Lecture #11: EXTERIOR DERIVATIVE

\[ d : \Lambda^p M \longrightarrow \Lambda^{p+1} M \quad (\star) \]

is defined in a coordinate system as

\[ d (\epsilon_i \cdots \epsilon_p dx^i \wedge \cdots \wedge dx^p) = (\partial_i \epsilon_i \cdots \epsilon_p) \ dx^1 \wedge \cdots \wedge dx^p \]

i.e.

\[ d = dx^i \frac{\partial}{\partial x^i} \wedge. \]

It obeys

(i) Linear: \[ d (\lambda \omega - \mu \beta) = \lambda d\omega - \mu d\beta \]

(ii) Graded Leibnitz:

\[ d (\omega \wedge \beta) = d\omega \wedge \beta + (-1)^p \omega \wedge d\beta \]

(iii) \[ d^2 = 0 \]

(iv) \[ \text{if } f : M \rightarrow \mathbb{R}, \quad df \text{ is the differential of } f \]

Notice (\star) holds because even though

\[ dx^i \partial_i X \]

is not a tensor, its totally antisymmetric part is.
Example $d=3$

**function** $f$ $\xrightarrow{d} \frac{\partial f}{\partial x_i} \text{ gradient}$

1-form $\omega = x dx + y dy + z dz \xrightarrow{d} (\partial_x y - \partial_y x) dx \wedge dy$

$+ (\partial_y z - \partial_z y) dy \wedge dz \text{ curl}$

$+ (\partial_z x - \partial_x z) dz \wedge dx \text{ divergence}$

2-form $\beta^x dy \wedge dz - \beta^y dz \wedge dx + \beta^z dx \wedge dy \xrightarrow{d} (\partial_x \beta^x + \partial_y \beta^y + \partial_z \beta^z) dx \wedge dy \wedge dz$

3-form $\gamma dx \wedge dy \wedge dz \xrightarrow{d} 0$

Then $d^2 = 0$ summarizes $\text{curl grad} = 0$

$\text{div curl} = 0$

**Note** If $\omega = df$ for some $f$ we call $\omega$ exact

If $d\omega = 0$ we call $\omega$ closed

The operator $d$ defines the de Rham complex

$0 \rightarrow \Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^k M \xrightarrow{d} \Lambda^k M \rightarrow 0$

Its cohomology $\text{closed forms} \xrightarrow{\text{exact forms}} = \text{de Rham cohomology}$
**Classical mechanics**

**Symplectic Manifold**

We call \( Z \) a symplectic manifold if it has a closed, non-degenerate 2-form \( \omega \). [Assume \( Z, \omega \) differentiable]

Non-degenerate means \( \omega = \omega_{AB} \, dz^A \wedge dz^B \)

\[
\text{det} \, \omega_{AB} \neq 0. \quad \Rightarrow \dim Z \text{ is even}
\]

**Poincaré-Lemma**

On any contractible domain, a closed \( p \)-form \( \alpha \) can be written

\[
\alpha = d\beta
\]

for some \( p \)-form \( \beta \).

* HW gives an idea for a proof
* \( \alpha \wedge \alpha = 0 \) follows freely since \( d^2 \beta = 0 \)
* \( \beta \) is not unique, \( \beta \to \beta + d\alpha \) is still a solution. "Gauge symmetry": \( \alpha \to \alpha + d\delta \) is called a "gauge for gauge" symmetry.
Example \( T^*M = \{ (p, \theta_p) : p \in M, \theta_p \in T^*_p M \} \)

The cotangent bundle is itself a differentiable manifold. At every point we assign

\[(p, \theta_p) \mapsto \theta_p\]

"Tautological one form".

In a coordinate system

\[(x^\mu, \theta_p = p_\mu dx^\mu) \mapsto p_\mu dx^\mu\]

\[\Omega = (x^\mu, p_\mu)\]

If we view this as a symplectic current

\[\Theta = p_\mu dx^\mu\]

then

\[\omega = d\Theta = d(p_\mu dx^\mu) = dp_\mu \wedge dx^\mu\]

Here use \(x^\mu\) (rather than \(x\), since may want to interpret one coordinate at a time).
Since \( dp^m \wedge dx^m = \frac{1}{2} dp^m \wedge dx^m - \frac{1}{2} dx^m \wedge dp^m \),

\[ = \omega_{AB} \, dz^A \wedge dz^B \]

we have

\[ \omega_{AB} = \left( \begin{array}{c|c} \frac{1}{i} \delta^m_v & \frac{1}{i} \delta^m_v \\ \hline - \frac{1}{i} \delta^m_v & \frac{1}{i} \delta^m_v \end{array} \right) \text{ invertible} \]

Actually, a basic theorem of symplectic geometry is that there always exists a coordinate chart around \( P \in \mathbb{Z} \) such that \( \omega = dx_i \wedge dy^i \) \( [\mathbb{R}^n \rightarrow (x^i, y^i)] \) called Darboux theorem. "Symplectic geometry is more rigid than Riemannian geometry."
Hamiltonian dynamics

$Z = \text{Symplectic manifold}$

"Phase space"
or "generalized positions & momenta"

Want to find integral curves or flows that are consistent with positions and momenta.

Suppose $H : Z \to \mathbb{R}$ is some function.

Then $dH$ is a 1-form so if we had a $(\frac{3}{2})$-tensor $\Omega$ (say)

$\Omega(\omega H, \cdot) \in \mathbb{R} Z$

because its second slot maps 1-forms to real numbers.
But on a symplectic manifold, there is a canonical choice for $\Omega$:

$$\Omega = (\omega_{AB})^{-1} \frac{\partial}{\partial z^A} \otimes \frac{\partial}{\partial z^B}$$

because $\omega$ is non-degenerate.

$\Omega$ is called the Poisson bivector.

The vector field

$$X_H = \Omega (dH, \cdot)$$

is called a Hamiltonian vector field.

Let $g : \mathbb{R} \to \mathbb{R}$. The job of a vector field is to compute the directional derivative of functions along integral curves to the field.
l.e. from the gradient of \( \mathbf{f} \), form a 1-form on which \( X_H \) can act.

In equations

\[
\nabla \mathbf{f} = \Omega (dH, df)
\]

\( \frac{d}{dt} \) along flows

These are called Hamilton's equations.

Usually they are written slightly differently by introducing the "Poisson bracket"

\[\{ \cdot, \cdot \}_PB = \Omega (d\cdot, d\cdot)\]

\[\Rightarrow \dot{f} = \{H, f\}_PB \quad \text{or} \quad \frac{d}{dt} = \{H, \cdot \}_PB\]

If you are familiar with quantum mechanics, the second equation should remind you of the Schrödinger equation.
Review Exercise: Find an example of a closed but non-exact differential form.