Affine Springer fibers, Hilbert schemes and knots

By

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To my grandfather, Ville Salomon Palomäki.

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Abstract

Symmetry is abundant in mathematics, and often appears in the guise of representation theory. It often appears in unexpected places, and due to its highly structured nature can be a powerful tool in the study of other mathematical objects. This thesis is concerned with examples of such phenomena in the realm of geometric representation theory. More specifically, we study Hilbert schemes on singular plane curves and smooth surfaces as well as affine Springer fibers from various points of view in three stand-alone chapters, which are more interrelated than might seem to the untrained eye. In Chapter II, we relate Haiman's isospectral Hilbert scheme of points on the plane to certain very unramified elements in the loop Lie algebra of a split reductive group. These elements have already been studied in the work of Goresky, Kottwitz, and MacPherson. This novel connection gives evidence for a conjecture by Bezrukavnikov. In Chapter III, we study a certain subalgebra of the Weyl algebra on $2m$ variables acting on the homology of Hilbert schemes of points of a reduced complex planar curve with m irreducible components. We compute the representation in the fundamental example of the node and furthermore compute parts of the action in the case of m lines intersecting in the plane. A majority of the results in Chapters II-III are original. In Chapter IV, we study (geometric) representations of the trigonometric DAHA in (twisted) type A with certain rational parameters. For untwisted type A, the chapter is mainly expository. The main technical result in the twisted case is a new combinatorial classification of the irreducible representations of the DAHA which follows from the study of a map defined by Lusztig.

Chapter 1

Introduction

This thesis contains several chapters dealing with different topics in geometric representation theory. Most of the chapters are standalone, but contain results closely related to each other and taken together shed light for example on the Oblomkov-Rasmussen-Shende conjecture for torus links and elucidate the connection between trigonometric Cherednik algebras and quantum affine algebras in type A.

In this introduction, we outline the main results of the various chapters, and discuss their motivation and how they relate to one another. For more detailed overviews of the chapters, including theorem statements and comparisons to results already in the literature, see the introduction to each individual chapter.

1.1 How this thesis is organized

In Chapter 2, we study the equivalued unramified affine Springer fibers also considered by Goresky-Kottwitz-Macpherson, and show these bear a close connection to Haiman's work on the Hilbert scheme of points on the plane. This chapter is mostly based on the results of [49].

In Chapter 3, we focus on Hilbert schemes of points on curves with locally planar singularities. We present an action of a subalgebra of a Weyl algebra on their homologies, and determine the corresponding representation in the case of a node. We also illustrate the connection to the Coulomb branches of Braverman-Finkelberg-Nakajima. The results of this chapter are based on [48] as well as the upcoming work [21].

In Chapter 4, we study the trigonometric double affine Hecke algebras in type A with rational equal parameters as well as in type B with certain rational unequal parameters, using geometry of nilpotent orbits of certain θ-representations. We also make explicit some of the combinatorial constructions of Lusztig-Yun in these cases and suggest further topics of study. The results of this chapter are mainly new, although in the type A case they mainly give a new interpretation to existing results of Cherednik, Vasserot, and Suzuki.

1.2 Borel-Moore homology of Affine Springer fibers

In Chapter 2, we consider the unramified equivalued affine Springer fibers studied by Goresky-Kottwitz-MacPherson [25] and their relation to the isospectral Hilbert scheme of Mark Haiman [38]. This may be thought of as the conceptual heart of this thesis, although we make little reference from the latter chapters to this one. Most of the results here are based on the paper [49], although an error in the proof of Theorem 1.2.1 pointed out to us by Eric Vasserot and Peng Shan has now been taken into account and some results been modified accordingly.

Let $G/\mathbb{C}$ be a connected reductive group and $T \subset B \subset G$ as usual. Denote $\mathfrak{g} := \mathrm{Lie}(G)$ and let $\mathcal{K} = \mathbb{C}((t)), \mathcal{O} = \mathbb{C}[[t]]$. If $\gamma \in \mathfrak{g}(\mathcal{K})$, the affine Springer fiber Sp_γ in the affine Grassmannian $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ is a sub-ind-scheme whose closed points are

$$\mathrm{Sp}_\gamma = \{gG(\mathcal{O}) | g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O})\}.$$

Replacing the affine Grassmannian by an affine flag variety, one obtains analogous sub-ind-schemes of partial affine flag varieties. While most of the results of this thesis concern the affine Grassmannian case, natural extensions of many notions to the partial (including the full) affine flag variety setting are straightforward and interesting variations of the main statements should be expected.

The element γ is unramified and equivalued iff it is conjugate to an element of the form $\gamma = at^d$, where $a \in \mathfrak{t}^{reg}$, $d \geq 0$ is an integer and $\mathfrak{t} = \mathrm{Lie}(T)$. This is why sometimes these elements are called *diagonal*.

Using the methods of Goresky-Kottwitz-MacPherson [25, 26], we compute the equivariant BM homology of $\mathrm{Sp}_\gamma^{\mathbf{P}}$ when $\mathbf{P}$ is a maximal compact subgroup. In this case, we simply denote $\mathrm{Sp}_\gamma^{\mathbf{P}} = \mathrm{Sp}_\gamma$. This is by definition a reduced sub-ind-scheme of the affine Grassmannian of G. Fix a maximal torus and a Borel subgroup $T \subset B \subset G$, and denote $\mathrm{Lie}(T) = \mathfrak{t}, \mathrm{Lie}(B) = \mathfrak{b}, \mathrm{Lie}(G) = \mathfrak{g}$. Let moreover the cocharacter lattice of T be $\Lambda := X_*(T) \cong \bigoplus_{i=1}^r \mathbb{Z}\epsilon_i$ for the fundamental weights $\{\epsilon_i\}_{i=1}^r$ determined by B and some ordering thereof. Here r is the rank of G. Denote by $\mathbb{C}[\Lambda] = \mathbb{C}[X_*(T)]$ the group algebra of the cocharacter lattice. This can be canonically identified with functions on the Langlands dual torus $T^\vee$, or as the $3d$ $\mathcal{N} = 4$ Coulomb branch for $(T, 0)$ as in [**?**].

Our first result is the following theorem, proved as Theorem 3.0.1.

Theorem 1.2.1. *Let $\Delta = \prod_\alpha y_\alpha \in H_T^*(pt)$ be the Vandermonde element. The equivariant Borel-Moore homology of $X_d := \mathrm{Sp}_{t^d z}$ for a reductive group G is up to multiplication by Δ^d canonically isomorphic as a (graded) $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$-module to the ideal*

$$J_G^{(d)} = \bigcap_{\alpha \in \Phi^+} J_\alpha^d \subset \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}].$$

In particular, there is a natural algebra structure on $\Delta^d H_*^T(\mathrm{Sp}_\gamma)$ inherited from $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$, and $J_G^{(d)}$ is a free module over $\mathbb{C}[\mathfrak{t}]$.

Throughout, $H_*^T(-)$ denotes the equivariant BM homology, see Section 2.3 for details. In a few places, we also use the ordinary T-equivariant homology as in [26]; it is denoted $H_{*,ord}^T(-)$.

In [25], the following theorem is proved.

Theorem 1.2.2. *Let $\gamma = at^d$. Then the $T(\mathbb{C})$-equivariant homology of Sp_γ is a $\mathbb{C}[T^*T^\vee] = \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$-module such that*

$$H_{*,ord}^T(\mathrm{Sp}_\gamma) = \frac{\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}^*]}{\sum_{\alpha \in \Phi^+} \sum_{k=1}^d (1 - \alpha^\vee)^k \mathbb{C}[\Lambda] \otimes \ker(\partial_\alpha^k)}.$$

The ordinary $T(\mathbb{C})$-equivariant homology of $\widetilde{\mathrm{Sp}}_\gamma$ is a $\mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[\mathfrak{t}]$-module such that

$$H_{*,ord}^T(\widetilde{\mathrm{Sp}}_\gamma) \cong \frac{\mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[\mathfrak{t}^*]}{\sum_{\alpha \in \Phi^+} \sum_{k=1}^d (1 - \alpha^\vee)^k \mathbb{C}[\widetilde{W}] \otimes \ker(\partial_\alpha^k) + \sum_{\alpha \in \Phi^+} \sum_{k=1}^d (1 - \alpha^\vee)^{k-1} (1 - s_\alpha) \mathbb{C}[\widetilde{W}] \otimes \ker(\partial_\alpha^k)}.$$

In our study of Hecke correspondences on singular plane curves, which in this thesis is written in Chapter 3, but historically came first, we encountered a similar space for the case of the singularity $\{xy = 0\}$, suggesting some direct relationship between the homology spaces of the Hilbert schemes of the latter and the equivariant homology of the affine Springer fiber. Further computations based on predictions from knot homology of full twists [17] and the Migliorini-Shende-Viviani formula as well as the knowledge of fine compactified Jacobians of reducible plane curves in [67] then solidified this belief and also made us look for a proof of the main Theorem 1.2.1. As explained further in Chapters 2 and 3, we conjecture the following for Hilbert schemes of points on n projective lines intersecting at a point on the plane:

Conjecture 1.2.3. *Let C be the unique projective model of $\{x^{dn} = y^n\}$ with rational components and no other singular points than the origin. Then as A_n-representations (see the next section) we have*

$$\bigoplus_n H_*(\mathrm{Hilb}^n(C)) \cong \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]}{\sum_{i<j} \sum_{k=1}^d (x_i - x_j)^k \ker((\partial_{y_i} - \partial_{y_j})^k)}.$$

Forgetting about knot-theoretic motivations, our results raise many interesting mathematical questions. In particular, Theorem 1.2.1 connects the Procesi bundle on the Hilbert scheme to the affine Springer fibers in question, and one expects a relationship to the Hilbert schemes of points as above as well. While not explicated here, in upcoming joint work with Gorsky and Oblomkov [29], we show that it is in fact possible to produce sheaves on the Hilbert scheme from any *elliptic* affine Springer fiber in the affine Grassmannian. Note that this class of examples is basically orthogonal to the considerations in this thesis. From an automorphic point of view, the elliptic elements are the most interesting ones one could look at and were exploited in Ngô Bao Chaû's (and Laumon's and Waldspurger's) monumental proof of the fundamental lemma [72].

1.3 Hilbert schemes of points of singular plane curves

Let us begin with some history. In [4, 37, 70], and later [71], Hecke correspondences given by elementary modifications of sheaves on smooth surfaces are studied. In particular, vertex algebras and similar structures arise in the K-theory and (co)homology of these moduli spaces, as predicted by e.g. the AGT conjecture in physics [1]. Although there is no clear mathematical explanation as to why various cohomology theories of moduli spaces of sheaves on surfaces should yield representation-theoretically interesting algebras, it is by now widely accepted that this is a fruitful way to study one through the other. Namely, the cohomological Hall algebras of recent years [45, 95] are a wide generalization of this framework to moduli of sheaves on (noncommutative) Calabi-Yau spaces.

We are intentionally being vague about what kind of sheaves one considers in the above setup. The one relevant to this thesis is the case of moduli of zero-dimensional ideal sheaves, in which case the moduli spaces are the familiar *Hilbert schemes of points* on surfaces. By a straightforward yet miraculous theorem of Fogarty, these moduli spaces are *smooth* for ideal sheaves of arbitary length, making their study at least a little more accessible.

In [83], the passage from smooth surfaces to possibly singular curves lying on these surfaces was initiated. In this case, the Hilbert schemes become singular (Lagrangian) subvarieties of the smooth Hilbert schemes on surfaces, and one wonders how much of the above action is retained. Getting rid of the fixed surface in the first place and studying correspondences on projective singular irreducible curves, Rennemo then recovers a Weyl algebra in two variables acting on

$$\bigoplus_{n\geq 0} H_*(C^{[n]}).$$

In [48], this result was extended to possibly reducible but nonetheless reduced curves, the precise result is in Chapter 3, Theorem 3.0.4. The essential technical tool in both our and Rennemo's approach is Fulton-MacPherson's bivariant Borel-Moore homology. More intestingly, in the simplest case of a reducible curve, that of the transverse intersection of two lines, the representation we recover is already quite nontrivial, and has an apparent relation to the results in Chapter 2, although we have not been able to make this precise.

As stated earlier, our approach relies on working without a fixed global surface and instead using some results of Migliorini-Shende and Maulik-Yun [65, 68] on the total families of relative Hilbert schemes of points on locally versal deformations of locally planar curves. It would be interesting to know if we can directly recover the action on the level of the singular spaces that appear, namely flag Hilbert schemes, and whether we could recover this action directly from that of Nakajima and Grojnowski. Physics seems to suggest a post-hoc motivation for these correspondence algebras [6]; in fact they should be subalgebras of the Coulomb branch of a $3d\mathcal{N}=4$ gauge theory with gauge group GL_n and matter $\mathrm{Ad}\oplus V$. We briefly discuss this relationship in Section **??**, based on upcoming joint work with Niklas Garner [21].

Lastly, we want to mention a relation to curve-counting on Calabi-Yau threefolds, one of the favorite topics in physical mathematics of the last twenty-five years. Suppose our curve lies in a CY3 X. As shown by Pandharipande and Thomas [78, Appendix B], there are integers n_g so that

$$q^{1-g_a}\sum_n \chi(C^{[n]})q^n = \sum_{g=g_c}^{g_a} n_g \frac{q^{1-g_a}}{(1-q)^{2-2g_a}},$$

where g_c, g_a are the geometric and arithmetic genera of C. The numbers n_g should then be then exactly the "local contributions" to the Gopakumar-Vafa invariants of X. In fact, at least if C is integral, the numbers n_g have a more direct interpretation as well, as coefficients in a certain change-of-variables in the generating function for $\chi(P^{\leq n}H_*(\overline{\mathrm{Jac}}(C))$, where $\overline{\mathrm{Jac}}(C)$ is the compactified Jacobian studied elsewhere in this thesis under the alias "affine Springer fiber", and $P^{\leq n}$ is a certain filtration on its cohomology, called the *perverse filtration*. For more details on the relevant curve-counting we refer to [78, 83], and for more on the perverse filtration we refer to [65, 68], as well as [15] in a more general setting.

1.4 Trigonometric DAHAs and Lusztig-Yun theory

In work of Lusztig-Yun [60–63] and Liu [53], geometric representations of the trigonometric DAHA $\mathcal{H}'(G)$ associated to a reductive group G are constructed using a Springer theory for $\mathbb{Z}/m$-graded Lie algebras. In particular, all the integrable representations can be constructed via *spiral inductions*, which are an analogue of parabolic induction in the affine setting, or alternatively a finite-dimensional model of parahoric induction.

The representation categories which then appear, in the guise of blocks of perverse sheaves on a graded piece of the nilcone, describe Whittaker modules $\mathcal{O}_{\{\lambda_0\}}$ for $\mathcal{H}'$ (in analogy to the Lie algebra case). More precisely, the index $\{\lambda_0\}$ corresponds to fixing the eigenvalues of the polynomial part of $\mathcal{H}'$ by a choice of facet in the building of $G(\mathcal{K})$.

In the special case when the $\mathbb{Z}/m$-grading is induced by an outer involution θ, the categories $\mathcal{O}_{\{\lambda\}}$ have recently been studied by Vilonen- Xue [91] in the form of character sheaves for complex symmetric pairs. Their methods, however, do not have a straightforward relationship to those of Lusztig-Yun. In particular, it is interesting to consider the block decomposition of Lusztig-Yun and try to compare it to the results in [91]. As a first step in this direction, we produce in Chapter 4 an explicit description of the block decomposition of Lusztig-Yun.

By a conjecture of Xue, the blocks can be described in this case via blocks of products of type A Hecke algebras at $q = -1$, and as the author has learned from her, this is indeed the case. The finite Hecke algebras at roots of unity that appear from the construction of [36] seem to be in a Koszul-like duality with the Cherednik algebras, and promising candidates for the Hecke algebras in question appear in recent work of Losev and Shelley-Abrahamson [54]. We hope to understand these connections in future work.

It is however not clear what the DAHA-theoretic meaning of these blocks is. While block theory of rational Cherednik algebras is fairly well understood [94], in the trigonometric case such a study is still lacking.

The paper [91] singles out objects such as *biorbital complexes* and *cuspidal character sheaves* in the relevant categories of perverse sheaves, following Lusztig. The nearby cycles construction of Grinberg-Vilonen-Xue [36] implies that cuspidal sheaves in the principal Lusztig-Yun block correspond to representations of certain finite Hecke algebras (with unequal parameters) at roots of unity. On the other hand, Vasserot's construction implies that these cuspidal character sheaves correspond to finite-dimensional representations of $\mathcal{H}'(G)$.

As to the biorbital complexes, it is known by work of Varagnolo-Vasserot that the categories $\mathcal{O}_{\{\lambda_0\}}$ admit a geometrically defined functor

$$KZ : \mathcal{O}_{\{\lambda_0\}} \to \mathbf{h} - mod,$$

where $\mathbf{h}$ is the affine Hecke algebra of the same type, and the biorbital complexes should be in bijection with irreducible finite-dimensional representations of the latter. This is true in type A, as shown in [89].

Section **??** is devoted to defining parabolic induction and restriction functors for trigonometric DAHAs. While we do not study properties of these functors further, they suggest a crystal structure on the irreducible representations of trigonometric Cherednik algebras of type A, given by induction and restriction. By results of [89], should such a crystal structure exist, one expects the resulting crystal to be a $\mathcal{U}_q(\widehat{\mathfrak{sl}}_m)$-crystal with a surjection to the crystal $\mathcal{U}_q(\widehat{\mathfrak{sl}}_m)$.

One of the attractive properties about Lusztig-Yun theory is the fact one can compute Jordan-Hölder multiplicities and (dimensions of) *Ext*-groups using certain hyperplane arrangements introduced by Lusztig in [61, 63]. While we do not explain this connection further, we compute the hyperplane arrangements in some examples and refer the interested reader to [90] for more explanations.

Chapter 2

Unramified affine Springer fibers

2.1 Introduction

In this chapter, we study a family of affine Springer fibers depending on a connected reductive group G over $\mathbb{C}$ and a positive integer d. Recall that an affine Springer fiber $\mathrm{Sp}_\gamma^{\mathbf{P}}$ is a sub-ind-scheme of a partial affine flag variety $\mathrm{Fl}^{\mathbf{P}}$ (see [96] and Section 2.2) that can be informally thought of as a zero-set of a vector field for an element of the loop Lie algebra of G, $\gamma \in \mathfrak{g} \otimes \mathbb{C}((t))$. For us, $\gamma = zt^d$, where z is any regular semisimple element in $\mathfrak{g}(\mathbb{C})$. Without loss of generality, we may take z to be an element of $\mathrm{Lie}(T)^{reg}$, where T is a fixed maximal torus of G. In fact, all of our results hold for $\gamma \in \mathrm{Lie}(T)^{reg} \otimes \mathbb{C}((t))$ that are equivalued, but for simplicity we only consider this case.

Using the methods of Goresky-Kottwitz-MacPherson [25, 26], we compute the equivariant BM homology of $\mathrm{Sp}_\gamma^{\mathbf{P}}$ when $\mathbf{P}$ is a maximal compact subgroup. In this case, we simply denote $\mathrm{Sp}_\gamma^{\mathbf{P}} = \mathrm{Sp}_\gamma$. This is by definition a reduced sub-ind-scheme of the affine Grassmannian of G. Fix a maximal torus and a Borel subgroup $T \subset B \subset G$, and denote $\mathrm{Lie}(T) = \mathfrak{t}, \mathrm{Lie}(B) = \mathfrak{b}, \mathrm{Lie}(G) = \mathfrak{g}$. Let moreover the cocharacter lattice of T be $\Lambda := X_*(T) \cong \bigoplus_{i=1}^r \mathbb{Z}\epsilon_i$ for the fundamental weights $\{\epsilon_i\}_{i=1}^r$ determined by B and some ordering thereof. Here r is the rank of G. Denote by $\mathbb{C}[\Lambda] = \mathbb{C}[X_*(T)]$ the group algebra of the cocharacter lattice. This can be canonically identified with functions on the Langlands dual torus $T^\vee$, or as the $3d$ $\mathcal{N} = 4$ Coulomb branch for $(T, 0)$ as in [**?**].

Our first result is the following theorem, proved as Theorem 3.0.1.

Theorem 2.1.1. *Let $\Delta = \prod_\alpha y_\alpha \in H^*_T(pt)$ be the Vandermonde element. The equivariant Borel-Moore homology of $X_d := \mathrm{Sp}_{t^d z}$ for a reductive group G is up to multiplication by Δ^d canonically isomorphic as a (graded) $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$-module to the ideal*

$$J_G^{(d)} = \bigcap_{\alpha \in \Phi^+} J_\alpha^d \subset \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}].$$

In particular, there is a natural algebra structure on $\Delta^d H_*^T(\mathrm{Sp}_\gamma)$ inherited from $\mathbb{C}[\Lambda]\otimes\mathbb{C}[\mathfrak{t}]$, and $J_G^{(d)}$ is a free module over $\mathbb{C}[\mathfrak{t}]$.

Throughout, $H_*^T(-)$ denotes the equivariant BM homology, see Section 2.3 for details. In a few places, we also use the ordinary T-equivariant homology as in [26]; it is denoted $H^T_{*,ord}(-)$.

2.1.1 Anti-invariants and subspace arrangements

Let W be the finite Weyl group associated with G and sgn be the one-dimensional representation of W where all reflections act by -1. Observe that there is a natural left action $W \times T \to T$, and therefore actions

$$W \times T^*T^\vee \to T^*T^\vee, W \times \mathbb{C}[T^*T^\vee] \to \mathbb{C}[T^*T^\vee].$$

Note that the cocharacter lattice $\Lambda = X_*(T)$ naturally identifies with the character lattice of $T^\vee$. In particular, $\mathbb{C}[\Lambda] \cong \mathbb{C}[T^\vee]$, where the left-hand side denotes group algebra and the right-hand side denotes ring of regular functions. The cotangent bundle of $T^\vee$ is trivial, and in particular has fibers $\mathfrak{t}$. Therefore $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[T^*T^\vee]$.

Using the description of the equivariant Borel-Moore homology given in Theorem 2.1.1, we expect a relationship between the cohomology of Sp_γ and the sgn-isotypic component of the natural diagonal W-action on $\mathbb{C}[T^*T^\vee]$. First of all, it is not hard to see the following result.

Theorem 2.1.2. *Let $I_G \subseteq \mathbb{C}[T^*T^\vee]$ be the ideal generated by W-alternating regular functions in $\mathbb{C}[T^*T^\vee]$ with respect to the diagonal action. Then there is an injective map*

$$I_G^d \hookrightarrow J_G^{(d)} = H_*^T(\mathrm{Sp}_\gamma).$$

Consequently, any W-alternating regular function on $T^*T^\vee$ has a unique expression as a cohomology class in $H_*^T(\mathrm{Sp}_\gamma)$, where $\gamma = zt$.

In the case when $G = GL_n$, this isotypic part for the corresponding action on $T^*\mathbf{t}^\vee$ was studied by Haiman [38] in his study of the Hilbert scheme of points on the plane. More specifically, he considered the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by the anti-invariant polynomials, and proved that it is first of all equal to $J = \bigcap_{i \neq j} \langle x_i - x_j, y_i - y_j \rangle$ and moreover free over the y-variables. Note that if $f \in \mathbb{C}[\mathbf{x}^\pm, \mathbf{y}]$, it is by definition of the form $f = \frac{g}{(x_1 \cdots x_n)^k}$ for some $g \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ and $k \geq 0$. Since the denominator is a symmetric polynomial, $g \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is alternating for the diagonal $\mathfrak{S}_n$-action if and only if f is so. In particular, in the localization $\mathbb{C}[\mathbf{x}^\pm, \mathbf{y}]$ we have that $I_\mathbf{x} \cong I_{GL_n}$ for I_G as in Theorem 2.1.2.

Let us quickly sketch how the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^2)$ enters the picture. Let $A \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ be the space of antisymmetric polynomials for the diagonal action of $\mathfrak{S}_n$. From for example [41, Proposition 2.6], we have that

$$\mathrm{Proj} \bigoplus_{m \geq 0} A^m \cong \mathrm{Hilb}^n(\mathbb{C}^2).$$

In addition,

$$\mathrm{Proj} \bigoplus_{m \geq 0} I^m \cong X_n,$$

where

$$X_n \cong (\mathbb{C}^{2n} \times_{\mathbb{C}^{2n}/\mathfrak{S}_n} \mathrm{Hilb}^n(\mathbb{C}^2))^{red}$$

is the so-called *isospectral Hilbert scheme*. The superscript *red* means that we are taking the reduced fiber product, or fiber product in category of varieties instead of schemes.

By results of [39], we have $I^m = \bigcap_{i \neq j} \langle x_i - x_j, y_i - y_j \rangle^m$, so that $I_\mathbf{x}^d \cong J_{GL_n}^{(d)}$. In Section 2.4, we prove our next main result following this line of ideas.

Theorem 2.1.3. *There is a graded algebra structure on*

$$\bigoplus_{d \geq 0} \Delta^d H_*^T(\mathrm{Sp}_{zt^d}).$$

When $G = GL_n$, we have

$$\mathrm{Proj} \bigoplus_{d \geq 0} \Delta^d H_*^T(\mathrm{Sp}_{zt^d}) \cong Y_n,$$

where Y_n is the isospectral Hilbert scheme on $\mathbb{C}^* \times \mathbb{C}$.

We next observe that the natural map $\rho : X_n \to \mathrm{Hilb}^n(\mathbb{C}^2)$ restricts to a map $Y_n \to \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$. Define the *Procesi bundle* on $\mathrm{Hilb}^n(\mathbb{C}^2)$ to be $\mathcal{P} := \rho_* \mathcal{O}_{X_n}$. By results of Haiman, this is a vector bundle of rank $n!$. We then have the following corollary to Theorem 2.1.3.

Corollary 2.1.4. *We have that*

$$H^0(\mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C}, \mathcal{P} \otimes \mathcal{O}(d)) = J^{(d)}_{GL_n} = \Delta^d \cdot H^T_*(\mathrm{Sp}_\gamma),$$

where $\gamma = zt^d$.

Our results can be at least interpreted in terms of the Coxeter arrangement for the root data of G or $G^\vee$. More precisely, $\mathbb{C}[X_*(T)]$ can be thought of as the ring of functions on the dual torus $T^\vee \cong (\mathbb{C}^*)^n$, which in turn is the complement of "coordinate hyperplanes" in $\mathfrak{t}^\vee \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ for the basis given by fundamental weights determined by B. Note that the resulting divisor is independent of B.

There is another hyperplane arrangement in this space, determined by $\Phi^\vee$, which is called the Coxeter arrangement, and can be viewed as the locus where at least one of the positive coroots vanishes. In the exponentiated notation, this is exactly the divisor

$$\mathcal{V} = \bigcup_\alpha \mathcal{V}_\alpha = \left\{ \prod_{\alpha \in \Phi^+} (1 - x^{\alpha^\vee}) = 0 \right\} \subset T^\vee.$$

Let us go back to $\mathfrak{t}^\vee$ for a while. We may "double" the Coxeter hyperplane arrangement inside $\mathfrak{t}^\vee$ to a codimension two arrangement in $\mathfrak{t} \oplus \mathfrak{t}^\vee$ as follows. Each $\alpha^\vee$ corresponds to a positive root α for G, whose vanishing locus is a hyperplane $\mathcal{V}_\alpha^\vee$ in $\mathfrak{t}$. Both $\alpha, \alpha^\vee$ also determine hyperplanes inside $\mathfrak{t} \oplus \mathfrak{t}^\vee$ by the same vanishing conditions, and by abuse of notation we will denote these also by $\mathcal{V}_\alpha, \mathcal{V}_\alpha^\vee$. By intersecting, we then get a codimension two subspace $\mathcal{V}_\alpha \cap \mathcal{V}_\alpha^\vee$. It is clear from the description that the union of these subspaces as α runs over Φ^+ is defined by the ideal

$$\bigcap_{\alpha \in \Phi^+} \langle y_\alpha, x_{\alpha^\vee} \rangle \subset \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}^\vee].$$

Here $x_{\alpha^\vee}$ and y_α are the linear functionals associated to $\alpha^\vee, \alpha$. Localizing away from the coordinate hyperplanes in $\mathfrak{t}^\vee$, we then see that the ideal $J_G \subset \mathbb{C}[T^*T^\vee]$ from earlier determines a doubled Coxeter arrangement inside $T^*T^\vee$. In fact, it is immediate from the description that its Zariski closure inside $T^*\mathfrak{t}^\vee$ equals $\bigcup_\alpha \mathcal{V}_\alpha \cap \mathcal{V}_\alpha^\vee$. In the GL_n case, this doubled subspace arrangement coincides with the one studied by Haiman. In [40, Problem 1.5(b)], Haiman poses the question of what happens for other root systems. Reinterpreting the doubling procedure to mean the root system and its (Langlands) dual in $T^*T^\vee$, instead of taking $\mathcal{V} \otimes \mathbb{C}^2 \subset \mathfrak{t} \otimes \mathbb{C}^2$, we have freeness of J_G in "half of the variables" by Theorem 2.1.1, which answers the question in *loc. cit.*

There are several other corollaries to Theorem 2.1.1 that we now illustrate.

