50

Describe this region using spherical coordinates.

Question 1 *(30 points)* Consider the domain D depicted below:



Let $\overline{OP}$ be the distance from the point $P \in D$ to the origin O. Define a function

$$f : D \longrightarrow \mathbb{R}$$

by

$$f(P) = \overline{OP}.$$

Now answer the following questions:

(i) Sketch ∂D.

(ii) Give (crude) upper and lower bounds on the area A of D.

(iii) Write down, but do not compute, an integral equaling A.

(iv) Propose numbers a and b such that

$$a < \int_D dA\, f(P) < b\,.$$

Briefly explain your choices.