**On Image of TQFT representations of mapping class groups**

By

Shuang Ming

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Approved:

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Greg Kuperberg (Chair)

---

Eric Babson

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Monica Vazirani

Committee in Charge

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Shuang Ming
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## Abstract

We study the image of TQFT representations of mapping class groups with boundary. Especially, we are looking for the irreducibilty and denseness of TQFT representations of mapping class groups of surfaces with boundary. We proved the representations are always irreducible for $SU(2)$-TQFT at prime level. We also show the representations are Zariski dense when $g = 0$ and quantum parameter $A$ is a transcendental number.

**Acknowledgments**

First of all, I would like to express my deepest gratitude to my advisor, Professor Greg Kuperberg, for not only leading me to the world of mathematics, but also giving me the confidence to further explore it. Besides mathematics, We also had enlightening discussions about culture, politics and history of mathematics. Without his guidance and patience, this dissertation would not have been possible.

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CHAPTER 1

# Introduction

A $(2+1)$-dimensional topological quantum field theory(TQFT) is a monoidal functor from the category of 3-dimensional cobordisms to the category of finite dimensional vector spaces. That is, The functor takes an oriented surface to a finite dimensional vector space and takes a cobordism connecting two surfaces to a linear transformation between the vector spaces that been assigned to the surfaces. Naturally, a $(2+1)$-dimensional TQFT gives us:

- A (quantum) invariant of closed 3-manifolds: A monoidal functor send tensor unit to tensor unit, so a TQFT sends the empty set to the base field $k$. A closed 3-manifold can be considered as a cobordism connecting two empty set. Thus the TQFT send it to a number in the base field $k$.
- A (projective) representation of the mapping class group of a closed surface: A TQFT assigns to each surface a vector space. Identify mapping classes as mapping cylinders. Then applying TQFT we get an action of the mapping class group on the vector space assigned to the surface.

$(2+1)$-dimensional TQFTs can be constructed from modular tensor categories. In our work, we consider the TQFT constructed by a modular category combinatorially defined using Kauffman bracket. It is also called $SU(2)$ Chern-Simons TQFT because the modular tensor category defined by Kauffman bracket have the same monoidal structure as the representation theory of quantum enveloping algebra of $\mathfrak{su}(2)$.

In general, TQFTs constructed using modular tensor categories are not anomaly-free. That is, when we compose two actions(equivalently, glue two cobordisms), a constant factor appears. Thus, by TQFT representations of mapping class groups, we actually mean projective representations of mapping class groups or linear representation of some central extensions of mapping class groups.

Kauffman bracket defines a braided tensor category. After taking the quantum parameter $A$ in the Kauffman relation be some root of unity, one can truncate the category at a certain level. This operation gives us a modular tensor category. Moreover, With some special choice of $A$. The modular tensor category is unitary. We will explain the notations of tensor categories in Chapter 2.

$SU(2)$-TQFT representations are important examples for studying mapping class groups. It is still an open problem that whether mapping class groups are linear, by linear, we mean existing a faithful finite dimensional representation. The only two known families of finite dimensional representations of mapping class groups are homology representations and TQFT representations. $SU(2)$-TQFT representations are the first interesting case of the latter case. It provide interesting examples in both geometric topology [**12**] and quantum algebra [**9**].

In this paper, we are studying the image of mapping class groups in their $SU(2)$-TQFT representations. Usually we understand this question in the following order: We start with irreducibility, so we can restrict the image into blocks. Then we check the (in)finiteness of the image. If the image is finite then we have a finite quotient (*e.g.*, Weil representation). Otherwise, we study its closure under Zariski topology or Analytic topology.

The historical results of this problem are listed below:

- When $A$ is a $4r$-th root of unity, and $r$ is an odd prime number. The representations are irreducible for closed surfaces. [**21**]
- When $A$ is a $4pq$-th root of unity, and $p$, $q$ are odd prime numbers. The representations are irreducible for closed surface of genus 2. [**13**]
- When $A$ is a $8r$-th root of unity, and $r$ is an odd prime number. The representation for the close surface of genus 2 decomposes into two irreducible sub-representations. [**13**]
- When $A$ is a $4r$-th root of unity. The representations are irreducible for all surfaces with boundary if one of the boundary component is colored by 1. [**11**]
- When $A$ is a $4r$-th root of unity, and $r$ is an odd prime number. The representation is irreducible for a one-holed torus.(The original work is on $SO(3)$-TQFT, but the proof generalize to $SU(2)$ word-by-word) [**18**]
- When $A$ is a $4r$-th root of unity, and $r \neq 2,3,4,6,10$ and $g \geq 2$. There exist an element in the mapping class group with genus greater than 1 has infinite order in the image. [**17**]
- When $A$ is a $4r$-th root of unity, and $r$ is an odd prime number. The finiteness of the image is known for one-holed torus. [**7**] [**17**]
- When $A$ is a $4r$-th root of unity, and $r \neq 2,3,4,6,10$. The braid group representations from the braid structure have analytic dense image.(The representation is also called Jones representation) [**6**] [**15**]

- When $A$ is generic. The braid group representations from the braid structure have Zariski dense image. [**15**]
- When $A$ is a $4r$-th root of unity, and $r$ is an odd prime number. The representations have dense image for closed surfaces with genus at least 2. [**16**]

Our results are on the representations of mapping class groups with boundary. To get a vector space, the boundary components of the surface should be colored by simple objects of the modular tensor category. Our result is the following:

THEOREM 1.0.1. *When A is a $4p$-th root of unity, and p is an odd prime number. The representations is irreducible for surfaces with boundary with arbitrary colors on the boundary components.*

THEOREM 1.0.2. *When A is a transcendental number, the representation have Zariski dense image for n-holed sphere with arbitrary colors on the boundary components.*

We remark that for surfaces have no genus, the mapping class groups are actually pure braid groups. To define an action, we only need braid structure on the defining monoidal category. Thus the above theorem make sense in this way.

The idea of our proof is by first looking at small surfaces. Then we glue them up to large surfaces in different ways. By comparing different decompositions, we can get information of the image of the representation for the large surface.