Let $G = GL_n$. It is a conjecture of Bezrukavnikov (private communication) that under the lattice action of Λ on $H^*(\widetilde{\mathrm{Sp}}_\gamma)$, where $\gamma = zt$, we also have

$$H^*(\widetilde{\mathrm{Sp}}_\gamma)^\Lambda \cong DH_n$$

and

$$H^*(\mathrm{Sp}_\gamma) \cong DH_n^{sgn}.$$

While we are not able to prove said conjecture, we are able to prove an analogous statement in Borel-Moore homology for the *coinvariants* under the lattice action, see Theorem 2.4.16.

Theorem 2.1.5. *We have*

$$H_*(\mathrm{Sp}_\gamma)_\Lambda \cong DH_n^{sgn}.$$

Let us then discuss the freeness over $\mathrm{Sym}(\mathfrak{t})$ of the ideals $J_G^{(d)}$ and related ideals in more detail. For example, in type A, it is clear that the simultaneous substitution $x_i \mapsto x_i + c, c \in \mathbb{C}, i = 1, \dots, n$ leaves J_G invariant, so that the freeness over $\mathrm{Sym}(\mathfrak{t})$ of $\bigcap_{i \neq j} \langle x_i - x_j, y_i - y_j \rangle \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ can be deduced from that of J_G. We remark that the results of Section 2.4.3 can also be used to show this statement.

Theorem 2.1.6. *Let $G = GL_n$ and $J = \bigcap_{i \neq j} \langle x_i - x_j, y_i - y_j \rangle \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$. Then we have $\Delta^d \cdot H_*^T(\mathrm{Sp}_\gamma) \cong J_{\mathbf{x}}^d \subset \mathbb{C}[\mathbf{x}^\pm, \mathbf{y}]$, where the subscript $\mathbf{x}$ denotes localization in the $\mathbf{x}$-variables. In particular, $J^d \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is free over $\mathbb{C}[\mathbf{y}] := \mathbb{C}[y_1, \dots, y_n]$.*

It is somewhat subtle that Theorem 2.1.1 does not immediately imply the freeness over $\mathrm{Sym}(\mathfrak{t})$ of the ideals in $\mathbb{C}[T^*T^\vee], \mathbb{C}[T^*\mathfrak{t}^\vee]$ generated by the anti-invariants, even in type A. Of course, one would hope for a similar description as Haiman's for arbitary G, but it seems likely some modifications are in order outside of type A [22, 24].

Haiman's original proof [39] of a related stronger statement, "the Polygraph Theorem", implying the freeness of the anti-invariant ideal I and its powers over $\mathbb{C}[\mathbf{y}]$, and thus freeness of $J^d = J^{(d)}$ over $\mathbb{C}[\mathbf{y}]$, involves rather subtle commutative algebra. Until recently, it has been the only way of showing the freeness of $J^{(d)}$ without giving a clear conceptual explanation. On the other hand, Theorem 2.1.6 gives a quite hands-on explanation of this phenomenon. It does not seem to be impossible to use the representation-theoretic interpretation of $J^{(d)}$ and the S_n-action on $H_*^T(\mathrm{Sp}_\gamma)$ to try to directly attack freeness of I^d.

In fact, recent work of Gorsky-Hogancamp [**?**] on knot homology gives another proof of Theorem 2.1.6. Their results also rest on results of Elias-Hogancamp [17] on the HOMFLY homology of (n, dn)-torus links, which involves some quite nontrivial computations with Soergel bimodules. In this chapter, the complexity of the freeness statement is hidden in the cohomological purity of Sp_γ as proved by Goresky- Kottwitz-MacPherson [27].

2.1.2 Relation to braids

Let us first consider a general connected reductive group G. *Any* $\gamma \in \mathfrak{g} \otimes \mathbb{C}((t))$ gives a nonconstant (polynomial) loop $[\gamma] \in \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[t^{\pm}], \mathfrak{t}^{reg}/W)$, through which we get a conjugacy class $\beta \in \pi_1(\mathfrak{t}^{reg}/W) \cong \mathfrak{B}r_W$. Note that we do not have a natural choice of basepoint, so that β is not a bona fide element of the braid group, but just a conjugacy class.

Let now $G = GL_n$. Then the braid closure $\overline{\beta}$ is a knot or link in S^3. For links in the three-sphere, it is natural to consider various link invariants, such as the triply graded Khovanov-Rozansky homology (or HOMFLY homology) [52]. This is an assignment

$$\beta \mapsto \mathrm{HHH}(\overline{\beta})$$

of $\mathbb{Z}^{\oplus 3}$-graded $\mathbb{Q}$-vector spaces to braids, which factors through Markov equivalence. The invariant $\mathrm{HHH}(-)$ was recently generalized to y-ified HOMFLY homology in [**?**]. It is an assignment of $\mathbb{Z}^{\oplus 3}$-graded $\mathbb{C}[y_1, \ldots, y_m]$-modules to braids, and has many remarkable properties. We will discuss these in more detail in Section 2.5.

We are mostly interested in $\mathrm{HY}(-)$ for the braid associated to $\gamma = zt^d$, following previous parts of this introduction. In this case, β is the (nd)th power of a Coxeter braid $\mathbf{cox}_n$ (positive lift of the Coxeter element in S_n). In particular, β is the (d)th power of the *full twist* braid $\mathbf{cox}_n^n$. Note that ince β is central, it is alone in its conjugacy class and thus an actual braid. Taking the braid closure of β, it is well-known that we recover the (n, dn) torus link $T(n, dn)$.

Remark 2.1.7. The closures of powers of the Coxeter braids $\mathbf{cox}_G^m$ and their relation to affine Springer theory has appeared in the literature in several places [32, 75, 89], in the case where m is prime to the Coxeter number of G. The case we consider is the one where m is a multiple of the Coxeter number.

Now, progress in knot homology theory by several people [**?**, 17, 31, 66] has lead to an identification of the Hochschild degree zero part of the y-ified HOMFLY homology of (n, nd)-torus links and the ideals $J^d = \bigcap_{i<j} \langle x_i - x_j, y_i - y_j \rangle$ from above. In particular, combining these results and Theorem 3.0.1, we get the following corollary, proved in Corollary 2.5.4.

Corollary 2.1.8. *There is an isomorphism of* $\mathbb{C}[\mathbf{x}^{\pm}, \mathbf{y}]$*-modules*

$$\Delta^d H_*^T(\mathrm{Sp}_\gamma) \cong \mathrm{HY}(\mathrm{FT}_n^d)^{a=0} \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^{\pm}]$$

for $\gamma = zt^d$.

Remark 2.1.9. Assuming the purity of affine Springer fibers, one is able to deduce further corollaries to our results. If

$$\gamma = \begin{pmatrix} a_1 t^{d_1} & & \\ & \ddots & \\ & & a_n t^{d_n} \end{pmatrix},$$

the construction above gives us a pure braid β whose braid closure has linking numbers $d_{ij} = \min(d_i, d_j)$ between components i, j.

By [**?**, Proposition 5.5], if β has "parity", ie. $\mathrm{HHH}(\overline{\beta})$ is only supported in even or odd homological degrees, we have the following isomorphism of bigraded vector spaces

$$\mathrm{HY}^{a=0}(\overline{\beta}) \cong \cap_{i<j}(x_i - x_j, y_i - y_j)^{d_{ij}}.$$

By equivariant formality of $H_*(\mathrm{Sp}_\gamma)$, we then have in analogy to the equivalued case that

$$\prod_{i<j}(y_i - y_j)^{d_{ij}} H_*^T(\mathrm{Sp}_\gamma) \cong \bigcap_{i<j}\langle x_i - x_j, y_i - y_j\rangle^{d_{ij}} \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^{\pm}] \cong \mathrm{HY}^{a=0}(\overline{\beta}) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^{\pm}].$$

Remark 2.1.10. It is not clear to us what the correct analogues, if any, of these link-theoretic notions are for other root data. While the definition of the HOMFLY homology as Hochschild homology of certain complexes of Soergel bimodules [50] certainly makes sense in all types, many aspects of the theory, including the y-ification process, are undeveloped at the time. Work in progress by Hogancamp and Makisumi addresses some of these questions.

It is also interesting whether the resulting (Hochschild) homology of the (complex corresponding to the) full twist is parity, or related to J_G for other types.

2.1.3 Hilbert schemes of points on curves

It is useful to think of the link $\overline{\beta}$ from the previous section as the link of the plane curve singularity which is the pullback along γ of the universal spectral curve over $\mathfrak{t}^{reg}/\mathfrak{S}_n$. Recall that the *link* of $C \subset \mathbb{C}^2$ at $p \in C$ is the intersection of C with a small three-sphere centered at p. In particular, $Link(C,p)$ is a compact one-manifold inside S^3, i.e. a link in the previous sense. Motivated by conjectures of Gorsky-Oblomkov-Rasmussen-Shende [32, 73] there should then be a relationship of the affine Springer fibers, Hilbert schemes of points on the plane and link homology to the Hilbert schemes of the plane curve singularities $\{x^n = y^{dn}\}$. Namely, for $G = GL_n$ and

$$\gamma = \begin{pmatrix} a_1 t^d & & \\ & \ddots & \\ & & a_n t^d \end{pmatrix}$$

the characteristic polynomial of γ is

$$P(x) = \prod_i (x - a_i t^d).$$

We may assume that $a_i = \zeta^i$ for ζ a primitive nth root of unity, in which case $P(x) = x^n - t^{dn}$. This determines a spectral curve in $\mathbb{A}^2$ with coordinates (x, t), with a unique singularity at zero. It has a unique projective model with rational components and no other singularities. Call this curve C.

The compactified Jacobian of any curve C, denoted $\overline{\mathrm{Jac}}(C)$, is by definition the moduli space of torsion-free rank one, degree zero sheaves on C. It is known by eg. [72] that in the case when C has at worst planar singularities (and is reduced), we have a homeomorphism of stacks

$$\overline{\mathrm{Jac}}(C) \cong \mathrm{Jac}(C) \times^{\prod_{x\in C^{sing}} \mathrm{Jac}(C_x)} \prod_{x\in C^{sing}} \overline{\mathrm{Jac}}(C_x), \tag{2.1.1}$$

where $\overline{\mathrm{Jac}}(C_x)$ is a local version of the compactified Jacobian at a closed point $x \in C$, sometimes also called the Jacobi factor. In the case when $C = \{x^n = t^{dn}\}$, we have just a unique singularity and rational components, so that Eq. (2.1.1) becomes a homeomorphism between the moduli of fractional ideals in $\mathrm{Frac}(\mathbb{C}[[x, y]]/x^n - y^{dn})$ and the compactified Jacobian. From the lattice description of the affine Grassmannian, it is not too hard to show that this former space actually equals Sp_γ [96].

It is an interesting problem to determine the Hilbert schemes of points $C^{[n]}$ on these curves. These are naturally related to the compactified Jacobians via an Abel-Jacobi map, which has a local version as well. In the case when C is integral, it is known that this map becomes a $\mathbb{P}^{n-2g}$-bundle for $g \gg 0$. In general we only know that it is so for a union of irreducible components of the compactified Jacobian, of which there are infinitely many in the case when C has locally reducible singularities.

In [**?**], we have initiated an approach to computing $H_*(C^{[n]})$ where C is reducible, using a certain algebra action on

$$V := \bigoplus_{n\geq 0} H_*(C^{[n]}).$$

Note that this is a bigraded vector space, where one of the gradings is given by the number of points $(n, 0)$, and the other one is given by the homological degree $(0, j)$.

Theorem 2.1.11 ([?]). *Let*

$$A_m := \mathbb{C}[x_1, \dots, x_m, \partial_{y_1}, \dots, \partial_{y_m}, \sum_i \partial_{x_i}, \sum_i y_m] \subset \mathit{Weyl}(\mathbb{A}^{2m}),$$

where x_i carries the bigrading $(1,0)$ and y_i the bigrading $(1,2)$. Suppose C is locally planar and has m irreducible components. Then there is a geometrically defined action $A_m \times V \to V$.

Roughly speaking, the action on V is given as follows. For a fixed component C_i of C, the operator $x_i : V \to V$ adds points, and ∂_{y_i} removes them. These are defined using a choice of a point $c_i \in C_i$ and a corresponding embedding $C^{[n]} \hookrightarrow C^{[n+1]}$. On the other hand, the operator $\sum_i \partial_{x_i} : V \to V$ removes a "floating" point and $\sum_i y_i$ adds a floating point. These are defined as Nakajima correspondences.

The original computation of T-equivariant homology of affine Springer fibers in [26] for $G = GL_2$ bears a striking resemblance to the second main result in [?]. In particular, if C is the union of two projective lines along a point,

$$V \cong \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{(x_1 - x_2)\mathbb{C}[x_1, x_2, y_1 + y_2]}.$$

Furthermore, when $G = GL_2$, we have

$$H^T_{*,ord}(\mathrm{Sp}_{tz}) = \frac{\mathbb{C}[x_1^\pm, x_2^\pm, y_1, y_2]}{(x_1 - x_2)\mathbb{C}[x_1^\pm, x_2^\pm, y_1 + y_2]}.$$

Here $H^T_{*,ord}(-)$ means the Borel construction of ordinary T-equivariant homology. See Theorem 2.6.5 for a more general statement.

Based on computations in [?] and some new examples in Section 2.6, we are lead to conjecture the following.

Conjecture 2.1.12. *Let C be the (unique) compactification with rational components and no other singularities of the curve $\{x^n = y^{dn}\}$. Then as a bigraded A_n-module, we have*

$$V := \bigoplus_{m \geq 0} H_*(C^{[m]}, \mathbb{Q}) \cong \frac{\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]}{\sum_{i \neq j} \sum_{k=1}^d (x_i - x_j)^k \ker(\partial_{y_i} - \partial_{y_j})^k}. \tag{2.1.2}$$

2.1.4 Organization

The organization of the chapter is as follows. In Section 2.2 we give background on affine Springer fibers. In Section 2.3 we compute the torus equivariant Borel-Moore homology of the affine Springer fibers we are interested in, following Goresky-Kottwitz-MacPherson and Brion. In Section 2.4, we give background on Hilbert schemes of points on the plane and relate results from the previous sections with those of Haiman. We also discuss our results and their implications in this direction for arbitrary G in Section 2.4.4. In Section 2.5, we relate the equivariant Borel-Moore homology of affine Springer fibers with braid theory, and in the type A case with the knot homology theories of Khovanov-Rozansky and Gorsky-Hogancamp. Finally, in Section 2.6 we compute some new examples and make a conjecture describing the structure of the homology of Hilbert schemes of points on the plane curves $\overline{\{x^n = y^{dn}\}}$.

2.2 Affine Springer fibers

In this section, we define the affine Springer fibers we are considering. For more details on the definitions, see the notes of Yun [96]. Let G be a connected reductive group over $\mathbb{C}$. Choose $T \subset B \subset G$ a maximal torus and a Borel subgroup as per usual. We denote the Lie algebras of G, B, T respectively by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$.

Denote the lattice of cocharacters $X_*(T) = \Lambda$ and the Weyl group W. Let the extended affine Weyl group be $\widetilde{W} := \Lambda \rtimes W$. We use this convention to align with [26].

If R is a $\mathbb{C}$-algebra and F represents an fpqc sheaf out of Aff$/\mathbb{C}$, we let $F(R)$ be the associated functor of points evaluated at R (for an excellent introduction to these notions in the context we are interested in, see notes of Zhu [97]). Often when $R = \mathbb{C}$, we omit it from the notation and simply refer by F to the closed points.

Denote the affine Grassmannian of G by Gr_G and its affine flag variety by Fl_G. These are naturally ind-schemes. If $G = GL_n$, we will often write just Gr_n and Fl_n. Write $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Then $\mathrm{Gr}_G(\mathbb{C}) = G(\mathcal{K})/G(\mathcal{O})$ and $\mathrm{Fl}(\mathbb{C}) = G(\mathcal{K})/\mathbf{I}$, where $\mathbf{I}$ is the Iwahori subgroup corresponding to the choice of B and the uniformizer t. Let $\widetilde{T} := T \rtimes \mathbb{G}_m^{rot}$ be the extended torus, where $a \in \mathbb{G}_m^{rot}$ scales t by $t \mapsto at$.

There is a left action of $T(\mathbb{C})$ on $\mathrm{Gr}_G(\mathbb{C})$ and $\mathrm{Fl}_G(\mathbb{C}) = G(\mathcal{K})/\mathbf{I}$. This action is topological in the analytic topology. Its fixed points are determined using the following Bruhat decompositions:

$$G(\mathcal{K}) = \bigsqcup_{\lambda\in\Lambda} \mathbf{I}t^\lambda G(\mathcal{O}) = \bigsqcup_{w\in\widetilde{W}} \mathbf{I}t^w\mathbf{I}.$$

Since $T(\mathbb{C})$ acts nontrivially on the real affine root spaces in $\mathbf{I}$, and fixes the cosets $t^\lambda G(\mathcal{O})$, $t^w\mathbf{I}$ respectively, we see that the fixed point sets are discrete, and in a natural bijection with $\Lambda, \widetilde{W}$.

Definition 2.2.1. Let $\gamma \in \mathrm{Lie}(G) \otimes_{\mathbb{C}} \mathcal{K}$. The affine Springer fibers $\mathrm{Sp}_\gamma \subset \mathrm{Gr}_G$ and $\widetilde{\mathrm{Sp}}_\gamma \subset \mathrm{Fl}_G$ are defined as the reduced sub-ind-schemes of Gr_G and Fl_G whose complex points are given by

$$\mathrm{Sp}_\gamma(\mathbb{C}) = \{gG(\mathcal{K})|g^{-1}\gamma g \in \mathrm{Lie}(G) \otimes_{\mathbb{C}} \mathcal{O}\}$$

$$\widetilde{\mathrm{Sp}}_\gamma(\mathbb{C}) = \{g\mathbf{I}|g^{-1}\gamma g \in \mathrm{Lie}(\mathbf{I})\}.$$

2.3 Equivariant Borel-Moore homology of affine Springer fibers

In this section, we prove the main theorem of this chapter, Theorem 3.0.1. We thank Eric Vasserot and Peng Shan for pointing out a mistake in the previous formulation and proof of Lemma 2.3.10.

2.3.1 Borel-Moore homology

We now review equivariant Borel-Moore homology. The paper [**?**] is the main reference for this section. For a projective (but not necessarily irreducible) variety X, one defines the Borel-Moore homology as $H_*(X) := H^{-*}(X, \omega_X)$, where ω_X is the Verdier dualizing complex in $D^b_c(X)$. Note that we use $H_*(-)$ for *Borel-Moore* homology, not the usual singular or étale homologies.

For a T-variety X, where $T \cong \mathbb{G}_m^n$ is a diagonalizable torus, imitating the Borel construction of equivariant (co)homology is not completely straightforward, as the classifying space BT is not a scheme-theoretic object. However, using approximation by m-skeleta as in [?], or a simplicial resolution of BT as in [?], one gets around the issue by defining

$$H_k^T(X) := H_{k+2mn}(X \times^T ET_m),\ m \geq \dim X - k/2.$$

Here $ET_m := (\mathbb{C}^{m+1} - 0)^d$ with the T-action $(t_1, \ldots, t_d) \cdot (v_1, \ldots, v_d) = (t_1v_1, \ldots, t_dv_d)$. This action is free, and the quotient $ET_m \to (\mathbb{P}^m)^d$ is a principal T-bundle.

The above definition of $H_k^T(X)$ is independent of m as follows from the Gysin isomorphism $H_{k+2m'n}(X \times^T ET_{m'}) \to H_{k+2mn}(X \times^T ET_m)$ for $m' \geq m \geq \dim X - k/2$. Note that $H_*^T(X)$ is a graded module over $H_T^*(X)$ via the cap product and in particular a graded module over $H_*^T(pt)$.

Recall that X is *equivariantly formal* (see [25, 26]) if the Leray spectral sequence

$$H^p(BT, H^q(X)) \Rightarrow H_T^{p+k}(X)$$

degenerates at E_2. If X is equivariantly formal, then $H_*^T(X)$ is a *free* $H_T^*(pt)$-module [?, Lemma 2].

The above definition of $H_*^T(-)$ enjoys some of the usual localization properties, as studied e.g. in [?]. For example, we have an "Atiyah-Bott" formula [?, Lemma 1].

Theorem 2.3.1. *Suppose the T-action on X has finitely many fixed points. Let i_* : $H_*^T(X^T) \to H_*(X)$ be the $\mathbb{C}[\mathfrak{t}]$-linear map given by the inclusion of the fixed-point set to X. Then i_* becomes an isomorphism after inverting finitely many characters of T.*

From the perspective of commutative algebra, it is useful to note the following from [?, Proposition 3].

Proposition 2.3.2. *If X is equivariantly formal, then*

$$H_*^T(X) \cong \mathrm{Hom}_{\mathbb{C}[\mathfrak{t}]}(H_T^*(X), \mathbb{C}[\mathfrak{t}]).$$

The map is given by

$$\alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha)),$$

where $p_X : X \times^T ET \to BT$ is the projection.

Another localization theorem was proved in [25, Theorem 7.2] for T-equivariant (co)homology. As in [?, Corollary 1], it is translated to Borel-Moore homology as follows.

Proposition 2.3.3. *Let X be an equivariantly formal T-variety containing only finitely many orbits of dimension ≤ 1. Then $H_*^T(X) \cong i_*^{-1} H_*^T(X) \subset H_*^T(X^T) \otimes \mathbb{C}(\mathfrak{t})$ consists of all tuples $(\omega_x)_{x \in X^T}$ of rational differential forms on $\mathfrak{t}$ satisfying the following conditions.*

1. *The poles of each ω_x are contained in the union of singular hyperplanes and have order at most one. Recall that a* singular hyperplane *in $\mathfrak{t}$ is the vanishing set of $d\chi$, where $X^{\ker \chi} \neq X^T$ and $\ker \chi$ is the codimension one subtorus of T defined by χ.*

2. *For any singular character χ and for any connected component Y of $X^{\ker \chi}$, we have*
$$\mathrm{Res}_{\chi=0}\left(\sum_{x \in Y^T} \omega_x\right) = 0.$$

As the number of orbits of dimension ≤ 1 is finite, and the closure of each one-dimensional orbit contains exactly two fixed points (see [25]), it is natural to form the graph whose vertices are the fixed points and edges correspond to one-dimensional orbits. We call the associated weighted graph whose edges are labeled by the differentials $d\chi$ of singular characters the *GKM graph*.

Note that it is easy to recover $H_*(X)$ from $H_*^T(X)$ for equivariantly formal varieties by freeness, as shown in [?, Proposition 1]. Namely, we have

Proposition 2.3.4. *Let $T' \subset T$ be a subtorus. Then*
$$H_*^{T'}(X) \cong \frac{H_*^T(X)}{Ann(\mathfrak{t}') \cdot H^*},$$
where $Ann(\mathfrak{t}') \subset \mathbb{C}[\mathfrak{t}]$ is the annihilator of $\mathfrak{t}' = \mathrm{Lie}(T')$. In particular, when T' is trivial, we get
$$H_*(X) = \frac{H_*^T(X)}{\mathbb{C}[\mathfrak{t}]_+ H_*^T(X)}.$$

Ultimately, we are interested in the equivariant Borel-Moore homology of the ind-projective varieties $\mathrm{Sp}_{t^d z}$. Suppose now that $X = \varinjlim X_i$ is an ind-scheme over $\mathbb{C}$ given by a diagram
$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

where the maps are T-equivariant closed immersions and each X_i is projective. By properness and the definition of $H_*^T(-)$, there are natural pushforwards

$$H_*^T(X_i) \to H_*^T(X_{i+1}),$$

using which we *define*

$$H_*^T(X) := \varinjlim H_*^T(X_i).$$

The usual (non-equivariant) Borel-Moore homology is defined similarly. Note that since the X_i are varieties we are still abusing notation and mean $X_i(\mathbb{C})$ when taking homology.

Remark 2.3.5. While $H_*(-)$ and $H_*^T(-)$ could be defined for any finite-dimensional locally compact, locally contractible and σ-compact topological space X using the sheaf-theoretic definition [**?**, Corollary V.12.21.], it is *not* true that this definition gives the same answer for $X(\mathbb{C})$ as the above definition (there's always a map in one direction). For example, if $X(\mathbb{C}) = \varinjlim[-m, m] \cong \mathbb{Z}$ is the colimit of the discrete spaces $[-m, m] \subset \mathbb{Z}$, which are of course also the $\mathbb{C}$-points of a disjoint union of $2m + 1$ copies of $\mathbb{A}^0$, then $H^{-*}(X, \omega_X) \cong \mathbb{C}^{\mathbb{Z}}$ is the homology of the one-point compactification of $\mathbb{Z}$ with the cofinite topology, while treating X as an ind-variety we get $H_*(X) \cong \mathbb{C}^{\oplus\mathbb{Z}}$.

Call a T-ind-scheme X *equivariantly formal* if each X_i is equivariantly formal and T-stable. Call it *GKM* if each X_i has finitely many orbits of dimension ≤ 1. We have the following corollary to Theorem 2.3.3.

Corollary 2.3.6. *Let X be an equivariantly formal GKM T-ind-scheme. Then $H_*^T(X) \subset H_*^T(X^T) \otimes \mathbb{C}(\mathfrak{t})$ consists of all tuples $(\omega_x)_{x\in X^T}$ of rational differential forms on $\mathfrak{t}$ satisfying the conditions in Theorem 2.3.3.*

Proof. By assumption, we have inclusions of T-fixed points $X_i^T \to X_{i+1}^T$ and their union is X^T. Taking the colimit of $H_*^T(X_i) \hookrightarrow H_*^T(X_i^T)$, we get by exactness

$$\iota : H_*^T(X) := \varinjlim H_*^T(X_i) \hookrightarrow to \varinjlim H_*^T(X_i^T) =: H_*^T(X^T),$$

which becomes an isomorphism when tensoring with $\mathbb{C}(\mathfrak{t})$. Any tuple $(\omega_x)_{x\in X^T}$ of rational differential forms (of top degree) on $\mathfrak{t}$ inside $\iota_*^{-1} H_*^T(X)$ has some i such that it is in the image of $\iota_*^{-1} H^T(X_i)$. By Proposition 2.3.3, it therefore satisfies the desired conditions.

□

Remark 2.3.7. While the number of fixed points and one-dimensional orbits might now be infinite, we may still form the (possibly infinite) GKM graph.

2.3.2 The SL_2 case

We first prove Theorem 3.0.1 in the case $G = SL_2$. Recall that $\widetilde{T} = T(\mathbb{C}) \times \mathbb{C}^* \subset G((t))$ denotes the extended torus. As shown in [26, Lemma 6.4], for $G = SL_2$ the one-dimensional $\widetilde{T}$-orbits of $X_d := \mathrm{Sp}_{t^d z}$ are given as follows. If we identify $\mathrm{Sp}_{t^d z}^{\widetilde{T}} = \mathbb{Z}$, then there is an orbit between $a, b \in \mathbb{Z}$ if and only if $|a - b| \leq d$. Moreover, $\widetilde{T}$ acts on this orbit through the character (in fact, real affine root) $(\alpha, a+b) \in X_*(\widetilde{T}) \cong \Lambda \times \mathbb{Z}$. Identify further the differential of this character by $y + (a+b)t \in \mathbb{C}[\widetilde{\mathfrak{t}}]$.

