Lecture 25  Action Principles

\[ S(C) = \int dt \, L \circ \tilde{c}(t) \]

where \( \tilde{c} \) is the lift of \( C \) by \( L : TN \otimes \mathbb{R} \to \mathbb{R} \) is the Lagrangian.

Varying \( S(C) \), we found \( \frac{\delta S}{\delta C} = 0 \) when

\[ G_i = \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial x_i} = 0 \]

Components of a 2-form on \( TM \).

Euler-Lagrange equations.

Time independent.
Example. The energy

\[ L(x^i, \dot{x}^i) = \frac{1}{2} g_{ij}(x^i, x^j) \dot{x}^i \dot{x}^j \]

\[ \frac{\partial L}{\partial x^i} = \pm \delta_{ijk} \dot{x}^j \dot{x}^k \]

\[ \frac{\partial L}{\partial \dot{x}^i} = g_{ij} \dot{x}^j \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = g_{ij} \dot{x}^j + \frac{i}{\hbar} \partial_k g_{ij} \dot{x}^k \dot{x}^j = g_{ij} \ddot{x}^j + \partial_k g_{ij} \dot{x}^k \dot{x}^j \]

So the Euler-Lagrange equations say

\[ 0 = g_{ij} \ddot{x}^j + \frac{1}{\hbar} \left( \partial_k g_{ij} \dot{x}^k \dot{x}^j - \partial_j g_{ij} \dot{x}^k \dot{x}^k \right) \]

\[ \Rightarrow 0 = \ddot{x}^j + \Gamma^j_{ik} \dot{x}^i \dot{x}^k \text{ [Geodesic equation]} \]

Local requirement to minimize length.
Other action principles

The energy integral was parameterization dependent. Can we repair this?

\[ \int_I \nabla \cdot \mathbf{F} \] \[ (M, g) \]

The energy integral pulls back to \( I \) via \( dC \).

Give \( I \) its own intrinsic metric \( \gamma \) which we can always write

\[ \gamma = \delta \otimes \delta \]

where the "einbein" \( c \) is a 1-form on \( I \).

\[ c = \delta dt \]

and under changes of coordinates \( t \to t'(t) \)

\[ c' = \frac{dt'}{dt} \cdot c \]

Thus \( \int_I \delta \circ \gamma \delta dt \) is reparameterization independent:

(Example of generalized text - can integrate top form over a manifold.)
But \( \frac{d}{dt} g(\dot{c}, \dot{c}) \) is also reparameterization invariant.

Hence study

\[
S = \frac{1}{2} \int \frac{d}{dt} g(\dot{c}, \dot{c}) dt + \frac{m^2}{2} \int dt \
\]

Varying \( c \) we still get geodesic equation, but
varying \( \tilde{e} \) we obtain

\[
\frac{dS}{d\tilde{e}} = -\frac{1}{2} \frac{d}{dt} g(\dot{c}, \dot{c}) + m^2 = 0 \quad \text{Euler-Lagrange equation for } \tilde{e}.
\]

\( \Rightarrow \) \( \tilde{e} = \frac{1}{m} \int g(\dot{c}, \dot{c}) \) plugging back in \( S \) gives

\( \Rightarrow S = \frac{1}{m} \int dt \sqrt{g(\dot{c}, \dot{c})} = L(c) \) the length!

Special case \( m = 0 \) , Euler-Lagrange eqn. for \( \tilde{e} \), say

\( g(\dot{c}, \dot{c}) = 0 \) nonsense for \( g \) proper

For Lorentzian manifolds this is the light like condition!

Finally note that since \( (x) \) is reparameterization invariant, we can choose coordinates \( t \) \( \dot{t} \) such that

\( \dot{t} = \text{constant} = 1 \) (say)

\( \Rightarrow S = \frac{1}{2} \int dt g(\dot{c}, \dot{c}) + \text{constant} \) the energy integral.
Lie algebra cohomology

(Fuchs, Cohomology of
Infinite dimensional
Lie Algebras)

arXiv: 0906.4914

Let $M$ be some module, typically sections of some bundle
over $M$. Examples might include $\Omega M, \Lambda M, \ldots$

