Example \[ \varphi : \mathbb{R}^2 \rightarrow (x, y) \]
\[ \varphi^{-1} : (x, y) \rightarrow (r, \theta) \]
\[ \psi \circ \varphi^{-1} : (x, y) \rightarrow (\sqrt{x^2+y^2}, \arctan \frac{y}{x}) \]

Relation between coordinate vectors (Jacobian)

\[ \frac{r}{\partial r} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{dilations} \]
\[ \frac{\partial}{\partial \theta} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \quad \text{rotations} \]

These operators are defined at all \( P \in \mathbb{R}^2 \), so are vector fields not just vectors.

Algebra of vector fields on \( \mathbb{R}^n \)

\[ \mathbb{R}^n \xrightarrow{\varphi} (x', \ldots, x^n) \]

Coordinate vectors \( e_i = \frac{2}{\partial x^i} \)

First moments \( g^i_j = x^i \frac{\partial}{\partial x^j} \)
These differential operators close under commutation

\[
\begin{align*}
[e_i, e_j] &= [e_i, e_j] = 0 \\
g^i e_k - e_k g^i &= [g^i, e_k] = \delta^i_k e_k \\
g^i g^j - g^j g^i &= [g^i, g^j] = \delta^i_j g^j - \delta^j_i g^i 
\end{align*}
\]

This is an example of the Lie bracket and a Lie Algebra \((\mathbb{R}^n \ltimes \mathfrak{gl}(n)).\) More later.

**The Differential \(d\psi\)**

**Idea** \(\psi : M \rightarrow N\) differentiable

\[d\psi : T_p M \rightarrow T_{\psi(p)} N\]
The differential $d\varphi$ is defined by

$$\frac{d}{dt} \bigg|_0 = \gamma'(0) \quad \overset{\text{along } \mathcal{N}}{\longrightarrow} \quad \frac{d}{dt} \bigg|_0 = (\varphi \circ \gamma)'(0) \quad \overset{\text{along } \mathcal{T}_p \mathcal{M}}{\longrightarrow} \quad (\varphi \circ \gamma)'(0)$$

Indeed

$$d\varphi(v_p)(\gamma(t)) = v_p(\varphi \circ \gamma)$$

$\frac{d}{dt} (\varphi \circ \gamma)$

directional derivative of $\varphi \circ \gamma : \mathcal{N} \to \mathbb{R}$ along $\gamma$ in $\mathcal{N}$

$\frac{d}{dt} (\varphi \circ \gamma)$

directional derivative of $\varphi \circ \gamma : \mathcal{M} \to \mathbb{R}$ along $\gamma$ in $\mathcal{M}$
The Jacobian of \( \varphi \) Consider

\[
\begin{align*}
M & \xrightarrow{\varphi} N \\
(x_1, \ldots, x_m) & \mapsto (y_1, \ldots, y^n)
\end{align*}
\]

i.e. \( y(n) = \varphi \circ \varphi \circ \varphi^{-1}(x) \)

\( T_p M \) has basis \( \left\{ \frac{\partial}{\partial x_1} \bigg|_p, \ldots, \frac{\partial}{\partial x_m} \bigg|_p \right\} \)

\( T_{\varphi(p)} N \) has basis \( \left\{ \frac{\partial}{\partial y_1} \bigg|_{\varphi(p)}, \ldots, \frac{\partial}{\partial y_n} \bigg|_{\varphi(p)} \right\} \)

The Jacobian of \( \varphi \) is the \( mxn \) matrix representing \( dy \) in these bases.
\[ J = (J_i^a) \quad i = 1, \ldots, m \quad a = 1, \ldots, n \quad \text{where} \]

\[ d\varphi \left( \frac{\partial}{\partial x} \right) = J_i^a \frac{\partial}{\partial y^a} \left| \varphi(\mathbf{y}) \right| \]

