250A Homework 2

Solution by Jaejeong Lee

Question 1 Let $G$ be a group and suppose $x^2 = 1 \forall x \in G$. Prove that $G$ is abelian.

Solution Let $a, b \in G$. By assumption, we have

$$aba^{-1}b^{-1} = aba^{-1}b^{-1} (ba)^2 = ab^2a = a^2 = 1,$$

that is, $ab = ba$. Thus, $G$ is abelian.

Question 2 Let $G = \{x_1, \ldots, x_n\}$ be a finite abelian group. Prove

$$(x_1 \cdots x_n)^2 = 1.$$ 

Solution Note that the map $I : G \to G$ defined by $I(g) = g^{-1}$ is a bijection. (It is an isomorphism in case $G$ is abelian.) Since $G$ is abelian, we have

$$(x_1 \cdots x_n)^2 = (x_1 \cdots x_n)(I(x_1) \cdots I(x_n))$$

$$= (x_1 I(x_1)) \cdots (x_n I(x_n))$$

$$= (x_1 x_1^{-1}) \cdots (x_n x_n^{-1})$$

$$= 1.$$ 

Question 3 Find all groups of order 7 or less.

Solution Groups of prime order are cyclic by Corollary 10 [p90, Dummit & Foote]. Suppose $|G| = 4$. If $G$ has an element of order 4, then $G$ is isomorphic to $Z_4$. If there is no element of order 4 in $G$, then every nontrivial elements of $G$ have order 2 and $G$ is isomorphic to the Klein 4-group $V_4$. Now suppose $|G| = 6$. As before, $G$ is isomorphic to $Z_6$ or $G$ has an element $r$ of order 3. (If every nontrivial elements of $G$ have order 2, consider $\langle a \rangle$ for some $1 \neq a \in G$. By Question 1, $G$ is abelian and thus $\langle a \rangle$ is normal in $G$. Now the quotient group $G/\langle a \rangle$ is of order 3 and hence cyclic, but it has no order 3 element; a contradiction!) Consider $\langle r \rangle$, which is normal being of index 2, and let $m \notin \langle r \rangle$. Since $G/\langle r \rangle = \{\langle r \rangle, m \langle r \rangle\} = \{\langle r \rangle, \langle r \rangle m\}$, $m$ cannot be of order 3, that is, $|m| = 2$, 

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and we have (1) \(mr = rm\) or (2) \(mr = r^2m\): (1) \(G\) is abelian and is isomorphic to \(Z_2 \times Z_3 \simeq Z_6\), (2) \(G = \langle r, m \mid r^3 = m^2 = 1, mrm^{-1} = r^{-1} \rangle \simeq S_3 \simeq D_3\). In summary, groups of order 7 or less are: \(\{1\}, Z_2, Z_3, Z_4, V_4, Z_5, Z_6, S_3, Z_7\).

**Question 4** Let \(\mathbb{R}^+\) and \(\mathbb{R}^*\) be the group of real numbers under addition and non-zero real numbers under multiplication, respectively. Show that these groups are not isomorphic.

**Solution** Every nontrivial element of \(\mathbb{R}^+\) is of infinite order. On the other hand, \(\mathbb{R}^*\) has a nontrivial element of order 2, namely, \(-1\). Therefore, these two groups cannot be isomorphic.

**Question 5** Show that the set of \(n\)th roots of unity \(G = \{z \in \mathbb{C} \mid z^n = 1, n \in \mathbb{N}\}\) is a group under multiplication but not addition.

**Solution** Let \(z, w \in G\). Then \(z^m = w^n = 1\) for some \(m, n \in \mathbb{N}\). Since
\[
(zw^{-1})^m = (z^m)(w^{-1})^{-m} = 1,
\]
we see \(zw^{-1} \in G\). By Proposition 1 [p47, Dummit & Foote], \((G, \cdot)\) is a subgroup of the group \(\mathbb{C}^*\) of non-zero complex numbers. In particular, \((G, \cdot)\) is a group in itself. On the other hand, \((G, +)\) is not a group because it is not closed under addition; \(1 + 1 \notin G\).

**Question 6** Let \(x \in G\) be a group. Show that \(|x| = n < \infty\) implies that \(1, x, x^2, \ldots, x^{n-1}\) are distinct.

**Solution** Suppose on the contrary that \(x^i = x^j\) for some \(0 \leq i < j \leq n - 1\). Then \(x^{j-i} = 1\); a contradiction since \(0 < j - i < n\).

**Question 7** Suppose \(\mathcal{N}\) is a nilpotent operator on a vector space \(V\). In this case \(V\) is a direct sum of cyclic subspaces. Show that the number of summands is \(\dim \ker \mathcal{N}\).

**Solution** Let \(V = V_1 \oplus V_2 \oplus \cdots \oplus V_k\) be a decomposition of \(V\) into a direct sum of cyclic subspaces of \(\mathcal{N}\). If we denote \(V_i = \langle e_i, \mathcal{N}e_i, \mathcal{N}^2e_i, \ldots, \mathcal{N}^{m_i-1}e_i \rangle\) for
1 \leq i \leq k, then \ker(\mathcal{N}|_{V_i}) = (\mathcal{N}^{m_i-1}e_i), so \dim \ker(\mathcal{N}|_{V_i}) = 1. Since

\begin{align*}
\ker \mathcal{N} &= \ker \mathcal{N} \cap V \\
&= \ker \mathcal{N} \cap (V_1 \oplus \cdots \oplus V_k) \\
&= (\ker \mathcal{N} \cap V_1) \oplus \cdots \oplus (\ker \mathcal{N} \cap V_k) \\
&= \ker(\mathcal{N}|_{V_1}) \oplus \cdots \oplus \ker(\mathcal{N}|_{V_k}),
\end{align*}

we have \dim \ker \mathcal{N} = k.