Marked Tableaux

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Abstract

This dissertation is an exploration of tableaux that may be formed using entries from the ordered alphabet $\bar{X}' = \cdots < \bar{2} < \bar{1} < 1' < 1 < 2' < 2 < \cdots$ and the symmetric functions and crystal structures that are related to them.

These ”marked tableaux,” have had, have in this paper, and undeniably will continue to have profound impacts in the areas of Schur and P-Schur positivity, Stanley symmetric functions, Grothendieck polynomials, crystal bases, and the theory of jeu de taqin.

We will find that for each of the Stanley symmetric functions which have been historically or currently considered there is a corresponding type of marked tableaux. These tableaux will represent certain connected components on what we see as natural crystal structures for Stanley symmetric functions. Then, we will analyze the nature of two types of marked tableaux in greater detail: marked staircase shape tableaux, and primed shifted tableaux.

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CHAPTER 1

Introduction

We consider the alphabet $\bar{X}' = \cdots < \bar{2} < \bar{1} < 1' < 1 < 2' < 2 < \cdots$. A valid tableau is formed using these entries if its rows and columns are weakly increasing, its columns contain no repeated unmarked entries, and its rows contain no repeated marked entries. Such tableaux are the central theme of all that follows.

The work is structured as follows. The first part analyzes straight-shape marked tableaux. These marked tableaux are the natural generalization of straightshape semistandard Young tableaux. Closely related to this concept is the concept of a double Stanley function which is the analogous generalization of the type A Stanley symmetric function. The main result of this section about the double Schur positivity of the double Stanley symmetric function (see Theorem 4.3.1). The work in this part is based on a paper, [**Haw18**], written by the current author.

The second part analyzes primed tableaux of a shifted shape. Continuing our analogy these tableaux correspond to the type C Stanley symmetric function. There is also a mention of semistandard unimodal tableaux in this part. One may object that (as we will see) these tableaux do not contained marked entries but are rather defined (in part) by having rows composed of hook words (that is, which at first strictly descend and then weakly ascend). However this condition is equivalent (up to a power of 2 to the number of rows) to the condition that the rows are made up of barred and unmarked entries. Thus the relation between primed and signed tableaux played in the straightshape case is analogous to the relationship between shifted primed tableaux and semistandard decomposition tableaux in the shifted shape case. The main takeaway here is the relationship between a crystal structure and an insertion algorithm of Haiman: this is made precise in Theorem 3.3.3 as well as the Schur expansion for the type c Stanley which falls out as a result. This part is based on portions of a [**HPS17**] coauthored by Kirill Paramanov and Anne Schilling. However, the theorems and proofs which are included were originally written by the author himself.

The third part consists in analyzing a third type of tableaux. In this setting we are working with only one type of marked entry, which would be best considered as primed entries. Here we have the additional condition that we have some fixed index set that tells us for each i whether we may use unmarked or marked entries. These tableaux will only be of interest for the staircase shape (although we also may consider skew staircase shapes). Here the relation to symmetric functions is found by comparing the interchanging of marked and unmarked entries to the interchanging of elementary and homogeneous symmetric functions. This part is based on a [**Haw17**].

The final part is of a slightly different nature: It does not relate directly to marked tableaux but analyzes the crystal structure that the set of shifted primed tableaux (or equivalently the set of semistandard unimodal tableaux) affords from an abstract standpoint. The goal is to give a characterization of the more general object whose character corresponds to that modeled by either of these types of tableau. This includes portions of [**GHPS18**] coauthored with Maria Gillepsie, Wencin Poh, and Anne Schilling. The particular parts that are included here represent primarily the mathematical work of the current author, although many figures and examples were created or aided by other authors.

CHAPTER 2

Primed and Signed Tableaux of Straight Shape: Double Stanley Symmetric Functions

This chapter is based on the work in [**Haw18**].

2.1. Introduction

Throughout this chapter, when some $k \in \mathbb{N}$ is specified x will refer to the list of variables $(x_1, \ldots, x_k)$ and y will refer to the list of variables $(y_1, \ldots, y_k)$. On the other hand $\mathbf{x}$ will refer to the infinite list of variables $(x_1, x_2, \ldots)$ and $\mathbf{y}$ will refer to the infinite list of variables $(y_1, y_2, \ldots)$. If the polynomial $P(x)$ or $P(x, y)$ is defined for arbitrary k then $P(\mathbf{x})$ or, respectively, $P(\mathbf{x}, \mathbf{y})$ will represent the corresponding function obtained by letting $k \to \infty$.

The A_n Coxeter system is defined as the Coxeter system with generators, $s_1, \ldots, s_n$ and relations $(s_i s_j)^{m_{ij}} = 1$ where m_{ij} is an integer determined as follows:

- If $|i - j| = 0$, $m_{ij} = 1$.
- If $|i - j| = 1$, $m_{ij} = 3$.
- If $|i - j| > 1$, $m_{ij} = 2$.

By abuse of notation, we will also refer to the corresponding Coxeter group of size $(n + 1)!$ as A_n. The C_n Coxeter system is defined as the Coxeter system with generators, $s_0, s_1, \ldots, s_n$ and relations $(s_i s_j)^{m_{ij}} = 1$ where m_{ij} is an integer determined as follows:

- If $|i - j| = 0$, $m_{ij} = 1$.
- If $i > 0$ and $j > 0$, and $|i - j| = 1$, $m_{ij} = 3$.
- If $i = 0$ or $j = 0$, and $|i - j| = 1$, $m_{ij} = 4$.
- If $|i - j| > 1$, $m_{ij} = 2$.

Similarly, we will sometimes refer to corresponding group of size $2^n(n+1)!$ itself as C_n. Given the relations above one can define two types of symmetric functions, indexed, respectively, by elements of A_n and C_n.

First, suppose $\omega \in A_n$. A *reduced word* for ω is an expression, u, for ω using the generators $s_1, \ldots, s_n$ such that no other such expression for ω is shorter than u. Given a fixed k, a *reduced increasing k-factorization* (RIF), v, for ω is a reduced word u, for ω along with a subdivision of u into k parts such that each part is increasing under the order $s_1 < \cdots < s_n$. The *weight* of v is the vector whose i^{th} entry records the number of generators in the i^{th} subdivision of v. The type A Stanley symmetric polynomial [**Sta84**] in k variables for ω is:

$$F_{\omega}^{A}(x) = \sum_{v \in RIF(\omega)} x^{wt(v)},$$

where $RIF(\omega)$ is the set of reduced increasing k-factorizations of ω, and $wt(v)$ is the weight of v. Letting $k \to \infty$ in the type A Stanley symmetric polynomial gives the type A Stanley symmetric function for ω.

Now suppose $\omega \in C_n$. A *reduced word* for ω is an expression, u, for ω using the generators $s_0, s_1, \ldots, s_n$ such that no other such expression for ω is shorter than u. A *reduced unimodal k-factorization* (RMF), v, for ω is a reduced word u, for ω along with a subdivision of u into k parts such that each part is unimodal (i.e., decreasing and then increasing) under the order $s_0 < s_1 < \cdots < s_n$. The *weight* of v is the vector whose i^{th} entry records the number of generators in the i^{th} subdivision of v. The type C Stanley symmetric polynomial [**BH95**], [**FK96**], in k variables for ω is:

$$F_{\omega}^{C}(x) = \sum_{v \in U(\omega)} 2^{ne(v)} x^{wt(v)},$$

where $ne(v)$ is the number of nonempty subdivisions of v, $U(\omega)$ is the set of reduced unimodal factorizations of ω, and $wt(v)$ is the weight of v. Letting $k \to \infty$ in the type C Stanley symmetric polynomial gives the type C Stanley symmetric function for ω.

Of course, for any $\omega \in A_n$ we may consider both F^A_ω and F^C_ω. Both functions are Schur positive, but it is not exactly clear how the one relates to the other. To do this, we will define a third function F^d_ω. We now consider the generators $s_{-n}, \ldots s_{-1}, s_1, \ldots, s_n$ and impose relations $(s_i s_j)^{m_{ij}} = 1$ where m_{ij} is an integer determined as follows:

- If $|(|i| - |j|)| = 0$, $m_{ij} = 1$.
- If $|(|i| - |j|)| = 1$, $m_{ij} = 3$.
- If $|(|i| - |j|)| > 1$, $m_{ij} = 2$.

Of course, the resulting system is not Coxeter, for instance, the relations imply that $s_{-i} = s_i$ holds, [1] so the generating set is obviously not minimal.

In this setting, a *reduced word* for ω is an expression, u, for ω using the generators $s_{-n}, \ldots, s_{-1}$ and $s_1, \ldots, s_n$ such that no other such expression for ω is shorter than u. A *reduced signed increasing k-factorization* ($RSIF$), v, for ω is a reduced word u, for ω along with a subdivision of u into k parts such that each part is increasing under the order $s_{-n} < \cdots s_{-1} < s_1 < \cdots < s_n$. The *double weight* of v, denoted $(dw(v,1), dw(v,2))$ is the pair (X, Y), where the i^{th} entry of X records the number of generators with negative index in the i^{th} subdivision of v, and the i^{th} entry of Y records the number of generators with positive index in the i^{th} subdivision of v. For instance, $v = (s_{-3}s_{-2}s_1)(s_{-5}s_2s_3)(s_{-4}s_{-3})$ is an $RSIF$ (with $k = 3$) for $\omega = s_3s_2s_1s_2s_3s_5s_4s_3$ with double weight $((2,1,2),(1,2,0))$. We define the *double Stanley symmetric polynomial* in k variables for ω to be:

$$F^d_\omega(x, y) = \sum_{v \in RSIF(\omega)} x^{dw(v,1)} y^{dw(v,2)},$$

where $RSIF(\omega)$ is the set of reduced signed increasing k-factorizations of ω. Letting $k \to \infty$ in the double Stanley symmetric polynomial gives the double Stanley symmetric function for ω. We will frequently use the shorthand i for s_i and $\bar{i}$ for $-s_i$ when it is clear we are discussing expressions of Coxeter elements. For instance, v above may be rewritten: $v = (\bar{3}\bar{2}1)(\bar{5}23)(\bar{4}\bar{3})$. Inside tableaux, barred entries will be represented using a small -, for example $\boxed{\text{-4}}\boxed{\text{-3}}\boxed{\text{-1}}\boxed{2}\boxed{3}$ represents the one row tableau with reading word $\bar{4}\bar{3}\bar{1}23$. The

[1] This makes sense on the level of Weyl groups: the reflection over the plane perpendicular to the i^{th} simple root is equal to the reflection over the plane perpendicular to the opposite of the i^{th} simple root.

entries inside an *Edelman-Greene* or *signed Edelman-Greene tableau* (defined later) of i and -i represent s_i and s_{-i} respectively.

It is not hard to check that $F^d_\omega(\mathbf{0},\mathbf{x}) = F^A_\omega(\mathbf{x}) = F^d_{\omega^{-1}}(\mathbf{x},\mathbf{0})$ and that $F^d_\omega(\mathbf{x},\mathbf{x}) = F^C_\omega(\mathbf{x})$. Whether there is more to this function than being a way of expressing $F^A_\omega(\mathbf{x})$ and $F^C_\omega(\mathbf{x})$ in the same framework, depends whether there is any symmetry to the function $F^d_\omega(\mathbf{x},\mathbf{y})$ in general. Amazingly, $F^d_\omega(\mathbf{x},\mathbf{y})$ turns out be symmetric in $\mathbf{x}$ and symmetric in $\mathbf{y}$. The former meaning that, for any composition β, the coefficient of $\mathbf{y}^\beta$ is a symmetric function in $\mathbf{x}$. In fact this coefficient is Schur positive. (The analogous result for the coefficient of $\mathbf{x}^\beta$ is also true, as can be noted by the equality $F^d_\omega(\mathbf{x},\mathbf{y}) = F^d_{\omega^{-1}}(\mathbf{y},\mathbf{x})$.)

2.2. Primed Signed Tableaux

In this section we introduce signed tableaux, primed tableaux, and an interpolation between the two, which we call primed signed tableaux. We first explicitly define primed signed tableaux, and then primed tableaux and signed tableaux as special cases. The main take-away will be Corollary 2.2.11.

Fix some $k \in \mathbb{N}$ for the remainder of this section. We will work over the following alphabets:

- $X_k = \{1 < 2 < 3 < \cdots < k\}$.
- $X'_k = \{1' < 1 < 2' < 2 < \cdots < k' < k\}$.
- $\bar{X}_k = \{\bar{k} < \cdots < \bar{2} < \bar{1} < 1 < 2 < \cdots < k\}$.
- $\bar{X}'_k = \{\bar{k} < \cdots < \bar{2} < \bar{1} < 1' < 1 < 2' < 2 < \cdots < k' < k\}$.

(For now, these letters bear no relation to s_i and s_{-i}.) An element in these alphabets is called *marked* if it is barred or it is primed, and called *unmarked* otherwise. Fix partitions $\mu \subseteq \lambda$. Fix vectors X and Y in $\mathbb{Z}^k_{\geq 0}$. Finally, fix j and l in $\mathbb{Z}_{\geq 0}$ such that $l \leq X(j)$. Our goal is to eventually define the set of *primed signed tableaux* corresponding to these parameters, which we will denote by $PST(\lambda,\mu,X,Y,j,l)$. It easiest to first define a larger set (call them pre-PSTs), and then specify which of these are PSTs. A tableaux T is in the set pre-$PST(\lambda,\mu,X,Y,j,l)$ if:

(1) T has shape λ/μ.

(2) T has entries from $\bar{X}'_k$.

(3) The rows and columns of T are weakly increasing.

(4) Each row of T has at most one marked i and each column has at most one unmarked i.

(5) T contains $Y(i)$ unmarked is.

(6) T contains $X(i)$ primed is for each $i < j$.

(7) T contains $X(i)$ barred is for each $i > j$.

(8) T contains l primed js and $X(j) - l$ barred js.

(9) The uppermost primed j in T is in a lower row than the lowermost barred j.

Example 2.2.1. The following are both elements of pre-$PST\left(\begin{bmatrix}4\\3\\2\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\\2\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}, 3, 1\right)$:

$$T_1 = \begin{array}{|c|c|c|c|}\hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' \\ \cline{1-3} -3 & 1 & 2 \\ \cline{1-3} 2 & 3' \\ \cline{1-2}\end{array}\ , \qquad T_2 = \begin{array}{|c|c|c|c|}\hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' \\ \cline{1-3} -3 & 2 & 2 \\ \cline{1-3} 1 & 3' \\ \cline{1-2}\end{array}\ .$$

To decide whether a pre-PST is a PST we need to use conversion. If T is in the set pre-$PST(\lambda, \mu, X, Y, j, l)$, the *inward conversion* [2] of T, denoted $\leftarrow T$ is defined as follows:

(1) Change the uppermost primed j in T to a barred j if it exists.

(2) Repeat the following procedure until either (a) all rows and columns are weakly increasing or (b) there are two barred js in some row: Switch the lowermost barred j with either the entry above it or to its left, determined as follows:

- If only one of the entries exists, take it.
- If these entries are not equal, take the larger.
- If they are equal and are unmarked, take the one above.
- If they are equal and are marked, take the one on the left.

(3) If the process stops because of (a), then $\leftarrow T$ is defined to be the current tableau. If it stops because of (b), then $\leftarrow T$ is undefined. (Note that if $l = 0$, $\leftarrow T = T$ is well-defined.)

Example 2.2.2. $\leftarrow \begin{array}{|c|c|c|c|}\hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' \\ \cline{1-3} -3 & 1 & 2 \\ \cline{1-3} 2 & 3' \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' \\ \cline{1-3} -3 & 1 & 2 \\ \cline{1-3} -3 & 2 \\ \cline{1-2}\end{array}$ whereas $\leftarrow \begin{array}{|c|c|c|c|}\hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' \\ \cline{1-3} -3 & 2 & 2 \\ \cline{1-3} 1 & 3' \\ \cline{1-2}\end{array}$ is undefined.

[2] A somewhat similar definition appears in [**Hai89**] for the case where letters may not be repeated. In fact, one can use a certain standardization process along with Haiman's mixed insertion, Haiman's conversion, and Haiman's Theorem 3.12, [**Hai89**] to derive Theorem 2.2.9 below for the specific case of $\mu = \emptyset$, i.e., for straightshape tableaux. However, the proofs needed to do this are somewhat more complicated than those employed below, and the result, of course, less general.

Similarly, if T is in the set pre-$PST(\lambda, \mu, X, Y, j, k)$, the *outward conversion* of T, denoted $T \rightarrow$ is defined as follows:

(1) Change the lowermost barred j in T to a primed j if it exists.

(2) Repeat the following procedure until either (a) all rows and columns are weakly increasing or (b) there are two primed js in some row: Switch the uppermost primed j with either the entry below it or to its right, determined as follows:

- If only one of the entries exists, take it.
- If these entries are not equal, take the smaller.
- If they are equal and are unmarked, take the one below.
- If they are equal and are marked, take the one on the right.

(3) If the process stops because of (a), then $T \rightarrow$ is defined to be the current tableau. If it stops because of (b), then $T \rightarrow$ is undefined. (Note that if $l = X(j)$, $T \rightarrow = T$ is well-defined.)

Definition 2.2.3. We say that $T \in PST(\lambda, \mu, X, Y, j, l)$ if and only if T is in the set pre-$PST(\lambda, \mu, X, Y, j, l)$ and both of the following hold:

(1) $\leftarrow T$ is defined.

(2) $T \rightarrow$ is defined.

The reader should have no difficulty verifying the lemmas below:

Lemma 2.2.4. *Suppose T is in the set pre-$PST(\lambda, \mu, X, Y, j, l)$. Then*

(1) If $\leftarrow T$ is defined, then $\leftarrow T$ is in the set pre-$PST(\lambda, \mu, X, Y, j, l-1)$.

(2) If $T \rightarrow$ is defined, then $T \rightarrow$ is in the set pre-$PST(\lambda, \mu, X, Y, j, l+1)$.

Lemma 2.2.5. *Suppose T is in the set pre-$PST(\lambda, \mu, X, Y, j, l)$, then the following are equivalent:*

(1) $\leftarrow T$ is defined.

(2) $T \rightarrow$ is defined.

Example 2.2.6. $T_1 = \begin{array}{|c|c|c|c|} \hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' & \\ \cline{1-3} -3 & 1 & 2 & \\ \cline{1-3} 2 & 3' & & \\ \cline{1-2} \end{array}$ is a PST, but $T_2 = \begin{array}{|c|c|c|c|} \hline -4 & -3 & 2' & 4 \\ \hline -4 & 1' & 2' & \\ \cline{1-3} -3 & 2 & 2 & \\ \cline{1-3} 1 & 3' & & \\ \cline{1-2} \end{array}$ is not. (These are the tableaux from the previous example).

Lemma 2.2.7. *Suppose* $T \in PST(\lambda, \mu, X, Y, j, l)$.

(1) *If* $0 < l$ *then* $(\leftarrow T) \rightarrow$ *is defined. In particular,* $(\leftarrow T) \rightarrow = T$.

(2) *If* $l < Y(j)$ *then* $\leftarrow (T \rightarrow)$ *is defined. In particular,* $\leftarrow (T \rightarrow) = T$.

The main result concerning conversion is the following:

Theorem 2.2.8. *Suppose* $T \in PST(\lambda, \mu, X, Y, j, l)$. *Then*

(1) *If* $0 < l$ *then* $\leftarrow T \in PST(\lambda, \mu, X, Y, j, l-1)$.

(2) *If* $l < X(j)$ *then* $T \rightarrow \in PST(\lambda, \mu, X, Y, j, l+1)$.

Proof. (1) We need to check $\leftarrow T$ is a PST, i.e., that $\leftarrow (\leftarrow T)$ or $(\leftarrow T) \rightarrow$ is defined (by Lemma 2). But $(\leftarrow T) \rightarrow = T$ by Lemma 3, and so is clearly defined.

(2) We need to check that $T \rightarrow$ is a PST, i.e., that $\leftarrow (T \rightarrow)$ or $(T \rightarrow) \rightarrow$ is defined (by Lemma 2). But $\leftarrow (T \rightarrow) = T$ by Lemma 3, and so is clearly defined.

□

Theorem 2.2.9. *Fix* λ, μ, X, *and* Y. *Then for any pairs* (j, l) *and* (j', l') *such* $0 \leq l \leq X(j)$ *and* $0 \leq l' \leq Y(j')$ *there is a bijection* $PST(\lambda, \mu, X, Y, j, l) \Rightarrow PST(\lambda, \mu, X, Y, j', l')$.

Proof. Since $PST(\lambda, \mu, X, Y, j, Y(j)) = PST(\lambda, \mu, X, Y, j+1, 0)$ by definition, it suffices to assume $l < X(j)$ and find a bijection $PST(\lambda, \mu, X, Y, j, l) \Rightarrow PST(\lambda, \mu, X, Y, j, l+1)$. But this is given by outward conversion (and the inverse by inward conversion). □

Definition 2.2.10. Let $X, Y \in \mathbb{Z}^k_{\geq 0}$.

(1) A *primed tableau* of shape λ/μ and double weight (X, Y) is an element of $PST(\lambda, \mu, X, Y, k, X(k))$.

(2) A *signed tableau* of shape λ/μ and double weight (X, Y) is an element of $PST(\lambda, \mu, X, Y, 0, 0)$.

Corollary 2.2.11. *Letting* $PT(\lambda/\mu)$ *denote the set of all primed tableaux of shape* λ/μ *and* $ST(\lambda/\mu)$ *denote the set of all signed tableaux of shape* λ/μ*, there is a double weight preserving bijection:* $PT(\lambda/\mu) \Rightarrow ST(\lambda/\mu)$.

We can now make the following definition with no concern of ambiguity:

$$R^d_{\lambda/\mu}(x,y) = \sum_{T\in PT(\lambda/\mu)} x^{dw(T,1)}y^{dw(T,2)} = \sum_{T\in ST(\lambda/\mu)} x^{dw(T,1)}y^{dw(T,2)}.$$

Again, letting $k \to \infty$ we obtain the corresponding function $R^d_{\lambda/\mu}(\mathbf{x},\mathbf{y})$. In some sense, this function is an interpolation between a Schur function and a Q-Schur function. Indeed, supposing that λ has n parts, if we set δ equal to the partition $(n, n-1, \ldots, 1)$, then the set of shifted semistandard tableaux of shape $(\delta+\lambda)/(\delta+\mu)$ is equivalent to the set $PT(\lambda/\mu)$. It follows that the skew Q-Schur function, $Q_{(\delta+\lambda)/(\delta+\mu)}(\mathbf{x})$, is equal to $R_{\delta/\mu}(\mathbf{x},\mathbf{x})$. On the other hand, we have $R_{\delta/\mu}(\mathbf{x},\mathbf{0}) = s_{\lambda'/\mu'}(\mathbf{x})$ and $R_{\delta/\mu}(\mathbf{0},\mathbf{x}) = s_{\lambda/\mu}(\mathbf{x})$.

2.3. Littlewood-Richardson Rules

Although it does not have applications to double Stanley symmetric functions, an immediate question which arises is whether there exist Littlewood-Richardson coefficients for $R^d_{\lambda/\mu}$. That is, are there coefficients $h^\lambda_{\mu\nu}$ such that $R^d_{\lambda/\mu} = \sum_\nu h^\lambda_{\mu\nu} R^d_\nu$? If $c^\lambda_{\mu\nu}$ denote the regular Littlewood-Richardson coefficients, then since $R^d_{\lambda/\mu}(\mathbf{x},\mathbf{0}) = s_{\lambda/\mu}(\mathbf{x})$ (and similarly $R^d_{\lambda/\mu}(\mathbf{0},\mathbf{y}) = s_{\lambda/\mu}(\mathbf{y})$) we see that the equation $R^d_{\lambda/\mu} = \sum_\nu h^\lambda_{\mu\nu} R^d_\nu$ holds when both sides are evaluated at $(\mathbf{x},\mathbf{0})$, (or at $(\mathbf{0},\mathbf{y})$) if and only if $h^\lambda_{\mu\nu} = c^\lambda_{\mu\nu}$. Thus if such $h^\lambda_{\mu\nu}$ exist, they must be equal to the regular Littlewood-Richardson coefficients. However, in general, it is not clear that the equation $R^d_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} R^d_\nu$ holds when both sides are evaluated at $(\mathbf{x},\mathbf{y})$. Similarly, one may ask if there exist coefficients, $l^\lambda_{\mu\nu}$ such that $R^d_\mu R^d_\nu = \sum_\lambda l^\lambda_{\mu\nu} R^d_\lambda$. Again, it is not clear if such coefficients exist, but if they do, they must, for analogous reasons to those above, be equal to the regular Littlewood-Richardson coefficients. As before, we will fix k (it will be convenient to assume $k > |\lambda|$) and state the explicit results and proofs for the variable set (x,y). Letting $k \to \infty$ we obtain the corresponding results for $(\mathbf{x},\mathbf{y})$.

Let $W(k,r)$ denote the set of words of length r from the alphabet X_k. Let $SW(k,r)$ denote the set of words of length r from the alphabet $\bar{X}_k$. The theory of Schensted insertion and *jeu de taquin* extend to signed tableaux in a natural way. For clarity, we will refer to these analogues as *signed insertion* and *signed jdt*. The easiest way to describe them is through a standardization process. A word w in $SW(k,r)$ induces a partial order on the positions $p_1,\ldots,p_r$ of the word w. We extend this to a total order by defining:

- $p_i \prec p_j$ if $i < j$ and the entries of p_i and p_j are equal to l for some $1 \le l \le k$.
- $p_j \prec p_i$ if $i < j$ and the entries of p_i and p_j are equal to $\bar{l}$ for some $1 \le l \le k$.

The permutation induced by this order is defined to be the standardization of w, $s(w)$. The reading word of a signed tableau, T, $rd(T)$, is the word composed of barred and unbarred letters formed by reading the rows from left to right, moving from bottom to top. In this way T may be considered as a word in $\bar{X}_k$ and the standardization of T, $s(T)$, is (the tableau) formed by standardizing this word. The *standardization map* is the injective map, sending T to the triple $(s(T), X, Y)$ where (X, Y) is the double weight of T, and is defined similarly for words. Notice that, for tableau, the inverse of the standardization map is only defined in certain cases. Hence, one should check, that in the definitions below, when the phrase "apply the inverse of the standardization map" is used, this is well-defined.

(1) The signed insertion of $w \in SW(k,r)$ is the tableau formed by first applying the standardization map to w, then applying Schensted insertion, and then applying the inverse of the standardization map.

(2) For $T \in ST(\lambda/\mu)$, the signed (inward or outward) *jdt* of T into a box b is done by applying the standardization map to T, applying regular (inward or outward) *jdt* into b and then applying the inverse of the standardization map. (Of course, signed *jdt* of T into b is only defined when regular *jdt* of $s(T)$ into b is defined.)

The following is an immediate consequence of the formulation of these definitions and the standard case:

Lemma 2.3.1. *Suppose $w \in SW(k,r)$. Then the signed insertion of w can be obtained by placing the elements of w along a southwest to northeast diagonal and then applying inward signed* jdt *until a normal shape is obtained.*

The usual type A crystal operators f_i and e_i for $1 \le i \le k-1$ are operators which map $W(k,r) \to W(k,r) \cup 0$ (see [**BS17**] for definitions). Below we define operators, $\hat{f}_{\bar{i}}$ and $\hat{f}_i$ for $1 \le i \le k-1$ and $\hat{f}_0$, which map $SW(k,r) \to SW(k,r) \cup 0$.

(1) $\hat{f}_{\bar{i}}(w)$: Let $-w$ be the word obtained by unbarring the barred entries of w and vice-versa. One can apply the usual type A operator e_i to $-w$ by ignoring the (now) barred entries. Define $\hat{f}_{\bar{i}}(w) = -e_i(-w)$. (We assume $-0 = 0$.)

(2) $\hat{f}_i(w)$: One can apply the usual type A operator f_i to w by ignoring the barred entries. Define $\hat{f}_i(w) = f_i(w)$.

(3) $\hat{f}_0(w)$: Among all entries which are $\bar{1}$ or 1, consider the leftmost one. If this entry is $\bar{1}$ change it to 1. Otherwise, $\hat{f}_0(w) = 0$.

We have

- $\hat{f}_{\bar{1}}(\bar{2}\bar{1}\bar{2}1\bar{2}) = \bar{2}\bar{1}\bar{1}1\bar{2}$
- $\hat{f}_0(\bar{2}\bar{1}\bar{2}1\bar{2}) = \bar{2}1\bar{2}1\bar{2}$
- $\hat{f}_1(\bar{2}\bar{1}\bar{2}1\bar{2}) = \bar{2}\bar{1}\bar{2}2\bar{2}$
- $\hat{f}_{\bar{1}}(\bar{2}21\bar{1}\bar{1}) = 0$
- $\hat{f}_0(\bar{2}21\bar{1}\bar{1}) = 0$
- $\hat{f}_1(\bar{2}21\bar{1}\bar{1}) = 0$

We define the operators $\hat{e}_i$, $\hat{e}_{\bar{i}}$, and $\hat{e}_0$ to be the respective inverses of $\hat{f}_i$,$\hat{f}_{\bar{i}}$, and $\hat{f}_0$. If all of the operators $\hat{e}_i$, $\hat{e}_{\bar{i}}$, and $\hat{e}_0$ are 0 for w, we say that w is highest weight. Similarly, if all of the operators $\hat{f}_i$,$\hat{f}_{\bar{i}}$, and $\hat{f}_0$ are 0 for w, we say that w is lowest weight.

Suppose $w \in SW(k, r)$ for some $r < k$, then w is lowest weight if and only if:

- w has no barred entries.
- Reading w from left to right one has that all times one has read no more is than $i + 1$s for each $1 \leq i \leq k - 1$.

and similarly, w is highest weight if and only if:

- w has only barred entries.
- Reading w from left to right one has that all times one has read no more $\bar{i}$s than $i \bar{+} 1$s for each $1 \leq i \leq k - 1$.

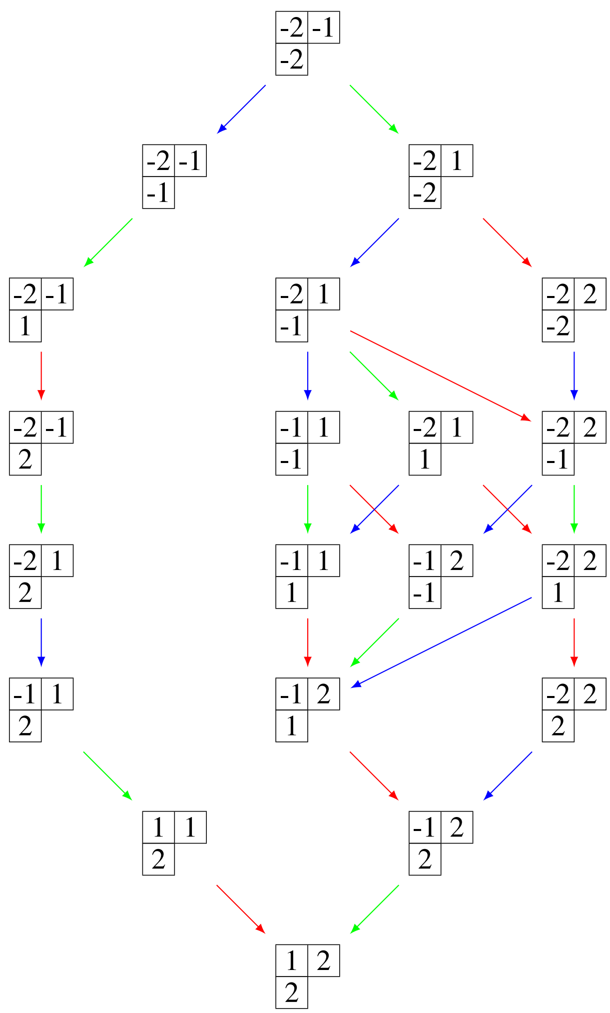
The operators $\hat{f}_{\bar{i}}$, $\hat{f}_i$, and $\hat{f}_0$ (and their inverses) are defined on signed tableaux by letting them act on the reading word (the operators f_i are defined on semistandard tableau in the same way). We note here that, on a signed tableau, T, $\hat{f}_{\bar{i}}$ has the following alternative description:

- $\hat{f}_{\bar{i}}(T)$: Transpose T. Then change each $\bar{i}$ to $-i$ and add $k + 1$ to all entries. Then apply the usual type A operator f_{-i+k}. Then subtract $k + 1$ from all entries, change each $-i$ to $\bar{i}$, and transpose.

It is clear from the descriptions above that for any normal shape signed tableau, there is a unique highest and lowest weight (Recall that we assume $k > |\lambda|$). Moreover, this fact along with the fact the signed insertion commutes with our operators (see below) means that for any connected component of the crystal on signed words has a unique highest and lowest weight.

Below, we include the entire crystal structure for $k = 2$ and $\lambda = (2, 1)$ using the color conventions:

- $\hat{f}_{\bar{1}}$: $\longrightarrow$
- $\hat{f}_{0}$: $\longrightarrow$
- $\hat{f}_{1}$: $\longrightarrow$



Theorem 2.3.2. *Let $x \in \{\overline{k-1} \cdots \bar{1}, 0, 1, k-1\}$. Suppose T' is obtained from T by performing (*reverse *or* forward*) signed* jdt *into box b. Then $\hat{f}_x(T') = 0$ if and only if $\hat{f}_x(T) = 0$. Otherwise, $\hat{f}_x(T')$ is obtained from $\hat{f}_x(T)$ by performing (*reverse *or* forward*) signed* jdt *into b.*

Proof. The result for $\hat{f}_i$ follows from the usual result for f_i. The result for $\hat{f}_{\bar{i}}$ can be obtained similarly using the alternative description of $\hat{f}_{\bar{i}}$. For f_0 note that the leftmost lowest 1 or $\bar{1}$ is preserved under signed *jdt* moves and that the whether the leftmost lowest 1 or $\bar{1}$ is a 1 or a $\bar{1}$ does not affect the structure of signed *jdt* at all. $\square$

By lemma 2.3.1 we have:

Corollary 2.3.3. *Let $x \in \{\overline{k-1} \cdots \bar{1}, 0, 1, k-1\}$. Suppose T is obtained from w by performing signed insertion. Then $\hat{f}_x(T) = 0$ if and only if $\hat{f}_x(w) = 0$, and otherwise, $\hat{f}_x(T)$, is obtained from $\hat{f}_x(w)$ by performing signed insertion.*

Corollary 2.3.4.

$$R^d_{\lambda/\mu}(x, y) = \sum_{\nu} c^{\lambda}_{\mu\nu} R^d_{\nu}(x, y) \tag{2.1}$$

$$R^d_{\mu}(x, y) R^d_{\nu}(x, y) = \sum_{\lambda} c^{\lambda}_{\mu\nu} R^d_{\lambda}(x, y) \tag{2.2}$$

Proof. First we establish 2.1. By theorem 2.3.2, the coefficient of $R^d_{\nu}(x, y)$ in $R^d_{\lambda/\mu}(x, y)$ is the number of lowest weight ST of shape λ/μ that rectify under signed *jdt* to shape ν. But lowest weight ST of shape λ/μ are exactly the lowest weight SSYT of shape λ/μ in the alphabet X^k. Moreover, semistandard *jdt* coincides with signed *jdt* for such tableaux. Thus, the coefficient of $R^d_{\nu}(x, y)$ in $R^d_{\lambda/\mu}(x, y)$ is also the number of lowest weight $SSYT$ of shape λ/μ that rectify under semistandard *jdt* to shape ν, which is $c^{\lambda}_{\mu\nu}$.