Suppose the (super) Lie algebra

\[ \mathfrak{g} = \{ g_A \}, \quad [g_A, g_B] = f_{AB}^C g_C \]

is represented by a set of operators

\[ g_A: M \rightarrow M \]

where

\[ (g_A g_B - (-)^{AB} g_B g_A - f_{AB}^C g_C) \omega = 0 \]

for $\omega \in \Omega M$. Examples include $g_A = L_A$

In vector fields on $\Omega M$ or the extrinsic derivative

\[ d \omega = 0 \quad \{ g_A \} = \{ e_i \}, \quad f_{ABC} = 0 \]
Consider the graded polynomial algebra
\[ k[A, b_A, c^A] / H \]
where \( H \) are the relations
\[
\begin{align*}
\{ b_A, b_B \} &= 0 \quad \{ \text{b, c have opposite grading} \} \\
[ b_A, c^B ] &= \delta^B_A \\
[ c^A, c^B ] &= 0
\end{align*}
\]
Claim the operator
\[ Q_j^c = c^A c^B f_{BA} c C \]
is nilpotent: \( Q_j^c \) is non-nilpotent.

You can check this using the (super) Jacobi identity for \( Q_j \) and the relations above.
Alternatively the relationship

$$g_A g_B - c^{AB} g_B g_A - f_{AB} g_c = 0$$

naturally implies an exact sequence:

Writing out any of these relationships,

$$g_1 g_2 - g_2 g_1 = g_3$$  (say)

says

$$(-g_2, g_1, -1) \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0$$

\[ \text{O} \rightarrow M \xrightarrow{Q_{m+1}} M \otimes \text{dim} \rightarrow Q_{m+1} \rightarrow M \otimes \left[ \text{dim} \right] \rightarrow \ldots \]

This complex is generated by

$$\otimes = c^A g_A + \frac{1}{2} c^A c^B f_{AB} b_c$$
Maxwell's and linearized Einstein's Equations from Lie Algebra Cohomology

The space of sections of $TM$ - differential forms $\Omega M$

or $\otimes TM$ - totally symmetric tensors $\Omega M$

are endowed with some useful operators:

Give them a unified description by viewing
differential & "symmetric" form

as (polynomial) functions of anticommuting/
commuting differentials $dx^M$

forms $dx^m dx^n = - dx^n dx^m$ [the wedge product]
symmetric forms $dx^m dx^n = + dx^n dx^m$

degree/number operator

$N = dx^m \frac{d}{(dx^m)}$ (No metric required.)
Gradient $\nabla^m \nabla

In using this is the exterior derivative $d$
for symmetric from call this operator grad
$\nabla^m \rightarrow \delta^{k+1} \nabla$
$\nabla \nabla \rightarrow \delta^{k+1} \nabla$
$\mu_1 \mu_2 \rightarrow \delta^{k+1} \nabla$

$N \nabla = \delta (N+1)
\delta \delta \delta = \frac{1}{2} d^2 = 0$

$N \nabla \nabla = \nabla \delta \delta (N+1)$

Divergence $\nabla^m \nabla$

In forms this operator is called $\delta$, the
codifferential
$\delta: \delta \nabla \rightarrow \delta^{k-1} \nabla$
The codifferential obey:

\[ N \delta = \delta N - 1 \]
\[ \delta^2 = 1 \]
\[ d \delta + \delta d = \Delta \]

where \( \Delta \) is the "form Laplacian". It is a curvature modification of the "Bochner Laplacian" \( g^{\mu \nu} \nabla_\mu \nabla_\nu \). The form Laplacian is central.

For symmetric forms, the above operator is the divergence \( \text{div} : S^{k+1} M \to S^k M \) \( w \)
\[ \xi_1 \ldots \xi_k \to \text{div} \xi_1 \xi_2 \ldots \xi_k \]

again there is a nice algebra (due to Lichnerowicz)

\[ N \text{div} = \text{div}(N - 1) \]

Bochner
\[ \nabla \]

\[ \text{div grad} - \text{grad div} = \Delta + \mathbb{R}^{###} \]

But the right hand side is generally not central. Curvature modification
For symmetric forms there are two other new operators

\[ g = \delta_{\mu \nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} : S^{k \nu} \rightarrow S^{k+2 \nu} \]

\[ \tau = \frac{\partial^2}{\partial x^\mu \partial x^\nu} : S^{k \nu} \rightarrow S^{k-2 \nu} \]

The operators \( g, \tau \) obey an sl(2) lie algebra

\[
\begin{align*}
[N, g] &= 2g \\
[N, \tau] &= -2\tau \\
[\tau, g] &= 4N + 2dimmM
\end{align*}
\]

Next time study lie algebra cohomology of these two algebras.
Review

1. We wrote an action principle to minimize lengths of curves. Study its generalization to
   (i) areas
   (ii) volumes.

2. Look up Hodge duality. In particular, find out how to map k-forms to (dim M - k)-forms when M has
   a metric ds^2.