The first part of the thesis will serve as a exposition on modular category and 3-dimensional TQFTs. The second part will be our work on this image problem, together with some possible problems for the future work and relations to other questions in the field. The detail structure of the thesis is listed below:

- In Chapter 2, we begin with the definition of modular tensor categories(MTC). Important computations and invariants of MTC will be introduced. We will also introduce the modular tensor category defined by Kauffman's bracket.
- In Chapter 3, we revisit the construction of the Reshetikhin-Turaev TQFT.
- In Chapter 4, we give a short introduction to mapping class groups. Then we will give a combinatorial description of the mapping classes on the $SU(2)$-representation.
- In Chapter 5, we first introduce different versions of cross-lamination lemma. Then we prove 1.0.1 using induction on both genus and number of boundary components.

- In Chapter 6, we study how irreducibility together with block-diagonal surjectivity implies surjectivity. then we proof 1.0.2 by induction on number of boundary components.
- In Chapter 7, we discuss some future works and connections to some other works in the field.

CHAPTER 2

# Modular Tensor Categories

A modular tensor category is a structure that generalizes the representation theory of quantum groups. The goal of this chapter is reviewing the formal definition of a modular tensor category, together with the computation of some important invariants of modular tensor categories. We end this chapter by introducing the modular tensor category from the Kauffman bracket, which will be used to construct the $SU(2)$-TQFT in later chapters. For a detailed treatment of materials in this chapter, we refer readers to Chapter 2 and 12 of Turaev's book [**23**] and Chapter 2 and 8 of [**4**].

## 2.1. Definition of a Modular Tensor Category

The goal of this section is to give the formal definition of a modular tensor category(MTC), from a category theoretical point of view. Roughly speaking, a modular tensor category is a semi-simple abelian category with an extra tensor product operation and with other structures. The tensor product can be realized as a categorical generalization of the tensor product in the category of vector spaces. A category with tensor product is called a monoidal category(in some of the literature, such a category is called a tensor category). We start this section with the formal definition of monoidal category.

DEFINITION 2.1.1. *A monoidal category is a category $\mathscr{C}$ with*

(1) *A bifunctor $\otimes$ called tensor product bifunctor, $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$*
(2) *A natural isomorphism a called the associator, $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$*
(3) *A object $\mathbf{1}$ called unit, with an isomorphism $\iota : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$*

*and subject to the following two axioms.*

(1) *The pentagon axiom: The diagram*

$$
\begin{array}{ccccc}
 & & ((W\otimes X)\otimes Y)\otimes Z & & \\
 & \swarrow{\scriptstyle a_{W,X,Y}\otimes \mathrm{id}_Z} & & \searrow{\scriptstyle a_{W\otimes X,Y,Z}} & \\
W\otimes(X\otimes Y)\otimes Z & & & & (W\otimes X)\otimes(Y\otimes Z) \\
\downarrow{\scriptstyle a_{W,X\otimes Y,Z}} & & & & \downarrow{\scriptstyle a_{W,X,Y\otimes Z}} \\
W\otimes((X\otimes Y)\otimes Z) & & \xrightarrow{\mathrm{id}_W\otimes a_{X,Y,Z}} & & W\otimes(X\otimes(Y\otimes Z))
\end{array}
$$

*is commutative for all objects $W,X,Y,Z$ in $\mathscr{C}$.*

(2) *The unit axiom: The functors $L_1 : X \to 1\otimes X$ and $R_1 : X \to X\otimes 1$ of left and right multiplication by $\mathbf{1}$ are autoequivalences of $\mathscr{C}$.*

The definition of monoidal category is a generalization of the category of vector spaces. The next definition generalize the dual space of a vector space.

DEFINITION 2.1.2. *An object $X^*$ in $\mathscr{C}$ is said to be a left dual of $X$ if there exist morphism $\mathbf{ev}_X : X^*\otimes X \to \mathbf{1}$ and $\mathbf{coev}_X : 1 \to X\otimes X^*$, called the evaluation and coevaluation, such that the compositions*

$$X \xrightarrow{\mathbf{coev}_X\otimes \mathrm{id}_X} (X\otimes X^*)\otimes X \xrightarrow{a_{X,X^*,X}} X\otimes(X^*\otimes X) \xrightarrow{\mathrm{id}_X\otimes \mathbf{ev}_X} X,$$

$$X^* \xrightarrow{\mathrm{id}_X\otimes \mathbf{coev}_X} X^*\otimes(X\otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^*\otimes X)\otimes X^* \xrightarrow{\mathbf{ev}_X\otimes \mathrm{id}_{X^*}} X^*$$

*are identity morphisms.*

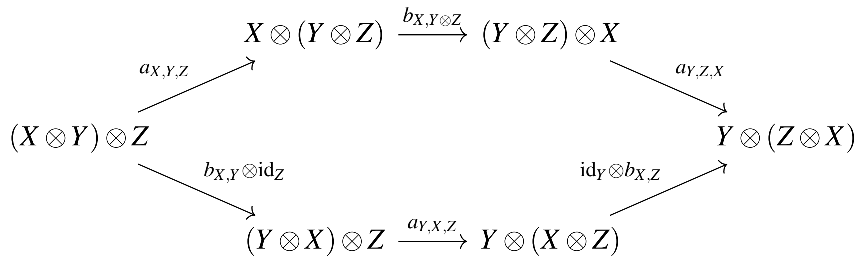
A modular tensor category is a fusion category with modular data. We start with formal definition of a fusion category.