Recall that the affine Grassmannian of SL_2 decomposes as the the disjoint union of finite-dimensional Schubert cells $\mathrm{Gr}_{SL_2}^m := SL_2(\mathcal{O})t^\lambda SL_2(\mathcal{O})$. Let $\mathrm{Gr}_{SL_2}^{\leq m} = \overline{\mathrm{Gr}_{SL_2}^m} = \bigsqcup_{l \leq m} \mathrm{Gr}_{SL_2}^l$. It is clear that the subvarieties $X_d^{\leq m} := (\mathrm{Sp}_{t^d z})^{\leq m} = \overline{\mathrm{Sp}_{t^d z} \cap \mathrm{Gr}_{SL_2}^{\leq m}}$ are $\widetilde{T}$-stable. The corresponding GKM graph is just the induced subgraph formed by the vertices $[-m, m] \subset \mathbb{Z}$. In particular, we may compute $H_*^{\widetilde{T}}(X_m)$ using Theorem 2.3.3 for the corresponding GKM graphs. Note that each such graph in this case is a chain of complete graphs on d vertices glued along $d-1$ vertices. Let us first practice the case when the length of the chain is one, i.e. we are computing the $\widetilde{T}$-equivariant Borel-Moore homology of the classical Springer fiber $sp_e \subset \mathrm{Gr}(2d, d)$, where e is the square of a regular nilpotent element (see [?]). This is essentially a projective space of dimension d.

Example 2.3.8. Let $d = 1$. Then the GKM graph of sp_e is two vertices joined by a line, with the character $y + t$. Theorem 2.3.3 then tells us that

$$i_* : H_*^T(sp_e^T) \to H_*^T(sp_e)$$

is injective and $(i_*)^{-1} H_*^T(sp_e)$ consists of rational differential forms (ω_0, ω_1) so that

$$\mathrm{Res}_{y=-t}(\omega_0 + \omega_1) = 0$$

with poles of order at most one and along $y = -t$. In particular, any polynomial linear combination of $a = (\frac{dydt}{y+t}, \frac{-dydt}{y+t})$ and $b = (dydt, 0)$ satisfies these requirements and is the most general choice, so we conclude $H_*^T(X)$ is a free $\mathbb{C}[y,t]$-module with basis a, b. As $sp_e = \mathbb{P}^1$ is smooth, we further use the Atiyah-Bott localization theorem to conclude that $a = [\mathbb{P}^1]$.

From now on, we will save notation and write each tuple of differential forms $(\omega_1, \dots, \omega_q) = (f_1 dydt, \dots, f_q dydt)$ simply as $(f_1, \dots f_q)$.

Let us now compute $H_*^T(\mathrm{Sp}_{tz})$ for $G = SL_2$ for illustrative purposes. This is very similar to Example 2.3.8.

Proposition 2.3.9. *If $d = 1$ and $G = SL_2$, then $H_*^{\widetilde{T}}(\mathrm{Sp}_{tz})$ is the $\mathbb{C}[t,y]$-linear span of*

$$a = (\dots, 0, 0, 1, 0, 0, \dots)$$

and

$$b_i = (\dots, 0, \frac{1}{(2i+1)t+y}, \frac{-1}{(2i+1)t+y}, 0, \dots),$$

where the 1 in a is at the 0th position and the nonzero entries in b_i are at the ith and $(i+1)$th positions, respectively. In particular,

$$\frac{H_*^{\widetilde{T}}(X)}{t \cdot H_*^{\widetilde{T}}(X)} \cong H_*^T(X)$$

is isomorphic to the $\mathbb{C}[y]$-linear span of a and $b_i' = (\dots, 0, 1/y, -1/y, 0, \dots)$.

Proof. By the discussion above, the GKM graph has vertices $\mathbb{Z}$ and edges exactly between $i, i+1$ for all i. Indeed, it is well-known that X_1 is just an infinite chain of projective lines. The weights of the edges for the $\widetilde{T}$-action are given by the character $(2i+1)t + y$ by [26, Lemma 6.4.]. Applying Corollary 2.3.6 we get the first claim. Setting t to zero recovers $H_*^T(X)$, so that we get the second result. $\square$

Lemma 2.3.10. *Let $d \geq 1$. Then the $\widetilde{T}$-equivariant Borel-Moore homology of $X_d = \mathrm{Sp}_{t^d z}$ is the $\mathbb{C}[t,y]$-linear span of*

$$
\begin{aligned}
a_0 &= (\dots, 0, 0, 1, 0, 0, \dots) \\
a_1 &= (\dots, 0, 0, \frac{1}{y+t}, \frac{-1}{y+t}, 0, \dots) \\
&\vdots \\
a_{d-1} &= (\dots, 0, 0, \frac{1}{\prod_{i=1}^{d-1}(y+it)}, \frac{-\binom{d-1}{1}}{(y+t)\prod_{i=2}^{d-1}(y+(i+1)t)}, \dots, \frac{(-1)^{d-1}\binom{d-1}{d-1}}{(\prod_{i=1}^{d-1}(y+(d-1+i)t)}, 0, \dots) \\
b_k &= (\dots, 0, 0, \frac{\binom{d}{0}}{f_k^{(1)}}, \frac{-\binom{d}{1}}{f_k^{(2)}}, \dots, \frac{(-1)^d\binom{d}{d}}{f_k^{(d)}}, 0, \dots), \ k \in \mathbb{Z},
\end{aligned}
$$

where

$$
f_k^{(j)} = \prod_{i=0}^{j-1}(y+(2k+i+j)t) \prod_{i=j+1}^{d}(y+(2k+i+j)t), \quad j = 1, \dots, d.
$$

Here the nonzero entries in a_i are at $0, \dots, i$ and the nonzero entries in b_k are at $k, \dots, k+d$.

In particular, letting $t = 0$,

$$
H_*^T(X_d) \subseteq H_*^T(\Lambda)
$$

is the $\mathbb{C}[y]$-linear span of

$$
\begin{aligned}
a_0' &= (\dots, 0, 0, 1, 0, 0, \dots) \\
a_1' &= (\dots, 0, 0, \frac{1}{y}, \frac{-1}{y}, 0, \dots) \\
&\vdots \\
b_k' &= (\dots, 0, 0, \frac{\binom{d}{0}}{y^d}, \frac{-\binom{d}{1}}{y^d}, \dots, \frac{(-1)^{d-1}\binom{d}{d-1}}{y^d} \frac{(-1)^d\binom{d}{d}}{y^d}, 0, \dots), \ k \in \mathbb{Z}.
\end{aligned}
$$

Note that if we write $\mathbb{C}[\Lambda] = \mathbb{C}[x^{\pm}]$, then in the monomial basis $a_0' = x^0$, $a_1' = \frac{1-x}{y}$, and $b_k' = x^k(1-x)^d/y^d$.

Proof. Let us first check the residue conditions of Corollary 2.3.6. Note that $a_0, \dots, a_{d-1}$ are just b_0 for some smaller d, in particular it is enough to check the conditions for b_k. There is an orbit between $k+j$ and $k+j'$ whenever $|j-j'| \le d$, and $\widetilde{T}$ acts on said orbit via $\chi = y + (2k+j+j')t$. In particular, we need to prove that

$$
\operatorname{Res}_{y=-(2k+j+j')t} \left(\frac{(-1)^j\binom{d}{j}}{f_k^{(j)}} + \frac{(-1)^{j'}\binom{d}{j'}}{f_k^{(j')}} \right) = 0.
$$

First, we compute that

$$f_k^{(j)} = \prod_{i \neq j, 1 \leq i \leq d} (y + (2k + i + j)t),$$

so the residue at $y = -(2k + j + j')t$ of $1/f_k^{(j)}$ equals

$$\frac{1}{\prod_{i \neq j, j'} (i - j')t} = \frac{(j - j')}{\prod_{i \neq j'} (i - j')t} = \frac{(j - j')}{(-1)^{j'} (j')!(d - j')!}.$$

If we multiply this by

$$(-1)^j \binom{d}{j},$$

we get

$$\frac{(j - j')d!}{(-1)^{j'+j} (j')!(d - j')!j!(d - j)!},$$

which is antisymmetric under switching j and j'. By linearity of taking residues, we get the result.

We need to show the reverse inclusion. Let sp_d be the Spaltenstein variety of d-planes in $\mathbb{C}^{2d}$ stable under the (d, d)-nilpotent element. From [?, page 448], we know that X_d is an infinite chain of sp_d glued along sp_{d-1}. In addition, $X_d^{\leq m}$ from the beginning of Section 2.3.2 is a chain of $2m$ copies of sp_d glued along sp_{d-1}. From the form of the GKM graph it is immediate that the T-equivariant Borel-Moore homology of $X_d^{\leq m}$ as a graded $\mathbb{C}[y, t]$-module looks like that of a chain of $2m$ copies of $\mathbb{P}^d$ consecutively glued along $\mathbb{P}^{d-1}$. In particular, $H_*^T(X_d^{\leq m})$ has rank 1 over $\mathbb{C}[y, t]$ in degrees $\leq 2d - 2$ and rank $2m$ in degree $2d$. Since the classes b_i for $i = -m, \ldots, m$ are linearly independent over $\mathbb{C}[y, t]$ and there are $2m$ of them, the b_i must span $H_{2d}^T(X_d^{\leq m})$. Taking the colimit, the first result follows. The second result is immediate from the form of $f_k^{(j)}$ and setting $t = 0$.

□

Remark 2.3.11. In [26, Section 12], the analogues of the classes b_k are played by the polynomials denoted $f_{k,d}$ in *loc. cit.* They are the ones attached to "constellations" of one-dimensional orbits.

Remark 2.3.12. In Proposition 2.3.9 and Lemma 2.3.10, the polynomials $f_k^{(j)}$ that appear seem to be related to the affine Schubert classes in $H_*^T(X_d)$ given by intersections by $G(\mathcal{O})$-orbits on Gr_{SL_2}. In case the components ($\cong sp_d$) are rationally smooth, which we suspect to be true but could not find a reference for, $f_k^{(j)}$ are exactly the inverses to $\widetilde{T}$-equivariant Euler classes of the kth irreducible component at the fixed point $j \in \Lambda$. Note that rational smoothness follows from for example Poincaré duality.

2.3.3 The general case

In this section, we prove Theorem 3.0.1. The GKM graph for $\widetilde{T}$ acting on $\mathrm{Sp}_{t^d z}$ is always infinite; indeed we have the following.

Lemma 2.3.13. *The vertices of the GKM graph of* $\mathrm{Sp}_{t^d z}$ *are* $\Lambda = X_*(T)$ *and there is an edge* $\lambda \to \mu$ *whenever* $\lambda - \mu = k\alpha$*, where* $\alpha \in \Phi^+$ *and* $k \leq d$*.*

Proof. From [26, Lemma 5.12], we know that the one-dimensional $\widetilde{T}$-orbits are $(\mathrm{Sp}_{t^d z})_1 = \bigcup_{\alpha \in \Phi^+} (\mathrm{Sp}^\alpha_{t^d z})_1$ and $\mathrm{Sp}^\alpha_{t^d z} \cap \mathrm{Sp}^\beta_{t^d z} = \Lambda$ unless $\beta = \alpha$. In particular, we are reduced to the semisimple rank 1 case which is reduced to the SL_2 case by [26, Lemma 8.1] and the SL_2 case is handled by Lemma 6.4 in *loc. cit.*. $\square$

We also need the following corollary to Lemma 2.3.10.

Corollary 2.3.14. *Let* $\alpha \in \Phi^+$*, and let* $y_\alpha \in \mathbb{C}[\mathfrak{t}] = H^*_T(pt)$ *be the linear functional corresponding to* α*. Denote* $X^\alpha_d := \mathrm{Sp}^\alpha_{zt^d} := \mathrm{Sp}_{zt^d} \cap \mathrm{Gr}_{H^\alpha}$*. For any* G *and* $\alpha \in \Phi^+(G,T)$*, we have*

$$y^d_\alpha H^T_*(X^\alpha_d) = J^d_\alpha = \langle y_\alpha, 1 - \alpha^\vee \rangle^d \subset H^T_*(\Lambda) = \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}].$$

Here $\langle S \rangle$ *means the ideal in* $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$ *generated by the subset* S*.*

Proof. Since X^α_d is an unramified affine Springer fiber of valuation d for a semisimple rank one group, it is a disjoint union of infinite chains of Spaltenstein varieties sp_d, as explained in Section 2.3.2. More precisely, it is a disjoint union of such over $\Lambda/\langle \alpha^\vee \rangle$ inside X_d. Identify $H^T_*(\Lambda)$ with $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$ and write its elements $\mathbb{C}[\mathfrak{t}]$-linear combinations of $x^\lambda := x^\lambda \otimes 1$. From Lemma 2.3.10 and [26, Lemma 6.4], we have that $H^*_T(X^\alpha_d) \subset H_*(\Lambda) \otimes \mathbb{C}(\mathfrak{t})$ is the $\mathbb{C}[\mathfrak{t}]$-linear span of

$$\frac{x^\lambda (1 - x^{\alpha^\vee})^d}{y^d_\alpha}$$

and

$$\frac{(1-x^{\alpha^\vee})^k}{y_\alpha^k}$$

for $k = 0, \ldots, d-1$. In particular, $y_\alpha^d H_T^*(\mathrm{Sp}_{zt}^\alpha) \subset \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$ is identified with the ideal

$$J_\alpha^d = \langle (1-x^{\alpha^\vee})^d, (1-x^{\alpha^\vee})^{d-1} y_\alpha, \ldots, (1-x^{\alpha^\vee}) y_\alpha^{d-1}, y_\alpha^d \rangle.$$

□

Theorem 2.3.15. *Let $\Delta = \prod_\alpha y_\alpha \in H_T^*(pt)$ be the Vandermonde element. The equivariant Borel-Moore homology of $X_d := \mathrm{Sp}_{t^d z}$ for a reductive group G is up to multiplication by Δ^d canonically isomorphic as a (graded) $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$-module to the ideal*

$$J^{(d)} = \bigcap_{\alpha \in \Phi^+} J_\alpha^d \subset \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}].$$

In particular, there is a natural algebra structure on $\Delta^d H_*^T(\mathrm{Sp}_\gamma)$ inherited from $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]$, and $J^{(d)}$ is a free module over $\mathbb{C}[\mathfrak{t}]$.

Proof. By [26, Lemma 5.12] and Corollary 2.3.6, we have that $H_*^T(X_d) = \bigcap_\alpha H_*^T(X_d^\alpha) \subset H_*^T(\Lambda) \otimes \mathbb{C}(\mathfrak{t})$. By equivariant formality and Corollary 2.3.6, we furthermore have that

$$\Delta^d \cdot H_*^T(X_d) \subset H_*^T(\Lambda)$$

is a free $\mathbb{C}[\mathfrak{t}]$-module. Since $J_\alpha^d = y_\alpha^d H_*^T(\mathrm{Sp}_{t^d z}^\alpha)$ contains Δ, we must have $\Delta^d \cdot H_*^T(X_d) \subseteq J_\alpha^d$ for all α. Inverting Δ, we see that

$$\Delta^d \cdot H_*^T(X_d)_\Delta \cong \left(\bigcap_\alpha J_\alpha^d \right)_\Delta .$$

But $\Delta^d \cdot H_*^T(\mathrm{Sp}_{tz})$ was free over $\mathbb{C}[\mathfrak{t}]$, so by [?, Lemma 6.14], we have that $J^{(d)} = \Delta^d \cdot H_*^T(X_d)$. □

Remark 2.3.16. A priori, it is not at all obvious that $H_*^T(\mathrm{Sp}_{t^d z}^\alpha)$ would be a $\mathbb{C}[\Lambda]$-submodule of $H_*^T(\Lambda)$. The product structure on $H_*^T(\Lambda)$, while obvious in the algebraic statements, is geometrically a *convolution product*. In fact, it is the convolution product on the affine Grassmannian of T, as discussed in [?], and more recently [?] in the guise of a "$3d$ $\mathcal{N} = 4$ Coulomb branch for $(T, 0)$". Moreover, it is also nontrivial that $y_\alpha^d H_*^T(\mathrm{Sp}_{t^d z}^\alpha)$ should have a natural subalgebra structure.

Remark 2.3.17. It seems difficult to carry out analysis similar to Remark 2.3.12 for the case of general G. Erik Carlsson has informed us that he has performed computations related to X_d using affine Schubert calculus (see also [**?**]). It would be interesting to relate the two approaches.

2.3.3.1 The affine flag variety

In this section, we consider $Y_d = \widetilde{\mathrm{Sp}}_\gamma$, where $\gamma = zt^d$. We focus on the case $d = 1$. The $\widetilde{T}$-fixed points of Y_d are in a natural bijection with $\widetilde{W} = \Lambda \rtimes W$. For $G = SL_2$, it is known that Y_1 is an infinite chain of projective lines again, and if we write elements of $\widetilde{W}$ as (k, w), $k \in \mathbb{Z}$, $w \in \{1, s\}$, there are one-dimensional orbits precisely between $(k, 1)$ and (k, s) as well as $(k + 1, 1)$ and (k, s), see [26, Section 13].

Lemma 2.3.18. *When $G = SL_2$, we have that $H_*^{\widetilde{T}}(Y_1) \subset H_*^{\widetilde{T}}(\widetilde{W})$ is the $\mathbb{C}[y, t]$-linear span of the classes*

$$\begin{aligned}
a_0 &= (\dots, 0, 0, 1, 0, 0, \dots) \\
b_k &= (\dots 0, 0, \frac{1}{y + 2kt}, -\frac{1}{y + 2kt}, 0, 0, \dots) \\
b'_k &= (\dots 0, 0, \frac{1}{y + (2k-1)t}, 0, 0, -\frac{1}{y + (2k-1)t}, 0, 0, \dots)
\end{aligned}$$

where b_k has nonzero entries at positions $(k, 1)$ and (k, s) and similarly b'_k has nonzero entries at $(k, 1)$ and $(k - 1, s)$. In particular, by setting $t = 0$, we get that $H_*^T(Y_1)$ is

$$\left\{\frac{1-s}{y}, \frac{1-x}{y}, 1\right\} \cdot \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}] \subset \mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[\mathfrak{t}].$$

Proof. The residue conditions needed to apply Corollary 2.3.6 are almost exactly the same as in Proposition 2.3.9. The second claim follows from the fact that in $\mathbb{C}[\widetilde{W}]$, we may compute

$$-(1-s)\cdot(\lambda, 1) + (1-\alpha^\vee)\cdot(\lambda, 1) = -(\lambda, 1) + (\lambda, s) + (\lambda, 1) - (\lambda+1, 1) = (\lambda, s) - (\lambda+1, 1) = -b'_k|_{t=0} y.$$

$\square$

Corollary 2.3.19. *Let $y_\alpha \in \mathbb{C}[\mathfrak{t}] = H_T^*(pt)$ be the linear functional corresponding to α and $Y_d^\alpha := \widetilde{\mathrm{Sp}}_{zt^d}^\alpha := \widetilde{\mathrm{Sp}}_{zt^d} \cap \mathrm{Fl}_{H^\alpha}$. For any G and $\alpha \in \Phi^+(G, T)$, we have*

$$\widetilde{J}_\alpha := y_\alpha H_*^T(Y_1^\alpha) = \left\{1 - s_\alpha, 1 - x^{\alpha^\vee}, y_\alpha\right\} \cdot \mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}] \subset \mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[\mathfrak{t}].$$

Proof. This is similar to Corollary 2.3.14 and [26, page 547]. The affine Springer fiber Y_1^α is again a disjoint union of infinite chains of projective lines indexed by $\Lambda/\langle\alpha^\vee\rangle$. From this fact and the previous Corollary, we get that $H_*^T(Y_1^\alpha)$ is the $\mathbb{C}[\mathfrak{t}]$-linear span of $\frac{x^\lambda(1-x_\alpha)}{y_\alpha}, \frac{(1-s_\alpha)x^\lambda}{y_\alpha}$ and 1. Multiplying by y_α, we get the result. □

Theorem 2.3.20. *For any reductive group G,*

$$\Delta \cdot H_*^T(Y_1) = \bigcap_\alpha \widetilde{J}_\alpha \subset \mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[\mathfrak{t}]$$

and furthermore $\widetilde{J}_G$ is a free module over $\mathbb{C}[\mathfrak{t}]$. Here $\Delta = \prod_\alpha y_\alpha$ as before.

Proof. The proof is entirely similar to Theorem 3.0.1. □

Remark 2.3.21. It is not at all clear from this description whether $\Delta \cdot H_*^{\widetilde{T}}(Y_1)$ has an algebra structure. Based on Conjecture 2.4.13 and the fact that there is a (noncommutative) algebra structure when $d = 0$, it seems that this could be the case.

2.3.3.2 Equivariant K-homology

In this section, we state a version of Theorem 3.0.1 in K-homology. We omit detailed proofs because they are entirely parallel to those in previous sections.

In [?], more general equivariant cohomology theories, such as the equivariant K-theory of (reasonably nice) T-varieties is studied from the GKM perspective. Let $K^T(X)$ be the equivariant (topological) K-theory of a T-variety X. Following Proposition 2.3.2, *define* the equivariant K-homology of X as

$$\mathrm{Hom}_{R(T)}(K^T(X), R(T)),$$

where $R(T)$ is the representation ring of T over $\mathbb{C}$. In particular, fixing an isomorphism $T \cong \mathbb{G}_m^n$, we have $R(T) \cong \mathbb{C}[y_1^\pm, \dots, y_n^\pm]$.

Adapting the description of [?, Theorem 3.1], Proposition 2.3.3, and Lemma 2.3.6, we have an analogue of Corollary 2.3.6 in K-homology.

Proposition 2.3.22. *Let X be an equivariantly formal GKM T-ind-scheme. Then $K^T(X) \subset K^T(X^T) \otimes \mathbb{C}(\mathfrak{t})$ consists of all tuples $(\omega_x)_{x \in X^T}$ of rational differential forms on T satisfying the following conditions.*

1. *The poles of each ω_x are contained in the union of singular divisors (i.e. of the form $\{y^\chi = 1\}$ and have order at most one.*

2. *For any singular character χ and for any connected component Y of $X^{\ker \chi}$, we have*

$$\mathrm{Res}_{y^\chi=1}\left(\sum_{x\in Y^T}\omega_x\right) = 0.$$

From this, it directly follows that we have the following complementary versions of Theorems 3.0.1 and 2.3.20.

Theorem 2.3.23. *Let $\Delta' = \prod_{\alpha\in\Phi^+}(1-y^\alpha) \in R(T)$ be the Vandermonde element. The equivariant K-homology of $X_d := \mathrm{Sp}_{t^d z}$ for a reductive group G is up to multiplication by $(\delta')^d$ canonically isomorphic as a $\mathbb{C}[\Lambda]\otimes R(T)$-module to the ideal*

$$(J')^{(d)} := \bigcap_{\alpha\in\Phi+} (J'_\alpha)^d \subset \mathbb{C}[\Lambda]\otimes R(T).$$

Here $J'_\alpha := \langle 1-y^\alpha, 1-x^{\alpha^\vee}\rangle$. The algebra structure on $(\Delta')^d H^T_*(\mathrm{Sp}_\gamma)$ is given by the convolution product on $K^T(\Lambda)$

Theorem 2.3.24. *For any reductive group G,*

$$\Delta\cdot K^T(Y_1) = \bigcap_\alpha \widetilde{J}'_\alpha \subset \mathbb{C}[\widetilde{W}]\otimes R(T).$$

Here

$$\widetilde{J}'_\alpha = \left\{1-x^{\alpha^\vee}, 1-y^\alpha, 1-s_\alpha\right\}\mathbb{C}[\Lambda]\otimes R(T) \subset \mathbb{C}[\widetilde{W}]\otimes R(T).$$

2.4 The isospectral Hilbert scheme

2.4.1 Definitions

In this section, we define the relevant Hilbert schemes of points and list some of their properties. We then discuss the relationship of the results in Section 2.2 to the Hilbert scheme of points and the isospectral Hilbert scheme.

Definition 2.4.1. The Hilbert scheme of points on the complex plane, denoted $\mathrm{Hilb}^n(\mathbb{C}^2)$, is defined as the moduli space of length n subschemes of $\mathbb{C}^2$. Its closed points are given by

$$\{I \subset \mathbb{C}[x,y] \mid \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\},$$

where I is an ideal.

Definition 2.4.2. The isospectral Hilbert scheme X_n is defined as the following reduced fiber product:

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\rho"] & \mathbb{C}^{2n} \arrow[d, "\cdot/\mathfrak{S}_n"] \\ \mathrm{Hilb}^n(\mathbb{C}^2) \arrow[r, "\sigma"] & \mathbb{C}^{2n}/\mathfrak{S}_n \end{tikzcd}$$

We have the following localized versions, of interest to us.

Definition 2.4.3. The Hilbert scheme of points on $\mathbb{C}^* \times \mathbb{C}$ is the moduli space of length n subschemes of $\mathbb{C}^* \times \mathbb{C}$.

Note that $\mathbb{C}^* \times \mathbb{C}$ is affine, so that the closed points of $\mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$ are given by $\{I \subset \mathbb{C}[x^{\pm}, y] | \dim_{\mathbb{C}} \mathbb{C}[x^{\pm}, y]/I = n, I \text{ ideal}\}$. In fact, $\mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$ is naturally identified with the preimage $\pi^{-1}((\mathbb{C}^* \times \mathbb{C})^n/\mathfrak{S}_n)$ under the Hilbert-Chow map

$$\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathbb{C}^{2n}/\mathfrak{S}_n.$$

Definition 2.4.4. The isospectral Hilbert scheme on $\mathbb{C}^* \times \mathbb{C}$ is denoted Y_n, and defined to be the following reduced fiber product:

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d, "\rho"] & (\mathbb{C}^* \times \mathbb{C})^n \arrow[d, "\cdot/\mathfrak{S}_n"] \\ \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C}) \arrow[r, "\sigma"] & (\mathbb{C}^* \times \mathbb{C})^n/\mathfrak{S}_n \end{tikzcd}$$

Let $A = \mathbb{C}[\mathbf{x}, \mathbf{y}]^{sgn}$ be the space of alternating polynomials. This is to be interpreted in two sets of variables, ie. taking the *sgn*-isotypic part for the diagonal action. We recall the following theorem of Haiman.

Theorem 2.4.5 ([39]). *Consider the ideal $I \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ generated by A. Then for all $d \geq 0$,*

$$I^d = J^{(d)} = \bigcap_{i \neq j} \langle x_i - x_j, y_i - y_j \rangle^d \subseteq \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}]. \tag{2.4.1}$$

Moreover, I^d is a free $\mathbb{C}[\mathbf{y}]$-module, and by symmetry, a free $\mathbb{C}[\mathbf{x}]$-module.

Remark 2.4.6. $J^{(d)}$ is not free over $\mathbb{C}[\mathbf{x}, \mathbf{y}]$.

We have the following corollary to Theorem 3.0.1, as stated earlier.

Corollary 2.4.7. *The ideal* $J^{(d)} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *is free over* $\mathbb{C}[\mathbf{y}]$.

The ideals $I^d = J^d = J^{(d)}$ and the space of alternating polynomials naturally emerge in the study of Hilbert schemes of points on the plane.

Theorem 2.4.8. *The schemes* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *and* X_n *admit the following descriptions:*

$$\mathrm{Hilb}^n(\mathbb{C}^2) \cong \mathrm{Proj}\left(\bigoplus_{d\geq 0} A^d\right) \tag{2.4.2}$$

and

$$X_n \cong \mathrm{Proj}\left(\bigoplus_{d\geq 0} J^d\right). \tag{2.4.3}$$

Proof. See [42, Proposition 2.6]. □

Corollary 2.4.9. *We have*

$$\mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C}) \cong \mathrm{Proj}\left(\bigoplus_{d\geq 0} A^d_{\mathbf{x}}\right) \tag{2.4.4}$$

and

$$Y_n \cong \mathrm{Proj}\left(\bigoplus_{d\geq 0} J^d_{\mathbf{x}}\right), \tag{2.4.5}$$

where the subscript $\mathbf{x}$ *denotes localization in the* x_i.

Proof. Both of these equations describe blow-ups; the first along the diagonals in $(\mathbb{C}^* \times \mathbb{C})^n/\mathfrak{S}_n$ and the second along the diagonals in $(\mathbb{C}^* \times \mathbb{C})^n$. Note that $(J^{(d)})_{\mathbf{x}} = J^{(d)}_{\mathbf{x}}$ since localization commutes with intersection. Since blowing up commutes with restriction to open subsets [84, Lemma 30.30.3], Theorem 2.4.8 gives the result. □

There are several relevant sheaves on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and X_n that relate to $H^T_*(\mathrm{Sp}_\gamma)$ and $H^T_*(\widetilde{\mathrm{Sp}}_\gamma)$ naturally. From the Proj construction we naturally get very ample line bundles $\mathcal{O}_?(1)$ on both $? = X_n$ and $? = \mathrm{Hilb}^n(\mathbb{C}^2)$. Note that it is immediate from the construction that

$$\mathcal{O}_{X_n}(1) = \rho^* \mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)}(1).$$

On $\mathrm{Hilb}^n(\mathbb{C}^2)$ there is also a tautological rank n bundle $\mathcal{T}$ whose fiber at I is given by $\mathbb{C}[\mathbf{x},\mathbf{y}]/I$. Its determinant bundle can be shown to equal $\mathcal{O}(1)$.

