Lecture #12  CLASSICAL MECHANICS

* Symplectic Manifold \((\mathcal{Z}, \omega)\) is an even dimensional manifold with a non-degenerate closed 2-form \(\omega\).

* \(T^* \mathcal{M} \supset U \rightarrow (p_\mu, x^\mu)\) and \(\omega = dp_\mu \wedge dx^\mu\)

Hamiltonian dynamics

\[ Z = \text{Symplectic manifold} \]

"Phase space"
or "generalized positions & momenta"

Want to find integral curves or flows that are consistent with positions and momenta

Suppose \( H: \mathcal{Z} \rightarrow \mathbb{R} \) is some function.

Then \( dH \) is a 1-form so if we had a \((3)\)-tensor \( \Omega \) (say)

\[ \Omega (\mathbf{v}, \mathbf{w}, \mathbf{x}) \in \mathcal{Z} \]

because its second slot maps 1-forms to real numbers.
But on a symplectic manifold, there is a canonical choice for $\Omega$:

$$\Omega = (\omega_{AB})^{-1} \frac{\partial}{\partial z^A} \otimes \frac{\partial}{\partial \bar{z}^B}$$

because $\omega$ is non-degenerate.

$\Omega$ is called the **Poisson bivector**.

The vector field

$$X_H = \Omega (dH, \cdot)$$

is called a **Hamiltonian vector field**.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$. The job of a vector field is to compute the directional derivative of functions along integral curves to the field.
\[ \frac{d}{dt} \text{ along flows} \]

These are called Hamilton's equations.

Usually they are written slightly differently by introducing the "Poisson bracket"

\[ \{ \cdot, \cdot \}_\text{PB} = \mathcal{L} (d\cdot, d\cdot) \]

\[ \dot{f} = \{ H, f \}_\text{PB} \quad \text{or} \quad \frac{d}{dt} = \{ H, \cdot \}_\text{PB} \]

If you are familiar with quantum mechanics, the second equation should remind you of the Schrödinger equation.
Properties of Poisson Bracket

In Darboux coordinates

\[ \Omega = \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial x^i} \]

and

\[ \{ f, g \}_{PB} = \Omega (df, dg) \]

\[ = \Omega (dx^i \frac{\partial f}{\partial x^i} + dy^i \frac{\partial f}{\partial y^i}, dx^j \frac{\partial g}{\partial x^j} + dy^j \frac{\partial g}{\partial y^j}) \]

\[ = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial y^i} \]

which is the formula you will find in Goldstein.

* \( \{ f, g \}_{PB} = - \{ g, f \}_{PB} \)
* \( \{ f, \{ g, h \}_{PB} \}_{PB} + \text{cyclic} = 0 \)
* \( \{ f, gh \}_{PB} = \{ f, g \}_{PB} h + g \{ f, h \}_{PB} \)

All follow from the definition.
Notice that $\{\cdot, \cdot\}_{PB}$ obeys properties similar to the Lie bracket. In fact, it makes the space of differentiable functions on $\mathbb{Z}$ into a Lie algebra.

**Remark** Poisson Brackets are the basis for quantization

$$\{\cdot, \cdot\}_{PB} \longrightarrow \frac{i}{\hbar} [\cdot, \cdot]$$

where functions on $\mathbb{Z}$ are replaced by operators acting on a Hilbert space.

**Example** $T^*\mathbb{M}$ with $\omega = dp^\mu \wedge dx^\mu$

and $H: T^*\mathbb{M} \longrightarrow \mathbb{R}$

Then $\Omega = \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial p^\mu} - \frac{\partial}{\partial p^\mu} \otimes \frac{\partial}{\partial x^\mu}$ [check coeff!]

Recall $\frac{d}{dt} = \{H, \cdot\}_{PB}$
So we can compute integral curves in $T^*M$ by calculating $\dot{\gamma} = (\dot{x}^\mu, \dot{p}_\mu)$

\[
\begin{align*}
\dot{x}^\mu &= \{ H, x^\mu \}_PB = \frac{\partial H}{\partial x^\mu} - \frac{\partial H}{\partial p_\nu} \frac{\partial x^\nu}{\partial x^\mu} = \frac{\partial H}{\partial p_\mu} \\
\dot{p}_\mu &= -\frac{\partial H}{\partial x^\mu} \\
\end{align*}
\]

are Hamilton's equations.

Next choose an interesting $H$. Suppose $M$ has a ($^2$) tensor $g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = g$

Take $2H = g(\theta, \theta) = g^{\mu\nu}(x) \dot{p}_\mu \dot{p}_\nu$
Then

\[
\begin{align*}
\dot{x}^\mu &= \frac{2}{\beta_p} \left( \frac{1}{2} \beta_p g^{\rho \sigma} p_\rho p_\sigma \right) \\
\dot{p}_\mu &= -\frac{2}{\beta_p} \frac{\partial}{\partial x^\mu} \left( \frac{1}{2} \beta_p g^{\rho \sigma} p_\rho p_\sigma \right)
\end{align*}
\]

\[
\Rightarrow \begin{cases}
\dot{x}^\mu = g^{\rho \sigma} p_\rho p_\sigma \equiv p^\rho \\
\dot{p}_\mu = -\left( \frac{\partial}{\partial x^\mu} g^{\rho \sigma} \right) p_\rho
\end{cases}
\]

Review Exercise

(i) Compute \( \dot{x}^\mu \)

(ii) Look up the "geodesic" equation and compare your result to what you find.