Lie algebra-valued 1-form

\[ \omega = d \log (\cdot) : T_0 G \rightarrow \mathfrak{g} \]

Example \( G = \text{GL}(n, \mathbb{R}) \) non-invertible matrices

\[ \mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) \] non matrices with bracket = matrix commutator

The mapping

\[ G \ni x \mapsto g^x(0) \in \mathbb{R}^{n^2} \]

gives \( G \) a differentiable structure

If \( X^A_B \) are the components of a vector at \( \mathfrak{g}_0 = T_0 \text{GL}(n) \) the Jacobian for left multiplication \( L_x \) is just \( g^A_B \)

i.e. The left invariant vector field with coordinates \( g^A_X \) (or in a matrix notation, just \( g X \)).

So a basis for left-invariant vector fields are the operators

\[ e^A_B = g^A_C \frac{\partial}{\partial g^B_C} \]

whose algebra is easy to calculate

\[ [e^A_B, e^C_D] = \delta^C_B e^A_D - \delta^A_D e^C_B \]
Their duals
\[ \omega^A_B \left( e^C_P \right) = \delta^A_D \delta^C_B \]
are given by
\[ \omega^A_B = (g^{-1})_C^A \, d g^C_B \]
because
\[ d g^C_B \left( \frac{\partial}{\partial x^C} \right) = \delta^C_D \delta^D_B \]
so \( g^{-1} \) removes the factor \( g \).

Often write \( \omega = g^{-1} \, d g \), the same computation works for any Lie group written as an embedding in \( GL(n, \mathbb{R}) \).

Notice, left invariance is obvious, under \( g \rightarrow h g \), \( \omega \rightarrow (h g)^{-1} d (h g) \)
\[ = g^{-1} \, h^{-1} \, h \, d g = \omega \]

Notice also that \( \omega \) obeys a "zero curvature" condition called the Maurer–Cartan equation:

Define a "covariant derivative" or "connexion"
\[ \nabla = Id + \omega \]
matrix-valued operator
Then
\[ \nabla^2 = 0 \iff d \omega + \omega \wedge \omega = 0 \]
because
\[ \nabla = d + g^{-1} (d g) = g^{-1} d g \text{ as an operator} \]
\[ \Rightarrow \nabla^2 = g^{-1} d g \, g^{-1} d g = g^{-1} d^2 g = 0 \]

Also \( \omega \) can be used to from left invariant tensors, such a metric
\[ d s^2 = \text{tr} (\omega \otimes \omega) \text{ also right-invariant} \]
\[ \Rightarrow \text{Right } G \text{-action are isometries} \]
**Lengths of Curves**

Given a curve \( \gamma : \mathbb{R} \to N \), the velocity vector
\[ v : t \to \frac{d\gamma}{dt} \in T_{\gamma(t)}N \]

i.e. \( v = \frac{d\gamma}{dt} \) viewing \( \mathbb{R} \) as a manifold.

Call the restriction \( \gamma \) to \( I = [a, b] \subset \mathbb{R} \)

a segment or a path \( \gamma \). Define the length of \( \gamma \) via
\[ L(\gamma) = \int_a^b \sqrt{g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})} \, dt \]

(assuming \( g \) is proper).

We can make \( (M, g) \) a metric space by defining
\[ d((\gamma, a), (\gamma, b)) = \inf_{\gamma} L(\gamma) \]

over piecewise differentiable segments of \( \gamma \).

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Notice that keeping \( v(\theta) \) unchanged but changing parameterizations

\[ \begin{array}{c}
\theta \\
\hline \\
I \\
\hline \\
[0,1] \\
\hline \\
\gamma \\
\hline \\
t \end{array} \]

\[ u \]

\[ \frac{dt}{du} = \frac{d\theta}{dt} \frac{dt}{du} \]

but this Jacobian factor is canceled by the change of integration variable

\[ dt = du \frac{dt}{du} \]

Le. \( L(\gamma) \) is a functional of **unparameterized paths**.

For many applications, the square root is inconvenient.
It is more convenient to study the energy integral

$$E(C) = \frac{1}{2} \int_a^b dt \, g(\dot{C}, \dot{C})$$

but $E(\gamma)$ is a functional of parameterized paths. Both $E(\gamma)$ and $\mathcal{L}(\gamma)$ are extremized by geodesics. For this we need to study variations and action principles. In fact, there is even a third action principle which gives $\mathcal{R}$ also a metric.

**Exercise:** Show $\omega^A_B = (g^{-1})^A_C \, dq^C_B$ is the Cartan Maurer-Cartan form for $G\mathfrak{g}_W$. 
Review Exercises

1. Show \( \text{div}(\text{Ric} - \frac{1}{2}g_s) = 0 \)

The tensor \( G \) is called the Einstein tensor. Look up "Einstein's Principle".

Why was the tensor \( G \) important for Einstein?