To prove 2.2, we need the notion of a tensor product for our operators. Given two signed tableaux, say S and T, the operators $\hat{f}_{\bar{i}}$, $\hat{f}_i$, $\hat{f}_0$ act on $S \otimes T$ by acting on the word obtained by concatenating the reading word of T onto the right end of the reading word of S. It follows from corollary 2.3.3, that the coefficient of $R^d_{\lambda}(x, y)$ in $R^d_{\mu}(x, y) R^d_{\nu}(x, y)$ is the number of pairs of (S, T) with shapes μ and ν respectively such that the tensor product $S \otimes T$ is lowest weight with double weight equal to $((0, \ldots, 0), (0, \ldots, \lambda_n, \ldots, \lambda_1))$. But $S \otimes T$ is lowest weight under the operators $\hat{f}_{\bar{i}}$, $\hat{f}_i$, $\hat{f}_0$ if and only if S and T are SSYT and the tensor product $S \otimes T$ is lowest weight under the type A operators, $\{f_i\}$. Thus the coefficient of $R^d_{\lambda}(x, y)$ in $R^d_{\mu}(x, y) R^d_{\nu}(x, y)$ is also the number of pairs of SSYT, (S, T) with shapes μ and ν respectively such that the type A tensor product of

SSYT's, $S \otimes T$, is lowest weight, with weight equal to $(0, \ldots, \lambda_n, \ldots, \lambda_2, \lambda_1)$, which is $c^{\lambda}_{\mu\nu}$.

$\square$

2.4. Double Stanley Symmetric Functions

The main purpose of this section is to show that the function:

$$F^d_{\omega}(\mathbf{x}, \mathbf{y}) = \sum_{v \in RSIF(\omega)} \mathbf{x}^{dw(v,1)} \mathbf{y}^{dw(v,2)},$$

is symmetric in $\mathbf{x}$ and symmetric in $\mathbf{y}$. That is, for any composition β, the coefficient of $\mathbf{y}^{\beta}$ is a symmetric function in $\mathbf{x}$ and the coefficient of $\mathbf{x}^{\beta}$ is a symmetric function in $\mathbf{y}$. Moreover, these coefficients are not only symmetric, but Schur positive.

Let $\omega \in S_n$. We use Edelman-Greene insertion, [**EG87**], to create a bijection between $RSIF(\omega)$ and pairs of tableaux, (P, Q), where P is an Edelman-Greene tableau for ω, and Q is a primed tableau of the same shape. This bijection is described below. Again let us fix $k < \infty$ for the discussion:

Definition 2.4.1. *Signed-Recording Edelman-Greene map.* Suppose $v \in RSIF(\omega)$. Create the insertion tableau P by applying Edelman-Greene insertion to $|v|$, the expression obtained by ignoring the subdivisions of v and replacing s_{-i} by s_i for each i. Create the recording tableau, Q, as follows: Each time a box is added to P say in position (i, j), add a box to Q in position (i, j) and fill it as follows: Suppose box (i, j) was added to P when $|v|_r$ was inserted. Let l be the subdivision of v in which v_r occurs in v. If v_r is barred in v, fill box (i, j) of Q with l'. If v_r is unbarred in v, fill box (i, j) of Q with l.

Example 2.4.2. Let $v = (\bar{3}\bar{2}14)(\bar{3}\bar{2})(\bar{4}13)$.

$$\boxed{3}, \boxed{1'} \to \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1'\\\hline 1'\\\hline\end{array} \to \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1'\\\hline 1'\\\hline 1\\\hline\end{array} \to \begin{array}{cc} 1 & 4\\ 2 & \\ 3 & \end{array}, \begin{array}{cc} 1' & 1\\ 1' & \\ 1 & \end{array} \to \begin{array}{cc} 1 & 3\\ 2 & 4\\ 3 & \end{array}, \begin{array}{cc} 1' & 1\\ 1' & 2'\\ 1 & \end{array} \to \begin{array}{cc} 1 & 2\\ 2 & 3\\ 3 & 4 \end{array}, \begin{array}{cc} 1' & 1\\ 1' & 2'\\ 1 & 2' \end{array}$$

$$\to \begin{array}{ccc} 1 & 2 & 4\\ 2 & 3 & \\ 3 & 4 & \end{array}, \begin{array}{ccc} 1' & 1 & 3'\\ 1' & 2' & \\ 1 & 2' & \end{array} \to \begin{array}{ccc} 1 & 2 & 4\\ 2 & 3 & \\ 3 & 4 & \\ 4 & & \end{array}, \begin{array}{ccc} 1' & 1 & 3'\\ 1' & 2' & \\ 1 & 2' & \\ 3 & & \end{array} \to \begin{array}{ccc} 1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 4 & \\ 4 & & \end{array}, \begin{array}{ccc} 1' & 1 & 3'\\ 1' & 2' & 3\\ 1 & 2' & \\ 3 & & \end{array}$$

Theorem 2.4.3. *The Signed-Recording Edelman-Greene map is a double weight preserving bijection:* $RSIF(\omega) \Rightarrow (P, Q)$*, where* P *is an Edelman-Greene tableau for* ω*, and* Q *is a primed tableau of the same shape. (The double weight of* (P, Q) *refers to the double weight of* Q*.)*

Proof. The proof relies on a basic fact of Edelman-Greene insertion of an unsigned reduced word v: If $v = v_1 \dots v_s$ is inserted under Edelman-Greene, then $v_r < v_{r+1}$ if and only if the box added to the insertion tableau in the r^{th} step is in a row weakly above the row where a box is added in the $(r+1)^{st}$ step. To see the map is well-defined: Certainly P is an Edelman-Greene tableau. Certainly Q has weakly increasing rows and columns. There is at most one unbarred i in each column for each i because of the forward direction of the basic fact. There is at most one barred i in each row for each i because of the backwards direction of the basic fact. The inverse is obtained by applying reverse Edelman-Greene insertion to P in the order prescribed by the standardization of Q [3]. Subdivisions and the signs of the indices are then added in the unique way such the resulting factorization has the same double weight as Q. Again, the basic fact implies that this inverse is well-defined.

$\square$

Combining Theorem 4.3.1 and Corollary 2.2.11 and letting $k \to \infty$ we get the Schur expansion we desire:

Theorem 2.4.4. *For any composition* β*, the coefficient of* $\mathbf{y}^\beta$ *in* $F^d_\omega(\mathbf{x}, \mathbf{y})$ *is given by:*

$$p(y^\beta, \omega) = \sum_{T \in E(\omega)} \sum_{\mu \subseteq \lambda(T)} K^{\lambda(T)}_{\mu\beta} s_{\mu'}(\mathbf{x}).$$

Here $E(\omega)$ *is the set of Edelman-Greene tableau for* ω*,* $\lambda(T)$ *is the shape of* T*, and* $K^\lambda_{\mu\beta}$ *is the number of skew SSYT of shape* λ/μ *and weight* β*. Moreover, the coefficient of* $\mathbf{x}^\beta$ *in* $F^d_\omega(\mathbf{x}, \mathbf{y})$ *is given by* $p(x^\beta, \omega) = p(y^\beta, \omega^{-1})$*.*

Proof. By theorem 4.3.1 we have

$$F^d_\omega(\mathbf{x}, \mathbf{y}) = \sum_{T \in E(\omega)} R^d_{\lambda(T)}(\mathbf{x}, \mathbf{y}),$$

[3]The reading word and standardization of a primed tableau are defined exactly as for signed tableau. Simply replace the word "barred" with "primed" everywhere it appears in these definitions.

and by Corollary 2.2.11 we can express $R^d_{\lambda(T)}(\mathbf{x},\mathbf{y})$, in terms of signed tableaux. In particular, the coefficient of $\mathbf{y}^\beta$ in $R^d_\lambda(\mathbf{x},\mathbf{y})$ is equal to:

$$\sum_{\mu\subseteq\lambda} K^\lambda_{\mu\beta} \sum_{S\in ST^-(\mu)} \mathbf{x}^{dw(S,1)},$$

where $ST^-(\mu)$ is the set of signed tableaux of shape μ with only barred entries. Such tableaux are clearly in weight preserving bijection with the set of SSYT of shape μ', and so the sum on the right side above may be replaced by $s_{\mu'}$. The second statement follows from the fact that $F^d_\omega(\mathbf{x},\mathbf{y}) = F^d_{\omega^{-1}}(\mathbf{y},\mathbf{x})$ as can easily be verified from the definition. □

2.5. Signed-insertion

Our current goal is to construct the obvious analog to the Signed-Recording Edelman-Greene map, i.e., the Signed-Insertion Edelman-Greene map, by defining a notion of signed Edelman-Greene insertion, and creating the recording tableau in the normal way.

A tableau-word, R is a pair (R_1, R_2) where R_1 (the tableau part) is a stack of left-justified rows whose entries come from the alphabet $\bar{X}$ and R_2 (the word part) is any word using the letters from $\bar{X}$. The reading word of R, $rd(R)$ is the word obtained by reading the rows of R_1 from left to right, moving from bottom to top, and then by reading R_2. R is a tableau-word for ω if its reading word is a reduced signed word for ω.

Let K be the map $[m-2]\times S_m \to S_m$ such that $K(i,\sigma_1\cdots\sigma_n) = \sigma_1\cdots\sigma_{i-2}xyz\sigma_{i+2}\cdots\sigma_n$ where xyz is the unique three letter sequence which is distinct from, but Knuth equivalent to, $\sigma_{i-1}\sigma_i\sigma_{i+1}$, if such a sequence exists, and equal to $\sigma_{i-1}\sigma_i\sigma_{i+1}$ otherwise.

Let $RS(\omega)$ denote the set of reduced signed words for ω and suppose $l(\omega) = m$. Suppose there exists some maps $\mathcal{K}$: $[m-2\times RS(\omega)\to RS(\omega)$, and $\mathcal{S}$: $RS(\omega)\to S_m$, with the following properties: Supposing $w = w_1\cdots w_m \in RS(\omega)$ and $2\le i\le m-1$:

- $\mathcal{S}(\mathcal{K}(i,w)) = K(i,\mathcal{S}(w))$
- $\mathcal{K}(i,(i,w)) = w$.
- Setting $\mathcal{S}(w) = \sigma_1\cdots\sigma_m$, we have $w_j < w_{j+1} \iff \sigma_j < \sigma_{j+1}, \forall j < m$.

Both $\mathcal{K}$ and $\mathcal{S}$ can be defined on tableau-words by acting on the reading word, and it follows that $\mathcal{S}(\mathcal{K}(i,R)) = K(i,\mathcal{S}(R))$ for a tableau-word R. A tableau-word, R, whose word part is empty and whose tableau part has the shape of a partition is called a signed tableau for ω. If in addition, its standardization, $\mathcal{S}(R)$, is a standard

Young tableau, it is called a signed Edelman-Greene tableau, SEG. We define a functional descent of a tableau-word to be a descent in its standardization.

We now define an insertion algorithm $\mathcal{I}$ which maps $RS(\omega) \to SEG(\omega)$.

- The map $\mathcal{I}$ is defined by starting with the tableau-word $(\emptyset, w)$ and applying the map $\mathcal{I}'$ a total of $l(w)$ times.
- The map $\mathcal{I}'$ is defined on any tableau-word (R_1, R_2) by first removing the first entry of R_2 and appending it to the first row of R_1 and then applying the map $\mathcal{I}''$ as many times as possible.
- The map $\mathcal{I}''$ can be applied to any tableau-word (R_1, R_2) that has exactly one row, r, that has a functional descent, and where that functional descent is the second to last entry in r. It is defined by applying the map $\mathcal{I}'''$ as many times as possible, and then removing the first entry of r and appending it to the right end of row $r + 1$.
- The map $\mathcal{I}'''$ can be applied to any tableau-word (R_1, R_2) which has exactly one row, r, that has a functional descent, and where that functional descent is not the first entry of r. Supposing the first entry of r is the a^{th} entry in the reading word order, and the last entry of r is the b^{th} entry in the reading word order, then the map $\mathcal{I}'''$ applied to R is equal to $\mathcal{K}(a + 1(\cdots(\mathcal{K}(b - 2, \mathcal{K}(b - 1, R)))\cdots))$.

One should check, that in the definition above, if $\mathcal{K}$ is replaced with K and we assume w has no barred or repeated entries, we recover the definition of RSK insertion.

Definition 2.5.1. *Signed-Insertion Edelman-Greene map*: Given $v \in RSIF(\omega)$, form $\hat{v}$ by ignoring the subdivisions of v, so $\hat{v} \in RS(\omega)$. We will build up a pair of tableaux, $(P, Q) = ((P, \emptyset), Q)$, where P is a SEG and Q is an SSYT of the same shape, starting from $((\emptyset, \hat{v}), \emptyset)$, by successively applying the map $\mathcal{I}'$ to the lefthand factor $l(w)$ times. Meanwhile we create the recording tableau, Q, simultaneously: Each time a box is added to the tableau part of lefthand factor add a box in the corresponding position in the righthand factor. If this occurs during the t^{th} application of $\mathcal{I}'$ and v_t is in the s^{th} subdivision of v, then fill this box with s.

Theorem 2.5.2. *Suppose that there exists some maps $\mathcal{K}$ and $\mathcal{S}$ satisfying the conditions mentioned. Then, for any element $\omega \in C_n$, the* signed-insertion Edelman-Greene map *is a weight-preserving bijection between $RSIF(\omega)$ and pairs (P, Q), where $P \in SEG(\omega)$ and Q is a semistandard Young tableau of the same shape as P. (The weight of (P, Q), is taken to be the weight of Q.) In particular, we have:*

$$F^C_\omega(\mathbf{x}) = \sum_\lambda \bar{E}^\lambda_\omega s_\lambda(\mathbf{x})$$

where $\bar{E}^\lambda_\omega$ is the number of signed Edelman Greene tableaux for ω that have shape λ.

Proof. Let $v \in RSIF(\omega)$, and suppose v maps to (P, Q). First we show $P \in SEG(\omega)$. Let I be defined just as $\mathcal{I}$, except that K is used in place of $\mathcal{K}$. It is easy to verify that I is really just RSK insertion, and so the tableau $I(\mathcal{S}(\hat{v}))$ is automatically a standard Young tableau. But since $\mathcal{K}$ is assumed to commute with $\mathcal{S}$, one can check that the standardization of P is the same as $I(\mathcal{S}(\hat{v}))$, and hence a standard Young tableau. By definition, this makes a P an SEG.

Suppose Q is not a semistandard Young tableau. The only way this could happen would be if for some i such that v_i and v_{i+1} are in the same subdivision, the box added at the $(i+1)^{st}$ step is below the box added at the i^{th} step. Now, by definition $v_i < v_{i+1}$, and hence by assumption $\sigma_i < \sigma_{i+1}$ where $\sigma = \mathcal{S}((\emptyset, \omega))$. By commutivity of $\mathcal{K}$ and $\mathcal{S}$, boxes are added to Q in the same order as in the recording tableau of RSK insertion of σ. But it is a standard fact of RSK insertion that $\sigma_i < \sigma_{i+1}$ implies the box added at the $(i+1)^{st}$ step is weakly above the box added at the i^{th} step.

Now we show the map is invertible. By similar logic as above, one may verify that if v inserts to (P, Q), then the order that the boxes are added to Q is the standardization of Q (using the regular definition of standardization of an SSYT. Thus, we may uniquely reverse the map $\mathcal{I}$ (this is possible because the map $\mathcal{K}$ can be inverted for fixed i, i.e., $\mathcal{K}(i, (i, w)) = w$), to get an element of $RS(\omega)$ that inserts to P. By adding subdivisions to this element as dictated by the weight of Q, we find an element that maps under signed-insertion Edelman-Greene map to (P, Q). The fact that this element is truly in $RSIF(\omega)$ (particularly that the subdivisions are increasing) follows from the converse of the statement in the paragraph above, namely, if the box added at the $(i+1)^{st}$ step is weakly above the box added at the i^{th} step then $v_i < v_{i+1}$.

□

In order to make the statement in Theorem 2.5.2 explicit we must construct maps $\mathcal{K}$ and $\mathcal{S}$ with the required properties. In particular, this defines explicitly what a SEG tableau is, and how to compute the coefficients $\bar{E}^\lambda_\omega$. We begin by doing this explicitly for a special subset of very simple elements of the Coxeter group C_n.

Definition 2.5.3. We say an element $\omega \in C_n$ is *untangled* if the following hold for some (equivalently all) reduced word w for ω.

(1) s_2 does not appear in w

(2) For $i > 2$, if s_i and s_{i+1} appear in w, and one of s_i or s_{i+1} appears more than once, then s_{i-1} and s_{i+2} do not appear in w.

For instance, the following are *untangled*: 1010434, 010434676, 10134587.

Theorem 2.5.4. *Suppose $\omega \in C_n$ is untangled*[4]*. Then writing*

$$F^C_\omega(\mathbf{x}) = \sum_\lambda \bar{E}^\lambda_\omega s_\lambda(\mathbf{x})$$

$\bar{E}^\lambda_\omega$ is the number of tableaux, T, of shape λ composed of entries from the alphabet $\bar{X}$ such that:

(1) $rd(T)$ is a reduced signed word for ω.

(2) The rows and columns of T are weakly increasing.

(3) Whenever $T_{ij} = T_{(i+1)j}$ and $T_{ij} \neq 0$, there exists $k > j$ such that $|T_{ik}| = |T_{ij}| + 1$ or there exists $l < j$ such that $|T_{(i+1)l}| = |T_{ij}| + 1$, or else we have both $T_{ij} = \bar{1} = T_{(i+1)j}$ and $T_{i(j+1)} = 0 = T_{(i+1)(j+1)}$.

Proof. We explicitly define the maps $\mathcal{K}$ and $\mathcal{S}$ for $\omega \in C_n$ *untangled*. First, we define $\mathcal{S}$, by creating a total order, $\prec$, on entries of a signed word w.

(1) If $|w_i| > 1$ or $|w_j| > 1$ then $w_i \prec w_j$ if and only if $w_i < w_j$ in the order $\bar{X}$, or $w_i = w_j$, and there is $i < k < j$ such that $|w_k| = |w_i| - 1$.

(2) If $|w_i| \leq 1$ and $|w_j| \leq 1$, we use the following explicit ordering on the entries of the subword of w which is composed of $\bar{1}$s, 0s, and 1s, to determine whether $w_i \prec w_j$ or $w_j \prec w_i$:

- $01 = 12$
- $0\bar{1} = 21$
- $10 = 21$
- $\bar{1}0 = 12$
- $101 = 213$
- $\bar{1}01 = 123$
- $10\bar{1} = 321$
- $\bar{1}0\bar{1} = 231$
- $010 = 132$
- $0\bar{1}0 = 312$
- $0101 = 1324$
- $0\bar{1}01 = 3124$
- $010\bar{1} = 2431$
- $0\bar{1}0\bar{1} = 2143$
- $1010 = 4231$
- $10\bar{1}0 = 4213$
- $\bar{1}010 = 1342$
- $\bar{1}0\bar{1}0 = 2413$

[4] It is likely this theorem holds on a much larger subset of C_n. This will be discussed in the next section

Now we define $\mathcal{K}$. Given $w = w_1 \cdots w_n \in RS(\omega)$, we define $\mathcal{K}(i, w)$ to be $\mathcal{W}(i + \delta, w)$ where δ is given as follows. Setting $\mathcal{S}(w) = \sigma_1 \cdots \sigma_n$:

(1) If $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ or $\sigma_{i+1} < \sigma_i < \sigma_{i-1}$, then $\delta = -i$.

(2) If $\sigma_{i-1} < \sigma_{i+1} < \sigma_i$ or $\sigma_i < \sigma_{i+1} < \sigma_{i-1}$, then $\delta = 0$.

(3) If $\sigma_i < \sigma_{i-1} < \sigma_{i+1}$ or $\sigma_{i+1} < \sigma_{i-1} < \sigma_i$, then $\delta = 1$.

If $i = 0$, $\mathcal{W}(i, w) = w$. If $i > 0$, then $\mathcal{W}(i, w) = w' = w'_1 \cdots w'_n$, where $w'_j = w_j$ for all j except where indicated below. (For $a \in \mathbb{Z}_{\geq 0}$ we consider a and $\bar{a}$ as elements of $\bar{X}$ and assume $\bar{\bar{a}} = a \in \bar{X}$, and $\bar{0} = 0 \in \bar{X}$.)

(1) If $||w_i| - |w_{i+1}|| > 1$, $w'_i = w_{i+1}$, $w'_{i+1} = w_i$.

(2) If $||w_i| - |w_{i+1}|| = 1$, $min(|w_i|, |w_{i+1}|) > 0$ and:

- There exists $k < i$ such that $w_k = |w_{i+1}|$, then $w'_i = w_{i+1}$, $w'_{i+1} = w_i$, $w'_k = |w_i|$.
- There exists $k > i + 1$ such that $w_k = |\bar{w_i}|$, then $w'_i = w_{i+1}$, $w'_{i+1} = w_i$, $w'_k = |\overline{w_{i+1}}|$.
- Otherwise, $w'_i = \bar{w}_i$, $w'_{i+1} = \overline{w_{i+1}}$.

(3) If $||w_i| - |w_{i+1}|| = 1$, $min(|w_i|, |w_{i+1}|) = 0$ and:

- There is $l < i$ and $k > i + 1$ such that $|w_l| \leq 1$ and $|w_k| \leq 1$. Then $w'_i = w_{i+1}$, $w'_{i+1} = w_i$, $w'_l = w_k$, $w'_k = w_l$.
- There is $l < k < i$ such that $|w_l| \leq 1$ and $|w_k| \leq 1$ and:
  - $w_l = 1$. Then $w'_i = \bar{w}_i$.
  - Otherwise, $w'_i = w_{i+1}$, $w'_{i+1} = w_i$, $w'_l = \bar{w}_k$, $w'_k = \bar{w}_l$.
- There is $i + 1 < l < k$ such that $|w_l| \leq 1$ and $|w_k| \leq 1$ and:
  - $w_k = 1$. Then $w'_{i+1} = \overline{w_{i+1}}$.
  - Otherwise, $w'_i = w_{i+1}$, $w'_{i+1} = w_i$, $w'_l = \bar{w}_k$, $w'_k = \bar{w}_l$.
- If none of the cases above occur, then there exists exactly one $k \notin \{i, i + 1\}$ such that $|w_k| \leq 1$. In this case, $w'_i = \bar{w}_i$, $w'_{i+1} = \overline{w_{i+1}}$, $w'_k = \bar{w}_k$.

One easily checks that for any $\omega \in C_n$ untangled with $l(\omega) = m$:

(1) $\mathcal{S}(\mathcal{K}(i, w)) = K(i, \mathcal{S}(w))$ for any $w \in RS(\omega)$ and $2 \leq i \leq m - 1$.

(2) $\mathcal{K}(i, (i, w)) = w$ for any $w \in RS(\omega)$ and $2 \leq i \leq m - 1$.

(3) If $w = w_1 \cdots w_m \in RS(\omega)$, and $\mathcal{S}(w) = \sigma_1 \cdots \sigma_m$, then for each $1 \leq i < m$ we have $w_i < w_{i+1}$ if and only if $\sigma_i < \sigma_{i+1}$.

Thus by Theorem 2.5.2 $\bar{E}^{\lambda}_{\omega}$ is the number of tableau-words for ω which have empty word part and whose standardization under $\mathcal{S}$ is a standard Young tableau. It is not difficult to check that this is equivalent to the four properties listed in the theorem. $\square$

Below, we use the maps $\mathcal{S}$ and $\mathcal{K}$ explicitly defined for untangled words in the proof above, and the insertion map $\mathcal{I}$, corresponding to them, to apply the Signed-Insertion Edelman-Greene map in two examples:

Example 2.5.5. Let $v = (13)(\bar{4}0)(\bar{3}1)$. The pair (P, Q) is obtained as follows:

$$\Big(\big(\emptyset : 13\bar{4}0\bar{3}1\big), \emptyset\Big) \to \Big(\big(\begin{array}{|c|}\hline 1 \\ \hline\end{array} : 3\bar{4}0\bar{3}1\big), \begin{array}{|c|}\hline 1 \\ \hline\end{array}\Big) \to \Big(\big(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} : \bar{4}0\bar{3}1\big), \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\Big) \to \Big(\big(\begin{array}{|c|c|}\hline -4 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} : 0\bar{4}1\big), \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\Big)$$

$$\to \Big(\big(\begin{array}{|c|c|}\hline -4 & 0 \\ \hline 1 & 3 \\ \hline\end{array} : \bar{3}1\big), \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\Big) \to \Big(\big(\begin{array}{|c|c|}\hline -4 & -3 \\ \hline 0 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} : 1\big), \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\Big) \to \Big(\big(\begin{array}{|c|c|c|}\hline -4 & -3 & 1 \\ \hline 0 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} : \emptyset\big), \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\Big).$$

Example 2.5.6. Let $v = (\bar{3}01)(04)(\bar{1}3)$. The pair (P, Q) is obtained as follows:

$$\Big(\big(\emptyset : \bar{3}0104\bar{1}3\big), \emptyset\Big) \longrightarrow \Big(\big(\begin{array}{|c|}\hline -3 \\ \hline\end{array} : 0104\bar{1}3\big), \begin{array}{|c|}\hline 1 \\ \hline\end{array}\Big) \longrightarrow \Big(\big(\begin{array}{|c|c|}\hline -3 & 0 \\ \hline\end{array} : 104\bar{1}3\big), \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\Big) \longrightarrow$$

$$\Big(\big(\begin{array}{|c|c|c|}\hline -3 & 0 & 1 \\ \hline\end{array} : 04\bar{1}3\big), \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}\Big) \longrightarrow \Big(\big(\begin{array}{|c|c|c|}\hline -3 & 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} : 403\big), \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\Big) \longrightarrow \Big(\big(\begin{array}{|c|c|c|c|}\hline -3 & 0 & 1 & 4 \\ \hline 1 \\ \cline{1-1}\end{array} : 03\big), \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\Big)$$

$$\longrightarrow \Big(\big(\begin{array}{|c|c|c|c|}\hline -3 & -1 & 0 & 4 \\ \hline 0 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} : 3\big), \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\Big) \to \Big(\big(\begin{array}{|c|c|c|c|}\hline -4 & -1 & 0 & 3 \\ \hline 0 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} : \emptyset\big), \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\Big).$$

For $\omega \in A_n$, the *t-Stanley symmetric function*:

$$F^t_{\omega}(\mathbf{x}, t) = \sum_{v \in RSIF(\omega)} \mathbf{x}^{w(v)} t^{ht(v)},$$

where $ht(v)$ is the number of barred entries in v and $w(v)$ is the vector whose i^{th} coordinate records the total number of entries in the i^{th} subdivision of v, is Schur-positive. This can be verified from the equation,

$$F^d_{\omega}(\mathbf{x}, \mathbf{y}) = \sum_{v \in RSIF(\omega)} \mathbf{x}^{dw(v,1)}, \mathbf{y}^{dw(v,2)},$$

by plugging in $\mathbf{y} = t\mathbf{x}$. Moreover, as w and $\mathcal{K}(i, w)$ always have the same number of barred entries in the type A_n case, we have:

Corollary 2.5.7. *Suppose $\omega \in A_n$ is untangled. Then*

$$F^t_\omega(\mathbf{x}) = \sum_\lambda \bar{E}^{\lambda r}_\omega s_\lambda(\mathbf{x}) t^r$$

where $\bar{E}^{\lambda r}_\omega$ is the number of tableaux, T, composed of entries from the alphabet $\bar{X}$ with shape λ such that:

1. *$rd(T)$ is a reduced signed word for ω with r barred entries.*
2. *The rows and columns of T are weakly increasing.*
3. *Whenever $T_{ij} = T_{(i+1)j}$, there exists $k > j$ such that $|T_{ik}| = |T_{ij}| + 1$ or there exists $l < j$ such that $|T_{(i+1)l}| = |T_{ij}| + 1$.*

For $\omega \in C_n$ such that no word for ω has more than one s_0, the parity of the number of barred entries in w and $\mathcal{K}(i, w)$ is always the same. Hence for such ω, which are also untangled the *even Stanley symmetric function* and *odd Stanley symmetric function*:

$$F^{even}_\omega(\mathbf{x}) = \sum_{v \in RSIF(\omega)} \left[\frac{-1^{ht(v)} + 1}{2}\right] \mathbf{x}^{w(v)} \qquad F^{odd}_\omega(\mathbf{x}) = -\sum_{v \in RSIF(\omega)} \left[\frac{-1^{ht(v)} - 1}{2}\right] \mathbf{x}^{w(v)}$$

are Schur-positive, and we have:

Corollary 2.5.8. *Suppose $\omega \in C_n$ is untangled and each word for ω has at most one s_0. Then*

$$F^{even}_\omega(\mathbf{x}) = \sum_\lambda \bar{E}^{\lambda +}_\omega s_\lambda(\mathbf{x}) \qquad F^{odd}_\omega(\mathbf{x}) = \sum_\lambda \bar{E}^{\lambda -}_\omega s_\lambda(\mathbf{x})$$

where $\bar{E}^{\lambda +}_\omega$ (resp., $\bar{E}^{\lambda -}_\omega$) is the number of tableaux, T, composed of entries from the alphabet $\bar{X}$ with shape λ such that:

1. *$rd(T)$ is a reduced signed word for ω with even (odd) number of barred entries.*
2. *The rows and columns of T are weakly increasing.*
3. *Whenever $T_{ij} = T_{(i+1)j}$, there exists $k > j$ such that $|T_{ik}| = |T_{ij}| + 1$ or there exists $l < j$ such that $|T_{(i+1)l}| = |T_{ij}| + 1$.*

2.6. Conjectures

Conjecture 2.6.1. The maps $\mathcal{K}$ and $\mathcal{S}$ assumed to exist in Theorem 2.5.2 exist for any $\omega \in C_n$. Moreover, these maps, and the set $SEG(\omega)$ defined with respect to $\mathcal{S}$ satisfy the following properties:

(1) Let $w \in RS(\omega)$. Then if any of $|w_{i-1}|, |w_i|, |w_{i+1}| > 1$, then w and $\mathcal{K}(i, w)$ have the same number of barred entries.

(2) Let $w \in RS(\omega)$. If there is no more than one 0 in w, then the parity of the number of barred entries of w and $\mathcal{K}(i, w)$ is the same.

(3) Let $w \in RS(\omega)$ and let $w' = \mathcal{K}(i, w)$, then either $w_1 \cdots w_{i-2} = w'_1 \cdots w'_{i-2}$ or $w_{i+2} \cdots w_n = w'_{i+2} \cdots w'_n$ as signed words, or both.

(4) $SEG(\omega)$ is a subset of the set of signed tableaux for ω with weakly increasing rows and columns.

(5) If T is a signed tableau for ω, then $\mathcal{I}(rd(T)) = T$ if and only if $T \in SEG(\omega)$.

Consider C_n with generators $\{s_0, s_1, s_2, \ldots, s_n\}$. We say $\omega \in C_n^{p-q}$ if $\omega \in C_n$ and each reduced word for ω contains p generators of which q are equal to s_0. For instance, C_n^{p-0} is the subset of length p elements in A_n.

Definition 2.6.2. We say an element $\omega \in C_n$ is *unknotted* if the following hold for all reduced words w for ω.

(1) If the sequence $s_0 s_1 s_0 s_1$ appears in w, then s_2 does not.

(2) For $i > 0$ if the sequence $s_i s_{i+1} s_i$ appears in w then s_{i+2} does not.

For instance 123454, 2101232, 1010343, and 213243 are *unknotted*. Clearly *unknotted* is a weaker condition than *untangled*. The subset of C_n^{p-q} composed of *unknotted* elements is denoted $C_n^{\bar{p}-q}$. We conclude by establishing the results in 2.5.4, 2.5.7, and 2.5.8 to for $C_n^{\bar{p}-q}$ for certain n, p, and q as equalities of polynomials in three variables. In order to do so for 2.5.8 it is first good to know:

Theorem 2.6.3. *If $\omega \in C_8^{8-1}$ then $F_\omega^{odd}(x_1, x_2, x_3)$ and $F_\omega^{even}(x_1, x_2, x_3)$ are symmetric and Schur positive.*

Proof. Computer verification. There are 4489 such ω to check. □

Conjecture 2.6.4. The above holds for all $\omega \in C_n^{p-1}$ with (x_1, x_2, x_3) replaced by $(\mathbf{x})$.

Proposition 2.6.5. *Suppose $\omega \in C_8^{\bar{8}-1}$ Then*

$$F^{even}_{\omega}(x_1,x_2,x_3) = \sum_{\lambda} \bar{E}^{\lambda+}_{\omega} s_{\lambda}(x_1,x_2,x_3) \qquad F^{odd}_{\omega}(x_1,x_2,x_3) = \sum_{\lambda} \bar{E}^{\lambda-}_{\omega} s_{\lambda}(x_1,x_2,x_3)$$

where $\bar{E}^{\lambda+}_{\omega}$ (resp., $\bar{E}^{\lambda-}_{\omega}$) is the number of tableaux, T, composed of entries from the alphabet $\bar{X}$ with shape $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that:

(1) $rd(T)$ is a reduced signed word for ω with even (odd) number of barred entries.

(2) The rows and columns of T are weakly increasing.

(3) Whenever $T_{ij} = T_{(i+1)j}$, there exists $k > j$ such that $|T_{ik}| = |T_{ij}| + 1$ or there exists $l < j$ such that $|T_{(i+1)l}| = |T_{ij}| + 1$.

Proof. Computer verification. There are 2511 such ω to check. □

Conjecture 2.6.6. The above holds for all $\omega \in C_n^{\bar{p}-1}$ with (x_1, x_2, x_3) replaced by $(\mathbf{x})$.

Example 2.6.7. For instance, if $\omega = s_1 s_0 s_1 s_2 s_3 s_4 s_3 s_6$ then $\bar{E}^{(4,3,1)-}_{\omega}$ counts the 2 tableaux:

-1	2	3	4
0	4	6	
1			

-6	-4	-3	4
0	1	2	
1			

.

Proposition 2.6.8. *If $\omega \in C_9^{\bar{9}-0}$. Then*

$$F^{t}_{\omega}(x_1,x_2,x_3) = \sum_{\lambda} \bar{E}^{\lambda r}_{\omega} s_{\lambda}(x_1,x_2,x_3) t^r$$

where $\bar{E}^{\lambda r}_{\omega}$ is the number of tableaux, T, composed of entries from the alphabet $\bar{X}$ with shape $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that:

(1) $rd(T)$ is a reduced signed word for ω with r barred entries.

(2) The rows and columns of T are weakly increasing.

