Question 1 (Internal direct sum/product) Show that if a ring $R = J_1 + \cdots + J_n$ and the ideals $J_i \cong R_i$ for rings $R_i$ and $J_i \cap \sum_{j \neq i} J_j = \{0\}$, then $R \cong \prod_{i=1}^n R_i$.

Solution Define $\varphi : \prod_{i=1}^n J_i = J_1 \times J_2 \times \cdots \times J_n \longrightarrow R$ by

$\{a_i\} = (a_1, a_2, \ldots, a_n) \longmapsto \sum a_i = a_1 + a_2 + \cdots + a_n$

and let $a_i, b_i \in J_i$ for $1 \leq i \leq n$.

(i) $\varphi$ preserves additive structure: $\varphi(\{a_i\} + \{b_i\}) = \varphi(\{a_i + b_i\}) = \sum (a_i + b_i) = \sum a_i + \sum b_i = \varphi(\{a_i\}) + \varphi(\{b_i\})$.

(ii) $\varphi$ preserves multiplicative structure: $\varphi(\{a_i\}\{b_i\}) = \varphi(\{a_ib_i\}) = \sum a_i b_i = \sum a_i \sum b_i = \varphi(\{a_i\}) \varphi(\{b_i\})$, where the third equality follows from $a_i b_j \in J_i \cap J_j = \{0\}$ whenever $i \neq j$.

(iii) $\varphi$ is injective: Suppose $\varphi(\{a_i\}) = \sum a_i = 0$. Then, for each $i$, we have $a_i = -\sum_{j \neq i} a_j \in J_i \cap \sum_{j \neq i} J_j = \{0\}$ and therefore $a_i = 0$.

(iv) $\varphi$ is surjective: This is obvious from $R = J_1 + \cdots + J_n$.

From (i)-(iv), we conclude $\varphi$ is an isomorphism of rings and $R \cong \prod_{i=1}^n J_i$. (cf. Theorem 9 [p171, Dummit & Foote])

Question 2 Look up Zorn’s lemma on partially ordered sets (e.g. Dummit and Foote Appendix I). Let $R$ be a ring with 1 and at least one proper ideal (an ideal other than $R$ or $\{0\}$). Now prove that $R$ has a maximal proper ideal (i.e. not contained in any other proper ideal).

Solution The proof goes exactly the same as that of Proposition 11 [p254, Dummit & Foote].

Question 3 Frobenius homomorphism Let $R$ be a commutative ring with identity and prime characteristic $p$. Show that the map

$\varphi : R \longrightarrow R, \quad r \longmapsto r^p$,

1Hint: study “chains” in $\{A | I \subseteq A \subseteq R\}$ for some proper ideal $I$. 

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is a ring homomorphism.

**Solution** See Proposition 35 [p548, Dummit & Foote], whose proof actually works for a commutative ring with identity of characteristic $p$. The field axiom is used only when to show injectivity.

**Question 4** Find all ideals in $\mathbb{Z}$.

**Solution** See Examples (2) [p243, Dummit & Foote] and Examples (2) [p252, Dummit & Foote]. Recall also Theorem 7 [p58, Dummit & Foote].