As noted before, $\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C})$ is the preimage under the Hilbert-Chow map of $(\mathbb{C}^*\times\mathbb{C})^n/\mathfrak{S}_n$, it is a (Zariski) open subset of $\mathrm{Hilb}^n(\mathbb{C}^2)$. Similarly, $Y_n=\rho^{-1}(\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C}))\subset X_n$ is an open subset. Restriction then gives very ample line bundles

$$\mathcal{O}_{Y_n}(1)=\mathcal{O}_{X_n}(1)|_{Y_n},\ \mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C})}(1)=\mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)}(1)|_{\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C})}.$$

Definition 2.4.10. Let $\mathcal{O}_{X_n}$ be the structure sheaf of the isospectral Hilbert scheme. Define the *Procesi bundle* $\mathcal{P}:=\rho_*\mathcal{O}_X$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$.

In particular, $H^0(\mathrm{Hilb}^n(\mathbb{C}^2),\mathcal{P}\otimes\mathcal{O}(d))=J^d$.

Theorem 2.4.11 (The $n!$ theorem, [39])**.** *The Procesi bundle is locally free of rank $n!$ on* $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Localizing the ideal J at $\mathbf{x}$, we get the following result.

Proposition 2.4.12. *Let $\gamma=zt^d\in\mathfrak{gl}_n\otimes\mathcal{K}$ as before. Then*

$$H^0(\mathrm{Hilb}^n(\mathbb{C}\times\mathbb{C}^*),\mathcal{P}\otimes\mathcal{O}(d))=J_{\mathbf{x}}^{(d)}\cong\Delta^d H_*^T(\mathrm{Sp}_\gamma). \tag{2.4.6}$$

Proof. We have by definition that

$$H^0(\mathrm{Hilb}^n(\mathbb{C}\times\mathbb{C}^*),\mathcal{P}\otimes\mathcal{O}(d))=H^0(Y_n,\mathcal{O}_{Y_n}(d)).$$

Since $Y_n\subset X_n$ is in fact a principal open subset determined by $\prod_{i=1}^n x_i\in\mathbb{C}[\mathbf{x}^\pm,\mathbf{y}]^{\mathfrak{S}_n}$, restriction to the open subset coincides with localization. So we get

$$H^0(Y_n,\mathcal{O}_{Y_n}(d))=J_{\mathbf{x}}^{(d)}.$$

By Theorem 3.0.1, we conclude

$$H^0(\mathcal{P}\otimes\mathcal{O}(d),\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C}))\cong\Delta^d H_*^T(\mathrm{Sp}_\gamma).$$

∎

Although it is not clear to us what the cohomology of the affine Springer fiber $\widetilde{\mathrm{Sp}}_\gamma$ in Fl_G describes in these terms, we make the following conjecture.

Conjecture 2.4.13. *As graded $\mathbb{C}[y_1, \ldots, y_n]$-modules, we have*

$$H^0(\mathcal{P} \otimes \mathcal{P}^* \otimes \mathcal{O}(d), \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})) \cong \Delta^d \cdot H_*^T(\widetilde{\mathrm{Sp}}_{zt^d}). \tag{2.4.7}$$

Example 2.4.14. When $d = 0$, the above conjecture states

$$H^0(\mathcal{P} \otimes \mathcal{P}^*, \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})) = \mathbb{C}[\widetilde{W}] \otimes \mathbb{C}[\mathbf{y}] = \mathbb{C}[\mathbf{x}^{\pm}, \mathbf{y}] \rtimes W \cong H_*^T(\widetilde{\mathrm{Sp}}_z).$$

If it is also true for $d = 1$, Theorem 2.3.20 implies that

$$H^0(\mathcal{P} \otimes \mathcal{P}^* \otimes \mathcal{O}(1), \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})) \cong \widetilde{J}_{GL_n}.$$

Remark 2.4.15. The motivation for Conjecture 2.4.13 is as follows. In [**?**], Gordon and Stafford relate $J_n^{(d)}$ and the Procesi bundle to the spherical representation of the trigonometric DAHA in type A. For $d = 1$, the antisymmetrized version of this representation has the same size (as an S_n-representation) as $\mathcal{P} \otimes \mathcal{P}$, as does $H_*^T(\widetilde{\mathrm{Sp}}_{tz})$. Since $H_*^T(\widetilde{\mathrm{Sp}}_{tz})$ also carries a trigonometric DAHA-action (at $c = 0$) by results of Oblomkov-Yun [75], it is plausible to conjecture that it is "the same" module as the Gordon-Stafford construction would give.

2.4.2 Diagonal coinvariants and Bezrukavnikov's conjecture

When $G = GL_n$, it is known that the fibers of the Procesi bundle $\mathcal{P}$, as introduced in the previous section, at torus-fixed points in $\mathrm{Hilb}^n(\mathbb{C}^2)$ afford the regular representation of $\mathfrak{S}_n$ [39], and in particular have dimension $n!$. On the other hand, they appear as quotients of the ring of *diagonal coinvariants* (sometimes also called diagonal harmonics)

$$DH_n := \mathbb{C}[\mathbf{x}, \mathbf{y}]/\mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{\mathfrak{S}_n},$$

which is now known to be $(n+1)^{n-1}$- dimensional. Additionally, it is known that the isotypic component DH_n^{sgn} has dimension C_n, where C_n is the nth Catalan number, and that its bigraded character is given by

$$(e_n, \nabla e_n).$$

Here $(-,-)$ is the Hall inner product on symmetric functions over $\mathbb{Q}(q,t)$ and e_j denotes the jth elementary symmetric function. The operator ∇ is the nabla operator introduced by Garsia and Bergeron [8].

As far as the relation with affine Springer theory goes, from work of Oblomkov-Yun, Oblomkov-Carlsson and Varagnolo-Vasserot [75], [**?**], [89], it follows that we have, up to regrading,

$$H^*(\widetilde{\mathrm{Sp}}_{\gamma'}) \cong DH_n, H^*(\mathrm{Sp}_{\gamma'}) \cong DH_n^{sgn},$$

where γ' is an endomorphism of $\mathcal{K}^n = \mathrm{span}\{e_1, \ldots, e_n\}_{\mathcal{K}}$ given by $\gamma'(e_i) = e_{i+1}, i = 1, \ldots, n-1$ and $\gamma'(e_n) = te_1$. Note that in this case, γ' is elliptic so that $\mathrm{Sp}_{\gamma'}$ and $\widetilde{\mathrm{Sp}}_{\gamma'}$ are projective schemes of finite type and thus their cohomologies are finite dimensional. In fact, after adding some equivariance to the picture the cohomologies in question become the finite-dimensional representations of the trigonometric and rational Cherednik algebras with parameter $c = \frac{n+1}{n}$.

It is a conjecture of Bezrukavnikov (private communication) that under the lattice action of Λ on $H^*(\widetilde{\mathrm{Sp}}_\gamma)$, where $\gamma = zt$, we also have

$$H^*(\widetilde{\mathrm{Sp}}_\gamma)^\Lambda \cong DH_n$$

and

$$H^*(\mathrm{Sp}_\gamma)^\Lambda \cong DH_n^{sgn}.$$

So far, we are not able to prove this conjecture, but can prove its analogue in Borel-Moore homology, where we replace invariants by coinvariants.

Theorem 2.4.16. *We have*

$$H_*(\mathrm{Sp}_\gamma)_\Lambda \cong DH_n^{sgn}.$$

Proof. Using Theorem 3.0.1, we compute that

$$H_*(\mathrm{Sp}_\gamma) \cong \frac{H_*^T(\mathrm{Sp}_\gamma)}{\langle \mathbf{y} \rangle}.$$

As the actions of $\mathbb{C}[\mathbf{x}^\pm]$ and $\mathbb{C}[\mathbf{y}]$ commute, the result is still a $\mathbb{C}[\mathbf{x}^\pm]$-module. Taking coinvariants, we have

$$H_*(\mathrm{Sp}_\gamma)_\Lambda := \frac{H_*(\mathrm{Sp}_\gamma)}{\langle 1-\mathbf{x} \rangle H_*(\mathrm{Sp}_\gamma)} \cong \frac{H_*^T(\mathrm{Sp}_\gamma)}{\langle 1-\mathbf{x}, \mathbf{y} \rangle H_*^T(\mathrm{Sp}_\gamma)}.$$

The last equality follows from the isomorphism theorems for modules. Here $\langle 1-\mathbf{x}\rangle$ means the set $\{1-x_1,\ldots,1-x_n\}$ and $\mathbf{y}$ means the set $\{y_1,\ldots,y_n\}$.

On the other hand,

$$J_{GL_n}/\langle x_1-1,\ldots,x_n-1,y_1,\ldots,y_n\rangle J_{GL_n}$$

may be identified with $J/\langle x_1-1,\ldots,x_n-1,y_1,\ldots,y_n\rangle J$, where

$$J := \bigcap_{i\neq j}\langle x_i-x_j, y_i-y_j\rangle \subset \mathbb{C}[\mathbf{x},\mathbf{y}]$$

since quotient and localization commute. Since J is translation-invariant with respect to $x_i \mapsto x_i + c, i = 1,\ldots,n$, so that

$$J/\langle x_1-1,\ldots,x_n-1,y_1,\ldots,y_n\rangle J \cong J/\langle \mathbf{x},\mathbf{y}\rangle J.$$

On the other hand, we have that $J/\langle \mathbf{x},\mathbf{y}\rangle J \cong DH_n^{sgn}$ by the fact that the left-hand side is the space of sections of $\mathcal{O}(1)$ on the zero-fiber of the Hilbert-Chow map inside $\mathrm{Hilb}^n(\mathbb{C}^2)$ [39, Proposition 6.1.5]. $\square$

Corollary 2.4.17. *One has*

$$\dim_{q,t} H_*(\mathrm{Sp}_\gamma)_\Lambda = \langle e_n, \nabla e_n\rangle,$$

and $\dim_{\mathbb{C}} H_*(\mathrm{Sp}_\gamma)_\Lambda = C_n$*, where* C_n *is the nth Catalan number.*

Remark 2.4.18. In the spirit of Conjecture 2.4.13, it seems likely that the approach from above can be used to show that $H_*(\widetilde{\mathrm{Sp}}_\gamma)_\Lambda \cong DH_n$. Both would follow from an explicit description of $H^0(\mathcal{P}\otimes\mathcal{P}, \mathrm{Hilb}^n(\mathbb{C}^2))$.

2.4.3 Rational and elliptic versions

We now comment on the relation of our results to $\mathrm{Hilb}^n(\mathbb{C}^2)$ and $\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C}^*)$. These are known to quantize to the full DAHA and the rational Cherednik algebra of $\mathfrak{gl}_n$. Let us start with the elliptic version. In Theorem 2.3.23, the description of the K-homology of Sp_γ is given. As blow-up commutes with restriction to opens, we have the following analogue to Theorem 2.4.8 and Corollary 2.4.9.

Corollary 2.4.19. *We have*

$$\mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C}^*) \cong \mathrm{Proj}\left(\bigoplus_d A^d_{\mathbf{x},\mathbf{y}}\right) \tag{2.4.8}$$

and

$$Y'_n \cong \mathrm{Proj}\left(\bigoplus_d (J')^d\right). \tag{2.4.9}$$

Here the subscript $\mathbf{x}, \mathbf{y}$ *denotes localization in* $\prod x_i$ *and* $\prod y_i$, *and* Y'_n *is the isospectral Hilbert scheme on* $\mathbb{C}^* \times \mathbb{C}^*$.

Analogously to Proposition 2.4.12, we have the following.

Proposition 2.4.20. *We have*

$$H^0(\mathcal{P} \otimes \mathcal{O}(d), \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C}^*)) \cong (\Delta')^d K^T(\mathrm{Sp}_{t^d z}) \tag{2.4.10}$$

Let now $\mathrm{Gr}+_{GL_n} := \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}^\lambda$ be the positive part of the affine Grassmannian. Let $\mathrm{Sp}_{t^d z} \cap \mathrm{Gr}^+_{GL_n}$ Then the T-fixed points in both are identified with Λ^+ and their classes in $\mathbb{C}[\Lambda]$ with the monomials without negative powers. Intersecting $\Delta^d H^T_*(\mathrm{Sp}_{t^d z})$ with $H^T_*(\Lambda^+)$ gives $J^{(d)} \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$. From the proof of Theorem 3.0.1, it is not hard to see that this agrees with $\Delta^d H^T_*(\mathrm{Sp}^+_{t^d z})$. In particular, we have

Theorem 2.4.21.

$$H^0(\mathcal{P} \otimes \mathcal{O}(d), \mathrm{Hilb}^n(\mathbb{C}^2)) \cong \Delta^d H^T_*(\mathrm{Sp}^+_{t^d z}).$$

2.4.4 Other root data

In this section, we consider a general connected reductive group G. As we will see, many things from the above discussion are not as straightforward.

In [39], Haiman discusses the extension of his $n!$ and $(n+1)^{n-1}$ conjectures to other groups. The naturally appearing space here is $T^*\mathfrak{t}$ with its diagonal W- action. In the case of a general reductive group, Gordon [24] has proved that there is a canonically defined doubly graded quotient ring R^W of the coinvariant ring

$$\mathbb{C}[T^*\mathfrak{t}]/\mathbb{C}[T^*\mathfrak{t}]^W_+$$

whose dimension is $(h+1)^r$ for the Coxeter number h and rank r. It is also known that $sgn \otimes R^W$ affords the permutation representation of W on $Q/(h+1)Q$ for Q the root lattice of G. It would be interesting to compare the lattice-invariant parts of $H^*(\mathrm{Sp}_\gamma)$ and $H^*(\widetilde{\mathrm{Sp}}_\gamma)$ to this quotient in other Cartan-Killing types.

We have now seen how the antisymmetric pieces of spaces of diagonal coinvariants appear from affine Springer fibers in the affine Grassmannian. On the other hand, we have seen that in type A, the antisymmetric part of $\mathbb{C}[\mathbf{x},\mathbf{y}]$ plays the main role in the construction of the isospectral Hilbert scheme X_n as a blow-up. From solely the point of view of Weyl group representations, it would be then natural to consider the sgn-isotypic part of $\mathbb{C}[T^*\mathfrak{t}], \mathbb{C}[T^*T^\vee]$.

We now restate and prove Theorem 2.1.2.

Theorem 2.4.22. *Let $I_G \subseteq \mathbb{C}[T^*T^\vee]$ be the ideal generated by W-alternating polynomials in $\mathbb{C}[T^*T^\vee]$ with respect to the diagonal action. Then there is an injective map*

$$I^d \hookrightarrow J_G^{(d)} = \Delta^d H_*^T(\mathrm{Sp}_\gamma).$$

Proof. Write $(\mathbf{x},\mathbf{y}) = (x_1,\ldots,x_r,y_1,\ldots,y_r)$ for the coordinates on $T^*T^\vee$ determined by $x_i = \exp(\epsilon_i)$ and where the y_i are the cotangent directions. Let $f(\mathbf{x},\mathbf{y}) \in I_G$ and let $\alpha \in \Phi^+$ be a positive root. Denote by s_α the corresponding reflection. Without loss of generality we may take $f(\mathbf{x},\mathbf{y})$ to be W-antisymmetric. Then at points $(\mathbf{x},\mathbf{y})$ where $\exp(\alpha^\vee) = 1$, $\partial_\alpha = 0$ we must have $s_\alpha \cdot f(\mathbf{x},\mathbf{y}) = -f(\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{y})$ for any s_α. Thus $f(\mathbf{x},\mathbf{y}) = 0$ on the subspace arrangement defined by J_G, and by the Nullstellensatz $f \in J_G$. Taking dth powers and observing that $J_G^d \subseteq J_G^{(d)}$ for any d gives the result. □

Proposition 2.4.23. *There is a natural graded algebra structure on*

$$\bigoplus_{d\geq 0} J_G^{(d)}$$

given by multiplication of polynomials:

$$J_G^{(d_1)} \times J_G^{(d_2)} \to J_G^{(d_1+d_2)}.$$

Proof. Suppose $f_i \in \bigcap_{\alpha\in\Phi^+} \langle 1-\alpha^\vee, y_\alpha\rangle^i$, $i=1,2$. Then $f_1 f_2 \in \langle 1-\alpha^\vee, y_\alpha\rangle^{d_1+d_2}$ for all α, so that $J_G^{(d_1)} J_G^{(d_2)} \subseteq J_G^{(d_1+d_2)}$. □

The following Theorem was communicated to the author by Mark Haiman.

Theorem 2.4.24.

$$Y_G := \mathrm{Proj}\left(\bigoplus_{d\geq 0} J_G^{(d)}\right)$$

is a normal variety.

Proof. The powers of an ideal generated by a regular sequence are integrally closed, as is an intersection of integrally closed ideals. Therefore, each of the ideals $J_G^{(d)}$ is integrally closed, and so is the algebra

$$\bigoplus_{d\geq 0} J_G^{(d)}.$$

By construction, the ring is an integral domain, so Y_G is by definition normal. See also [39, Proposition 3.8.4] for the proof of this statement in type A. □

Remark 2.4.25. This Proj-construction is sometimes called the symbolic blow-up. Since we do not know if $J_G^d = J_G^{(d)}$, and likely this is not the case, the ring $\bigoplus_{d\geq 0} J_G^{(d)}$ is not generated in degree one. However, if we did have translation invariance in the Λ-direction in this case, we could deduce results about the geometry of the double Coxeter arrangement in $T^*\mathfrak{t}^\vee$ by similar arguments as in type A. It would be reasonable to suspect Y_G also has a map to the "W-Hilbert scheme" or some crepant resolution but we do not discuss these possibilities any further. It should be mentioned that in [22], Ginzburg studies the "isospectral commuting variety". He has proved that its normalization is Cohen-Macaulay and Gorenstein. It would be interesting to know how this variety relates to the variety Y_G.

2.5 Relation to knot homology

Gorsky and Hogancamp have recently defined y-ified Khovanov-Rozansky homology $\mathrm{HY}(-)$ [?]. It is a deformation of the triply-graded knot homology theory of Khovanov and Rozansky [52], which is often dubbed HOMFLY homology, for it categorifies the HOMFLY polynomial. In this section, we discuss the relationship of the results in previous sections to these link homology theories.

Recall that the HOMFLY homology of a braid closure $\overline{\beta}$ can be defined [52] as the Hochschild homology of a certain complex of Soergel bimodules called the Rouquier complex. We denote the triply graded homology of $\overline{\beta}$ by $\mathrm{HHH}(\overline{\beta})$.

As stated above, there exists a nontrivial deformation of this theory, called y-ification, which takes place in an enlarged category of curved complexes of y-ified "Soergel bimodules". It was defined in [**?**] and in practice is still defined as the Hocschild homology of a deformed Rouquier complex. We denote the y-ified homology groups of a braid closure $L = \overline{\beta} \subset S^3$ by $\mathrm{HY}(L)$. They are triply graded modules over a superpolynomial ring $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m, \theta_1, \ldots, \theta_m]$, where m is the number of components in L. The θ-grading comes from Hochschild homology, and we will mainly be interested in the Hochschild degree zero part. We will denote this by $\mathrm{HY}(L)^{a=0}$. See [**?**, Definition 3.4] for the precise definitions.

Definition 2.5.1. Let $\mathbf{cox}_n \in \mathfrak{B}r_n$ be the positive lift of the Coxeter element of $\mathfrak{S}_n$. The dth power of the *full twist* is the braid $\mathrm{FT}_n^d := \mathbf{cox}_n^{nd}$.

Remark 2.5.2. The element FT_n is a central element in the braid group and it is known to generate the center.

Theorem 2.5.3 ([**?**]). *We have* $\mathrm{HY}(\mathrm{FT}_n^d)^{a=0} \cong J^d \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$.

Corollary 2.5.4. *There is an isomorphism of* $\mathbb{C}[\mathbf{x}^{\pm}, \mathbf{y}]$*-modules*

$$\Delta^d H_*^T(\mathrm{Sp}_\gamma) \cong \mathrm{HY}(\mathrm{FT}_n^d)^{a=0} \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^{\pm}]$$

for $\gamma = zt^d$.

Following Theorem 3.0.1 for $G = GL_n$, it is interesting to consider the homologies of the powers of the full twist as $d \to \infty$. By [43], it is known that the $a = 0$ part of the ordinary HOMFLY homology of FT_n^∞ is given by a polynomial ring on generators $g_1, \ldots, g_n$ of degrees $1, \ldots, n$, which coincide with the exponents of G, and in particular with the equivariant BM homology of the affine Grassmannian. In the context of *loc. cit.* the corresponding algebra appears as the endomorphism algebra of a categorified

Jones-Wenzl projector. The corresponding statement in y-ified homology is stronger, and states

$$\mathrm{HY}(\mathrm{FT}_n^\infty) \cong \mathbb{C}[g_1, \ldots, g_n, y_1, \ldots, y_n].$$

Theorem 2.5.5. *Consider the system of maps*

$$\Delta \circ i_* : \Delta^d H_*^T(\mathrm{Sp}_{t^d z}) \to \Delta^{d+1} H_*^T(\mathrm{Sp}_{t^{d+1} z}).$$

Taking the colimit in the category of $\mathbb{C}[\mathbf{x}^\pm, \mathbf{y}]$*-modules, we have*

$$\varinjlim \Delta^d H_*^T(\mathrm{Sp}_{t^d z}) \cong \mathbb{C}[g_1, \ldots, g_n, y_1, \ldots, y_n] \subset \mathbb{C}[\mathbf{x}^\pm, \mathbf{y}],$$

where g_i *is of* y*-degree* i*. In particular,*

$$\varinjlim \Delta^d H_*^T(\mathrm{Sp}_{t^d z}) \cong \mathrm{HY}(\mathrm{FT}_n^\infty).$$

Proof. We have

$$\varinjlim \Delta^d H_*^T(\mathrm{Sp}_{t^d z}) = \left(\bigoplus_d \Delta^d H_*^T(\mathrm{Sp}_{t^d z}) \right) / \Delta\alpha_i = \alpha_{i+1} = \varinjlim_d H_*^T(\mathrm{Sp}_{t^d z}) = H_*^T(\mathrm{Gr}_{GL_n}).$$

□

Remark 2.5.6. Note that the algebra from above is exactly the localization of the projective coordinate ring of Y_n at Δ. On the other hand, this is the coordinate ring of the open affine where the points on the (isospectral) Hilbert scheme have distinct y-coordinates, and by [39, Section 3.6], this has coordinate ring $\mathbb{C}[g_1, \ldots, g_n, y_1, \ldots, y_n]$.

We record the following theorem from [**?**, Theorem 1.14], relating commutative algebra in $2n$ variables to the link-splitting properties of $\mathrm{HY}(-)$.

Theorem 2.5.7. *Suppose that a link* L *can be transformed to a link* L' *by a sequence of crossing changes between different components. Then there is a homogeneous "link splitting map"*

$$\Psi : \mathrm{HY}(L) \to \mathrm{HY}(L')$$

which preserves the $\mathbb{Q}[\mathbf{x}, \mathbf{y}, \theta]$*-module structure. If, in addition,* $HY(L)$ *is free as a* $\mathbb{Q}[\mathbf{y}]$*-module, then* Ψ *is injective. If the crossing changes only involve components* i *and* j*, then the link splitting map becomes a homotopy equivalence after inverting* $y_i - y_j$*, where* i, j *label the components involved.*

The cohomological purity of Sp_γ should be compared to the parity statements in [**?**, Definitions 1.16, 3.18, 4.9]. Namely, we have the following Theorem.

Theorem 2.5.8 ([?], Theorem 1.17)**.** *If an r-component link L is parity then*

$$\mathrm{HY}(L) \cong \mathrm{HHH}(L) \otimes \mathbb{C}[\mathbf{y}]$$

is a free $\mathbb{C}[\mathbf{y}]$-module.

In particular, $\mathrm{HY}(L)/\mathbf{y}\,\mathrm{HY}(L) \cong \mathrm{HHH}(L)$ *as triply graded vector spaces.*

Consequently any link splitting map identifies $\mathrm{HY}(L)$ *with a* $\mathbb{Q}[\mathbf{x}, \mathbf{y}, \theta]$*- submodule of* $\mathrm{HY}(\mathit{split}(L))$*.*

In the case of the powers of the full twist, Theorem 2.5.7 is easy to understand. Namely, inverting $y_i - y_j$ we simply remove the ideal $(x_i - x_j, y_i - y_j)$ from the intersection J. This also clearly holds for $J^{(m)}$. Let us consider similar properties for the anti-invariants, following Haiman [41].

Lemma 2.5.9. *The ideal I factorizes locally as the product of I for parabolic subgroups of $\mathfrak{S}_n$.*

Proof. Let g be a generator of

$$I' = I(x_1, y_1, \ldots, x_r, y_r)I(x_{r+1}, y_{r+1}, \ldots, x_n, y_n),$$

alternating in the first r and last $n-r$ indices. Let h be any polynomial which belongs to the localization J_Q at every point $Q \neq P$ in the $\mathfrak{S}_n$-orbit of P, but doesn't vanish at P. Then $f = \mathrm{Alt}(gh)$ belongs to I. The terms of f corresponding to $w \in \mathfrak{S}_n$ not stabilizing P belong to J_P, by construction of h. Since g alternates with respect to the stabilizer of P, the remaining terms sum to a unit times g, or more precisely $g \sum_{wP=P} wh$. Hence $g \in I_P$. This means that I and I^m factorize locally as products of the corresponding ideals in the first r and last $n-r$ indices. □

It is curious to note that a similar property holds for the affine Springer fibers. As shown in [26, Theorem 10.2], we have the following relationship between equivariant (co)homology of Sp_γ and the corresponding affine Springer fiber of an "endoscopic" group. This is to say, G' has a maximal torus isomorphic to T and its roots with respect to this torus can be identified with a subset of $\Phi(G,T)$. If G' is such a group for $G = GL_n$ (which in this case can just be identified with a subgroup of G), we have an isomorphism

$$H_i^T(\mathrm{Sp}_\gamma;\mathbb{C})_S \cong H_{i-2r}^T(X_{\gamma_T}^T;\mathbb{C})_S, \tag{2.5.1}$$

where S is the multiplicative subset generated by $(1-\alpha^\vee)$, where the coroots $\alpha^\vee$ run over all coroots *not* corresponding to G'. If we denote this set by $\Phi(G)^+ - \Phi(G')^+$, then r is the cardinality of this finite set times d. For general diagonal γ, or alternatively the pure braids discussed in the introduction, r is the degree of the corresponding product of Vandermonde determinants, or in the automorphic form terminology the homological transfer factor. The fact that this localization corresponds exactly to link splitting in y-ified homology (after using the Langlands duality $\mathbf{x} \leftrightarrow \mathbf{y}$) is in the author's opinion quite beautiful and deep.

2.6 Hilbert schemes of points on planar curves

2.6.1 Hilbert schemes on curves and compactified Jacobians

In the case $G = GL_n$, which we will assume to be in from now on, the affine Grassmannian has a description as the space of lattices:

$$G(\mathcal{K})/G(\mathcal{O}) = \{\Lambda \subseteq \mathcal{K}^n | \Lambda \otimes_{\mathcal{O}} \mathcal{K} = \mathcal{K}^n, \Lambda \text{ a projective } \mathcal{O}^n\text{-module}\}$$

We may think of Sp_γ as $\{\Lambda | \gamma\Lambda \subseteq \Lambda\}$. If γ is regular semisimple, the characteristic polynomial of γ determines a polynomial $P_\gamma(x)$ in $\mathcal{O}[x]$, which equals the minimal polynomial of γ. Denote $A = \mathcal{O}[x]/P_\gamma(x)$, $F = \mathrm{Frac}(A)$. As a vector space, we then have $F = \mathcal{K}[x]/P_\gamma(x) \cong \mathcal{K}^n$, and Sp_γ can be identified with the space of fractional ideals in F. On the other hand, this is by definition the Picard factor or local compactified Picard associated to the germ $\mathcal{O}[[x]]/P_\gamma(x)$ of the plane curve $C = \{P_\gamma(x) = 0\}$ [2].

