250A Homework 4

Solution by Jaejeong Lee

Question 1 Find all quotient groups for D_8. Draw the lattice diagram and indicate which subgroups are normal. Also, compute and compare all composition series of D_8. The same for S_4.

Solution Let $D_8 = \langle r, s \mid r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle$ be the dihedral group of order 8. The lattice of subgroups of D_8 is given on [p69, Dummit & Foote]. All order 4 subgroups and $\langle r^2 \rangle$ are normal. Thus all quotient groups of D_8 over order 4 normal subgroups are isomorphic to $\mathbb{Z}_2$ and $D_8/\langle r^2 \rangle = \{1\{1, r^2\}, r\{1, r^2\}, s\{1, r^2\}, rs\{1, r^2\}\} \simeq D_4 \simeq V_4$. All possible series of subgroups of length 3, e.g. $1 < \langle r^2 s \rangle < \langle s, r^2 \rangle < D_8$, give rise to composition series in which each factors are isomorphic to $\mathbb{Z}_2$.

A_4 is the only order 12 subgroup of S_4 (being the only normal subgroup of order 12 by Homework 3). To find all order 8 subgroups, which are Sylow 2-subgroups of S_4, let $(r, s) = ((1234), (24)), ((1243), (23))$or $((1324), (34))$. Then $\langle r, s \rangle$ is an order 8 subgroup of S_4 which is isomorphic to D_8. The third Sylow theorem [p139, Dummit & Foote] says the number of Sylow 2-subgroups of S_4 is 1 or 3, but we have already found three such groups and thus they are all order 8 subgroups of S_4. Since there is no order 6 element in S_4, all order 6 subgroups of S_4 must be isomorphic to S_3, hence $S_{\{1,2,3\}}, S_{\{1,2,4\}}, S_{\{1,3,4\}}$, and $S_{\{2,3,4\}}$. Subgroups of order 4 are

$$\langle (1234) \rangle, \langle (1243) \rangle, \langle (1324) \rangle,$$
$$\{(1), (12), (34), (12)(34)\}, \{(1), (13), (24), (13)(24)\}, \{(1), (14), (23), (14)(23)\},$$
$$\text{and } \{(1), (12)(34), (13)(24), (14)(23)\}.$$

Subgroups of order 3 are $\langle (123) \rangle$,$\langle (124) \rangle$,$\langle (134) \rangle$ and $\langle (234) \rangle$. Finally, subgroups of order 2 are

$$\langle (12) \rangle, \langle (13) \rangle, \langle (14) \rangle, \langle (23) \rangle, \langle (24) \rangle, \langle (34) \rangle,$$
$$\langle (12)(34) \rangle, \langle (13)(24) \rangle, \text{and } \langle (14)(23) \rangle.$$

Of course, $S_4/A_4 \simeq \mathbb{Z}_2$. Let $V = \{(1), (12)(34), (13)(24), (14)(23)\}$. Then $S_4/V = \{(1)V, (12)V, (13)V, (23)V, (123)V, (132)V\} \simeq S_3$. All composition series of S_4 are of the form

$$1 < \langle (12)(34) \rangle < V < A_4 < S_4$$

with factors $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2$.

Question 2 Show that $H \trianglelefteq G$ and $K \trianglelefteq G \Rightarrow H \cap K \trianglelefteq G$ and $HK \trianglelefteq G$. Suppose the normal subgroups K and H are both maximal in G, what does this imply for their product KH?

Solution It is clear that $H \cap K$ is normal in G. By Corollary 15 [p94, Dummit & Foote], HK is a subgroup of G. Let $h \in H$, $k \in K$ and $g \in G$. Since $g(hk)g^{-1} = (ghg^{-1})(gkg^{-1}) \in HK$, HK is normal in G, too. Suppose K and H are two different maximal normal subgroups of G. Since $K \lneq KH$ and $KH \trianglelefteq G$, we must have $KH = G$ by the maximality of K.

Question 3 If $N \trianglelefteq G$ and G/N is an infinite cyclic group, prove that G has a normal subgroup of index n $\forall n \in \mathbb{N}$.

Solution We use Theorem 20 [p99, Dummit & Foote]. Since G/N is infinite cyclic, it has a subgroup $\bar{A}$ of index n for all $n \in \mathbb{N}$ and $\bar{A}$ is of the form A/N for some subgroup A of G containing N. By (2), $[G : A] = [G/N : A/N] = n$. Since G/N is abelian, $A/N \trianglelefteq G/N$ and by (5), $A \trianglelefteq G$.

Question 4 Show that Z_8 has Z_2 and Z_4 as homomorphic images. What can you say about the homomorphic images of cyclic groups? What about abelian groups?

Solution Let A be an abelian group and $\varphi : A \to G$ an epimorphism for some G. By the first isomorphism theorem, $G \simeq A/\ker\phi$ and $\ker\phi$ is a (normal) subgroup of A. On the other hand, for every subgroup N of A, N is normal in A (since A is abelian) and there is a canonical epimorphism $\pi : A \to A/N$. In conclusion, homomorphic images of A are completely determined by subgroups of A. In particular, if A is infinite cyclic, then its subgroups are trivial or infinite cyclic of index $n \geq 1$ and thus its homomorphic images are isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$. If $A \simeq \mathbb{Z}_m$, then its subgroups are isomorphic to $\mathbb{Z}_k$ for $k|m$ and its homomorphic images are isomorphic to $\mathbb{Z}_{m/k}$. For example, let $\mathbb{Z}_8 = \langle a \mid a^8 = 1\rangle$. Then $\mathbb{Z}_8/\langle a^2\rangle \simeq \mathbb{Z}_2$ and $\mathbb{Z}_8/\langle a^4\rangle \simeq \mathbb{Z}_4$.

Question 5 Decompose A_4 and A_5 into conjugacy classes. (Hint, what happens to the cyclic form of a permutation under conjugation?)

Solution The conjugacy classes of A_4 have orders 1,4,4 and 3 with representatives

$$(1), (123), (132) \text{ and } (12)(34),$$

respectively. One easily verifies that the conjugacy class of (123) is

$$\{(123), (214), (341), (432)\}$$

and the conjugacy class of (132) is

$$\{(132), (241), (314), (423)\}.$$

As shown in the proof of Theorem 12 [p128, Dummit & Foote], the conjugacy classes of A_5 have orders 1,20,15,12 and 12 with representatives

$$(1), (123), (12)(34), (12345) \text{ and } (13524),$$

respectively. One checks that the conjugacy class of (12345) is

$$\begin{array}{c}\{(12345), (15432), (13254), (14523), (42513), (52431), \\ (21354), (45312), (35142), (53241), (21435), (34125)\}\end{array}$$

and the conjugacy class of (13524) is

$$\begin{array}{c}\{(13524), (14253), (12435), (15342), (45321), (54123), \\ (23415), (43251), (31254), (52134), (24513), (31542)\}.\end{array}$$

Question 6 Construct explicit examples of both 1st and 2nd isomorphism theorems.

Solution $Gl(n, \mathbb{R})/Sl(n, \mathbb{R}) = Gl(n, \mathbb{R})/\ker(\det) \simeq \mathbb{R}^{\times}$, where $\det : Gl(n, \mathbb{R}) \to \mathbb{R}^{\times}$ is an epimorphism. Let $G = S_4$, $A = \langle (1234), (24) \rangle$ and $B = A_4$. Then $AB = G$ and $A \cap B = V = \{(1), (12)(34), (13)(24), (14)(23)\}$. Of course, $S_4/A_4 \simeq A/V \simeq \mathbb{Z}_2$.