A Distance for Circular Heegaard Splittings

By

KEVIN MATTHEW LAMB

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Approved:

Abigail Thompson, Chair

Francisco Javier Arsuaga

Joel Hass

Committee in Charge

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To my family and friends,
whose endless encouragement, patience, and understanding
were indispensible in making this dream a reality.

“If you want to build a ship, don’t herd people together to collect wood and don’t assign them tasks and work, but rather teach them to long for the endless immensity of the sea.”

- Antoine de Saint-Exupery

“The clock on the clubhouse wall says it’s time for fun with math!”

- James R. Mueller

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Abstract

For a knot $K \subset S^3$, its exterior $E(K) = S^3 \backslash \eta(K)$ has a singular foliation by Seifert surfaces of K derived from a circle-valued Morse function $f\colon E(K) \to S^1$. When f is self-indexing and has no critical points of index 0 or 3, the regular levels that separate the index-1 and index-2 critical points decompose $E(K)$ into a pair of compression bodies. We call such a decomposition a *circular Heegaard splitting* of $E(K)$. We define the notion of *circular distance* (similar to Hempel distance) for this class of Heegaard splitting and show that it can be bounded under certain circumstances. Specifically, if the circular distance of a circular Heegaard splitting is too large: (1) $E(K)$ can't contain low-genus incompressible surfaces, (2) a minimal-genus Seifert surface for K is unique up to isotopy, and (3) the tunnel number of K must be large.

Acknowledgements

I believe that my character and achievements are the sum total of internal and external forces, no matter how small, for good or bad. Thank you to everyone - teachers, friends, family, terrible influences - for being the nudges in my life that added up to make me who I am and brought me to this point. I couldn't have done it without you. There are a few names that deserve special recognition and thanks:

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If the above names deserve special recognition, then the next few names should get an award to go with my gratitude. Were it not for any one of them, my dream may not have become a reality.

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There is no love or support that parallels that of one's parents. There are no words to describe it, nor is there a way to express the gratitude that it deserves. It is unwavering and inexplicable, and yet it is always needed and welcome. To my parents, Steve and Teri Lamb, I can only tell you

how much your love, support, patience, and understanding mean to me; and that will never do any of it justice. These margins cannot contain my gratitude and love for you.

The final remarks belong to my wife, Kellan. I could just as easily go on about your undying love, support, patience, and understanding. This journey has been an ebb and flow of the tide, but you have always kept me anchored - careful not to let me slip too far adrift. I've doubted myself countless times, but you've never given up on me. Moreover, you've reminded me of the good I've relished, the bad I've conquered, and the future that awaits us. For all the times that I've struggled, I've never had to tell you what I needed. You've always known - a smile to brighten my day, a tug on the arm to go do something fun, the right words at the right times. It's not what you do that I need, but rather who you are. Thank you for being what I need when I need it. I love you.

You all hold a special place in my heart. Because you have all built some small piece of me, this is your achievement as well. We did it!

- Kevin Lamb, June 2018

CHAPTER 1

Introduction

In the study of classical knot theory, it is often the case that questions about a knot can be translated into questions about a 3-manifold (and vice-versa). Hence, we can study knots with tools from the theory of 3-manifolds. In particular, for a knot K in a 3-manifold M, we study its *complement* $C(K) = M \backslash K$ or its (compact) *exterior* $E(K) = M \backslash \eta(K)$, where $\eta(K)$ is an open regular neighborhood of K in M. The classification of 3-manifolds is a central problem in low-dimensional topology, and decomposition theorems lie at its heart.

Decompositions of 3-manifolds are typically created via surfaces. Hierarchies are a primary example of this methodology (cf. [**Ja**]), although the present work concerns itself with *Heegaard splittings* of 3-manifolds. This is a decomposition of a 3-manifold M into two *handlebodies* H_1, H_2 via a closed separating surface Σ (called a *Heegaard surface*) such that $H_1 \cap H_2 = \partial H_1 = \partial H_2 = \Sigma$ and $H_1 \cup H_2 = M$. By a theorem of Moise [**Mo**], every closed, orientable 3-manifold admits a triangulation $\mathcal{T}$. We can then construct a Heegaard splitting of M using the second barycentric subdivision of $\mathcal{T}$. Hence, every closed, orientable 3-manifold has a Heegaard splitting.

Although Heegaard surfaces separate a 3-manifolds into standard components, they aren't "elementary" surfaces of 3-manifolds. A Heegaard surface Σ for a 3-manifold M can be *compressed* (via properly embedded *compression disks*) into a less complicated surface through standard cut-and-paste techniques. In fact, Σ can be compressed to a 2-sphere S^2 on both sides (though, not necessarily simultaneously) since Σ bounds handlebodies to both sides. A more useful property for a surface to have is to not compress entirely to a 2-sphere. In other words, we'd like for there to be a surface $F \subset M$ that can't be compressed (to either side if F is orientable). Such a surface is called *incompressible* and can be thought of as an elementary part of M.

Although they aren't incompressible surfaces, we can study how compressible Heegaard surfaces are to both sides simultaneously. Harvey [**Harv**] described a way to assign a simplicial complex to a compact, orientable surface F. The vertices of this complex are isotopy classes of simple closed curves on F that do not bound disks in F and are not parallel into ∂F (we call these curves *essential* in F). We draw an edge between two distinct vertices if they have representatives that are disjoint on F. The remainder of this simplicial complex $\mathcal{C}(F)$ is built as a flag complex (filling in simplices for vertices with representatives in F that are pairwise disjoint from one another in F), though we will only concern ourselves with its 1-skeleton $C(F)$.

For a Heegaard splitting $H_1 \cup_\Sigma H_2 = M$, Hempel [**Hemp**] defined a measure of incompressibility of Σ in M. A pair of compression disks $D_i \subset H_i$ $(i = 1, 2)$ have boundary curves that lie on Σ. These curves are representatives of vertices in $C(\Sigma)$, and we can use the graph distance to measure how far separated they are in $C(\Sigma)$. The minimum distance in $C(\Sigma)$ between the boundaries of all pairs of compression disks $D_1 \subset H_1$, $D_2 \subset H_2$ is called the *(Hempel) distance* of Σ, denoted $d(\Sigma)$. We can define standard classifications of Heegaard splittings using this notion. We say M is: (1) *reducible* if $d(\Sigma) = 0$, (2) *weakly reducible* if $d(\Sigma) = 1$, and (3) *strongly irreducible* if $d(\Sigma) \geq 2$. Distance has been used in many settings (cf. [**BS**], [**Bir**], [**E**], [**Thom**]) to prove various results both algebraic and geometric in nature.

In [**Hart**], Hartshorn showed that incompressible surfaces can be used to bound the distance of Heegaard splittings of closed, orientable 3-manifolds. In particular, he shows that for any 3-manifold M that contains an incompressible surface of genus $g \geq 1$, the distance $d(\Sigma)$ of any Heegaard splitting Σ of M is bounded by $2g$. Scharlemann and Tomova [**ScTom**] elaborate on this idea and use Cerf theoretical arguments to show that there is a similar bound if we replace the incompressible surface with a strongly irreducible Heegaard splitting P; that is, $d(\Sigma) \leq 2 - \chi(P) = 2g(P)$. We recall the notion of distance of a Heegaard splitting and state these theorems precisely later.

Let $K \subset S^3$ be a knot. In [**MG**], Manjarréz-Gutiérrez employed a circle-valued Morse function $f \colon E(K) \to S^3$ to induce a *circular handle decomposition* of the exterior $E(K)$ of K. She then adapted the ideas of [**ST2**] to create a notion of *circular thin position* of such decompositions of $E(K)$. We elaborate on these ideas and develop notions of *circular Heegaard splittings* and of *circular distance* of circular Heegaard splittings.

Circular Heegaard splittings are different from the usual Heegaard splittings of 3-manifolds in two important ways: (i) the Heegaard surface is not closed and (ii) the Heegaard surface is comprised of two connected components. Both of these pose a challenge to the definition of a circular distance.

We show that methods of Hartshorn carry over to this setting, but much care needs to be taken in their adaptation. We show how the shift from Heegaard splittings of closed manifolds to circular Heegaard splittings of knot exteriors restricts these methods. We also provide analogous technical lemmas for this new setting. Once these have laid a solid foundation for Hartshorn-like methods, we define the circular distance of a circular Heegaard splitting and show that it behaves as expected using these methods.

We then prove the main theorem:

THEOREM 4.1.1 (Main Theorem). *Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. If its exterior $E(K)$ contains a closed, orientable, essential surface G of genus g, then $cd(F, S) \leq 2g$.*

An immediate corollary follows and bears striking resemblance to Hartshorn's theorem for closed 3-manifolds; if we consider essential surfaces disjoint from F, we get a bound in the usual curve complex for S:

COROLLARY 4.1.1. *Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. If its exterior $E(K)$ contains a closed, orientable, essential surface G of genus g that can be isotoped to be disjoint from F, then $d(S) \leq 2g$.*

Once this has been shown, we demonstrate other applications:

THEOREM 4.2.1. *Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. If F' is an incompressible Seifert surface for K with genus g that is not isotopic to F, then $d(S) \leq 2g + 1$.*

COROLLARY 4.2.1. *Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is a Seifert surface of K that realizes the Seifert genus $g(K)$. If $d(S) > 2g(K)+1$, then F is the unique Seifert surface of minimal genus for K up to isotopy.*

THEOREM 4.3.1. *Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a locally thin circular Heegaard splitting (F, S). If Σ is a strongly irreducible Heegaard splitting of genus g for $E(K)$, then $cd(F, S) \leq 2g$.*

Corollary 4.3.1. *Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a locally thin circular Heegaard splitting (F, S), and suppose $\tau = \{\tau_1, \ldots, \tau_t\}$ is a tunnel system for K. If the Heegaard splitting Σ induced by τ is strongly irreducible, then $cd(F, S) \leq 2t + 2$. In particular, if K has tunnel number $t(K)$ and possesses such a strongly irreducible Σ, then $cd(F, S) \leq 2t(K) + 2$.*

CHAPTER 2

Background and Definitions

2.1. Knots and Links

For a positive integer n, let $S_n = S^1 \sqcup \cdots \sqcup S^1$ be a disjoint union of n circles. Let M^3 be a compact 3-manifold. Two smooth embeddings $f, g\colon S_n \to M$ are called *isotopic* if there exists a homotopy $h\colon S^1 \times [0,1] \to M$ such that

(1) $f = h|_{S_n \times \{0\}}$
(2) $g = h|_{S_n \times \{1\}}$
(3) for all $0 \le t \le 1$, $h|_{S_n \times \{t\}}$ is an embedding.

We define a *link in* M to be the isotopy class of a smooth embedding $f\colon S_n \to M$, and we set n to be the number of *components* of the link. A *knot in* M is defined to be a link with one component.

For a link $L \subset M$, the 3-manifold $C(L) = M \backslash L$ is called the *complement of* L *in* M. We distinguish this from the compact *exterior of* L *in* M, denoted by $E(L) = M \backslash \eta(L)$.

For a link $L \subset M$, a compact, orientable surface $F \subset M$ whose boundary ∂F is the link will be called a *Seifert surface* of L. It should be noted that such a surface always exists for any link in S^3.

A *handlebody* is closed regular neighborhood of a graph Γ. A graph that is a deformation retract of a handlebody will be called a *spine* of the handlebody. See Figure 2.5.

For a link $L \subset M$, a *tunnel system for* L is a collection $\tau = \{\tau_1, \ldots, \tau_n\}$ of arcs properly embedded in $M \backslash L$ such that $M \backslash \eta(L \cup \tau)$ is a handlebody. The smallest number of arcs needed to construct a tunnel system for L is called the *tunnel number* $t(L)$ *of* L. A tunnel system that realizes the tunnel number of L will be called *minimal*. In should be noted that every link in S^3 has a tunnel system. This can be proven using a link diagram in the plane, adding a vertical tunnel at each crossing.

A link $L \subset M$ is said to be *fibered* if there exists a fibration $f\colon S^3\backslash L \to S^1$ such that each component L_i has a neighborhood framed as $S^1 \times D^2$ (with $L_i \cong S^1 \times \{0\}$) where $f_{S^1\times(D^2\backslash\{0\})}$ is the map $(x, y) \mapsto \frac{y}{|y|}$. This framing condition is in place to specify behavior near L since $M\backslash L$ is not compact.

2.2. 3-Manifolds and Heegaard Splittings

Compressibility and Irreducibility

Let M be a 3-manifold and $F \subset M$ be a compact, orientable surface properly embedded in M. We say that F *is compressible in* M if there exists a disk $D \subset M$ to either side of F such that

(1) $F \cap D = \partial D \subset F$,
(2) ∂D is not ∂-parallel into ∂F, and
(3) ∂D does not bound a disk in F; that is, ∂D is a curve *essential in* F.

Such a disk D is called a *compressing disk for* F *in* M. See Figure 2.1. If there are no compressing disks for F in M, we say that F is *incompressible in* M. An incompressible surface F is called *essential in* M if it also not ∂-parallel into ∂M.

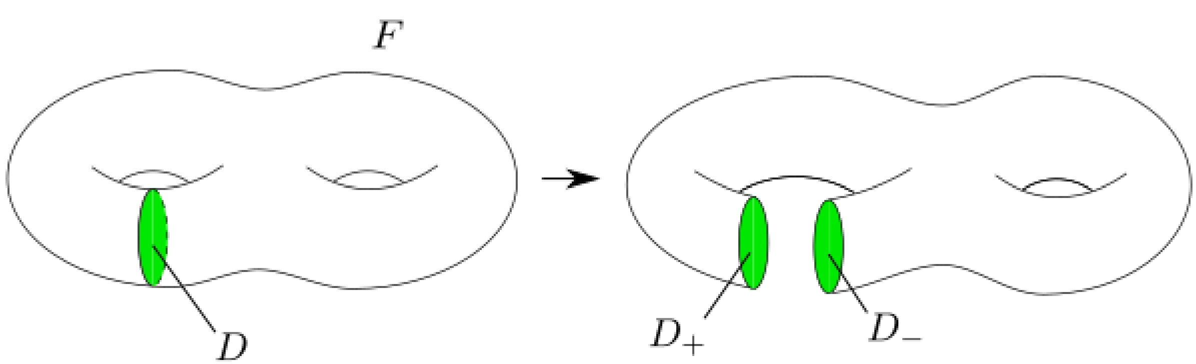

Figure 2.1. A compression of the surface F via the compressing disk D.

Let $\eta(D) \subset M$ be an open regular neighborhood of a compressing disk D for a surface $F \subset M$. Denote by D_+ and D_- the two disk components of $\partial\eta(D)$ so that $A = \partial\eta(D)\backslash(D_+ \cup D_-) \subset F$ is an annulus. Then the surface $(F\backslash A) \cup (D_+ \cup D_-)$ will be called a *compression of* F *in* M. The process of replacing F with a compression of F will be called *compressing* F *in* M.

If M has non-empty boundary and for a compact surface F we have $(F, \partial F) \subset (M, \partial M)$, we say F is *properly embedded in* M if both F is embedded in M and ∂F is embedded in ∂M. A properly embedded surface $F \subset M$ is said to be *∂-compressible* if there exists a disk $D \subset M$ such that

(1) $D \cap F = \alpha \subset \partial D$ and $D \cap \partial M = \beta \subset \partial D$,

(2) $\alpha \cup \beta = \partial D$ with $\alpha \cap \beta = \partial\alpha = \partial\beta$, and

(3) α does not cobound a disk with another arc in ∂F; that is, α is an arc *essential in* F.