(3) Whenever $T_{ij} = T_{(i+1)j}$, there exists $k > j$ such that $|T_{ik}| = |T_{ij}| + 1$ or there exists $l < j$ such that $|T_{(i+1)l}| = |T_{ij}| + 1$.

Proof. Computer verification. There are 3167 such ω to check. □

Conjecture 2.6.9. The above holds for all $\omega \in C_n^{\bar{p}-0}$ with (x_1, x_2, x_3) replaced by $\mathbf{x}$.

Example 2.6.10. For instance, if $\omega = s_3s_4s_8s_2s_3s_5s_6s_1s_5 \in C_9^{\bar{9}-0}$ then $\bar{E}_\omega^{(3,3,3)1}$ counts the 2 tableaux:

$$\begin{array}{|c|c|c|}\hline -6&1&3\\\hline 2&4&5\\\hline 3&6&8\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline -6&1&5\\\hline 2&3&5\\\hline 3&4&8\\\hline\end{array}.$$

Proposition 2.6.11. *Suppose $\omega \in C_7^{\bar{7}-q}$ for any q. Then*

$$F_\omega^C(x_1, x_2, x_3) = \sum_\lambda \bar{E}_\omega^\lambda s_\lambda(x_1, x_2, x_3)$$

where $\bar{E}_\omega^\lambda$ is the number of tableaux, T, of shape $\lambda = \lambda_1, \lambda_2, \lambda_3$ composed of entries from the alphabet $\bar{X}$ such that:

(1) $rd(T)$ is a reduced signed word for ω.

(2) The rows and columns of T are weakly increasing.

(3) Whenever $T_{ij} = T_{(i+1)j}$ and $T_{ij} \neq 0$, there exists $k > j$ such that $|T_{ik}| = |T_{ij}| + 1$ or there exists $l < j$ such that $|T_{(i+1)l}| = |T_{ij}| + 1$, or else we have both $T_{ij} = \bar{1} = T_{(i+1)j}$ and $T_{i(j+1)} = 0 = T_{(i+1)(j+1)}$.

Proof. Computer verification. There are 1414 such ω to check. □

Conjecture 2.6.12. The above holds for all $\omega \in C_n^{\bar{p}-q}$ with (x_1, x_2, x_3) replaced by $\mathbf{x}$.

Example 2.6.13. For instance, if $\omega = s_1s_0s_1s_2s_1s_0s_1 \in C_7^{\bar{7}-2}$ then $\bar{E}_\omega^{(4,3,0)} = \bar{E}_\omega^{(4,2,1)} = \bar{E}_\omega^{(3,3,1)} = \bar{E}_\omega^{(3,3,2)} = 1$ and all others are 0. The corresponding tableaux are:

$$\begin{array}{|c|c|c|c|}\hline -2&-1&0&1\\\hline -1&0&1\\\cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline -2&-1&0&1\\\hline 0&1\\\cline{1-2} 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline -1&0&1\\\hline 0&1&2\\\hline 1\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline -2&0&1\\\hline 0&1\\\cline{1-2} 1&2\\\cline{1-2}\end{array}.$$

Example 2.6.14. For instance, if $\omega = s_1s_0s_1s_0s_3s_4s_5 \in C_7^{\bar{7}-2}$ then $\bar{E}_\omega^{(4,3,0)}$ counts the 6 tableaux:

$$\begin{array}{|c|c|c|c|}\hline -4&-1&0&5\\\hline -3&-1&0\\\cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline -4&0&1&5\\\hline 0&1&3\\\cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline -4&-1&0&5\\\hline -1&0&3\\\cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline -5&-1&0&1\\\hline 0&3&4\\\cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline -5&-1&0&1\\\hline -3&0&4\\\cline{1-3}\end{array} \quad \begin{array}{|c|c|c|c|}\hline -1&0&1&5\\\hline 0&3&4\\\cline{1-3}\end{array}.$$

CHAPTER 3

Primed and Signed Tableaux of Shifted Shape: Type C Stanley Symmetric Functions

This chapter is based on the work in [**HPS17**].

3.1. Introduction

In this chapter, we carry out a crystal analysis of the Stanley symmetric functions $F^C_w(\mathbf{x})$ of type C, indexed by a Coxeter group element w. In particular, we use Kraśkiewicz insertion [**Kra89**, **Kra95**] and Haiman's mixed insertion [**Hai89**] to find a crystal structure on shifted semistandard tableaux, which in turn implies a crystal structure $\mathcal{B}_w$ on reduced unimodal factorizations (as defined in chapter 1) of w for which $F^C_w(\mathbf{x})$ is a character. Moreover, we present a type A crystal isomorphism $\Phi\colon \mathcal{B}_w \to \bigoplus_\lambda \mathcal{B}_\lambda^{\oplus g_{w\lambda}}$ for some combinatorially defined nonnegative integer coefficients $g_{w\lambda}$; here $\mathcal{B}_\lambda$ is the type A highest weight crystal of highest weight λ . This implies the desired decomposition $F^C_w(\mathbf{x}) = \sum_\lambda g_{w\lambda} s_\lambda(\mathbf{x})$ (see Corollary 3.3.10) and similarly for type B.

Recall the *Coxeter group* W_C of type C_n as defined in chapter 2. It is often convenient to write down an element of a Coxeter group as a sequence of indices of s_i in the product representation of the element. For example, the element $w = s_2s_1s_2s_1s_0s_1s_0s_1$ is represented by the word $\mathbf{w} = 2120101$. A word of shortest length ℓ is referred to as a *reduced word* and $\ell(w) := \ell$ is referred as the length of w.

Recall from chapter 2 that the [**BH95**, **FK96**, **Lam95**] *type C Stanley symmetric function* associated to $w \in W_C$ is defined as

$$F^C_w(\mathbf{x}) = \sum_{\mathbf{A}\in U(w)} 2^{\mathrm{nz}(\mathbf{A})}\mathbf{x}^{\mathrm{wt}(\mathbf{A})}. \tag{3.1}$$

Here $\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $\mathbf{x}^{\mathbf{v}} = x_1^{v_1}x_2^{v_2}x_3^{v_3}\cdots$.

In Section 3.2 we describe our crystal isomorphism by combining a slight generalization of the Kraśkiewicz insertion [**Kra89**, **Kra95**] and Haiman's mixed insertion [**Hai89**]. The main result regarding the crystal structure under Haiman's mixed insertion is stated in Theorem 3.3.3. The combinatorial interpretation of the coefficients $g_{w\lambda}$ is given in Corollary 3.3.10. In Section 3.4, we provide an alternative interpretation of the coefficients $g_{w\lambda}$ in terms of semistandard unimodal tableaux. Appendices A.1 and A.2 are reserved for the proofs of Theorems 3.3.3 and 3.3.6.

3.2. Crystal isomorphism

In this section, we combine a slight generalization of the Kraśkiewicz insertion, reviewed in Section 3.2.1, and Haiman's mixed insertion, reviewed in Section 3.2.2, to provide an isomorphism of crystals between the crystal of words $\mathcal{B}^h$ and certain sets of primed tableaux of shifted shape.

3.2.1. Kraśkiewicz insertion. In this section, we describe the Kraśkiewicz insertion. To do so, we first need to define the *Edelman–Greene insertion* [**?**]. It is defined for a word $\mathbf{w} = w_1 \dots w_\ell$ and a letter k such that the concatenation $w_1 \dots w_\ell k$ is an A-type reduced word. The Edelman–Greene insertion of a letter k into an *increasing* word $\mathbf{w} = w_1 \dots w_\ell$, denoted by $\mathbf{w} \leftsquigarrow k$, is constructed as follows:

(1) If $w_\ell < k$, then $\mathbf{w} \leftsquigarrow k = \mathbf{w}'$, where $\mathbf{w}' = w_1 w_2 \dots w_\ell\, k$.

(2) If $k > 0$ and $k\,k+1 = w_i\, w_{i+1}$ for some $1 \leqslant i < \ell$, then $\mathbf{w} \leftsquigarrow k = k+1 \leftsquigarrow \mathbf{w}$.

(3) Else let w_i be the leftmost letter in $\mathbf{w}$ such that $w_i > k$. Then $\mathbf{w} \leftsquigarrow k = w_i \leftsquigarrow \mathbf{w}'$, where $\mathbf{w}' = w_1 \dots w_{i-1}\, k\, w_{i+1} \dots w_\ell$.

In the cases above, when $\mathbf{w} \leftsquigarrow k = k' \leftsquigarrow \mathbf{w}'$, the symbol $k' \leftsquigarrow \mathbf{w}'$ indicates a word $\mathbf{w}'$ together with a "bumped" letter k'.

Next we consider a reduced unimodal word $\mathbf{a} = a_1 a_2 \dots a_\ell$ with $a_1 > a_2 > \dots > a_v < a_{v+1} < \dots < a_\ell$. The *Kraśkiewicz row insertion* [**Kra89, Kra95**] is defined for a unimodal word $\mathbf{a}$ and a letter k such that the concatenation $a_1 a_2 \dots a_\ell k$ is a C-type reduced word. The Kraśkiewicz row insertion of k into $\mathbf{a}$ (denoted similarly as $\mathbf{a} \leftsquigarrow k$), is performed as follows:

(1) If $k = 0$ and there is a subword 101 in $\mathbf{a}$, then $\mathbf{a} \leftsquigarrow 0 = 0 \leftsquigarrow \mathbf{a}$.

(2) If $k \neq 0$ or there is no subword 101 in $\mathbf{a}$, denote the decreasing part $a_1 \dots a_v$ as $\mathbf{d}$ and the increasing part $a_{v+1} \dots a_\ell$ as $\mathbf{g}$. Perform the Edelman-Greene insertion of k into $\mathbf{g}$.

   (a) If $a_\ell < k$, then $\mathbf{g} \leftsquigarrow k = a_{v+1} \dots a_\ell k =: \mathbf{g}'$ and $\mathbf{a} \leftsquigarrow k = \mathbf{dg} \leftsquigarrow k = \mathbf{d}\,\mathbf{g}' =: \mathbf{a}'$.

   (b) If there is a bumped letter and $\mathbf{g} \leftsquigarrow k = k' \leftsquigarrow \mathbf{g}'$, negate all the letters in $\mathbf{d}$ (call the resulting word $-\mathbf{d}$) and perform the Edelman-Greene insertion $-\mathbf{d} \leftsquigarrow -k'$. Note that there will always be a bumped letter, and so $-\mathbf{d} \leftsquigarrow -k' = -k'' \leftsquigarrow -\mathbf{d}'$ for some decreasing word $\mathbf{d}'$. The result of the Kraśkiewicz insertion is: $\mathbf{a} \leftsquigarrow k = \mathbf{d}[\mathbf{g} \leftsquigarrow k] = \mathbf{d}[k' \leftsquigarrow \mathbf{g}'] = -[-\mathbf{d} \leftsquigarrow -k']\,\mathbf{g}' = [k'' \leftsquigarrow \mathbf{d}']\mathbf{g}' = k'' \leftsquigarrow \mathbf{a}'$, where $\mathbf{a}' := \mathbf{d}'\mathbf{g}'$.

Example 3.2.1.

$$31012 \leftsquigarrow 0 = 0 \leftsquigarrow 31012, \quad 3012 \leftsquigarrow 0 = 0 \leftsquigarrow 3102,$$

$$31012 \leftsquigarrow 1 = 1 \leftsquigarrow 32012, \quad 31012 \leftsquigarrow 3 = 310123.$$

The insertion is constructed to "commute" a unimodal word with a letter: If $\mathbf{a} \leftsquigarrow k = k' \leftsquigarrow \mathbf{a}'$, the two elements of the type C Coxeter group corresponding to concatenated words $\mathbf{a}\,k$ and $k'\mathbf{a}'$ are the same.

The type C Stanley symmetric functions (3.1) are defined in terms of unimodal factorizations. To put the formula on a completely combinatorial footing, we need to treat the powers of 2 by introducing signed unimodal factorizations. A *signed unimodal factorization* of $w \in W_C$ is a unimodal factorization $\mathbf{A}$ of w, in which every non-empty factor is assigned either a $+$ or $-$ sign. Denote the set of all signed unimodal factorizations of w by $U^{\pm}(w)$.

For a signed unimodal factorization $\mathbf{A} \in U^{\pm}(w)$, define $\mathrm{wt}(\mathbf{A})$ to be the vector with i-th coordinate equal to the number of letters in the i-th factor of $\mathbf{A}$. Notice from (3.1) that

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U^{\pm}(w)} \mathbf{x}^{\mathrm{wt}(\mathbf{A})}. \tag{3.1}$$

We will use the Kraśkiewicz insertion to construct a map between signed unimodal factorizations of a Coxeter group element w and pairs of certain types of tableaux $(\mathbf{P}, \mathbf{T})$. We define these types of tableaux next.

A *shifted diagram* $\mathcal{S}(\lambda)$ associated to a partition λ with distinct parts is the set of boxes in positions $\{(i, j) \mid \ 1 \leqslant i \leqslant \ell(\lambda),\ i \leqslant j \leqslant \lambda_i + i - 1\}$. Here, we use English notation, where the box $(1, 1)$ is always top-left.

Let X_n° be an ordered alphabet of n letters $X_n^\circ = \{0 < 1 < 2 < \cdots < n-1\}$, and let X_n' be an ordered alphabet of n letters together with their primed counterparts as $X_n' = \{1' < 1 < 2' < 2 < \cdots < n' < n\}$.

Let λ be a partition with distinct parts. A *unimodal tableau* $\mathbf{P}$ of shape λ on n letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet X_n° such that the word P_i obtained by reading the ith row from the top of $\mathbf{P}$ from left to right, is a unimodal word, and P_i is the longest unimodal subword in the concatenated word

$P_{i+1}P_i$ [**BHRY14**] (cf. also with decomposition tableaux [**Ser10, Cho13**]). The *reading word* of a unimodal tableau $\mathbf{P}$ is given by $\pi_{\mathbf{P}} = P_\ell P_{\ell-1} \dots P_1$. A unimodal tableau is called *reduced* if $\pi_{\mathbf{P}}$ is a type C reduced word corresponding to the Coxeter group element $w_{\mathbf{P}}$. Given a fixed Coxeter group element w, denote the set of reduced unimodal tableaux $\mathbf{P}$ of shape λ with $w_{\mathbf{P}} = w$ as $\mathcal{UT}_w(\lambda)$.

A *shifted primed tableau* $\mathbf{T}$ of shape λ on n letters (cf. semistandard Q-tableau [**Lam95**]) is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet X'_n such that:

(1) The entries are weakly increasing along each column and each row of $\mathbf{T}$.

(2) Each row contains at most one i' for every $i = 1, \dots, n$.

(3) Each column contains at most one i for every $i = 1, \dots, n$.

Denote the set of shifted primed tableaux of shape λ by $\mathcal{PT}^{\pm}(\lambda)$. Given an element $\mathbf{T} \in \mathcal{PT}^{\pm}(\lambda)$, define the weight of the tableau $\mathrm{wt}(\mathbf{T})$ as the vector with i-th coordinate equal to the total number of letters in $\mathbf{T}$ that are either i or i'.

Example 3.2.2. $\left(\begin{array}{ccccc} 4 & 3 & 2 & 0 & 1 \\ & 2 & 1 & 2 & \\ & & 0 & & \end{array} \;,\; \begin{array}{ccccc} 1 & 1 & 2' & 3' & 3 \\ & 2' & 2 & 3' & \\ & & 4 & & \end{array} \right)$ is a pair consisting of a unimodal tableau and a shifted primed tableau both of shape $(5, 3, 1)$.

For a reduced unimodal tableau $\mathbf{P}$ with rows $P_\ell, P_{\ell-1}, \dots, P_1$, the Kraśkiewicz insertion of a letter k into tableau $\mathbf{P}$ (denoted again by $\mathbf{P} \leftsquigarrow k$) is performed as follows:

(1) Perform Kraśkiewicz insertion of the letter k into the unimodal word P_1. If there is no bumped letter and $P_1 \leftsquigarrow k = P'_1$, the algorithm terminates and the new tableau $\mathbf{P}'$ consists of rows $P_\ell, P_{\ell-1}, \dots, P_2, P'_1$. If there is a bumped letter and $P_1 \leftsquigarrow k = k' \leftsquigarrow P'_1$, continue the algorithm by inserting k' into the unimodal word P_2.

(2) Repeat the previous step for the rows of $\mathbf{P}$ until either the algorithm terminates, in which case the new tableau $\mathbf{P}'$ consists of rows $P_\ell, \dots, P_{s+1}, P'_s, \dots, P'_1$, or, the insertion continues until we bump a letter k_e from P_ℓ, in which case we then put k_e on a new row of the shifted shape of $\mathbf{P}'$, so that the resulting tableau $\mathbf{P}'$ consists of rows $k_e, P'_\ell, \dots, P'_1$.

Example 3.2.3.

$$\begin{array}{ccccc} 4 & 3 & 2 & 0 & 1 \\ & 2 & 1 & 2 & \\ & & 0 & & \end{array} \leftsquigarrow 0 = \begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \\ & 2 & 1 & 0 & \\ & & 0 & 1 & \end{array},$$

since the insertions row by row are given by $43201 \leftsquigarrow 0 = 0 \leftsquigarrow 43210$, $212 \leftsquigarrow 0 = 1 \leftsquigarrow 210$, and $0 \leftsquigarrow 1 = 01$.

Lemma 3.2.4. *[**Kra89**] Let $\mathbf{P}$ be a reduced unimodal tableau with reading word $\pi_{\mathbf{P}}$ for an element $w \in W_C$. Let k be a letter such that $\pi_{\mathbf{P}}k$ is a reduced word. Then the tableau $\mathbf{P}' = \mathbf{P} \leftsquigarrow k$ is a reduced unimodal tableau, for which the reading word $\pi_{\mathbf{P}'}$ is a reduced word for ws_k.*

Lemma 3.2.5. *[**Lam95**, Lemma 3.17] Let $\mathbf{P}$ be a unimodal tableau, and $\mathbf{a}$ a unimodal word such that $\pi_{\mathbf{P}}\mathbf{a}$ is reduced. Let $(x_1, y_1), \dots, (x_r, y_r)$ be the (ordered) list of boxes added when $\mathbf{P} \leftsquigarrow \mathbf{a}$ is computed. Then there exists an index v, such that $x_1 < \cdots < x_v \geqslant \cdots \geqslant x_r$ and $y_1 \geqslant \cdots \geqslant y_v < \cdots < y_r$.*

Let $\mathbf{A} \in U^{\pm}(w)$ be a signed unimodal factorization with unimodal factors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. We recursively construct a sequence $(\emptyset, \emptyset) = (\mathbf{P}_0, \mathbf{T}_0), (\mathbf{P}_1, \mathbf{T}_1), \dots, (\mathbf{P}_n, \mathbf{T}_n) = (\mathbf{P}, \mathbf{T})$ of tableaux, where $\mathbf{P}_s \in \mathcal{UT}_{(\mathbf{a}_1\mathbf{a}_2\dots\mathbf{a}_s)}(\lambda^{(s)})$ and $\mathbf{T}_s \in \mathcal{PT}^{\pm}(\lambda^{(s)})$ are tableaux of the same shifted shape $\lambda^{(s)}$.

To obtain the *insertion tableau* $\mathbf{P}_s$, insert the letters of $\mathbf{a}_s$ one by one from left to right, into $\mathbf{P}_{s-1}$. Denote the shifted shape of $\mathbf{P}_s$ by $\lambda^{(s)}$. Enumerate the boxes in the skew shape $\lambda^{(s)}/\lambda^{(s-1)}$ in the order they appear in $\mathbf{P}_s$. Let these boxes be $(x_1, y_1), \dots, (x_{\ell_s}, y_{\ell_s})$.

Let v be the index that is guaranteed to exist by Lemma 3.2.5 when we compute $\mathbf{P_{s-1}} \leftsquigarrow \mathbf{a_s}$. The *recording tableau* $\mathbf{T}_s$ is a shifted primed tableau obtained from $\mathbf{T}_{s-1}$ by adding the boxes $(x_1, y_1), \dots, (x_{v-1}, y_{v-1})$, each filled with the letter s', and the boxes $(x_{v+1}, y_{v+1}), \dots, (x_{\ell_s}, y_{\ell_s})$, each filled with the letter s. The special case is the box (x_v, y_v), which could contain either s' or s. The letter is determined by the sign of the factor $\mathbf{a}_s$: If the sign is $-$, the box is filled with the letter s', and if the sign is $+$, the box is filled with the letter s. We call the resulting map the *primed Kraśkiewicz map* KR'.

Example 3.2.6. Given a signed unimodal factorization $\mathbf{A} = (-0)(+212)(-43201)$, the sequence of tableaux is

$$(\emptyset, \emptyset), \quad \left(\boxed{0}, \boxed{1'}\right), \quad \left(\begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1' & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}\right), \quad \left(\begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 2 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 0 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1' & 2' & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3' & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}\right).$$

If the recording tableau is constructed, instead, by simply labeling its boxes with $1, 2, 3, \dots$ in the order these boxes appear in the insertion tableau, we recover the original Kraśkiewicz map [**Kra89**,**Kra95**], which is a bijection

$$\mathrm{KR}\colon R(w) \to \bigcup_{\lambda} [\mathcal{UT}_w(\lambda) \times \mathcal{ST}(\lambda)],$$

where $\mathcal{ST}(\lambda)$ is the set of *standard shifted tableau* of shape λ, i.e., the set of fillings of $\mathcal{S}(\lambda)$ with letters $1, 2, \ldots, |\lambda|$ such that each letter appears exactly once, each row filling is increasing, and each column filling is increasing.

Theorem 3.2.7. *The primed Kraśkiewicz map is a bijection*

$$\mathrm{KR}'\colon U^{\pm}(w) \to \bigcup_{\lambda} [\mathcal{UT}_w(\lambda) \times \mathcal{PT}^{\pm}(\lambda)].$$

Proof. First we show that the map is well-defined: Let $\mathbf{A} \in U^{\pm}(w)$ such that $\mathrm{KR}'(A) = (\mathbf{P}, \mathbf{Q})$. The fact that $\mathbf{P}$ is a unimodal tableau follows from the fact that KR is well-defined. On the other hand, $\mathbf{Q}$ satisfies Condition (1) in the definition of shifted primed tableaux since its entries are weakly increasing with respect to the order the associated boxes are added to $\mathbf{P}$. Now fix an s and consider the insertion $\mathbf{P_{s-1}} \leftsquigarrow \mathbf{a_s}$. Refer to the set-up in Lemma 3.2.5. Then, $y_1 < \cdots < y_v$ implies there is at most one s' in each row and $y_v \geqslant \cdots \geqslant y_{\ell_s}$ implies there is at most one s in each column, so Conditions (2) and (3) of the definition have been verified, implying that indeed $\mathbf{Q}$ is a shifted primed tableau.

Now suppose $(\mathbf{P}, \mathbf{Q}) \in \bigcup_{\lambda} [\mathcal{UT}_w(\lambda) \times \mathcal{PT}^{\pm}(\lambda)]$. The ordering of the alphabet X' induces a partial order on the set of boxes of $\mathbf{Q}$. Refine this ordering as follows: Among boxes containing an s', box b is greater than box c if box b lies below box c. Among boxes containing an s, box b is greater than box c if box b lies to the right of box c. Let the standard shifted tableau induced by the resulting total order be denoted $\mathbf{Q}^*$.

Let $w = \mathrm{KR}^{-1}(\mathbf{P}, \mathbf{Q}^*)$. Divide w into factors, where the size of the s-th factor is equal to the s-th entry in $\mathrm{wt}(\mathbf{Q})$. Let $\mathbf{A} = \mathbf{a}_1 \ldots \mathbf{a}_n$ be the resulting factorization, where the sign of $\mathbf{a}_s$ is determined as follows: Consider the lowest leftmost box in $\mathbf{Q}$ that contains an s or s' (such a box must exist if $\mathbf{a}_s \neq \emptyset$). If this box contains an s give $\mathbf{a}_s$ a positive sign, and otherwise a negative sign. Let $b_1, \ldots, b_{|\mathbf{a}_s|}$ denote the boxes of $\mathbf{Q}^*$ corresponding to $\mathbf{a}_s$ under KR^{-1}. The construction of $\mathbf{Q}^*$ and the fact that $\mathbf{Q}$ is a shifted primed tableau imply that the coordinates of these boxes satisfy the hypothesis of Lemma 3.2.5. Since these are exactly the boxes that appear when we compute $\mathbf{P_{s-1}} \leftsquigarrow \mathbf{a}_s$, Lemma 3.2.5 implies that $\mathbf{a}_s$ is unimodal. It follows that $\mathbf{A}$ is a signed unimodal factorization mapping to $(\mathbf{P}, \mathbf{Q})$ under KR'. It is not hard to see $\mathbf{A}$ is unique. □

Theorem 3.2.7 and Equation (3.1) imply the following relation:

$$F_w^C(\mathbf{x}) = \sum_\lambda \left|\mathcal{UT}_w(\lambda)\right| \sum_{\mathbf{T}\in\mathcal{PT}^\pm(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}. \tag{3.2}$$

Remark 3.2.8. The sum $\sum_{\mathbf{T}\in\mathcal{PT}^\pm(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}$ is also known as the Q-Schur function. The expansion (3.2), with a slightly different interpretation of Q-Schur function, was shown in [**BH95**].

At this point, we are halfway there to expand $F_w^C(\mathbf{x})$ in terms of Schur functions. In the next section we introduce a crystal structure on the set $\mathcal{PT}(\lambda)$ of shifted semistandard tableaux.

3.2.2. Mixed insertion. Set $\mathcal{B}^h = \mathcal{B}^h_\infty$. Similar to the well-known RSK-algorithm, mixed insertion [**Hai89**] gives a bijection between $\mathcal{B}^h$ and the set of pairs of tableaux $(\mathbf{T}, \mathbf{Q})$, but in this case $\mathbf{T}$ is shifted primed tableau of shape λ and $\mathbf{Q}$ is a standard shifted tableau of the same shape.

An *(shifted primed tableau* of shape λ (cf. semistandard P-tableau [**Lam95**] or semistandard marked shifted tableau [**Cho13**]) is a shifted primed tableau $\mathbf{T}$ of shape λ with only unprimed elements on the main diagonal. Denote the set of shifted primed tableaux of shape λ by $\mathcal{PT}(\lambda)$. The weight function $\mathrm{wt}(\mathbf{T})$ of $\mathbf{T} \in \mathcal{PT}(\lambda)$ is inherited from the weight function of shifted primed tableaux, that is, it is the vector with i-th coordinate equal to the number of letters i' and i in $\mathbf{T}$. We can simplify (3.2) as

$$F_w^C(\mathbf{x}) = \sum_\lambda 2^{\ell(\lambda)}\left|\mathcal{UT}_w(\lambda)\right| \sum_{\mathbf{T}\in\mathcal{PT}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}. \tag{3.3}$$

Remark 3.2.9. The sum $\sum_{\mathbf{T}\in\mathcal{PT}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}$ is also known as a P-Schur function.

Given a word $b_1 b_2 \dots b_h$ in the alphabet $X = \{1 < 2 < 3 < \cdots\}$, we recursively construct a sequence of tableaux $(\emptyset, \emptyset) = (\mathbf{T}_0, \mathbf{Q}_0), (\mathbf{T}_1, \mathbf{Q}_1), \dots, (\mathbf{T}_h, \mathbf{Q}_h) = (\mathbf{T}, \mathbf{Q})$, where $\mathbf{T}_s \in \mathcal{PT}(\lambda^{(s)})$ and $\mathbf{Q}_s \in \mathcal{ST}(\lambda^{(s)})$. To obtain the tableau $\mathbf{T}_s$, insert the letter b_s into $\mathbf{T}_{s-1}$ as follows. First, insert b_s into the first row of $\mathbf{T}_{s-1}$, bumping out the leftmost element y that is strictly greater than b_i in the alphabet $X' = \{1' < 1 < 2' < 2 < \cdots\}$.

(1) If y is not on the main diagonal and y is not primed, then insert it into the next row, bumping out the leftmost element that is strictly greater than y from that row.
(2) If y is not on the main diagonal and y is primed, then insert it into the next column to the right, bumping out the topmost element that is strictly greater than y from that column.

(3) If y is on the main diagonal, then it must be unprimed. Prime y and insert it into the column on the right, bumping out the topmost element that is strictly greater than y from that column.

If a bumped element exists, treat it as a new y and repeat the steps above – if the new y is unprimed, row-insert it into the row below its original cell, and if the new y is primed, column-insert it into the column to the right of its original cell.

The insertion process terminates either by placing a letter at the end of a row, bumping no new element, or forming a new row with the last bumped element.

Example 3.2.10. Under mixed insertion,

$$\begin{array}{|c|c|c|c|}\hline 2 & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array} \leftarrow 1 = \begin{array}{|c|c|c|c|}\hline 1 & 2' & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}.$$

Let us explain each step in detail. The letter 1 is inserted into the first row bumping out the 2 from the main diagonal, making it a $2'$, which is then inserted into the second column. The letter $2'$ bumps out 2, which we insert into the second row. Then 3 from the main diagonal is bumped from the second row, making it a $3'$, which is then inserted into third column. The letter $3'$ bumps out the 3 on the second row, which is then inserted as the first element in the third row.

The shapes of $\mathbf{T}_{s-1}$ and $\mathbf{T}_s$ differ by one box. Add that box to $\mathbf{Q}_{s-1}$ with a letter s in it, to obtain the standard shifted tableau $\mathbf{Q}_s$.

Example 3.2.11. For a word 332332123, some of the tableaux in the sequence $(\mathbf{T}_i, \mathbf{Q}_i)$ are

$$\left(\begin{array}{|c|c|}\hline 2 & 3' \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \end{array}\right), \quad \left(\begin{array}{|c|c|c|c|}\hline 2 & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}, \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}\right), \quad \left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2' & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 3' & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 9 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\right).$$

Theorem 3.2.12. *[**Hai89**] The construction above gives a bijection*

$$\mathrm{HM}\colon \mathcal{B}^h \to \bigcup_{\lambda \vdash h} [\mathcal{PT}(\lambda) \times \mathcal{ST}(\lambda)].$$

The bijection HM is called a *mixed insertion*. If $\mathrm{HM}(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$, denote $P_{\mathrm{HM}}(\mathbf{b}) = \mathbf{T}$ and $R_{\mathrm{HM}}(\mathbf{b}) = \mathbf{Q}$.

3.3. Explicit crystal operators on shifted primed tableaux

We consider the alphabet $X' = \{1' < 1 < 2' < 2 < 3' < \cdots\}$ of primed and unprimed letters. It is useful to think about the letter $(i+1)'$ as a number $i + 0.5$. Thus, we say that letters i and $(i+1)'$ differ by half a unit and letters i and $(i+1)$ differ by a whole unit.

Given a shifted primed tableau $\mathbf{T}$, we construct the *reading word* $\mathrm{rw}(\mathbf{T})$ as follows:

(1) List all primed letters in the tableau, column by column, from top to bottom within each column, moving from the rightmost column to the left, and with all the primes removed (i.e. all letters are increased by half a unit). (Call this part of the word the *primed reading word*.)
(2) Then list all unprimed elements, row by row, left to right within each row, moving from the bottommost row to the top. (Call this part of the word the *unprimed reading word*.)

To find the letter on which the crystal operator f_i acts, apply the bracketing rule for letters i and $i+1$ within the reading word $\mathrm{rw}(\mathbf{T})$. If all letters i are bracketed in $\mathrm{rw}(\mathbf{T})$, then $f_i(\mathbf{T}) = \mathbf{0}$. Otherwise, the rightmost unbracketed letter i in $\mathrm{rw}(\mathbf{T})$ corresponds to an i or an i' in $\mathbf{T}$, which we call *bold unprimed* i or *bold primed* i respectively.

If the bold letter i is unprimed, denote the cell it is located in as x.

If the bold letter i is primed, we *conjugate* the tableau $\mathbf{T}$ first.

The *conjugate* of a shifted primed tableau $\mathbf{T}$ is obtained by reflecting the tableau over the main diagonal, changing all primed entries k' to k and changing all unprimed elements k to $(k+1)'$ (i.e. increase the entries of all boxes by half a unit). The main diagonal is now the North-East boundary of the tableau. Denote the resulting tableau as $\mathbf{T}^*$.

Under the transformation $\mathbf{T} \to \mathbf{T}^*$, the bold primed i is transformed into bold unprimed i. Denote the cell it is located in as x.

Given any cell z in a shifted primed tableau $\mathbf{T}$ (or conjugated tableau $\mathbf{T}^*$), denote by $c(z)$ the entry contained in cell z. Denote by z_E the cell to the right of z, z_W the cell to its left, z_S the cell below, and z_N the cell above. Denote by z^* the corresponding conjugated cell in $\mathbf{T}^*$ (or in $\mathbf{T}$). Now, consider the box x_E (in $\mathbf{T}$ or in $\mathbf{T}^*$) and notice that $c(x_E) \geqslant (i+1)'$.

Crystal operator f_i on shifted primed tableaux:

(1) If $c(x_E) = (i+1)'$, the box x must lie outside of the main diagonal and the box immediately below x_E cannot contain $(i+1)'$. Change $c(x)$ to $(i+1)'$ and change $c(x_E)$ to $(i+1)$ (i.e. increase the entry in cell x and x_E by half a unit).

(2) If $c(x_E) \neq (i+1)'$ or x_E is empty, then there is a maximal connected ribbon (expanding in South and West directions) with the following properties:

  (a) The North-Eastern most box of the ribbon (the tail of the ribbon) is x.

  (b) The entries of all boxes within a ribbon besides the tail are either $(i+1)'$ or $(i+1)$.

  Denote the South-Western most box of the ribbon (the head) as x_H.

  (a) If $x_H = x$, change $c(x)$ to $(i+1)$ (i.e. increase the entry in cell x by a whole unit).

  (b) If $x_H \neq x$ and x_H is on the main diagonal (in case of a tableau $\mathbf{T}$), change $c(x)$ to $(i+1)'$ (i.e. increase the entry in cell x by half a unit).

  (c) Otherwise, $c(x_H)$ must be $(i+1)'$ due to the bracketing rule. We change $c(x)$ to $(i+1)'$ and change $c(x_H)$ to $(i+1)$ (i.e. increase the entry in cell x and x_H by half a unit).

In the case when the bold i in $\mathbf{T}$ is unprimed, we apply the above crystal operator rules to $\mathbf{T}$ to find $f_i(\mathbf{T})$

Example 3.3.1. We apply operator f_2 on the following tableaux. The bold letter is marked if it exists:

(1) $\mathbf{T} = \begin{array}{cccc} 1 & 2' & 2 & 3' \\ & 2 & 3' & 3 \end{array}$, $\mathrm{rw}(\mathbf{T}) = 3322312$, thus $f_2(\mathbf{T}) = \mathbf{0}$;

(2) $\mathbf{T} = \begin{array}{cccc} 1 & 2' & \mathbf{2} & 3' \\ & 2 & 3' & 4 \end{array}$, $\mathrm{rw}(\mathbf{T}) = 3322412$, thus $f_2(\mathbf{T}) = \begin{array}{cccc} 1 & 2' & 3' & 3 \\ & 2 & 3' & 4 \end{array}$ by Case (1).