DEFINITION 2.1.3. *Let $\mathscr{C}$ be a finite $\mathbf{k}$-linear abelian rigid semisimple monoidal category. We call $\mathscr{C}$ a fusion category if*

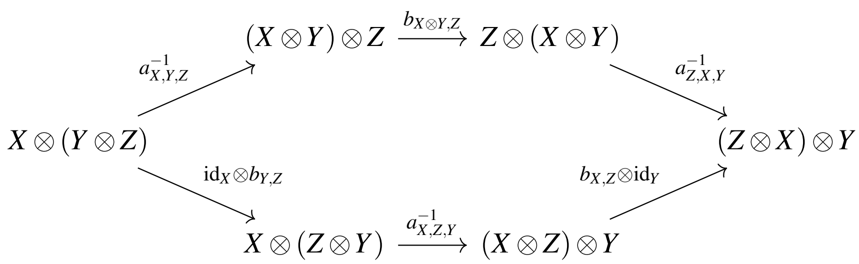
- *bifunctor $\otimes$ is bilinear on morphisms,*
- $\mathrm{End}(\mathbf{1},\mathbf{1}) \cong \mathbf{k}$.

Examples of fusion categories include category of vector spaces and representation theories of finite dimensional Hopf algebras. To define modular data, we need the category to be braided and ribbon. By word braided and ribbon, we denote natural isomorphisms that represent 'crossing' and 'full twist' respectively.

DEFINITION 2.1.4 (Braiding). *A braiding on a monoidal category $\mathscr{C}$ is a natural isomorphism $b_{X,Y}$ : $X \otimes Y \to Y \otimes X$ such that the hexagonal diagrams*




*and*




*commute for all $X, Y$ and $Z$ in $\mathscr{C}$. A braided monoidal category is a pair consisting of a monoidal category and a braiding.*

DEFINITION 2.1.5 (Ribbon). *A twist on a braided rigid monoidal category $\mathscr{C}$ is $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathscr{C}})$ such that*

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ b_{X,Y} \circ b_{Y,X}$$

*for all $X$ and $Y$ in $\mathscr{C}$. A ribbon tensor category is a braided rigid monoidal category equipped with a twist satisfying $(\theta_X)^* = \theta_{X^*}$.*

Before we give the definition of a modular tensor category. We discuss the notion *trace* in rigid monoidal categories. In the context of the category of finite dimensional $k$-vector spaces. we can define trace as a linear map from $\mathrm{Hom}(V,V) \to k \cong \mathrm{Hom}(\mathbf{1},\mathbf{1})$. But in an arbitrary rigid monoidal category, there is no such a canonical map from $\mathrm{Hom}(V,V)$ to $\mathrm{Hom}(\mathbf{1},\mathbf{1})$. The most natural analoge of trace is the map left quantum trace $\mathrm{Tr}^L : \mathrm{Hom}(V,V^{**}) \to \mathrm{Hom}(\mathbf{1},\mathbf{1})$ and right quantum trace $\mathrm{Tr}^R : \mathrm{Hom}(V,{}^{**}V) \to \mathrm{Hom}(\mathbf{1},\mathbf{1})$ defined as following,

$$\mathrm{Tr}^L(a) : \mathbf{1} \xrightarrow{coev_v} V \otimes V^* \xrightarrow{a\otimes \mathrm{id}_{V^*}} V^{**} \otimes V^* \xrightarrow{ev_{V^*}} \mathbf{1}$$

$$\mathrm{Tr}^R(a) : \mathbf{1} \xrightarrow{coev_{*V}} {}^*V \otimes V \xrightarrow{\mathrm{id}_{*V} \otimes a} {}^*V \otimes {}^{**}V \xrightarrow{ev_{**V}} \mathbf{1}$$

In the ribbon category, one can construct a natural isomorphism $\phi_V : V \to V^{**}$ such that for any $a \in \mathrm{Hom}(V,V)$ one has

$$\mathrm{Tr}^L(\phi_V a) = \mathrm{Tr}^R(a \phi_V^{-1}) \tag{2.1}$$

In some literature, the natural isomorphism $\psi_V$ are called a spherical structure on category $\mathscr{C}$. For a ribbon category, we can define a trace map from $\mathrm{Hom}(V,V)$ to $\mathrm{Hom}(\mathbf{1},\mathbf{1}) \cong k$.

The construction of $\psi$ is given as the following composition. For proof of Equation 2.1 we refer readers to Section 8.10 of [**4**].

$$\psi_V := V \xrightarrow{\theta_V} V \xrightarrow{\mathrm{id}_V \otimes coev_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{b_{V,V^*} \otimes \mathrm{id}_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{ev_V \otimes \mathrm{id}_{V^{**}}} V^{**} \tag{2.2}$$

Let $\mathscr{C}$ be a ribbon category. We define the $S$-matrix of $\mathscr{C}$ to be $S := (s_{XY})_{X,Y \in Irr(\mathscr{C})}$ where $s_{XY} = \mathrm{Tr}(b_{Y,X} \circ b_{X,Y})$.

DEFINITION 2.1.6. *A pre-modular tensor category is a ribbon fusion category. It is said to be modular if its S-matrix is non-degenerate.*

We end this section by giving the statement of the Mac Lane strictness theorem for monoidal categories. One important corollary of the theorem is that one can use tensor networks to present tensors and compositions of morphisms of a monoidal category.

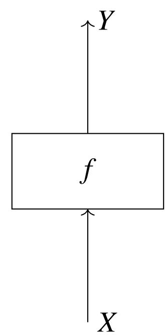
THEOREM 2.1.7. *(Mac Lane strictness theorem)We call a monoical category strict if the natural transformation a, $L_1$ and $R_1$ defined in Definition 2.1.1 are identities. A monoidal category is monoidally equivalent to a strict monoidal category.*

### 2.2. Modular Invariant and Computations

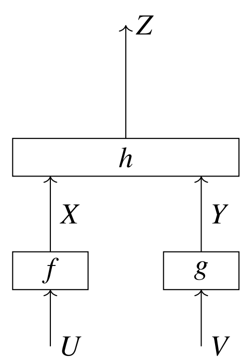
In this section, we will first introduce the graphical calculus for a modular tensor category. Then we define some invariant of modular tensor cateogries which will be used to define topological invariant and tqft in later sections.

**2.2.1. Graphical calculus.** One of the most important corollary of the Mac Lane strictness theorem is that we can remove parentheses and identity objects in a product of objects of a monoidal category by assuming the category is strict. With this property, we can present morphisms by tensor networks.