Such a disk D above is called a *∂-compressing disk* for F in M. See Figure 2.2 If there are no ∂-compressing disks for F in M, we say that F is *∂-incompressible in* M.

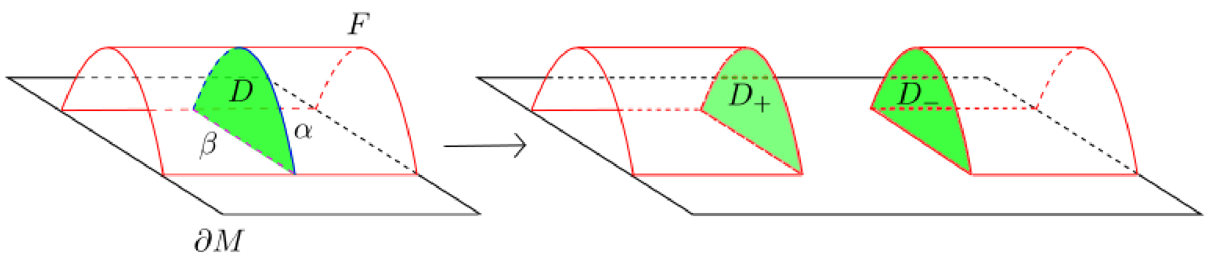

Figure 2.2. A ∂-compression of the surface F via the compressing disk D.

Let $\eta(D) \subset M$ be an open regular neighborhood of a ∂-compressing disk D for a surface $F \subset M$. Denote by D_+ and D_- the two disk components of $\partial\eta(D)$ parallel to D so that $\partial\eta(D)\backslash(D_+ \cup D_-) = \eta(\alpha) \cup \eta(\beta)$ is an annulus. Then the surface $(F\backslash\eta(\alpha) \cup (D_+ \cup D_-)$ will be called a *∂-compression of F in M*. The process of replacing F with a ∂-compression of F will be called *∂-compressing F in M*.

A 3-manifold M is called *irreducible* if every $S^2 \subset M$ bounds a 3-ball to at least one side. We say an irreducible 3-manifold M is *Haken* if it contains an incompressible, orientable surface F with positive genus.

Theorem 2.2.1 (Asphericity). For any knot $K \subset S^3$, $\pi_n(E(K))$ is trivial for $n \geq 2$.

This follows from the sphere theorem and a theorem of Whitehead. A full proof can be found in [**Rolf**]. In particular, the Asphericity Theorem tells us that knot complements (and exteriors) in

S^3 contain no essential 2-spheres; that is, for a knot $K \subset S^3$, both $C(K)$ and $E(K)$ are irreducible 3-manifolds.

Morse Theory and Heegaard Splittings

Let $0 \leq k \leq n$ be integers. An *n-dimensional k-handle* is an n-ball $B^n = B^k \times B^{n-k}$. When the dimension is immaterial or clear from context, we abbreviate this to a *k-handle*. The term $B^k \times \{0\}$ will be called the *core* of a k-handle, and the term $\{0\} \times B^{n-k}$ will be called its *co-core*. We see in the boundary

$$S^{n-1} = \partial B^n = [(\partial B^k) \times B^{n-k}] \cup [B^k \times (\partial B^{n-k})] = [S^{k-1} \times B^{n-k}] \cup [B^k \times S^{n-k-1}].$$

The term $S^{k-1} \times B^{n-k}$ is referred to as the *attaching region* of the k-handle. In the case $k = 0$, we take $S^{-1} = \emptyset$ and "attaching a 0-handle" is the same as adding a disjoint B^n. In our context, a handlebody as described in the section above will be regarded as a single 3-dimensional 0-handle with 1-handles glued to its boundary along their attaching regions. If we connect via radial line segements the cores of the 1-handles to the core of the 0-handle, the resulting graph may be regarded as a spine of the handlebody. See Figure 2.3. The four types of 3-dimensional handles are shown in Figure 2.4.

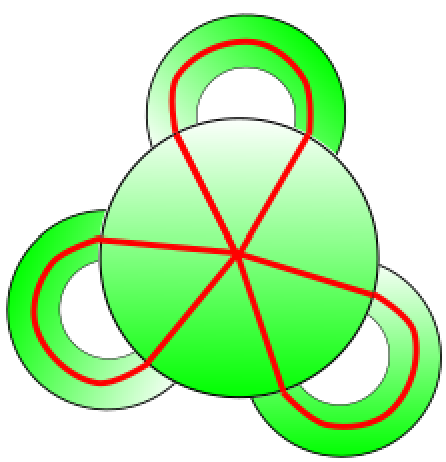

Figure 2.3. Using the cores of 1-handles to define a spine of a 3-dimensional handlebody.

Let F be a compact, connected, orientable surface. Define $W_0 = (F \times I) \cup T$, where T is a collection of 2-handles attached to $F \times \{1\}$. If there are any S^2 components in ∂W_0, we fill them

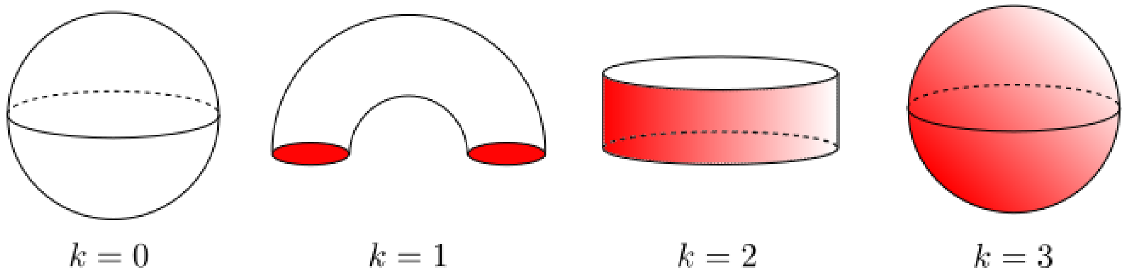

Figure 2.4. The four types of 3-dimensional handles and their attaching regions (in red).

with 3-handles. The resulting 3-manifold W is called a *compression body*. We denote by $\partial_- W$ the surface $F \times \{0\}$ and $\partial_+ W = \partial W \backslash \partial_- W$. The union of a graph and the inner boundary $\partial W \backslash (F \times \{0\})$ that is a deformation retract of W is called a *spine* of W. A handlebody is considered a trivial compression body in the sense that $\partial_+ W = \emptyset$. See Figure 2.5.

Let M be a compact, orientable n-manifold. A smooth function $f\colon M \to [0,1]$ is called *Morse* if all of its critical points are non-degenerate and occur at different critical values. The non-degeneracy condition implies that at each critical point $p \in M$ of f there is a coordinate patch $x = (x_1, \dots, x_n)$ with p identified with the origin such that

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

We call $k \in \{0, 1, \dots, n\}$ the *index* of the critical point p. Given a Morse function $f\colon M \to [0,1]$, it is well-known (cf. [**Mi**]) that M has the homotopy type of a CW-complex with one cell of dimension k for each critical point of f with index k. In particular, M has the homotopy type of a collection of n-dimensional k-handles, one for each index-k critical point of f, glued together. It is easy to see in this description that an index-k critical point of f is an index-$(n-k)$ critical point of $-f$. In this sense, a k-handle is thought of as an "upside-down" $(n-k)$-handle, and this can also be seen as $B^n = B^k \times B^{n-k} = B^{n-k} \times B^k$ from the definition of a k-handle.

For a Morse function $f\colon M \to [0,1]$, if the critical values of all index-k critical points are less than the critical values of all index-ℓ critical points whenever $k < \ell$, then we say f is *self-indexing*. In the context of compact, orientable 3-manifolds, a self-indexing Morse function (with $\partial M \subset f^{-1}\{0,1\}$) defines a decomposition of M into two compression bodies V_0, V_1, where V_0 contains the 0- and 1-handles and V_1 contains the 2- and 3-handles. Any regular level separating the 1-handles from

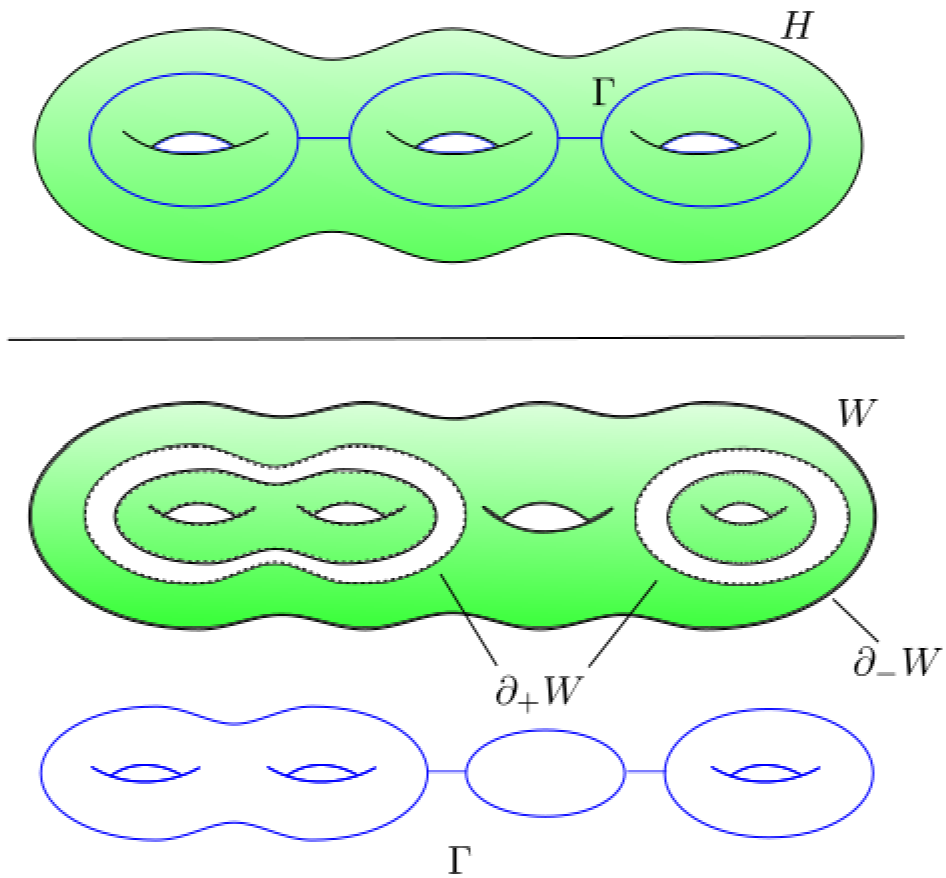

Figure 2.5. A handlebody H and a compression body W with representative spines for each.

the 2-handles is a closed surface $\Sigma \subset M$. The triple $(\Sigma; V_0, V_1)$ is called a *Heegaard splitting* of M. We call Σ a *Heegaard surface* for M. We often abbreviate the triple $(\Sigma; V_0, V_1)$ as Σ when the compression bodies are immaterial or clear from context. Because every compact, orientable 3-manifold M admits a self-indexing Morse function [**Mi**], a Heegaard splitting $(\Sigma; V_0, V_1)$ always exists for M.

Let $(\Sigma; V_0, V_1)$ be a Heegaard splitting for the compact, orientable 3-manifold M. Let σ_0 and σ_1 be spines for V_0 and V_1, respectively. We see that $M \backslash (\sigma_0 \cup \sigma_1) \cong \Sigma \times (0,1)$. We can use this product structure to define a function $h\colon M \to [0,1]$ where $h^{-1}(t) \cong \Sigma$ for all $0 < t < 1$ and $h^{-1}(i) = \sigma_i$ for $i = 1, 2$. Such a function h is called a *sweep-out of M induced by Σ* (or just a *sweep-out* when M and Σ are clear from context).

There is one class of Heegaard splitting that will be of particular interest. A knot $K \subset S^3$ always has a tunnel system τ so that $H = S^3 \backslash \eta(K \cup \tau)$ is a handlebody. Then $V = S^3 \backslash \mathrm{int}(H)$ is a compression body with $\partial_+ V = \partial\eta(K)$. If we take $\Sigma = \partial H = \partial_- V$, we see that $(\Sigma; H, V)$ is a Heegaard splitting of $E(K)$. We say that Σ is the Heegaard splitting *induced by the tunnel system* τ.

Surfaces in Handlebodies and Compression Bodies

Let Δ be a collection of compressing disks for a compression body W. If compressing $\partial_- W$ in W along all disks in Δ produces $\partial_+ W$ and possibly a collection of S^2 components, then Δ is called a *complete disk system for* W. If there are no S^2 components in the resulting compression, then Δ is called a *minimal complete disk system* for W.

In a compression body W, let γ be a curve in $\partial_+ W$. Using the product structure of W, we stretch γ across W to $\partial_- W$ to create an annulus $\gamma \times [0, 1]$. An annulus created in this fashion is called a *spanning annulus* of W. If γ is essential in $\partial_+ W$, we say that $\gamma \times [0, 1]$ is an *essential spanning annulus* of W. See Figure 2.6.

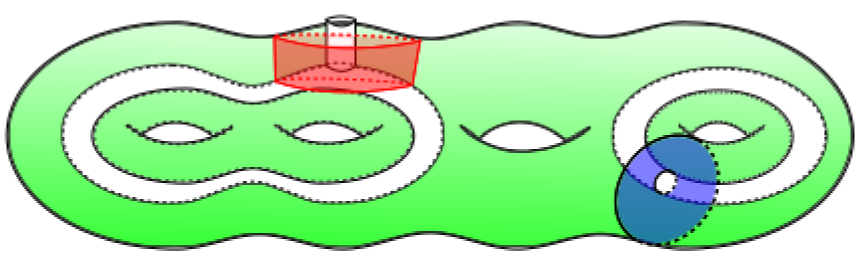

Figure 2.6. An essential (blue) spanning annulus and a inessential, ∂-parallel spanning annulus (red) for a compression body. The boundary curves of the inessential ∂-parallel on the inner and outer boundary components of the compression body.

Theorem 2.2.2 (cf. [**Ja**]). The only compact, connected, orientable, incompressible, ∂-incompressible, surfaces in a compression body that aren't parallel into the boundary are essential disks and essential spanning annuli. In a handlebody, such surfaces can only be essential disks.

2.3. The Curve Complex

Constructing the Complex of Curves

As outlined in the introduction, the following construction is due to Harvey [**Harv**] as he studied the asymptotic geometry of a surface's Teichmúller space. To a compact, orientable (possibly disconnected) surface F, we assign a 1-dimensional simplicial complex $C(F)$ as follows:

(1) For each isotopy class of closed curves essential in F, add a vertex.

(2) Add an edge between distinct vertices if there are representatives of each isotopy class (one from each vertex) that are disjoint in F.

This construction may be continued to construct a higher-dimensional simplicial complex by requiring that this complex be a flag complex. The result is called the *curve complex* of F. We will only concern ourselves with the 1-skeleton $C(F)$ of this complex and still refer to it as the "curve complex" of F. See Figure 2.7.