(3) $\mathbf{T} = \begin{array}{cccc} 1 & 1 & 2 & \mathbf{2} \\ & 3 & 4' & 4 \end{array}$, $\mathrm{rw}(\mathbf{T}) = 4341122$, thus $f_2(\mathbf{T}) = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ & 3 & 4' & 4 \end{array}$ by Case (2a).

(4) $\mathbf{T} = \begin{array}{ccccc} 1 & 1 & 2' & \mathbf{2} & 3 \\ & 2 & 2 & 3' & \\ & & 3 & 3 & \end{array}$, $\mathrm{rw}(\mathbf{T}) = 3233221123$, thus $f_2(\mathbf{T}) = \begin{array}{ccccc} 1 & 1 & 2' & 3' & 3 \\ & 2 & 2 & 3' & \\ & & 3 & 3 & \end{array}$ by Case (2b).

(5) $\mathbf{T} = \begin{array}{ccccc} 1 & 1 & 1 & \mathbf{2} & 3 \\ & 2 & 2 & 3' & \\ & & 3 & 4' & \end{array}$, $\mathrm{rw}(\mathbf{T}) = 3432211123$, thus $f_2(\mathbf{T}) = \begin{array}{ccccc} 1 & 1 & 1 & 3' & 3 \\ & 2 & 2 & 3 & \\ & & 3 & 4' & \end{array}$ by Case (2c).

In the case when the bold i is primed in $\mathbf{T}$, we first conjugate $\mathbf{T}$ and then apply the above crystal operator rules on $\mathbf{T}^*$, before reversing the conjugation. Note that Case (2b) is impossible for $\mathbf{T}^*$, since the main diagonal is now on the North-East.

Example 3.3.2.

$$\text{Let } \mathbf{T} = \begin{array}{|c|c|c|c|}\hline 1 & \mathbf{2'} & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 4' & \multicolumn{1}{|c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 4 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}, \quad \text{then } \mathbf{T}^* = \begin{array}{|c|c|c|}\cline{1-1} 2' & \multicolumn{2}{|c}{} \\ \cline{1-2} \mathbf{2} & 4' & \multicolumn{1}{|c}{} \\ \hline 3' & 4 & 5' \\ \hline 4' & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \quad \text{and } f_2(\mathbf{T}) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 4' & \multicolumn{1}{|c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 4 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}.$$

Theorem 3.3.3. *For any* $\mathbf{b} \in \mathcal{B}^h$ *with* $P_{\mathrm{HM}}(\mathbf{b}) = \mathbf{T}$ *and* $f_i(\mathbf{b}) \neq \mathbf{0}$*, the operator* f_i *defined on above satisfies*

$$P_{\mathrm{HM}}(f_i(\mathbf{b})) = f_i(\mathbf{T}).$$

Also, $f_i(\mathbf{b}) = \mathbf{0}$ *if and only if* $f_i(\mathbf{T}) = \mathbf{0}$.

The proof of Theorem 3.3.3 is quite technical and is relegated to Appendix A.1. However, from it we obtain:

Theorem 3.3.4. *The recording tableau* $R_{\mathrm{HM}}(\cdot)$ *is constant on each connected component of the crystal* $\mathcal{B}^h$.

Proof. Given a word $\mathbf{b} = b_1 \dots b_h$, let $\mathbf{b}' = f_i(\mathbf{b}) = b'_1 \dots b'_h$, so that $b_m \neq b'_m$ for some m and $b_i = b'_i$ for any $i \neq m$. We show that $Q_{\mathrm{HM}}(\mathbf{b}) = Q_{\mathrm{HM}}(\mathbf{b}')$.

Denote $\mathbf{b}^{(s)} = b_1 \dots b_s$ and similarly $\mathbf{b}'^{(s)} = b'_1 \dots b'_s$. Due to the construction of the recording tableau Q_{HM}, it suffices to show that $P_{\mathrm{HM}}(\mathbf{b}^{(s)})$ and $P_{\mathrm{HM}}(\mathbf{b}'^{(s)})$ have the same shape for any $1 \leqslant s \leqslant h$.

If $s < m$, this is immediate. If $s \geqslant m$, note that $\mathbf{b}'^{(s)} = f_i(\mathbf{b}^{(s)})$. Using Theorem 3.3.3, one can see that $P_{\mathrm{HM}}(\mathbf{b}'^{(s)}) = P_{\mathrm{HM}}(f_i(\mathbf{b}^{(s)})) = f_i(P_{\mathrm{HM}}(\mathbf{b}^{(s)}))$ has the same shape as $P_{\mathrm{HM}}(\mathbf{b}^{(s)})$. □

The next step is to describe the raising operators $e_i(\mathbf{T})$. Consider the reading word $\mathrm{rw}(\mathbf{T})$ and apply the bracketing rule on the letters i and $i+1$. If all letters $i+1$ are bracketed in $\mathrm{rw}(\mathbf{T})$, then $e_i(\mathbf{T}) = \mathbf{0}$. Otherwise, the leftmost unbracketed letter $i+1$ in $\mathrm{rw}(\mathbf{T})$ corresponds to an $i+1$ or an $(i+1)'$ in $\mathbf{T}$, which we will call bold unprimed $i+1$ or bold primed $i+1$, respectively. If the bold $i+1$ is unprimed, denote the cell it is located in by y. If the bold $i+1$ is primed, conjugate $\mathbf{T}$ and denote the cell with the bold $i+1$ in $\mathbf{T}^*$ by y.

Crystal operator e_i on shifted primed tableaux:

(1) If $c(y_W) = (i+1)'$, then change $c(y)$ to $(i+1)'$ and change $c(y_W)$ to i (i.e. decrease the entry in cell y and y_W by half a unit).

(2) If $c(y_W) < (i+1)'$ or y_W is empty, then there is a maximal connected ribbon (expanding in North and East directions) with the following properties:

   (a) The South-Western most box of the ribbon (the head of the ribbon) is y.

   (b) The entry in all boxes within a ribbon besides the tail is either i or $(i+1)'$.

   Denote the North-Eastern most box of the ribbon (the tail) as y_T.

   (a) If $y_T = y$, change $c(y)$ to i (i.e. decrease the entry in cell y by a whole unit).

   (b) If $y_T \neq y$ and y_T is on the main diagonal (in case of a conjugate tableau $\mathbf{T}^*$), then change $c(y)$ to $(i+1)'$ (i.e. decrease the entry in cell y by half a unit).

   (c) If $y_T \neq y$ and y_T is not on the diagonal, the entry of cell y_T must be $(i+1)'$ and we change $c(y)$ to $(i+1)'$ and change $c(y_T)$ to i (i.e. decrease the entry of cell y and y_T by half a unit).

When the bold $i+1$ is unprimed, $e_i(\mathbf{T})$ is obtained by applying the rules above to $\mathbf{T}$. When the bold $i+1$ is primed, we first conjugate $\mathbf{T}$, then apply the raising crystal operator rules on $\mathbf{T}^*$, and then reverse the conjugation.

Proposition 3.3.5.

$$e_i(\mathbf{b}) = \mathbf{0} \quad \textit{if and only if} \quad e_i(\mathbf{T}) = \mathbf{0}.$$

Proof. According to Lemma A.1.1, the number of unbracketed letters i in $\mathbf{b}$ is equal to the number of unbracketed letters i in $\mathrm{rw}(\mathbf{T})$. Since the total number of both letters i and $j = i+1$ is the same in $\mathbf{b}$ and in $\mathrm{rw}(\mathbf{T})$, that also means that the number of unbracketed letters j in $\mathbf{b}$ is equal to the number of unbracketed letters j in $\mathrm{rw}(\mathbf{T})$. Thus, there are no unbracketed letters j in $\mathbf{b}$ if and only if there are no unbracketed letters j in $\mathbf{T}$. □

Theorem 3.3.6. *Given a shifted primed tableau* $\mathbf{T}$ *with* $f_i(\mathbf{T}) \neq \mathbf{0}$*, for the operators* e_i *defined above we have the following relation:*

$$e_i(f_i(\mathbf{T})) = \mathbf{T}.$$

The proof of Theorem 3.3.6 is relegated to Appendix A.2.

Corollary 3.3.7. *For any* $\mathbf{b} \in \mathcal{B}^h$ *with* $\mathrm{HM}(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$*, the operator* e_i *defined above satisfies*

$$\mathrm{HM}(e_i(\mathbf{b})) = (e_i(\mathbf{T}), \mathbf{Q}),$$

given the left-hand side is well-defined.

The consequence of Theorem 3.3.3, as discussed in Section 3.2.2, is a crystal isomorphism $\Psi_\lambda \colon \mathcal{PT}(\lambda) \to \bigoplus \mathcal{B}_\mu^{\oplus h_{\lambda\mu}}$. Now, to determine the nonnegative integer coefficients $h_{\lambda\mu}$, it is enough to count the highest weight elements in $\mathcal{PT}(\lambda)$ of given weight μ.

Proposition 3.3.8. *A shifted primed tableau* $\mathbf{T} \in \mathcal{PT}(\lambda)$ *is a highest weight element if and only if its reading word* $\mathrm{rw}(\mathbf{T})$ *is a Yamanouchi word. That is, for any suffix of* $\mathrm{rw}(\mathbf{T})$*, its weight is a partition.*

Thus we define $h_{\lambda\mu}$ to be the number of shifted primed tableaux $\mathbf{T}$ of shifted shape $\mathcal{S}(\lambda)$ and weight μ such that $\mathrm{rw}(\mathbf{T})$ is Yamanouchi.

Example 3.3.9. Let $\lambda = (5, 3, 2)$ and $\mu = (4, 3, 2, 1)$. There are three shifted primed tableaux of shifted shape $\mathcal{S}((5, 3, 2))$ and weight $(4, 3, 2, 1)$ with a Yamanouchi reading word, namely

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2' \\ \none & 2 & 2 & 3' \\ \none & \none & 3 & 4' \end{ytableau}\ , \quad \begin{ytableau} 1 & 1 & 1 & 1 & 3' \\ \none & 2 & 2 & 2 \\ \none & \none & 3 & 4' \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 1 & 1 & 1 & 4' \\ \none & 2 & 2 & 2 \\ \none & \none & 3 & 3 \end{ytableau}\ .$$

Therefore $h_{(5,3,2)(4,3,2,1)} = 3$.

We summarize our results for the type C Stanley symmetric functions as follows.

Corollary 3.3.10. *The expansion of* $F_w^C(\mathbf{x})$ *in terms of Schur symmetric functions is*

$$F_w^C(\mathbf{x}) = \sum_\lambda g_{w\lambda} s_\lambda(\mathbf{x}), \quad \text{where} \quad g_{w\lambda} = \sum_\mu 2^{\ell(\mu)} \left|\mathcal{UT}_w(\mu)\right| h_{\mu\lambda}\ . \tag{3.1}$$

Replacing $\ell(\mu)$ by $\ell(\mu) - o(w)$ gives the Schur expansion of $F_w^B(\mathbf{x})$. Note that since any row of a unimodal tableau contains at most one zero, $\ell(\mu) - o(w)$ is nonnegative. Thus the given expansion makes sense combinatorially.

Example 3.3.11. Consider the word $w = 0101 = 1010$. There is only one unimodal tableau corresponding to w, namely $\mathbf{P} = \begin{ytableau} 1 & 0 & 1 \\ \none & 0 \end{ytableau}$, which belongs to $\mathcal{UT}_{0101}(3, 1)$. Thus, $g_{w\lambda} = 4h_{(3,1)\lambda}$. There are only three possible highest weight shifted primed tableaux of shape $(3, 1)$, namely $\begin{ytableau} 1 & 1 & 1 \\ \none & 2 \end{ytableau}$, $\begin{ytableau} 1 & 1 & 2' \\ \none & 2 \end{ytableau}$ and $\begin{ytableau} 1 & 1 & 3' \\ \none & 2 \end{ytableau}$, which implies that $h_{(3,1)(3,1)} = h_{(3,1)(2,2)} = h_{(3,1)(2,1,1)} = 1$ and $h_{(3,1)\lambda} = 0$ for other weights λ. The expansion of $F_{0101}^C(\mathbf{x})$ is thus

$$F_{0101}^C = 4s_{(3,1)} + 4s_{(2,2)} + 4s_{(2,1,1)}.$$

3.4. Signed tableaux of shifted shape: Semistandard unimodal tableaux

As mentioned in the introduction, the proper notion of a signed tableau of shifted shape is manifested by tableaux known as semistandard unimodal tableaux. Many of the results of given earlier in this chapter have counterparts that involve the notion of semistandard unimodal tableaux in place of shifted primed tableaux. We give a brief overview of these results, mostly without proof. The proofs, when written out in detail, mirror the approach to shifted primed tableaux.

First, let us define semistandard unimodal tableaux. We say that a word $a_1a_2\dots a_h \in \mathcal{B}^h$ is *weakly unimodal* if there exists an index v, such that

$$a_1 > a_2 > \cdots > a_v \leqslant a_{v+1} \leqslant \cdots \leqslant a_h.$$

A *semistandard unimodal tableau* $\mathbf{P}$ of shape λ is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet X such that the i^{th} row of $\mathbf{P}$, denoted by P_i, is weakly unimodal, and such that P_i is the longest weakly unimodal subword in the concatenated word $P_{i+1}P_i$. Denote the set of semistandard unimodal tableaux of shape λ by $\mathcal{SUT}(\lambda)$.

Let $\mathbf{a} = a_1 \dots a_h \in \mathcal{B}^h$. The alphabet X imposes a partial order on the entries of $\mathbf{a}$. We can extend this to a total order by declaring that if $a_i = a_j$ as elements of X, and $i < j$, then as entries of $\mathbf{a}$, $a_i < a_j$. For each entry a_i, denote its numerical position in the total ordering on the entries of $\mathbf{a}$ by n_i and define the *standardization* of $\mathbf{a}$ to be the word with superscripts, $n_1^{a_1}\dots n_h^{a_h}$. Since its entries are distinct, $n_1\dots n_h$ can be considered as a reduced word. Let $(\mathbf{R},\mathbf{S})$ be the Kraśkiewicz insertion and recording tableaux of $n_1\dots n_h$, and let $\mathbf{R}^*$ be the tableau obtained from $\mathbf{R}$ by replacing each n_i by a_i. One checks that setting $\mathrm{SK}(\mathbf{a}) = (\mathbf{R}^*,\mathbf{S})$ defines a map,

$$\mathrm{SK}\colon \mathcal{B} = \bigoplus_{h\in\mathbb{N}} \mathcal{B}^h \to \bigcup_{\lambda} [\mathcal{SUT}(\lambda)\times\mathcal{ST}(\lambda)].$$

In fact, this map is a bijection [**Ser10**, **Lam95**]. It follows that the composition $\mathrm{SK}\circ\mathrm{HM}^{-1}$ gives a bijection

$$\bigcup_{\lambda} [\mathcal{PT}(\lambda)\times\mathcal{ST}(\lambda)] \to \bigcup_{\lambda} [\mathcal{SUT}(\lambda)\times\mathcal{ST}(\lambda)].$$

The following remarkable fact, which appears as [**Ser10**, Proposition 2.23], can be deduced from [**Lam95**, Theorem 3.32], which itself utilizes results of [**Hai89**].

Theorem 3.4.1. *For any word* $\mathbf{a} \in \mathcal{B}^h$, $Q_{\mathrm{SK}}(\mathbf{a}) = Q_{\mathrm{HM}}(\mathbf{a})$.

This allows us to define a bijective map $\Phi_{\mathbf{Q}} \colon \mathcal{PT}(\lambda) \to \mathcal{SUT}(\lambda)$ as follows. Choose a standard shifted tableau $\mathbf{Q}$ of shape λ. Then, given a shifted primed tableau $\mathbf{P}$ of shape λ set $(\mathbf{R}, \mathbf{Q}) = \mathrm{SK}(\mathrm{HM}^{-1}(\mathbf{P}, \mathbf{Q}))$, and let $\Phi_{\mathbf{Q}}(\mathbf{P}) = \mathbf{R}$.

For any filling of a shifted shape λ with letters from X, associating this filling to its reading word (the element of $\mathcal{B}^{|\lambda|}$ obtained by reading rows left to right, bottom to top) induces crystal operators on the set of all fillings of this shape. In particular, we can apply these induced operators to any element of $\mathcal{SUT}(\lambda)$ (although, a priori, it is not clear that the image will remain in $\mathcal{SUT}(\lambda)$). We now summarize our main results for SK insertion and its relation to this induced crystal structure.

Theorem 3.4.2. *For any* $\mathbf{b} \in \mathcal{B}^h$ *with* $\mathrm{SK}(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$ *and* $f_i(\mathbf{b}) \neq \mathbf{0}$, *the induced operator* f_i *described above satisfies*

$$\mathrm{SK}(f_i(\mathbf{b})) = (f_i(\mathbf{T}), \mathbf{Q}).$$

Also, $f_i(\mathbf{b}) = \mathbf{0}$ *if and only if* $f_i(\mathbf{T}) = \mathbf{0}$.

Corollary 3.4.3. $\mathcal{SUT}(\lambda)$ *is closed under the induced crystal operators described above.*

Replacing HM by SK in the proof of Theorem 3.3.4, or by combining Theorem 3.3.4 with Theorem 3.4.1 yields:

Theorem 3.4.4. *The recording tableau under* SK *insertion is constant on each connected component of the crystal* $\mathcal{B}^h$.

The upshot of all this is the following theorem.

Theorem 3.4.5. *With respect to the crystal operators we have defined on semistandard tableaux and the induced operators on semistandard unimodal tableaux described above, the map* Φ_Q *is a crystal isomorphism.*

Proof. This says no more than that Φ_Q is a bijection (which we have established) and that it commutes with the crystal operations on semistandard tableaux and semistandard unimodal tableaux. But this is simply combining Theorem 3.3.4 with Theorem 3.4.4. $\square$

Theorem 3.4.5 immediately gives us another combinatorial interpretation of the coefficients $g_{w\lambda}$. Let $k_{\mu\lambda}$ be the number of semistandard unimodal tableaux of shape μ and weight λ, whose reading words are Yamanouchi (that is, tableaux that are the highest weight elements of $\mathcal{SUT}(\mu)$).

Corollary 3.4.6. *The expansion of $F_w^C(\mathbf{x})$ in terms of Schur symmetric functions is*

$$F_w^C(\mathbf{x}) = \sum_{\lambda} g_{w\lambda} s_\lambda(\mathbf{x}), \quad \textit{where} \quad g_{w\lambda} = \sum_{\mu} 2^{\ell(\mu)} \left| \mathcal{UT}_w(\mu) \right| k_{\mu\lambda}\,.$$

Again, replacing $\ell(\mu)$ by $\ell(\mu) - o(w)$ gives the Schur expansion of $F_w^B(\mathbf{x})$.

Example 3.4.7. According to Example 3.3.11, we should find three highest weight semistandard unimodal tableaux of shape $(3,1)$, one for each of the weights $(3,1)$, $(2,2)$, and $(2,1,1)$. These are $\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline & 1 & \\ \cline{2-2} \end{array}$, $\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline & 2 & \\ \cline{2-2} \end{array}$ and $\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline & 1 & \\ \cline{2-2} \end{array}$.

CHAPTER 4

Marked Tableaux of Staircase Shape: The Schur function $s_{\delta/\mu}$

This chapter is based on the work in [**Haw17**].

4.1. Introduction

The ring of symmetric functions, Λ, has a $\mathbb{Z}$-basis composed of Schur functions. Hence we can define an invertible linear operator ω, by the formula $\omega(s_\lambda) = s_{\lambda'}$. We will call $f \in \Lambda$ a fixed point of ω if $\omega(f) = f$. Clearly s_λ is a fixed point for any self-conjugate partition λ. Moreover, one can show that $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$, [**Sta99**] meaning that $s_{\lambda/\mu}$ is a fixed point for any self-conjugate partitions $\mu \subseteq \lambda$. In particular $s_{\delta/\mu}$, where $\delta = (n, n-1, \ldots 1)$ is a fixed point for any self-conjugate $\mu \subseteq \delta$. A priori there is little reason to expect that for any $\mu \subseteq \delta$ (not necessarily self-conjugate) $s_{\delta/\mu}$ would still be a fixed point. The fact that δ is self-conjugate and $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$ means that the statement above is equivalent to $s_{\delta/\mu} = s_{\delta/\mu'}$. However, this will be an immediate consequence of the symmetry of generalized staircase tableaux.

Besides being a fixed point of ω, the function $s_{\delta/\mu}$ is interesting in its relation to shifted Schur functions. For one, it is known, [**AS12**] that $s_{\delta/\mu}$ is P-Schur positive. We do not recreate this result here but do obtain the result that $s_{\delta/\mu}(x_1, x_1, x_2, x_2, \ldots)$ is Q-Schur positive. (Note that P-Schur positivity is immediately guaranteed as a corollary of the result of [**AS12**], but not necessarily Q-Schur positivity.) In particular, we derive that $s_{\delta/\mu}(x_1, x_1, x_2, x_2, \ldots)$ is equal to a certain skew Q-Schur function, from which the result follows by [**Wor84**], or [**Ste89**].

Our last application is deriving the equality of the skew Q-Schur functions $Q_{\lambda+\delta/\mu+\delta} = Q_{\lambda'+\delta/\mu'+\delta}$. On the way to accomplishing this we define a t-deformation of Q-Schur functions and show that, in certain cases, it is symmetric and Schur positive, and give a combinatorial interpretation of the Schur coefficients. This is accomplished with the help of a certain crystal structure introduced in [**HPS17**]. We then prove that certain permutations to the order of the alphabet $1' < 1 < 2' < 2\cdots$, which is typically used to define shifted semistandard tableaux (e.g. [**Ser09**]), may be made when the partition does not touch the diagonal.

Lastly, we note that the name generalized staircase tableaux (GST) is a bit deceptive, as we will define them for arbitrary shapes. However, non-staircase GST are simply used to aid in our proofs, and no interesting results involve them.

4.2. Definitions

In what follows, we fix some n, and write $\delta = \delta = (n, n-1, \ldots, 1)$. Any partition denoted by μ which appears henceforth will be assumed to satisfy $\mu \subseteq \delta$. Moreover, whenever λ is mentioned we will assume also that $\mu \subseteq \lambda$ and that $l(\lambda) \leq n$.

A generalized staircase tableau (GST) of shape λ/μ and set $I \subseteq \mathbb{N}$ is a filling of the Young diagram λ/μ with natural numbers such that:

(1) The rows and columns are weakly increasing.
(2) If $i \in I$ then each row has at most one i.
(3) If $i \notin I$, then each column has at most one i.

Let $\mathbf{G}(\lambda/\mu, I)$ denote the set of all GST of shape λ/μ and set I. For instance, $\mathbf{G}(\lambda/\mu, \emptyset)$ is the set of skew semistandard Young tableau of shape λ/μ, and the set $\mathbf{G}(\lambda/\mu, 2\mathbb{N}-1)$ is in in bijection with the set of skew shifted semi-standard tableaux of shape $(\delta+\lambda)/(\delta+\mu)$.

Suppose T is a GST of shape λ/μ . We will denote the box in row i and column j by b_{ij}. We will denote the value inside b_{ij} by $c(b_{ij})$ or by the "content of b_{ij}." For indexing purposes we will allow the coordinates of i and j to be any non-negative integers, although box b_{ij} is always empty whenever $i = 0$ or $j = 0$. We define the content of these border boxes to be $-\infty$. We also define an empty box inside of μ to have content $-\infty$. On the other hand, an empty box outside of λ (and with i and j positive) is defined to have content ∞. The weight of T, denoted by $wt(T)$ is defined to be the vector whose i^{th} coordinate is equal to the number of is appearing in T.

Suppose that T is a GST of shape λ/μ and that $b_{ij} \in \mu$ such that $\mu - \{b_{ij}\}$ is a partition. We define *forward jdt* into b_{ij} as follows.

(1) Between the boxes $b_{i(j+1)}$ and $b_{(i+1)j}$ select the box whose content is lesser. If the contents are equal, then select $b_{i(j+1)}$ if $c(b_{i(j+1)}) = c(b_{(i+1)j}) \in I$ and $b_{(i+1)j}$ otherwise.
(2) If an empty box was selected, the algorithm terminates. Otherwise move the content of the selected box into b_{ij}.

(3) Re-index so that the newly emptied box is b_{ij} and return to step 1.

Similarly, we define *reverse jdt* into a box $b_{ij} \notin \lambda$ such that $\lambda \cup \{b_{ij}\}$ is a partition as follows:

(1) Between the boxes $b_{i(j-1)}$ and $b_{(i-1)j}$ select the box whose content is greater. If the contents are equal, then select $b_{i(j-1)}$ if $c(b_{i(j-1)}) = c(b_{(i-1)j}) \in I$ and $b_{(i-1)j}$ otherwise.

(2) If an empty box was selected, the algorithm terminates. Otherwise move the content of the selected box into b_{ij}.

(3) Re-index so that the newly emptied box is b_{ij} and return to step 1.

Note that in both cases a valid GST is returned. Moreover, our jdt satisfies the familiar properties of classic jdt:

J1 If T' is obtained from T by *forward jdt* into b_{ij}, and $b_{i'j'}$ is the last box to be emptied, then T can be obtained from T' by *reverse jdt* into $b_{i'j'}$.

J2 If T' is obtained from T by *reverse jdt* into b_{ij}, and $b_{i'j'}$ is the last box to be emptied, then T can be obtained from T' by *forward jdt* into $b_{i'j'}$.

J3 If $i \geq k$ and $j < l$ and it is possible to *forward jdt* into b_{kl} and then *forward jdt* into b_{ij}, and the boxes emptied by doing this are (in order) $b_{k'l'}$ and $b_{i'j'}$ then $i' \geq k'$ and $j' < l'$.

J4 If $i < k$ and $j \geq l$ and it is possible to *forward jdt* into b_{kl} and then *forward jdt* into b_{ij}, and the boxes emptied by doing this are (in order) $b_{k'l'}$ and $b_{i'j'}$ then $i' < k'$ and $j' \geq l'$.

J5 If $i \geq k$ and $j < l$ and it is possible to *reverse jdt* into b_{ij} and then *reverse jdt* into b_{kl}, and the boxes emptied by doing this are (in order) $b_{i'j'}$ and $b_{k'l'}$ then $i' \geq k'$ and $j' < l'$.

J6 If $i < k$ and $j \geq l$ and it is possible to *reverse jdt* into b_{ij} and then *reverse jdt* into b_{kl}, and the boxes emptied by doing this are (in order) $b_{i'j'}$ and $b_{k'l'}$ then $i' < k'$ and $j' \geq l'$.

4.3. Results

Theorem 4.3.1. *If I and I' are any subsets of the natural numbers, there is a weight preserving bijection from $\mathbf{G}(\delta/\mu, I)$ to $\mathbf{G}(\delta/\mu, I')$.*

Proof. It suffices to show that for any I and any $i \notin I$ there is a weight preserving bijection from $\mathbf{G}(\delta/\mu, I)$ to $\mathbf{G}(\delta/\mu, I \cup i)$. Let $\mu \subseteq \nu \subseteq \lambda$, be the partition consisting of all boxes of μ and all boxes of λ with content less than i. It will suffice to find a weight-preserving bijection from $\mathbf{G}(\delta/\nu, I_{\geq i})$ to $\mathbf{G}(\delta/\nu, I_{\geq i} \cup i)$ or equivalently from $\mathbf{G}(\delta/\nu, J)$ to $\mathbf{G}(\delta/\nu, J \cup 1)$ where $J = [(-i+1) + I]_{\geq 1}$.

We therefore assume that in the statement of the theorem, $i = 1 \notin I$, and $I' = I \cup 1$. Let $T \in \mathbf{G}(\delta/\mu, I)$. First erase all of the 1s which appear in T. The result is a horizontal strip of empty boxes on the inside of the tableau. Now, *forward jdt* into each one of these empty boxes starting with the rightmost and moving left. By property J3, the boxes which are emptied along the outside of the tableau will form a horizontal strip, and they will be emptied from right to left. However, since δ is the staircase shape, this strip is in fact also a vertical strip. Now, *reverse jdt* into each of the boxes of this vertical strip, starting with the highest and moving down. By property J6, the boxes emptied along the inside of the tableau will form a vertical strip, and they will be emptied starting with the highest and moving down. Put a 1 into each of the newly emptied boxes. This produces a tableau in $\mathbf{G}(\delta/\mu, I \cup 1)$ which we define to be $\phi(T)$. Now, given any $T \in \mathbf{G}(\delta/\mu, I \cup 1)$ define $\phi^{-1}(T) = \phi(T^t)^t$, where the superscript t stands for row/column transposition. It is not hard to check that $\phi^{-1}(T) \in \mathbf{G}(\delta/\mu, I)$ and that both $\phi \circ \phi^{-1} = Id$ and $\phi^{-1} \circ \phi = Id$.

□

Theorem 4.3.2. *$s_{\delta/\mu}$ is a fixed point of the involution ω.*

Proof. By the comments in the introduction, this is equivalent to showing that $s_{\delta/\mu} = s_{\delta/\mu'}$. But this equality is equivalent to the equality:

$$\sum_{T \in \mathbf{G}(\delta/\mu, \emptyset)} \mathbf{x}^{wt(T)} = \sum_{T \in \mathbf{G}(\delta/\mu, \mathbb{N})} \mathbf{x}^{wt(T)},$$

which is true because $\mathbf{G}(\delta/\mu, \emptyset)$ and $\mathbf{G}(\delta/\mu, \mathbb{N})$ are in weight preserving bijection.

□

Our next application relates Schur and Q-Schur functions of certain shapes: For our purposes we will define a Q-tableau to be a filling of the shape λ/μ using letters from the ordered alphabet $1' < 1 < 2' < 2 \ldots$ such that:

(1) The rows and columns are weakly increasing.
(2) No primed number appears more than once in any row.
(3) No unprimed number appears more than once in any column.

We define the reading word of a Q-tableau to be the word obtained by reading the primed entries in T down columns from right to left and then reading the the unprimed entries left to right across rows, starting with the lowest row and working up. We consider this word as a word in the alphabet $\{1, 2, 3, \ldots\}$ by ignoring the primes which appear above the entries at the beginning of the word. This is the same definition as for the reading word of "primed tableaux" in [**HPS17**], except that here, our tableaux are not shifted.

We define a function:

$$Q^{tr}_{\lambda/\mu} = \sum_{T} \mathbf{x}^{wt(T)} t^{P(T)} r^{U(T)},$$

where the sum is over all Q-tableaux of shape λ/μ, where $wt(T)$ is the vector whose i^{th} coordinate counts the number of times either i or i' appears in T, and where $P(T)$ (resp. $U(T)$) counts the number of times a primed (resp. unprimed) entry appears in T. Notice that, by definition, $Q^{tr}_{\lambda/\mu}$ at $t = 1 = r$ is the Q-Schur function $Q_{\lambda+\delta/\mu+\delta}$.

Theorem 4.3.3.

$$Q^{tr}_{\lambda/\mu} = \sum_{k} \left(\sum_{\nu} c^{\nu,k}_{\lambda/\mu} s_{\nu} \right) t^{k} r^{|\lambda|-|\mu|-k}$$

Where $c^{\nu,k}_{\lambda/\mu}$ is the number of Q-tableau of shape λ/μ and weight ν which have exactly k of their entries primed, and whose reading word is Yamanouchi.

Proof. The crystal operators on primed tableau given in [**HPS17**] induce crystal operators on skew primed tableaux in the natural way. Notice that the set of skewed primed tableaux of shape $\lambda + \delta/\mu + \delta$ is cannonically equivalent to the set of all Q-tableaux of shape λ/μ, and so we obtain a crystal structure on the latter. Moreover, when the set of Q-tableaux of shape λ/μ inherits this structure, the highest weight elements of this crystal will be those Q-tableaux whose reading word is Yamanouchi. This follows directly from the description of highest weight primed tableaux given in [**HPS17**]. In order to prove the theorem, it remains to show that $P(T)$ is constant on connected components of the induced crystal on Q-tableaux. However, one may check that the crystal operator f_i in [**HPS17**] preserves the number of primes in a given primed tableau whenever this tableau has no is or $(i+1)$s on the diagonal. However, note that we are associating Q-tableaux of shape λ/μ to primed tableaux of shifted skew shape $\lambda + \delta/\mu + \delta$, and that the latter shape has no boxes on the diagonal. Thus, it is the case that for all i, we are always applying the operator f_i to a (skew) primed tableaux with no is or $(i + 1)$s on the diagonal. Thus, the induced operators on Q-tableaux also preserve the number of primes in a given Q-tableau. ◻

Theorem 4.3.4.

$$s_{\delta/\mu}(tx_1, rx_1, tx_2, rx_2, \ldots) = Q^{tr}_{\delta/\mu} = \sum_k \left(\sum_\nu c^{\nu,k}_{\delta/\mu} s_\nu \right) t^k r^{|\delta|-|\mu|-k}$$

*In particular, $s_{\delta/\mu}(x_1, x_1, x_2, x_2, \ldots)$ is Q-Schur positive (since it is equal to the skew Q-Schur function $Q_{\lambda+\delta/\mu+\delta}$ which is Q-Schur positive by [**Wor84**] or [**Ste89**]), and we have that $s_{\delta/\mu}(tx_1, x_1, tx_2, x_2, \ldots)$ is Schur positive.*

Proof.

$$s_{\delta/\mu}(tx_1, rx_1, tx_2, rx_2, \ldots) = \sum_T \mathbf{x}^{wt(T)} t^{P(T)} r^{|\delta|-|\mu|-P(T)},$$

where we claim the sum can be taken over any of the following:

(1) Over all SSYT of shape δ/μ, where $wt(T)$ is the vector whose i^{th} coordinate counts the number of times either $2i-1$ or $2i$ appears in T, and where $P(T)$ counts the number of times an odd entry appears in T.

(2) Over $\mathbf{G}(\delta/\mu, \emptyset)$, where $wt(T)$ is the vector whose i^{th} coordinate counts the number of times either $2i-1$ or $2i$ appears in T, and where $P(T)$ counts the number of times an odd entry appears in T.

(3) Over $\mathbf{G}(\delta/\mu, 2\mathbb{N}-1)$, where $wt(T)$ is the vector whose i^{th} coordinate counts the number of times either $2i-1$ or $2i$ appears in T, and where $P(T)$ counts the number of times an odd entry appears in T.

(4) Over all Q-tableaux of shape δ/μ, where $wt(T)$ is the vector whose i^{th} coordinate counts the number of times either i or i' appears in T, and where $P(T)$ counts the number of times a primed entry appears in T.

(1) is true by definition. (1) $\implies$ (2) by the definition of GST. (2) $\implies$ (3) by 4.3.1. (3) $\implies$ (4) by relabeling the alphabet, and (4) corresponds to the statement in the theorem.

□

Corollary 4.3.5. *$Q^{tr}_{\delta/\mu}$ is symmetric in t and r, and $Q^{tr}_{\delta/\mu} = Q^{tr}_{\delta/\mu'}$. In particular, we have the equality of skew Q-Schur functions $Q_{\delta+\delta/\mu+\delta} = Q_{\delta+\delta/\mu'+\delta}$.*

In fact, more generally we have:

Proposition 4.3.6. $Q^{tr}_{\lambda/\mu}(\mathbf{x};t,r) = Q^{tr}_{\lambda'/\mu'}(\mathbf{x};r,t)$.

