**Lecture #6**

**Diffeomorphisms**

**Theorem.** If \( \varphi : M \rightarrow N \) is a local diffeomorphism at \( P \), then
\[
d\varphi : T_P M \rightarrow T_{\varphi(P)} N
\]
is a vector space isomorphism.

Conversely, if \( d\varphi : T_P M \rightarrow T_{\varphi(P)} N \) is an isomorphism
for some differentiable map at \( P \),
then \( \varphi \) is a local diffeomorphism at \( P \).

For the converse apply the inverse function theorem to the differentiable map
\[
\tilde{\varphi} : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{via} \quad \tilde{\varphi} = \varphi \circ \Phi \circ \Psi
\]
where \((\Psi, \Phi, \Psi)\) is a chart at \( P \) and \( \Phi(P) \)
and use the fact that the Jacobian is the matrix
of the map \( d\varphi \) in the coordinate basis.

For the forward direction, \( d\varphi^{-1} \) is easily checked
to be the required inverse \( \Phi^{-1} \).

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**The Tangent Bundle is a Manifold**

Let \( TM = \{(P, V_P) : P \in M, V_P \in T_PM\} \)

We call \( TM \) the tangent bundle. To see
that \( TM \) is a manifold we need to build
an atlas. Start with an atlas \( \{(U_a, (x_1, x_2, \ldots, x_n))\} \)
for \( M \). At each \( P \in M \), \( T_PM \) has a basis \( \left\{ \frac{\partial}{\partial x^i} \right\} \).

Take as chart map
\[
\psi_a = (P, V_P = \left( \frac{\partial}{\partial x^i} \right)_P) \rightarrow (x^1, \ldots, x^n; v_1, \ldots, v^n)
\]

\[V_a = \{(P, V_P) : P \in U_a, V_P \in T_PM\} \subseteq \mathbb{R}^{2n}\]

From chart to chart \( \psi_i^a \) is a
linear transformation
\[
\text{Clearly differentiable.}
\]

The Jacobian of a diffeomorphism \( \mathbb{R}^n \rightarrow \mathbb{R}^n \)
Vector Fields

"Sections of the tangent bundle.

A vector field is a map

\[ X: M \rightarrow TM \]

\[ \mu \]

\[ P \rightarrow (p, v_p) \]

and say \( X \) is differentiable if it is differentiable as a map between manifolds.

In a chart \( \varphi = (x^1, \ldots, x^n) \)

\[ X(p) = X^i(p) \frac{\partial}{\partial x^i}(p) \]

and differentiability of \( X \) requires differentiability of the component functions

\[ p \rightarrow X^i(p) \]

Notice that a vector field defines a mapping

\[ X: \mathbb{R} \rightarrow \mathcal{F}_M \] via \( (Xf)(p) = X^i(p) \frac{\partial f}{\partial x^i}(p) \)

the directional derivative of \( f \) in the direction of \( X \) at \( p \).

Suppose \( X, Y \) are differentiable vector fields and \( f: M \rightarrow \mathbb{R} \) is \( C^\infty \) then \( X(f) \) and \( Y(f) \) are both differentiable functions. Hence we may consider \( X(Y(f)) \).
Calculate in a chart

\[
X(YH) = X(\frac{\partial Y}{\partial x})
\]

\[
= X\frac{\partial}{\partial x} \left( Y\frac{\partial}{\partial x} \right)
\]

\[
= X\frac{\partial}{\partial x} Y + X\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right)
\]

\[
\Rightarrow (XY - YX) = [X, Y]
\]

\[
= (x\frac{\partial}{\partial x} Y - y\frac{\partial}{\partial y} x) \frac{\partial}{\partial x}
\]

is a vector field. The vector field \([X, Y]\)

is called the (Lie) bracket of \(X\) and \(Y\).

It is differentiable when \(X\) and \(Y\) are.

Properties of \([\cdot, \cdot]\) : \(\Lambda^2 \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)\)

(i) \([X, Y] = -[Y, X]\) antisymmetry

(ii) \([aX + bY, Z] = a[X, Z] + b[Y, Z]\) linear

(iii) \([X, Y], Z] + cyclic = 0\) Jacobi

(iv) \([fX, gY] = f g [X, Y] + fX(g) Y - gX(f) X\)

Leibnitz

Given a vector field \(X\)

we can search for integral curves such

that for any \(p \in \mathfrak{X},\) \(\nu_p = \phi'(t)\) where \(\phi(t) = p\)
To solve this problem we must study a system of ODE: In coordinates \( x^i \)
\[
X = x^i \frac{\partial}{\partial x^i} \quad \text{and} \quad x^i(0) = \phi^i \circ \tau(\sigma^i)
\]

so components are
\[
x_i(x(0)) = \frac{dx^i(t)}{dt}
\]

with some initial condition, the start of the integral curve \( x^i(0) = \phi^i \circ \tau(\sigma^i) \).

Small time existence of solutions to the system \( X^i = \dot{x}^i \) is guaranteed by Picard's theorem.

The curves \( \tau_t(\sigma) \) allow us to define a Lie derivative
\[
(\mathcal{L}_X \tau)(\sigma) = \lim_{t \to 0} \frac{\tau(t) \circ \tau(\sigma) - \tau(\sigma)}{t}
\]

Exercises
1. Find out what it means for a manifold to be orientable.
   Give examples of an orientable and a non-orientable differentiable structure.
2. Check the properties of the Lie bracket.