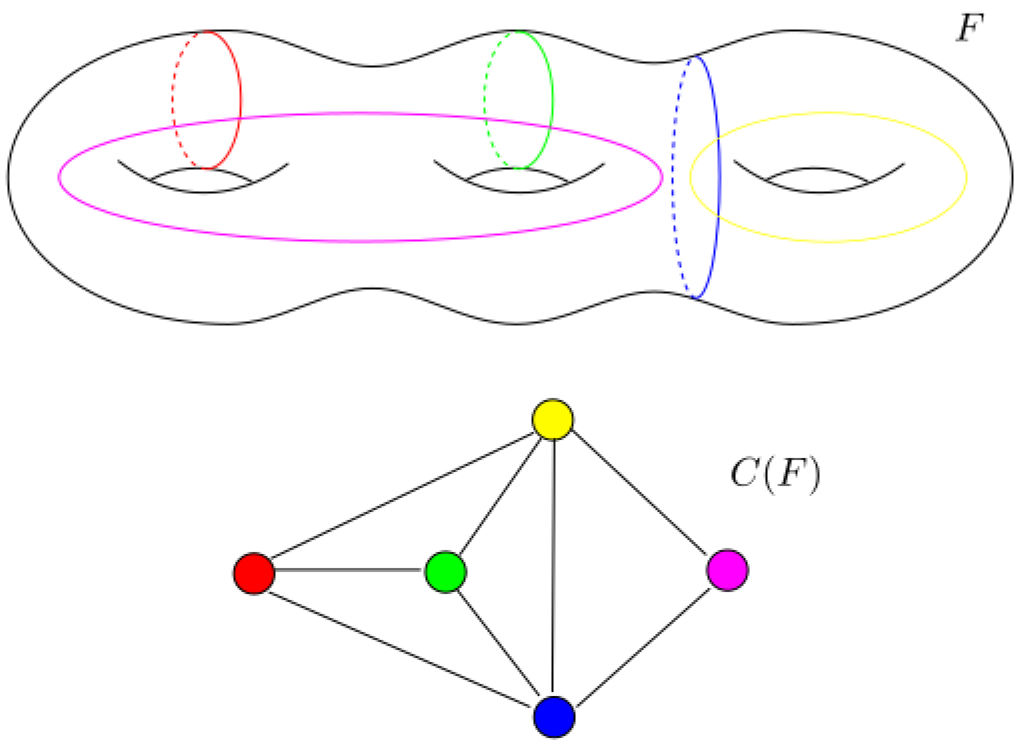

Figure 2.7. A surface F and the subgraph of its curve complex $C(F)$ induced by the vertices representing the colored curves.

For a pair of essential curves γ_1, γ_2 in the same connected component of F, we define the *distance* $d_{\mathcal{C}}(\gamma_1, \gamma_2)$ between γ_1 and γ_2 to be the graph distance in $C(F)$ between the vertices $[\gamma_1]$ and $[\gamma_2]$. If γ_1 and γ_2 lie in different connected components, then we take the convention $d_{\mathcal{C}}([\gamma_1], [\gamma_2]) = \infty$. We also use the convention that $d_{\mathcal{C}}([\gamma], \emptyset) = 0$ for any essential curve $\gamma \subset F$. We often abbreviate and abuse the notation $d_{\mathcal{C}}(\gamma_1, \gamma_2)$ to denote the distance between γ_2 and γ_2, omitting the brackets denoting isotopy classes of the curves.

Types of Heegaard Splittings

Let Σ be a Heegaard splitting of a closed, orientable 3-manifold M. The surface Σ is compressible in many ways to both sides in the handlebodies H_1 and H_2. Our goal is to measure how compressible Σ is to both sides. We collect all compressing disks of Σ in H_1 into the set Γ_1 and all compressing disks of Σ in H_2 into the set Γ_2. In order to compare these sets, we appeal to the curve complex $C(\Sigma)$ of the Heegaard surface; that is, where the two sets could potentially intersect.

Each disk in $\Gamma_1 \cup \Gamma_2$ has a boundary curve $\gamma \subset \Sigma$ that is essential in Σ. The isotopy class $[\gamma]$ is represented by a vertex in $C(\Sigma)$. Denote by $V_i \subset C(\Sigma)$ the collection of vertices of $C(\Sigma)$ defined in this way by the set Γ_i (for $i = 1, 2$).

The *(Hempel) distance* $d(\Sigma)$ of a Heegaard splitting Σ [**Hemp**] is the minimum graph distance $d_{\mathcal{C}}$ along $C(\Sigma)$ from an element of $[\gamma_1] \in V_1$ to an element of $[\gamma_2] \in V_2$; that is,

$$d(\Sigma) = \min\{d_{\mathcal{C}}(\partial\alpha, \partial\beta) \ \mid\ \alpha \in \Gamma_1, \beta \in \Gamma_2\}.$$

If $d(\Sigma) = 0$, there exist compressing disks of Σ to both sides whose boundaries are isotopic in Σ. If we isotope these disks so that their boundary curves coincide, then the disks can be glued together to form a S^2 that intersects Σ in a curve essential in Σ. In particular, the Heegaard splitting must be *reducible* or is a genus-one Heegaard splitting of $S^1 \times S^2$.

If $d(\Sigma) \leq 1$, there exist compressing disks of Σ to both sides whose boundaries can be isotoped to be disjoint. This implies the Heegaard splitting is *weakly reducible* as defined in [**CG**]; that is, there exist essential disks $D_i \subset V_i$ for $i = 1, 2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$. Observe that reducible Heegaard splittings are indeed weakly reducible splittings since we can isotope the boundary of one disk to be disjoint from the other.

If $d(\Sigma) \geq 2$, Σ is *strongly irreducible* [**CG**]; that is, each essential disk V_1 intersects each essential disk in V_2.

Hartshorn [**Hart**] and Scharlemann-Tomova [**ScTom**] relate the existence of essential surfaces to strongly irreducible Heegaard surfaces, respectively, to the distance of a Heegaard splitting. We state these theorems below:

THEOREM 3.0.3 ([**Hart**]). *Let M be a Haken 3-manifold containing an orientable incompressible surface of genus g. Then any Heegaard splitting of M has distance at most $2g$.*

THEOREM 2.3.1 ([**ScTom**]). Suppose P and Q are Heegaard splittings of a closed 3-manifold M. Then either $d(P) \leq 2g(Q)$ or Q is isotopic to P or to a stabilization of P.

2.4. Circular Handle Decompositions

In the following section, much of the notation and definitions follow from Manjarréz-Guti'errez [**MG**].

Let $K \subset S^3$ be a knot. Let $F\colon S^3 \backslash K \to S^1$ be a Morse function and define $f\colon E(K) \to S^1$ to be the restriction $F|_{E(K)}$. Since a fundamental cobordism of f can be isotoped to have no local maxima or minima [**Mi**], we assume that all critical points of f have index 1 or 2.

We construct a handle decomposition of $E(K)$ from f as follows: Choose R a regular level of f between an index-1 and an index-2 critical point. There are many such choices: We assume R is chosen to have smallest genus among all choices. If f has no critical points, then K is fibered and there is only one choice for R. Otherwise, we see that the critical points of f define collections $N = \{N_1, \dots, N_k\}$ and $T = \{T_1, \dots, T_k\}$ of 1- and 2-handles, respectively. We assume the handles in N_1 appear first after R and, moreover, that the handles in N_i appear before the handles in T_i and that the handles in T_i appear before the handles in N_{i+1} (taking all indices modulo k where necessary). Construct the compact manifold

$$H = (R \times I) \cup (N_1 \cup T_1) \cup \cdots \cup (N_k \cup T_k)$$

by flowing R along $E(K)$ via the gradient of f, attaching handles from N and T as prescribed by the critical points of f. Choose a regular level S_i separating N_i from T_i; that is,

$$S_i \cong \overline{\partial[(R \times I) \cup (N_1 \cup T_1) \cup \cdots \cup N_i] \backslash [\partial E(K) \cup (R \times \{0\})]}.$$

Call S_i a *thick level* of f and set $\mathcal{S} = \cup_{i=1}^{k} S_i$. Similarly, choose a regular level F_i separating T_i from N_{i+1}; that is,

$$F_i = \overline{\partial[(R \times I) \cup (N_1 \cup T_1) \cup \cdots \cup (N_i \cup T_i)] \backslash [\partial E(K) \cup (R \times \{0\})]}.$$

Call F_i a *thin level* of f and set $\mathcal{F} = \cup_{i=1}^{k} F_i$.

We also define

$$W_i = (\text{collar of } F_i) \cap (N_i \cup T_i),$$

which is a 3-manifold with boundary $\partial W_i = F_i \cup F_{i+1} \cup (W_i \cap \partial E(K))$. The thick level S_i defines a compact (but not closed!) Heegaard surface for W_i, dividing it into compression bodies

$$A_i = (\text{collar of } F_i) \cup N_i \quad \text{and} \quad B_i = (\text{collar of } S_i) \cup T_i.$$

The boundary ∂A_i can be seen as the union of three components; that is, $\partial A_i = S_i \cup F_i \cup \partial_v A_i$, where $\partial_v A_i = \partial A_i \cap \partial E(K)$. We call $\partial_v A_i$ the *vertical boundary of* A_i. We can similarly define $\partial_v B_i = \partial B_i \cap \partial E(K)$ to be the vertical boundary of B_i. Note that the vertical boundary is an annulus.

Observe that F_k is diffeomorphic to R. The function f defines a diffeomorphism $\phi\colon R \to R$. When K is fibered, ϕ is called the *monodromy* of K. In this case, we see that

$$E(K) = H/(R \times \{0\}) = \phi(R \times \{1\}).$$

See Figure 2.8. This construction is well-defined - a different choice of R merely translates the labels of the sets N and T by the same amount.

Definition 2.4.1. The collection $\mathcal{D} = \{(W_i; A_i, B_i)\}_{i=1}^{k}$ will be called the *circular handle decomposition of* $E(K)$ *induced by* f.

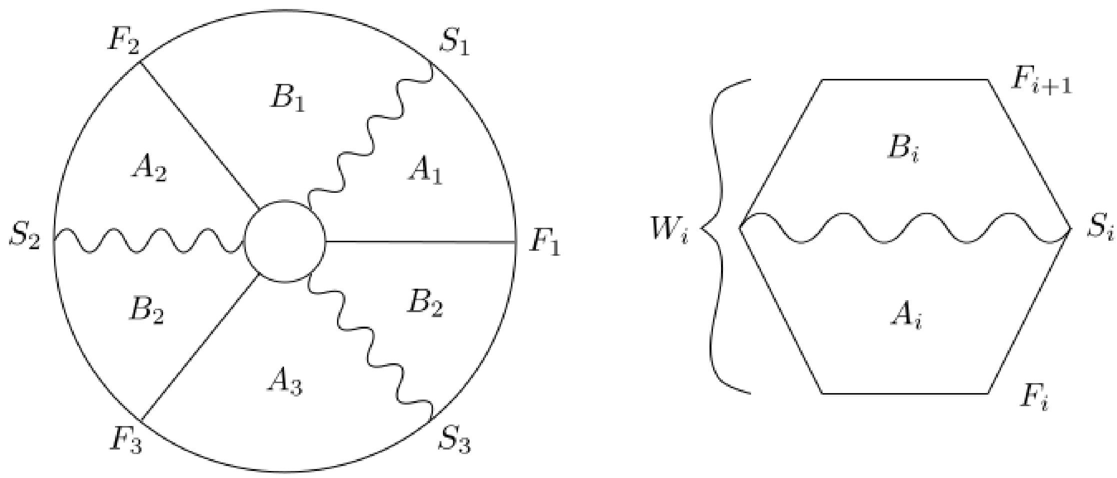

Figure 2.8. A circular handle decomposition of a knot exterior and an arbitrary section W_i.

Definition 2.4.2. For a closed, connected surface $S \neq S^2$ define its *complexity* $c(S) = 1 - \chi(S)$. If S has a nonempty boundary, define $c(S) = 1 - \chi(\overline{S})$, where $\overline{S}$ denotes S with its boundary components capped off with disks. We define $c(S^2) = 0$ and $c(D^2) = 0$. If S is disconnected, define $c(S) = \sum c(S_i)$, where $S = \coprod_i S_i$ and each S_i is connected.

Definition 2.4.3. For a knot $K \subset S^3$ with circular handle decomposition $\mathcal{D}$ for $E(K)$, the *circular width $cw(\mathcal{D})$ of the decomposition* $\mathcal{D}$ is the multiset of integers $\{c(S_i)\}_{i=1}^k$, and $|cw(\mathcal{D})| = k$ is the number of thick levels in $\mathcal{D}$. The *circular width $cw(E(K))$ of the knot exterior* $E(K)$ is defined to be the minimal circular width among all circular handle decompositions of $E(K)$. The minimum is taken using the lexicographic ordering on multisets of integers.

The pair $(E(K), \mathcal{D})$ is in *circular thin position* if $\mathcal{D}$ realizes the circular width of $E(K)$. When K is a fibered knot, we define $cw(E(K)) = \emptyset$, so $|cw(E(K))| = 0$. If $|cw(E(K))| = 1$, we say that K is *almost-fibered*.

Manjarréz-Gutiérrez [**MG**] examined knot exteriors using this setup in her doctoral dissertation. In addition to analyzing the behavior of circular width under some common knot operations, she showed:

Theorem 2.4.1 ([**MG**]). Let $K \subset S^3$ be a knot. At least one of the following holds:

(1) K is fibered;

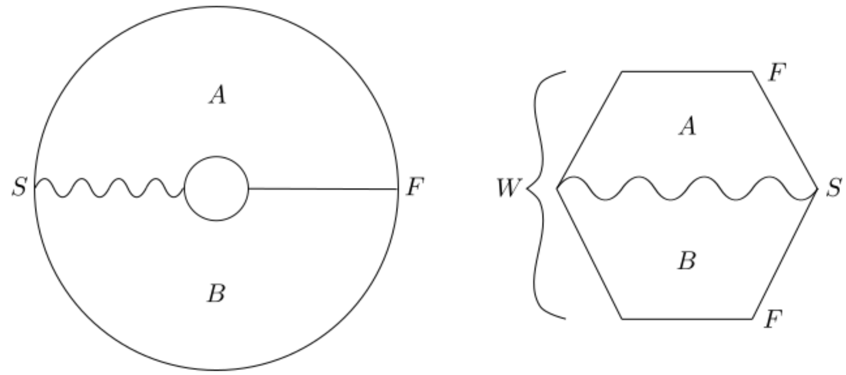

Figure 2.9. An admissible circular handle decomopsition of a knot exterior and its only section W. The pair (F, S) describes a circular Heegaard splitting of the knot exterior.

(2) K is almost-fibered;

(3) K contains a closed essential surface in its complement. Moreover, this closed essential surface is in the complement of an incompressible Seifert surface of K;

(4) K has at least two non-isotopic, incompressible Seifert surfaces.

The point we emphasize here is that circular thin position can be used to show the existence of multiple, non-isotopic, incompressible Seifert surfaces for a knot in S^3.

One other result from her work follows from Scharlemann-Thompson [**ST2**] in their work on thin position for 3-manifolds:

THEOREM 2.4.2 ([**MG**]). If $(E(K), \mathcal{D})$ is in circular thin position, then

(1) Each Heegaard splitting S_i of W_i is strongly irreducible.

(2) Each F_i is incompressible in $E(K)$.

(3) Each S_i is a weakly incompressible surface in $E(K)$.

The converse of this theorem is not true in general. That is, a circular handle decomposition satisfying the three properties above need not be thin.

DEFINITION 2.4.4. A circular handle decomposition $\mathcal{D}$ is said to be *locally thin* if the thin levels F_i are incompressible and the thick levels S_i are weakly incompressible.

A circular handle decomposition most resembles a Heegaard splitting when there is only one thick level and one thin level. We will focus on this case, which we call *admissible*:

Definition 2.4.5. Let $K \subset S^3$ be a knot with exterior $E(K)$, and let $\mathcal{D}$ be a circular handle decomposition of $E(K)$. We say that $\mathcal{D}$ is *admissible* if $\mathcal{D} = \{(W; A, B)\}$; that is, $\mathcal{D}$ contains only one thick level S and one thin level F. For an admissible decomposition, the pair (F, S) will be called a *circular Heegaard splitting of* $E(K)$.

