Under a change of coordinates a vector \( \mathbf{u} = u^i \mathbf{e}_i \) becomes

\[
\mathbf{u} = u^i(x) \frac{\partial}{\partial x^i} = u^j(x') \frac{\partial}{\partial x'^j} = u'^i(x) \frac{\partial x'^i}{\partial x^j} \frac{\partial}{\partial x'^j}
\]

\[
\Rightarrow u'^i = \frac{\partial x'^i}{\partial x^j} u^j
\]

\[
\Rightarrow \frac{\partial u^i}{\partial x^k} = \frac{\partial x'^i}{\partial x^l} \frac{\partial}{\partial x'^l} \left( \frac{\partial x^j}{\partial x^l} u^j \right)
\]

\[
= \frac{\partial x'^i}{\partial x^k} \frac{\partial}{\partial x'^l} \left( \frac{\partial x^j}{\partial x^l} u^j \right)
\]

\[
= \frac{\partial x'^i}{\partial x^k} \frac{\partial}{\partial x'^l} \left( \frac{\partial x^j}{\partial x^l} u^j \right) + \frac{\partial x'^i}{\partial x^l} \frac{\partial}{\partial x'^j} \left( \frac{\partial x^j}{\partial x^k} u^j \right)
\]

Transformation of a \((1,1)\)-tensor

So modify

\[
\nabla \mathbf{u} = (\partial_i u^i + T^i_{\ jk} u^j)
\]
Where

\[ T^i_{\ j\ k} = \frac{\partial x^i}{\partial x^l} \frac{\partial x^m}{\partial x^j} \frac{\partial x^n}{\partial x^k} \frac{1}{(\frac{\partial x^i}{\partial x^l})(\frac{\partial x^j}{\partial x^m})(\frac{\partial x^n}{\partial x^k})} \]

\text{(1) - TENSOR TRANSFORMATION}

Note if \( J = \left( \frac{\partial x^i}{\partial x^j} \right) \) is the Jacobian and we call \( T = \left( T^i_{\ j} \right) = \delta x^i \delta x^j \), then

\[ T = J^{-1} TTJ - J^{-1} dJ \]

or even sweeter

\[ T - d = J^{-1}(T - d)J \]

as an operator statement.

This is called the transformation of a \( \text{g} \)-valued connection.
A (particular) solution for $T$ is the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)$$

which are not the components of a $(2)$-tensor but to the partial derivatives of $g$.

We can add any $(l)$-tensor to $\Gamma$ and obtain a solution for $T$ transforming correctly (homogeneous solution).

When $T = \Gamma$, we call $\nabla$ the affine connection, else we say that $\nabla$ is torsionful.

Torsion can result in two evils

* Failure of parallelogram to close
* Failure of metric to be parallel.
Example: You have actually been using a torsion-free covariant derivative $\nabla\mu$ where $\nabla = dx^\mu \nabla_\mu$ for a long time without knowing it. This is because in Cartesian coordinates $\Gamma^i_{jk} = 0$.

Even in polar coordinates this is no longer so:

\[(R^2, ds^2 = dr^2 + r^2 d\theta^2)\]

\[
\begin{align*}
dr^2 &= dr \cdot g_{rr} \cdot dr + d\theta \cdot g_{\theta\theta} \cdot d\theta \\
&= \begin{cases}
g_{rr} = 1 & \Rightarrow (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \\
g_{\theta\theta} = r^2 & \end{cases}
\end{align*}
\]

Non-vanishing Christoffels:

\[
\begin{align*}
2 \Gamma^r_{r\theta\theta} &= 2 \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta} = -2r & \Rightarrow \Gamma^r_{\theta\theta} = -r \\
2 \Gamma^\theta_{r\theta} &= \partial_r g_{\theta\theta} + \partial_\theta g_{r\theta} - \partial_\theta g_{\theta\theta} = \partial_r r^2 = 2r & \Rightarrow \Gamma^\theta_{r\theta} = \frac{1}{r}
\end{align*}
\]

Let's compute the Laplacian, consider a function $f$. 
Then \( \nabla f = df \) is a 1-form. \( g^{-1}(\nabla f, \cdot) \) is a vector.

Explicitly:

\[
g^{-1}(\nabla f, \cdot) = (\frac{2}{\sqrt{r}} \frac{\partial f}{\partial r} + \frac{i}{r} \frac{\partial f}{\partial \theta})\left(\frac{df}{dr} + \frac{r}{2} \frac{df}{d\theta}\right) = \frac{2}{\sqrt{r}} \frac{df}{dr} + \frac{i}{r} \frac{df}{d\theta} \quad \text{a vector}
\]

Now \( D : \text{vectors} \to (1',1') \)-tensor

But there is a canonical method to produce a scalar from a \((1',1')\)-tensor called \( \text{trace} \)

\[
T'_{ij} dx^i \otimes \frac{\partial}{\partial x^j}, \quad \text{trace} \Rightarrow T'_{ij} \langle dx^i, \frac{\partial}{\partial x^j} \rangle = T'_{ii}
\]

For a general vector \( u \), \( \text{trace}(Du) = \text{div}(u) = D_i u^i \)

We call \( \text{div}(g^{-1}(\nabla f, \cdot)) = \Delta f = D_i D^i f \) the \text{Laplacian of } f.

Here \( \text{div}(u) = D_r u^r + D_\theta u^\theta \)

\[
= \partial_r u^r + \Gamma^2_{rk} u^k + \partial_\theta u^\theta + \Gamma_{\theta r}^\theta u^r
\]

\[
= \partial_r u^r + \partial_\theta u^\theta + \frac{i}{r} u^r
\]
But putting \((u, w) = (\vartheta + r, \frac{1}{r} \vartheta + f)\) gives

\[
\Delta f = \partial_r^2 f + \frac{1}{r^2} \partial_\vartheta^2 f + \frac{1}{r} \vartheta f
\]

\[
= \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_\vartheta^2 f
\]

The standard \underline{Laplaceian in Riemannian coordinates}

Note that \(\Delta f = \text{div} (g^{-1}(\nabla f, \cdot))\) generalizes to any tensor and is called the \underline{Bochner Laplacian}.

\[\text{LINEAR CONNIEXION / GRADIENT}\]

\[
\nabla : \Gamma (T^2 \mathcal{M}) \rightarrow \Gamma (T^2 \mathcal{M})
\]

\[
\psi \rightarrow \nabla \psi
\]

such that

\[
\text{i)} \quad \nabla (T + S) = \nabla T + \nabla S
\]

\[
\text{ii)} \quad \nabla (f T) = df \otimes T + f \nabla T
\]

\[
\text{iii)} \quad \nabla (T \otimes S) = \nabla T \otimes S + T \otimes \nabla S
\]

\[
\text{iv)} \quad \nabla \langle w, w \rangle = \langle \nabla_\nu w, w \rangle + \langle w, \nabla_\nu w \rangle
\]
If \( \{e_a\} \) and \( \{o^b\} \) are dual, but not necessarily orthonormal bases,

\[
\nabla e_a = \Gamma^c_{ab} o^b \otimes e_c
\]

where the functions \( \Gamma^c_{ab} \) are called connection coefficients.

* When \( e_a = \frac{\partial}{\partial x^a} \), coordinate vector fields, and \( \nabla \) is torsion free, \( \Gamma^c_{ab} = \Gamma^c_{ba} \), the Christoffel symbols.

* \( \Gamma^c_{ab} \) are not tensor components.