Lecture #14  Riemannian Geometry

$$g = ds^2 = (\cdot, \cdot) : \mathbb{C}^2 \otimes TM \rightarrow \mathbb{R}$$ (non-degenerate, differentiable)

gives $T_pM$ an inner product and norm

(when $g$ is proper).

At any $p \in M$ we can diagonalize $g_{ij} = (ei, ej)$
and by $gl(dimM)$ transformations achieve

$$g_{ij} = \text{diag}(\pm 1, \pm 1, \ldots, \pm 1)$$

The signature of $g$ is the number of plus or minus signs (it is an invariant because $g$ is non-degenerate and $gl(dimM)$ transformations are denoted by the sign of their determinant).

For example: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

is signature $(-+++)$.
Pseudo-Riemannian structures are endowed with causal structures.

We call a vector $\mathbf{v}$

(i) timelike $g(\mathbf{v}, \mathbf{v}) > 0$
(ii) spacelike $g(\mathbf{v}, \mathbf{v}) < 0$
(iii) null/lightlike $g(\mathbf{v}, \mathbf{v}) = 0$

Then we say curves are timelike, spacelike or null if their tangent vectors are everywhere timelike, spacelike or null.
An inner product induces a canonical isomorphism b/w a vector space and its dual:

\[ \mathfrak{m} \xrightarrow{\mathfrak{n}} \mathfrak{m}^* \]

\[ (0) - \text{tensor} \]
\[ (0) - \text{covariant vector} \]

Notice this is the pullback again:

\[ \mathfrak{T}_{p \mathcal{M}} \xrightarrow{\mathfrak{l}} \mathfrak{T}_{p \mathcal{M}}^* \]

\[ \mathfrak{u}^* = \mathfrak{u} \circ \mathfrak{v} \]
If \( \{e_i\} \) is a basis for \( T_p M \) and \( \{\theta^i\} \) the dual basis for \( T^*_p M \). Then

\[
u(e_i) = g(e_i, \cdot) = w_i^j \theta^j \quad \text{for some } w_i^j
\]

To compute \( w_i^j \), notice

\[
(i(e_i))(e_j) = g(e_i, e_j) = \delta^i_j
\]

but

\[
(w_i^k \theta^k)(e_j) = w_i^k \theta^k(e_j) = w_i^k \delta^k_j = w_i^j
\]

\[
\Rightarrow \nu(e_i) = g_{ij} \theta^j
\]

so \( \nu : e_i \rightarrow g_{ij} \theta^j \quad \text{"lowering an index"} \)

Similarly

\[
\nu^{-1} : \theta^i \rightarrow g^{ij} e_j
\]

\( g^{ij} = (g_{ij})^{-1} \quad \text{"raising an index"} \)
In general, we can now biject tensor of types \((n^m)\) to types \((n^{m+n_{+m}})\), \(\ldots\), \((n^{0})\) using \(c\) and \(c^{-1}\).

For example

\[
T = T_{ij}^{k} e_{i} \otimes e_{j} \otimes \omega_{k} \rightarrow T_{ij}^{k} \omega_{i} \otimes \omega_{j} \otimes \omega_{k}
\]

**Representations**

The symmetric group can be used to break \(gl(d\text{im}M)\) representations into irreducibles. To further decompose into \(so(d\text{im}M)\) representations, the metric can be used:

\[
\begin{align*}
\text{EX} & \quad X_{ij} = X[ij] + X(ij) \quad \text{where} \quad \left\{ \begin{array}{c}
[1 \ldots n] = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
1 & 2 & \ldots & n \\
1 & 2 & \ldots & n
\end{array} \right) \\
(1 \ldots n) = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{array} \right)
\end{array} \right. \\
& \quad = X[ij] + X_{ij} + \frac{1}{\text{dim} M} g_{ij} g^{kl} X_{kl}
\end{align*}
\]
Coordinate vs. non-coordinate bases

An orthonormal frame is a set of \( n \)-vector fields \( \{ e_a \in \mathcal{X}M \} \) such that

\[ g(e_a, e_b) = \delta_{ab} \]

i.e.

\( e_a \) are orthonormal basis for \( T_p M \) at each \( p \in M \).

Can always construct an orthonormal frame for an open set \( U \subset M \), but globally there is an obstruction measured by 2nd Strichartz-Whitney class.
Ex $\mathbb{R}^2$ with $ds^2 = dx^2 + dy^2$.

\[ \begin{cases} e_x = \frac{1}{\delta_x} , & e_y = \frac{2}{\delta_y} \end{cases} \]

is an orthonormal frame because $g(e_x, e_x) = (dx \circ dx)(e_x, e_x) = 1$

$g(e_x, e_y) = 0$

$g(e_y, e_y) = (dy \circ dy)(e_y, e_y) = 1$

Now in general $\mathcal{U} \subseteq \mathbb{R}^n$ open and $\{e_a\}$ is an orthonormal frame on $\mathcal{U}$, then

$e_a = e_a^i \frac{\partial}{\partial x^i}$

components of vectors $e_a$ in basis $\partial_i$.

But $\delta_{ab} = g(e_a, e_b) = e_a^i e_b^j g(\partial_i, \partial_j)$

$= e_a^i e_b^j$ $\delta_{ij}$

$\Rightarrow \begin{bmatrix} e_a^i \end{bmatrix} e_b^j g_{ij} = \delta_{ab}$

The functions $e_a^i$ are often called the inverse "vielbein" (Gell-Mann).
Notice: Even though the metric components in the basis \{e_a\} are just \(g_{ab}\) (i.e. \(g_{ab} = g(e_a, e_b) = \delta_{ab}\)), this basis is usually not a coordinate basis.

i.e. \(l_{e_a e_b} = [e_a, e_b] = c_{ab}^\text{c c c} \neq 0\) if \(\{e_a\}\) is a basis

and the structure functions \(c_{abc}\) are in general not zero while \(L_{d_i} (e_j) = \{d_i, e_j\} = 0\) in a coordinate basis.
Review Exercise 1. Show that the vielbein
\[ e^a_i = (e_a e^a)^{-1} \]
is the "square root" of
the metric tensor in the sense
\[ e_i a e_j b \delta_{ab} = g_{ij} \]

2. Consider the metric
\[ ds^2 = dr^2 + r^2 d\theta^2 \]
for \( \mathbb{R}^2 \). Construct vielbeine for this metric.
Is this a coordinate basis?