Remark 2.4.1. If $K \subset S^3$ is a fibered knot, we can construct a circular Heegaard splitting (F_0, F_1) of $E(K)$ using a parallel copy F_1 of the fiber surface F_0 and setting the thick level $S = F_1$. In this sense, fibered knots also admit circular Heegaard splittings even when the circle-valued Morse function has no critical points.

CHAPTER 3

Elementary Compressions and Circular Distance

Let M be a closed, orientable 3-manifold with a Heegaard splitting $(H_1, H_2; \Sigma)$. Suppose M contains an essential, orientable surface G of genus g. Hartshorn [**Hart**] has shown that the distance of Σ is bounded by twice the genus of G; that is, $d(\Sigma) \leq 2g$. He uses what he calls an *elementary compression* of $G \cap H_1$ into H_2 to define a sequence of isotopies of G across Σ that controls the distance of Σ. Since we will use these ideas extensively, we outline his argument below:

Theorem 3.0.3 ([**Hart**]). Let M be a Haken 3-manifold containing an orientable incompressible surface of genus g. Then any Heegaard splitting of M has distance at most $2g$.

Outline of proof:

(1) Isotope G so that it intersects each H_i $(i = 1, 2)$ in incompressible, properly embedded components.
(2) Show that an elementary compression G' of G preserves the incompressibility of G in H_1. Specifically, if each component of $G \cap H_1$, say, is incompressible in H_1, then so is each component of $G' \cap H_1$. We note that we may lose the incompressibility of some components of $G' \cap H_2$.
(3) Show that the distance in the curve complex $\mathcal{C}(\Sigma)$ between any curve in $G \cap \Sigma$ and any curve in $G' \cap \Sigma$ is at most one.
(4) Use a result of Kobayashi [**K**] to isotope G so that the components of $G \cap H_1$ contain exactly one essential disk.
(5) Use elementary compressions to produce a sequence of isotopies of G such that the final embedding G' is such that no component of $G' \cap H_1$ is a disk and that exactly one component of $G' \cap H_2$ is an essential disk. Performing these isotopies in such a minimal fashion bounds $d(\Sigma)$ by the number of elementary compressions in this sequence.
(6) Finally, show that the number of elementary compressions necessary for this kind of sequence of isotopies is no greater than $2g$.

□

We will adapt this method to circular Heegaard splittings of knot complements and the closed, essential surfaces they contain. We address the first three steps in this chapter while the last three steps are addressed in the next.

3.1. Surfaces in Compression Bodies

Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S). Suppose $G \subset E(K)$ is an essential, orientable surface of genus g. Consider its intersection $G \cap A$ with the compression body A. If there is a ∂-compressible component G_0 of $G \cap A$, then there exists a disk D properly embedded in A so that $D \cap A = \partial D = \alpha \cup \beta$, where $\alpha \subset G_0 \cap A$ and $\beta \subset \partial A = (S \cup F) \cup (A \cap \partial E(K))$.

Suppose $\beta \cap \partial(E(K)) = \emptyset$. Let $\eta(\alpha) \subset G_0$ be an open regular neighborhood of α. Push $\overline{\eta(\alpha)}$ along D and slightly into B. This effectively removes a (two-dimensional) 1-handle from G_0 and attaches a 1-handle in B to G_0. The result is an isotopy of G. See Figure 3.1.

Definition 3.1.1. The move just described is called a *∂^*-compression of G from A into B*. It is also known as an *isotopy of type A* of G across $S \cup F$.

If G admits a ∂^*-compression from A into B, we say G is *∂^*-compressible from A into B*.

If G isn't ∂^*-compressible from A into B, then we say G is *∂^*-incompressible from A*.

Remark 3.1.1. Unless otherwise specified, a definition or claim about the compression body A will also hold symmetrically for the compression body B.

Lemma 3.1.1. Suppose G' is the result of a ∂^*-compression of G from A into B. Then $\chi(G' \cap A) = \chi(G \cap A) + 1$, and $\chi(G' \cap B) = \chi(G \cap B) - 1$.

Proof. Notice that, in the definition of ∂^*-compression, the removal of the neighborhood $\eta(\alpha)$ from G_0 removes an open disk and two edges from G_0. The net effect on $\chi(G_0)$ is an addition of 1. Similarly, for $G \cap B$, we have added that open disk and two edges to $G \cap B$. The net effect on $\chi(G \cap B)$ is the subtraction of 1. □

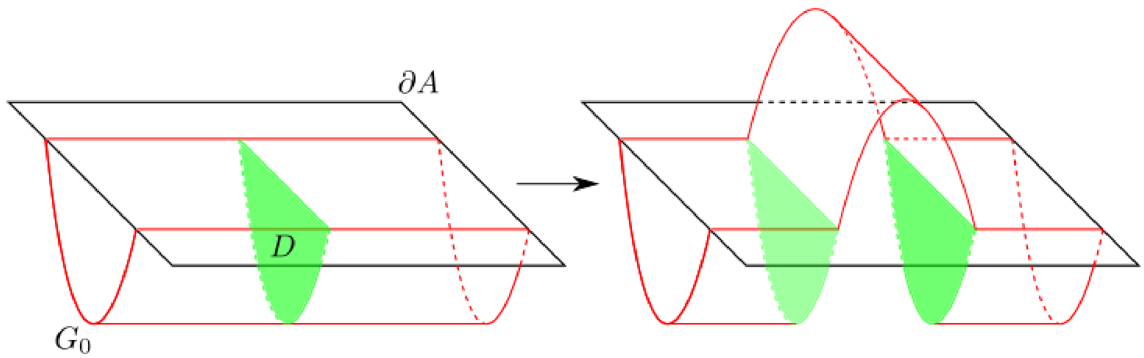

Figure 3.1. A ∂-compression of G from A into B.

In light of the definition of distance of a Heegaard splitting in the previous chapter, we would like to omit any ∂^*-compressions that create inessential disk components. This happens only if $G \cap A$ has a ∂-parallel annulus component. However, since ∂^*-compressions avoid $\partial(E(K))$, this ∂-parallel annulus would be parallel into either S or F but not $\partial(E(K))$.

Definition 3.1.2. Let $G \cap A$ contain an annular component G_0 that is parallel into either S or F. This annulus will be called a *∂^*-parallel annulus.* A ∂^*-compression of A_0 will be called an *annular compression from A into B.*

Any ∂^*-compression of G from A into B that is not an annular compression will be called an *elementary compression.*

Remark 3.1.2. Unless otherwise stated, annular compressions of G from A into B will always be followed by the isotopy carrying the resulting inessential disk across $S \cup F$. The net effect is to push the ∂^*-annulus out of A. In particular, $\chi(G \cap A) = \chi(G' \cap A)$ after an annular compression of G from A into B.

The incompressible and ∂-incompressible surfaces in compression bodies have been classified [**BO**]. These surfaces are ∂-parallel or they are essential disks or essential spanning annuli. We need a similar classification of incompressible and ∂^*-incompressible surfaces in the compression bodies defined by a circular Heegaard splitting of $E(K)$. This classification allows for one additional type of surface.

Recall that we can identify $A \cong (F \times I) \cup N$, where N is a collection of 1-handles attached to $F \times \{1\}$. Choose an arc γ properly embedded in $F \times \{1\}$ that is disjoint from the attaching disks for N. Thus, $\gamma \times I \subset A$ is a properly embedded disk D in A that is disjoint from N.

Definition 3.1.3. The disk $D = \gamma \times I$ constructed above is called a *product disk* of A. The arc γ will be called a *spanning arc* of F. We call D *inessential in* A if γ is inessential in F. Otherwise, D is said to be *essential in* A. See Figure 3.2.

A product disk of B is defined similarly by identifying $B \cong (S \times I) \cup T \cong (F \times I) \cup N'$, where T is a collection of 2-handles and N' is a collection of 1-handles dually equivalent to T.

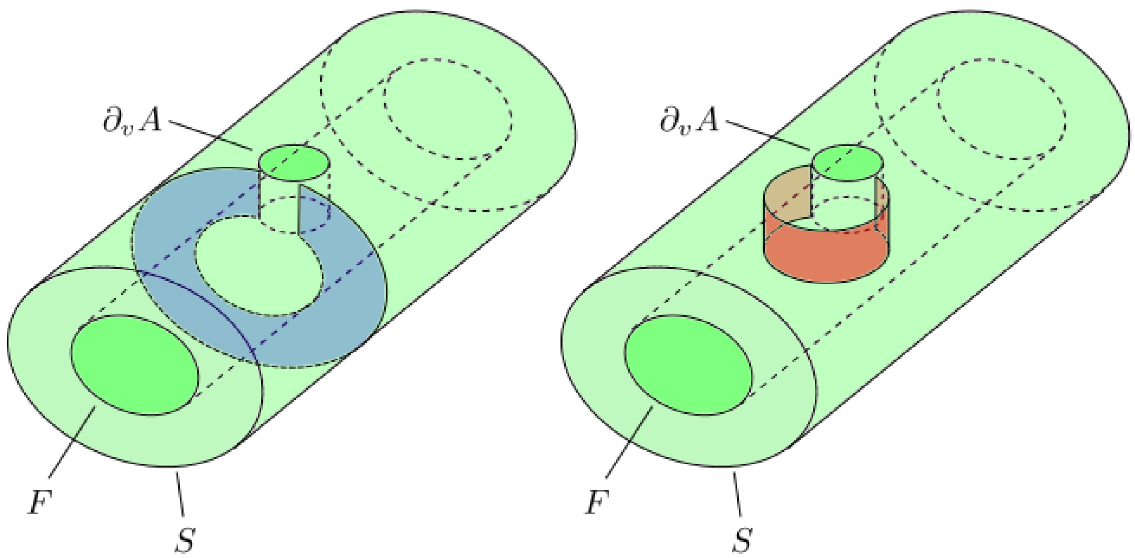

Figure 3.2. A piece of A showing an essential (blue) and inessential (red) product disk of A.

Lemma 3.1.2. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition $\mathcal{D}$. Suppose G is a compact, connected, non-∂-parallel surface that is ∂^*-incompressible from A and can be isotoped to be disjoint from $\partial_v A$. Then G is either an essential disk or an essential spanning annulus.

Proof. The proof is similar to [**BO**]. We include it here for completeness.

First isotope G so that $G \cap \partial_v A = \emptyset$. Let G be some component of $G \cap A$. For the sake of contradiction, suppose that G_0 is neither an essential disk nor an essential spanning annulus. Let

Δ be a minimal, complete disk system for A (this can be taken to be the co-cores of the 1-handles in the set N of $\mathcal{D}$). Through a standard innermost-disk/outermost-arc argument, we may isotope G_0 so that $G_0 \cap \Delta = \emptyset$. Cutting A along Δ yields a manifold $V \cong F \times I$.

If $(\partial V) \cap G = \emptyset$, then G is closed. This would imply that G is compressible; hence, we conclude that $(\partial V) \cap G \neq \emptyset$. Without loss of generality, let's say that $F \cap G \neq \emptyset$.

Because G_0 is not a spanning annulus, we know that $\pi_1(G_0, S \cap G_0, \star)$ is non-trivial for some basepoint $\star \in S$. We also observe that $\pi_1(V, F)$ is trivial. These observations together show that there must exist an essential arc α properly embedded in S cobounding a disk D with some essential arc β in G_0. In particular, G_0 is ∂-compressible by another innermost-disk/outermost-arc argument. Using this disk, either it is a ∂-compressing disk for G_0 or we can use it to find such a disk. Since we assumed that $G_0 \cap \partial_v A = \emptyset$, this ∂-compression may be taken to be a ∂^*-compression.

From this contradiction, we conclude that G_0 can only be an essential disk or an essential spanning annulus. $\square$

Lemma 3.1.3. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition $\mathcal{D}$. Suppose G is a compact, connected, non-∂-parallel surface that is ∂^*-incompressible from A and cannot be isotoped to be disjoint from $\partial_v A$. Then G is an essential disk.

Proof. First isotope G to intersect $\partial_v A$ in a minimal number of arcs. The intersection $\sigma = G \cap \partial_v A$ will be a collection of at least two parallel, essential spanning arcs of $\partial_v A$. Furthermore, we can isotope σ to be vertical with respect to the identification $A \cong (F \times I) \cup N$. Specifically, each arc in σ map be considered to be identified with $\{x\} \times I$ for some $x \in \partial F$. Unless otherwise specified, we now take all isotopies of G to fix σ.

Now, as in the previous lemma, we may find a minimal, complete disk system Δ for A. We then use a standard innermost-disk/outermost-arc argument to isotope G disjoint from Δ so that G lies in a manifold $V \cong F \times I$.

Choose a collection of arcs Γ properly embedded in $F \times \{1\}$ that are disjoint from the attaching disks of the 1-handles in N. Furthermore, choose Γ so that $G \backslash \eta(\Gamma)$ is a disk. Observe that each arc $\gamma_i \in \Gamma$ is a spanning arc for some essential product disk D_i of A. These resulting product disks $\mathcal{E} = \{D_i\}_{i=1}^N$ constitute a minimal complete (product) disk system for the handlebody V.

For sake of contradiction, we assume G is not a product disk. Then G is either (i) $\chi(G) \leq 0$ or (ii) a disk that is not a product disk. Suppose $\chi(G) \leq 0$ and that the genus of G is at least one. Then we can choose an essential arc in G with both endpoints in S. An argument similar to Lemma 3.1.2 shows that there is a ∂^*-compression disk for G from A. This contradiction implies that the genus of G is zero.

In this case, ∂G is comprised of at least two components; otherwise, we are in case (ii) above. Then G contains an essential arc whose endpoints lie in different components of ∂G and in the same surface, either F or S. Once again, an argument similar to Lemma 3.1.2 yields a ∂^*-compression disk for G from A. Therefore, G must have only one boundary component; that is, G is a disk so that $\chi(G) = 1$. We conclude that G can only be a disk. □

Corollary 3.1.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S). The only compact, connected, orientable, properly embedded surfaces in the compression body A that are not ∂-parallel, are incompressible, and are ∂^*-incompressible from A are essential disks and essential spanning annuli.

3.2. Behavior of Elementary Compressions

Recall the setting at the beginning of Hartshorn's argument. There is an incompressible surface G of genus g inside a 3-manifold M, and M has a Heegaard splitting $(H_1, H_2, ; \Sigma)$. Hartshorn established that ∂-compressions of $G \cap H_1$ from H_1 don't alter the incompressibility of the components of $G \cap H_1$. We show the same of $G \cap A$ and $G \cap B$ if the circular Heegaard splitting is circular locally thin.

Our first objective is to show that an essential surface $G \subset E(K)$ can be isotoped to intersect $S \cup F$ in curves that are essential in both S and F. This is not immediately obvious due to the fact that S is weakly incompressible.

Lemma 3.2.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a locally thin circular handle decomposition $\mathcal{D}$. Suppose that $E(K)$ contains a closed, orientable incompresible surface G that intersects $S \cup F$ in a minimal number of curves. Then

(i) $G \cap A$ and $G \cap B$ are incompressible in A and B, respectively.
(ii) $G \cap (S \cup F)$ is a collection of simple closed curves that are essential in $S \cup F$.

(iii) there are no ∂-parallel annulus components of either $G \cap A$ or $G \cap B$. In particular, there are no ∂^*-parallel annuli in $G \cap A$ or $G \cap B$.

Proof. The proof is nearly identical to [**Hart**]. We include the argument here for completeness.