By eg. Ngô's product theorem [72], there is a homeomorphism of stacks

$$\overline{\mathrm{Jac}}(C) \cong \mathrm{Jac}(C) \times^{\prod_{x\in C^{sing}} \mathrm{Jac}(C_x)} \prod_{x\in C^{sing}} \overline{\mathrm{Jac}}(C_x).$$

Call γ elliptic if it has anisotropic centralizer over $\mathcal{K}$, or equivalently $P_\gamma(x)$ is irreducible over $\mathcal{K}$. There has been a lot of work in determining the compactified Jacobians of C, in particular in the cases where $P_\gamma(x) = t^n - x^m$, $\gcd(m, n) = 1$ [30, 59, 76, 79].

There is always an Abel-Jacobi map $AJ : C^{[n]} \to \overline{\mathrm{Pic}}(C)$ given by $\mathcal{I}_Z \mapsto \mathcal{I}_Z \otimes \mathcal{O}(ny)$, where y is any smooth point on C. It is known that for elliptic γ this becomes a $\mathbb{P}^{n-2g}$-bundle for $n > 2g$. For nonelliptic γ as we are interested in, there is no such stabilization. On the local factors it is known AJ is an isomorphism for $n > 2g$, and in the nonelliptic case it is known that AJ is a dominant map to a union of irreducible components of $\overline{\mathrm{Pic}}(C)_x$.

In addition to the relationship of $C^{[n]}$ with the compactified Jacobians, conjectures of Oblomkov-Rasmussen-Shende [73, 74] predict that they in fact determine the knot homologies of the links of singularities of C and vice versa. For simplicity, assume C has a unique singularity at zero, and let $C_0^{[n]}$ be the punctual Hilbert scheme of subschemes of length n in C supported at zero.

Then [73, Conjecture 2] states

Conjecture 2.6.1.

$$V_0 := \bigoplus_{n \geq 0} H_*(C_0^{[n]}) \cong \mathrm{HHH}^{a=0}(L).$$

Remark 2.6.2. On the level of Euler characteristics, this is known to be true by [64].

We should mention that there is yet another reason to care about $C^{[n]}$; the Hilbert schemes and their Euler characteristic generating functions are closely related to BPS/DT invariants as shown in [77,78]. In [77] some of the examples we are interested in are studied.

In earlier work [**?**], the author considered the Hilbert schemes of points on reducible, reduced planar curves $C/\mathbb{C}$. The main result in *loc. cit* is as follows.

Theorem 2.6.3 ([**?**], Theorem 1.1)**.** *If $C = \bigcup_{i=1}^m C_i$ is a decomposition of C into irreducible components, the space $V = \bigoplus_{n \geq 0} H_*(C^{[n]}, \mathbb{Q})$ carries a bigraded action of the algebra*

$$A = A_m := \mathbb{Q}[x_1, \ldots, x_m, \partial_{y_1}, \ldots, \partial_{y_m}, \sum_{i=1}^m y_i, \sum_{i=1}^m \partial_{x_i}],$$

where $V = \bigoplus_{n,d\geq 0} V_{n,d}$ is graded by number of points n and homological degree d. Moreover, the operators x_i have degree $(1,0)$ and the operators ∂_{y_i} have degree $(-1,-2)$ in this bigrading. In effect, the operator $\sum y_i$ has degree $(1,2)$ and the operator $\sum \partial_{x_i}$ has degree $(-1,0)$.

Example 2.6.4. In the case $x^2 = y^2$, we have

$$V = \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{(x_1 - x_2)\mathbb{C}[x_1, x_2, y_1 + y_2]}$$

as $\mathbb{C}[x_1, x_2, y_1 + y_2, \partial_{x_1} + \partial_{x_2}, \partial_{y_1}, \partial_{y_2}]$-modules.

2.6.2 Conjectural description in the case $C = \{x^n = y^{dn}\}$

As discussed in the introduction, the representation in Example 2.6.4 very similar to the main result in [26] when $G = GL_2$ and $d = 1$. We now recall said theorem.

Theorem 2.6.5. *Let G be a connected reductive group and $\gamma = zt^d$ as before. Then the ordinary (i.e. not Borel-Moore) T-equivariant homology of Sp_γ is a $\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}^*]$-module, where $\mathfrak{t}$ acts by derivations, and*

$$H^T_{*,ord}(\mathrm{Sp}_\gamma) \cong \frac{\mathbb{C}[\Lambda] \otimes \mathbb{C}[\mathfrak{t}]}{\sum_{\alpha\in\Phi^+} \sum_{k=1}^{d} (1 - x^{\alpha^\vee})^k \mathbb{C}[\Lambda] \otimes \ker(\partial_\alpha^k)}.$$

Example 2.6.6. If $G = GL_2$, $d = 1$, we have

$$H^T_{*,ord}(\mathrm{Sp}_\gamma) = \frac{\mathbb{C}[x_1^\pm, x_2^\pm, y_1, y_2]}{(1 - x_1 x_2^{-1}\mathbb{C}[x_1^\pm, x_2^\pm, y_1 + y_2]}.$$

The above examples, as well as Examples 2.6.17, 2.6.16 and Theorem 3.0.1 motivate us to conjecture the following.

Conjecture 2.6.7. *Let $C = \overline{\{x^n = y^{dn}\}}$ be the compactification with unique singularity and rational components of the curve defined by the affine equation $\{x^n = y^{dn}\}$. Then as a bigraded A_n-module (see Theorem 2.6.3), we have*

$$V := \bigoplus_{n\geq 0} H_*(C^{[n]}, \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]}{\sum_{i\neq j} \sum_{k=1}^{d} (x_i - x_j)^k \otimes \ker(\partial_{y_i} - \partial_{y_j})^k}. \tag{2.6.1}$$

Remark 2.6.8. In some sense, passing from the equivariant homology of affine Springer fibers to the Borel-Moore version involves only half of the variables, namely the equivariant parameters. It is not immediate from the construction of the A_m- action in [**?**] what the analogous procedure would be to pass to $H^*(C^{[n]})$ from $H_*(C^{[n]})$. It would be interesting to know, at least on the level of bigraded Poincaré series, how to compare V to the ideal $J^d \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$, assuming that Conjecture 2.6.7 is true. The q, t-character of J^d is by work of Haiman [39] known to be given by the following inner product of symmetric functions:

$$\dim_{q,t} J^d = (\nabla^d p_1^n, e_n).$$

Thanks to work of Gorsky and Hogancamp [**?**] we then also know that (up to regrading) the bigraded character of $HY^{a=0}(T(n, dn))$ is given by the same formula.

For some support for the conjecture, let us consider the following examples.

Theorem 2.6.9 ([**?**]). *When $C = \overline{\{x^2 = y^2\}}$, we have that*

$$V = \bigoplus_{n \geq 0} H_*(C^{[n]}) \cong \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{\mathbb{C}[x_1, x_2, y_1 + y_2](x_1 - x_2)} \tag{2.6.2}$$

as an A_2-module, where

$$A_2 = \mathbb{C}[x_1, x_2, \partial_{x_1} + \partial_{x_2}, y_1 + y_2, \partial_{y_1}, \partial_{y_2}] \subset \mathit{Weyl}(\mathbb{A}^4).$$

Remark 2.6.10. Note that we get an extremely similar looking result for $H_*^H(\mathrm{Sp}_{\mathrm{diag}(t,-t)})$ and $H_*(C^\bullet)$, where $C^\bullet = \bigsqcup_{n \geq 0} C^{[n]}$ is the Hilbert scheme of points on the curve $C = \{x^2 = y^2\} \subset \mathbb{P}^2$.

Remark 2.6.11. We are no longer using equivariant homology, but have replaced the equivariant parameters by the fundamental classes of the components of the global curve C. It does make sense to consider the equivariant cohomology for the Hilbert schemes of points on $C = \{x^n = y^{dn}\}$, but we do not know how to produce a nice action of a rank n torus in this case and whether it would agree with expectations. Note that there is a natural $(\mathbb{C}^*)^2$-action on C and its Hilbert schemes, coming from the $(\mathbb{C}^*)^2$-action with weights $(d, 1)$ on the plane.

Remark 2.6.12. In general, we may describe the Hilbert schemes $C^{[2]}$ explicitly for $C = \overline{\{x^n = y^{dn}\}}$. Fix a decomposition into irreducible components $C = \bigcup_{i=1}^n C_i$. Since C has n rational components, there is a component $M_i \cong \mathrm{Sym}^2\mathbb{P}^1 \cong \mathbb{P}^2$ for each i, and for each $i < j$ we have a component $N_{ij} \cong \mathrm{Bl}_{pt}(\mathbb{P}^1 \times \mathbb{P}^1)$, see [**?**, Example 5.9]. The $\binom{n}{2}$ components N_{ij} all intersect along an exceptional $\mathbb{P}^1$ that can be identified with $\mathrm{Hilb}^2(\mathbb{C}^2, 0)$. Denote this line by E. We have $M_i \cap M_j = \emptyset$ for all $i \neq j$, and $M_i \cap N_{jk} \cong \mathbb{P}^1$ if $i = j$ or $i = k$, and $M_i \cap N_{jk} = \emptyset$ otherwise. Denote these lines of intersection by L_i. It is helpful to picture them as naturally isomorphic to C_i. The L_i do not intersect each other, but intersect $\mathrm{Hilb}^2(\mathbb{C}^2, 0)$ at points corresponding to the slopes of the corresponding lines C_i.

The homology of $C^{[2]}$ in degree two is spanned by $[L_i], i = 1, \ldots, n$ and E. Denote the fundamental class $[C_i] \in H_2(C^{[1]})$ by y_i. Using the A_n-action, we have elements

$$x_i y_i = [L_i] \in H_2(C^{[2]}), i = 1, \ldots, n, \quad \text{and } x_i y_j = [L_j] - [E], i \neq j.$$

Hence we have the relations

$$\begin{aligned} (x_i - x_j)(y_i + y_j) &= 0 \ \forall i, j \\ (x_i - x_j) y_k &= 0 \ k \neq i, j. \end{aligned}$$

Using these relations, we may express all the classes $[L_i], i = 1, \ldots, n$ and $[E]$ as linear combinations of $x_i y_i$ and for example $x_1 y_2$. Since

$$\dim_{\mathbb{C}} H_2(C^{[2]}) = n + 1,$$

there cannot be any other relations in this degree. This verifies equation (2.6.1) of Conjecture 2.6.7 in degree $q^2 t^2$.

2.6.3 Compactified Jacobians and the MSV formula

Homologically, we have the following relationship between the cohomology of the compactified Jacobians and the Hilbert schemes of points $C^{[n]}$, proved independently by Maulik-Yun and Migliorini-Shende.

Theorem 2.6.13 ([65, 68]). *Let $\pi : \mathcal{C} \to B$ be a locally versal deformation of C, and $\pi^{[n]} : \mathcal{C}^{[n]} \to B, \pi^J : \overline{\mathrm{Jac}}(\mathcal{C}) \to B$ be the relative Hilbert schemes of points and compactified Jacobians of π. Then, inside $D^b_c(B)[[q]]$, we have*

$$\bigoplus_{n\geq 0} q^n R\pi^{[n]}_* \mathbb{C} = \frac{\oplus q^i \ {}^pR^i\pi^J_*\mathbb{C}}{(1-q)(1-q\mathbb{L})},$$

where $\mathbb{L}$ is the Lefschetz motive (ie. the constant local system on B in this case.)

For reducible curves, the bigraded structure can be also computed from the theorem of Migliorini-Shende-Viviani [69, Theorem 1.16].

Theorem 2.6.14. *Let $\{C_S \to B_S\}_{S\subset[m]}$ be an independently broken family of reduced planar curves (see [69] for the definition), such that all the $C_S \to B_S$ are H-smooth, ie. their relative Hilbert schemes of points have smooth total spaces, and such that the families $C_S \to B_S$ admit fine compactified Jacobians $\overline{J(C_S)} \to B_S$. Then, inside $D^b_c(\bigsqcup B_S)[[q]]$, we have:*

$$(q\mathbb{L})^{1-g} \bigoplus_{n\geq 0} q^n R\pi^{[n]}_*\mathbb{C} = Exp\left((q\mathbb{L})^{1-g}\frac{\bigoplus q^i IC(\Lambda^i R^1\pi_{sm*}\mathbb{C}[-i])}{(1-q)(1-q\mathbb{L})}\right) \tag{2.6.3}$$

$$= Exp\left((q\mathbb{L})^{1-g}\frac{\bigoplus q^i \ {}^pR^i\pi^J_*\mathbb{C}}{(1-q)(1-q\mathbb{L})}\right). \tag{2.6.4}$$

Here, $g : B_S \to \mathbb{N}$ is the upper semicontinuous function giving the arithmetic genus of the fibers, and $\mathbb{L}$ is the Lefschetz motive.

Remark 2.6.15. Later, we will use the substitution $\mathbb{L} \mapsto t^2$, which recovers the Poincaré polynomial.

We turn to a more complicated example of $C^{[n]}$.

Example 2.6.16. Consider the (projective completion with unique singularity of the) curve $\{x^3 = y^3\}$, ie. three lines on a projective plane intersecting at a point.

We are interested in computing the stalk of the left hand side of (2.6.3) at the central fiber. On the right, the exponential map is a sum over all distinct decompositions of $C = C_1 \cup C_2 \cup C_3$ into subcurves. By symmetry, there are only three fundamentally different ones: the decomposition into three disjoint lines, the decomposition into a node and a line, and the trivial decomposition. Since we know that the fine compactified Jacobians of nodes and lines are points [69], these terms on the right hand side are relatively easy to compute. Namely, for the three lines we have $\left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^3$, and $\left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^2$ for the decompositions to a node plus a line.

As to the last term on the right, C has arithmetic genus one, so is its own fine compactified Jacobian, as shown by Melo-Rapagnetta-Viviani [**?**]. Moreover, C can be realized as a type III Kodaira fiber in a smooth elliptic surface $f : E \to T$, where T is a smooth curve. Let Σ be the singular locus of f. By the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [5], we have from eg. [16, Example 1.8.4]

$$Rf_*\mathbb{Q}_E[2] = \mathbb{Q}_T[2] \oplus (IC(R^1f_*^{sm}\mathbb{Q}_E) \oplus \mathcal{G}) \oplus \mathbb{Q}_T$$

where $\mathcal{G}$ is a skyscraper sheaf on Σ with stalks $H_2(f^{-1}(s))/\langle[f^{-1}(s)]\rangle$. Note that the rank of this sheaf is the number of irreducible components of the fiber *minus one*.

The terms in the above direct sum are ordered so that we first have the second perverse cohomology sheaf ${}^p\mathcal{H}^2(Rf_*\mathbb{Q}_E[2])$, then the first one inside the parentheses and lastly the zeroth perverse cohomology sheaf. Since the base is smooth $IC(R^1) = R^1$ and its stalk is zero at the central fiber. This gives that the numerator of our last term is $1 + 2q\mathbb{L} + q^2\mathbb{L}$. In total, we have

$$\sum_{n\geq 0} q^n H^*(C^{[n]}) = \left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^3 + 3\left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^2 + \tag{2.6.5}$$

$$\frac{1+2q\mathbb{L}+q^2\mathbb{L}}{(1-q)(1-q\mathbb{L})}, \tag{2.6.6}$$

which we compute to be

$$\frac{q^6\mathbb{L}^3 - 2q^5\mathbb{L}^2 + q^4\mathbb{L}^2 + q^3\mathbb{L}^2 + q^4\mathbb{L} - 2q^3\mathbb{L} + q^2\mathbb{L} + q^2 - 2q + 1}{(1-q)^3(1-q\mathbb{L})^3} \tag{2.6.7}$$

Let us now consider the simplest example where $d > 1$.

Example 2.6.17. Similarly, we may consider the projective model of the curve $C = \{x^4 = y^2\}$, which has two rational components that are parabolas. This also has arithmetic genus one and by the same line of reasoning as above we have

$$\sum_{n\geq 0} q^n H^*(C^{[n]}) = \left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^2 + \frac{1+q\mathbb{L}+q^2\mathbb{L}}{(1-q)(1-q\mathbb{L})}$$
$$= \frac{q^4\mathbb{L}^2 - q^3\mathbb{L} + q^2\mathbb{L} - q + 1}{(1-q)^2(1-q\mathbb{L})^2}.$$

Let us now compute the Hilbert series, as predicted by Conjecture 2.6.7, in the cases of Examples 2.6.16, 2.6.17.

Example 2.6.18. In the case of Example 2.6.16, write

$$U_i = (x_j - x_k)\mathbb{C}[x_1, x_2, x_3, y_j + y_k, y_i],\ k \neq i \neq j \neq k.$$

Denote by $\operatorname{gr}\dim V$ the (q,t)-graded dimension of a bigraded vector space V. Then

$$gr\dim(U_1 + U_2 + U_3) = \operatorname{gr}\dim(U_1) + \operatorname{gr}\dim(U_2) + \operatorname{gr}\dim(U_3)$$
$$- \operatorname{gr}\dim((U_1+U_2)\cap U_3) - \operatorname{gr}\dim(U_1\cap U_2)$$

and we compute that:

$$(U_1 + U_2)\cap U_3 = (x_1 - x_3)\mathbb{C}[x_1, x_2, x_3, y_1 + y_2 + y_3]$$
$$+ (x_1 - x_2)(x_2 - x_3)y_3\mathbb{C}[x_1, x_2, x_3, y_1 + y_2 + y_3],$$
$$U_1 \cap U_2 = (x_1 - x_2)\mathbb{C}[x_1, x_2, x_3, y_1 + y_2 + y_3].$$

We then have

$$\operatorname{gr}\dim(U_1 + U_2)\cap U_3 = \frac{q + q^4t^2}{(1-q)^3(1-qt^2)}$$

and

$$\operatorname{gr}\dim(U_1\cap U_2) = \frac{q^2}{(1-q)^3(1-qt^2)}.$$

Hence

$$\operatorname{gr}\dim(V) = \frac{1}{(1-q)^3(1-qt^2)^3} - 3\frac{q}{(1-q)^3(1-qt^2)^2} + \frac{q+q^2+q^4t^2}{(1-q)^3(1-qt^2)},$$

which can be checked to equal the right-hand side of (2.6.7).

Example 2.6.19. In the case of Example 2.6.17, write

$$\begin{aligned} U =&(x_1 - x_2)\mathbb{C}[x_1, x_2, y_1 + y_2], \\ U' =&(x_1 - x_2)^2 \left(\mathbb{C}[x_1, x_2, y_1 + y_2] \oplus \mathbb{C}[x_1, x_2, y_1 + y_2](y_1 - y_2)\right). \end{aligned}$$

Then $U \cap U' = (x_1 - x_2)^2\mathbb{C}[x_1, x_2, y_1 + y_2]$, and we have that the right hand side of (2.6.1) equals

$$\frac{1}{(1-q)^2(1-q\mathbb{L})^2} - \frac{q}{(1-q)^2(1-q\mathbb{L})^2} - \frac{q^2(1+q\mathbb{L})}{(1-q)^2(1-q\mathbb{L})}$$
$$+ \frac{q^2}{(1-q)^2(1-q\mathbb{L})} = \frac{q^4\mathbb{L}^2 - q^3\mathbb{L} + q^2\mathbb{L} - q + 1}{(1-q)^2(1-q\mathbb{L})^2}.$$

As a continuation of Examples 2.6.16, 2.6.17, let us verify that the Poincaré series agrees with the Oblomkov-Rasmussen-Shende conjectures in both cases, since this result does not appear in the literature.

Proposition 2.6.20. *If $C = \{x^3 = y^3\}$, then under the substitutions*

$$q\mathbb{L} \mapsto T^{-1},\ q \mapsto Q,$$

we have the following equality in $\mathbb{Z}[[q,t]]$:

$$\sum_{n \geq 0} q^n H^*(C_0^{[n]}) = f_{000}(Q, 0, T),$$

where $f_{000}(Q, A, T)$ denotes the triply graded Poincaré series of

$$\mathrm{HHH}(T(3,3)).$$

Note that we are considering the punctual Hilbert schemes $C_0^{[n]}$ here.

Proof. From [17, page 9], we have

$$\begin{aligned} f_{000}(Q, A, T) = \frac{1+A}{(1-Q)^3}\Big(&(T^3Q^2 + Q^3T^2 - 2T^2Q^2 - 2TQ^3 - 2QT^3 \\ &+ T^3 + Q^3 + TQ^2 + QT^2 + TQ) + (T^2Q^2 \\ &- 2TQ^2 - 2QT^2 + T^2 + Q^2 + TQ + T + T)A + A^2\Big). \end{aligned}$$

It is quickly verified that letting $A = 0$ and doing the substitution above gives the result. $\square$

Remark 2.6.21. In fact, [17] compute the polynomials $f_v(A,Q,T)$ corresponding to HOMFLY homologies of certain complexes C_v, where v is any binary sequence, using a recursive description. All of these complexes are supported in even degree, and it would be interesting to know how the corresponding pure braids are realized as affine Springer fibers. It would also be interesting to understand these recursions either on $\mathrm{Hilb}^n(\mathbb{C}^2)$ or in terms of affine Springer fibers for GL_n.

The case $C = \{x^2 = y^4\}$ is slightly more straightforward.

$$\sum_{n\geq 0} q^n H^*(C_0^{[n]}) = (1-\mathbb{L}^2)^2 \sum_{n\geq 0} q^n H^*(C^{[n]})$$

can be checked to equal with the Poincaré polynomial of $\mathrm{HHH}^{a=0}(T(2,4))$ for example as follows. From [73, Corollary 15], we have

$$P(\mathrm{HHH}^{a=0}(T(2,4))) = \frac{Q^2 + (1-Q)(T^2+QT)}{(1-Q)^2T^2}.$$

Chapter 3

Hilbert schemes of points on singular curves

In this chapter, we turn to another incarnation of Hilbert schemes, namely those corresponding to Hilbert schemes of points on singular plane curves. This chapter is essentially a reproduction of [48].

Let C be a complex, reduced, locally planar curve. We are interested in studying the homologies of the Hilbert schemes of points $C^{[n]}$. In the case when C is integral, work of Rennemo, Migliorini-Shende, Maulik-Yun [65, 68, 83] relates these homologies to the homology of the compactified Jacobian of C equipped with the perverse filtration. Furthermore, work of Migliorini-Shende-Viviani [69] considers an extension of these results to reduced but possibly reducible curves.

Following Rennemo, we approach the problem of computing the homologies of the Hilbert schemes in question from the point of view of representation theory. In [83], a Weyl algebra in two variables acting on $V := \bigoplus_{n\geq 0} H_*^{BM}(C^{[n]})$ was constructed for integral locally planar curves, and V was described in terms of the representation theory of the Weyl algebra. The superscript BM denotes Borel-Moore homology. When C has m irreducible components, we construct an algebra A acting on V, where A is an explicit subalgebra of the Weyl algebra in $2m$ variables. The main result is the following.

Theorem 3.0.1. *If $C = \bigcup_{i=1}^m C_i$ is a decomposition of C into irreducible components, the* *space* $V = \bigoplus_{n\geq 0} H_*(C^{[n]}, \mathbb{Q})$ *carries a bigraded action of the algebra*

$$A = A_m := \mathbb{Q}[x_1, \ldots, x_m, \partial_{y_1}, \ldots, \partial_{y_m}, \sum_{i=1}^m y_i, \sum_{i=1}^m \partial_{x_i}],$$

where $V = \bigoplus_{n,d\geq 0} V_{n,d}$ *is graded by number of points* n *and homological degree* d*. Moreover, the operators* x_i *have degree* $(1,0)$ *and the operators* ∂_{y_i} *have degree* $(-1,-2)$ *in this bigrading. In effect, the operator* $\sum y_i$ *has degree* $(1,2)$ *and the operator* $\sum \partial_{x_i}$ *has degree* $(-1,0)$.

Remark 3.0.2. The algebra A does not depend on C, but only on the number of components m.

Remark 3.0.3. An argument similar to [83, Theorem 1.2] shows that V is free over $\mathbb{Q}[x_i]$ for any $i = 1, \ldots, m$, and also over $\mathbb{Q}[\sum_{i=1}^m y_i]$. Through the ORS conjectures (see below), this may be seen as a version of Rasmussen's remark in [82] that the triply-graded homology of the link L of C is free over the homology of an unlink corresponding to a component of L.

In Section 3.1 we will discuss the relevant geometry, namely the deformation theory of locally planar curves. In particular, we prove that the relative families of (flag) Hilbert schemes have smooth total spaces, which is crucial for applying a bivariant homology formalism, described in Section 3.2.1.

We then define the action of the generators of A on V by explicit geometric constructions in Section 3.2, and prove the commutation relations in Section 3.3.

In Section 3.4, we describe the representation V of the algebra A_2 in the example of the node. More precisely, we have

Theorem 3.0.4. *When* $C = \{x^2 = y^2\} \subset \mathbb{P}^2$*, we have that*

$$V \cong \frac{\mathbb{Q}[x_1, x_2, y_1, y_2]}{\mathbb{C}[x_1, x_2, y_1 + y_2](x_1 - x_2)} \tag{3.0.1}$$

as an A*-module, where*

$$A = \mathbb{Q}[x_1, x_2, \partial_{x_1} + \partial_{x_2}, y_1 + y_2, \partial_{y_1}, \partial_{y_2}] \subset \mathrm{Weyl}(\mathbb{A}^4).$$

Remark 3.0.5. Although seeing the algebra A for the first time immediately raises the question whether we can define the operators ∂_{x_i} or multiplication by y_i separately, i.e. extend this action to the whole Weyl algebra, this example shows that it is in fact not possible to do this while retaining the module structure for V.

Locally planar curve singularities are connected naturally to topics ranging from the Hitchin fibration [72] to HOMFLY-PT homology of the links of the singularities [32, 73]. For example, from [73] we have

Conjecture 3.0.6. *If C has a unique singularity at 0, its* link *is by definition the intersection of C with a small three-sphere around 0. There is an isomorphism*

$$V_0^c \cong HHH_{a=0}(\textit{Link of } C),$$

where $V_0^c = \bigoplus_{n\geq 0} H^*(C_0^{[n]})$ is the cohomology of the punctual Hilbert scheme, and $HHH(-)$ is the triply graded HOMFLY-PT homology of Khovanov and Rozansky [51].

This conjecture is still wide open. Recently, advances on the knot homology side have been made by Hogancamp, Elias and Mellit [17, 44, 66], who compute the HOMFLY-PT homologies of for example (n, n)-torus links using algebraic techniques. As the (n, n)-torus links appear as the links of the curves $C = \{x^n = y^n\} \subset \mathbb{P}^2$, a partial motivation for this work was to study the Hilbert schemes of points on these curves.

Remark 3.0.7. There are many natural algebras acting on $HHH(-)$, for example the positive half of the Witt algebra as proven in [52]. It might be possible that the actions of the operators $\mu_+ = \sum_i \partial_{x_i}$ and x_i on V are related to this action.

In the case where $C = \{x^p = y^q\}$ for coprime p and q there is an action of the spherical rational Cherednik algebra of SL_n with parameter $c = p/q$ on the cohomology of the compactified Jacobian of C [75, 89], or rather its associated graded with respect to the perverse filtration, which is intimately related to the space V. For arbitrary torus links, it might still be true that V or its variants carry some form of an action of a rational Cherednik algebra.

3.1 Geometry of Hilbert schemes of points

We describe the general setup for this chapter. Fix $C/\mathbb{C}$ a locally planar reduced curve and let $C = \bigcup_{i=1}^m C_i$ be a decomposition of C to irreducible components. We will be working with versal deformations of C.

Definition 3.1.1. If X is a projective scheme, a *versal deformation* of X is a map of germs $\pi : \mathcal{X} \to B$ such that B is smooth, $\pi^{-1}(0) = X$, and given $\pi' : \mathcal{X}' \to B'$ with $\pi'^{-1}(b') = X$ there exists $\phi : B' \to B$ such that $b' \mapsto 0$ and π' is the pullback of π along ϕ. If T_0B coincides with the first-order deformations of X, or in other words the base B is of minimal dimension, we call π a *miniversal* deformation.

We call a family of locally planar reduced complex algebraic curves over a smooth base B *locally versal* at $b \in B$ if the induced deformations of the germs of the singular points of $\pi^{-1}(b)$ are versal. We are interested in smoothness of relative families of Hilbert schemes of points for such deformations, needed for example for Lemma 3.3.3.

Definition 3.1.2. If $\mathcal{X} \to B$ is any family of projective schemes, and $P(t)$ is any Hilbert polynomial, we denote the *relative Hilbert scheme* of this family by $\mathcal{X}^{P(t)}$. By definition, Hilbert schemes are defined for families [35], and we note here that at closed points $b \in B$ the fibers of the relative Hilbert scheme are exactly $\operatorname{Hilb}^{P(t)}(\mathcal{X}_b)$.