Before proving this, we introduce a generalization of Q-tableau. Let $I \subseteq \mathbb{N}$ and define the total order $\leq_I$ on the alphabet $\{1', 1, 2', 2, \ldots\}$ by

(1) If $i < j$ then $i <_I j$, $i <_I j'$, $i' <_I j$, $i' <_I j'$
(2) If $i \in I$ then $i <_I i'$
(3) If $i \notin I$ then $i' <_I i$

We define a generalized Q-tableau of shape λ/μ and set I to be a filling of this shape using $\{1', 1, 2', 2, \ldots\}$ such that:

(1) The rows and columns are weakly increasing under $\leq_I$.
(2) No primed number appears more than once in any row.
(3) No unprimed number appears more than once in any column.

The set of all such tableaux is denoted $\mathbf{Q}(\lambda/\mu, I)$.

Theorem 4.3.7. *For any subsets of the natural numbers, I, and I', there is a bijection from $\mathbf{Q}(\lambda/\mu, I)$ to $\mathbf{Q}(\lambda/\mu, I')$ which preserves $wt(T)$ and $P(T)$.*

Proof. It suffices to suppose that $I' = I \cup i$ for some $i \notin I$. Let $T \in \mathbf{Q}(\lambda/\mu, I)$ and define $\psi(T)$ as follows. First, write down T. Notice that the is and i's in T form a set of connected ribbons. Within each of these connected ribbons, cycle every entry one position: to the left if the box to its left is in the ribbon, downwards if the box below it is in the ribbon, or, if neither is the case, i.e., it is at the bottom left end of the ribbon, move it to the upper right end of the ribbon. ψ^{-1} is defined similarly, but by cycling the other direction. □

We can now prove 4.3.6.

Proof. We seek a weight preserving bijection from $\mathbf{Q}(\lambda'/\mu', \emptyset)$ to $\mathbf{Q}(\lambda/\mu, \emptyset)$ which interchanges $P(T)$ and $U(T)$. Let $T \in \mathbf{Q}(\lambda'/\mu', \emptyset)$. Tranpose T and then prime the unprimed elements and unprime the primed elements. This gives a weight preserving bijection from $\mathbf{Q}(\lambda'/\mu', \emptyset)$ to $\mathbf{Q}(\lambda/\mu, \mathbb{N})$ which interchanges $P(T)$ and $U(T)$, and by 4.3.7, this is sufficient. □

Corollary 4.3.8. *We have the equality of skew Q-Schur functions $Q_{\lambda+\delta/\mu+\delta} = Q_{\lambda'+\delta/\mu'+\delta}$.*

(Here we make the additional assumption that $\lambda_1 \leq n$.)

CHAPTER 5

Primed Tableaux of Shifted Shape Revisited: Crystal Characterization

This chapter is based on a portion of the work in [**GHPS18**].

5.1. Introduction

One of the major advances in the theory of crystals for simply-laced Lie algebras was the discovery by Stembridge [**Ste03**] of local axioms that uniquely characterize the crystal graphs corresponding to Lie algebra representations. These local axioms provide a completely combinatorial approach to the theory of crystals; this viewpoint was taken in [**BS17**].

A theory of highest weight crystals for the queer superalgebra $\mathfrak{q}(n)$ was recently developed by Grantcharov et al. [**GJK$^+$15**]. They provide an explicit combinatorial realization of the highest weight crystal bases in terms of semistandard decomposition tableaux and show how these crystals can be derived from a tensor product rule and the vector representation. Independently, Hiroshima [**Hir18**] and Assaf and Oguz [**AKO18a**, **AKO18b**] defined a queer crystal structure on semistandard shifted tableaux, extending the type A crystal structure of [**HPS17**] on these tableaux.

In this chapter, we provide a characterization of the queer supercrystals in analogy to Stembridge's [**Ste03**] characterization of crystals associated to classical simply-laced root systems. Assaf and Oguz [**AKO18a**, **AKO18b**] conjecture a local characterization of queer crystals in the spirit of Stembridge [**Ste03**], which involves local relations between the odd crystal operator f_{-1} with the type A_{n-1} crystal operators f_i for $1 \leqslant i < n$. However, we provide a counterexample to [**AKO18b**, Conjecture 4.16], which conjectures that these local axioms uniquely characterize the queer supercrystals. Instead, we define a new graph $G(C)$ on the relations between the type A components of the queer supercrystal C, which together with Assaf's and Oguz' local queer axioms and further new axioms uniquely fixes the queer crystal structure (see Theorem 5.5.1). We provide a combinatorial description of $G(C)$ by providing the combinatorial rules for all odd queer crystal operators f_{-i} and e_{-i} on certain highest weight elements for $1 \leqslant i < n$.

This chapter is structured as follows. In Section 5.2, we review the combinatorial definition of the queer supercrystals by [**GJK⁺15**]. In Section 5.3, we state the local queer axioms by Assaf and Oguz [**AKO18a**, **AKO18b**] and provide a counterexample to [**AKO18b**, Conjecture 4.16]. The graph $G(C)$ is introduced in Section 5.4 which together with the local queer axioms of Definition 5.3.1 and new connectivity axioms of Definition 5.4.3 uniquely characterize the queer crystals as stated in Theorem 5.5.1.

5.2. Queer supercrystals

5.2.1. Definition of queer supercrystals. An *(abstract) crystal* of type A_n is a nonempty set B together with the maps

$$\begin{aligned} e_i, f_i &\colon B \to B \sqcup \{0\} \qquad \text{for } i \in I, \\ \mathrm{wt} &\colon B \to \Lambda, \end{aligned} \tag{5.1}$$

where $\Lambda = \mathbb{Z}_{\geqslant 0}^{n+1}$ is the weight lattice of the root of type A_n and $I = \{1, 2, \ldots, n\}$ is the index set, subject to several conditions. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I$ the simple roots of type A_n, where ϵ_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$. Then we require:

A1. For $b, b' \in B$, we have $f_i b = b'$ if and only if $b = e_i b'$. In this case $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.

For $b \in B$, we also define

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid f_i^k(b) \neq 0\} \qquad \text{and} \qquad \varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid e_i^k(b) \neq 0\}.$$

For further details, see for example [**BS17**, Definition 2.13].

There is an action of the symmetric group S_n on a type A_n crystal B given by the operators

$$s_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geqslant 0, \\ e_i^{-k}(b) & \text{if } k < 0, \end{cases} \tag{5.2}$$

for $b \in B$, where $k = \varphi_i(b) - \varepsilon_i(b)$.

An element $b \in B$ is called *highest weight* if $e_i(b) = 0$ for all $i \in I$. Similarly, b is called *lowest weight* if $f_i(b) = 0$ for all $i \in I$. For a subset $J \subseteq I$, we say that b is J-highest weight if $e_i(b) = 0$ for all $i \in J$ and similarly b is J-lowest weight if $f_i(b) = 0$ for all $i \in J$.

We are now ready to define an abstract queer crystal.

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] \arrow[r, dashed, swap, "-1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n"] & \boxed{n+1} \end{tikzcd}$$

FIGURE 5.1. $\mathfrak{q}(n+1)$-queer crystal of letters $\mathcal{B}$

DEFINITION 5.2.1. [**GJK$^+$14**, Definition 1.9] An *abstract $\mathfrak{q}(n+1)$-crystal* is a type A_n crystal B together with the maps $e_{-1}, f_{-1}\colon B \to B \sqcup \{0\}$ satisfying the following conditions:

Q1. $\mathrm{wt}(B) \subset \Lambda$;

Q2. $\mathrm{wt}(e_{-1}b) = \mathrm{wt}(b) + \alpha_1$ and $\mathrm{wt}(f_{-1}b) = \mathrm{wt}(b) - \alpha_1$;

Q3. for all $b, b' \in B$, $f_{-1}b = b'$ if and only if $b = e_{-1}b'$;

Q4. if $3 \leqslant i \leqslant n$, we have

(a) the crystal operators e_{-1} and f_{-1} commute with e_i and f_i;

(b) if $e_{-1}b \in B$, then $\varepsilon_i(e_{-1}b) = \varepsilon_i(b)$ and $\varphi_i(e_{-1}b) = \varphi_i(b)$.

Given two $\mathfrak{q}(n+1)$-crystals B_1 and B_2, Grantcharov et al. [**GJK$^+$14**, Theorem 1.8] provide a crystal on the tensor product $B_1 \otimes B_2$, which we state here in reverse convention. It consists of the type A_n tensor product rule (see for example [**BS17**, Section 2.3]) and the *tensor product rule* for $b_1 \otimes b_2 \in B_1 \otimes B_2$

$$\begin{aligned} e_{-1}(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes e_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ e_{-1}b_1 \otimes b_2 & \text{otherwise,} \end{cases} \\ f_{-1}(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes f_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ f_{-1}b_1 \otimes b_2 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3}$$

The crystals of interest are the *crystals of words* $\mathcal{B}^{\otimes \ell}$, where $\mathcal{B}$ is the $\mathfrak{q}(n+1)$-queer crystal of letters depicted in Figure 5.1.

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5.2.2. Properties of queer supercrystals.

REMARK 5.2.2. The operators f_i for $i \in I_0$ have an easy combinatorial description on $b \in \mathcal{B}^{\otimes \ell}$ given by the *signature rule*, which can be directly derived from the tensor product rule (see for example [**?**, Section 2.4]). One can consider b as a word in the alphabet $\{1, 2, \ldots, n+1\}$. Consider the subword of b consisting only of the letters i and $i+1$. Pair (or bracket) any consecutive letters $i+1, i$ in this order, remove this pair, and repeat. Then f_i changes the rightmost unpaired i to $i+1$; if there is no such letter $f_i(b) = 0$. Similarly, e_i changes the leftmost unpaired $i+1$ to i; if there is no such letter $e_i(b) = 0$.

Remark 5.2.3. From (5.3), one may also derive a simple combinatorial rule for f_{-1} and e_{-1}. Consider the subword v of $b \in \mathcal{B}^{\otimes \ell}$ consisting of the letters 1 and 2. The crystal operator f_{-1} on b is defined if the leftmost letter of v is a 1, in which case it turns it into a 2. Otherwise $f_{-1}(b) = 0$. Similarly, e_{-1} on b is defined if the leftmost letter of v is a 2, in which case it turns it into a 1. Otherwise $e_{-1}(b) = 0$.

5.3. Local axioms

In [**AKO18b**, Definition 4.11], Assaf and Oguz give a definition of regular queer crystals. In essence, their axioms are rephrased in the following definition, where $\tilde{I} := I_0 \cup \{-1\}$.

Definition 5.3.1. Let C be a graph with labeled directed edges given by f_i for $i \in I_0$ and f_{-1}. If $b' = f_j b$ for $j \in \tilde{I}$ define e_j by $b = e_j b'$.

LQ1. The subgraph with all vertices but only edges labeled by $i \in I_0$ is a type A_n Stembridge crystal.

LQ2. $\varphi_{-1}(b), \varepsilon_{-1}(b) \in \{0, 1\}$ for all $b \in C$.

LQ3. $\varphi_{-1}(b) + \varepsilon_{-1}(b) > 0$ if $\mathrm{wt}(b)_1 + \mathrm{wt}(b)_2 > 0$.

LQ4. Assume $\varphi_{-1}(b) = 1$ for $b \in C$.

(a) If $\varphi_1(b) > 2$, we have

$$f_1 f_{-1}(b) = f_{-1} f_1(b),$$
$$\varphi_1(b) = \varphi_1(f_{-1}(b)) + 2,$$
$$\varepsilon_1(b) = \varepsilon_1(f_{-1}(b)).$$

(b) If $\varphi_1(b) = 1$, we have

$$f_1(b) = f_{-1}(b).$$

LQ5. Assume $\varphi_{-1}(b) = 1$ for $b \in C$.

(a) If $\varphi_2(b) > 0$, we have

$$f_2 f_{-1}(b) = f_{-1} f_2(b),$$

$$\varphi_2(b) = \varphi_2(f_{-1}(b)) - 1,$$

$$\varepsilon_2(b) = \varepsilon_2(f_{-1}(b)).$$

(b) If $\varphi_2(b) = 0$, we have

$$\begin{array}{lcl} \varphi_2(b) = \varphi_2(f_{-1}(b)) - 1 = 0, & \text{or} & \varphi_2(b) = \varphi_2(f_{-1}(b)) = 0, \\ \varepsilon_2(b) = \varepsilon_2(f_{-1}(b)), & & \varepsilon_2(b) = \varepsilon_2(f_{-1}(b)) + 1. \end{array}$$

LQ6. Assume that $\varphi_{-1}(b) = 1$ and $\varphi_i(b) > 0$ with $i \geqslant 3$ for $b \in C$. Then

$$f_i f_{-1}(b) = f_{-1} f_i(b),$$

$$\varphi_i(b) = \varphi_i(f_{-1}(b)),$$

$$\varepsilon_i(b) = \varepsilon_i(f_{-1}(b)).$$

Proposition 5.3.2 ([**AKO18b**]). *The queer crystal of words $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 5.3.1.*

Proof. This can easily checked using the combinatorial interpretation of the operators outlined in the remarks above. □

In [**AKO18b**, Conjecture 4.16], Assaf and Oguz conjecture that every regular queer crystal is a normal queer crystal. In other words, every connected graph satisfying the local queer axioms of Definition 5.3.1 is isomorphic to a connected component in some $\mathcal{B}^{\otimes \ell}$. We provide a counterexample to this claim in Figure 5.2. In the figure, the I_0-components of the $\mathfrak{q}(3)$-crystal of highest weight $(4, 2, 0)$ are shown. Some of the f_{-1}-arrows are drawn in green. The remaining arrows can be filled in using the axioms of local queer axioms in a consistent manner. If the dashed green arrow from 331131 to 332131 and the dashed green arrow from 331132 to 332132 are replaced by the dashed purple arrow from 331131 to 331231 and the dashed purple arrow from 331132 to 332231, respectively, all axioms of Definition 5.3.1 are still satisfied with the remaining f_{-1}-arrows filled in. However, the I_0-component with highest weight element 132121 has become disconnected and hence the two crystals are not isomorphic.

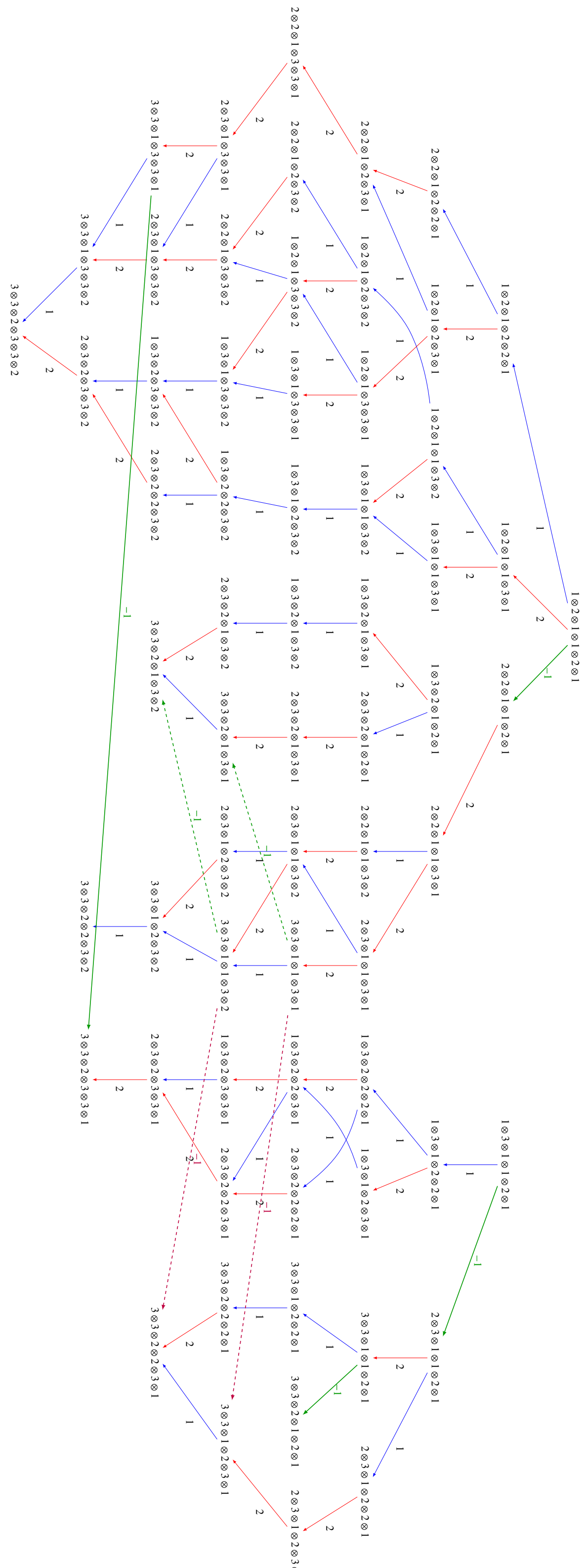

FIGURE 5.2. Counterexample to the unique characterization of the local queer axioms of Definition 5.3.1.

The problem is demonstrated by the counterexample in Figure 5.2: switching components with the same I_0-highest weights can cause non-uniqueness. In fact, if $f_{-1}b$ is determined for all $b \in C$ such that

$$\varphi_i(b) = 0 \quad \text{for all } i \in I_0 \setminus \{1\} \text{ and } \quad \varphi_1(b) = 2, \tag{5.1}$$

then, by the relations between f_{-1} and f_i for $i \in I_0$ of Definition 5.3.1, f_{-1} is determined on all elements in C.

Lemma 5.3.3. *Let $v \in \mathcal{B}^{\otimes \ell}$ be an I_0-lowest weight element, that is, $\varphi_i(v) = 0$ for all $i \in I_0$. Then every $b \in \mathcal{B}^{\otimes \ell}$ satisfying* (5.1) *is of the form*

$$g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v \qquad \text{for some } 1 \leqslant j \leqslant k \leqslant n. \tag{5.2}$$

Conversely, every $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k \leqslant n$ satisfies (5.1).

Proof. The statement of the lemma is a statement about type A_n crystals and hence can be verified by the tableaux model for type A_n crystals (see for example [**BS17**]). The element v is I_0-lowest weight and hence as a tableau in French notation contains the letter $n+1$ at the top of each column, the letter n in the second to top box in each column, and in general the letter $n+2-i$ in the i-th box from the top in its column. If there is a letter $k+1$ in the first row of v, then $(e_1 \cdots e_k)$ applies to v and $b' = (e_1 \cdots e_k)v$ satisfies $\varphi_i(b') = 0$ for $i \in I_0 \setminus \{1\}$ and $\varphi_1(b') = 1$. The element b' has several changed entries in the first row, and otherwise the entries above the first row all have letter $n+2-i$ in the i-th box from the top in their column. If b' has a letter $j+1$ in the first row with $1 \leqslant j \leqslant k$, then $(e_1 \cdots e_j)$ applies to b' and $b = g_{j,k} = (e_1 \cdots e_j)b'$ satisfies (5.1). Note that if $j > k$, then the last e_1 would no longer apply and hence $b = 0$. This proves that $g_{j,k} \neq 0$ as in (5.2) satisfies (5.1). If conversely b satisfies (5.1), then as a tableau it contains two extra 1's in the first row that have a 3 or bigger above them rather than a 2 in their columns, and for entries higher than the first row the i-th box from the top in its column contains $n+2-i$. It is not hard to check that then $(f_k \cdots f_1)(f_j \cdots f_1)b = v$ for some $1 \leqslant j \leqslant k \leqslant n$. Hence b is of the form (5.2). □

In the next section, we introduce a new graph just on I_0-highest weight elements and new connectivity axioms (see Definition 5.4.3) that uniquely characterizes queer crystals (see Theorem 5.5.1).

5.4. Graph on type A components

Let C be a crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. In this section, we define a graph for C labeled by the type A_n components of C. We draw an edge from vertex C_1 to vertex C_2 in this graph if there is an element b_1 in the component C_1 and an element b_2 in the component C_2 such that $f_{-1}b_1 = b_2$. We also provide new axioms in Definition 5.4.3 that will be used in Section 5.5 to provide a unique characterization of queer crystals.

Definition 5.4.1. Let C be a crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. We define the *component graph* of C, denoted by $G(C)$, as follows. The vertices of $G(C)$ are the type A_n components of C (typically labeled by their highest weight elements). There is an edge from vertex C_1 to vertex C_2 in this graph, if there is an element b_1 in the component C_1 and an element b_2 in the component C_2 such that

$$f_{-1}b_1 = b_2.$$

Example 5.4.2. Let C be the connected component in the $\mathfrak{q}(3)$-crystal $\mathcal{B}^{\otimes 6}$ with highest weight element $1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1$ of highest weight $(4, 2, 0)$. The graph $G(C)$ is given in Figure 5.3 on the left (disregarding the labels on the edges). The graph $G(C')$ for the counterexample C' in Figure 5.2 is given in Figure 5.3 on the right. Since the two graphs are not isomorphic as unlabeled graphs, this confirms that the purple dashed arrows in Figure 5.2 do not give the queer crystal even though the induced crystal satisfies the axioms in Definition 5.3.1.

Next we introduce new axioms.

Definition 5.4.3. Let C be a connected crystal satisfying the local queer axioms of Definition 5.3.1. Let $v \in C$ be an I_0-lowest weight element and $u =\uparrow v$. As in (5.2), define $g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v$ for $1 \leqslant j \leqslant k \leqslant n$.

- **C0.** $\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.
- **C1.** Suppose that $G(C)$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$. For all $h < j \leqslant k$ such that $g_{j,k} \neq 0$, we require that $f_{-1}g_{j,k} \neq 0$ and

$$f_{-1}g_{j,k} = (e_2 \cdots e_j)(e_1 \cdots e_h)v',$$

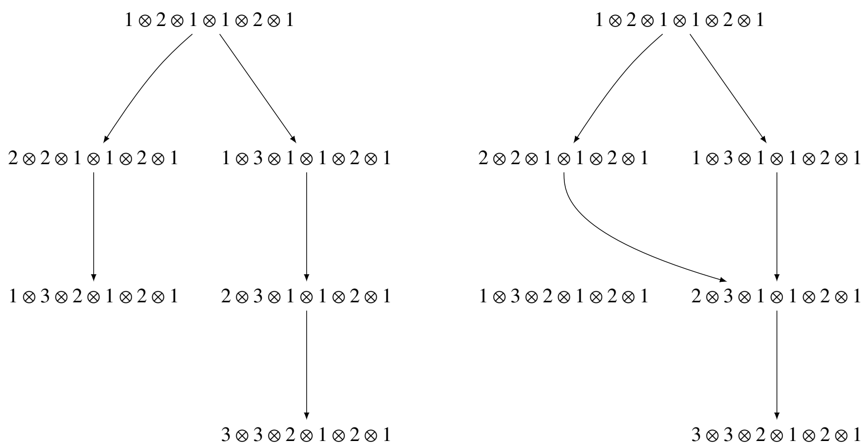

FIGURE 5.3. **Left:** The correct $G(C)$. **Right:** $G(C')$ for the counterexample of Example 5.4.2.

where v' is I_0-lowest weight with $\uparrow v' = u'$.

C2. Suppose that either (a) $G(C)$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$ or (b) no such edge exists in $G(C)$. For all $1 \leqslant j \leqslant h$ in case (a) and all $1 \leqslant j \leqslant k$ in case (b) such that $g_{j,k} \neq 0$ and $f_{-1}g_{j,k} \neq 0$, we require that

$$f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v.$$

REMARK 5.4.4. Condition **C0** can be replaced by the following condition:

LQ7. If $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$ for $b \in C$ with $\varepsilon_2(b) > 0$, then $\varphi_{-1}(b) \leqslant \varphi_{-1}(e_1e_2(b))$.

This condition indeed implies **C0**. Suppose $\varphi_{-1}(e_1 \cdots e_k v) = 1$. Then for $b = (e_3 \cdots e_j)(e_1 \cdots e_k)v$, we have $\varphi_{-1}(b) = 1$. However, b satisfies $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$, so the above condition implies that $\varphi_{-1}(e_1e_2(b)) = 1$ as well. But $e_1e_2(b) = g_{j,k}$. Hence $\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

Moreover, in $\mathcal{B}^{\otimes \ell}$ the conditions in **LQ7** are satisfied. Namely, the condition $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$ implies that $e_2(b) \neq 0$ and $e_1e_2(b) \neq 0$. Moreover, this condition implies that e_1 acts on $e_2(b)$ in a position weakly to the left of where e_2 acts on b. Thus if $\varphi_{-1}(b) = 1$, it immediately follows that $\varphi_{-1}(e_1e_2(b)) = 1$ which proves the statement.

THEOREM 5.4.5. *The $\mathfrak{q}(n+1)$-queer crystal $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 5.4.3.*

The proof of Theorem 5.4.5 is given in Appendix B.2.

5.5. Characterization of queer crystals

Our main theorem gives a characterization of the queer supercrystals.

Theorem 5.5.1. *Let C be a connected component of a generic abstract queer crystal (see Definition 5.2.1). Suppose that C satisfies the following conditions:*

(1) C satisfies the local queer axioms of Definition 5.3.1.

(2) C satisfies the connectivity axioms of Definition 5.4.3.

(3) $G(C)$ is isomorphic to $G(\mathcal{D})$, where $\mathcal{D}$ is some connected component of $\mathcal{B}^{\otimes \ell}$.

Then the queer supercrystals C and $\mathcal{D}$ are isomorphic.

Theorem 5.5.1 states that the local queer axioms, the connectivity axioms, and the component graph uniquely characterize queer crystals. Before we give its proof, we need the following statement. Recall that $g_{j,k} = (e_1 \cdots e_j)(e_1 \cdots e_k)v$ was defined in (5.2), where v is an I_0-lowest weight vector.

Lemma 5.5.2. *In a crystal satisfying the local queer axioms of Definition 5.3.1 and* **C0** *of Definition 5.4.3, we have for any $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k$*

$$\varphi_{-1}(g_{j,k}) = 0 \quad \textit{if and only if} \quad \varphi_{-1}(e_1 \cdots e_k v) = 0.$$

Proof. The condition **C0** requires that $\varphi_{-1}(g_{j,k}) = 0$ implies $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

For the converse direction, note that $\mathrm{wt}(e_1 \cdots e_k v)_1 > 0$. Hence

$$\varphi_{-1}(e_1 \cdots e_k v) = 0 \quad \Leftrightarrow \quad \varepsilon_{-1}(e_1 \cdots e_k v) = 1.$$

By the local queer axioms **LQ6** and **LQ5** of Definition 5.3.1 (see also Figure **??**), we have

$$\varepsilon_{-1}(e_1 \cdots e_k v) = 1 \quad \Leftrightarrow \quad \varepsilon_{-1}((e_3 \cdots e_j)(e_1 \cdots e_k)v) = 1 \quad \Rightarrow \quad \varepsilon_{-1}((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1.$$

It can be easily checked that $\varphi_1((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1$ for $j \leqslant k$ (for example using the tableaux model for type A_n crystals). Hence by the local queer axioms

$$\varepsilon_{-1}((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1 \quad \Leftrightarrow \quad \varepsilon_{-1}((e_1 \cdots e_j)(e_1 \cdots e_k)v) = 1.$$

This proves that $\varphi_{-1}(e_1 \cdots e_k v) = 0$ implies $\varphi_{-1}(g_{j,k}) = 0$. □

APPENDIX A

Appendix 1: Proofs for Type A crystal on primed tableaux

A.1.

In this appendix, we provide the proof of Theorem 3.3.3.

A.1.1. Preliminaries. We use the fact from [**Hai89**] that taking only elements smaller or equal to $i+1$ from the word $\mathbf{b}$ and applying the mixed insertion corresponds to taking only the part of the tableau $\mathbf{T}$ with elements $\leqslant i+1$. Thus, it is enough to prove the theorem for a "truncated" word $\mathbf{b}$ without any letters greater than $i+1$. To shorten the notation, we set $j = i+1$ in this appendix. We sometimes also restrict to just the letters i and j in a word w. We call this the *$\{i, j\}$-subword* of w.

First, in Lemma A.1.1 we justify the notion of the reading word $\mathrm{rw}(\mathbf{T})$ and provide the reason to use a bracketing rule on it. After that, in Section A.1.2 we prove that the action of the crystal operator f_i on $\mathbf{b}$ corresponds to the action of f_i on $\mathbf{T}$ after the insertion.

Given a word $\mathbf{b}$, we apply the crystal bracketing rule for its $\{i, j\}$-subword and globally declare the rightmost unbracketed i in $\mathbf{b}$ (i.e. the letter the crystal operator f_i acts on) to be a bold i. Insert the letters of $\mathbf{b}$ via Haiman insertion to obtain the insertion tableau $\mathbf{T}$. During this process, we keep track of the position of the bold i in the tableau via the following rules. When the bold i from $\mathbf{b}$ is inserted into $\mathbf{T}$, it is inserted as the rightmost i in the first row of $\mathbf{T}$ since by definition it is unbracketed in $\mathbf{b}$ and hence cannot bump a letter j. From this point on, the tableau $\mathbf{T}$ has a *special* letter i and we track its position:

(1) If the special i is unprimed, it is always the rightmost i in its row. When a letter i is bumped from this row, only one of the non-special letters i can be bumped, unless the special i is the only i in the row. When the non-diagonal special i is bumped from its row to the next row, it will be inserted as the rightmost i in the next row.

(2) When the diagonal special i is bumped from its row to the column to its right, it is inserted as the bottommost i' in the next column.

(3) If the special i is primed, it is always the bottommost i' in its column. When a letter i' is bumped from this column, only one of the non-special letters i' can be bumped, unless the special i' is the only i' in the column. When the primed special i is bumped from its column to the next column, it is inserted as the bottommost i' in the next column.

(4) When i is inserted into a row with the special unprimed i, the rightmost i becomes special.

(5) When i' is inserted into a column with the special primed i, the bottommost primed i becomes special.

Lemma A.1.1. *Using the rules above, after the insertion process of* $\mathbf{b}$*, the special i in* $\mathbf{T}$ *is the same as the rightmost unbracketed i in the reading word* $\mathrm{rw}(\mathbf{T})$ *(i.e. the definition of the bold i in* $\mathbf{T}$*). Moreover, the number of unbracketed letters i in* $\mathbf{b}$ *is equal to the number of unbracketed letters i in* $\mathrm{rw}(\mathbf{T})$.

Proof. First, note that since both the number of letters i and the number of letters j are equal in $\mathbf{b}$ and $\mathrm{rw}(\mathbf{T})$, the fact that the number of unbracketed letters i is the same implies that the number of unbracketed letters j must also be the same. We use induction on $1 \leqslant s \leqslant h$, where the letters $b_1 \dots b_s$ of $\mathbf{b} = b_1 b_2 \dots b_h$ have been inserted using Haiman mixed insertion with the above rules. That is, we check that at each step of the insertion algorithm the statement of our lemma stays true.

The induction step is as follows: Consider the word $b_1 \dots b_{s-1}$ with a corresponding insertion tableau $\mathbf{T}^{(s-1)}$. If the bold i in $\mathbf{b}$ is not in $b_1 \dots b_{s-1}$, then $\mathbf{T}^{(s-1)}$ does not contain a special letter i. Otherwise, by induction hypothesis assume that the bold i in $b_1 \dots b_{s-1}$ by the above rules corresponds to the special i in $\mathbf{T}^{(s-1)}$, that is, it is in the position corresponding to the rightmost unbracketed i in the reading word $\mathrm{rw}(\mathbf{T}^{(s-1)})$. Then we need to prove that for $b_1 \dots b_s$, the special i in $\mathbf{T}^{(s-1)}$ ends up in the position corresponding to the rightmost unbracketed i in the reading word of $\mathbf{T}^{(s)} = \mathbf{T}^{(s-1)} \leftsquigarrow b_s$. We also need to verify that the second part of the lemma remains true for $\mathbf{T}^{(s)}$.

Remember that we are only considering "truncated" words $\mathbf{b}$ with all letters $\leqslant j$.

Case 1. Suppose $b_s = j$. In this case j is inserted at the end of the first row of $\mathbf{T}^{(s-1)}$, and $\mathrm{rw}(\mathbf{T}^{(s)})$ has j attached at the end. Thus, both statements of the lemma are unaffected.

Case 2. Suppose $b_s = i$ and b_s is unbracketed in $b_1 \dots b_{s-1} b_s$. Then there is no special i in tableau $\mathbf{T}^{(s-1)}$, and b_s might be the bold i of the word $\mathbf{b}$. Also, there are no unbracketed letters j in $b_1 \dots b_{s-1}$, and thus all j in $\mathrm{rw}(\mathbf{T}^{(s-1)})$ are bracketed. Thus, there are no letters j in the first row of $\mathbf{T}^{(s-1)}$, and i is inserted in the first row of $\mathbf{T}^{(s-1)}$, possibly bumping the letter j' from column c into an empty column $c+1$ in the process. Note that if j' is bumped, moving it to column $c+1$ of $\mathbf{T}^{(s)}$ does not change the reading word, since column c of $\mathbf{T}^{(s-1)}$ does not contain any primed letters other than j'. The reading word of $\mathbf{T}^{(s)}$ is thus the same as $\mathrm{rw}(\mathbf{T}^{(s-1)})$ except for an additional unbracketed i at the end. The number of unbracketed letters i in both $\mathrm{rw}(\mathbf{T}^{(s)})$ and $b_1 \dots b_{s-1} b_s$ is thus increased by one compared to $\mathrm{rw}(\mathbf{T}^{(s-1)})$ and $b_1 \dots b_{s-1}$. If b_s is the bold i of the word $\mathbf{b}$, the special i of tableau $\mathbf{T}^{(s)}$ is the rightmost i on the first row and corresponds to the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T}^{(s)})$.

Case 3. Suppose $b_s = i$ and b_s is bracketed with a j in the word $b_1 \dots b_{s-1}$. In this case, according to the induction hypothesis, $\mathrm{rw}(\mathbf{T}^{(s-1)})$ has an unbracketed j. There are two options.

Case 3.1. If the first row of $\mathbf{T}^{(s-1)}$ does not contain j, b_s is inserted at the end of the first row of $\mathbf{T}^{(s-1)}$, possibly bumping j' in the process. Regardless, $\mathrm{rw}(\mathbf{T}^{(s)})$ does not change except for attaching an i at the end (see Case 2). This i is bracketed with one unbracketed j in $\mathrm{rw}(\mathbf{T}^{(s)})$. The special i (if there was one in $\mathbf{T}^{(s-1)}$) does not change its position and the statement of the lemma remains true.

Case 3.2. If the first row of $\mathbf{T}^{(s-1)}$ does contain a j, inserting b_s into $\mathbf{T}^{(s-1)}$ bumps j (possibly bumping j' beforehand) into the second row, where j is inserted at the end of the row. So, if the first row contains $n \geqslant 0$ elements i and $m \geqslant 1$ elements j, the reading word $\mathrm{rw}(\mathbf{T}^{(s-1)})$ ends with $\dots i^n j^m$, and $\mathrm{rw}(\mathbf{T}^{(s)})$ ends with $\dots j i^{n+1} j^{m-1}$. Thus, the number of unbracketed letters i does not change and if there was a special i in the first row, it remains there and it still corresponds to the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T}^{(s)})$.