(i) Without loss of generality, assume that $G \cap A$ has a component compressible in A. Then there is a disk D properly embedded in A such that ∂D doesn't bound a disk in $G \cap A$. Because G is incompressible in $E(K)$, there is a second disk D' in G such that $\partial D' = \partial D$. Observe here that $D' \cap (S \cup F) \neq \emptyset$ since ∂D doesn't bound a disk in $G \cap A$. Because $E(K)$ is irreducible, the disks D and D' cobound a ball in $E(K)$. We can then isotope G so that D' may be pushed through this ball and entirely out of A, into B, and off of $S \cup F$. This reduces the number of components in $G \cap (S \cup F)$, contradicting the assumption of minimality.

(ii) Without loss of generality, suppose that c is a curve in $G \cap \partial A$ that is inessential in ∂A. Then there is a disk D in ∂A with $\partial D = c$. Because G is incompressible in $E(K)$, there is another disk D' in G such that $\partial D' = c$ as well.

Consider the surface $G' = (G \backslash D') \cup D$. Since $E(K)$ is irreducible, we see that G is actually isotopic to G'. Hence, we may push D from G' slightly off of $S \cup F$ to make G' intersect $S \cup F$ in fewer components.

(iii) Without loss of generality, assume that A has a ∂-parallel annulus component G_0. Because G is closed, $G \cap \partial E(K) = \emptyset$ so that $G \cap \partial_v A = \emptyset$. Hence, any ∂-parallel component is actually ∂^*-parallel. Therefore, any such component can be pushed out of A and into B so as to reduce the number of components in $G \cap (S \cup F)$.

□

Now that G intersects $S \cup F$ in such a desirable fashion, we would like to show that elementary compressions preserve this structure. Moreover, we need to show that they exist in the first place.

Lemma 3.2.2. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible, locally thin circular handle decomposition (F, S). Suppose that $G \subset E(K)$ is a closed, connected, orientable, incompressible surface. If G intersects $S \cup F$ in a minimal number of curves and $\chi(G \cap A) < 0$, then there is an elementary compression of $G \cap A$ from A into B.

Proof. Since G is incompressible in $E(K)$, it follows that $G \cap \partial A \neq \emptyset$. By Lemma 3.2.1, $G \cap A$ is incompressible in A. If all components of $G \cap A$ are disks and annuli, then $\chi(G \cap A) \geq 0$. Our assumption that $\chi(G \cap A) < 0$ implies that there is some component of $G \cap A$ that is neither a disk nor an annulus. In particular, $G \cap A$ has a ∂-compressible component. If this ∂-compression isn't a ∂^*-compression, we can assume that this component is ∂^*-incompressible (otherwise, we are done). By Lemma 3.1.2, this component must be either a disk or an annulus. Neither such surface is ∂-compressible since no component of $G \cap A$ is a ∂^*-parallel annulus. We conclude then that there must be a ∂^*-compression of some component of $G \cap A$. Because this ∂^*-compression doesn't take place on a ∂^*-parallel annulus, it is an elementary compression of G from A into B. $\square$

Lemma 3.2.3. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S). Suppose that G is a compact, orientable, incompressible surface properly embedded in A with no ∂^*-parallel components such that $G \cap \partial_v A$ has at most one component and that $G \cap \partial A \neq \emptyset$. If $\chi(G \cap A) < 0$, then there is an elementary compression of $G \cap A$ from A into B.

Proof. If $G \cap A$ has no such elementary compression, then by 3.1.1 each of its components is either an essential disk or essential spanning annulus. But then we would have $\chi(G \cap A) \geq 0$, and this contradicts our hypothesis. $\square$

Lemma 3.2.4. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S). Suppose that G is a compact, orientable surface properly embedded in $E(K)$ so that $G \cap A$ is incompressible in A and that each component of $G \cap (S \cup F)$ is essential in ∂A. Denote by G' the embedding of G after an elementary compression from A into B. Then:

(i) G' is also incompressible in A.
(ii) The components of $G' \cap A$ are also essential in ∂A.

Proof. This proof is nearly identical to that found in [**Hart**]. We include it here for completeness.

(i) Suppose G is compressed in A via the disk $D \subset A$ with boundary $\partial D = \alpha \cup \beta$, where $\alpha \subset G$ and $\beta \subset \partial A \backslash \partial_v A$. We can choose a neighborhood $D \times I$ of this disk in such a way that

$(\partial D) \times I = (\alpha \cup \beta) \times I$ keeps $\alpha \times I \subset G$ and $\beta \times I \subset \partial A \backslash \partial_v A$. The elementary compression of G along D replaces $\alpha \times I$ with two disks $D \times (\partial I)$. Name these disks $D_0 = D \times \{0\}$ and $D_1 = D \times \{1\}$. We can then consider D_0 and D_1 as submanifolds of G'.

Assume that G' is compressible in A, and let $c \subset G'$ be the boundary curve of some compressing disk $D' \subset A$. We can isotope c to be disjoint from D_0 and D_1 so that an innermost-disk argument moves D' disjoint from D_0 and D_1 as well. Reversing the elementary compression from above, we can view D' now as a compressing disk from G, thereby contradicting the incompressibility of $G \cap A$.

(ii) Let $\Sigma = \partial A = S \cup F$. Suppose instead that there is a component $c' \subset G' \cap \Sigma$ that bounds a disk $D \subset \Sigma$. Then c' must come from an elementary compression of G from A into B; otherwise, c' is an inessential curve in $G \cap \Sigma$ and we contradict our assumption that $G \cap (\Sigma)$ was a collection of curves essential in Σ. Thus, we take $\beta \subset \Sigma$ to be the defining arc of this ∂^*-compression. The proof now proceeds by cases dependent on the number of components of $G \cap \Sigma$ that β joins.

One component - If β joins only one component of $G \cap \Sigma$, the compression breaks c into two components. One of these components is the curve c' from above, and we name the other curve c''. Either the disk D contains c'' or it doesn't.

We first assume that $c'' \subset D$. Then c'' itself must bound a disk in D. Hence, c' is isotopic to the original curve c so that c is inessential in Σ. This contradiction implies that c'' lies outside of the disk D. If this is the case, we can isotope D so that its boundary can be decomposed into two arcs in Σ: the arc β from the elementary compression and the arc $\gamma = \overline{c' \backslash \beta}$. If D' is the disk realizing the compression of $G \cap \Sigma$ with $\partial D' = \alpha \cup \beta$ ($\alpha \in G$), then the union $D'' = D \cup D'$ is a disk such that $\partial D'' = \alpha \cup \gamma \subset G$. If we push the interior of D'' slightly into A, then D'' constitutes a compressing disk for G in A. This contradicts the incompressibility of G in A. See Figure 3.3.

Two components - If β instead joins two components c_0, c_1 of $G \cap \Sigma$, then observe that c' is the only curve of $G' \cap \Sigma$ affected by the compression. There are two ways that c' can bound D: either D contains β or it doesn't.

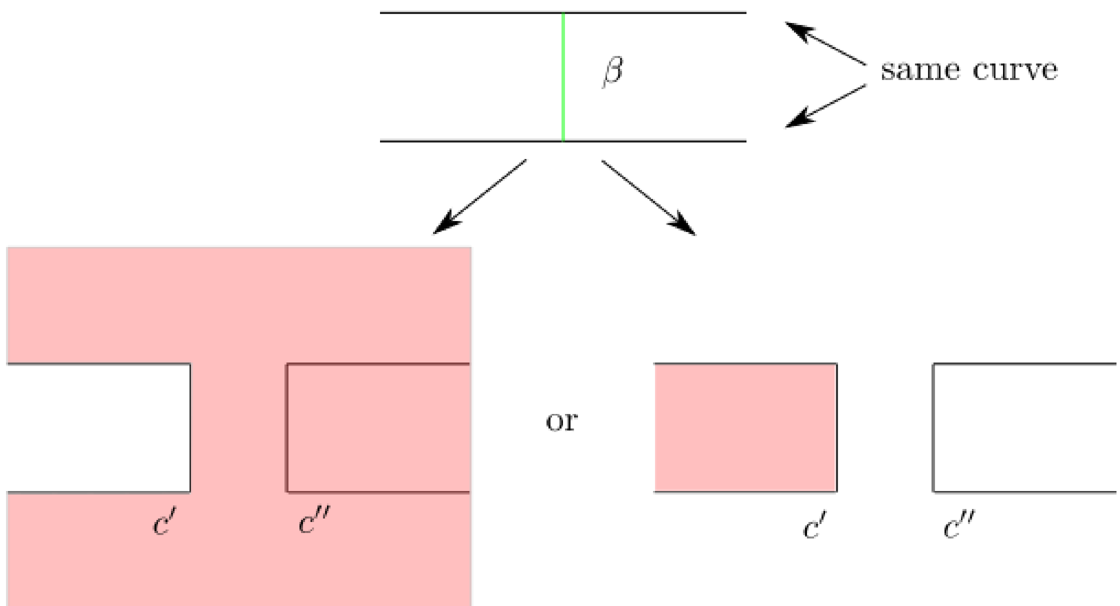

Figure 3.3. The local result of compressing G along the curve β connecting a single component of ∂G. The disk D mentioned in the proof of Lemma 3.2.4 is shaded red.

If D contains β, then reversing the compression pinches D into two disks with boundaries c_0 and c_1, respectively. Hence, both c_0 and c_1 are contained inside a disk and themselves bound disks. This contradicts our assumption that c_0 and c_1 were both essential in Σ.

If β lies outside D, then reversing the compression is the same as gluing a (two-dimensional) 1-handle onto D, thereby creating two boundary components (namely, c_0 and c_1). That is, D is converted into an annulus. From our arguments in (i) above, this annulus must be inessential in A so that it is ∂-parallel. Hence, the ∂^*-compression must have been used instead for an annular compression rather than our assumed elementary compression. See Figure 3.4.

□

3.3. Circular Distance

The necessity of the requirement that the intersections $G \cap (S \cup F)$ be essential in $S \cup F$ stems from our utilization of the curve complex of $S \cup F$. In the sequence of isotopies that we eventually

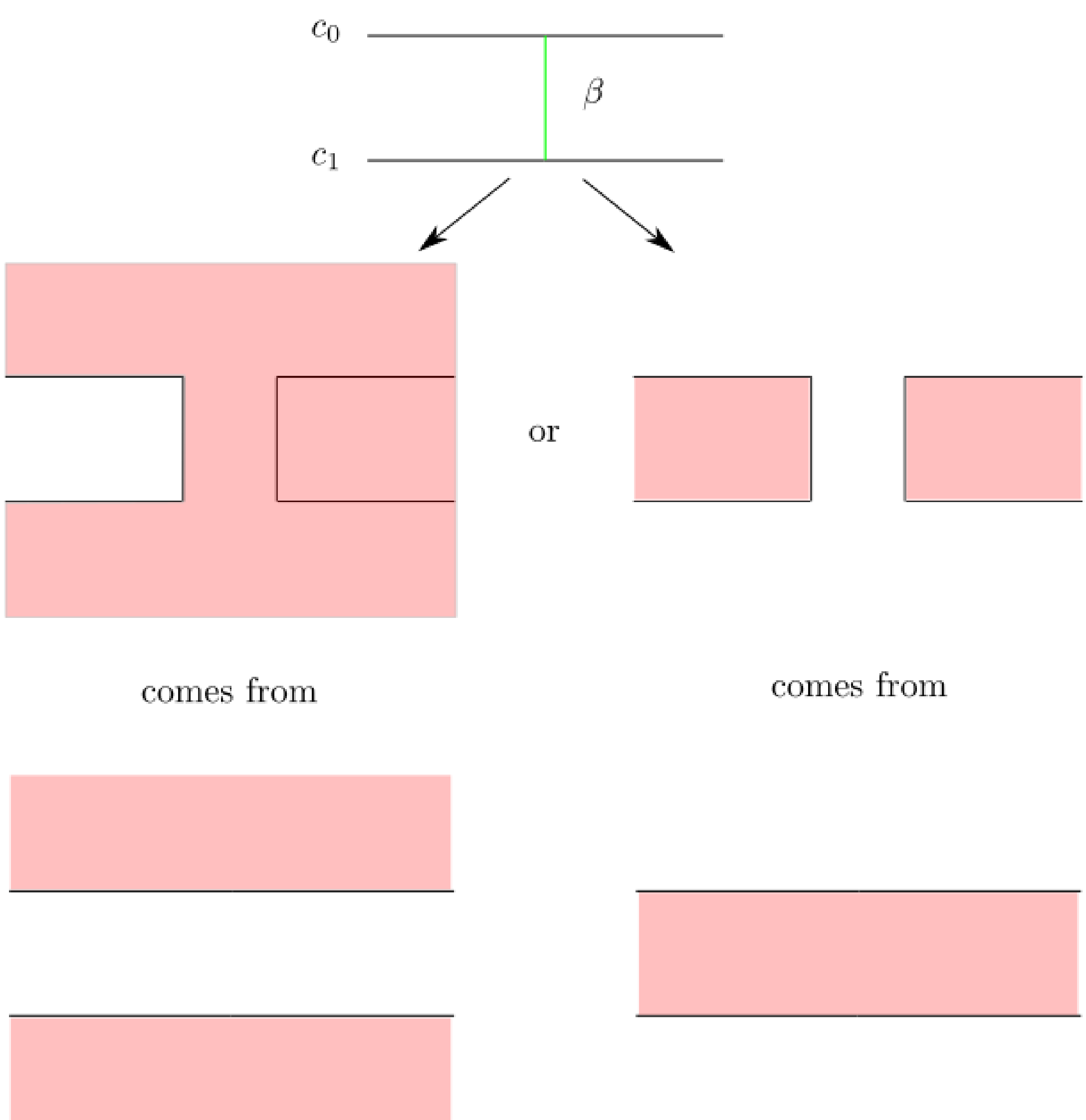

Figure 3.4. The local result of compressing G along the curve β connecting two components $c_0, c_1 \subset \partial G$. The disk D mentioned in the proof of Lemma 3.2.4 is shaded red.

create, we need to be able to compare intersection curves from one term of the sequence to the intersection curves of the next. The following lemma makes this idea more precise.

Lemma 3.3.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S). Suppose that G is a compact, orientable surface properly embedded in $E(K)$.

Denote by G' the embedding of G after a ∂^*-compression from A into B, and let $c \subset G \cap (S \cup F)$ and $c' \subset G' \cap (S \cup F)$ be curves essential in $S \cup F$. Then $d_{\mathcal{C}}(c, c') \leq 1$.

Proof. The argument again follows very closely to [**Hart**]. We include it here for completeness.

Call $\Sigma = S \cup F$. Let D be the disk that realizes the ∂^*-compression with $\partial D = \alpha \cup \beta$, where $\alpha \subset G \cap A$ and $\beta \subset \Sigma$.

If the ∂^*-compression is annular, then c is entirely removed from $S \cup F$ as a result. Hence, any c' chosen from $G' \cap \Sigma$ must be disjoint from c so that $d_{\mathcal{C}}(c, c') = 1$. We now assume that the ∂^*-compression is, in fact, an elementary compression of G from A into B.

If c is not affected by the compression, then c can be made disjoint from c' so that $d_{\mathcal{C}}(c, c') \leq 1$. Therefore, we assume that c is indeed affected by the compression. We proceed by cases dependent on whether the arc β joins one component or two different components of $G \cap \Sigma$.