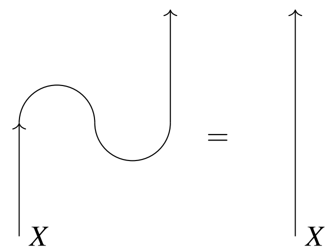
The tensor network should be read upward, and from left to right. Objects are represented by colored strands and morphisms are represented by their name in boxes. For example, a morphism $f : X \to Y$ is represented by




We represent composition of morphisms by stacking a new tensor network on the top. Tensoring a new object by putting a colored strand on the right. For example, the composition $U \otimes V \xrightarrow{f\otimes g} X \otimes Y \xrightarrow{h} Z$ is represented by

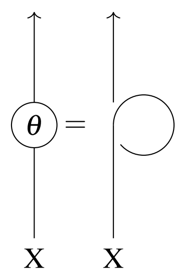



If the category is rigid, we use a upward or downward arrow to define $X$ and dual of $X$, and use U-turn to define evaluations and co-evaluations. For example, the identity $X \xrightarrow{\mathrm{id}\otimes \mathbf{coev}} X \otimes {}^*X \otimes X \xrightarrow{\mathbf{ev}\otimes \mathrm{id}} X = X \xrightarrow{\mathrm{id}} X$ is represented by




If the category is braided, we present braiding in its most natural way, i.e. by overcrossing and undercrossing. We call category with $b$ braided because the morphism represented by the tensor network with crossings is up to isotopy, Reidemeister move 2 and Reidemeister move 3. This is straightforward by the hexagon axiom, the definition of functor $b$ and the functoriality of $b$ respectively.

The twist $\theta_X$ is represented by the following graph



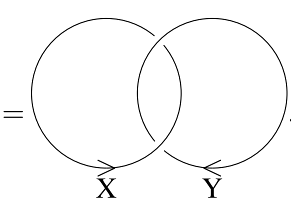

For our purpose, $\mathbf{k}$ will be a algebraic closed field with characteristic 0. In this case, when $X$ is irreducible, $\mathrm{Hom}(X,X)$ is 1-dimensional. We will also denote the scalar by $\theta_X$.

Notice that from the definition of ribbon category. Given an object $X$ in a ribbon category, we can get:

(1) A framed link invariant: We can realize the framed knot as a directed ribbon colored by $X$, which can be realized as an element in $\mathrm{Hom}(\mathbf{1},\mathbf{1}) = \mathbf{k}$.

(2) A projective representation of braid group $B_k$: The $B_k$ acting on $\mathrm{Hom}(X^{\otimes k}, X^{\otimes k})$ by stacking the morphism on the bottom. The hexagon axiom and functoriality of $b$ ensure that the representation is well defined.

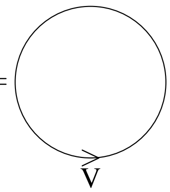
We close this subsection by a graph representing the entries of $S$-matrix:

$s_{XY} =$  .

**2.2.2. Modular Invariants.** In this subsection, we define some modular invariants for modular tensor cateogories. These invariants are used to define the Reshetikhin-Turaev invariant for 3-manifold in the next chapter.

In a modular tensor category $\mathscr{C}$, we define quantum dimension of an object $V$ to be the trace of the identity map.

$$\dim(V) = \mathrm{Tr}(\mathrm{id}_V) :=$$




Notice that in a modular tensor category, $\dim V = \dim V^*$. The total dimension of a modular tensor category is defined to be

$$D = \sqrt{\sum_{i \in \mathrm{Irr}(\mathscr{C})} \dim^2 V_i}$$

In the rest of the section, we will adopt the convention that uncolored strand are considered as summation over all irreducibles in the modular tensor category with the weight of $V_i$ be $d_i = \dim V_i$. Sometimes, we also call it $\Omega$ color.

We define

$$p^{\pm} := \sum_{i \in I} \theta_i^{\pm} d_i^2$$

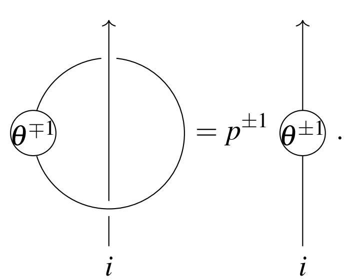
and $T$-matrix to be

$$T_{ij} = \delta_{ij} \theta_i.$$

By computation, we have the following lemma which is crucial in later sections.

LEMMA 2.2.1. *The following identities hold:*

(2.3)



## 2.3. MTC from the Kauffman Bracket

In this section, we will introduce the modular tensor category generated by the Kauffman bracket. We first introduce a skein theoretic spherical monoidal category. Then we make the category abelian by taking the Karoubi envelope. Then we define a braid structure that is compatible with the spherical structure of the category by Kauffman bracket. In the Kauffman bracket, there is an coefficient $A$. We show that for generic $A$, the category is semisimple. If $A$ is a $4r$-th root of unity, the category have a modular quotient.

**2.3.1. Quantum Integers.** In this section and the rest of the paper. We will adopt the notation of quantum integer and quantum factorial. They are defined as following:

$$[n] = \frac{A^n - A^{-n}}{A - A^{-1}}$$

$$[n]! = [n][n-1]...[1].$$

We remark that in the most literature of quantum algebra, people use letter $q$ as the quantum parameter instead of $A$ we used here. The relation between $q$ and $A$ are $q = A^2$. The only advantage of using $A$ here is that one can avoid fractional exponents.

### 2.3.2. The Temperley-Lieb Category.

DEFINITION 2.3.1. *The Temperley-Lieb category is a strict monoidal $\mathbb{C}$-category. The objects are natural numbers and $n \in \mathbb{N}$ is represented by $n$ dots on the real line. The morphism space $Hom(m,n)$ are $\mathbb{C}$-span of planar tangle diagrams in $\mathbb{R} \times [0,1]$ with $n$ boundary components on $\mathbb{R} \times \{1\}$ and $m$ boundary components on $\mathbb{R} \times \{0\}$. The composition of morphisms are represented by putting one diagram on top of the other and evaluate closed circle by a factor of $-A^2 - A^{-2}$. Tensoring with an object(resp. morphisms) is represented by putting dots(resp. diagram) on the right side.*

*We denote the Temperley-Lieb category with coefficient A by $\mathscr{C}_A$, and the tensor generating object by $X$.*

The tensor generating object $X$ in the Temperley-Lieb category is self dual. The coevaluation morphism and the evaluation morphism are given by the cup diagram and the cap diagram. The dual of a morphism is defined to be the flip of the diagram. By definition, coevaluation is cosymmetric and evaluation is symmetric. The category $\mathscr{C}_A$ is not abelian because the kernel and cokernel of some morphisms do not exist. For instance, $X \otimes X$ is not isomorphic to $\mathbf{1}$. but the kernel of the evaluation morphism does not exist.