Case 4. Suppose $b_s < i$. Inserting b_s could change both the primed reading word and unprimed reading word of $\mathbf{T}^{(s-1)}$. As long as neither i nor j is bumped from the diagonal, we can treat primed and unprimed changes separately.

Case 4.1. Suppose neither i nor j is not bumped from the diagonal during the insertion. This means that there are no transitions of letters i or j between the primed and the unprimed parts of the reading word. Thus, it is enough to track the bracketing relations in the unprimed reading word; the bracketing relations in the primed reading word can be verified the same way via the transposition. After we make sure that the number of unbracketed letters i and j changes neither in the primed nor unprimed reading word, it is enough to consider the case when the special i is unprimed, since the case when it is primed can again be checked using the transposition. To avoid going back and forth, we combine these two processes together in each subcase to follow.

Case 4.1.1. If there are no letters i and j in the bumping sequence, the unprimed $\{i, j\}$-subword of $\mathrm{rw}(\mathbf{T}^{(s)})$ is the same as in $\mathrm{rw}(\mathbf{T}^{(s-1)})$. The special i (if there is one) remains in its position, and thus the statement of the lemma remains true.

Case 4.1.2. Now consider the case when there is a j in the bumping sequence, but no i. Let that j be bumped from the row r. Since there is no i bumped, row r does not contain any letters i. Thus, bumping j from row r to the end of row $r+1$ does not change the $\{i, j\}$-subword of $\mathrm{rw}(\mathbf{T}^{(s-1)})$, so the statement of the lemma remains true.

Case 4.1.3. Consider the case when there is an i in the bumping sequence. Let that i be bumped from the row r.

Case 4.1.3.1. If there is a (non-diagonal) j in row $r+1$, it is bumped into row $r+2$ (j' may have been bumped in the process). Note that in this case the i bumped from row r could not have been a special one. If there are $n \geqslant 0$ elements i and $m \geqslant 1$ elements j in row r, the part of the reading word $\mathrm{rw}(\mathbf{T}^{(s-1)})$ with $\ldots i^n j^m i \ldots$ changes to $\ldots j i^{n+1} j^{m-1} \ldots$ in $\mathrm{rw}(\mathbf{T}^{(s)})$. The bracketing relations remain the same, and if row $r+1$ contained a special i, it would remain there and would correspond to the rightmost i in $\mathrm{rw}(\mathbf{T}^{(s)})$.

Case 4.1.3.2. If there are no letters j in row $r+1$, and j' in row $r+1$ does not bump a j, the $\{i, j\}$-subword does not change and the statement of the lemma remains true.

Case 4.1.3.3. Now suppose there are no letters j in row $r+1$ and j' from row $r+1$ bumps a j from another row. This can only happen if, before the i was bumped, there was only one i in row r of $\mathbf{T}^{(s-1)}$, there is a j' immediately below it, and there is a j in the column to the right of i and in row $r' \leqslant r$.

If $r' = r$, then after the insertion process, i and j are bumped from row r to row $r+1$. Since there was only one i in row r and there are no letters j in row $r+1$, the $\{i, j\}$-subword of $\mathrm{rw}(\mathbf{T}^{(s-1)})$ does not change and the statement of the lemma remains true.

Otherwise $r' < r$. Then there are no letters i in row r' and by assumption there is no letter j in row $r+1$. Thus, moving i to row $r+1$ and moving j to the row $r'+1$ does not change the $\{i, j\}$-subword of $\mathrm{rw}(\mathbf{T}^{(s-1)})$ and the statement of the lemma remains true.

Case 4.2. Suppose i or j (or possibly both) are bumped from the diagonal in the insertion process.

Case 4.2.1. Consider the case when the insertion sequence ends with $\cdots \to z \to j[j']$ with $z < i$ and possibly $\to j$ right after it. Let the bumped diagonal j be in column c. Then columns $1, 2, \ldots, c$ of $\mathbf{T}^{(s-1)}$ could only contain elements $\leqslant z$, except for the j on the diagonal. Thus, the bumping process just moves j from the unprimed reading word to the primed reading word without changing the overall order of the $\{i, j\}$-subword.

Case 4.2.2. Consider the case when the insertion sequence ends with $\cdots \to i' \to i \to j[j']$ and possibly $\to j$. Let the bumped diagonal j be in row (and column) r. Note that r must be the last row of $\mathbf{T}^{(s-1)}$. Then i has to be bumped from row $r-1$ (and, say, column c) and i' also has to be in row $r-1$ (moreover, it has to be the only i' in column $c-1$). Also, since there are no letters j' in column c (otherwise it would be in row r, which is impossible), bumping i' to column c does not change the $\{i, j\}$-subword of $\mathrm{rw}(\mathbf{T}^{(s-1)})$. Note that after i' moves to column c, there are no i' or j' in columns $1, \ldots, r$, and thus priming j and moving it to column $r+1$ does not change the $\{i, j\}$-subword. If the last row r contains n elements j, the $\{i, j\}$-subword of $\mathbf{T}^{(s-1)}$ contains $\ldots j^n i \ldots$ and after the insertion it becomes $\ldots j i j^{n-1} \ldots$, where the left j is from the primed subword. Thus, the number of bracketed letters i does not change. Also, if we moved the special i in the process, it could only have been the bumped i'. Its position in the reading word is unaffected.

Case 4.2.3. The case when the insertion sequence does not contain i', does not bump i from the diagonal, but contains i and bumps j from the diagonal is analogous to the previous case.

Case 4.2.4. Suppose both i and j are bumped from the diagonal. That could only be the case with diagonal i bumped from row (and column) r, bumping another letter i from the row r and column $r+1$, and bumping j from row (and column) $r+1$ (and possibly bumping j to row $r+2$ at the end). Let the number of letters i' in column $r+1$ be n and let the number of letters j in row $r+1$ be m.

Case 4.2.4.1 Let $m \geqslant 2$. Then the $\{i, j\}$-subword of $\mathrm{rw}(\mathbf{T}^{(s-1)})$ contains $\ldots i^n j^m i i \ldots$ and after the insertion it becomes $\ldots j i^{n+1} j i j^{m-2} \ldots$. The number of unbracketed letters i stays the same. Since $m \geqslant 2$, the special i of $\mathbf{T}^{(s-1)}$ could not have been involved in the bumping procedure. However, the special i might have been the bottommost i' in column $r+1$ of $\mathbf{T}^{(s-1)}$, and after the insertion the special i would still be the bottommost i' in column $r+1$ and would correspond to the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T}^{(s)})$:

$$\begin{array}{|c|c|c|c|}
\hline
\cdot & \cdot & i' & \cdot \\
\hline
 & i & i & \cdot \\
\hline
 & & j & j \\
\hline
\end{array}
\quad \mapsto \quad
\begin{array}{|c|c|c|c|}
\hline
\cdot & \cdot & i' & \cdot \\
\hline
 & \cdot & i' & \cdot \\
\hline
 & & i & j' \\
\hline
 & & & j \\
\hline
\end{array}$$

Case 4.2.4.2. Let $m = 1$. Then the $\{i, j\}$-subword of $\mathbf{T}^{(s-1)}$ contains $\dots i^n jii \dots$ and after the insertion it becomes $\dots ji^{n+1}i$. The number of unbracketed letters i stays the same. If the special i was in row r and column $r+1$, then after the insertion it becomes a diagonal one, and it would still correspond to the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T}^{(s)})$.

Case 4.2.5. Suppose only i is bumped from the diagonal (let that i be on row and column r). Note that there cannot be an i' in column r.

Case 4.2.5.1. Suppose i from the diagonal bumps another i from column $r + 1$ and row r. In that case there are no letters j in row $r+1$. No letters j or j' are affected and thus the $\{i, j\}$-subword of $\mathbf{T}^{(s)}$ does not change, and the special i in $\mathbf{T}^{(s)}$ (if there is one) still corresponds to the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T}^{(s)})$.

Case 4.2.5.2. Suppose i from the diagonal bumps j' from column $r + 1$ and row r. Note that j' must be the only j' in column $r + 1$. Suppose also that there is one j in row $r + 1$. Denote the number of letters i' in column $r + 1$ of $\mathbf{T}^{(s-1)}$ by n. If there is a j in row $r + 1$ of $\mathbf{T}^{(s-1)}$, then the $\{i, j\}$-subword of $\mathbf{T}^{(s-1)}$ contains $\dots i^n jji \dots$ and after the insertion it becomes $\dots ji^{n+1} j \dots$. If there is no j in row $r + 1$ of $\mathbf{T}^{(s-1)}$, then the $\{i, j\}$-subword of $\mathbf{T}^{(s-1)}$ contains $\dots i^n ji \dots$ and after the insertion it becomes $\dots ji^{n+1} \dots$. The number of unbracketed letters i is unaffected. If the special i of $\mathbf{T}^{(s-1)}$ was the bottommost i' in column $r + 1$ of $\mathbf{T}^{(s-1)}$, after the insertion the special i is still the bottommost i' in column $r + 1$ and corresponds to the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T}^{(s)})$. □

Corollary A.1.2.

$$f_i(\mathbf{b}) = \mathbf{0} \quad \textit{if and only if} \quad f_i(\mathbf{T}) = \mathbf{0}.$$

A.1.2. Proof of Theorem 3.3.3. By Lemma A.1.1, the cell x in the definition of the operator f_i corresponds to the bold i in the tableau $\mathbf{T}$. Furthermore, we know how the bold i moves during the insertion procedure. We assume that the bold i exists in both $\mathbf{b}$ and $\mathbf{T}$, meaning that $f_i(\mathbf{b}) \neq \mathbf{0}$ and $f_i(\mathbf{T}) \neq \mathbf{0}$ by Corollary A.1.2. We prove Theorem 3.3.3 by induction on the length of the word $\mathbf{b}$.

Base. Our base is for words $\mathbf{b}$ with the last letter being a bold i (i.e. rightmost unbracketed i). Let $\mathbf{b} = b_1 \dots b_{h-1} b_h$ and $f_i(\mathbf{b}) = b_1 \dots b_{h-1} b'_h$, where $b_h = i$ and $b'_h = j$. Denote the mixed insertion tableau of $b_1 \dots b_{h-1}$ as $\mathbf{T}_0$, the insertion tableau of $b_1 \dots b_{h-1} b_h$ as $\mathbf{T}$, and the insertion tableau of $b_1 \dots b_{h-1} b'_h$ as $\mathbf{T}'$. Note that $\mathbf{T}_0$ does not have letters j in the first row. If the first row of $\mathbf{T}_0$ ends with $\dots j'$, then the first row of $\mathbf{T}$ ends with $\dots \mathbf{i} j'$ and the first row of $\mathbf{T}'$ ends with $\dots j' j$. If the first row of $\mathbf{T}_0$ does not contain j', the first row of $\mathbf{T}$ ends with $\ldots \mathbf{i}$ and the first row of $\mathbf{T}'$ ends with $\ldots j$, and the cell x_S is empty. In both cases $f_i(\mathbf{T}) = \mathbf{T}'$.

Induction step. Now, let $\mathbf{b} = b_1 \ldots b_h$ with operator f_i acting on the letter b_s in $\mathbf{b}$ with $s < h$. Denote the mixed insertion tableau of $b_1 \ldots b_{h-1}$ as $\mathbf{T}$ and the insertion tableau of $f_i(b_1 \ldots b_{h-1})$ as $\mathbf{T}'$. By induction hypothesis, we know that $f_i(\mathbf{T}) = \mathbf{T}'$. We want to show that $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$. In Cases 1-3 below, we assume that the bold letter i is unprimed. Since almost all results from the case with unprimed i are transferrable to the case with primed bold i via the transposition of the tableau $\mathbf{T}$, we just need to cover the differences in Case 4.

Case 1. Suppose $\mathbf{T}$ falls under Case (1) of the rules for f_i: the bold i is in the non-diagonal cell x in row r and column c and the cell x_E in the same row and column $c+1$ contains the entry j'. Consider the insertion path of b_h.

Case 1.1. If the insertion path of b_h in $\mathbf{T}$ contains neither cell x nor cell x_E, the insertion path of b_h in $\mathbf{T}'$ also does not contain cells x and x_E. Thus, $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 1.2. Suppose that during the insertion of b_h into $\mathbf{T}$, the bold i is row-bumped by an unprimed element $d < i$ or is column-bumped by a primed element $d' \leqslant i'$. This could only happen if the bold i is the unique i in row r of $\mathbf{T}$. During the insertion process, the bold i is inserted into row $r+1$. Since there are no letters i in row r of $\mathbf{T}'$, inserting b_h into $\mathbf{T}'$ inserts d in cell x, bumps j' to cell x_E, and bumps j into row $r+1$. Thus we are in a situation similar to the induction base. It is easy to check that row $r+1$ does not contain any letters j in $\mathbf{T}$. If it contains j', this j' is bumped back into row $r+1$. Similar to the induction base, $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 1.3. Suppose that during the insertion of b_h into $\mathbf{T}$, an unprimed i is inserted into row r. Note that in this case, row r in $\mathbf{T}$ must contain a j (or else the i from row r would not be the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T})$). Thus inserting i into row r in $\mathbf{T}$ shifts the bold i to column $c+1$, shifts j' to column $c+2$ and bumps j to row $r+1$. Inserting i into row r in $\mathbf{T}'$ shifts j' to column $c+1$ with a j to the right of it, and bumps j into row $r+1$. Thus $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 1.4. Suppose that during the insertion of b_h into $\mathbf{T}$, the j' in cell x_E is column-bumped by a primed element d' and the cell x is unaffected. Note that in order for $\mathbf{T} \leftsquigarrow b_h$ to be a valid shifted primed tableau, i

must be smaller than d', and thus d' could only be j'. On the other hand, j' cannot be inserted into column $c+1$ of $\mathbf{T}'$ in order for $\mathbf{T}' \leftsquigarrow b_h$ to be a valid shifted primed tableau. Thus this case is impossible.

Case 2. Suppose tableau $\mathbf{T}$ falls under Case (2a) of the crystal operator rules for f_i. This means that for a bold i in cell x (in row r and column c) of tableau $\mathbf{T}$, the cell x_E contains the entry j or is empty and cell x_S is empty. Tableau $\mathbf{T}'$ has all the same elements as $\mathbf{T}$, except for a j in the cell x. We are interested in the case when inserting b_h into either $\mathbf{T}$ or $\mathbf{T}'$ bumps the element from cell x.

Case 2.1. Suppose that the non-diagonal bold i in $\mathbf{T}$ (in row r) is row-bumped by an unprimed element $d < i$ or column-bumped by a primed element $d' < j'$. Element d (or d') bumps the bold i into row $r+1$ of $\mathbf{T}$, while in $\mathbf{T}'$ (since there are no letters i in row r of $\mathbf{T}'$) it bumps j from cell x into row $r+1$. Thus we are in the situation of the induction base and $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 2.2. Suppose x is a non-diagonal cell in row r, and during the insertion of b_h into $\mathbf{T}$, an unprimed i is inserted into the row r. In this case, row r in $\mathbf{T}$ must contain a letter j. The insertion process shifts the bold i one cell to the right in $\mathbf{T}$ and bumps a j into row $r+1$, while in $\mathbf{T}'$ it just bumps j into the row $r+1$. We end up in Case (2a) of the crystal operator rules for f_i with bold i in the cell x_E.

Case 2.3. Suppose that during the insertion of b_h into $\mathbf{T}'$, the j in the non-diagonal cell x is column-bumped by a j'. This means that j' was previously bumped from column $c-1$ and row $\geqslant r$. Thus the cell x_{SW} (cell to the left of an empty x_S) is non-empty. Moreover, right before inserting j' into the column c, the cell x_{SW} contains an entry $< j'$. Inserting j' into column c of $\mathbf{T}$ just places j' into the empty cell x_S. Inserting j' into column c of $\mathbf{T}'$ places j' into x, and bumps j into the empty cell x_S. Thus, we end up in Case (2c) of the crystal operator rules after the insertion of b_h with $y = x_S$.

Case 2.4. Suppose that x in $\mathbf{T}$ is a diagonal cell (in row r and column r) and that it is row-bumped by an element $d < i$. Note that in this case there cannot be any letter j in row $r+1$. Also, since d is inserted into cell x, there cannot be any letters i' in columns $1, \ldots, r$, and thus there cannot be any letters j' in column $r+1$ (otherwise the i in cell x would not be bold). The bumped bold i in tableau $\mathbf{T}$ is inserted as a primed bold i' into the cell z of column $r+1$.

Case 2.4.1. Suppose that there are no letters i in column $r+1$ of $\mathbf{T}$. In this case, the cell z in $\mathbf{T}$ either contains j (and then that j would be bumped to the next row) or is empty. Inserting b_h into tableau $\mathbf{T}'$ bumps the diagonal j in cell x, which is inserted as a j' into cell z, possibly bumping j after that. Thus,

$\mathbf{T} \leftsquigarrow b_h$ falls under Case (2a) of the "primed" crystal rules with the bold i' in cell z (note that there cannot be any j' in cell $(z*)_E$ of the tableau $(\mathbf{T} \leftsquigarrow b_h)*$). Since $\mathbf{T} \leftsquigarrow b_h$ and $\mathbf{T}' \leftsquigarrow b_h$ differ only by the cell z, $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 2.4.2. Suppose that there is a letter i in cell z of column $r+1$ of $\mathbf{T}$. Note that cell z can only be in rows $1, \ldots, r-1$ and thus z_{SW} contains an element $< i$. Thus, during the insertion process of b_h into $\mathbf{T}$, diagonal bold i from cell x is inserted as bold i' into cell z, bumping the i from cell z into cell z_S (possibly bumping j afterwards). On the other hand, inserting b_h into $\mathbf{T}'$ bumps the diagonal j from cell x into cell z_S as a j' (possibly bumping j afterwards). Thus, $\mathbf{T} \leftsquigarrow b_h$ falls under Case (1) of the "primed" crystal rules with the bold i' in cell z, and so $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 2.5. Suppose that x is a diagonal cell (in row r and column r) and that during the insertion of b_h into $\mathbf{T}$, an unprimed i is inserted into row r. In this case, the entry in cell x_E has to be j and the diagonal cell x_{ES} must be empty. Inserting i into row r of $\mathbf{T}$ bumps a j from cell x_E into cell x_{ES}. On the other hand, inserting i into row r of $\mathbf{T}'$ bumps a j from the diagonal cell x, which in turn is inserted as a j' into cell x_E, which bumps j from cell x_E into cell x_{ES}. Thus, $\mathbf{T} \leftsquigarrow b_h$ falls under Case (2b) of the crystal rules with bold i in cell x_E and $y = x_{ES}$, and so $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 3. Suppose that $\mathbf{T}$ falls under Case (2b) or (2c) of the crystal operator rules. That means x_E contains the entry j or is empty and x_S contains the entry j' or j. There is a chain of letters j' and j in $\mathbf{T}$ starting from x_S and ending on a box y. According to the induction hypothesis, y is either on the diagonal and contains the entry j or y is not on the diagonal and contains the entry j'. The tableau $\mathbf{T}' = f_i(\mathbf{T})$ has j' in cell x and j in cell y. We are interested in the case when inserting b_h into $\mathbf{T}$ affects cell x or affects some element of the chain. Let r_x and c_x be the row and the column index of cell x, and r_y, c_y are defined accordingly. Note that during the insertion process, j' cannot be inserted into columns $c_y, \ldots, c_x$ and j cannot be inserted into rows $r_x+1, \ldots, r_y$, since otherwise $\mathbf{T} \leftsquigarrow b_h$ would not be a shifted primed tableau.

Case 3.1. Suppose the bold i in cell x (of row r_x and column c_x) of $\mathbf{T}$ is row-bumped by an unprimed element $d < i$ or column-bumped by a primed element $d' < i$. Note that in this case, bold i in row r_x is the only i in this row, so row r_x+1 cannot contain any letter j. Therefore the entry in cell x_S must be j'. In tableau $\mathbf{T}$, the bumped bold i is inserted into cell x_S and j' is bumped from cell x_S into column c_x+1, reducing the chain of letters j' and j by one. Notice that since x_E either contains a j or is empty, j' cannot be bumped into a position to the right of x_S, so Case (1) of the crystal rules for $\mathbf{T} \leftsquigarrow b_h$ cannot occur. As for $\mathbf{T}'$, inserting d into row r_x (or inserting d' into column c_x) just bumps j' into column $c_x + 1$, thus reducing the length of the chain by one in that tableau as well. Note that in the case when the length of the chain is one (i.e. $y = x_S$), we would end up in Case (2a) of the crystal rules after the insertion. Otherwise, we are still in Case (2b) or (2c). In both cases, $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 3.2. Suppose a letter i is inserted into the same row as x (in row r_x). In this case, x_E must contain a j (otherwise the bold i would not be in cell x). After inserting b_h into $\mathbf{T}$, the bold i moves to cell x_E (note that there cannot be a j' to the right of x_E) and j from x_E is bumped to cell x_{ES}, thus the chain now starts at x_{ES}. As for $\mathbf{T}'$, inserting i into the row r_x moves j' from cell x to the cell x_E and moves j from cell x_E to cell x_{ES}. Thus, $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 3.3. Consider the chain of letters j and j' in $\mathbf{T}$. Suppose an element of the chain $z \neq x, y$ is row-bumped by an element $d < j$ or is column-bumped by an element $d' < j'$. The bumped element z (of row r_z and column c_z) must be a "corner" element of the chain, i.e. in $\mathbf{T}$ the entry in the boxes must be $c(z) = j'$, $c(z_E) = j$ and $c(z_S)$ must be either j or j'. Therefore, inserting b_h into $\mathbf{T}$ bumps j' from box z to box z_E and bumps j from box z_E to box z_{ES}, and inserting b_h into $\mathbf{T}'$ has exactly the same effect. Thus, there is still a chain of letters j and j' from x_S to y in $\mathbf{T}$ and $\mathbf{T}'$, and $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 3.4. Suppose $\mathbf{T}$ falls under Case (2c) of the crystal rules (i.e. y is not a diagonal cell) and during the insertion of b_h into $\mathbf{T}$, j' in cell y is row-bumped (resp. column-bumped) by an element $d < j'$ (resp. $d' < j'$). Since y is the end of the chain of letters j and j', y_S must be empty. Also, since it is bumped, the entry in y_E must be j. Thus, inserting b_h into $\mathbf{T}$ bumps j' from cell y to cell y_E and bumps j from cell y_E into row $r_y + 1$ and column $\leqslant c_y$. On the other hand, inserting b_h into $\mathbf{T}'$ bumps j from cell y into row $r_y + 1$ and column $\leqslant c_y$. The chain of letters j and j' now ends at y_E and $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 3.5. Suppose $\mathbf{T}$ falls under Case (2b) of the crystal rules (i.e. y with entry j is a diagonal cell) and during the insertion of b_h into $\mathbf{T}$, j in cell y is row-bumped by an element $d < j$. In this case, the cell y_E must contain the entry j. Thus, inserting b_h into $\mathbf{T}$ bumps j from cell y (making it j') to cell y_E and bumps j from cell y_E to the diagonal cell y_{ES}. On the other hand, inserting b_h into $\mathbf{T}'$ has exactly the same effect. The chain of letters j and j' now ends at the diagonal cell y_{ES}, so $\mathbf{T} \leftsquigarrow b_h$ falls under Case (2b) of the crystal rules and $f_i(\mathbf{T} \leftsquigarrow b_h) = \mathbf{T}' \leftsquigarrow b_h$.

Case 4. Suppose the bold i in tableau $\mathbf{T}$ is a primed i. We use the transposition operation on $\mathbf{T}$, and the resulting tableau $\mathbf{T}^*$ falls under one of the cases of the crystal operator rules. When b_h is inserted into $\mathbf{T}$, we can easily translate the insertion process to the transposed tableau $\mathbf{T}^*$ so that $[\mathbf{T}^* \leftsquigarrow (b_h+1)'] = [\mathbf{T} \leftsquigarrow b_h]^*$: the letter $(b_h + 1)'$ is inserted into the first column of $\mathbf{T}^*$, and all other insertion rules stay exactly same, with one exception – when the diagonal element d' is column-bumped from the diagonal cell of $\mathbf{T}^*$, the element d' becomes $(d - 1)$ and is inserted into the row below. Notice that the primed reading word of $\mathbf{T}$ becomes an unprimed reading word of $\mathbf{T}^*$. Thus, the bold i in tableau $\mathbf{T}^*$ corresponds to the rightmost unbracketed i in the *unprimed* reading word of $\mathbf{T}^*$. Therefore, everything we have deduced in Cases 1-3 from the fact that bold i is in the cell x will remain valid here. Given $f_i(\mathbf{T}^*) = \mathbf{T}'^*$, we want to make sure that $f_i(\mathbf{T}^* \leftsquigarrow (b_h + 1)') = \mathbf{T}'^* \leftsquigarrow (b_h + 1)'$.

The insertion process of $(b_h + 1)'$ into $\mathbf{T}^*$ falls under one of the cases above and the proof of $f_i(\mathbf{T}^* \leftsquigarrow (b_h + 1)') = \mathbf{T}'^* \leftsquigarrow (b_h + 1)'$ is exactly the same as the proof in those cases. We only need to check the cases in which the diagonal element might be affected differently in the insertion process of $(b_h + 1)'$ into $\mathbf{T}^*$ compared to the insertion process of $(b_h + 1)'$ into $\mathbf{T}'^*$. Fortunately, this never happens: in Case 1 neither x nor x_E could be diagonal elements; in Cases 2 and 3 x cannot be on the diagonal, and if x_E is on diagonal, it must be empty. Following the proof of those cases, $f_i(\mathbf{T}^* \leftsquigarrow (b_h + 1)') = \mathbf{T}'^* \leftsquigarrow (b_h + 1)'$.

A.2.

This appendix provides the proof of Theorem 3.3.6. In this section we set $j = i + 1$. We begin with two preliminary lemmas.

A.2.1. Preliminaries.

Lemma A.2.1. *Consider a shifted tableau* $\mathbf{T}$.

(1) Suppose tableau $\mathbf{T}$ *falls under Case (2c) of the* f_i *crystal operator rules, that is, there is a chain of letters* j *and* j' *starting from the bold* i *in cell* x *and ending at* j' *in cell* x_H*. Then for any cell* z *of the chain containing* j*, the cell* z_{NW} *contains* i*.*

(2) Suppose tableau $\mathbf{T}$ *falls under Case (2b) of the* f_i *crystal operator rules, that is, there is a chain of letters* j *and* j' *starting from the bold* i *in cell* x *and ending at* j *in the diagonal cell* x_H*. Then for any cell* z *of the chain containing* j *or* j'*, the cell* z_{NW} *contains* i *or* i' *respectively.*

$$
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \boldsymbol{i} \\
 & \cdot & \cdot & \cdot & i & i & i & j' \\
 & & \cdot & \cdot & j' & j & j & j \\
 & & & \cdot & j' & & &
\end{array}
\qquad
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & i' & \boldsymbol{i} \\
 & \cdot & i' & i & i & j' \\
 & & i & j' & j & j \\
 & & & j & &
\end{array}
$$

Proof. The proof of the first part is based on the observation that every j in the chain must be bracketed with some i in the reading word $\mathrm{rw}(\mathbf{T})$. Moreover, if the bold i is located in row r_x and rows $r_x, r_x+1, \ldots, r_z$ contain n letters j, then rows $r_x, r_x+1, \ldots, r_z-1$ must contain exactly n non-bold letters i. To prove that these elements i must be located in the cells to the North-West of the cells containing j, we proceed by induction on n. When we consider the next cell z containing j in the chain that must be bracketed, notice that the columns $c_z, c_z+1, \ldots, c_x$ already contain an i, and thus we must put the next i in column c_z-1; there is no other row to put it than r_z-1. Thus, z_{NW} must contain an i.

This line of logic also works for the second part of the lemma. We can show that for any cell z of the chain containing j, the cell z_{NW} must contain an i. As for cells z containing j', we can again use the fact that the corresponding letters j in the primed reading word of $\mathbf{T}$ must be bracketed. Notice that these letters j' cannot be bracketed with unprimed letters i, since all unprimed letters i are already bracketed with unprimed letters j. Thus, j' must be bracketed with some i' from a column to its left. Let columns $1, 2, \ldots, c_z$ contain m elements j'. Using the same induction argument as in the previous case, we can show that z_{NW} must contain i'. □

Next we need to figure out how y in the raising crystal operator e_i is related to the lowering operator rules for f_i.

Lemma A.2.2. *Consider a pair of tableaux* $\mathbf{T}$ *and* $\mathbf{T}' = f_i(\mathbf{T})$.

(1) *If tableau* $\mathbf{T}$ *(in case when bold i in* $\mathbf{T}$ *is unprimed) or* $\mathbf{T}^*$ *(if bold i is primed) falls under Case (1) of the* f_i *crystal operator rules, then cell y of the* e_i *crystal operator rules is cell* x_E *of* $\mathbf{T}'$ *or* $(\mathbf{T}')^*$*, respectively.*

(2) *If tableau* $\mathbf{T}$ *(in case when bold i in* $\mathbf{T}$ *is unprimed) or* $\mathbf{T}^*$ *(if bold i is primed) falls under Case (2a) of the* f_i *crystal operator rules, then cell y of the* e_i *crystal operator rules is located in cell x of* $\mathbf{T}'$ *or* $(\mathbf{T}')^*$*, respectively.*

(3) *If tableau* $\mathbf{T}$ *falls under Case (2b) of the* f_i *crystal operator rules, then cell y of the* e_i *crystal operator rules is cell* x^* *of* $(\mathbf{T}')^*$.

(4) *If tableau* $\mathbf{T}$ *(in case when bold i in* $\mathbf{T}$ *is unprimed) or* $\mathbf{T}^*$ *(if bold i is primed) falls under Case (2c) of the* f_i *crystal operator rules, then cell y of the* e_i *crystal operator rules is cell* x_H *of* $\mathbf{T}'$ *or* $(\mathbf{T}')^*$*, respectively.*

Proof. In all the cases above, we need to compare reading words rw($\mathbf{T}$) and rw($\mathbf{T}'$). Since f_i affects at most two boxes of $\mathbf{T}$, it is easy to track how the reading word rw($\mathbf{T}$) changes after applying f_i. We want to check where the bold j under e_i ends up in rw($\mathbf{T}'$) and in $\mathbf{T}'$, which allows us to determine the cell y of the e_i crystal operator rules.

Case 1.1. Suppose $\mathbf{T}$ falls under Case (1) of the f_i crystal operator rules, that is, the bold i in cell x is to the left of j' in cell x_E. Furthermore, f_i acts on $\mathbf{T}$ by changing the entry in x to j' and by changing the entry in x_E to j. In the reading word rw($\mathbf{T}$), this corresponds to moving the j corresponding to x_E to the left and changing the bold i (the rightmost unbracketed i) corresponding to cell x to j (that then corresponds to x_E). Moving a bracketed j in rw($\mathbf{T}$) to the left does not change the $\{i, j\}$ bracketing, and thus the j corresponding to x_E in rw($\mathbf{T}'$) is still the leftmost unbracketed j. Therefore, this j is the bold j of $\mathbf{T}'$ and is located in cell x_E.

Case 1.2. Suppose the bold i in $\mathbf{T}$ is primed and $\mathbf{T}^*$ falls under Case (1) of the f_i crystal operator rules. After applying lowering crystal operator rules to $\mathbf{T}^*$ and conjugating back, the bold primed i in cell x^* of $\mathbf{T}$ changes to an unprimed i, and the unprimed i in cell $(x^*)_S$ of $\mathbf{T}$ changes to j'. In terms of the reading word of $\mathbf{T}$, it means moving the bracketed i (in the unprimed reading word) corresponding to $(x^*)_S$ to the left so that it corresponds to x^*, and then changing the bold i (in the primed reading word) corresponding to x^* into the letter j corresponding to $(x^*)_S$. The first operation does not change the bracketing relations between i and j, and thus the leftmost unbracketed j in rw($\mathbf{T}'$) corresponds to $(x^*)_S$. Hence the bold unprimed j is in cell x_E of $(\mathbf{T}')^*$.

Case 2.1. If $\mathbf{T}$ falls under Case (2a) of the f_i crystal operator rules, f_i just changes the entry in x from i to j. The rightmost unbracketed i in the reading word of $\mathbf{T}$ changes to the leftmost unbracketed j in rw($\mathbf{T}'$). Thus, the bold j in rw($\mathbf{T}'$) corresponds to cell x.

Case 2.2. The case when $\mathbf{T}^*$ falls under Case (2a) of the f_i crystal operator rules is the same as the previous case.

Case 3. Suppose $\mathbf{T}$ falls under Case (2b) of f_i crystal operator rules. Then there is a chain starting from cell x (of row r_x and column c_x) and ending at the diagonal cell z (of row and column r_z) consisting of elements j and j'. Applying f_i to $\mathbf{T}$ changes the entry in x from i to j'. In $\mathrm{rw}(\mathbf{T})$ this implies moving the bold i from the unprimed reading word to the left through elements i and j corresponding to rows $r_x, r_x + 1, \dots, r_z$, then through elements i and j in the primed reading word corresponding to columns $c_z - 1, \dots, c_x$, and then changing that i to j which corresponds to cell x. But according to Lemma A.2.1, the letters i and j in these rows and columns are all bracketed with each other, since for every j or j' in the chain there is a corresponding i or i' in the North-Western cell. (Notice that there cannot be any other letter j or j' outside of the chain in rows $r_x + 1, \dots, r_z$ and in columns $c_z - 1, \dots, c_x$.) Thus, moving the bold i to the left in $\mathrm{rw}(\mathbf{T})$ does not change the bracketing relations. Changing it to j makes it the leftmost unbracketed j in $\mathrm{rw}(\mathbf{T}')$. Therefore, the bold j in $\mathrm{rw}(\mathbf{T}')$ corresponds to the primed j in cell x of $\mathbf{T}'$, and the cell y of the e_i crystal operator rules is thus cell x^* in $(\mathbf{T}')^*$.

Case 4.1. Suppose $\mathbf{T}$ falls under Case (2c) of the f_i crystal operator rules. There is a chain starting from cell x (in row r_x and column c_x) and ending at cell x_H (in row r_H and column c_H) consisting of elements j and j'. Applying f_i to $\mathbf{T}$ changes the entry in x from i to j' and changes the entry in x_H from j' to j. Moving j' from cell x_H to cell x moves the corresponding bracketed j in the reading word $\mathrm{rw}(\mathbf{T})$ to the left, and thus does not change the $\{i, j\}$ bracketing relations in $\mathrm{rw}(\mathbf{T}')$. On the other hand, moving the bold i from cell x to cell x_H and then changing it to j moves the bold i in $\mathrm{rw}(\mathbf{T})$ to the right through elements i and j corresponding to rows $r_x, r_x + 1, \dots, r_H$, and then changes it to j. Note that according to Lemma A.2.1, each j in rows $r_x + 1, r_x + 2, \dots, r_H$ has a corresponding i from rows $r_x, r_x + 1, \dots, r_H - 1$ that it is bracketed with, and vise versa. Thus, moving the bold i to the position corresponding to x_H does not change the fact that it is the rightmost unbracketed i in $\mathrm{rw}(\mathbf{T})$. Thus, the bold j in $\mathrm{rw}(\mathbf{T}')$ corresponds to the unprimed j in cell x_H of $\mathbf{T}'$.