Two components - If β joins two components c_0, c_1 of $G \cap \Sigma$, then we can choose closed collar neighborhoods η_0 and η_1, respectively, so that $\beta \cap \mathrm{int}(\eta_0 \cup \eta_1) = \emptyset$. Let $\hat{c}_0$ and $\hat{c}_1$ be the boundary components of η_0 and η_1, respectively, where $\beta \cap (\hat{c}_0 \cup \hat{c}_1) = \emptyset$. Then $\hat{c}_0$ is isotopic to c_0 and $\hat{c}_1$ is isotopic to c_1, and the curve δ resulting from the compression is disjoint from both $\hat{c}_0$ and $\hat{c}_1$. Hence, because c is either c_0 or c_1, any component c' of $G' \cap \Sigma$ can be chosen so that it is disjoint from c. Therefore, $d_{\mathcal{C}}(c, c') \leq 1$. See Figure 3.5.

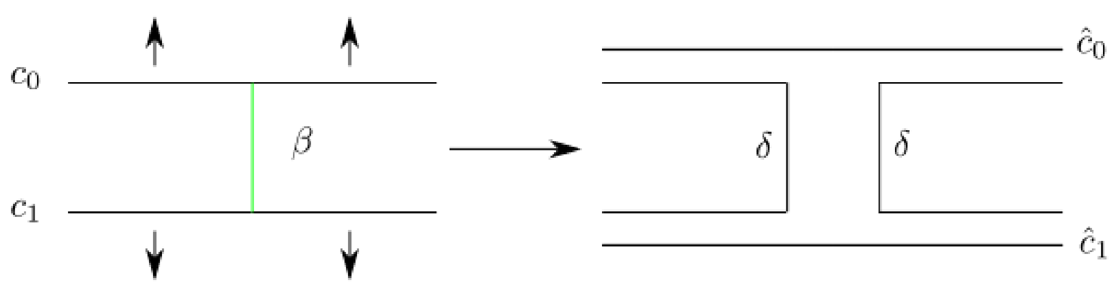

Figure 3.5. The local result of compressing G along the curve β connecting two components $c_0, c_1 \subset \partial G$.

One component - If β joins the same component c of $G \cap \Sigma$, then we consider a normal push-off $\mathcal{G}$ of G so that $G \cap \mathcal{G} = \emptyset$. In particular, $\mathcal{G}$ can be chosen so that it is disjoint from the ∂^*-compression disk D. Denote by $c_\epsilon \subset \mathcal{G} \cap \Sigma$ the image of c under the push-off.

We then perform the elementary compression of S from A into B. The curve c is thereby pinched into the pair of curves $c_0', c_1' \subset G' \cap \Sigma$, both of which are now disjoint from c_ϵ. Hence, c is isotopic to a curve that is disjoint from any component c' of $G' \cap \Sigma$. Thus, $d_{\mathcal{C}}(c, c') = 1$. See Figure 3.6.

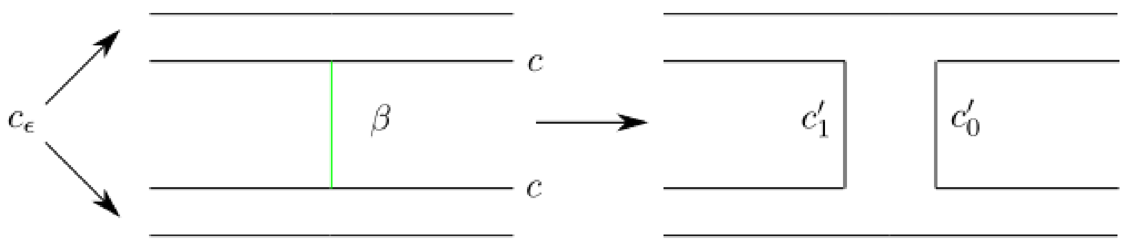

Figure 3.6. The local result of compressing G along the curve β connecting two components $c_0, c_1 \subset \partial G$.

□

DEFINITION 3.3.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S). Denote by Γ_A the set of all essential disks and essential spanning annuli of A. Similarly, define the set Γ_B for B.

For the circular Heegaard splitting (F, S), we define its *circular distance* to be

$$cd(F,S) = \min\{d_{\mathcal{C}}(\partial_S\alpha, \partial_S\beta) + d_{\mathcal{C}}(\partial_F\alpha, \partial_F\beta) \ \mid \ \alpha \in \Gamma_A, \beta \in \Gamma_B\}.$$

and the *thick distance of S* to be

$$td(S) = \min\{d_{\mathcal{C}}(\partial_S\alpha, \partial_S\beta) \ \mid \ \alpha \in \Gamma_A, \beta \in \Gamma_B\}.$$

In the case of a fibered knot $K \subset S^3$, we immediately find $td(S) = 0$ for any circular Heegaard splitting (F, S) of $E(K)$. This is realized by a vertical, essential annulus $\mathcal{A}$ in $E(K)\backslash\eta(F_0)$. This annulus is cut by S into a pair of essential spanning annuli $\alpha \subset A$ and $\beta \subset B$ with $\partial_S\alpha = \partial_S\beta$. Hence, the only non-zero contribution to the circular distance $cd(F, S)$ comes from its thin level F.

From a theorem of Johnson's [**Jo**], we know that $d_{\mathcal{C}}(\partial_F\alpha, \partial_F\beta) \leq 4$. Hence, this annulus $\mathcal{A}$ also gives an upper bound to any circular Heegaard splitting (F, S) of $E(K)$ of a fibered knot:

LEMMA 3.3.2. Let $K \subset S^3$ be a fibered knot and (F, S) be a circular Heegaard splitting for its exterior $E(K)$. Then $cd(F, S) \leq 4$.

REMARK 3.3.1. Because we have this bound, we will now only consider those knots in S^3 that are not fibered. In particular, *the results that follow do not necessarily hold for fibered knots.*

When we remove the essential spanning annuli and product disks from Γ_A and Γ_B and regard S as a Heegaard splitting of $E(K)\backslash\eta(F)$, the thick distance $td(S)$ is the usual Hempel distance $d(S)$. In some sense, we can view the thick distance as a generalization of Hempel distance. As such, the usual notions of reducibility and weak reducibility of Heegaard splittings are extended to include essential spanning annuli and essential product disks. Specifically, for a circular Heegaard splitting (F, S), we say that (F, S) is:

- *reducible* if $cd(F, S) = 0$,
- *weakly reducible* if $cd(F, S) = 1$, and
- *strongly irreducible* if $cd(F, S) \geq 2$.

For non-fibered knots, we also note that we have the inequalities

$$0 \leq td(S) \leq cd(F, S) \leq d(S) \leq td(S) + 2.$$

The last inequality follows since, for any essential spanning annulus in a compression body, there is an essential disk that is disjoint from it.

CHAPTER 4

Bounding Circular Distance

This first part of this chapter is devoted to the proof of the Main Theorem 4.1.1. We go on to give several applications of circular distance. These include a partial validation of a conjecture of Majarréz-Gutiérrez [**MG**] as well as a sketch of a proof of a theorem analagous to a result of Scharlemann and Tomova [**ScTom**].

4.1. Proof of the Main Theorem

In this section, we complete our proof of the analog to Hartshorn's theorem for an incompressible surface $G \subset E(K)$ and a circular Heegaard splitting (F, S) of $E(K)$. We recall that we must isotope G so that $G \cap A$ contains exactly one essential disk or essential spanning annulus α. Our goal is to then isotope G across S so that $G \cap B$ contains exactly one essential disk or essential spanning annulus β. Then α and β realize an upper bound for the circular distance $cd(F, S)$.

To compute this bound explicitly, our isotopy of G across S must control the distances of the curves of intersection $G \cap (S \cup F)$. If G' is the result of an elementary compression of $G \cap A$ from A into B, then Lemma 3.3.1 says the distance between a curve $c' \in G' \cap (S \cup F)$ and a curve $c \in G \cap (S \cup F)$ is at most one. Hence, the upper bound we compute is exactly the number of elementary compressions of $G \cap A$ from A into B across S in order to produce the essential disk or essential spanning annulus β.

Our first goal, then, is to produce the essential disks or essential spanning annuli α and β.

Lemma 4.1.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S) such that F is incompressible and $td(S) \geq 2$. Suppose that G is a closed, connected, orientable, essential surface in $E(K)$ such that each component of $G \cap (S \cup F)$ is essential in either S or F and that each component of $G \cap A$ is incompressible in A. If $G \cap B$ contains an essential disk or an essential spanning annulus, then there is a sequence of isotopies

$$G \simeq G_0 \simeq G_1 \simeq \cdots \simeq G_k \simeq G_{k+1} \simeq \cdots \simeq G_n$$

of G such that

- Each component of $G_i \cap A$ is incompressible in A_i for each $0 \leq i \leq n$;
- Each component of $G_i \cap (S \cup F)$ is essential in either S or F;
- For any choice of components $c_i \in G_i \cap (S \cup F)$ that belong to the same component of $S \cup F$, $d_{\mathcal{C}}(c_i, c_{i+1}) \leq 1$ for $0 \leq i \leq n-1$.
- For $0 \leq i \leq k$, at least one component of $G_i \cap B$ is an essential disk or essential spanning annulus.
- For $k+1 \leq i \leq n-1$, no component of either $G_i \cap A$ or $G_i \cap B$ is an essential disk or essential spanning annulus;
- The final isotopy from G_{n-1} to G_n ensures that $G_n \cap A$ contains exactly one essential disk or essential spanning annulus component;
- We must have $k \leq n-2$.

Proof. First, remove from $G \cap A$ any ∂^*-parallel annuli via annular compressions to form G_0. If none exist, we take $G_0 = G$. Because $td(S) \geq 2$, no component of $G \cap A$ is an essential disk or essential spanning annulus so that $\chi(G_0 \cap A) < 0$. Some component of G_0 must meet S in a curve essential in S since, otherwise, G may be passed through S so as to lie entirely in A or B. Hence, there is an elementary compression of $G_0 \cap A$ from A into B across S. Performing this elementary compression creates $\hat{G}_1$, an isotopy of G_0 that differs only by an elementary compression. Now remove from $\hat{G}_1 \cap A$ any ∂^*-parallel annuli via annular compressions. As before, if none exist, we taken $G_1 = \hat{G}_1$. Continue in this fashion to create the remaining G_i for $2 \leq i \leq k$.

We choose k to be the greatest integer such that $G_k \cap B$ contains an essential disk or essential spanning annulus. Such an integer exists by Lemma 3.1.1. Starting with G_k, continue the procedure above to create G_i for $k+1 \leq i \leq n$, choosing n to be the smallest integer such that $G_n \cap A$ contains an essential disk or essential spanning annulus component. We note here that $k \leq n-2$ because $td(S) \geq 2$. The first three bullet points are shown using Lemmas 3.2.1 and 3.3.1.

If $G_n \cap A$ contains any essential disk components, then there is only one such component; otherwise, $\chi(G_n \cap A) \geq \chi(G_{n-1} \cap A) + 2$ which contradicts Lemma 3.1.1.

It is possible, however, that $G_n \cap A$ contains two essential spanning annuli. If this is indeed the case, then these annuli must have come from a pair-of-pants component of $G_{n-1} \cap A$. Observe that there is an alternative elementary compression of this component across F. See Figure 4.1. Performing this elementary compression produces a single essential spanning annulus in $G_n \cap A$. This shows the fourth, fifth, and sixth bullet points.

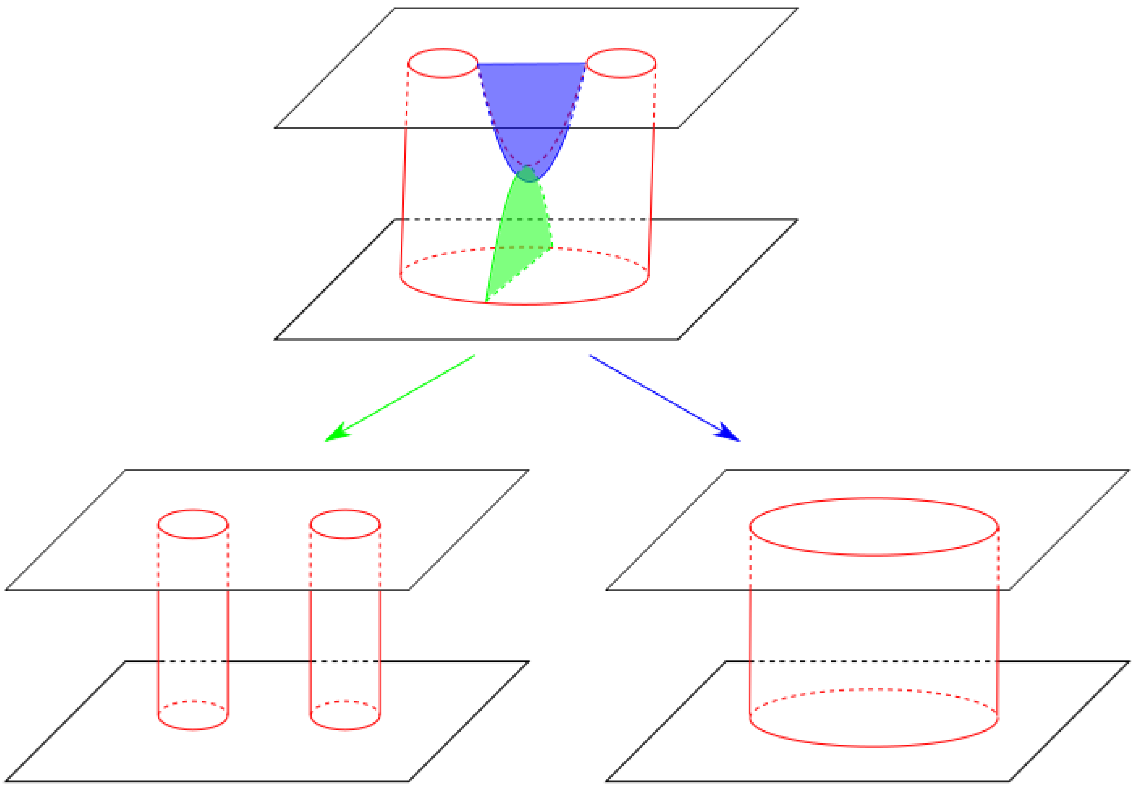

Figure 4.1. If there is an elementary compression through F coming from a pair of pants, then there is an alternative elementary compression through S that we make instead.

To show the final bullet point, we notice that $G_i \cap A$ and $G_i \cap B$ cannot both contain an essential disk and an essential annulus since $td(S) \geq 2$ and F is incompressible. If $k = n-1$, there must be components $c_k \subset G_k \cap S$ and $c_n = c_{k+1} \subset G_n \cap S$ such that c_k bounds an essential disk or an essential spanning annulus in $G_n \cap A$. The third bullet point shows that $d_{\mathcal{C}}(c_k, c_n) \leq 1$. Hence, we would have $td(S) \leq 1$ and contradict our assumption that $td(S) \geq 2$. Then $k \leq n-2$ and we have shown the final bullet point. $\square$

Again, we emphasize the symmetry between A and B in this lemma. That is, we could have started with an essential disk or essential spanning annulus in $G \cap A$ instead. We could then perform our sequence of isotopies via elementary compressions of G from B into A. We also take a moment here to observe that either $\chi(G_n \cap A) \leq 1$ or $\chi(G_n \cap B) \leq 1$.

If no such essential disk or essential annulus exists in either $G \cap A$ or $G \cap B$, then we can still produce one in the same fashion as above.