The Temperley-Lieb category can be made abelian without adding new morphisms. The correct construction is called Karoubi envelope.

**2.3.3. Karoubi Envelope.** In a category $\mathscr{C}$, a idempotent of $\mathscr{C}$ is an endomorphism $e : A \to A$ with $e \circ e = e$. The Karoubi envelope of category $\mathscr{C}$ is the category of idempotents in $\mathscr{C}$. The formal definition of the Karoubi envelope of a preadditive category is given below.

DEFINITION 2.3.2. *Let $\mathscr{C}$ be a preadditive category, its Karoubi envelope $KAR(\mathscr{C})$ is defined as follows:*

- *Objects are idempotent morphisms $e : X \to X$ in $\mathscr{C}$;*
- *Morphisms between $(X, e_1)$ and $(Y, e_2)$ are morphisms $f \in Mor(X,Y)$ such that $f = e_2 \circ f \circ e_1$. The composition of morphisms in $KAR(\mathscr{C})$ is identical as the composition of morphisms in $\mathscr{C}$.*

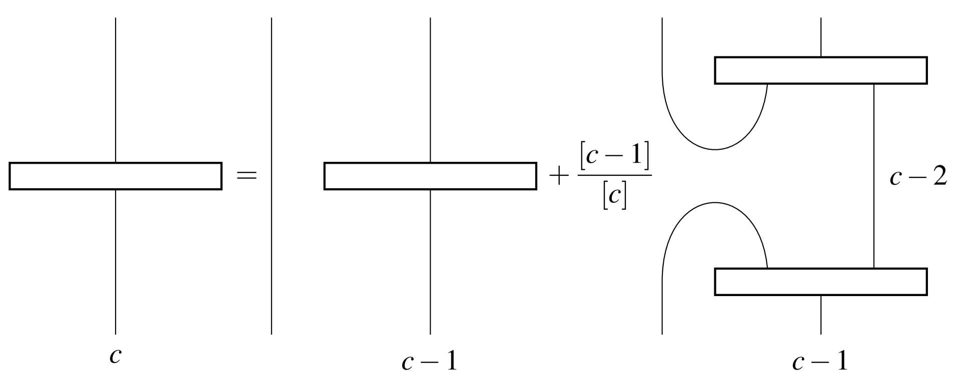
In general, the Karoubi envelope of a preadditive category is pseudo-abelian(i.e every idempotent morphism admit a kernel) which is weaker than an abelian category. But in the case of Temperley-Lieb category, the Karoubi envelope of $\mathscr{C}_A$ is an abelian category after the additive completion.

DEFINITION 2.3.3. *We denote the additive completion of a category $\mathscr{C}$ by $Mat(\mathscr{C})$. The objects of $Mat(\mathscr{C})$ are direct sum of objects in $\mathscr{C}$. The morphisms are matrices with entries morphisms in $\mathscr{C}$.*

To state the theorem of the abelianess of $Mat(Kar(\mathscr{C}_A))$, we need to first introduce a set of projectors which are called Jones-Wenzl's projectors.

DEFINITION 2.3.4. *The c-th Jones-Wenzl's projector is an endomorphism of $X^{\otimes c}$ defined recursively as follows.*

(2.4)




The properties of the Jones-Wenzl projectors are summarized in the following proposition.

PROPOSITION 2.3.5. *The c-th Jones-Wenzl projector $p_c$ satisfies the following properties:*

(1) $\mathrm{id}_{X^{\otimes c}}$ *has coefficient* 1 *in the expansion of* $p_c$.

(2) *Evaluation and coevaluation of adjacent strand annihilate* $p_c$.

(3) $p_c^2 = p_c$.

The proof of the above proposition is not trivial. We refer readers to Wenzl's original paper [**24**] for the computation.

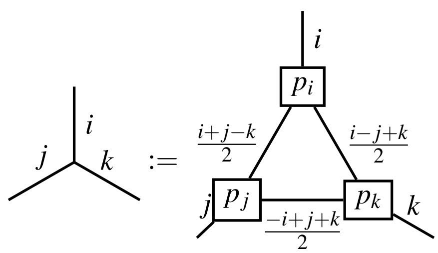
The following theorem says that the additive completion of the Karoubi envelope of $\mathscr{C}_A$ is semisimple.

THEOREM 2.3.6. *Let A be generic or a transcendental complex number. $Mat(Kar(\mathscr{C}_A))$ is a semisimple. Moreover, Jones-Wenzl projectors gives a complete set of isomorphism classes.*

The Proposition 2.3.5 implies

(1) The Jones-Wenzl projectors are simple objects in $Mat(Kar(\mathscr{C}_A))$.

(2) $\dim(\mathrm{Hom}(p_i \otimes p_j, p_k)) \leq 1$ with equality hold if and only if $|i-j| \leq k \leq i+j$ and $i+j+k$ is an even number. Moreover, $\mathrm{Hom}(p_i \otimes p_j, p_k)$ is generated by the following diagram:




The above theorem is proved by expanding $\mathrm{id}_{p_i \otimes p_j}$ in terms of the above morphisms. The computation in [**2**] shows that when $A$ is generic or transcendental, the coefficients are nonzero for all admissible $k$, which implies:

$$p_i \otimes p_j = p_{|i-j|} \otimes p_{|i-j|+2} \otimes \cdots \otimes p_{|i+j|}.$$

However, When $A$ is a $4r$-th root of unity, the nonzero coefficients property fails. To get a semisimple category, we need to kill the ideal of negligible objects and morphisms.