Case 4.2. Suppose $\mathbf{T}$ has a primed bold i and $\mathbf{T}^*$ falls under Case (2c) of the f_i crystal operator rules. This means that there is a chain (expanding in North and East directions) in $\mathbf{T}$ starting from i' in cell x^* and ending in cell x_H^* with entry i consisting of elements i and j'. The crystal operator f_i changes the entry in cell x^* from i' to i and changes the entry in x_H^* from i to j'. For the reading word $\mathrm{rw}(\mathbf{T})$ this means moving the bracketed i in the unprimed reading word to the right (which does not change the bracketing relations) and moving the bold i in the primed reading word through letters i and j corresponding to columns $c_x, c_x + 1, \dots, c_H$, which are bracketed with each other according to Lemma A.2.1. Thus, after changing the bold i to j makes it the leftmost unbracketed j in $\mathrm{rw}(\mathbf{T}')$. Hence the bold primed j in $\mathbf{T}'$ corresponds to cell x_H^*. Therefore y from the e_i crystal operator rules is cell x_H of $(\mathbf{T}')^*$. $\square$

A.2.2. Proof of Theorem 3.3.6. Let $\mathbf{T}' = f_i(\mathbf{T})$.

Case 1. If $\mathbf{T}$ (or $\mathbf{T}^*$) falls under Case (1) of the f_i crystal operator rules, then according to Lemma A.2.2, e_i acts on $\mathbf{T}'$ (or on $(\mathbf{T}')^*$) by changing the entry in cell $y_W = x$ back to i and changing the entry in $y = x_E$ back to j'. Thus, the statement of the theorem is true.

Case 2. If $\mathbf{T}$ (or $\mathbf{T}^*$) falls under Case (2a) of the f_i crystal operator rules, then according to Lemma A.2.2, e_i acts on $\mathbf{T}'$ (or on $(\mathbf{T}')^*$) by changing the entry in the cell $y = x$ back to i. Thus, the statement of the theorem is true.

Case 3. If $\mathbf{T}$ falls under Case (2b) of the f_i crystal operator rules, then according to Lemma A.2.2, e_i acts on cell $y = x^*$ of $(\mathbf{T}')^*$. Note that according to Lemma A.2.1, there is a maximal chain of letters i and j' in $(\mathbf{T}')^*$ starting at y and ending at a diagonal cell y_T. Thus, e_i changes the entry in cell $y = x^*$ in $(\mathbf{T}')^*$ from j to j', so the entry in cell x in $\mathbf{T}'$ goes back from j' to i. Thus, the statement of the theorem is true.

Case 4. If $\mathbf{T}$ (or $\mathbf{T}^*$) falls under Case (2c) of the f_i crystal operator rules, then according to Lemma A.2.2, e_i acts on cell $y = x_H$ of $\mathbf{T}'$ (or of $(\mathbf{T}')^*$). Note that according to Lemma A.2.1, there is a maximal (since $c(x_E) \neq j'$ and $c(x_E) \neq i$) chain of letters i and j' in $\mathbf{T}'$ (or $(\mathbf{T}')^*$) starting at y and ending at cell $y_T = x$. Thus, e_i changes the entry in cell $y = x_H$ in $(\mathbf{T}')^*$ from j back to j' and changes the entry in $y_T = x$ from j' back to i. Thus, the statement of the theorem is true.

APPENDIX B

Appendix 2: Proofs for characterization of queer crystals

B.1.

In this appendix we prove Theorem 5.5.1

Proof. By Proposition 5.3.2 and Theorem 5.4.5, $\mathcal{D}$ satisfies the local queer axioms and the connectivity axioms and hence all conditions of the theorem.

By **LQ1** of the local queer axioms of Definition 5.3.1, each type A_n-component of C is a Stembridge crystal and hence is uniquely characterized by [**Ste03**]. By assumption $G(C) \cong G(\mathcal{D})$. In particular, the vertices of $G(C)$ and $G(\mathcal{D})$ agree. This proves that C and $\mathcal{D}$ are isomorphic as A_n crystals.

Next we show that all (-1)-arrows also agree on C and $\mathcal{D}$. As discussed just before Lemma 5.3.3, given the local queer axioms of Definition 5.3.1, it suffices to show that f_{-1} acts in the same way in C and $\mathcal{D}$ on the almost lowest elements satisfying (5.1) or equivalently by Lemma 5.3.3 on every $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k \leqslant n$. For the remainder of this proof, fix $g_{j,k} \neq 0$ in the I_0-component u.

Let us first assume that $G(C)$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ for some $h < k$. If $h < j \leqslant k$, then $f_{-1}g_{j,k}$ is determined by **C1** of Definition 5.4.3. If $j \leqslant h$, pick $h < j' \leqslant k$ such that $g_{j',k} \neq 0$. Such a j' must exist since there is an edge $u \to u'$ in $G(C)$. By **C1**, we have $\varphi_{-1}(g_{j',k}) = 1$ and hence by Lemma 5.5.2 also $\varphi_{-1}(g_{j,k}) = 1$. Hence $f_{-1}g_{j,k}$ is determined by **C2**(a).

Next assume that $G(C)$ does not contain an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$.

Claim: If $g_{k,k} \neq 0$, then $f_{-1}g_{j,k} = 0$.

Proof. Suppose $f_{-1}g_{k,k} \neq 0$. By **C2**(b), we have $f_{-1}g_{k,k} = (e_2 \cdots e_k)(e_1 \cdots e_k)v = f_1 g_{k,k}$. But this contradicts the local queer axioms of Definition 5.3.1 since $\varphi_1(g_{k,k}) > 1$. Hence $\varphi_{-1}(g_{k,k}) = 0$ and by Lemma 5.5.2 also $\varphi_{-1}(g_{j,k}) = 0$, which proves the claim. □

If $g_{k,k} = 0$, we have $j < k$ since by assumption $g_{j,k} \neq 0$.

Claim: Suppose $g_{k,k} = 0$.

(1) Suppose there is an edge $\overline{u} \to u$ in $G(C)$ such that $\mathrm{wt}(u)$ is obtained from $\mathrm{wt}(\overline{u})$ by moving a box from row $n+1-\overline{k}$ to row $n+1-\overline{h}$ such that $\overline{h} < k \leqslant \overline{k}$. Then $f_{-1}g_{j,k} = 0$.

(2) Suppose $G(C)$ does not contain an edge as in (1). Then $f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v$.

Proof. Suppose that the conditions in (1) are satisfied. Then by **C1** there must exist

$$\overline{g}_{\overline{j},\overline{k}} := (e_1 \cdots e_{\overline{j}})(e_1 \cdots e_{\overline{k}})\overline{v} \neq 0,$$

where $\overline{h} < \overline{j} \leqslant \overline{k}$ and $\overline{v}$ is the I_0-lowest weight element in the component of $\overline{u}$, such that

$$f_{-1}\overline{g}_{\overline{j},\overline{k}} = (e_2 \cdots e_{\overline{j}})(e_1 \cdots e_{\overline{h}})v. \tag{B.1}$$

Since $g_{j,k} \neq 0$, we have in particular that $(e_1 \cdots e_k)v \neq 0$. Since $\mathrm{wt}(u)$ is obtained from $\mathrm{wt}(\overline{u})$ by moving a box from row $n+1-\overline{k}$ to row $n+1-\overline{h}$, this hence also implies that $\overline{g}_{k,\overline{k}} = (e_1 \cdots e_k)(e_1 \cdots e_{\overline{k}})\overline{v} \neq 0$. Hence by **C1** Equation (B.1) holds for $\overline{j} = k$.

If $f_{-1}g_{\overline{h},k} = 0$, we also have $f_{-1}g_{j,k} = 0$ by Lemma 5.5.2 as claimed. Hence we may assume that $f_{-1}g_{\overline{h},k} \neq 0$. Then by **C2**(b) we have

$$f_{-1}g_{\overline{h},k} = (e_2 \cdots e_k)(e_1 \cdots e_{\overline{h}})v.$$

But then $f_{-1}\overline{g}_{k,\overline{k}} = f_{-1}g_{\overline{h},k} = (e_2 \cdots e_k)(e_1 \cdots e_{\overline{h}})v$, which contradicts the fact that the crystal operator f_{-1} has a partial inverse since $\overline{g}_{k,\overline{k}} \neq g_{\overline{h},k}$. This proves (1).

Now suppose that the conditions in (2) are satisfied. Recall that by assumption $g_{j,k} \neq 0$ with $j < k$. This implies that $y := (e_2 \cdots e_k)(e_1 \cdots e_j)v \neq 0$, $\varphi_i(y) = 0$ for $i \in I_0 \setminus \{2\}$ and $\varphi_2(y) = 1$. By the local queer axioms of Definition 5.3.1, this implies that $x := e_{-1}y \neq 0$ with $\varphi_1(x) \in \{1,2\}$ and $\varphi_i(x) = 0$ for $i \in I_0 \setminus \{1\}$. Thus we may write $x = (e_1 \cdots e_s)(e_1 \cdots e_t)\overline{v}$, where $0 \leqslant s \leqslant t$ and $\overline{v} \in C$ is some I_0-lowest weight vector. This yields the equality

$$f_{-1}(e_1 \cdots e_s)(e_1 \cdots e_t)\overline{v} = (e_2 \cdots e_k)(e_1 \cdots e_j)v.$$

If $\overline{v} \neq v$, then by the connectivity axioms of Definition 5.4.3 this means that $j < k = s \leqslant t$ and there is an edge in $G(C)$ from $\uparrow \overline{v}$ to $u = \uparrow v$, moving a box from row $n+1-t$ to row $n+1-j$. This contradicts the

$$\begin{tikzcd}
1 \otimes 1 \otimes 1 \otimes 1 \arrow[d] \\
2 \otimes 1 \otimes 1 \otimes 1 \arrow[d] \\
3 \otimes 2 \otimes 1 \otimes 1 \arrow[d] \\
4 \otimes 3 \otimes 2 \otimes 1
\end{tikzcd}$$

FIGURE B.1. The graph $G(C)$ for the example in Remark B.1.1.

assumptions of (2). Hence we must have $\overline{v} = v$. By **C2**(b) we have $f_{-1}g_{s,t} = (e_2 \cdots e_t)(e_1 \cdots e_s)v$, so that $k = t$ and $j = s$. This implies $f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v$, proving the claim. $\square$

We have now shown that $f_{-1}g_{j,k}$ is determined in all cases, which proves the theorem. $\square$

REMARK B.1.1. Consider the $\mathfrak{q}(4)$-queer crystal $\mathcal{B}^{\otimes 4}$. The elements 4114 and 4113 both lie in the same $\{1, 2, 3\}$-component of highest weight $(3, 1)$. The highest (resp. lowest) weight element in this component is $u = 2111$ (resp. $v = 4344$). Both 4114 and 4113 satisfy (5.1). In fact, $4114 = (e_1e_2)(e_1e_2e_3)v = g_{2,3}$ and $4113 = (e_1e_2e_3)(e_1e_2e_3)v = g_{3,3}$. In the component of u there is no sequence of crystal operators that would induce the action of f_{-1} on 4114 from the action of f_{-1} on 4113 using the local queer axioms of Definition 5.3.1.

This suggests that the connectivity axioms of Definition 5.4.3 are indeed necessary. However, in this example the graph $G(C)$, where C is the connected component in $\mathcal{B}^{\otimes 4}$ containing 2111, is linear and hence forces 4114 and 4113 to be mapped to the same $\{1, 2, 3\}$-component by f_{-1}, see Figure B.1.

REMARK B.1.2. Consider the connected component C of 111212121 in the $\mathfrak{q}(6)$-queer crystal $\mathcal{B}^{\otimes 9}$. The $\{1, 2, 3, 4, 5\}$-component containing 321312121 is connected to the components 421312121, 431312121, and 432312121 in $G(C)$. The elements $g_{4,5} = 651615464$ and $g_{3,5} = 651615465$ in the component of 321312121 are mapped to the same component 432312121 by **C1** of Definition 5.4.3. However, the element $g_{4,5}$ is connected to 431413131 in the crystal using only arrows that commute with f_{-1} and the element $g_{3,5}$ is connected to 431413143 in the crystal using only arrows that commute with f_{-1}. However, these two components (containing 431413131 resp. 431413143 using only crystal operators f_i and e_i with $i \in I_0$ that commute with f_{-1}) are disjoint. This suggests that **C1** of Definition 5.4.3 is necessary for uniqueness.

B.2.

In this appendix we prove Theorem 5.4.5.

We use the shorthand notation $e_1^k := e_1 \cdots e_k$, $e_{\bar{1}}^k := e_{-1}e_2 \cdots e_k$, $f_k^1 := f_k \cdots f_1$, and $f_k^{\bar{1}} := f_k \cdots f_2 f_{-1}$.

Lemma B.2.1. *In $\mathcal{B}^{\otimes \ell}$, condition **C0** of Definition 5.4.3 holds.*

Proof. This follows from Remark 5.4.4. □

The connectivity axioms **C1** and **C2** of Definition 5.4.3 are implied by the following conditions. Here v is an I_0-lowest weight vector in C:

C1'. If $h < k$ and there exists some $j \in (h, k]$ such that $f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v)$ is I_0-lowest weight, then for any $j' \in (h, k]$ with $e_1^{j'} e_1^k(v) \neq 0$ we have $f_{j'}^{\bar{1}} e_1^{j'} e_1^k(v) = f_j^{\bar{1}} e_1^j e_1^k(v)$.

C2'. If $j \leqslant k$ and $f_{-1} e_1^j e_1^k(v) \neq 0$, then either:

(a) $j \neq k$ and $f_j^1 f_k^{\bar{1}} e_1^j e_1^k(v) = v$, or

(b) $f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v)$ is I_0-lowest weight for some $h < j$.

Proposition B.2.2. *In $\mathcal{B}^{\otimes \ell}$, condition **C2'** holds.*

The proof of Proposition B.2.2 is given in Section B.2.1.

Proposition B.2.3. *In $\mathcal{B}^{\otimes \ell}$, condition **C1'** holds.*

We will prove a seemingly weaker statement:

Lemma B.2.4. *In $\mathcal{B}^{\otimes \ell}$, condition **C1'** holds for $j = n - 1$, $j' = k = n$ and for $j = k = n$, $j' = n - 1$.*

The proof of Lemma B.2.4 is given in Sections B.2.2 and B.2.3.

Proposition B.2.5. *Lemma B.2.4 implies Proposition B.2.3.*

Proof. We first assume that $h < j < j' \leqslant k$ and the assumptions in **C1'** hold. Then we have

$$\begin{aligned} f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v) &= f_h^1 f_j^{\bar{1}} (f_{j'} \cdots f_{j+2})(e_{j+2} \cdots e_{j'}) e_1^j e_1^k(v) \\ &= (f_{j'} \cdots f_{j+2}) f_h^1 f_j^{\bar{1}} e_1^j (e_{j+2} \cdots e_{j'}) e_1^k(v) \\ &= (f_{j'} \cdots f_{j+2}) f_h^1 f_j^{\bar{1}} e_1^j e_1^{j+1}(v'), \end{aligned}$$

where $v' = (e_{j+2} \cdots e_{j'})(e_{j+2} \cdots e_k)(v)$. Here we have used Stembridge relations to commute crystal operators and in the last step also that the operators are acting on an I_0-lowest weight element. Note that v' is $\{1, \dots, j+1\}$-lowest weight. Moreover, $f_h^1 f_j^{\bar{1}} e_1^j e_1^{j+1}(v')$ is $\{1, \dots, j+1\}$-lowest weight. Since $e_1^{j+1} e_1^{j+1}(v') = e_1^{j'} e_1^k(v) \neq 0$, we may apply Lemma B.2.4 with $n = j+1$. This implies

$$\begin{aligned} (f_{j'} \cdots f_{j+2}) f_h^1 f_j^{\bar{1}} e_1^j e_1^{j+1}(v') &= (f_{j'} \cdots f_{j+2}) f_h^1 f_{j+1}^{\bar{1}} e_1^{j+1} e_1^{j+1}(v') \\ &= f_h^1 f_{j'}^{\bar{1}} e_1^{j+1} e_1^{j+1} (e_{j+2} \cdots e_{j'})(e_{j+2} \cdots e_k)(v) \\ &= f_h^1 f_{j'}^{\bar{1}} e_1^{j'} e_1^k(v), \end{aligned}$$

which proves the claim.

Next assume that $h < j' < j \leqslant k$. Then

$$f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v) = f_h^1 f_j^{\bar{1}} e_1^{j'+1} e_1^{j'+1} (e_{j'+2} \cdots e_j)(e_{j'+2} \cdots e_k)(v) = (f_j \cdots f_{j'+2}) f_h^1 f_{j'+1}^{\bar{1}} e_1^{j'+1} e_1^{j'+1}(v'),$$

where $v' = (e_{j'+2} \cdots e_j)(e_{j'+2} \cdots e_k)(v)$. In this case, both v' and $f_h^1 f_{j'+1}^{\bar{1}} e_1^{j'+1} e_1^{j'+1}(v')$ are $\{1, \dots, j'+1\}$-lowest weight. Since $e_1^{j'} e_1^{j'+1}(v') \neq 0$, we may apply Lemma B.2.4 with $n = j'+1$ to obtain

$$f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v) = (f_j \cdots f_{j'+2}) f_h^1 f_{j'}^{\bar{1}} e_1^{j'} e_1^{j'+1}(v') = f_h^1 f_{j'}^{\bar{1}} e_1^{j'} e_1^k(v),$$

proving the claim. □

B.2.1. Proof of Proposition B.2.2. Given a word $w = w_1 \cdots w_\ell$ in the letters $\{1, \dots, n+1\}$ we write $w^\# = \overline{w_\ell} \cdots \overline{w_1}$, where $\overline{w_i} = n + 2 - w_i$. Suppose that $x = g_{j,k} = e_1^j e_1^k(v) \in \mathcal{B}^{\otimes \ell}$, where v is I_0-lowest weight and $1 \leqslant j \leqslant k \leqslant n$, so that by Lemma 5.3.3 we have $\varphi_1(x) = 2$ and $\varphi_i(x) = 0$ for all $i > 1$. The RSK insertion tableau for $x^\#$, denoted by $P(x^\#)$, can be constructed as follows: Construct the semistandard Young tableau with weight and shape equal to the weight of $v^\#$. Change the rightmost $n+1-k$ in row $n+1-k$ and the rightmost $n+1-j$ in row $n+1-j$ to $n+1$.

For instance, suppose $n = 8$ and $x = 198199887766$. Then $x = e_1^6 e_1^8(v)$, where $v = 998799887766$ is I_0-lowest weight and $v^\# = 443322113211$ has weight $(4,3,3,2)$. Hence the tableau $P(x^\#)$ is obtained from the tableau of shape and weight equal to $(4,3,3,2)$ by changing the rightmost 1 in row 1 to 9 and the rightmost 3 in row 3 to 9:

$$\begin{array}{|c|c|cc}\hline 4 & 4 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & \multicolumn{1}{c|}{3} & \\ \hline 2 & 2 & \multicolumn{1}{c|}{2} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|cc}\hline 4 & 4 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & \multicolumn{1}{c|}{9} & \\ \hline 2 & 2 & \multicolumn{1}{c|}{2} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{9} \\ \hline \end{array} \; .$$

Below, we consider the entries of a tableau to be linearly ordered in the row reading order. If $f_{-1}(x) \neq 0$ there are two possibilities:

(1) The recording tableau of $x^\#$ is the same as the recording tableau of $(f_{-1}(x))^\#$. This implies that during the insertion of $x^\#$, the final two $(n+1)$'s to be inserted are at no point in the same row. (Note that this is clearly impossible if $j = k$.) This means, that after the insertion of the final two $(n+1)$'s, the rightmost $n+1$ is never inserted into another row containing an $n+1$, and, moreover, there is never an n being inserted into the row containing the rightmost $n+1$ (since after the insertion of the final two $(n+1)$'s, the rightmost n or $n+1$ is always $n+1$). In this case, $P((f_{-1}(x)^\#)$ is obtained from $P(x^\#)$ by changing the $n+1$ in row $n+1-k$ into an n. Since $x^\#$ and $(f_{-1}(x))^\#$ have the same recording tableau, x and $f_{-1}(x)$ are in the same connected component. Since it is evident from $P((f_{-1}(x)^\#)$ that $f_j^1 f_k \cdots f_2(f_{-1}(x))$ must be I_0-lowest weight, it follows that $v = f_j^1 f_k^{\bar{1}} e_1^j e_1^k(v)$. This is precisely what happens in the example above; $P((f_{-1}(x)^\#)$ is obtained from $P(x^\#)$ by:

$$\begin{array}{|c|c|cc}\hline 4 & 4 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & \multicolumn{1}{c|}{9} & \\ \hline 2 & 2 & \multicolumn{1}{c|}{2} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{9} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|cc}\hline 4 & 4 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & \multicolumn{1}{c|}{9} & \\ \hline 2 & 2 & \multicolumn{1}{c|}{2} & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{8} \\ \hline \end{array} \; .$$

Hence **C2'**(a) holds.

(2) The recording tableau of $x^\#$ differs from the recording tableau of $(f_{-1}(x))^\#$. This implies that during the insertion of $x^\#$, there is some point at which the final two $(n+1)$'s to be inserted are in the same row. Call this row r and suppose that this occurs during the insertion of the i^{th} letter of $x^\#$. Let P_i be the tableau obtained from inserting the first i letters of $x^\#$ and let P'_i be the tableau obtained from inserting the first i letters of $(f_{-1}(x))^\#$. Then P'_i is obtained from P_i by changing the second to rightmost $n+1$ to n and moving the rightmost $n+1$ from row r to some row $s > r$.

Now continue with the insertion of the $(i+1)^{st}$ letter in each case. Since the $(n, n+1)$-subword of $x^{\#}$ ends with two $(n+1)$'s, and these are the only $(n, n+1)$-unbracketed $(n+1)$'s in this subword, the same is true of the $(n, n+1)$-subword of each of $P_i, P_{i+1}, \ldots, P_\ell$. This implies that at no point in the rest of the insertion of $x^{\#}$ is the second to rightmost $n+1$ inserted into a row containing another $n+1$, and moreover at no point is an n inserted into the row containing the second to rightmost $n+1$ (since after the insertion of the final two $(n+1)$'s, the two rightmost entries which are either n or $n+1$ must both be $n+1$).

It follows that, if we ignore, the rightmost $n+1$ in $P((f_{-1}(x)^{\#})$ and $P(x^{\#})$, then they have the same shape, and the second differs from the first only by changing its rightmost n to $n+1$. Adding back the rightmost $n+1$ to $P(x^{\#})$, we see that it must go somewhere to the right of this position (by definition), and adding back the rightmost $n+1$ to $P(f_{-1}(x^{\#}))$, we see that it must go somewhere to the left of this position (otherwise $P((f_{-1}(x)^{\#})$ would have an $(n, n+1)$-unbracketed $n+1$.)

It follows that $P((f_{-1}(x)^{\#})$ is obtained from $P(x^{\#})$ by eliminating the (rightmost) $n+1$ in row $n-k+1$, changing the (leftmost) $n+1$ in row $n-j+1$ to n and adding an $n+1$ to some row $n-h+1$ for $h < j$. It follows that $v' = f_h^1 f_j^{\bar{1}} e_1^j e_1^k(v)$ and v are both (distinct) I_0-lowest weight elements. Hence **C2'**(b) holds.

To see an example of the second case, let $v = 99889$. Then $v^{\#} = 12211$, $(e_1^7 e_1^8(v))^{\#} = 29911$, $(f_{-1}e_1^7 e_1^8(v))^{\#} = 29811$, and $(f_6^1 f_7^{\bar{1}} e_1^7 e_1^8(v))^{\#} = 23211$ have the following insertion tableaux:

$$\begin{array}{|c|c|c}\cline{1-2} 2 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c}\cline{1-2} 2 & 9 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{9} \\ \hline\end{array} \longrightarrow \begin{array}{|c|c}\cline{1-1} 9 & \\ \hline 2 & \multicolumn{1}{c|}{8} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \longrightarrow \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}.$$

B.2.2. Proof of Lemma B.2.4 for $j = n-1$ and $j' = n$. Define $X = (e_1 \cdots e_n)v$. For $1 \leqslant i \leqslant n+1$, set $A_i = (e_i \cdots e_n)X$ and $B_i = (e_i \cdots e_{n-1})X$. For $2 \leqslant i \leqslant n+1$, set $A_{-i} = (f_{(i-1)} \cdots f_2 f_{-1})A_1$ and $B_{-i} = (f_{(i-1)} \cdots f_2 f_{-1})B_1$. (So $A_1 = A_{-1}$ and $B_1 = B_{-1}$. Moreover, $B_{n+1} = B_n$.) By assumption $(f_h \cdots f_1)(B_{-n})$ is I_0-lowest weight, so $f_n(f_h \cdots f_1)(B_{-n}) = 0$ and hence $B_{-(n+1)} = 0$.

Let x_i be the integer which represents the position where A_{i+1} and A_i differ, and y_i be the integer which represents the position where B_{i+1} and B_i differ. Also, let x_{-i} be the integer which represents the position where A_{-i} and $A_{-(i+1)}$ differ, and let y_{-i} be the integer which represents the position where B_{-i} and $B_{-(i+1)}$ differ. Note that y_n and y_{-n} are undefined.

Recall that $v \in \mathcal{B}^{\otimes \ell}$. Suppose W is any word of length ℓ in the letters $\{1, \ldots, n+1\}$. If $1 \leqslant p \leqslant \ell$, we define $W(p)$ to be the p^{th} entry of W. If $1 \leqslant p \leqslant q \leqslant \ell$ are integers, then the notation $W(p : q)$ will be used to refer to the word $W(p)W(p+1) \ldots W(q-1)W(q)$.

If $1 \leqslant i \leqslant n$, we define the $i/(i+1)$-subword of W to be the word composed of the symbols $\{i, i+1, \_\}$ which is obtained from W by changing each entry that is neither i nor $i+1$ to the symbol $\_$. For instance the 2/3-subword of 241432143 is $2\_\_\_32\_\_3$. When we speak of erasing an i or $i+1$, we mean changing that entry to $\_$; similarly, when we speak of adding an i or $i+1$, we mean changing some $\_$ to i or $i+1$. Moving an i or $i+1$ from p to q means erasing an i or $i+1$ from position p and adding an i or $i+1$ to position q. The notation $W(p : q)$ is used in the same way for subwords as it is for words. For instance, if W=$3\_\_\_32\_\_3$ then $W(3 : 7) = \_\_32\_$.

Claim B.2.6. For $2 \leqslant i \leqslant n$, we have $x_i \geqslant x_{i-1}$. For $2 \leqslant i \leqslant n-1$, we have $y_i \geqslant y_{i-1}$.

Proof. If $x_i < x_{i-1}$, then it follows that $f_i A_{i-1} \neq 0$. But this is the statement that

$$f_i(e_{i-1}e_i \cdots e_n)(e_1 \cdots e_n)v \neq 0$$

for some integer $2 \leqslant i \leqslant n$, which is absurd since v is I_0-lowest weight. If $y_i < y_{i-1}$, then it follows that $f_i B_{i-1} \neq 0$. But this is the statement that

$$f_i(e_{i-1}e_i \cdots e_{n-1})(e_1 \cdots e_n)v \neq 0$$

for some integer $2 \leqslant i \leqslant n-1$, which is also absurd. □

Claim B.2.7. We have $x_1 > x_{-1}$ and $y_1 > y_{-1}$. (In particular, $f_{-1}(A_1) \neq 0$, so x_{-1} is well-defined.)

Proof. By the definition of the operator f_{-1} we have $y_1 \geqslant y_{-1}$. Since v and $v^* := f_h^1 f_{n-1}^{\bar{1}} e_1^{n-1} e_1^n v$ are both I_0-lowest weight and have different weights, we cannot have $y_1 = y_{-1}$. Thus $y_1 > y_{-1}$. Now $B_n(1 : y_{-1}) = B_1(1 : y_{-1})$. Therefore, there are no 1's or 2's in $B_n(1 : y_{-1} - 1)$ and we have $B_n(y_{-1}) = 1$ since these statements must be true of B_1. If $x_1 > y_{-1}$, then $A_1(1 : y_{-1}) = B_1(1 : y_{-1})$ and so $A_{-2} \neq 0$ with $x_{-1} = y_{-1}$. If $x_1 < y_{-1}$, then $A_1(1 : x_1 - 1) = B_n(1 : x_1 - 1)$ contains no 1's or 2's and $A_1(x_1) = 1$. Thus $A_{-2} \neq 0$ with $x_{-1} = x_1$. It is clearly impossible for $x_1 = y_{-1}$. Therefore, we have established that $A_{-2} = f_{-1}(A_1) \neq 0$. In the notation of Proposition B.2.2, we have for $j = k = n$, that $f_{-1}e_1^j e_1^k(v) \neq 0$. Hence we must be in case **C2'**(b) from which we deduce that $f_{-1}(A_1)$ lies in a different I_0-connected component than A_1. From this it follows that $x_1 > x_{-1}$. □

Claim B.2.8. For $2 \leqslant i \leqslant n$, we have $x_{-(i-1)} \leqslant x_{-i}$. For $2 \leqslant i \leqslant n$, we have $y_{-(i-1)} \leqslant y_{-i}$. (In particular, $A_{-3}, \ldots, A_{-(n+1)}$ are nonzero, so $x_{-2}, \ldots, x_{-n}$ are well-defined.)

Proof. Again, case **C2'**(b) applies to $f_{-1}(A_1)$ and so the parenthetical statement is immediate. First, it is clear from the definitions of the f_{-1} and f_2 operators that $x_{-1} \leqslant x_{-2}$ and that $y_{-1} \leqslant y_{-2}$. If $x_{-(i-1)} > x_{-i}$ for $i > 2$, then it follows that $f_i A_{-(i-1)} \neq 0$. But this is the statement that $f_i(e_{i-1}e_i \cdots e_n)(e_1 \cdots e_g)\hat{v} \neq 0$ for some I_0-lowest weight element $\hat{v}$ and integers $3 \leqslant i \leqslant n$ and $0 \leqslant g < n$ which is absurd. If $y_{-(i-1)} > y_{-i}$ for $i > 2$, then it follows that $f_i(B_{-(i-1)}) \neq 0$. But this is the statement that $f_i(e_{i-1}e_i \cdots e_{n-1})(e_1 \cdots e_g)v^* \neq 0$ for some integers $3 \leqslant i \leqslant n$ and $0 \leqslant g < n$ which is equally absurd. □

So far, we have the following situation:

$$x_n \geqslant \cdots \geqslant x_2 \geqslant x_1 > x_{-1} \leqslant x_{-2} \leqslant \cdots \leqslant x_{-n} \qquad \text{and}$$

$$y_{n-1} \geqslant \cdots \geqslant y_2 \geqslant y_1 > y_{-1} \leqslant y_{-2} \leqslant \cdots \leqslant y_{-(n-1)}.$$

Claim B.2.9. We have $x_{-1} = y_{-1}$.

Proof. Since $x_1 = y_{-1}$ is impossible and since $x_1 < y_{-1}$ would imply that $x_{-1} = x_1$, which contradicts $x_1 > x_{-1}$, we may assume $x_1 > y_{-1}$. However, in this case we have $A_1(1 : y_{-1}) = B_1(1 : y_{-1})$. Since f_{-1} acts on B_1 in position y_{-1}, it follows that f_{-1} acts on A_1 in position y_{-1} as well. This implies $x_{-1} = y_{-1}$. □

Claim B.2.10. For $1 \leqslant i \leqslant n-1$, we have $x_i \leqslant y_i$.

Proof. First we show that $x_{n-1} \leqslant y_{n-1}$. Now y_{n-1} represents the position of the leftmost $(n-1, n)$-unbracketed n in B_n. This n is also unbracketed in A_n because the $(n-1)/n$-subword of A_n is obtained from the $(n-1)/n$-subword of B_n by inserting an n. Hence the leftmost $(n-1, n)$-unbracketed n in A_n is weakly to the left of position y_{n-1}, so $x_{n-1} \leqslant y_{n-1}$. Next, suppose that $x_{i+1} \leqslant y_{i+1}$ but $x_i > y_i$. The $i/(i+1)$-subword of A_{i+1} only differs from the $i/(i+1)$-subword of B_{i+1} by moving an $i+1$ to the left from y_{i+1} to x_{i+1}. Since $y_i < x_{i+1}$ by assumption, the $i+1$ which appears in $B_{i+1}(y_i)$ still appears in $A_{i+1}(y_i)$ and is $(i, i+1)$-unbracketed. This implies $x_i \leqslant y_i$. Induction completes the proof. □

Claim B.2.11. For $1 \leqslant i \leqslant n$, we have $x_i \geqslant x_{-i}$. For $1 \leqslant i \leqslant n-1$, we have $y_i \geqslant y_{-i}$.

Proof. We already know that $x_1 \geqslant x_{-1}$. So assume that $x_{i-1} \geqslant x_{-(i-1)}$ but $x_i < x_{-i}$. The $i/(i+1)$-subword of A_i is obtained from the $i/(i+1)$-subword of A_{-i} by moving an i to the right from $x_{-(i-1)}$ to x_{i-1}. Since $A_{-i}(x_{-i})$ contains an $(i, i+1)$-unbracketed i and $x_{i-1} < x_{-i}$, we see that $A_i(x_{-i})$ still contains an $(i, i+1)$-unbracketed i. This implies that $x_i \geqslant x_{-i}$. Induction completes the proof. The second statement is proved in the same way. □

From the previous result, we have the following situation:

$$\begin{array}{ccccccccccccccc} \cdots & \geqslant & x_3 & \geqslant & x_2 & \geqslant & x_1 & > & x_{-1} & \leqslant & x_{-2} & \leqslant & x_{-3} & \leqslant & \cdots \\ & & \rotatebox{90}{$\geqslant$} & & \rotatebox{90}{$\geqslant$} & & \rotatebox{90}{$\geqslant$} & & \| & & & & & & \\ \cdots & \geqslant & y_3 & \geqslant & y_2 & \geqslant & y_1 & > & y_{-1} & \leqslant & y_{-2} & \leqslant & y_{-3} & \leqslant & \cdots \end{array}$$

where every entry on the left side of the array is $\geqslant$ to its mirror image on the right side of the array. From now on, let j be minimal such that $x_j < y_j$; if no such j exists, set $j = n$.

Claim B.2.12. We have $x_i = y_i$ for all $i < j$ and $x_{i+1} < y_i$ for all $j \leqslant i < n$.