We now consider the tangent spaces to (relative) Hilbert schemes.

Lemma 3.1.3. *For any projective scheme X and a flag of subschemes $X_1 \subset \cdots \subset X_k$ in X with fixed Hilbert polynomials $P_1(t), \ldots, P_k(t)$, the Zariski tangent space is given by*

$$T_{(X_1,\ldots,X_n)} \operatorname{Hilb}^{\overrightarrow{P(t)}}(X) \cong H^0(X, \mathcal{N}_{(X_1,\ldots,X_m)/X}),$$

where the sections of the normal sheaf $\mathcal{N}_{(X_1,\ldots,X_m)/X} \subseteq \bigoplus_{i=1}^k \mathcal{N}_{X_i/X}$ *are tuples $(\xi_1, \ldots, \xi_k)$ of normal vector fields such that $\xi_i|_{X_j} = \xi_j$ modulo $\mathcal{N}_{X_j/X_i}$ whenever $X_i \supseteq X_j$. The normal sheaf is by definition the sheaf of germs of commutative diagrams of homomorphisms of $\mathcal{O}_X$-modules of the form*

$$\begin{array}{ccccccccc} \mathcal{I}_k & \subset & \mathcal{I}_{k-1} & \subset & \cdots & \subset & \mathcal{I}_1 \\ \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \sigma_{k-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\ \mathcal{O}_{X_k} & \to & \mathcal{O}_{X_{k-1}} & \to & \cdots & \to & \mathcal{O}_{X_1} \end{array} \qquad (3.1.1)$$

Proof. Note that from first-order deformation theory it immediately follows that if $k = 1$ we have $T_{X_1} \operatorname{Hilb}^{P(t)}(X) \cong H^0(\mathcal{N}_{X_1/X}, X) = H^0(X, \operatorname{Hom}_{\mathcal{O}_{X_1}}(\mathcal{I}_1/\mathcal{I}_1^2, \mathcal{O}_{X_1}))$, where $\mathcal{I}_1$ is the ideal sheaf of X_1. For the proof of the result for flag Hilbert schemes we refer to [85, Proposition 4.5.3]. □

The following proposition is proved in e.g. [86, Proposition 17], and we reprove it here for convenience of the reader.

Proposition 3.1.4. *Let $\pi : \mathcal{C} \to B'$ be a versal deformation of C, a reduced locally planar curve. Then the total space of the family $\pi^{[n]} : \mathcal{C}^{[n]} \to B'$ is smooth.*

Proof. Let $B \subset \mathbb{C}[x, y]$ be a finite dimensional smooth family of polynomials containing the local equation for C and all polynomials of degree at most n, such that the associated deformation is versal. Consider the family of curves over B given by $\mathcal{C}_B := \{(f \in B, p \in \mathbb{C}^2) f(p) = 0\}$. Denote the fiber over f by C_f and let $Z \subset C_f$ be a subscheme of length n. By e.g. [85, Section 4], there is always an exact sequence

$$0 \to H^0(\mathcal{N}_{Z/C_f}, Z) \to T_Z C_B^{[n]} \to T_f B \to \operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{I}_Z, \mathcal{O}_Z)$$

For squarefree f, there is always some open neighborhood U of f such that $C_U^{[n]}$ is reduced of pure dimension $n + \dim B$ [65, Proposition 3.5]. Since B is smooth and $H^0(\mathcal{N}_{Z/C_f}, Z)$ has dimension n (see e.g. [12]), it is enough to prove that the last Ext-group vanishes to get smoothness of the total space $C_U^{[n]}$ at Z.

Now from the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{C_f} \to \mathcal{O}_Z \to 0$$

taking Hom to $\mathcal{O}_Z$ we have

$$\cdots \to \operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{O}_{C_f}, \mathcal{O}_Z) \to \operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{I}_Z, \mathcal{O}_Z) \to \operatorname{Ext}^2_{\mathcal{O}_{C_f}}(\mathcal{O}_Z, \mathcal{O}_Z) = 0 \to \cdots$$

As $\operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{O}_{C_f}, \mathcal{O}_Z) \cong H^1(\mathcal{O}_Z, C_f) = 0$ and the sequence is exact, we must have

$$\operatorname{Ext}^1_{\mathcal{O}_{C_f}}(\mathcal{I}_Z, \mathcal{O}_Z) = 0$$

as well. So the total space is smooth.

Now if $\overline{\mathcal{C}} \to \overline{B}$ is the miniversal deformation, by versality there are compatible isomorphisms $\mathcal{C} \cong \overline{\mathcal{C}} \times (\mathbb{C}^t, 0)$ and $B \cong \overline{B} \times (\mathbb{C}^t, 0)$ for some t, see e.g. [33]. Hence we have smoothness for any versal family.

$\square$

We now consider the relative flag Hilbert scheme of a versal deformation. If S is a smooth complex algebraic surface, its nested Hilbert scheme of points $S^{[n,n+1]}$ is smooth by results of [12, 88].

Remark 3.1.5. This nested Hilbert scheme $S^{[n,n+1]}$, together with the ordinary Hilbert scheme of n points $S^{[n]}$ are the only flag Hilbert schemes of points on S that are smooth, as shown in [12, 88].

We also have the following result.

Proposition 3.1.6. *The total space of the relative family $\mathcal{C}^{[n,n+1]} \to B$ is smooth.*

Proof. This is a local question, so we can assume that C is the germ of a plane curve singularity in $\mathbb{C}^2$.

From Lemma 3.1.3, we have that at $(J \subset I) \in \mathcal{C}^{[n,n+1]}$ the tangent space is $\ker(\phi - \psi)$, where

$$\phi : \mathrm{Hom}_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/I) \to \mathrm{Hom}_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/I), \quad \text{and}$$
$$\psi : \mathrm{Hom}_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/J) \to \mathrm{Hom}_{\mathbb{C}[x,y]}(J, \mathbb{C}[x,y]/I)$$

are the induced maps given by restriction and further quotient. Here $\phi - \psi$ is the difference of the maps from the direct sum. This is precisely the requirement needed for the normal vector fields in question.

Suppose again that $B \subset \mathbb{C}[x,y]$ is a finite dimensional smooth family of polynomials containing the local equation for C and all polynomials of degree at most $n+1$, such that the associated deformation is versal.

Consider the inclusion $\mathcal{C}_B^{[n,n+1]} \hookrightarrow B \times (\mathbb{C}^2)^{[n,n+1]}$. We have an exact sequence

$$0 \to T_{f,J\subset I}\mathcal{C}_B^{[n,n+1]} \to T_f B \times T_{J\subset I}(\mathbb{C}^2)^{[n,n+1]} \to \mathbb{C}[x,y]/J$$

where the last map is given by $\left(f+\epsilon g, \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right) \mapsto (\phi\eta_1)(f) - g \bmod J$. By the assumption on B the last map is surjective, hence $T_{f,J\subset I}\mathcal{C}_B^{[n,n+1]}$ has dimension $\dim(\mathbb{C}^2)^{[n,n+1]} + \dim B - n + 1 = n + 1 + \dim B$, as expected, and the total space is smooth.

Again, if $\overline{\mathcal{C}} \to \overline{B}$ is the miniversal deformation, by versality there are compatible isomorphisms $\mathcal{C} \cong \overline{\mathcal{C}} \times (\mathbb{C}^t, 0)$ and $B \cong \overline{B} \times (\mathbb{C}^t, 0)$ for some t, see e.g. [33]. Hence we have smoothness for any versal family.

$\square$

Another result we will also need is the description of the components and dimensions of the irreducible components of $C^{[n]}$.

Proposition 3.1.7. *For any locally planar reduced curve $C = \bigcup_{i=1}^m C_i$, the irreducible components of $C^{[n]}$ are given by*

$$\overline{(C_1^{sm})^{[r_1]} \times \cdots \times (C_m^{sm})^{[r_m]}}, \quad \sum_i r_i = n.$$

Here C_i^{sm} denotes the smooth locus of C_i. In particular, there are $\binom{n+m-1}{n}$ irreducible components of $C^{[n]}$, all of dimension n.

Proof. That $(C^{sm})^{[n]}$ is dense in $C^{[n]}$ can be found e.g. in [67, Fact 2.4]. The schemes $(C_1^{sm})^{[r_1]} \times \cdots \times (C_m^{sm})^{[r_m]}$ are disjoint. They are of dimension n, smooth and connected, so irreducible, and as $(r_1, \ldots, r_m)$ runs over all possibilities, cover $(C^{sm})^{[n]}$. Taking closures we get the result. $\square$

3.2 Definition of the algebra A

Let $V = \bigoplus_{i\geq 0} H_*(C^{[n]}, \mathbb{Q})$, where we take singular homology in the analytic topology. This is mostly for simplicity, a majority of the results work with $\mathbb{Z}$ coefficients. A notable exception is Section 3.4, where $\mathbb{Q}$-coefficients are essential. From now on we will be suppressing the coefficients from our notation. V is naturally a bigraded $\mathbb{Q}$-vector space, graded by the number of points n and homological degree d. We denote by $V_{n,d}$ the (n,d)-graded piece of V. We define the following operators on V, following ideas of Rennemo [83] (and that originally go back to Nakajima and Grojnowski [37, 70]).

Definition 3.2.1.
1. Let $c_i \in C_i^{sm}$ be fixed smooth points and $\iota_i : C^{[n]} \to C^{[n+1]}$ be the maps $Z \mapsto Z \cup c_i$. Let $x_i : V \to V$ be the operators given by $(\iota_i)_*$. These are homogeneous of degree $(1,0)$ and only depend on the component the points c_i lie in, see Lemma 3.2.2 below.

2. Let $d_i : V \to V$ be the operators given by the Gysin/intersection pullback map $(\iota_i)^!$. These are homogeneous of degree $(-1,-2)$, and well defined since the ι_i are regular embeddings. See Lemma 3.2.3 below for a proof of this latter fact.

Lemma 3.2.2. *The maps ι_{c_i} and $\iota_{c_i'}$ are homotopic whenever $c_i, c_i' \in C_i^{sm}$. In particular the corresponding pushforwards induce the same operators x_i on V.*

Proof. Take any path $c_i(t)$ from c_i to c_i' and consider the homotopy $C^{[n]} \times [0,1] \to C^{[n+1]}$ given by $(Z,t) \mapsto Z \cup c_i(t)$. □

Lemma 3.2.3. *The map ι_x is a regular embedding.*

Proof. This is a property which is local in the analytic topology ([3, Chapter 2, Lemma 2.6]). Suppose $Z \subset C$ is a subscheme of length n which contains x with multiplicity k.

If U is an analytic open set around x such that the only component of Z contained in $\overline{U}$ (closure in the analytic topology) is x. Then locally around Z the morphism is isomorphic to

$$U^{[k]} \times (C \backslash \overline{U})^{[n-k]} \hookrightarrow (U)^{[k+1]} \times (C \backslash \overline{U})^{[n-k]},$$

where the map is given on factors by adding x and the identity map, respectively. The first map in local coordinates looks exactly like the inclusion $\mathbb{C}^{[k]} \to \mathbb{C}^{[k+1]}$ given as follows. If we identify coordinates on $\mathbb{C}^{[k]}$ with symmetric functions a_i in the roots of some degree k polynomial, i.e. coefficients of a monic polynomial of degree k, the map is given by $\sum_{i=0}^{k-1} a_i z^i \mapsto (z-x)\sum_{i=0}^{k-1} a_i z^i$. But this last map is linear in the a_i and of rank k, in particular a regular embedding. □

Consider the following diagram.

$$\begin{tikzcd} & C^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C^{[n]} & & C^{[n+1]} \end{tikzcd} \tag{3.2.1}$$

To define the operators μ_+ and μ_-, we want to define correspondences in homology between $C^{[n]}$ and $C^{[n+1]}$. This is done as follows. By Propositions 3.1.4, 3.1.6, we may embed C into a smooth locally versal family $\pi : \mathcal{C} \to B$ so that the relative family $\mathcal{C}^{[n]}$ is smooth and $\pi^{-1}(0) = C$. After possibly doing an étale base extension, we may also assume that the family also has sections $s_i : B \to \mathcal{C}$ hitting only the smooth loci of the fibers and so that $s_i(0) = c_i$.

Consider now the diagram

$$\begin{tikzcd}
 & C^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "i"] & \\
C^{[n]} \arrow[dd, "i"] & & C^{[n+1]} \arrow[dd, "i"] \\
 & \mathcal{C}^{[n,n+1]} \arrow[dl, "\widetilde{p}"'] \arrow[dr, "\widetilde{q}"] & \\
\mathcal{C}^{[n]} & & \mathcal{C}^{[n+1]}
\end{tikzcd} \tag{3.2.2}$$

where i is the inclusion of the central fiber. Since $\mathcal{C}^{[n]}$ is smooth, from Property 7 in Section 3.2.1, we have that

$$H^*(\mathcal{C}^{[n,n+1]} \to \mathcal{C}^{[n]}) \cong H^{BM}_{*-2n-\dim B}(\mathcal{C}^{[n,n+1]}).$$

Denote the fundamental class of $\mathcal{C}^{[n,n+1]}$ under this isomorphism as $[\widetilde{p}]$. Then pulling back $[\widetilde{p}]$ along i to $H^*(C^{[n,n+1]} \to C^{[n]})$ gives us a canonical orientation, using which we define $p^! : H_*(C^{[n]}) \to H_*(C^{[n,n+1]})$ as $p^!(\alpha) = \alpha \cdot i^*([\widetilde{p}])$. The definition of $q^!$ is identical, where we replace $C^{[n]}$ by $C^{[n+1]}$.

We are finally ready to define the operators μ_+ and μ_-.

Definition 3.2.4. Let $\mu_\pm : V \to V$ be the *Nakajima correspondences* $\mu_+ = q_* p^!$, and $\mu_- = p_* q^!$. These are operators of respective bidegrees $(1, 2)$ and $(-1, 0)$.

Remark 3.2.5. The n-degree in the above maps is easy to see from the definition. The homological degrees follow from the definition of the Gysin maps using $i^*[\widetilde{p}]$, which sits in homological degree $2n + 2$, and the fact that degrees are additive under the bivariant product.

We are now ready to define the algebra(s) A.

Theorem 3.2.6. *The operators defined in Definitions 3.2.1, 3.2.4 satisfy the following commutation relations:* $[d_i, \mu_+] = [\mu_-, x_i] = 1$*, and the rest are trivial.*

Remark 3.2.7. For $m = 1$ we recover Theorem 1.2 in [83].

Definition 3.2.8. Fix $m \geq 1$. Let A_m be the $\mathbb{Q}$-algebra generated by the symbols

$$x_1, \dots, x_m, d_1, \dots, d_m, \mu_+, \mu_-$$

with the relations

$$[d_i, \mu_+] = [\mu_-, x_i] = 1, [x_i, x_j] = [x_i, d_i] = [x_i, \mu_+] = [d_i, \mu_-] = 0.$$

Remark 3.2.9. We can realize A_m inside Weyl($\mathbb{A}^{2m}_{\mathbb{Q}}$) as follows: Let $\mathbb{A}^{2m}$ have coordinates

$$x_1, \dots, x_m, y_1, \dots, y_m$$

and $d_i = \partial_{y_i}, k = \sum_{i=1}^m \partial_{x_i}, j = \sum_{i=1}^m y_i$. Then from the commutation relations, we immediately have that A_m is isomorphic to the subalgebra $\langle x_i, \partial_{y_i}, \sum_{i=1}^m \partial_{x_i}, \sum_{i=1}^m x_i \rangle \subset$ Weyl($\mathbb{A}^{2m}_{\mathbb{Q}}$).

Remark 3.2.10. Although A_m depends on m we will be suppressing the subscript from the notation from here on. It should be evident from the context which m we are considering.

Let us give an outline of the proof of Theorem 3.2.6. We first prove the trivial commutation relations in Subsections 3.3.1, 3.3.2. We then prove in Subsection 3.3.3 that $[d_i, \mu_+] = 1$ with the aid of the bivariant homology formalism, and then in a similar vein that $[\mu_-, x_i] = 1$.

3.2.1 Bivariant Borel-Moore homology

We now describe the bivariant Borel-Moore homology formalism from [20]. Suppose we are in a category of "nice" spaces; for example those that can be embedded in some $\mathbb{R}^n$. We will not define bivariant homology here, but for us the most essential facts about it are the following ones:

1. The theory associates to maps $X \xrightarrow{f} Y$ a graded abelian group $H^*(X \xrightarrow{f} Y)$. We will be working over $\mathbb{Q}$ throughout also with bivariant homology.

2. Given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a product homomorphism

$$H^i(X \xrightarrow{f} Y) \otimes H^j(Y \xrightarrow{g} Z) \to H^{i+j}(X \xrightarrow{g\circ f} Z).$$

For $\alpha \in H^i(X \xrightarrow{f} Y)$ and $\beta \in H^j(Y \xrightarrow{g} Z)$ we thus get a product $\alpha \cdot \beta \in H^{i+j}(X \xrightarrow{g\circ f} Z)$.

3. For any proper map $X \xrightarrow{f} Y$ and any map $Y \xrightarrow{g} Z$ there is a pushforward homomorphism $f_* : H^*(X \xrightarrow{g\circ f} Z) \to H^*(Y \xrightarrow{g} Z)$.

4. For any cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

there is a pullback homomorphism $H^*(X \xrightarrow{f} Y) \to H^*(X' \xrightarrow{g} Y')$. (Recall that a cartesian square is a square where $X' \cong X \times_Y Y'$.)

5. Product and pullback commute: Given a tower of cartesian squares

$$\begin{tikzcd} X' \arrow[r, "h''"] \arrow[d, "f'"] & X \arrow[d, "f", "\alpha"'] \\ Y' \arrow[r, "h'"] \arrow[d, "g'"] & Y \arrow[d, "g", "\beta"'] \\ Z' \arrow[r, "h"] & Z \end{tikzcd}$$

we have $h^*(\alpha \cdot \beta) = h'^*(\alpha) \cdot h^*(\beta)$ in $H^*(X' \xrightarrow{g'\circ f'} Z')$.

6. Product and pushforward commute: Given

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with $\alpha \in H^*(X \xrightarrow{g\circ f} Z)$ and $\beta \in H^*(Z \xrightarrow{h} W)$, we have $f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot \beta$ in $H^*(Y \xrightarrow{h\circ g} W)$.

7. For any space X, the groups $H^i(X \to \mathrm{pt})$ and $H^i(X \xrightarrow{\mathrm{id}} X)$ are by construction canonically identified with $H^{BM}_{-i}(X)$ and $H^i(X)$, respectively. These are called the associated covariant and contravariant theories, respectively. Note that the three bivariant operations recover the usual homological operations of cup and cap product, proper pushforwards in homology and arbitrary pullbacks in cohomology.

8. If Y is a nonsingular variety and $f : X \to Y$ is any morphism, the induced homomorphism
$$H^*(X \xrightarrow{f} Y) \to H^{*-2\dim Y}(X \to \mathrm{pt}) = H^{BM}_{2\dim Y - *}(X)$$
given by taking the product with $[Y] \in H^{-2\dim Y}(Y \to \mathrm{pt})$ is an isomorphism. Again the last equality is given by the associated covariant theory. In such a situation we will frequently identify $H^*(X \to Y)$ with $H^{\mathrm{BM}}_{2\dim Y - *}(X)$. In particular, if X has a fundamental class $[X] \in H^{BM}_{2\dim X}(X)$, this induces a class $[X] \in H^{2(\dim Y - \dim X)}(X \to Y)$.

9. Any class $\alpha \in H^i(X \xrightarrow{f} Y)$ defines a Gysin pull-back map $f^! : H^{BM}_*(Y) \to H^{BM}_{*-i}(X)$ by
$$f^!(\beta) := \alpha \cdot \beta, \qquad \forall \beta \in H^{\mathrm{BM}}_*(Y).$$

3.3 Proof of the commutation relations

3.3.1 Proof of the trivial commutation relations for x_i and d_i

We now show that $[x_i, x_j] = 0$ for all i, j. This is fairly easy; under either composition $\iota_i \circ \iota_j, \iota_j \circ \iota_i$ we map $Z \mapsto Z \cup x_i \cup x_j$ and as $(\iota_i \circ \iota_j)_* = x_i x_j$, we get $x_i x_j = x_j x_i$.

The next step is to describe the Gysin maps and their commutation relations. Denote these as before by $d_i = (\iota_{x_i})^! : H_*(C^{[n]}) \to H_{*-2}(C^{[n-1]})$.

Proposition 3.3.1. *We have $[d_i, d_j] = 0$ and $[d_i, x_j] = 0$.*

Proof. Let $\alpha \in H_*(C^{[n]})$. As we saw before, $\iota_{x_i} \circ \iota_{x_j} = \iota_{x_j} \circ \iota_{x_i}$. By functoriality of the Gysin maps, $[d_i, d_j] = 0$.

Now choosing a representative for α, we see that

$$d_i x_j(\alpha) = (\iota_j)^![\iota_i(\alpha)] = [Z \in \alpha | c_i, c_j \subset Z].$$

However, we also have

$$x_j d_i(\alpha) = x_j[Z' \in \alpha | c_i \subset Z'] = [Z \in \alpha | c_i, c_j \subset Z].$$

When the points are equal in the above, that is to say $i = j$, we pick a linearly equivalent point c_i' near c_i. Since the inclusion maps $\iota_{c_i}, \iota_{c_i'}$ are homotopic in this case by Lemma 3.2.2, we still have $[d_i, x_j] = 0$.

□

3.3.2 Proof of the trivial commutation relations for Nakajima operators

As we saw before, the definition of the Nakajima operators requires making sense of the Gysin morphisms $p^!$ and $q^!$, which is done using the bivariant homology formalism. Recall that we are working with a fixed family $\mathcal{C} \to B$ as in Section 3.2, guaranteeing smoothness of $\mathcal{C}^{[n]}$ and $\mathcal{C}^{[n,n+1]}$. In this section and later on, all commutative diagrams should be thought of as commutative diagrams of topological spaces (corresponding to the analytic spaces of the varieties under consideration) and living over B, such that we may restrict to the central fiber and obtain similar squares with calligraphic $\mathcal{C}$s replaced by regular Cs, i.e. our curve of interest. We will denote these restrictions in homology computations by the subscript 0.

Proposition 3.3.2. *We have $[x_i, \mu_+] = 0$ and $[d_i, \mu_-] = 0$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} & \mathcal{C}^{[n,n+1]} \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[rr, "\iota_i''"] & & \mathcal{C}^{[n+1,n+2]} \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ \mathcal{C}^{[n]} \arrow[rr, "\iota_i"] & & \mathcal{C}^{[n+1]} \arrow[rr, "\iota_i'"] & & \mathcal{C}^{[n+2]} \end{tikzcd} \tag{3.3.1}$$

Here ι_i, ι_i' are defined as adding points at the sections s_i to $\mathcal{C}^{[n]}, \mathcal{C}^{[n+1]}$ respectively, and ι_i'' is adding points at the section s_i as follows: $(Z_1 \subset Z_2) \mapsto (Z_1 \cup s_i \subset Z_2 \cup s_i)$.

We have by definition that $x_i\mu_+ = ((\iota'_{s_i})_* q_* p^!)_0$, where the subscript 0 denotes restriction to the central fiber.

In Diagram 3.3.1, the square formed by the maps ι'_i, q, q' and ι''_i is commutative, so on homology we have $(\iota'_i)_* q_* p^! = q'_*(\iota''_i)_* p^!$. Similarly, the square formed by the maps $\iota_i, p', p, \iota''_i$ is commutative. Because of the fact that pushforward and the Gysin maps in bivariant homology also commute in this case, as explained in Section 3.2.1 (Property 7), we get $q'_*(\iota''_i)_* p^! = q'_*(p')^!(\iota_i)_*$. Restricting to C we have $(q'_*(p')^!(\iota_i)_*)_0 = \mu_+ x_i$.

Similarly, for the other commutation relation we have $\mu_- d_i = (p_* q^!(\iota'_i)^!)_0 = (p_*(\iota''_i)^!(q')^!)_0 = ((\iota_i)^! p'_*(q')^!)_0 = d_i\mu_-$.

Let us explain the restriction to central fiber once and for all. For example, in the last computation if $\alpha \in H_*(C^{[n]})$, we have

$$\mu_- d_i \alpha = i^*[\widetilde{p}] \cdot \iota_i^! \alpha = i^*[\widetilde{p}] \cdot i^*[\widetilde{d}_i] \cdot \alpha,$$

where $i^*[\widetilde{d}_i] = [d_i]$ is the fundamental class corresponding to the Gysin map $\iota_i^!$ and $[\widetilde{d}_i]$ is the corresponding class in the family. Since the product and pullback commute in the bivariant theory,

$$i^*[\widetilde{p}] \cdot i^*[\widetilde{d}_i] \cdot \alpha = i^*([\widetilde{p}] \cdot [\widetilde{d}_i]) \cdot \alpha.$$

Similarly, $d_i\mu_-\alpha = [d_i] \cdot i^*[\widetilde{p}] \cdot \alpha = i^*([\widetilde{d}_i] \cdot [\widetilde{p}]) \cdot \alpha$. So composing our operators in the family and then deducing the result for C is justified.

□

3.3.3 Proof that $[\mu_-, x_i] = [d_i, \mu_+] = 1$

To compute the desired commutation relation, we compare the composition of the operators d_i, μ_+ on V in either order. By abuse of notation we will first consider d_i and μ_+ as operators acting on the space $\mathcal{V} = \bigoplus_{n \geq 0} H_*^{BM}(\mathcal{C}^{[n]})$, and then use the properties of the bivariant theory, more precisely the ability to pull back in cartesian squares (Property 5 in Section 3.2.1), to restrict to the special fiber and get an action on V.

Consider the diagrams

$$
\begin{tikzcd}
 & \mathcal{C}^{[n,n+1]} \arrow[dl, "\theta", "p"'] \arrow[d, "\kappa"', "q"] & X_i \arrow[l, hook', "\widetilde{\lambda}_i"', "\widetilde{\iota}_i"] \arrow[d, "\widetilde{\kappa}_i"', "\widetilde{q}_i"] \\
\mathcal{C}^{[n]} & \mathcal{C}^{[n+1]} & \mathcal{C}^{[n]} \arrow[l, hook', "\lambda_i"', "\iota_i"]
\end{tikzcd}
\tag{3.3.2}
$$

and

$$
\begin{tikzcd}
 & & \mathcal{C}^{[n-1,n]} \arrow[dl, "\theta'"', "p'"] \arrow[dr, "q'", "\kappa'"'] & \\
\mathcal{C}^{[n]} & \mathcal{C}^{[n-1]} \arrow[l, hook', "\lambda'_i"', "\iota'_i"] & & \mathcal{C}^{[n]}
\end{tikzcd}
\tag{3.3.3}
$$

where the two labels on the arrows denote the corresponding map $f : Y \to Z$ and a bivariant class $\alpha \in H^*(X \xrightarrow{f} Y)$

In the first diagram $X_i = \mathcal{C}^{[n,n+1]} \times_{\mathcal{C}^{[n+1]}} \mathcal{C}^{[n]}$ is the fiber product, and the square containing X_i is cartesian. The morphisms ι_i, ι'_i correspond to adding a point at the sections $s_i : B \to \mathcal{C}$. The bivariant classes $\theta, \lambda_i, \kappa$ and their primed versions are the ones defined by fundamental classes, using the fact that the targets are smooth. The classes $\widetilde{\lambda_i}$ and $\widetilde{\kappa_i}$ are the cartesian pullbacks of λ_i and κ, respectively.

Let $\alpha \in H_*(\mathcal{C}^{[n]})$. We first compute

$$
d_i\mu_+(\alpha) = \lambda'_i \cdot q_*(\theta \cdot \alpha) = \widetilde{q}_*(\widetilde{\lambda}_i \cdot \theta \cdot \alpha) \tag{3.3.4}
$$

$$
\mu_+ d_i(\alpha) = q'_*(\theta' \cdot \lambda'_i \cdot \alpha). \tag{3.3.5}
$$

Let us elaborate on the first computation a little bit. Here θ is the fundamental class and the isomorphism $H^{BM}_{*-n-1}(\mathcal{C}^{[n,n+1]}) \cong H^*(\mathcal{C}^{[n,n+1]} \to \mathcal{C}^{[n]})$ of the Borel-Moore homology group with the bivariant one is given by product with the fundamental class θ. On the other hand, the Gysin pullback $\iota_i^!$ is by definition equal to the product in the bivariant theory with λ_i. In the first equation of (3.3.4), we then use that the diagram in (3.3.2) in is cartesian.