Lemma 4.1.2. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition (F, S) such that F is incompressible and $td(S) \geq 2$. Suppose that G is a closed, connected, orientable, essential surface in $E(K)$ such that each component of $G \cap (S \cup F)$ is essential in either S or F and that each component of $G \cap A$ is incompressible in A. If neither $G \cap A$ nor $G \cap B$ contain an essential disk or an essential spanning annulus, then there is a sequence of isotopies

$$G \simeq G_{-m} \simeq G_{1-m} \simeq \cdots \simeq G_0 \simeq G_1 \simeq \cdots \simeq G_n$$

of G such that

- Each component of $G_i \cap A$ is incompressible in A_i for $-m \leq i \leq 0$ and $G_i \cap B$ is incompressible in B_i for $0 \leq i \leq n$;
- Each component of $G_i \cap (S \cup F)$ is essential in either S or F;
- For any choice of components $c_i \in G_i \cap (S \cup F)$ that belong to the same component of $S \cup F$, $d_{\mathcal{C}}(c_i, c_{i+1}) \leq 1$ for $-m \leq i \leq n-1$.
- There is exactly one component of $G_{-m} \cap A$ and $G_n \cap B$ that is either an essential disk or an essential spanning annulus;
- For $1-m \leq i \leq n-1$, no component of either $G_i \cap A$ or $G_i \cap B$ is an essential disk or essential spanning annulus.

Proof. Since both $G \cap A$ and $G \cap B$ contain no essential disks or essential spanning annuli, it follows that both $\chi(G \cap A) < 0$ and $\chi(G \cap B) < 0$. Hence, $G \cap A$ has an elementary compression from A into B across S. We define the sequence of isotopies as before to get the G_i for $-m \leq i \leq -1$.

Then $G_{-m} \cap A$ contains exactly one essential disk or essential spanning annulus. Now starting at G_0, but reversing the roles of A and B in the previous lemma, gives us the G_i surfaces for $1 \leq i \leq n$. Then $G_n \cap B$ contains exactly one essential disk or essential spanning annulus. All the noted properties of the sequence are now satisfied via the proof of Lemma 4.1.1. □

THEOREM 4.1.1 (Main Theorem). Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. If its exterior $E(K)$ contains a closed, orientable, essential surface G of genus g, then $cd(F, S) \leq 2g$.

PROOF. We may isotope G so that it intersects $S \cup F$ in a minimal number of curves. We divide the argument into two cases: (i) $td(S) \geq 2$ and (ii) $td(S) \leq 1$

We first assume that $td(S) \geq 2$. By Lemma 3.2.1, this embedding of G satisfies the conditions of Lemmas 4.1.1 and 4.1.2. From these lemmas, we conclude that there must exist a sequence of isotopies $G_0 \simeq \cdots G_n$ such that $G_0 \cap A$ and $G_n \cap B$ both contain exactly one essential disk or essential spanning annulus each. Call these components $P_A \subset G_0 \cap A$ and $P_B \subset G_n \cap B$, respectively. We also choose a sequence of curves $c_i \subset G_i \cap (S \cup F)$ such that $c_0 \in \partial P_A$ and $c_n \in \partial P_B$

We bound n by first observing that $\chi(G) = \chi(G_0 \cap A) + \chi(G_0 \cap B)$ because $G \simeq G$ and $G \cap (S \cup F)$ is a collection of circles with Euler characteristic zero. Since $G_0 \cap A$ contains exactly one essential disk or essential spanning annulus, we use Lemma 3.1.1 to show inductively that

$$\begin{aligned}
\chi(G) &= \chi(G_0 \cap A) + \chi(G_0 \cap B) \\
2 - 2g &\leq 1 + \chi(G_0 \cap B) \\
1 - 2g &\leq \chi(G_0 \cap B) \\
1 - 2g &\leq \chi(G_n \cap B) - n \\
1 - 2g &\leq 1 - n \\
n &\leq 2g.
\end{aligned}$$

Since n was chosen to be the smallest integer such that $G_n \cap B$ contains an essential disk or essential spanning annulus, we then need n elementary compressions of G to move from having an essential disk or essential spanning annulus in A to having one in B. Hence, we can bound the circular distance of the decomposition as

$$cd(F,S) \leq d_{\mathcal{C}}(c_0, c_n) \leq n \leq 2g.$$

We now assume that $td(S) \leq 1$. The remarks following Definition 3.3.1 show that $cd(F,S) \leq 3$. If $cd(F,S) \leq 2$, then $cd(F,S) \leq 2g$ trivially. Hence, the only other case we need to consider is when $cd(F,S) = 3$ with $g = 1$. We show now that this is impossible.

Observe that $td(S) = 1$ and $d(S) = 3$ when $cd(F,S) = 3$ and S is strongly irreducible. Suppose that $G \cap B$ contains an essential disk or essential spanning annulus. Then we are able to recover the sequence of isotopies constructed in Theorem 4.1.1 until we reach the final paragraph of the proof. The last bullet point follows from the fact that $cd(F,S) = 3$; that is, $G_i \cap A$ and $G_i \cap B$ cannot both contain essential disks and essential spanning annuli for any $0 \leq i \leq n$. Hence, $n > k$.

If $k = n-1$, there would exist components $c_k \subset G_k \cap S$ and $c_n \subset G_n \cap S$ such that c_k bounds an essential disk or essential spanning annulus $P_B \subset G_n \cap B$ and that c_n bounds an essential disk or essential spanning annulus $P_A \subset G_k \cap A$. Moreover, because $td(S) = 1$, we see that $d_{\mathcal{C}}(c_k, c_n) = 1$.

The strong irreducibility of S shows P_A and P_B cannot both be disks. A similar contradiction in $cd(F,S) = 3$ is found, without loss of generality, if P_A is a disk and P_B is an annulus. We then see that P_A and P_B are both annuli. However, $d_{\mathcal{C}}(\partial_F P_A, \partial_F P_B) \leq 1$ so that $cd(F,S) \leq 2$ again. We conclude that $k \leq n-2$.

If neither $G \cap A$ nor $G \cap B$ contain an essential disk or essential spanning annulus, then we are able to recover the sequence constructed in Theorem 4.1.2 in its entirety. The Euler characteristic argument above then shows that $cd(F,S) \leq 2$, thereby contradicting the requirement that $cd(F,S) = 3$. □

Corollary 4.1.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F,S) such that F is incompressible. If its exterior $E(K)$ contains a closed, orientable, essential surface G of genus g that can be isotoped to be disjoint from F, then $d(S) \leq 2g$.

Proof. The proof of Theorem 4.1.1 holds. Moreover, we can realize the bound on $cd(F,S)$ without using essential spanning annuli. □

4.2. Circular Distance Bound via an Alternate Seifert Surface

It was conjectured in [**MG**] that it may be possible that Seifert surfaces of minimal genus appear as the thin levels of thin circular handle decompositions. We provide a partial affirmation of this conjecture; however, because Seifert surfaces are not closed, more justification is needed in order to prove this fact. We first state a uniqueness theorem as a corollary to the following theorem:

THEOREM 4.2.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. If F' is an incompressible Seifert surface for K with genus g that is not isotopic to F, then $d(S) \leq 2g + 1$.

COROLLARY 4.2.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is a Seifert surface of K that realizes the Seifert genus $g(K)$. If $d(S) > 2g(K)+1$, then F is the unique Seifert surface of minimal genus for K up to isotopy.

We adapt our work from the previous section to the case where the closed surface $G \subset E(K)$ is instead taken to be properly embedded with non-empty boundary. Chapter 3 classifies the compact, connected, incompressible, ∂^*-incompressible surfaces that may appear in the compression bodies A and B, and Lemma 3.2.3 indicates when elementary compressions exist.

The next step is to mimic Theorems 4.1.1 and 4.1.2. We do this by replacing the closed surface G with an incompressible Seifert surface F' for the knot K. We first notice that Lemma 3.2.1 is still applicable in this setting; that is, we can still isotope F' to (i) intersect S in curves that are essential in S and (ii) intersect A and B so that $F' \cap A$ and $F' \cap B$ are incompressible in A and B, respectively. In addition, Scharlemann and Thompson [**ST1**] show that F' may be assumed to be disjoint from the thin level F (as F, too, is an incompressible Seifert surface for K).

Observe that if F' is disjoint from F, then $F' \cap A$ and $F' \cap B$ contain no essential spanning annuli. Hence, we first consider the case where one of either $F' \cap A$ or $F' \cap B$ contains an essential disk component, and then we consider the case where neither $F' \cap A$ nor $F' \cap B$ contain any essential disks.

LEMMA 4.2.1. Let $K \subset S^3$ be a non-fibered knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. Suppose further that F' is an incompressible Seifert surface for K disjoint from and non-isotopic to F such that $F' \cap S$ is a collection of simple closed curves that are essential in S.

If $F' \cap A$ contains an essential disk and each component of $F' \cap A$ is incompressible in A, then there exists a sequence of isotopies

$$F' \simeq F'_0 \simeq F'_1 \simeq \cdots \simeq F'_m \simeq \cdots \simeq F'_n$$

such that

- Each component of $F'_i \cap S$ is essential in S;
- Each component of $F'_i \cap A$ is incompressible in A;
- For any choice of components $c_i \in F'_i \cap S$ and $c_{i+1} \in F'_{i+1} \cap S$, we have $d_{\mathcal{C}}(c_i, c_{i+1}) \leq 1$ for $0 \leq i \leq n-1$;
- $F'_n \cap B$ contains exactly one essential disk component;
- $F'_i \cap A$ contains an essential disk component for each $0 \leq i \leq m$, and neither $F'_i \cap A$ nor $F'_i \cap B$ contain any essential disk components for $m+1 \leq i \leq n-1$.

Proof. If $F' \cap A$ contains any ∂^*-parallel annuli, use annular compressions to remove them from B and define the resulting surface F'_0; if not, then define $F'_0 = F'$. Because $td(S) \geq 2$ and $F'_0 \cap F = \emptyset$, we see that $F'_0 \cap B$ contains no essential disks or essential spanning annuli. This imples that $F'_0 \cap B$ is ∂^*-compressible. Perform an elementary compression to form $\hat{F}'_1$. Now proceed inductively (as in the proof of Theorem 4.1.1) and let m be the smallest integer such that either (i) $F'_m \cap B$ contains an essential disk component or (ii) $\chi(F'_m \cap B) = 0$ and F'_m contains no essential disk components.

If $F'_m \cap B$ contains an essential disk component, recall that $\chi(F' \cap B) = \chi(F'_{m-1} \cap B) + 1$ by Lemma 3.1.1. If $F'_m \cap B$ were to contain more that one essential disk component, then it would be the case that $\chi(F'_m \cap B \geq \chi(F'_{m-1} \cap B) + 2$. This contradicition dictates that $m = n$ in this case.

If $\chi(F'_m \cap B) = 0$ and $F'_m \cap B$ contains no essential disk components, then $F'_m \cap B$ must be a collection of annuli. Note that none of these annuli are ∂^*-parallel by construction. If all of these annuli intersect $\partial_v B$, they would also intersect S since $F'_m \cap F = \emptyset$. We may then isotope F'_m to lie entirely in A. We then see that F'_m is a properly embedded surface in B that is connected, incompressible, and disjoint from both S and F. Additionally, $\partial F'_m \subset \partial_v B$ has exactly one component. Therefore, F'_m must be isotopic to F, which contradicts the assumption of the lemma.

We can now choose an annulus in $F'_m \cap B$ that is disjoint from $\partial_v B$. Being disjoint from F, this annulus must be ∂^*-compressible. We perform the elementary compression to create $\hat{F}'_{m+1}$ and then form F'_{m+1} by removing any ∂^*-parallel annuli via annular compressions as before. Set $n = m + 1$. We then find $\chi(F'_n \cap B) = 1$ so that $F'_n \cap B$ contains exactly one essential disk component.

It cannot be the case that both $F' \cap A$ nor $F' \cap B$ contain essential disk components as S is strongly irreducible. We conclude $n > m$. If $m = n - 1$, then there exist curves $c_m \subset F'_m \cap S$ and $c_{m+1} = c_n \subset F'_n \cap S$ so that c_m bounds an essential disk in $F'_m \cap A$ and c_n bounds an essential disk in $F'_n \cap B$. Then $d(S) \leq 1$ so that this contradiction allows us to conclude that $m \leq n - 2$.

The remaining points of the lemma follow as in the proof of Lemma 4.1.1. □

Lemma 4.2.2. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular Heegaard splitting (F, S) such that F is incompressible. Suppose further that F' is an incompressible Seifert surface for K disjoint from and non-isotopic to F such that $F' \cap S$ is a collection of simple closed curves that are essential in S.

If neither $F' \cap A$ nor $F' \cap B$ contains any essential disk components, and if each component of $F' \cap A$ and $F' \cap B$ is essential in A and B, respectively, then there exists a sequence of isotopies

$$F'_{-m} \simeq F'_{-m+1} \simeq \cdots \simeq F'_0 \simeq F' \simeq F')1 \simeq \cdots \simeq F'_n$$

such that

- Each component of $F'_i \cap S$ is essential in S;
- Each component of $F'_i \cap A$ and $F'_i \cap B$ is essential in A and B, respectively;
- For any choice of components $c_i \in F'_i \cap S$ and $c_{i+1} \in F'_{i+1} \cap S$, we have $d_{\mathcal{C}}(c_i, c_{i+1}) leq 1$ for $-m \leq i \leq n - 1$;
- Both $F'_{-m} \cap A$ and $F'_n \cap B$ contain exactly one essential disk component, and no $F'_i \cap A$ or $F'_i \cap B$ contains any essential disk components for $-m + 1 \leq i \leq n - 1$.

Proof. This is nearly identical to the proof of Lemma 4.1.2. □

We now prove Theorem 4.2.1:

Proof of Theorem 4.2.1. This proof is similar to the Main Theorem (4.1.1). If $d(S) \leq 1$, then the theorem follows; hence, we assume $d(S) \geq 2$.

First, isotope F' so that it intersects S in a minimal number of components. Regardless of whether or not $F' \cap A$ and $F' \cap B$ contain essential disk components, Lemmas 4.2.1 and 4.2.2 prove there exists a sequence $F'_0, \ldots, F'_n$ of Seifert surfaces isotopic to F' such that $F'_0 \cap B$ and $F'_n \cap A$ each contain exactly one essential disk component. Moreover, $F'_i \cap S$ is a collection of simple closed curves that are essential in S for all $0 \leq i \leq n$, and $\chi(F'_{i+1} \cap A) = \chi(F'_i \cap A) + 1$.

In any case, let c_B be the boundary of the essential disk in $F'_0 \cap B$ and c_A be the boundary of the essential disk in $F'_n \cap A$. For $0 \leq i \leq n-1$, we have $d_{\mathcal{C}}(c_i, c_{i+1}) \leq 1$. By setting $c_0 = c_B$ and $c_n = c_A$, repeated application of the triangle inequality shows that $d_{\mathcal{C}}(c_A, c_B) \leq n$ (cf. the proof of the Main Theorem). Moreover, we find

$$\begin{aligned}
\chi(F') &= \chi(F'_0 \cap A) + \chi(F'_0 \cap B) \\
1 - 2g &\leq 1 + \chi(F'_0 \cap B) \\
-2g &\leq \chi(F'_0 \cap B) \\
-2g &\leq \chi(F'_n \cap B) - n \\
-2g &\leq 1 - n \\
n &\leq 2g + 1.
\end{aligned}$$

so that

$$d(S) \leq d_{\mathcal{C}}(c_A, c_B) \leq n \leq 2g + 1.$$

□

4.3. Circular Distance Bound via a Strongly Irreducible Heegaard Splitting

For a closed 3-manifold M, suppose we are given two strongly irreducible Heegaard splittings P and Q. Scharlemann and Tomova [**ScTom**] showed that (assuming trivial intersection and stabilization cases don't occur) the Hempel distance of P is bounded by Q similar to Hartshorn's bound for incompressible surfaces. That is, $d(P) \leq 2 - \chi(Q) = 2g(Q)$. We outline an analogous theorem in this section.