Proof. The first claim is immediate. Next we note that $x_i < y_i$ for all $i \geqslant j$. (Otherwise $x_i = y_i$ for some $i \geqslant j$. This implies that $x_k = y_k$ for all $k \leqslant i$, and, in particular, $x_j = y_j$.) By definition, we have $B_{i+1}(y_i) = i+1$ and $A_{i+2}(x_{i+1}) = i+2$. From the latter, it follows that $B_{i+2}(x_{i+1}) \geqslant i+2$ and, since $y_{i+1} > x_{i+1}$ (or y_{i+1} is undefined) that $B_{i+1}(x_{i+1}) \geqslant i+2$. Therefore, we have $x_{i+1} \neq y_i$. If $x_{i+1} > y_i$, we must have $x_i < x_{i+1}$ and $y_i < y_{i+1}$ from which it follows that $A_{i+1}(1 : y_i) = B_{i+1}(1 : y_i)$. But this makes $x_i < y_i$ impossible. By contradiction, we conclude that $x_{i+1} < y_i$. □

Claim B.2.13. For $i < j$ we have $x_{-i} = y_{-i}$. Also, $x_j > x_{j-1}$.

Proof. Since the restrictions of A_{j-1} and B_{j-1} to the alphabet $\{1, 2, \ldots, j-1\}$ are identical, and since the operators $e_{j-2}, \ldots, e_1, f_{-1}, f_2, \ldots, f_{j-2}$ only depend on and effect these letters, it follows that for $i \leqslant j-2$ we have $x_{-i} = y_{-i}$. Now we must show $x_{-(j-1)} = y_{-(j-1)}$. We have $A_{j+1}(x_j) = j+1$ and thus $B_{j+1}(x_j) \geqslant j+1$, and hence by $x_j < y_j$, $B_j(x_j) \geqslant j+1$. Since $B_j(y_{j-1}) = j$, this yields $x_j \neq y_{j-1}$. In light of $x_{j-1} = y_{j-1}$ this gives $x_j \neq x_{j-1}$. From this it follows that $A_j(1 : x_{j-1}) = B_j(1 : x_{j-1})$. By the minimality of j and by the result for $i \leqslant j-2$ this implies that $A_{-(j-1)}(1 : x_{j-1}) = B_{-(j-1)}(1 : x_{j-1})$. Since we have both $x_{-(j-1)} \leqslant x_{j-1}$ and $y_{-(j-1)} \leqslant y_{j-1}$, the previous equality implies that $x_{-(j-1)} = y_{-(j-1)}$. □

If $1 < i < n$, let $\#(A_{-i}(p : q))$ denote the number of i's minus the number of $(i + 1)$'s which appear in $A_{-i}(p : q)$. Define $\#(B_{-i}(p : q))$ analogously. Set $AB_i(p : q) = \#(A_{-i}(p : q)) - \#(B_{-i}(p : q))$.

Claim B.2.14. Suppose $1 < i < n$.

(1) If $x_{-i} < y_{-i}$, then $AB_i(1 : x_{-i}) > 0$.
(2) If $x_{-i} > y_{-i}$, then $AB_i(1 : y_{-i}) < 0$.
(3) If $x_{-i} < y_{-i}$, then $AB_i(x_{-i} + 1 : y_{-i}) < 0$.
(4) If $x_{-i} < y_{-i}$, $x_{-i} = x_i$, $x_i \neq x_{i+1}$, and $x_i \neq y_i$, then $AB_i(x_{-i} + 1 : y_i) < -1$.

Proof. Once again, **C2'**(b) applies to $f_{-1}(A_1)$ and so we may write $A_{-i} = e_i \cdots e_n e_1^{h'}(v')$ for some I_0-lowest weight element v' and some $h' < n$. It follows that A_{-i} has exactly one $(i, i + 1)$-unbracketed i and it occurs in x_{-i}. In addition, case **C2'**(b) applies to $f_{-1}(B_1)$ by assumption, so $B_{-i} = e_i \cdots e_{n-1} e_1^h(v^*)$ for an I_0-lowest weight element v^*. Hence B_{-i} has exactly one $(i, i + 1)$-unbracketed i and it occurs in y_{-i}. Thus we have $\#(A_{-i}(1 : x_{-i})) > 0$ and $\#(B_{-i}(1 : y_{-i})) > 0$. If $x_{-i} < y_{-i}$ then $\#(B_{-i}(1 : x_{-i})) \leqslant 0$, while if $x_{-i} > y_{-i}$ then $\#(A_{-i}(1 : y_{-i})) \leqslant 0$. Together this proves the first two statements. For the third statement we have $\#(A_{-i}(x_{-i} + 1 : y_{-i})) \leqslant 0$ and $\#(B_{-i}(x_{-i} + 1 : y_{-i})) > 0$. For the fourth statement, again, we have $\#(A_{-i}(x_{-i} + 1 : y_{-i})) \leqslant 0$, but now note that $A_{i+1}(x_i) = i + 1$. Since $x_i \neq x_{i+1}$, also, $A_{i+2}(x_i) = i + 1$, whence $B_{i+1}(x_i) = i+1$, and, by, $x_i \neq y_i$, we have $B_i(x_i) = i+1$. This now implies that $B_{-i}(x_i) = i+1$ or $B_{-i}(x_{-i}) = i+1$. Since the i in $B_{-i}(y_i)$ must be $(i, i + 1)$-unbracketed this implies that $\#(B_{-i}(x_{-i} + 1 : y_{-i})) > 1$. □

Claim B.2.15. Fix an interval $[p, q]$. We define the function $[t]$ by $[t] = 1$ if $t \in [p, q]$ and $[t] = 0$ otherwise. With this notation, we have that

$$AB_i(p : q) = [x_{-(i-1)}] - [x_{i-1}] + 2[x_i] - [x_{i+1}] + [y_{i+1}] - 2[y_i] + [y_{i-1}] - [y_{-(i-1)}].$$

Proof. This is a straightforward computation. □

Claim B.2.16. Suppose $j < n$. If either $x_j > x_{-j}$ or $y_j > y_{-j}$, then both $x_j > x_{-j}$ and $y_j > y_{-j}$. In this case we have $x_{-j} = y_{-j}$.

Proof. If $j = 1$, the conclusions of the claim have already been proven in previous claims. Thus assume $j > 1$. First note that, since $x_{-(j-1)} = y_{-(j-1)}$ and $x_{j-1} = y_{j-1}$, we have $AB_j(p : q) = 2[x_j] - [x_{j+1}] + [y_{j+1}] -$

$2[y_j]$. To prove the first statement, we will show that both (1) $x_j > x_{-j}$ and $y_j = y_{-j}$ and (2) $y_j > y_{-j}$ and $x_j = x_{-j}$ are impossible.

First suppose that $x_j > x_{-j}$ and that $y_j = y_{-j}$. Since $x_{-j} < x_j < y_j = y_{-j}$, we have by Claim B.2.14 that $AB_j(1 : x_{-j}) > 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> x_{-j}$ so by Claim B.2.15 we have $AB_j(1 : x_{-j}) = 0$. Hence, $x_j > x_{-j}$ and $y_j = y_{-j}$ is impossible.

Now suppose that $y_j > y_{-j}$ and that $x_j = x_{-j}$.

Case 1: $y_{-j} < x_{-j}$. Since $y_{-j} < x_{-j}$ we have by Claim B.2.14 that $AB_j(1 : y_{-j}) < 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> y_{-j}$ so by Claim B.2.15 we have $AB_j(1 : y_{-j}) = 0$.

Case 2: $y_{-j} = x_{-j}$. We have $A_{j+1}(x_j) = j + 1$ and so $B_{j+1}(x_j) \geqslant j + 1$. Hence by $x_j < y_j$ we have $B_j(x_j) \geqslant j + 1$ which gives $B_{-j}(x_j) \geqslant j + 1$. However, by definition $B_{-j}(y_{-j}) = j$ so this makes $x_{-j} = y_{-j}$ impossible in light of $x_j = x_{-j}$.

Case 3a: $y_{-j} > x_{-j}$ and $x_j = x_{j+1}$. Since $y_{-j} > x_{-j}$ we have by Claim B.2.14 that $AB_j(x_{-j} + 1 : y_{-j}) < 0$. However, x_j, x_{j+1} are each $< x_{-j}+1$ and y_j, y_{j+1} are each $> y_{-j}$ so by Claim B.2.15 we have $AB_j(1 : y_{-j}) = 0$.

Case 3b: $y_{-j} > x_{-j}$ and $x_j < x_{j+1}$. Since $y_{-j} > x_{-j} = x_j$, $x_j \neq x_{j+1}$, and $x_j \neq y_j$, we have by Claim B.2.14 that $AB_j(x_{-j} + 1 : y_{-j}) < -1$. However, $x_j < x_{-j} + 1$ and y_j, y_{j+1} are each $> y_{-j}$ so by Claim B.2.15 we have $AB_j(x_{-j} + 1 : y_{-j}) \in \{-1, 0\}$.

Hence $y_j > y_{-j}$ and $x_j = x_{-j}$ is impossible. This establishes that if either $x_j > x_{-j}$ or $y_j > y_{-j}$, then both $x_j > x_{-j}$ and $y_j > y_{-j}$.

Now assume that both $x_j > x_{-j}$ and $y_j > y_{-j}$. If $x_{-j} < y_{-j}$, we have by Claim B.2.14 that $\#_j(A_{-j}(1 : x_{-j})) > 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> x_{-j}$ so by Claim B.2.15 we have $\#_j(A_{-j}(1 : x_{-j})) = 0$. If $x_{-j} > y_{-j}$, we have by Claim B.2.14 that $\#_j(A_{-j}(1 : y_{-j})) < 0$. However, $x_j, x_{j+1}, y_{j+1}, y_j$ are each $> x_{-j}$ so by Claim B.2.15 we have $\#_j(A_{-j}(1 : y_{-j})) = 0$. Hence $x_{-j} = y_{-j}$. □

Claim B.2.17. If $x_j < x_{-j}$ or $y_j < y_{-j}$, then for $j \leqslant i < n$ we have $y_{-i} < y_i$ and $y_{-i} \leqslant x_{-i}$.

Proof. We proceed by induction. By the first statement of Claim B.2.16, we can be sure that $y_{-j} < y_j$. By the second statement of Claim B.2.16 we can be sure that $y_{-j} = x_{-j}$, so in particular, $y_{-j} \leqslant x_{-j}$. Therefore the claim holds for $i = j$. Now let $i > j$ and suppose that the claim holds for $i - 1$ so that $y_{-(i-1)} < y_{i-1}$ and $y_{-(i-1)} \leqslant x_{-(i-1)}$. We will show that under this assumption, each of (1) $y_{-i} = y_i$ and $y_{-i} > x_{-i}$, (2) $y_{-i} < y_i$ and $y_{-i} > x_{-i}$, and (3) $y_{-i} = y_i$ and $y_{-i} \leqslant x_{-i}$ is impossible.

First suppose that $y_{-i} = y_i$ and that $y_{-i} > x_{-i}$.

Case 1: $x_{-i} < x_i$. Since $y_{-i} > x_{-i}$ by Claim B.2.14 we have $AB_i(1 : x_{-i}) > 0$. However, by assumption $x_i, x_{i+1}, y_{i+1}, y_i, y_{i-1}$ are each $> x_{-i}$ and $x_{-(i-1)} = y_{-(i-1)}$ so the only possible relevant change is at x_{i-1}. Thus by Claim B.2.15 we have $AB_i(1 : y_{-i}) \in \{-1, 0\}$.

Case 2a: $x_{-i} = x_i$ and $x_i = x_{i+1}$. Since $y_{-i} > x_{-i}$ by Claim B.2.14 we have $AB_i(1 : x_{-i}) > 0$. By assumptions, each of $x_{-(i-1)}, x_{i-1}, x_i, x_{i+1}, y_{-(i-1)}$ are $< x_{-i} + 1$. Clearly $y_i = y_{-i} \in [x_{-i} + 1 : y_{-i}]$. Moreover, $y_{i-1} \leqslant y_i = y_{-i}$ and $y_{i-1} > x_i = x_{-i}$, so $y_{i-1} \in [x_{-i} + 1 : y_{-i}]$. Without computing the value of $[y_{i+1}]$ we may conclude by Claim B.2.15 that $AB_i(1 : y_{-j}) \in \{-1, 0\}$.

Case 2b: $x_{-i} = x_i$ and $x_i < x_{i+1}$. Since $y_{-i} > x_{-i}$, $x_{-i} = x_i$, $x_i \neq x_{i+1}$, and $x_i \neq y_i$ we have by Claim B.2.14 that $AB_i(x_{-i} + 1 : y_{-i}) < -1$. By assumptions, each of $x_{-(i-1)}, x_{i-1}, x_i, y_{-(i-1)}$ are $< x_{-i} + 1$. Again, we know that $y_i, y_{i-1} \in [x_{-i} + 1 : y_{-i}]$. Without computing the value of $[y_{i+1}]$ and $[x_{i+1}]$ we may compute by Claim B.2.15 that $AB_i(x_{-i} + 1 : y_{-i}) \in \{-1, 0, 1\}$.

Hence it is impossible that $y_{-i} = y_i$ and that $y_{-i} > x_{-i}$. Now suppose that $y_{-i} < y_i$ and that $y_{-i} > x_{-i}$.

Case 1a: $x_{-i} < x_i$ and $x_i \leqslant y_{-i}$. Since $y_{-i} > x_{-i}$, we have by Claim B.2.14 that $AB_i(x_{-i} + 1 : y_{-i}) < 0$. We have that $x_{-(i-1)}, y_{-(i-1)}$ are both $< x_{-i} + 1$, that $x_i \in [x_{-i} + 1 : y_{-i}]$ and that y_i, y_{i+1} are both $> y_{-i}$. Without computing $[x_{i-1}], [x_{i+1}], [y_{i-1}]$ we may determine by Claim B.2.15 that $AB_i(x_{-i} + 1 : y_{-i}) \in \{0, 1, 2, 3\}$.

Case 1bi: $x_{-i} < x_i$, $x_i > y_{-i}$, and $x_{i-1} \leqslant x_{-i}$. Since $y_{-i} > x_{-i}$, we have by Claim B.2.14 that $AB_i(x_{-i} + 1 : y_{-i}) < 0$. By assumption each of $x_{-(i-1)}, x_{i-1}, y_{-(i-1)}$ are $< x_{-i} + 1$ and $x_{i+1}, x_i, y_i, y_{i+1}$ are $> y_{-i}$. Without computing $[y_{i-1}]$ we may determine by Claim B.2.15 that $AB_i(x_{-i} + 1 : y_{-i}) \in \{0, 1\}$.

Case 1bii: $x_{-i} < x_i$, $x_i > y_{-i}$, and $x_{i-1} > x_{-i}$. Since $y_{-i} > x_{-i}$, we have by Claim B.2.14 that $AB_i(1 : x_{-i}) < 0$. By assumption $x_{-(i-1)}, y_{-(i-1)}$ are $\leqslant x_{-i}$ whereas each of $x_{i-1}, x_i, x_{i+1}, y_{i-1}, y_i, y_{i+1}$ are $> x_{-i}$. Thus by Claim B.2.15, we have $AB_i(1 : x_{-i}) = 0$.

Case 2a: $x_{-i} = x_i$ and $x_i = x_{i+1}$. Since $y_{-i} > x_{-i}$ we have by Claim B.2.14 that $AB_i(x_{-i} + 1 : y_{-i}) < 0$. By assumption each of $x_{-(i-1)}, x_{i-1}, x_i, x_{i+1}, y_{-(i-1)}$ are $< x_{-i} + 1$ and y_i, y_{i+1} are $> y_{-i}$. Without computing $[y_{i-1}]$ we may determine by Claim B.2.15 that $AB_i(x_{-i} + 1 : y_{-i}) \in \{0, 1\}$.

Case 2b: $x_{-i} = x_i$ and $x_i < x_{i+1}$. Since $y_{-i} > x_{-i}$, $x_{-i} = x_i$, $x_i \neq x_{i+1}$, and $x_i \neq y_i$ we have by Claim B.2.14 that $AB_i(x_{-i}+1 : y_{-i}) < -1$. By assumption each of $x_{-(i-1)}, x_{i-1}, x_i, y_{-(i-1)}$ are $< x_{-i}+1$ and y_i, y_{i+1} are $> y_{-i}$. Without computing $[y_{i-1}]$ and $[x_{i-1}]$ we may determine by Claim B.2.15 that $AB_i(x_{-i}+1 : y_{-i}) \in \{-1, 0, 1\}$.

Hence $y_{-i} < y_i$ and $y_{-i} > x_{-i}$ is impossible. Now suppose $y_{-i} = y_i$ and $y_{-i} \leqslant x_{-i}$. This would imply $y_i = y_{-i} \leqslant x_{-i} \leqslant x_i < y_i$ which is absurd. The three possibilities listed in the beginning of the proof are thus impossible, and the only remaining one is $y_{-i} < y_i$ and $y_{-i} \leqslant x_{-i}$. □

Supposing $j = 3$, and $n = 5$, and $x_j > x_{-j}$ our situation would look as follows:

$$\begin{array}{ccccccccccccccccc}
x_5 & \geqslant & x_4 & \geqslant & \mathbf{x_3} & > & x_2 & \geqslant & x_1 & > & x_{-1} & \leqslant & x_{-2} & \leqslant & x_{-3} & \leqslant & x_{-4} \geqslant x_{-5} \\
\wedge & & \wedge & & & & \| & & \| & & \| & & \| & & \| & & \not\wedge \\
\mathbf{y_4} & \geqslant & \mathbf{y_3} & & \geqslant & & y_2 & \geqslant & y_1 & > & y_{-1} & \leqslant & y_{-2} & \leqslant & y_{-3} & \leqslant & y_{-4}
\end{array}$$

where again every entry on the left side of the array is $\geqslant$ its mirror image on the right side of the array, and the bold entries are bigger than their mirror image.

Claim B.2.18. If $x_j = x_{-j}$, then $A_{-(n+1)} = B_{-n}$.

Proof. We have for all $i < j$ that $x_i = y_i$ and $x_{-i} = y_{-i}$. Since by assumption $x_j = x_{-j}$, we have for all $i \geqslant j$, $x_i = x_{-i}$. Moreover, if $j < n$ then by Claim B.2.16 $y_j = y_{-j}$ and for all $i \geqslant j$, we have $y_i = y_{-i}$. If ℓ is the length of the word v and $1 \leqslant p \leqslant \ell$, define the vector $\vec{p}$ to be the vector of length ℓ, which has a 1 in position p and 0's elsewhere. Then recalling that $A_{n+1} = X = B_n$, we have the equalities:

$$A_{-(n+1)} = X - \sum_{i=1}^{n} \vec{x}_i + \sum_{i=1}^{n} \vec{x}_{-i} = X - \sum_{i=1}^{j-1} \vec{x}_i + \sum_{i=1}^{j-1} \vec{x}_{-i} = X - \sum_{i=1}^{j-1} \vec{y}_i + \sum_{i=1}^{j-1} \vec{y}_{-i}$$
$$= X - \sum_{i=1}^{n-1} \vec{y}_i + \sum_{i=1}^{n-1} \vec{y}_{-i} = B_{-n}.$$

□

Claim B.2.19. We have $x_j = x_{-j}$.

Proof. Suppose $x_j > x_{-j}$.

Case 1: $j = n$. By the definition of j, we have $x_{n-1} = y_{n-1}$ and by Claim B.2.13 we have $x_{-(n-1)} = y_{-(n-1)}$. Since $x_{-n} < x_n$, this implies $A_{-n}(1 : x_{-n}) = B_{-n}(1 : x_{-n})$. Since A_{-n} contains an $(n, n+1)$-unbracketed n in position x_{-n}, so does B_{-n}. Therefore, $f_n(B_{-n}) \neq 0$ which contradicts $B_{-(n+1)} = 0$.

Case 2a: $j < n$ and $x_{n-1} = x_{-(n-1)}$. We have $y_{-(n-1)} \leqslant x_{-(n-1)} \leqslant x_n$. Since $x_n < y_{n-1}$ this means that we cannot have $y_{-(n-1)} = x_n$, so we must have $y_{-(n-1)} < x_n$. Since $x_{n-1} = x_{-(n-1)}$ and $y_{n-1} > x_n$, the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$ is obtained from the $n/(n+1)$-subword of $A_n(1 : x_{-n})$ by:

(1) Erasing an n from x_n and adding an n in $y_{-(n-1)}$. (Note $y_{-(n-1)} < x_n$.)

(2) Adding an $n+1$ to x_n.

Therefore, since the $n/(n+1)$-subword of $A_{-n}(1 : x_n)$ contains an $(n, n+1)$-unbracketed n and each one of these two steps does not change that property, the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$ also does. This implies $f_n(B_{-n}) \neq 0$ which contradicts $B_{-(n+1)} = 0$.

Case 2b: $j < n$ and $x_{n-1} > x_{-(n-1)}$. Since, $x_{n-1}, y_{n-1} \in [1 : x_{n-1}]$ and $x_{n-1}, x_n \in [x_{n-1}+1 : x_n]$ and $y_{n-1} > x_n$, the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$ is obtained from the $n/(n+1)$-subword of $A_{-n}(1 : x_n)$ by:

(1) Erasing an n from $x_{-(n-1)}$ and adding an n in $y_{-(n-1)}$. (Note $y_{-(n-1)} \leqslant x_{-(n-1)}$).

(2) Adding an n to x_{n-1} and erasing an n from x_n. (Note $x_{n-1} \leqslant x_n$).

(3) Adding an $n+1$ to x_n.

Therefore, since the $n/(n+1)$-subword of $A_{-n}(1 : x_n)$ contains an $(n, n+1)$-unbracketed n and each one of these three steps does not change that property, so does the $n/(n+1)$-subword of $B_{-n}(1 : x_n)$. This implies $f_n(B_{-n}) \neq 0$ which contradicts $B_{-(n+1)} = 0$. □

Since, indeed $x_j = x_{-j}$, we have $A_{-(n+1)} = B_{-n}$ by Claim B.2.18, which completes the proof of Lemma B.2.4.

B.2.3. Proof of Lemma B.2.4 for $j = n$ and $j' = n-1$.

Lemma B.2.20. *Suppose v is I_0-lowest weight and $h < n-1$. Suppose that $(e_2 \cdots e_{n-1})e_1^h(v) \neq 0$ and $e_2 \cdots e_n e_1^h(v) \neq 0$. If $f_n^1 f_n^1 e_{\bar{1}}^n e_1^n(v)$ is I_0-lowest weight, then $f_n^1 f_{n-1}^1 e_{\bar{1}}^{n-1} e_1^n(v)$ is I_0-lowest weight.*

Proof of Lemma B.2.20. Suppose v and $v' = f_n^1 f_n^1 e_{\bar{1}}^n e_1^h(v)$ are I_0-lowest weight and $(e_2 \cdots e_{n-1})e_1^h(v) \neq 0$. We must show that $f_n^1 f_{n-1}^1 e_{\bar{1}}^{n-1} e_1^h(v)$ is I_0-lowest weight.

Claim B.2.21. Given a word W, define $L(W)$ to be the length of the longest weakly increasing subsequence of W. If V is I_0-lowest weight, and W and V are in the same I_0-connected component, then the number of $(n+1)$'s which appear in V is equal to $L(W)$.

Proof. This easily follows from analyzing the RSK insertion tableaux of the words. $\square$

Claim B.2.22. We have $L(e_{\overline{1}}^{n-1}e_1^h(v)) \geqslant L(e_{\overline{1}}^{n}e_1^h(v))$.

Proof. Since $Y = e_2 \cdots e_{n-1}e_1^h(v) \neq 0$, by inspection of the insertion tableaux of v and Y we observe that $\varphi_1(Y) = 0$, $\varphi_2(Y) = 1$, and $\varphi_k(Y) = 0$ for all $k > 2$. This implies that Y contains a letter 2 which precedes all letters 1. Hence $e_{\overline{1}}^{n-1}e_1^h(v) = e_{-1}(Y) \neq 0$, so the statement $L(e_{\overline{1}}^{n-1}e_1^h(v)) \geqslant L(e_{\overline{1}}^{n}e_1^h(v))$ is well-defined.

We will now recycle notation from the proof of Section B.2.2 with slight changes. Let $X = e_1^h(v)$. For $2 \leqslant i \leqslant n+1$, set $A_i = (e_i \cdots e_n)(X)$ and $B_i = (e_i \cdots e_{n-1})(X)$. Set $A_1 = e_{-1}(A_2)$ and $B_1 = e_{-1}(B_2)$. Let x_i be the integer which represents the position, where A_{i+1} and A_i differ and y_i be the integer which represents the position where B_{i+1} and B_i differ.

Suppose that v contains r letters $(n+1)$. It follows from weight considerations that v' contains $(r+1)$ letters $(n+1)$. This implies that $L(e_{\overline{1}}^{n}e_1^h(v)) = r+1$ whereas $L(e_2 \cdots e_n e_1^h(v)) = r$. This is to say $L(A_1) = r+1$ and $L(A_2) = r$. So A_1 contains a weakly increasing subsequence of length $r+1$, specified by the indices $i_1^0, \ldots, i_1^r$. We must have that $i_1^0 = x_1$ and that $A_1(i_1^1) = 1$, otherwise the same indices would specify a weakly increasing subsequence of A_2 of length $r+1$. It follows that A_2 has a weakly increasing subsequence given by the indices $i_2^1, \ldots, i_2^r$ where $A_2(i_2^1) = 1$. Now suppose $2 \leqslant k \leqslant n$ and A_k has a weakly increasing subsequence given by the indices $i_k^1, \ldots, i_k^r$, where $A_k(i_k^1) = 1$. If $x_k \notin \{i_k^1, \ldots, i_k^r\}$, then A_{k+1} has such a subsequence specified by the same indices.

Now suppose that $x_k \in \{i_k^1, \ldots, i_k^r\}$. Create a list of indices as follows:

(1) If $i_k^j \leqslant x_k$ or $A_k(i_k^j) \neq k$, then $i_{k+1}^j = i_k^j$.
(2) If $i_k^j > x_k$ and $A_k(i_k^j) = k$, then $A_k(i_k^j)$ is $(k, k+1)$-bracketed with some $k+1$ in a position between x_k and i_k^j. Let i_{k+1}^j denote this position.

This creates a set $\{i_{k+1}^1, \ldots, i_{k+1}^r\}$, which, after a possible reordering into increasing order, specifies a weakly increasing subsequence of A_{k+1} with $A_{k+1}(i_{k+1}^1) = 1$.

By induction $B_n = A_{n+1} = X$ has a weakly increasing subsequence specified by the indices $\{i'^1_n, \ldots, i'^r_n\}$, with $B_n(i'^1_n) = 1$. Let $k > 1$ and assume B_{k+1} has a weakly increasing subsequence specified by the indices

$\{i'^1_{k+1}, \ldots, i'^r_{k+1}\}$, with $B_{k+1}(i'^1_{k+1}) = 1$. If $y_k < i'^1_{k+1}$, then the same is true of B_k with the same indices. If $y_k > i'^1_{k+1}$ then $B_k = e_k(B_{k+1}) = [B_{k+1}(1 : i'^1_{k+1})\ e_k(B_{k+1}(i'^1_{k+1} + 1 : \ell))]$. Since $B_{k+1}(i'^1_{k+1} + 1 : \ell)$ has a weakly increasing subsequence of length $r - 1$, $e_k(B_{k+1}(i'^1_{k+1} + 1 : \ell))$ does as well. Thus $B_k = [B_{k+1}(1 : i'^1_{k+1})\ e_k(B_{k+1}(i'^1_{k+1} + 1 : \ell))]$ has a weakly increasing subsequence of length r specified by some indices $\{i'^1_k, \ldots, i'^r_k\}$, with $B_k(i'^1_k) = 1$ (where $i'^1_k = i'^1_{k+1}$). By induction this is true for $k = 2$. Since $e_{-1}(B_2) = B_1$ is defined and since $B_2(i'^1_2) = 1$, we have $y_1 < i'^1_2$ and so $\{y_1, i'^1_2, \ldots, i'^r_2\}$ is a list of indices which give a weakly increasing subsequence of length $r + 1$ in B_1. □

We want to show that $f_n^1 f_{n-1}^1 e_{\bar{1}}^{n-1} e_1^h(v)$ is I_0-lowest weight. Now $e_{-1}(Y)$ is obtained from $Y = e_2 \cdots e_{n-1} e_1^h(v)$ by changing its first 2 to 1. As a result $\varphi_1(e_{-1}(Y)) \in \{1, 2\}$ and $\varphi_k(e_{-1}(Y)) = 0$ for all $k > 1$. Therefore, we may write $e_{-1}(Y) = e_{\bar{1}}^s e_1^t(v^*)$ for some I_0-lowest weight element v^*, and $s \geqslant 0$ and $t > 0$ with $t \geqslant s$ (using Lemma 5.3.3 when $\varphi_1(e_{-1}(Y)) = 2$). This gives $v^* = f_t^1 f_s^1 e_{\bar{1}}^{n-1} e_1^h(v)$. Since v' contains one more $n+1$ than v, it follows from Claims B.2.21 and B.2.22 that v^* contains at least one more $n+1$ than v, which means we must have $t = n$. This also means that v and v^* are not in the same connected I_0-component. But if $v = f_h^1 f_{n-1}^{\bar{1}} e_1^s e_1^n(v^*)$ is in a different connected I_0-component than v^*, then **C2'**(b) applies which forces $s = n - 1$. Thus $v^* = f_n^1 f_{n-1}^1 e_{\bar{1}}^{n-1} e_1^h(v)$.

This concludes the proof of Lemma B.2.20. □

Proposition B.2.23. *Lemma B.2.4 with $j = n - 1$ and $j' = n$ and Lemma B.2.20 imply Lemma B.2.4.*

Proof. We need to show that if v is I_0-lowest weight, $e_1^{n-1} e_1^n(v) \neq 0$, $e_1^n e_1^n(v) \neq 0$, and $v^* = f_h^1 f_n^{\bar{1}} e_1^n e_1^n(v)$ is I_0-lowest weight, then $f_{n-1}^{\bar{1}} e_1^{n-1} e_1^n(v) = f_n^{\bar{1}} e_1^n e_1^n(v)$. Now $v = f_n^1 f_n^1 e_{\bar{1}}^n e_1^h(v^*)$ is I_0-lowest weight (in particular, $e_2 \cdots e_n e_1^h(v^*) \neq 0$). Now we show that $e_2 \cdots e_{n-1} e_1^h(v^*) \neq 0$. By definition, $e_1^h(v^*) \neq 0$. Either v^* has more n's than $(n-1)$'s so that $e_2 \cdots e_{n-1} e_1^h(v^*) \neq 0$, or else v^* has the same number of n's as $(n-1)$'s and $h = n - 2$ in which case also $e_2 \cdots e_{n-1} e_1^h(v^*) \neq 0$. Therefore, by Lemma B.2.20 $v' = f_n^1 f_{n-1}^1 e_{\bar{1}}^{n-1} e_1^h(v^*)$ is I_0-lowest weight. Rewriting this as $v^* = f_h^1 f_{n-1}^{\bar{1}} e_1^{n-1} e_1^n(v')$ and noting that $\mathrm{wt}(v) = \mathrm{wt}(v')$ implies $e_1^n e_1^n(v') \neq 0$ Lemma B.2.4 with $j = n - 1$ and $j' = n$ gives $v^* = f_h^1 f_n^{\bar{1}} e_1^n e_1^n(v')$. This implies that $v = v'$ and that hence that $f_{n-1}^{\bar{1}} e_1^{n-1} e_1^n(v) = f_n^{\bar{1}} e_1^n e_1^n(v)$. □

Bibliography

[AKO18a] S. Assaf and E. Kantarci Oguz, *Crystal graphs for shifted tableaux*, Sém. Lothar. Combin. **80B** (2018), Art. 26, 12.

[AKO18b] ———, *A local characterization of crystals for the quantum queer superalgebra*, (2018), preprint arXiv:1803.06317v1.

[AS12] F. Ardilla and L. Serrano, *Staircase skew Schur functions are Schur P-positive*, J. Algebraic Combin. **36** (2012), no. 3, 409–423.

[BH95] S. Billey and M. Haiman, *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), no. 2, 443–482.

[BHRY14] S. Billey, Z. Hamaker, A. Roberts, and B. Young, *Coxeter-Knuth graphs and a signed Little map for type B reduced words*, Electron. J. Combin. **21** (2014), no. 4, Paper 4.6, 39.

[BS17] D. Bump and A. Schilling, *Crystal Bases: Representations and Combinatorics*, World Scientific, 2017.

[Cho13] S. Cho, *A new Littlewood-Richardson rule for Schur P-functions*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 939–972.

[EG87] P. Edelman and C. Greene, *Balanced tableaux*, Adv. in Math. **63** (1987), no. 1, 42–99.

[FK96] S. Fomin and A. N. Kirillov, *Combinatorial B_n-analogues of Schubert polynomials*, Trans. Amer. Math. Soc. **348** (1996), no. 9, 3591–3620.

[GHPS18] M. Gillespie, G. Hawkes, W. Poh, and A. Schilling, *Characterization of queer supercrystals*, (2018), preprint arXiv:1809.04647.

[GJK+14] D. Grantcharov, J. H. Jung, S.-J. Kang, M. Kashiwara, and M. Kim, *Crystal bases for the quantum queer superalgebra and semistandard decomposition tableaux*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 457–489.

[GJK+15] ———, *Crystal bases for the quantum queer superalgebra*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 7, 1593–1627.

[Hai89] M. Haiman, *On mixed insertion, symmetry, and shifted Young tableaux*, J. Combin. Theory Ser. A **50** (1989), no. 2, 196–225.

[Haw17] G. Hawkes, *Generalized staircase tableaux: Symmetry and applications*, (2017), preprint arXiv:1809.04434.

[Haw18] ———, *Symmetric function theory at the border of a and c*, (2018), preprint arXiv:1809.04433.

[Hir18] T. Hiroshima, $\mathfrak{q}$*-crystal structure on primed tableaux and on signed unimodal factorizations of reduced words of type B*, (2018), preprint arXiv:1803.05775.

[HPS17] G. Hawkes, K. Paramonov, and A. Schilling, *Crystal analysis of type C Stanley symmetric functions*, Electron. J. Combin. **24** (2017), no. 3, Paper 3.51, 32.

[Kra89] W. Kraśkiewicz, *Reduced decompositions in hyperoctahedral groups*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 16, 903–907.

[Kra95] ———, *Reduced decompositions in Weyl groups*, European J. Combin. **16** (1995), no. 3, 293–313.

[Lam95] T. K. Lam, *B and D analogues of stable Schubert polynomials and related insertion algorithms*, ProQuest LLC, Ann Arbor, MI, 1995, Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Ser09] L. Serrano, *The shifted plactic monoid (extended abstract)*, 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), Discrete Math. Theor. Comput. Sci. Proc., AK, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009, pp. 757–768.

[Ser10] ———, *The shifted plactic monoid*, Math. Z. **266** (2010), no. 2, 363–392.

[Sta84] R. P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5** (1984), no. 4, 359–372.

[Sta99] R. Stanley, *Enumerative Combinatorics Volume 2*, Cambridge University Press, 1999.

[Ste89] J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), no. 1, 87–134.

[Ste03] ———, *A local characterization of simply-laced crystals*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 4807–4823.

[Wor84] D. R. Worley, *A theory of shifted Young tableaux*, MIT, 1984, Thesis (Ph.D.)–Massachusetts Institute of Technology.