Let $f_i : \mathcal{C}^{[n-1,n]} \to X_i$ be given by $(Z_1 \subset Z_2) \mapsto (Z_1 \cup s_i \subset Z_2 \cup s_i, Z_1 \cup s_i)$, and $g_i : \mathcal{C}^{[n]} \to X_i$ be given by $Z \mapsto (Z \subset Z \cup s_i, Z)$.

Lemma 3.3.3. *For all i, $[X_i] = (f_i)_*([\mathcal{C}^{[n-1,n]}]) + (g_i)_*([\mathcal{C}^{[n]}])$.*

Proof. By Proposition 3.1.4 and Proposition 3.1.6 the total spaces of the relative families $\mathcal{C}^{[n]} \to B$ and $\mathcal{C}^{[n,n+1]} \to B$ are smooth.

Consider the fiber product X_i. The images of f_i and g_i cover all of X_i. On the level of points (of the fibers) this is easy to see; we are looking at pairs consisting of a flag of subschemes of lengths $n, n+1$ and a subscheme of length n that project to the same length $n+1$ subscheme in the cartesian square (3.3.2). Since the points in the image contain s_i, either the above pairs come from adding s_i to both parts of the flag as well as taking the second factor to be $Z_1 \cup s_i$, or by creating a new flag by adding s_i to Z and taking Z to be the second factor.

By [83, Lemma 3.4], the intersection of the images of f_i and g_i is codimension one in X_i. Consider a point$(Z \subset Z \cup s_i, Z) \in \mathrm{Im}(f_i) \cap \mathrm{Im}(g_i)$, which can also be written as $(Z' \cup s_i \subset Z' \cup s_i \cup s_i, Z' \cup s_i)$. We can then remove the smooth point s_i from both of the factors unambiguously. So the intersection is isomorphic to $\mathcal{C}^{[n-1]}$. On the complement of the intersection the maps f_i, g_i are scheme-theoretic isomorphisms, because we can unambiguously remove the point s_i from $(Z \subset Z \cup s_i, Z)$ or $(Z_1 \cup s_i \subset Z_2 \cup s_i, Z_1 \cup s_i)$. Hence, the images of f_i, g_i yield a partition of X_i to irreducible components. In particular, the fundamental class $[X_i]$ is the sum of the fundamental classes of the images, which are by definition the pushforwards in question. □

Corollary 3.3.4. *We have* $[X_i] = (f_i)_*(\theta' \cdot \lambda_i') + (g_i)_*[\mathcal{C}^{[n]}]$.

Proof. By Lemma 3.3.3 $[X_i] = (f_i)_*([\mathcal{C}^{[n-1,n]}]) + (g_i)_*([\mathcal{C}^{[n]}])$. Rewrite $[\mathcal{C}^{[n-1,n]}]$ as follows. First of all note that

$$\theta' \cdot \lambda_i' \in H^{2\dim \mathcal{C}^{[n]} - 2\dim \mathcal{C}^{[n-1,n]}}(\mathcal{C}^{[n-1,n]} \to \mathcal{C}^{[n]}) \cong H^{BM}_{2\dim \mathcal{C}^{[n-1,n]}}(\mathcal{C}^{[n,n+1]}).$$

The last isomorphism is given by Property 8 in Section 3.2.1, ie. taking the product with $[\mathcal{C}^{[n]}] \in H^*(\mathcal{C}^{[n]} \to \mathrm{pt})$. On the other hand, we know that $\theta' \cdot \lambda_i' \cdot [\mathcal{C}^{[n]}]$ has to be $[\mathcal{C}^{[n-1,n]}]$ by the same isomorphism. Plugging this into the result of Lemma 3.3.3 gives

$$[X_i] = (f_i)_*(\theta' \cdot \lambda_i') + (g_i)_*([\mathcal{C}^{[n]}]).$$

□

Using Corollary 3.3.4, we have

$$d_i\mu_+(\alpha) = (\widetilde{q}_i)_*(\widetilde{\lambda}_i \cdot \theta \cdot \alpha) = (\widetilde{q}_i)_*([X_i] \cdot \alpha) = (\widetilde{q}_i)_*((f_i)_*(\theta' \cdot \lambda_i') \cdot \alpha) + (\widetilde{q}_i)_*((g_i)_*[\mathcal{C}^{[n]}] \cdot \alpha). \tag{3.3.6}$$

The last equality follows by linearity of the pushforward. From Property 6 in Section 3.2.1, we have that the pusforward is also functorial and commutes with products. Hence we have

$$(\widetilde{q}_i)_*((f_i)_*(\theta' \cdot \lambda_i') \cdot \alpha) + (\widetilde{q}_i)_*((g_i)_*[\mathcal{C}^{[n]}] \cdot \alpha) = (\widetilde{q}_i \circ (f_i))_*(\theta' \cdot \lambda_i' \cdot \alpha) + (\widetilde{q}_i \circ (g_i))_*(\alpha). \tag{3.3.7}$$

Since $\widetilde{q}_i \circ f_i = q'$, we get

$$(\widetilde{q}_i \circ (f_i))_*(\theta' \cdot \lambda_i' \cdot \alpha) = q'_*(\theta' \cdot \lambda_i' \cdot \alpha).$$

Finally, since $\widetilde{q}_i \circ (g_i) = \mathrm{id}$,

$$(\widetilde{q}_i \circ (g_i))_*(\alpha) = \mathrm{id}_*(\alpha).$$

Substituting these into Eq. (3.3.7), we get

$$(\widetilde{q}_i \circ (f_i))_*(\theta' \cdot \lambda_i' \cdot \alpha) + (\widetilde{q}_i \circ (g_i))_*(\alpha) = q'_*(\theta' \cdot \lambda_i' \cdot \alpha) + \mathrm{id}_*(\alpha) = \mu_+ d_i(\alpha) + \alpha. \tag{3.3.8}$$

Suppose now that α_0 is a class in $H_*(C^{[n]})$. Then by the fact that pushforward, pullback, and the product in the bivariant theory commute, $((\widetilde{q}_i)_0)_*((\widetilde{\lambda}_i)_0 \cdot \theta_0 \cdot \alpha_0) = (q_0')_*((\theta')_0 \cdot (\lambda_i')_0 \cdot \alpha_0) + \mathrm{id}_*(\alpha_0)$ and $d_i\mu_+ = \mu_+ d_i + \mathrm{id} : V \to V$, as desired.

The case of $[\mu_-, x_i]$ is much similar; here we have

$$\mu_- x_i(\alpha) = (p)_*(\kappa \cdot (\iota_i)_*(\alpha)) = (p \circ \widetilde{\iota}_i)_*(\widetilde{\kappa}_i \cdot \alpha)$$

and

$$x_i\mu_-(\alpha) = (\iota_i' \circ p')_*(\kappa' \cdot \alpha).$$

Under the identification of $H^*(X \xrightarrow{\widetilde{q}} \mathcal{C}^{[n]})$ with $H^{\mathrm{BM}}_{*+2\dim \mathcal{C}^{[n]}}(X)$ we have $\widetilde{\kappa}_i = [X_i]$. This follows from

$$\widetilde{\kappa}_i \cdot [\mathcal{C}^{[n]}] = \widetilde{\kappa}_i \cdot \lambda_i \cdot [\mathcal{C}^{[n+1]}] = \widetilde{\lambda}_i \cdot \kappa[\mathcal{C}^{[n+1]}] = \widetilde{\lambda}_i \cdot [\mathcal{C}^{[n,n+1]}] = [X_i],$$

where the last equality is the fact that X_i is a Cartier divisor in $\mathcal{C}^{[n,n+1]}$. Using Lemma 3.3.3 we get

$$\widetilde{\kappa}_i = [X_i] = (f_i)_*[\mathcal{C}^{[n-1,n]}] + (g_i)_*[\mathcal{C}^{[n]}] = (f_i)_*(\kappa') + (g_i)_*[\mathcal{C}^{[n]}].$$

A computation similar to (3.3.6) now shows $\mu_- x_i(\alpha) = x_i\mu_-(\alpha) + \alpha$ as needed, and the restriction to the special fiber works exactly the same way. This finishes the proof of Theorem 3.2.6 and thus of Theorem 3.0.1.

3.4 Example: The node

In this section we describe the representation V for the the curve $\{xy = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$, which is the first nontrivial curve singularity with two components.

3.4.1 Geometric description of $C^{[n]}$

One first thing we may ask is how the components in Proposition 3.1.7 look like? Ran [80] describes the geometry of the Hilbert scheme of points on (germs of) nodal curves very thoroughly. For $n = 0, 1$ we get a point and C itself, whereas $C^{[2]}$ is a chain of three rational surfaces, that intersect their neighbors transversely along projective lines. More generally, $C^{[n]}$ is a chain of $n+1$ irreducible components of dimension n, consecutive members of which meet along codimension one subvarieties.

Lemma 3.4.1. *Denote by $M_{n,k}$ the irreducible component of $C^{[n]}$, where generically we have k points on the component $y = 0$ of C, and on the component $x = 0$ we have $n - k$ points. Then*

$$M_{n,k} \cong Bl_{\mathbb{P}^{k-1}\times\mathbb{P}^{n-k-1}}(\mathbb{P}^k \times \mathbb{P}^{n-k}).$$

Proof. First of all, $\mathbb{P}^k \times \mathbb{P}^{n-k}$ has natural coordinates given by coefficients of polynomials $(a(x), b(y))$ of degrees k and $n - k$. It is also natural to identify the roots of these polynomials with the corresponding subschemes in $C_1, C_2 \cong \mathbb{P}^1$. From $(a(x), b(x))$ we construct an ideal in the homogeneous coordinate ring of C by taking the product

$$I = (y, a(x))(x, b(x)) = (xy, xa(x), yb(y), a(x)b(y)).$$

This determines a length n subscheme so a point in $M_{n,k}$. Note that this map is invertible outside the locus where we have at least one point from each axis at the origin.

We can further write $a(x)b(y) = a_0b_0 + a_0b'(y) + b_0a'(x) \bmod (xy)$, where $a(x) = a_0 + a'(x)$, $b(y) = b_0 + b'(y)$ and $a'(x), b'(x)$ have no constant term. Consider now the limit of $I = I_1$ as the products of the coordinates of the roots of $a(x), b(y)$ separately go to zero linearly, i.e. let $t \to 0$ in $a_0 = At$, $b_0 = Bt$ and in the corresponding family of ideals I_t. Since this is a flat family, the limiting ideal $I_0 = \lim_{t\to 0} I_t$ has the same colength and support on the locus where at least one point from each axis is at the origin. In particular $(a_0b_0 + a_0b'(y) + b_0a'(x))/t \to Ab'(y) + Ba'(x)$ as $t \to 0$, and

$$\lim_{t\to 0} I_t = (xy, xa'(x), yb'(y), Ab'(y) + Ba'(x)).$$

Since all ideals in the locus of $M_{n,k}$ with at least one point from each axis at the origin can be written in this form, and $(A : B) \in \mathbb{P}^1$ determines the limiting ideal completely, we can identify $(A : B)$ with the normal coordinates $(a_0 : b_0)$ and the natural map

$$\pi : M_{n,k} = \overline{(C_1^{sm})^{[k]} \times (C_2^{sm})^{[n-k]}} \to \mathbb{P}^k \times \mathbb{P}^{n-k}$$

is the blowup along the locus where both $a(x)$ and $b(y)$ have zero as a root.

See also [81] for a similar blow-up description.

□

The intersections of the components can also be seen in this description.

Lemma 3.4.2. *We have $E^n_{k,k+1} = M_{n,k} \cap M_{n,k+1} \cong \mathbb{P}^{n-k-1} \times \mathbb{P}^k$ and all the other intersections are trivial.*

Proof. We continue in the notation of the proof of Lemma 3.4.1. Denote the locus where at least one point from either axis is at the origin in $M_{n,k}$ by $L_{n,k}$. Suppose then that we are outside $L_{n,k} \cup L_{n,\widetilde{k}}$ inside $M_{n,k} \cap M_{n,\widetilde{k}}$. Then only one point is at the origin and the this locus is identified naturally with the complement of the corresponding locus in $\mathbb{P}^{n-k-1} \times \mathbb{P}^k$ if $k + 1 = \widetilde{k}$ and is empty otherwise. We are thus left to studying the loci $L_{n,k}$.

Consider again points $I = (xy, xa'(x), yb'(y)), Ab'(y) + Ba'(x))$ in $L_{n,k}$ and points $\widetilde{I} = (xy, x\widetilde{a}'(x), y\widetilde{b}'(y)), \widetilde{A}\widetilde{b}'(y) + \widetilde{B}\widetilde{a}'(x))$ in $L_{n,\widetilde{k}}$.

First, restrict I to the x-axis i.e. let $y = 0$. Then $I|_{y=0} = (xa'(x), Ba'(y))$. If $B = 0$ this has colength $k + 1$ since $a'(x)$ is of degree k. If $B \neq 0$ the colength is k.

Similarly, we get the colengths of $\widetilde{I}|_{y=0}$ to be $\widetilde{k}$ or $\widetilde{k} + 1$ depending on whether $\widetilde{B}$ is nonzero or not. Without loss of generality we can assume $\widetilde{k} > k$. In this case the only possibility for $I, \widetilde{I}$ to be in the intersection $M_{n,k} \cap M_{n,k+1}$ is to have $k + 1 = \widetilde{k}, B = 0$ and $\widetilde{B} \neq 0$.

A similar analysis for the y-axis shows that we must have $\widetilde{A} = 0$ and $A \neq 0$. So in particular, the intersections $E^n_{k,\widetilde{k}}$ are isomorphic to $\mathbb{P}^{n-k-1} \times \mathbb{P}^k$ if $k + 1 = \widetilde{k}$ and empty otherwise.

□

Now one may compute what V is. There is a natural stratification of a blowup to the exceptional divisor and its complement. These both come with affine pavings, so a particularly easy way to compute the cohomologies of $C^{[n]}$, or at least the Betti numbers, is to count these cells. We have

Proposition 3.4.3. *The bigraded Poincaré series for the space* $V = \bigoplus_{n \geq 0} H_*(C^{[n]})$ *is given by*

$$P_V(q,t) = \frac{q^2t^2 - q + 1}{(1-q)^2(1-qt^2)^2}.$$

The grading corresponding to t is the homological degree, whereas q keeps track of the grading given by number of points.

Proof. It is easily confirmed that the Poincaré polynomials of the components are given by

$$P_{M_{n,k}}(t) = t^2 \left(\sum_{i=0}^{k-1} t^{2i}\right)\left(\sum_{i=0}^{n-k-1} t^{2i}\right) + \left(\sum_{i=0}^{n-k} t^{2i}\right)\left(\sum_{i=0}^{k} t^{2i}\right).$$

Similarly, the Poincaré polynomials of the intersections are given by

$$Q_{E^n_{k,k+1}}(t) = \left(\sum_{i=0}^{k} t^{2i}\right)\left(\sum_{i=0}^{n-k-1} t^{2i}\right), \ k \leq n-1,$$

	0	**2**	**4**	**6**	**8**	**10**	$\cdots$
0	1	0	0	0	0	0	$\cdots$
1	1	2	0	0	0	0	$\cdots$
2	1	3	3	0	0	0	$\cdots$
3	1	4	5	4	0	0	$\cdots$
4	1	5	7	7	5	0	$\cdots$
5	1	6	9	10	9	6	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 3.1. The dimensions $V_{n,d}$ i.e. the Betti numbers of $C^{[n]}$. The columns are labeled by homological degree d and the rows by the number of points n.

and $\sum_{k=0}^{n} P_{M_{n,k}}(t) - \sum_{k=0}^{n-1} Q_{E^n_{k,k+1}}(t)$ is by Mayer-Vietoris the Poincaré polynomial of $C^{[n]}$. It is easy to see that $\sum_{n\geq 0} q^n \left(\sum_{k=0}^{n} P_{M_{n,k}}(t) - \sum_{k=0}^{n-1} Q_{E^n_{k,k+1}}(t)\right) = P_V(q,t)$. $\square$

Figure 3.1 shows the graded dimension of V as a bigraded vector space.

3.4.2 Computation of the A-action

We will now investigate the action of the algebra $A = \langle x_1, x_2, \mu_+, \mu_-, d_1, d_2\rangle_{\mathbb{Q}}$ on V.

Consider $V_{\bullet,2n} = \bigoplus_i V_{i,2n} \subset V$, i.e. all the classes in homological degree $2n$. Denote by $[M_{n,k}]$ the fundamental class of the irreducible component $M_{n,k}$ of $C^{[n]}$ as described in the previous subsection.

Theorem 3.4.4. *The fundamental classes $[M_{n,k}] \in V_{n,2n}$ generate $V_{\bullet,2n}$ as a $\mathbb{Q}[x_1,x_2]$-module.*

Proof. This is equivalent to proving that the maps $x_i|_{V_{k,2n}}$ are jointly surjective for $k \geq n$. Dualizing the maps to pullbacks $x_i^*|_{V_{k,2n}}$ in cohomology, this condition is to say that the operators $x_i^* : H^{<2k+2}(C^{[k+1]}) \to H^*(C^{[k]})$ must satisfy $\bigcap \ker x_i^* = 0$.

We have the following diagram for the components of $C^{[n]}$ and their intersections.

$$\begin{tikzcd}[column sep=small]
& E^n_{0,1} \arrow[dl, hook'] \arrow[dr, hook] & & E^n_{1,2} \arrow[dl, hook'] \arrow[dr, hook] & & E^n_{2,3} \arrow[dl, hook'] \arrow[dr, hook] & \cdots & \\
M_{n,0} & & M_{n,1} & & M_{n,2} & & M_{n,3} & \cdots
\end{tikzcd}$$

Since $C^{[n]}$ is a chain of the components $M_{k,n-k}$ intersecting transversally, without triple intersections, the Mayer- Vietoris sequence in homology for unions splits to short exact sequences:

$$0 \to \bigoplus_{k=0}^{n-1} H_i(E^n_{k,k+1}) \to \bigoplus_{k=0}^{n} H_i(M_{n,k}) \to H_i(C^{[n]}) \to 0$$

Dually, we have an exact sequence the other way around in cohomology. By our blow-up description of $\pi : M_{n,k} \mapsto \mathbb{P}^{n-k} \times \mathbb{P}^k$, we have as graded vector spaces that (see e.g. [34], Chapter 6):

$$H^*(M_{n,k}) = \frac{\pi^* H^*(\mathbb{P}^{n-k} \times \mathbb{P}^k) \oplus H^*(\mathbb{P}(\mathcal{N}_{\mathbb{P}^{n-k}\times\mathbb{P}^k/\mathbb{P}^{n-k-1}\times\mathbb{P}^{k-1}}))}{\pi^* H^*(\mathbb{P}^{n-k-1} \times \mathbb{P}^{k-1})}.$$

In particular, we can write

$$H^*(\mathbb{P}^{n-k} \times \mathbb{P}^k) = \mathbb{Q}[a_{n,k}, b_{n,k}]/(a_{n,k}^{n-k+1}, b_{n,k}^{k+1})$$

as well as

$$H^*(\mathbb{P}(\mathcal{N}_{\mathbb{P}^{n-k}\times\mathbb{P}^k/\mathbb{P}^{n-k-1}\times\mathbb{P}^{k-1}})) = \mathbb{Q}[a'_{n,k}, b'_{n,k}, \zeta_{n,k}]/(a'^{n-k}_{n,k}, b'^{k}_{n,k}, (\zeta_{n,k} - a'_{n,k})(\zeta_{n,k} - b'_{n,k})),$$

where $\zeta_{n,k} = c_1(\mathcal{O}(1))$.

Since the classes

$$a''^{i}_{n,k} b''^{j}_{n,k} \in \pi^* H^*(\mathbb{P}^{n-k-1} \times \mathbb{P}^{k-1}) \cong \mathbb{Q}[a''_{n,k}, b''_{n,k}]/(a''^{n-k}_{n,k}, b''^{k}_{n,k})$$

are in $\pi^* H^*(\mathbb{P}^{n-k} \times \mathbb{P}^k)$ identified with $a^i_{n,k} b^j_{n,k}$ where $i < n-k$, $j < k$ and in the exceptional divisor with $a'^{i}_{n,k} b'^{j}_{n,k}$, the quotient map as graded $\mathbb{Q}$-modules is the one identifying $a^i_{n,k} b^j_{n,k}$ with $a'^{i}_{n,k} b'^{j}_{n,k}$.

All in all, we can write as a graded $\mathbb{Q}$-vector space that

$$H^*(M_{n,k}) = \langle 1, a_{n,k}, b_{n,k}, \ldots, a^{n-k}_{n,k} b^k_{n,k}, \zeta_{n,k}, \ldots, \zeta_{n,k} a^{n-k-1}_{n,k} b^{k-1} \rangle_{\mathbb{Q}}. \tag{3.4.1}$$

Example 3.4.5. We have that $H^*(M_{2,0}) = \mathrm{Span}\{1, a_{3,0}, a_{3,0}^2\}$, as expected, since $M_{2,0} \cong \mathbb{P}^2_{\mathbb{C}}$. We also have that $H^*(M_{2,1}) = \mathrm{Span}\{1, a_{2,1}, a_{2,1}^2, b_{2,1}, a_{2,1}b_{2,1}, a_{2,1}^2 b_{2,1}, \zeta_{2,1}, \zeta_{2,1}a_{2,1}\}$.

Having described the cohomology of the components $M_{n,k}$ we can get back to our exact sequence. Identify

$$H^*(\mathbb{P}^{n-k-1} \times \mathbb{P}^k) \cong \mathbb{Q}[\mu_{n,k}, \nu_{n,k}]/(\mu_{n,k}^{n-k}, \nu_{n,k}^{k+1}).$$

Lemma 3.4.6. *Under the inclusion $E^n_{k,k+1} \hookrightarrow M_{n,k}$ we have $\mu_{n,k} \mapsto a_{n,k} - \zeta_{n,k}$ and $\nu_{n,k} \mapsto b_{n,k} - \zeta_{n,k}$ in cohomology. Similarly, under the inclusion $E^n_{k,k+1} \hookrightarrow M_{n,k+1}$ we have $\mu_{n,k} \mapsto a_{n,k+1} - \zeta_{n,k+1}$ and $\nu_{n,k} \mapsto b_{n,k+1} - \zeta_{n,k+1}$.*

Proof. The class of $\mu_{n,k}$ in the intersection is the class dual to the the line $L^y_{n,k}$, where we fix all points in $E^n_{k,k+1}$ at the origin except for one at the y-axis. Similarly the class of $\nu_{n,k}$ is the line $L^x_{n,k}$ where we have but one point on the x-axis. Under the blowup $\pi_{n,k} : M_{n,k} \to \mathbb{P}^{n-k} \times \mathbb{P}^k$ the class of $L^y_{n,k}$ in $M_{n,k}$ is given by the total transform, which satisfies $[L^y_{n,k}] + \zeta_{n,k} = a_{n,k}$. The computation for the other three cases is nearly identical and we omit it. $\square$

Example 3.4.7. When $n = 2$ the Hilbert scheme $C^{[2]}$ has the following components: $M_{2,0} \cong M_{2,2} \cong \mathbb{P}^2$ and $M_{2,1} \cong Bl_{pt}(\mathbb{P}^1 \times \mathbb{P}^1)$. The intersections are $E^2_{0,1} \cong E^2_{1,2} \cong \mathbb{P}^1$. The fundamental class of the first intersection is denoted $\mu_{2,0}$ and that of the second one is denoted $\mu_{2,1}$. Under the inclusion $E^2_{0,1} \hookrightarrow M_{2,0}$, the class $\mu_{2,0}$ is identified with $a_{2,0}$, and under the inclusion $E^2_{0,1} \hookrightarrow M_{2,1}$ with $a_{2,1} - \zeta_{2,1}$. Similarly, under the inclusion $E^2_{1,2} \hookrightarrow M_{2,1}$ the class $\mu_{2,1}$ is identified with $a_{2,1} - \zeta_{2,1}$, whereas under the inclusion $E^2_{1,2} \hookrightarrow M_{2,2}$ it is identified with $a_{2,2}$.

As follows from the definition of the maps $\iota_i : C^{[n]} \to C^{[n+1]}$, we can consider them as restricted to $M_{n,k}$. They induce, by abuse of notation, maps in cohomology $x_i^* : H^*(M_{n+1,k+i-1}) \to H^*(M_{n,k})$. We can describe these maps explicitly.

Lemma 3.4.8. *In the basis of* (3.4.1), *we have*

$$x_1^* : a^i_{n+1,k} b^j_{n+1,k} \mapsto a^i_{n,k} b^j_{n,k}$$

and

$$\zeta_{n+1,k} a_{n+1,k}^i b_{n+1,k}^j \mapsto \zeta_{n,k} a_{n,k}^i b_{n,k}^j.$$

Similarly,

$$x_2^* : a_{n+1,k+1}^i b_{n+1,k+1}^j \mapsto a_{n,k}^i b_{n,k}^j$$

and

$$\zeta_{n+1,k+1} a_{n+1,k+1}^i b_{n+1,k+1}^j \mapsto \zeta_{n,k} a_{n,k}^i b_{n,k}^j.$$

Proof. We are adding one fixed smooth point i.e. embedding $C^{[n]} \hookrightarrow C^{[n+1]}$ as a divisor. Blowing down the components it is immediate that the a-classes go to the a-classes and the b-classes go to the b-classes. We can treat the classes in the exceptional divisor separately, where everything reduces again to embedding products of projective spaces as above. In addition, we need that $x_1^* \zeta_{n+1,k+1} = \zeta_{n,k}$, which is saying that the normal bundle of the exceptional divisor of $M_{n+1,k+1}$ restricts to that of the exceptional divisor of $M_{n,k}$ under the embedding $\iota_1 : M_{n,k} \to M_{n+1,k+1}$. In the notation of Lemma 3.4.1 we have that the map ι_1 is on $M_{n,k}$ given by multiplying $a(x)$ by $x - c$ for some fixed $c \neq 0$. In particular, the centers of the blowups become identified, and the restriction of the normal bundle of the exceptional divisor of $M_{n+1,k+1}$ is the normal bundle of $M_{n,k}$. $\square$

Having the above lemmas at our hands, we want to prove that the intersections of the kernels of the x_i^* are only the fundamental classes.

The basic object of study here is the commutative diagram

$$\begin{array}{ccc} \bigoplus_{k=0}^{n} H^{<2n}(M_{n,k}) & \longleftarrow & H^{<2n}(C^{[n]}) \\ \downarrow & & \downarrow \\ \bigoplus_{i=0}^{n-1} H^*(M_{n-1,i}) & \longleftarrow & H^*(C^{[n-1]}) \end{array} \tag{3.4.2}$$

We can explicitly describe the kernels on the left: for each $M_{n,k}$, only the classes $\zeta_{n,k} a_{n,k}^{k-1} b_{n,k}^{n-k-1}$ are in their intersection. In particular the intersection of the kernels is nonempty. But this can be remedied on the right, as follows. By Lemma 3.4.6 and the

Mayer-Vietoris sequence, inside the intersection we can check that

$$\begin{aligned}
x_2^*(\sum_k \lambda_k \zeta_{n,k} a_{n,k}^{k-1} b_{n,k}^{n-k-1}) &= \sum_k \lambda_k x_2^*(\zeta_{n,k}) a_{n-1,k}^{k-1} b_{n-1,k}^{n-k-1} \\
&= \sum_k \lambda_k x_2^*(a_{n,k} - a_{n,k+1} + \zeta_{n,k+1}) a_{n-1,k}^{k-1} b_{n-1,k}^{n-k-1} \\
&= \sum_k \lambda_k (a_{n-1,k-1} - a_{n-1,k}) a_{n-1,k}^{k-1} b_{n-1,k}^{n-k-1} \\
&= \sum_k \lambda_k a_{n-1,k-1} a_{n-1,k}^{k-1} b_{n-1,k}^{n-k-1},
\end{aligned}$$

which is 0 if and only if $\lambda_k = 0$ for all k. Repeating this for x_1^* we have

$$x_1^*(\sum_k \lambda_k \zeta_{n,k} a_{n,k}^{k-1} b_{n,k}^{n-k-1}) = \sum_k \lambda_k b_{n-1,k-1} a_{n-1,k}^{k-1} b_{n-1,k}^{n-k-1} = 0$$

if and only if $\lambda_k = 0$ for all k. In particular, we see that the image of the fundamental class of the exceptional divisor is also nonzero, i.e. it is not in the kernel and $\bigcap_i \ker x_i^* = 0$ on the right.

This finishes the proof of Theorem 3.4.4.

