Coordinate transformations

Just as a differentiable curve $\gamma$ through $p$ does not uniquely specify the tangent vector $v_p$, (but rather a representative of one equivalence class) neither does a set of components $v_i$:

Suppose $(U, \varphi)$ & $(V, \psi)$ are compatible charts at $p$:

Let $\varphi : M \to \mathbb{R}^n\quad \text{and} \quad \psi = \varphi \circ \varphi^{-1} \quad \tilde{\varphi} = \varphi \circ \psi^{-1}$
Observe \( \overline{f} = f \circ \Phi^{-1} \)

\[ \Rightarrow \frac{\partial \overline{f}}{\partial x^i} = \frac{\partial f}{\partial y^i} \left[ \frac{\partial y^i}{\partial x^j} \right] \]

where \( y(x) = \Phi \circ \Phi^{-1}(x) \)

Chain rule

Thus if \( \psi_\Phi \) has components \( v^i \) in chart \((u^j, \Phi)\)

Then

\[ \psi_\Phi(f) = v^i \frac{\partial f}{\partial y^i} \]

\[ = v^i \frac{\partial y^i}{\partial x^j} \frac{\partial f}{\partial x^j} \]

New components in the chart \((u^j, \Phi)\) are

\[ \underbrace{v^j = \frac{\partial y^j}{\partial x^i} v^i}_{(x)} \]

This rule is called the transformation of a contravariant vector. In many contexts the set of all components \( v^i \) with equivalence \((x)\) defined by \((x)\) is taken as the definition of a tangent vector.
We saw tangent vectors were

1. equivalence classes of curves
2. directional derivative
3. contravariant components

Another viewpoint

Let \( A \) be an algebra over a field \( F \)

We call a linear operator \( D \) a derivation of \( A \) if

\[
D(xy) = D(x)y + xD(y) \quad \forall x, y \in A
\]

We say that differentiable functions

\[
f, g : M \rightarrow \mathbb{R}
\]

have the same germ at \( p \in M \) if \( f \) and \( g \) are on an open neighborhood of \( p \) on which \( f = g \)

(Only need local knowledge of \( f \) to compute derivatives)
We can use "having the same germ" to define an equivalence relation on differentiable functions.

* The equivalence class of \( f : M \to \mathbb{R} \) is called the germ of \( f \) at \( p \).

* The set of all germs forms an algebra because we can multiply & add functions.

We can now define tangent vectors as derivations on the algebra of germs of differentiable functions at \( p \).

**Tangent Vector Space**

**Summary:** A tangent vector \( v_p \) is a linear functional \( \mathcal{D}(p) \to \mathbb{R} \) subject to

1. \( v_p(\alpha f + \beta g) = \alpha v_p(f) + \beta v_p(g) \)

\( f, g \in \mathcal{D}(p); \alpha, \beta \in \mathbb{R} \)

2. \( v_p(fg) = v_p(f)g + f v_p(g) \)
The space $T_p(M)$ of tangent vectors to $M$ at $p$ is called the tangent vector space.

A manifold equipped with tangent spaces at all $p \in M$ is called the tangent bundle.

The natural basis for $T_pM$

Given a coordinate system $x^i$ we can construct differentiable curves $\gamma^j$ only 1st slot non-vanishing

$$\phi_\gamma \circ \gamma^j : t \rightarrow (0, \ldots, t, \ldots) \quad \text{i.e.} \quad x^i(t) = \delta^i_j t$$

$$\Rightarrow \frac{d}{dt}(\phi_\gamma) = \frac{\partial \phi_\gamma}{\partial x^i} \frac{d(x^i)}{dt}$$

$$= \frac{\partial}{\partial x^i} \delta^i_j$$

We denote the natural vectors/coordinate vectors by $e^i_j = \frac{\partial}{\partial x^j}$ with components $e^i_j = \delta^i_j$

They form a basis for $T_pM$
Example \( \mathbb{R}^2 \ni \varphi \to (x, y) \)

coordinate vectors are \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \).

Polar coordinates: new chart \((\mathbb{R}^2, \varphi)\)

transition map

\[ \varphi \circ \psi^{-1} : (x, y) \to \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan \frac{y}{x} \end{cases} \]

\[ \frac{d}{dx} = \frac{\partial r}{\partial x} \frac{d}{dr} + \frac{\partial \theta}{\partial x} \frac{d}{d \theta} \]

\[ = \frac{x}{r} \frac{d}{dr} - \frac{y}{r^2} \frac{d}{d \theta} \]

\[ = \frac{1}{r} \left( \begin{array}{c} x \\ -y/r \end{array} \right) \left( \begin{array}{c} \frac{1}{r} \\ 0 \end{array} \right) = \frac{1}{r} \left( x \frac{d}{dx} + y \frac{d}{dy} \right) \]

\[ \frac{\partial}{\partial \theta} = x \frac{d}{dy} - y \frac{d}{dx} \]
The space of vectors at $p$ is represented by the vector space $\mathbb{R}^n \ni \mathbf{v}_i$.

Notice the above formula says

$$\mathbf{v}_p (\mathbf{t}) = \sum_i \mathbf{v}_i \frac{\partial \mathbf{t}}{\partial x^i}$$

and vector gradient is the $\mathbb{R}^n$ directional derivative.

Einstein summation convention

In nearly all our formulae, we can unambiguously discard the summation sign when repeated upper & lower indices appear so

$$\mathbf{v}_i \frac{\partial}{\partial x^i} \equiv \sum_{i=1}^n \mathbf{v}_i \frac{\partial}{\partial x^i}$$
Coordinate transformations

Just as a differentiable curve \( \gamma \) through \( p \) does not uniquely specify the tangent vector \( v_p \), (but rather a representative of one equivalence class) neither does a set of components \( v^i \):

Suppose \( (U, \phi) \) & \( (V, \psi) \) are compatible charts at \( p \):

\[
\begin{align*}
\phi &: \mathbb{R}^n \\
\psi &: \mathbb{R}^n
\end{align*}
\]

Let

\[ f : M \rightarrow \mathbb{R} \]

and

\[ \overline{f} = f \circ \phi^{-1} \quad \widetilde{f} = f \circ \psi^{-1} \]
Observe \( \overline{F} = \overline{F} \circ \psi \circ \varphi^{-1} \)

\[
\Rightarrow \frac{\partial \overline{F}}{\partial x^i} = \frac{\partial}{\partial x^i} \left[ \overline{F}(y^j(x)) \right]
\]

where \( y^j(x) = \psi \circ \varphi^{-1}(x) \)

Thus if \( U^p \) has components \( U^i \) in chart \( (u^i, \varphi) \)

then

\[
U^p(f) = U^i \frac{\partial \overline{F}}{\partial x^i} = U^i \frac{\partial y^j}{\partial x^i} \frac{\partial \overline{F}}{\partial y^j}
\]

New components in the chart \( (u^i, \varphi) \) are

\[
U^j = \frac{\partial y^j}{\partial x^i} U^i \quad \text{--- (x)}
\]

This rule is called the transformation of a contravariant vector. In many contexts the set of all components \( U^i \) with equivalence \((x)\) defined by \((x)\) is taken as the definition of a tangent vector.