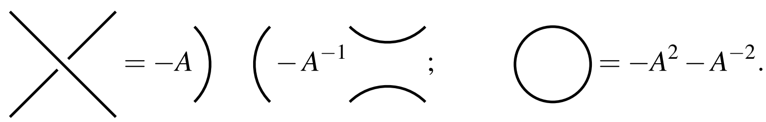
DEFINITION 2.3.7. *Let $\mathscr{C}$ be a spherical monoidal category. A morphism $f$ in $\mathscr{C}$ is called negligible if* $\mathrm{Tr}(g \circ f) = 0$ *for all g. An object X is called negligible if* $\mathrm{id}_X$ *is negligible.*

It is easy to check that negligible objects is a monoidal ideal. After taking the quotient of the negligible ideal. We denote $\overline{\mathscr{C}_A}$ be the quotient $Mat(Kar(\mathscr{C}_A))/\{negligibles\}$.

If $A$ is generic or transcendental, the negligible ideal is trivial, and the following theorem summarizes the property of $\overline{\mathscr{C}_A}$ when $A$ is a $4r$-th root of unity.

THEOREM 2.3.8. *Let $A$ be a $4r$-th root of unity. The quotient category $\overline{\mathscr{C}_A}$ is semisimple. Moreover, $\{p_0, p_1, \ldots, p_{r-2}\}$ gives a complete set of isomorphism classes.*

**2.3.4. Kauffman bracket and modular structure.** The category $\overline{\mathscr{C}_A}$ admit a nontrivial braiding structure, given by the Kauffman bracket below:




It is easy to check that the braid structure generated by the Kauffman's bracket satisfies the hexagon relation and the full twist is compatible with the spherical structure. Moreover, if $A$ is a $4r$-th root of unity, the category $\overline{\mathscr{C}_A}$ is modular.

THEOREM 2.3.9. *[23] Let $r \geq 3$ be an integer, and $A$ be a primitive $4r$-th root of unity. Then the pair $(\overline{\mathscr{C}_A}, \{V_i\}_{0 \leq i \leq n-2})$ is a modular tensor category with $S$ matrix*

$$S_{ij} = (-1)^{i+j}[(i+1)(j+1)].$$

**2.3.5. Hermitian structure of the category.** When $A$ be a $4r$-th root of unity, The category $(\overline{\mathscr{C}_A}$ admit a Hermitian structure, that is, its morphism spaces are Hermitian vector spaces. In this subsection, we construct the Hermitian structure and construct sets of orthogonal basis of the Hermitian vector spaces using trivalent graphs.

Before we define the Hermitian space, we define the conjugation of a diagram $D \in \mathbb{R} \times [0.1]$ be its mirror reflection respect to line $\mathbb{R} \times 1/2$. We denote the conjugation of $D$ by $-D$. The conjugation extend to a conjugation in the morphism space $\mathrm{Hom}(v, w)$ in $\overline{\mathscr{C}_A}$ as follows:

$$\overline{\sum_{i=1}^{n} k_i D_i} = \sum_{i=1}^{n} \overline{k_i}(-D_i).$$

One can check the conjugation defined above is compatible with the crossings and the full twists.

The Hermitian product of morphisms $f, g \in \mathrm{Hom}(V, W)$ is defined as follows:

$$\langle f, g \rangle = \mathrm{Tr}(f \circ \overline{g}).$$

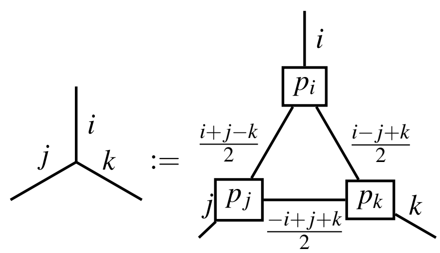
It is easy to check that $\langle \cdot, \cdot \rangle$ is sesquilinear.

In the rest of the section, we consider the Hermitian vector spaces of form

$$\mathrm{Hom}(X_{c_1} \otimes X_{c_2} \otimes ... X_{c_n}, \mathbf{1})$$

, where $X_\ell$ is isomorphic to the $\ell$-th Jones-Wenzl's projector.

We use trivalent graphs to present vector in the morphism space. Locally, a trivalent vertex is equivalent to the following diagram as earlier sections:




We have numerical restrictions for the adjacent projectors $(i, j, k)$ because of the following lemma about the fusion rule of $\overline{\mathscr{C}_A}$.

LEMMA 2.3.10. *Assume A be a* $4n$*-th root of unity, set n be* $\infty$ *if A is transcendental. Let* $X_i$ *be the object represented by i-th Jones-Wenzl projector. We have*

$$Hom(X_i \otimes X_j \otimes X_k, \mathbf{1}) \leq 1$$

*with equality hold if* $i + j + k \equiv 0 \mod 2$ *with following numerical conditions:*

i) $|i - j| \leq k \leq i + j$,

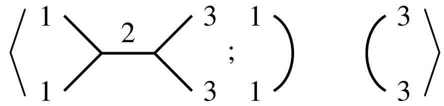
ii) $i + j + k \leq 2n - 4$.

*We call a triple* $(i, j, k)$ *admissible if the above conditions are satisfied.*

Any morphism in $\overline{\mathscr{C}_A}$ is tensor generated by such 'trivalent' morphisms. Based on this fact, we define a set of basis of the morphism space of $\overline{\mathscr{C}_A}$ as following.

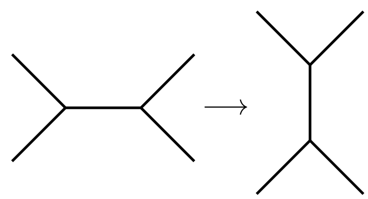
Let $n \geq 3$ and $\vec{c} = (c_1, c_2, \dots, c_n)$ be a integer valued vector with $0 \leq c_i \leq n - 2$. Define $V_{0,n;\vec{c}}$ be the vector space $\mathrm{Hom}(X_{c_1} \otimes X_{c_2} \otimes .. \otimes X_{c_n}, \mathbf{1})$. Let $\Gamma$ be a uni-trivalent tree with $n$ vertex with degree one. The edges adjacent to the one-edged vertices are colored by $c_1, c_2, \dots, c_n$ respectively. An admissible coloring of the graph $\Gamma$ gives a vector in $V_{0,n;\vec{c}}$. The set of admissible colorings is a set of basis of $V_{0,n;\vec{c}}$.