THEOREM 4.3.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a locally thin circular Heegaard splitting (F, S). If Σ is a strongly irreducible Heegaard splitting of genus g for $E(K)$, then $cd(F, S) \leq 2g$.

COROLLARY 4.3.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular locally thin circular Heegaard splitting (F, S), and suppose $\tau = \{\tau_1, \dots, \tau_t\}$ is a tunnel system for K. If the Heegaard splitting Σ induced by τ is strongly irreducible, then $cd(F, S) \leq 2t + 2$. In particular, if K has tunnel number $t(K)$ and possesses such a strongly irreducible Σ, then $cd(F, S) \leq 2t(K) + 2$.

PROOF. This follows from Theorem 4.3.1 by observing that the genus of Σ is $t + 1$. $\square$

COROLLARY 4.3.2. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a circular locally thin circular Heegaard splitting (F, S), and suppose $t(K) = 1$. Then $cd(F, S) \leq 4$.

PROOF. For any tunnel τ realizing $t(K)$, the Heegaard splitting induced by τ is strongly irreducible and has genus two. $\square$

LEMMA 4.3.1. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has an admissible circular handle decomposition $\{(W; A, B)\}$. Any Heegaard splitting Σ of $E(K)$ must intersect both the thin level F and the thick level S.

PROOF. Note that Σ splits $E(K)$ into a handlebody H and a compression body C that contains the peripheral torus $\partial E(K)$. Observe that a meridian curve $\gamma \subset \partial E(K)$ of $\eta(K)$ must intersect F and S since these are both Seifert surfaces of K. A push-off of γ into $E(K)$ can be taken to be part of a spine for H so that Σ must intersect both F and S. $\square$

We now extend a standard *1-parameter sweep-out* argument (cf. [**Schul**]) to compact surfaces G with non-empty boundary and circular Heegaard splittings.

LEMMA 4.3.2. Let M be a compact 3-manifold with a strongly irreducible Heegaard splitting Σ. Suppose $G \subset M$ is a compact, orientable, incompressible, properly embedded surface that

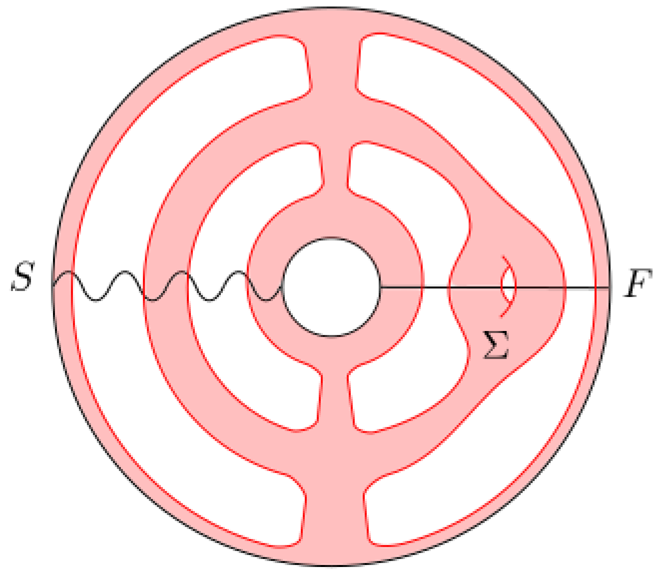

Figure 4.2. A Heegaard surface Σ juxtaposed with a circular Heegaard splitting (F, S) of $E(K)$. The shaded region constitutes the compression body component of the Heegaard splitting of $E(K)$ defined by Σ.

intersects ∂M in curves essential in ∂M. Then Σ may be isotoped to intersect G in curves that are essential (or ∂-parallel) in both Σ and G.

Proof. Let σ_0 and σ_1 be spines for the compression bodies in the Heegaard splitting defined by Σ. Let $h\colon M \to [0,1]$ be a sweep-out of M induced by $\sigma_0 = h^{-1}(0)$ and $\sigma_1 = h^{-1}(1)$, where $h^{-1}\left(\frac{1}{2}\right) = \Sigma$. We denote $\Sigma_t = h^{-1}(t)$, $A_t = h^{-1}([0,t])$, and $B_t = h{-}1([t,1])$. Possibly after an isotopy of G, we can assume that σ_0 and σ_1 transversally intersect G in a finite collection of points and curves essential in ∂M. We also note here that, for any $0 < t < 1$, each component of $\Sigma_t \cap G$ is essential or ∂-parallel in G because G is incompressible.

For small $\epsilon > 0$, observe that A_ϵ and $B_{1-\epsilon}$ intersect G in some essential spanning annuli and possibly some compression disks for Σ_ϵ and $\Sigma_{1-\epsilon}$. If there are no disks in $G \cap A_t$ for any time $0 < t < 1$, then there are no disks in $G \cap B_t$ either. We then take $\Sigma_{1/2}$ to be the required isotopy of Σ. So we assume that, for some times t_A and t_B where $0 < t_A < t_B < 1$, there is at least one disk in $G \cap A_{t_A}$ and at least disk in $G \cap B_{t_B}$.

As t increases from 0 to 1, the disks and annuli in $G \cap A_t$ will merge and split in various ways; but, for any $0 \le t \le 1$, there cannot be disks in $A_t \cap G$ and $B_t \cap G$ simultaneously since $\Sigma_t \simeq \Sigma$

is strongly irreducible. Because there is an essential disk in $A_{t_A} \cap G$, there aren't any in B_{t_A}. Similarly, there are essential disks in $B_{t_B} \cap G$, so there can't be any in $A_{t_B} \cap G$. Therefore, for some value $T \in (t_A, t_B)$, there can't be any essential disks in either A_T or B_T. Hence, Σ_T is an isotopy of Σ that intersects G in curves that are essential in Σ and are either essential or ∂-parallel in G. $\square$

Lemma 4.3.3. Let $K \subset S^3$ be a knot whose exterior $E(K)$ has a locally thin admissible circular handle decomposition $\{(W; A, B)\}$. Suppose $G \subset W$ is a compact, orientable, incompressible, properly embedded surface such that $\partial G \subset \partial_+ A \cup \partial_+ B$ and neither $\partial G \cap \partial_+ A$ nor $\partial G \cap \partial_+ B$ is empty. Further assume that each curve in $\partial G \cap \partial W$ is essential in $\partial W \backslash \partial_v W$ and that, for any spines of A and B, G intersects the graph portions of both spines. Then the thick level S may be isotoped to intersect G in curves that are either essential or ∂-parallel in both S and G.

Proof. We appeal to a sweep-out argument as in the previous lemma, using the thick level S to sweep out the compact 3-manifold W. We keep the same notation but note that, for each $i = 0, 1$, the spine σ_i is a graph joined to a a surface homeomorphic to the thin level F. The presence of essential disks to both sides of Σ_t near σ_0 and σ_1 is guaranteed by the intersection of G with their graph portions. The proof then follows identically as before since $\partial G \subset h^{-1}(\{0, 1\})$. $\square$

We can now sketch a proof of Theorem 4.3.1:

Sketch of proof of Theorem 4.3.1. First observe that Lemma 4.3.1 shows that Σ must intersect both F and S. We can then apply Lemma 4.3.2 to Σ and the incompressible Seifert surface F so that $\Sigma \cap F$ is a collection of curves essential in both Σ and F (and possibly some ∂-parallel in F).

Let $N = E(K) \backslash \eta(F)$. We either have $\Sigma' = \Sigma \cap N$ compressible in N or we don't.

Σ' is incompressible in N: If Σ' is incompressible in N, then we apply Lemma 4.3.3 to Σ' and the strongly irreducible Heegaard splitting S of N to get a new embedding of S (which we will still call S) such that $S \cap \Sigma'$ is a collection of curves that are essential or ∂-parallel in S.

We would like to employ Lemmas 4.1.1 and 4.1.2, but they do not account for ∂-parallel curves in S. There are new cases that occur for a ∂^*-compression to join together or pinch apart curves of

$\Sigma' \cap S$ together when we allow for ∂-parallel curves of S. These cases aren't already accommodated by Lemmas 4.1.1 and 4.1.2:

(1) we join one ∂-parallel curve to another to form an inessential curve,

(2) we join an essential curve to itself to form a ∂-parallel curve and another essential curve,

(3) we join one ∂-parallel curve to an essential curve to form another essential curve

(4) we join one ∂-parallel curve to itself to form a pair of essential curves,

(5) we join a pair of essential curves to form a ∂-parallel curve.

The first case describes an annular compression since we only have one boundary component of F and S. We perform all possible annular compressions before performing an elementary compression. The only case of an elementary compression which does not maintain at least one essential curve is the last one. Every other case offers elements of $C(S)$ for us to use to compare essential disks and essential spanning annuli of A and B. We need to avoid the final case so as not to remove all essential curves from $\Sigma' \cap S$.

If we can avoid this final case, the proof of Lemmas 4.1.1 and 4.1.2 can then be applied to the compact surface with boundary Σ'. We can then apply the proof of the Main Theorem (4.1.1) to Σ in $E(K)$ so that $cd(F, S) \leq 2g(\Sigma)$.

If we cannot avoid the final case, then there is an isotopy of Σ' in the sequence such that $\Sigma' \cap S$ is entirely comprised of curves that are ∂-parallel in S. Hence, either we can use elementary compressions to end up with an essential spanning annulus in A, or the only spanning annuli we find are ∂-parallel.

If there is a ∂-parallel annulus $\mathcal{A}$ in A, then it cannot abut a another ∂-parallel annulus in B; otherwise, Σ would be ∂-parallel in $E(K)$. Therefore, $\mathcal{A}$ abuts a surface $\mathcal{G}$ that is incompressible in N such that $\chi(\mathcal{G} < 0$. Then $\mathcal{G}$ admits an elementary compression across S that splits a ∂-parallel curve of $\Sigma' \cap S$ into at least one essential curve in S. Furthermore, performing this elementary compression turns $\mathcal{A}$ into a pair of pants in A. Hence, there is another elementary compression this pair of pants across F that results in a pair of essential spanning annuli (cf. Figure 4.1). Therefore, a ∂-parallel annulus can be used to produce an essential spanning annulus, call it α, in A.

We now wish to produce an essential disk or essential spanning annulus β in B. Either we can use elementary compressions of Σ' from B into A to eventually produce such a surface, or we eventually end up with ∂-parallel annuli in B. We can use one of these ∂-parallel annuli to produce a pair of essential spanning annuli. Choose one of them to be β. The proof of the Main Theorem 4.1.1 can then be applied so that we need at most $2g(\Sigma')$ elementary compressions to start with α and isotope Σ' so that $\Sigma' \cap B$ contains β. Hence, we obtain the bound $cd(F, S) \leq 2g(\Sigma)$.

Σ' is compressible in N: Now Σ' is either compressible to one side or to both sides (but not simultaneously since it is weakly incompressible). The proof proceeds by cases:

Subcase - Σ' compressible only to one side: Let V be the region in N bounded by Σ' that contains a compressing disk. Take Δ to be a collection of compressing disks for Σ' in N that is maximal in the sense that no two disks in Σ' are parallel in V. If we compress Σ' along Δ to get the surface Σ'', then Σ'' is incompressible to one side (as Δ was taken to be maximal). Observe that the 2-handle Addition Lemma makes Σ'' incompressible to the other side as well. This is due to the fact that Σ is a strongly irreducible Heegaard splitting of $E(K)$, and a reversal of the compressions along Δ would produce a pair of disjoint compressing disks (one to each side of Σ). Hence, we have just shown that $E(K)$ is Haken so that we can apply the Main Theorem (4.1.1) and find (again loosely) that $cd(F, S) \leq 2g(\Sigma)$.

Subcase - Σ' compressible to both sides: Compressing Σ' in N as much as possible to one side yields a surface Σ'_1, and compressing Σ' in N as much as possible to the other side yields another surface Σ'_2. We find all components of Σ'_1 and Σ'_2 must be incompressible to both sides since Σ has been maximally compressed and is a strongly irreducible Hegaard splitting of $E(K)$. We then apply Corollary 4.1.1 to bound $cd(F, S) \leq 2g(\Sigma)$. □

Bibliography

[BS] D. Bachman, S. Schleimer. *Distance and bridge position*, Pac. Jour. Math., vol. 219, no. 2, pp 221-235 (2005).

[Bir] J. Birman. *The topology of 3-manifolds, Heegaard distance and the mapping class group of a 2-manifold*, Proc. Symp. Pure Mathematics, vol 74 (2006).

[BO] F. Bonahon, J. P. Otal. *Scindements de Heegaard des espaces lenticulaires*, Ann. Sci. E.N.S., vol. 16, pp 451-466 (1983).

[CG] A. Casson, C. McA. Gordon. *Reducibility of Heegaard Splittings*, Topol. Appl. vol. 27, pp. 275-283 (1987).

[E] T. Evans. *High distance Heegaard splittings of 3-manifolds*, Topol. Appl., vol. 15, pp. 2631-2647 (2006).

[Hart] K. Hartshorn. *Heegaard splittings of Haken manifolds have bounded distance*, Pac. Jour. Math., vol. 204, no. 1, pp 61-75 (2002).

[Harv] W. J. Harvey, *Geometric structure of surface mapping clas groups*, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 255-269.

[Hemp] J. Hempel, *3-manifolds as viewd from the curve complex*, Topology 40 (2001) pp. 631-567. MR1838999

[Ja] W. H. Jaco. Lectures on three-manifold topology. CBMS Regional Conference Series in Mathematics, 43. *American Mathematical Society, Providence, R.I.*, 1980. xii+251 pp. ISBN: 0-8218-1693-4

[Jo] J. Johnson, *Heegaard splittings and open books*, October 10, 2011, preprint arXiv:1110.2142v1 [math.GT].

[K] T. Kobayashi. *Heights of simple loops and pseudo-Anosov homeomorphisms*, in 'Braids (Santa Cruz, CA, 1986),' Am. Math. Soc., Providence, RI, pp 327-338, MR 89m:57015, Zbl 0663.57010 (1988).

[MG] F. Manjarrez-Gutiérrez. *Circular thin position for knots in S^3*, Algebraic & Geometric Topology, vol. 9, pp 429-454 (2009).

[Mi] J. Milnor. *Lectures on the h-cobordism theorem*, Princeton University Press. Princeton, N.J., 1963.

[Mo] E. E. Moise. "Affine Structures in 3-Manifolds: V. The Triangulation Theorem and Hauptermutung." *Annals of Mathematics*, vol. 56, no. 1, 1952, pp. 96-114.

[Rolf] D. Rolfsen. *Knots and Links*, Publish or Perish. Berkeley, CA (1976).

[ST1] M. Scharlemann, A. Thompson. *Finding disjoint Seifert surfaces*, Bull. Lon. Math. Soc. 20, pp 61-64 (1988).

[ST2] M. Scharlemann, A. Thompson. *Thin position for 3-manifolds*, Contemp. Math., vol. 164, AMS, pp 231-238 (1994).

[ScTom] M. Scharlemann, M. Tomova. *Alternate Heegaard genus bounds distance*, Geom. Topol., vol 10, no. 1 (2006).

[Schul] J. Schultens. *Additivity of tunnel number for small knots*, Comment. Math. Helv. (2000) 75:353. https://doi.org/10.1007/s000140050131

[Thom] A. Thompson. *The disjoint curve property and genus 2 manifolds*, Topol. Appl., vol. 93 (3), pp. 273-279 (1999).