\text{Compute } J_i^a

\[ d\varphi \left( \frac{\partial}{\partial x} \right) \mid_{\varphi^{-1}} (\mathbf{x}) = J_i^a \frac{\partial}{\partial y^a} \left( \frac{\varphi \circ \varphi^{-1}}{\varphi(\mathbf{y})} \right) \]

\[ = \frac{\partial}{\partial x} \left( \frac{\varphi \circ \varphi^{-1}}{\varphi^{-1}(\mathbf{x})} \right) \frac{\partial}{\varphi^{-1}(\mathbf{x})} \]

\[ = \frac{\partial}{\partial x} \varphi \big( \varphi^{-1}(\mathbf{x}) \big) \]

\[ = \frac{\partial}{\partial x} \varphi \left( \mathbf{y}(\mathbf{x}) \right) \]

chain rule

\[ S = \frac{\partial}{\partial y^a} \left( \varphi \circ \varphi \circ \varphi^{-1} \right)^a \]

\[ J_i^a = \frac{\partial}{\partial x} \left( \varphi \circ \varphi \circ \varphi^{-1} \right)^a = \frac{\partial y^a}{\partial x^i} \]
1-forms & the Cotangent space

Let \( f: M \to \mathbb{R} \) be differentiable.

Then \( df: T_p M \to T_{f(p)} \mathbb{R} \cong \mathbb{R} \) is a linear functional on the vector space \( T_p M \).

The set of all linear functionals \( \{ T_p M \to \mathbb{R} \} = T_p^* M \) the dual space to \( T_p M \). It is called the cotangent space.

Coordinate differentials

Let \( \varphi_\alpha: P \to (x^1, \ldots, x^n)(p) \) be a chart map at \( p \).

We call \( x^i(p) \) a coordinate function \( x^i: U_\alpha \to \mathbb{R} \) regarding \( x^i \) as a differentiable map from manifold \( U_\alpha \to \mathbb{R} \) we may consider \( dx^i \).
In fact, the coordinate differentials $dx^i$ are dual to the coordinate vectors $\frac{\partial}{\partial x^i}$:

$$dx^i \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$$

and form a basis for $T^*_pM$. 
Exercises

1. Show that if \( f : M \to \mathbb{R} \)

\[
df \left( \frac{\partial}{\partial x^i} \big|_p \right) = \frac{\partial f}{\partial x^i} \big|_p
\]

i.e. the components of the gradient of \( f \).

2. Deduce the transformation properties of the components \( w_i \) of a 1-form \( \omega = w_i \, dx^i \in T^*_p M \) under changes of coordinates. Use this result to show \( \sum_i w_i \, \xi^i \) (where \( \xi^i \) are the components of a vector at \( p \)) is invariant.
Observe \( \tilde{f} = f \circ \psi \circ \varphi^{-1} \)

\[ \Rightarrow \frac{\partial \tilde{f}}{\partial x^i} = \frac{\partial}{\partial \tilde{y}^j} \left[ \tilde{f}(\psi(y(x))) \right] \]

\[ S = \frac{\partial \tilde{f}}{\partial \tilde{y}^j} \frac{\partial \tilde{y}^j}{\partial x^i} \]

**Chain rule**

Thus if \( \psi_\alpha \) has components \( \psi^i \) in chart \((u \cup v, \varphi)\)

then

\[ \psi_\alpha(f) = \psi^i \frac{\partial \tilde{f}}{\partial \tilde{y}^i} \]

\[ = \psi^i \frac{\partial \tilde{y}^j}{\partial x^i} \frac{\partial \tilde{f}}{\partial \tilde{y}^j} \]

New components in the chart \((u \cup v, \Psi)\) are

\[ \psi^i = \frac{\partial \tilde{y}^i}{\partial x^i} \psi^j \]

---

This rule is called the **transformation of a contravariant vector**. In many contexts the set of all components \( \psi^i \) with equivalence \((*)\) defined by \((*)\) is taken as the definition of a tangent vector.