For example,

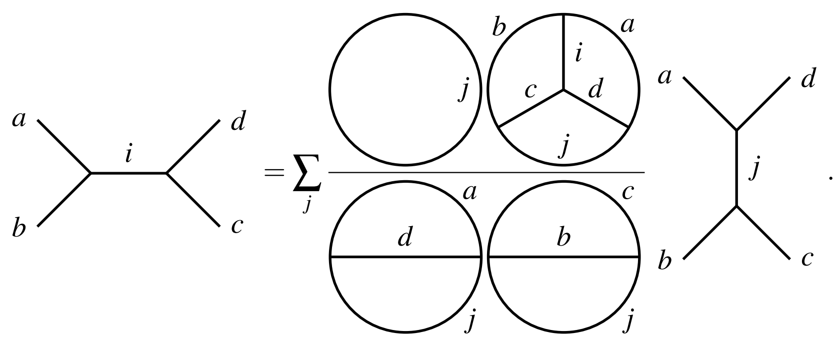
$V_{0,4;1,1,3,3} =$ 


PROPOSITION 2.3.11. *For a given uni-trivalent* $\Gamma$ *with n one-edged vertices with their adjacent edges colored by* $\vec{c}$*, the set of admissible colorings is a set of orthogonal basis of* $V_{n,\vec{c}}$.

For each $\Gamma$, we get a set of orthogonal basis for the morphism space. The change of basis formula can be derived as the following. For any pair of tree $\Gamma$ and $\Gamma'$ associated to $V_{n;\vec{c}}$, we can get $\Gamma'$ from $\Gamma$ by a sequence of the following move:



Locally, the change of basis formula is given by




Notice that the closed graphs are scalars.

In [**3**], theorem 4.11. they checked that when $A = e^{\frac{i\pi}{2r}}$, The Hermitian form is either positive definite or negative definite, depending on the projectors on the boundary.

**2.3.6. Relation with representation theory of quantum $\mathfrak{sl}(2)$.** The representation category of $U_q(\mathfrak{sl}(2))$ and $\overline{\mathscr{C}_A}$ have the same fusion rule when taking $q = A^2$. This two categories are both modular but they are not equivalent as modular categories. One can easily see this by looking at the quantum dimension(trace of the identity morphism) of the generating object. In $\overline{\mathscr{C}_A}$, the quantum dimension of the generating object is $-(A^2 + A^{-2})$, but in $\text{Rep}(U_q(\mathfrak{sl}(2)))$, the quantum dimension of the generation object is $q + q^{-1}$.

In $\text{Rep}(U_q(\mathfrak{sl}(2)))$, the generating object $X$ is self dual, that is, there exist an isomorphism $s : X \to X^*$. However, this morphism is not canonical in the sense that it can not extend to a autoequivalence of the modular category. One can get $\overline{\mathscr{C}_A}$ by negating the evaluations and coevaluations of the odd dimensional irreducible representations, by doing so, the quantum dimension of the odd dimensional irreducible representations is also negated. The sign trick make it possible to remove the arrows in the diagrams for certain modular tensor categories.

CHAPTER 3

# Three Dimensional Topological Quantum Field Theory

## 3.1. Reshetikhin-Turaev invariant

In this section, we revisit the Reshetikhin-Turaev invariant for closed 3-manifolds. Details of this part of the material can be found in the Reshetikhin and Turaev's original paper [**19**] or Turaev's book [**23**]. For our purpose, we only concern TQFTs over the field of complex numbers $\mathbb{C}$.

**3.1.1. Link surgeries in the 3-sphere.** Let $L$ be a framed link in $S^3$ with components $K_1, K_2, \dots K_m$ and $B$ be a 4-ball bounded by the $S^3$. For each of the link component $K_i$, we take $U_i$ as its neighborhood and glue a 2-handle $B^2 \times B^2$ along its boundary. The glueing map is taken to be:

$$U_i = S^1 \times B^2 = \partial B^2 \times B^2.$$

Let the resulting 4-manifold be $W_L$ and its boundary be $M_L$. A theorem by Thom shows that each closed oriented 3-manifold can be realized as the boundary of a 4-manifold. Thus every 3-manifold can be constructed by the surgery along some link $L$ in a 3-sphere.

**3.1.2. Invariant of closed 3-manifolds.** We can always realize a link in 3-sphere as a link in $\mathbb{R}^3$. Given a modular tensor category $\mathscr{C}$ describe in the last section. We can evaluate a colored link in $\mathbb{R}^3$. Thus, we want to make this link invariant a 3-manifold invariant.

To define the invariant of the link, we define a new color called $\Omega$ color for the tensor category.

$$\Omega = \Sigma_{V_i \in I} (\dim V_i) V_i$$

where $I$ is the set of isomorphism classes of irreducible objects.

THEOREM 3.1.1. *$\mathscr{D}^{-m-1}\langle L_\Omega \rangle$ is a topological invariant of pairs $(W, \partial W = M)$, where $(W, M)$ is the result of the link surgery of $L$ in $S^3$. Moreover, $\mathscr{D}^{-m-1}\langle L_\Omega \rangle$ is an invariant of framed 3-manifold.*

Every 3-manifold have a canonical framing, so the theorem above actually gives a 3-manifold invariant by choosing the canonical framing.

COROLLARY 3.1.2. *Let $\sigma(L)$ be the signature of the 4-manifold constructed above.*

$$RT(M) = (\frac{p_+}{p_-})^{\sigma(L)} \mathscr{D}^{-m-1} \langle L_\Omega \rangle$$

*is a 3-manifold invariant.*

Notice that we realize a 3-manifold as an $\Omega$-colored link diagram in $S^3$. Thus it is natural to extend the definition of Reshetikhin-Turaev invariant to oriented 3-manifolds with colored links inside. We end the section with the definition of extended Reshetikhin-Turaev invariant.