□

Having Theorem 3.4.4 at our hands, we can finally restate Theorem 3.0.4:

Theorem 3.4.9. *Consider the following bigraded vector space: let $V'' = \mathbb{Q}[x_1, x_2, y_1, y_2]$ with x_i in degree $(1,0)$ and y_i in degree $(1,2)$. Consider the action of $A' = \mathbb{Q}\langle x_i, \partial_{y_i}, \sum y_i, \sum \partial_{x_i}\rangle$ on this space as differential operators, and let U be the submodule $\mathbb{Q}[x_1, x_2, y_1+y_2](x_1-x_2)$. Define $V' = V''/U$. Then $V \cong V'$ as $A = \mathbb{Q}[x_1, x_2, d_1, d_2, \mu_+, \mu_- \cong A'$-modules.*

Proof. That A' is isomorphic to A in this case follows from the commutation relations when sending $x_i \mapsto x_i$, $\partial_{y_i} \mapsto d_i$ and $\sum y_i \mapsto \mu_+, \sum \partial_{x_i} \mapsto \mu_-$. We can identify V' and V as A-modules as follows: let the monomial $y_1^i y_2^j / i!j!$ correspond to the fundamental class of $M_{i+j,i}$. It is then clear that on the diagonal $\bigoplus_{n\geq 0} V_{n,2n}$ the operators d_1, d_2, μ_+, μ_- act as the corresponding differential operators in A'. Namely, the Gysin maps d_1, d_2 are given by intersection, from which it follows that $d_1[M_{i+j,i}] = [M_{i+j-1,i-1}]$ and $d_2[M_{i+j,i}] = [M_{i+j-1,i}]$. This can be compared to the fact that for example $\partial_{y_1} y_1^i y_2^j / i!j! = y_1^{i-1} y_2^j / (i-1)!j!$.

By the commutation relations, $[d_1^{i+1}, \mu_+] = (i+1)d_1^i$, so

$$[d_1^{i+1}, \mu_+]y_1^i y_2^j/i!j! = (i+1)y_2^j/j!,\ [d_2^{j+1}, \mu_+]y_1^i y_2^j/i!j! = (j+1)y_1^i/i!$$

and in particular

$$d_1^{i+1}\mu_+ y_1^i y_2^j/i!j! = (i+1)y_2^j/j!,\ d_2^{j+1}\mu_+ y_1^i y_2^j/i!j! = (j+1)y_1^i/i!.$$

Since $\mu_+ y_1^i y_2^j/i!j! = \sum_{k=0}^{i+j+1} c_k y_1^k y_2^{i+j+1-k}$ for some constants c_k, we must have $c_k = 0$ unless $k = i$ or $k = i+1$, in which case we have $c_k = 1/i!j!$. This shows that μ_+ can be identified with multiplication by $y_1 + y_2$ on the diagonal, and below the diagonal since it commutes with the action of x_1, x_2. The operator μ_- acts on the diagonal as zero by degree reasons. An argument similar to the above shows that μ_- acts below the diagonal by $\partial_{x_1} + \partial_{x_2}$.

Since the maps x_i are jointly surjective on the rows by Theorem 3.4.4, we get a surjection $\phi : \mathbb{Q}[x_1, x_2, y_1, y_2] \twoheadrightarrow V$. This is an A-module homomorphism by above. Its kernel contains U, since $(x_1 - x_2) \cdot 1 = 0$ and the actions of x_i and μ_+ commute with the x_i.

Consider then the graded dimensions/Poincaré series of V' and V. We have

$$P_{V'}(q,t) = P_{V''}(q,t) - P_U(q,t) = \frac{1}{(1-q)^2(1-qt^2)} - \frac{q(1-qt^2)}{(1-q)^2(1-qt^2)^2} = P_V(q,t),$$

and since $\ker\phi \supseteq U$, we must have $\ker\phi = U$. $\square$

3.5 Coulomb branches and correspondence algebras

In joint work with Garner [21], we study these Hilbert schemes as generalized affine Springer fibers in the sense of [7]. In this Section together with Section **??** give an introduction to this circle of ideas. Let $G/\mathbb{C}$ be reductive, $\mathfrak{g} = \mathrm{Lie}(G)$, and V be a(n algebraic) representation of G. Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let Gr_G be the affine Grassmannian of G.

Definition 3.5.1. Let $v \in V(\mathcal{K})$. Define the *generalized affine Springer fiber* associated to v as the reduced ind-scheme

$$M_v(\mathbb{C}) := \{g \in G(\mathcal{K}) | g^{-1}.v \in V(\mathcal{O})\}/G(\mathcal{O}).$$

Remark 3.5.2. Note that the definition of M_v also depends on G, V. Since we will only be working with $G = GL_n, V = \mathrm{Ad} \oplus \mathbb{C}^n$, we omit these from the notation.

Remark 3.5.3. The "classical" affine Springer fibers are the case when $V = \mathrm{Ad}$.

3.5.1 Springer representations

In [7], associated to the datum (G, V), certain convolution algebras were defined as follows.

Definition 3.5.4. Define the *BFN space* of (G, V) as

Note that the $\mathbb{C}$-points are given by the set

$$\mathcal{R}_{G,V}(\mathbb{C}) = \{(g, s) \in G(\mathcal{K}) \times V(\mathcal{O})\}/G(\mathcal{O}).$$

Theorem 3.5.5 (Braverman-Finkelberg-Nakajima)**.** *There is a natural convolution product on $\mathcal{A}_{G,V} := H_*^{G(\mathcal{O})}(\mathcal{R}_{G,V})$, making $\mathcal{A}_{G,V}$ an associative, commutative algebra with unit.*

3.6 Hilbert schemes of points on curve singularities

Let $\widehat{C} := \operatorname{Spec} R$ be a germ of a reduced plane curve singularity and write $R = \frac{C[[x,t]]}{f}$.

Definition 3.6.1. The *Hilbert scheme of N points on $\widehat{C}$* is defined as the reduced scheme

$$\widehat{C}^{[n]} := \mathrm{Hilb}^n(\widehat{C}) := \{\text{colength } n \text{ ideals in } R\}.$$

Remark 3.6.2. In particular, the reduced scheme

$$\mathrm{Hilb}^\bullet(\widehat{C}) := \bigsqcup_{N \geq 0} \mathrm{Hilb}^N(\widehat{C})$$

is naturally the moduli space of nonzero ideals on $\widehat{C}$.

We now state and prove our main theorem.

Theorem 3.6.3. *For any $\widehat{C}$, there is a generalized* $\mathrm{Ad} \oplus \mathbb{C}^n$*-affine Springer fiber $M_v \subset$ Gr_G so that there is an isomorphism of schemes*

$$\varphi : M_v \to \mathrm{Hilb}^\bullet(C)$$

Proof. Note that we can interpret $\widehat{C}$ and $\widehat{C}^{[n]}$ as follows. If $f(x, y)$ is a degree n polynomial in x, by reducedness of $\widehat{C}$ we may write as $\mathbb{C}[[t]] = \mathcal{O}$-modules that

$$\frac{\mathbb{C}[[x, t]]}{f} = \langle 1, x, \ldots, x^{n-1} \rangle_{\mathcal{O}}, \tag{3.6.1}$$

where $\langle S \rangle_{\mathcal{O}}$ denotes the free $\mathcal{O}$-module generated by a set S.

Taking the total ring of fractions of R, we see that as $\mathbb{C}((t)) = \mathcal{K}$-vector spaces $\mathrm{Frac}(R) \cong \prod_{i=1}^{d} F_i \cong \mathcal{K}^n$ where d is the number of irreducible factors over $\mathcal{K}$ of f and F_i are finite extensions of $\mathcal{K}$ so that $\sum_i [F_i : \mathcal{K}] = n$. There is a natural injection $R \hookrightarrow \mathrm{Frac}(R)$, and we choose an isomorphism $\phi : \mathrm{Frac}(R) \cong \mathcal{K}^n$ identifying R with $\mathcal{O}^n$ and $1 \in R$ with the vector $e_1 = (1, 0, \ldots, 0)$ in $\mathcal{K}^n$. We may moreover choose ϕ so that in the standard basis of $\mathcal{K}^n$, x has the form

$$\gamma = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \ddots & 0 & a_3 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & a_n \end{pmatrix}$$

so that $e_1, \gamma e_1, \ldots, \gamma^{n-1} e_1$ is a $\mathcal{O}$-basis of $\mathcal{O}^n$. Recall that a matrix of the above form is called the *companion matrix* of the polynomial $a_1 + a_2 t + \cdots + a_n t^n$.

By definition, $\mathcal{O}$-lattices in $\mathrm{Frac}(R)$ stable under x are identified with (nonzero) fractional R-ideals. The variety of (nonzero) ideals in R is then identified with fractional ideals in $\mathrm{Frac}(R)$ contained in R, and under ϕ, we get

$$\mathrm{Hilb}^{\bullet}(\widehat{C}) \cong X := \{\Lambda \subset \mathcal{O}^n | \gamma\Lambda \subset \Lambda\}.$$

Now for any Λ, there is an element $g \in G(\mathcal{K})$ so that $g\Lambda = \mathcal{O}^n$. It is well defined up to the stabilizer of $\mathcal{O}^n$ which is $G(\mathcal{O})$. In particular, $g^{-1} e_1 \in \Lambda$. Because $\gamma g^{-1} \mathcal{O}^n \subset g^{-1} \mathcal{O}^n$, we have $g\gamma g^{-1} \in \mathrm{Lie}(G(\mathcal{O}))$.

By sending Λ to $(g, g\gamma g^{-1}, g^{-1} e_1)$, get a map from X to the scheme

$$M_v = \{(g, \gamma', e) | g^{-1}\gamma' g = \gamma, ge = e_1\}/G(\mathcal{O}).$$

Conversely, to any (g, γ', e) satisfying these conditions we can associate

$$\Lambda = \langle e, \gamma' e, \dots, \gamma'^{n-1} e \rangle_{\mathcal{O}} \subset \mathcal{O}^n.$$

Then $\gamma\Lambda = \gamma g^{-1}\mathcal{O}^n = g^{-1}\gamma'\mathcal{O}^n \subset g^{-1}\mathcal{O}^n = \Lambda$. As these constructions are inverse to each other, we have $X \cong M_v$.

Finally, composing with the isomorphism to $\mathrm{Hilb}^\bullet(\widehat{C})$ we get that

$$\mathrm{Hilb}^\bullet(\widehat{C}) \cong M_v.$$

By Definition 3.5.1 the space M_v is the generalized $\mathrm{Ad} \oplus \mathbb{C}^n$-affine Springer fiber for $v = (\gamma, e_1)$.

$\square$

Chapter 4

Geometric representation theory of trigonometric DAHA

In this final chapter, which is slightly separate from the rest of the thesis, we give an introduction to the work of Lusztig-Yun and provide examples (most of which are not found or are scattered around the literature) of their theory in type A. This ties most closely to the theory of affine Springer fibers and we spend some time elucidating this connection.

4.1 Graded Lie algebras

Let G be a semisimple simply connected algebraic group over an algebraically closed field $\mathbf{k}$. Let $\mathfrak{g}$ be its Lie algebra and θ an order $m \in \mathbb{N} \cup \{\infty\}$ semisimple automorphism of G. Then $d\theta$ induces a $\mathbb{Z}/m$-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i$. Let $\mathcal{N}_i = \mathfrak{g}_i \cap \mathcal{N}$, where $\mathcal{N}$ is the cone of nilpotent elements in $\mathfrak{g}$. Without loss of generality we can let $i = 1$. There is an action $K \times \mathfrak{g}_i \to \mathfrak{g}_i$ by conjugation, called a θ-representation in the literature. Lusztig and Yun have studied the $K := G^\theta$-equivariant constructible sheaves on $\mathcal{N}_1$. Denote the derived category of constructible sheaves on $\mathcal{N}_i$ by $D_K(\mathcal{N}_i)$.

Theorem 4.1.1 ([60], Theorem 0.6)**.** *There is a canonical direct sum decomposition into full subcategories*

$$D_K(\mathcal{N}_i) = \bigoplus_{(M, \mathfrak{m}_*, C)} D_K(\mathcal{N}_i)_{(M, \mathfrak{m}_*, C)},$$

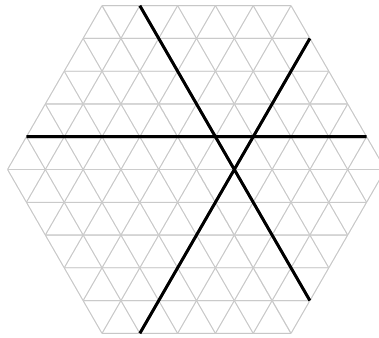

Figure 4.1. The hyperplane arrangement $\mathfrak{H}_1$ for A_2, $c = \frac{1}{n}$.

where M is a pseudo-Levi subgroup in G, $\mathfrak{m}_*$ is a $\mathbb{Z}$-grading on its Lie algebra compatible with the $\mathbb{Z}/m$-grading of $\mathfrak{g}$, and C is a cuspidal local system. The equivalence classes of triples $(M, \mathfrak{m}_*, C)$ are called admissible systems, and the full subcategories $D_K(\mathcal{N}_i)_{(M,\mathfrak{m}_*,C)}$ are called blocks.

Denote the category of K-equivariant perverse sheaves with middle perversity on $\mathcal{N}_i$ by $\mathcal{P}_K(\mathcal{N}_i)$. Let $I(\mathcal{N}_i)$ be the set of isomorphism classes of simple objects in this category, and $\mathfrak{B}$ the set of blocks.

In this chapter, we are mainly interested in the principal block, with the notable exception of Section 4.4.

We will review the type A edge cases where θ is either inner or the Cartan involution.

4.2 Spirals and spiral induction

Let A be a

4.3 Trigonometric double affine Hecke algebras

In this section, we review trigonometric double affine Hecke algebras and the classification of their irreducible representations at a rational parameter in type A, due to Cherednik, Suzuki, and Vasserot [13, 87, 90].

Let $G/\mathbb{C}$ be a connected reductive group. Fix a pinning $T \subset B \subset G$ and denote the associated Weyl group W. Define the extended affine Weyl group as $\widetilde{W} := W \ltimes X_*(T)$. The affine Weyl group W^{aff} is a canonical subgroup so that $\widetilde{W}/W^{aff}$ is a finite group. In

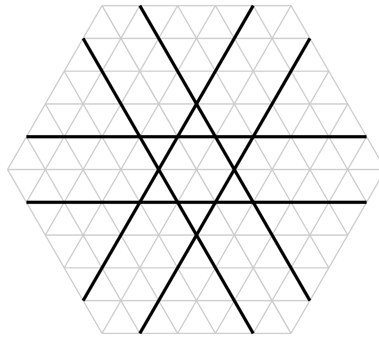

Figure 4.2. The hyperplane arrangement $\mathfrak{H}_1$ for A_2, $c = 1$.

fact, it is a cyclic group unless we are in type D_4, which we will for simplicity of exposition not consider in this thesis. Denote a lift of a generator of this cyclic group by π, and the the set of simple reflections in W^{aff} determined by the pinning S. We choose π so that $\pi s_i = s_{i+1}\pi$.

Definition 4.3.1. The *trigonometric DAHA* $\mathcal{H}$ of $\widetilde{W}$ is defined as follows. As a vector space, it is the tensor product

$$\mathbb{C}[\widetilde{W}] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[u].$$

Moreover,

1. The factors $\mathbb{C}[\widetilde{W}]$ and $\mathbb{C}[\mathfrak{t}]$ under their natural embeddings are subalgebras.
2. The element u is central.
3. For all $v \in \mathfrak{t}^*, s_i \in S$, we have $s_i v - {}^{s_i}v s_i = c_i u\langle \alpha_i, v\rangle$, and $\pi v = {}^{\pi}v\pi$.

The grading is defined by assigning u degree 2 and $\widetilde{W}$ degree zero (implying $\mathfrak{t}^*$ has degree 2). The numbers c_i are in general images of a conjugacy-invariant function $S \to \mathbb{C}$.

Definition 4.3.2. The specialization of $\mathcal{H}$ at $c \in \mathbb{C}$, is defined as

$$\mathcal{H}_c := \mathcal{H}/(u + c).$$

Note that this is only a filtered algebra. We will also call $\mathcal{H}_c$ the trigonometric DAHA as no confusion is likely to arise.

Remark 4.3.3. Below we will simply speak of $\mathcal{H}$ associated to G and in Section 4.4, to the pair (G, θ). In these cases the numbers c_i are as in [62, Section 2.5]. In particular if $G = GL_n$, $c_i = 1$ for all i, and if $(G, \theta) = (GL_n, g \mapsto g^{-t})$, the number for the short roots is 1 and for the long roots it's 1/2.

4.3.1 The case $G = GL_n$

In this subsection, we consider $G = \mathrm{GL}_n$ and parameters of the form $c = 1/m$, where m is an integer. We will state a classification of all $\mathcal{H}_c$-modules in "category $\mathcal{O}$", due to Cherednik, Vasserot, and Suzuki [13, 87, 90].

Remark 4.3.4. In [87], Suzuki uses slightly different notation. Namely, his "parameter of the degenerate double affine Hecke algebra" is the reciprocal of our c, or in other words the parameter usually called "t" in the literature. This can easily be seen by comparing Definition 4.3.2 and [87, Definition 3.1]. Note also that in the work of Lusztig-Yun [60] and in earlier work of Lusztig [56], the numbers c_i are related to ours by a factor of two (having to do with the normalization in pairing of roots and coroots).

Definition 4.3.5. Let $\mathcal{O}_c$ be the full subcategory of $\mathcal{H}_c$-fgmod consisting of those $\mathcal{H}_c$-modules M so that

1. M is locally $\mathbb{C}[\mathfrak{t}]$-finite (note that this implies that we have a generalized weight space decomposition $M = \oplus_{\zeta \in \mathfrak{t}^*} M_\zeta$),
2. The generalized weight spaces are contained in $X_*(T) + mu^*$, where u^* is the element in $X_*(T) \oplus \mathbb{Z}$ dual to u.

Theorem 4.3.6. *The set of isomorphism classes of irreducible representations in $\mathcal{O}_c$ is in bijection with the set of dimension n nilpotent representations of the cyclic quiver with m vertices.*

Proof. This is proved e.g. in [87, Theorem 7.2], but we give another proof here using results of [53, 60, 90], which boils essentially down to a tautology for those familiar with Springer theory. In *loc. cit.*, it is shown that the irreducible modules in $\mathcal{O}_c$ can be identified with irreducible G_0-equivariant perverse sheaves on $\mathfrak{g}_1^{nil}$. Since $G = \mathrm{GL}_n$, it is clear that these are parameterized by G_0-orbits on $\mathfrak{g}_1^{nil}$. On the other hand, these are by definition the nilpotent representations of dimension n of the cyclic quiver with m vertices. $\square$

4.4 Block decomposition for the split symmetric pair in type A

The following corollary is an immediate consequence of Theorem 4.1.1

Corollary 4.4.1. *There is a map*

$$\Psi : I(\mathcal{N}_i) \to \mathfrak{B}$$

associating to each simple perverse sheaf its block.

We will determine this map (called "map 3.5" in [60]) in the case $G = SL_N$, $\theta : g \mapsto g^{-t}$, N odd. The even N case is similar but we omit it for notational simplicity.

In principle, the methods here are generalizable to arbitary symmetric spaces, but as Lemma 4.4.2 is somewhat ad hoc and Lusztig's results largely apply classification results, we leave this for the future.

Lemma 4.4.2. *Let* $\lambda = (m \cdot i_m + (m-1) \cdot i_{m-1} + \cdots + 1 \cdot i_1)$. *Consider the sequence* $z_1 \leq z_2 \leq \cdots \leq z_{m'}$ *where* $m - (2i+1)$ *appears exactly* i_m *times for* $i = 0, \ldots, \lfloor m/2 \rfloor$. *Then suppose that*

$$h = \mathrm{diag}(z_1, \ldots, z_{m'}, -z_{m'}, \ldots, -z_1),$$

e is the upper-triangular matrix with nonzero entries 1 *exactly at* (i,j) *wherever* $z_i - z_j = 2$, *and f is the unique element so that* $[e,f] = h$. *Then* (e,h,f) *is a normal* $\mathfrak{sl}_2$*-triple in* $\mathfrak{g}$ *and* $e \in \mathcal{O}_\lambda$.

Proof. The involution θ is conjugate to the transpose along the antidiagonal. It is then clear $h \in \mathfrak{g}_0$ and $e, f \in \mathfrak{g}_1$. As to checking the Jordan type, we are immediately reduced to the case of a regular nilpotent, in which case this is standard. $\square$

Having the lemma at hand, we can now easily compute the algebra $\mathfrak{l}$. By definition, $\mathfrak{l}_\lambda := \bigoplus_j \{g \in \mathfrak{g}_{\underline{j}} | [h,g] = 2jg\}$

Lemma 4.4.3. *Let λ be written as above. For $e \in \mathcal{O}_\lambda$,*

$$\mathfrak{l} = \mathrm{SO}(\sum_j (2j+1) \cdot i_{2j+1}) \times \mathrm{Sp}(\sum_j 2j \cdot i_{2j})$$

Proof. Let $\Phi' = \{(\alpha, i) \in \Phi \times \mathbb{Z}/2 | \mathfrak{g}_i(\alpha) \neq 0\}$. Label the positive roots of GL_n by $(i,j), 1 \leq i < j \leq n$. Then $\mathfrak{g}_0(i,j) = E_{i,i+1} - E_{j,j+1}$ and $\mathfrak{g}_1(i,j) = E_{i,i+1} + E_{j,j+1}$, where E_{ij} is the elementary symmetric matrix. By construction, $\mathfrak{g}_i(\alpha)$ for $(\alpha, i) \in \Phi'$ are eigenspaces for the adjoint action of h. It is clear that transposing gives the corresponding negative eigenspaces. In particular, $\mathfrak{l}_k$ is the sum of $\mathfrak{g}_{\underline{k}}(ij)$ where $z_i - z_j = 2k$. Since $[\mathfrak{g}_{i_1}(\alpha_1), \mathfrak{g}_{i_2}(\alpha_2)] \subseteq \mathfrak{g}_{i_1+i_2}(\alpha_1 + \alpha_2)$, the subspaces $\mathfrak{l}_{odd} = \oplus_{i \equiv 1 \bmod 2} \mathfrak{l}_i, \mathfrak{l}_{even} = \oplus_{i \equiv 0 \bmod 2} \mathfrak{l}_i$ are Lie subalgebras. Moreover, they are of the form $\mathfrak{sp}(a)$ and $\mathfrak{so}(b)$, respectively. □

Example 4.4.4. We have

$$\mathfrak{l}_{1^5} = \mathfrak{g}_0$$

$$\mathfrak{l}_5 = \mathrm{SO}(5) = \mathfrak{g}_0$$

$$\mathfrak{l}_{311} = \mathrm{SO}(5)$$

$$\mathfrak{l}_{32} = \mathrm{SO}(3) \times \mathrm{Sp}(2)$$

Remark 4.4.5. In [92, 93], Vinberg defined carrier algebras for nilpotent orbits in θ-representations to classify them. The algebra $\mathfrak{l}$ is a canonical subalgebra of the carrier algebra. The computer program [14] is very useful in computing carrier algebras.

We view this association in terms of [60, Section 2.9]. Namely, we have the following Lemma.

Lemma 4.4.6. *To each $(\mathcal{O}, \mathcal{L})$ it is possible to associate a pseudo-Levi subgroup L and compatible $\mathbb{Z}$-grading $\mathfrak{l}_*$ of its Lie algebra by the method loc. cit. Up to conjugacy this only depends on λ, and we get an actual representative by choosing $x \in \mathcal{O}_\lambda$.*

Then $x \in \mathfrak{l}_1^\circ$, where $\circ$ denotes the open L_0-orbit. Restriction of $\mathcal{L}$ to $\mathfrak{l}_1^\circ$ gives a shifted irreducible L_0-equivariant local system on $\mathfrak{l}_1^\circ$. Its IC extension (after shifting) gives a simple L_0-equivarint perverse sheaf $\mathcal{L}_1$ on $\mathfrak{l}_1$.

By [60, Section 1.5] and [57, Section 7.5] we can associate to $\mathcal{L}_1$ a Levi subgroup M of L and a compatible grading of its Lie algebra.

4.4.1 Some combinatorics of symbols

In this subsection, we review Lusztig symbols for $\mathrm{Sp}(2n)$ and $\mathrm{SO}(N)$ following [58, Chapters 11-13]. We then use symbols and the classical Springer correspondence to prove Theorem 4.4.11.

4.4.1.1 $\mathrm{Sp}(2n)$

Suppose $G = \mathrm{Sp}(2n)$. Consider the set of all ordered pairs (A, B), where $A \subset \{0, 1, 2, \ldots\}$ and $B \subset \{1, 2, 3, \ldots\}$ are finite. The reader is advised to visualize these as abaci on two runners. Consider further the subset $\widetilde{\Psi}_{2n}$, where

1. $\{i, i+1\} \not\subset A, B$ for arbitary i,
2. $|A| + |B|$ is odd,
3. $\sum_{a \in A} a + \sum_{b \in B} b = n + \binom{|A|+|B|}{2}$.

Consider the equivalence relation $(A, B) \sim (\{0\} \cup (A+2), \{1\} \cup (B+2))$. The set of equivalence classes for this relation is denoted $\Psi_{\mathrm{Sp}(2n)}$.

Lemma 4.4.7 (Classical Springer correspondence for $\mathrm{Sp}(2n)$)**.** *There is a bijection*

$$\Psi_{\mathrm{Sp}(2n)} \to I(\mathcal{N})$$

that restricts to bijections

$$\Psi_{\mathrm{Sp}(2n)}^{(i)} \to I(\mathcal{N})^{(i)}.$$

Remark 4.4.8. Here and in the next section, it is useful to think about the similarity classes.

Proposition 4.4.9. *There is an injection $\mathcal{P}_{oe}(2n) \hookrightarrow \Psi_{\mathrm{Sp}(2n)}$ given by*

4.4.1.2 $\mathrm{SO}(2n+1)$

Suppose $G = \mathrm{SO}(2n+1)$. Consider the set of all unordered pairs $\{A, B\}$, where $A, B \subset \{0, 1, 2, \ldots\}$ are finite. Consider the subset given by the conditions

1. $\{i, i+1\} \not\subset A, B$ for arbitary i,

2. $\sum_{a\in A} a + \sum_{b\in B} b = n + \frac{(|A|+|B|)^2-2(|A|+|B|)}{2}$.

Consider the equivalence relation given by $\{A, B\} \sim \{\{0\}\cup(A+2), \{0\}\cup(B+2))$, and let the set of equivalence classes for this relation be Ψ'_{2n+1}.

Lemma 4.4.10 (Classical Springer correspondence for $\mathrm{SO}(2n+1)$)**.** *There is a bijection*

$$\Psi'_{\mathrm{SO}(2n+1)} \to I(\mathcal{N})$$

that restricts to bijections

$$\Psi^{(i)}_{\mathrm{SO}(2n+1)} \to I(\mathcal{N})^{(i)}.$$

4.4.1.3 Determination of the blocks

Theorem 4.4.11. *There is a commutative square*

$$\begin{array}{ccc} I(\mathcal{N}_1) & \xrightarrow{\quad\quad\Phi\quad\quad} & \mathfrak{B} \\ \downarrow & & \downarrow \\ \{(\lambda,\tau)\} & \xrightarrow{\Phi'} & \{(a,b)|a(a+1)+b^2 \le N\} \end{array}$$

where the vertical maps are bijections and Φ is combinatorially determined by the algorithm in Proposition 4.4.12.

Proposition 4.4.12. *The map*

$$\Phi' : \{(\lambda,\tau)\} \to \{(a,b)|a(a+1)+b^2 \le N\}$$

coincides with [60, 3.5.].

1. *Given (λ,τ), associate the algebra $\mathfrak{l}$ as in Lemma 4.4.6 to λ. Then $\mathfrak{l}$ is of the form $\mathfrak{so}(a)\times\mathfrak{so}(b)\times\mathfrak{gl}(n)$.*
2. *As in Lemma 4.4.6, associate an L_0-equivariant IC complex on $\mathfrak{l}_1$ as in the Lemma. On the level of component groups, it is given by restriction of τ to the carrier, as shown in [60].*
3. *This gives us two symbols σ_1, σ_2, one for $\mathfrak{so}(a)$ and one for $\mathfrak{sp}(b)$.*
4. *As in* loc. cit.*, we can now use the graded Springer correspondence of [57] to determine the admissible system (λ,τ) is coming from by determining the cores of σ_1, σ_2.*

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