DEFINITION 3.1.3. *The extended RT-invariant of a colored link $K$ in a 3-manifold $M_L$ is defined to be*

$$RT(M_L, K) = (\frac{p_+}{p_-})^{\sigma(L)} \mathscr{D}^{-m-1} \langle L_\Omega \cup K \rangle$$

### 3.2. Universal Construction of TQFT

In this section, we will give the definition of a TQFT and describe the universal construction. By universal construction, one can reconstruct the topological quantum field theory from a multiplicative invariant of closed 3-manifolds if the invariant comes from a non-degenerate topological quantum field theory.

An $(n+1)$d-topological quantum field theory(TQFT) is a symmetric tensor functor from the category of $(n+1)$-dimensional cobordisms to the category of finite dimensional vector spaces. We fix the notation of our cobordism category before giving the formal definition of $(n+1)$-d TQFT.

DEFINITION 3.2.1. *We denote $Cob_{(n+1)}$ to be the category of $n+1$-dimensional cobordisms with:*

- *Objects: Oriented n-dimensional closed manifold $\Sigma$.*
- *Morphisms: Oriented $n+1$-dimensional manifolds with orientations on their boundary components(i.e. a choice of a trivialization of their normal bundle). A morphism from $\Sigma_1$ to $\Sigma_2$ is denoted by $(M, \partial_-, f_- : \partial_- \xrightarrow{\sim} \Sigma_1, \partial_+, f_+ : \partial_+ \xrightarrow{\sim} \Sigma_2)$.*

DEFINITION 3.2.2. *A $(n+1)$-d topological quantum field theory $Z$ contains the following data:*

- *Assigning each n-dimensional oriented closed manifold $\Sigma$ a finite dimensional vector space $Z(\Sigma)$.*
- *Assigning each $n+1$-dimensional cobordism $(M, \partial_-, f_- : \partial_- \xrightarrow{\sim} \Sigma_1, \partial_+, f_+ : \partial_+ \xrightarrow{\sim} \Sigma_2)$ a linear transformation $Z(M)$ from $Z(\Sigma_1)$ to $Z(\Sigma_2)$.*

*Together with the following Axioms:*

- $Z(\emptyset) = k$, $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.
- $Z(\Sigma^{rev}) = Z(\Sigma)^*$
- $Z(\Sigma \times [0,1], \Sigma, \mathrm{id}, \Sigma, \mathrm{id}) = \mathrm{id}_{Z(\Sigma)}$
- $Z(M_1 \sqcup_f M_2) = Z(M_1) \circ Z(M_2)$.

Notice that if we have an $(n+1)$-d topological quantum field theory, we get a topological invariant for closed $(n+1)$-manifolds instantly because an $(n+1)$-dimensional manifold can be realized as a cobordism connecting two emptyset. Thus, the TQFT functor $Z$ send the manifold to an element of $\mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$. In the rest of this section, we answer the following question: If we have a topological invariant of $(n+1)$-manifolds, when does this invariant come from a $(n+1)$-dimensional TQFT? If so, can we recover the TQFT from the invariant?

To construct an $(n+1)$-dimensional TQFT from a multiplicative invariant $\tau$, we need to associate a vector space to each $n$-manifold $\Sigma$. The idea of the construction is by first taking the vector space spanned by all $(n+1)$-manifolds with $\Sigma$ as their boundary. We denote this large vector space by $\overline{\tau}(\Sigma)$. The invariant gives us a bilinear form:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \overline{\tau}(\Sigma) \times \overline{\tau}(\overline{\Sigma}) &\to \mathbb{C} \\ (H_1, H_2) &\to \tau(H_1 \cup_\Sigma H_2) \end{aligned}$$

Let $\tau(\Sigma)$ denotes the reduced vector space by taking the quotient of the kernel of the bilinear form. The construction above does not always derive a TQFT, even the topological invariant comes from a TQFT. Because the resulting vector space may not be the original one. We noticed that if the resulting functor $\tau$ is a TQFT, then $\tau$ must be cobordism generated.($\tau(\Sigma)$ is spanned by cobordisms with $\Sigma$ as boundary.)

The following uniqueness theorem answers the question that when we can recover the TQFT from the invariant.

THEOREM 3.2.3. *(Turaev) If $Z_1, Z_2$ are $(n+1)$-TQFTs which coincide as invariants on $(n+1)$-dimensional closed manifolds and $Z_1$ is cobordism generated, then $Z_1$ and $Z_2$ are equivalent.*

From the construction, The following proposition is straight forward:

PROPOSITION 3.2.4. *[2] Let $Z$ be a non-degenerate n-TQFT and suppose that for each $(n+1)$-dimensional cobordism it holds $\overline{Z(M)} = Z(\overline{M})$. Then for each $\Sigma$ there is a $\mathbb{C}$-antilinear isomorphism $i : Z(\Sigma) \to Z^*(\Sigma)$ defined by extending $\mathbb{C}$-antilinearly the map $\bar{\cdot} : Mor(\emptyset, \Sigma) \to Mor(\overline{\Sigma}, \emptyset)$ defined by $[M] \to [\overline{M}]$. This equips $Z(\Sigma)$ with a* $\mathrm{MCG}(\Sigma)$*-invariant Hermitian form $\langle \cdot, \cdot \rangle$.*

### 3.3. TQFT from Kauffman Bracket

In this section, we apply the universal construction to the Reshetikhin-Turaev invariant defined over the MTC from Kauffman bracket(See section 2.3). We will not get a TQFT instantly because the vector space associated to a torus is infinite dimensional [**8**]. The solution to this problem is by taking a central extension to the cobordism category. The universal construction gives a TQFT after the central extension.

We define the extended 3-dimensional cobordism category $\widetilde{Cob_3}$ as the following

DEFINITION 3.3.1. *Denote $\widetilde{Cob_3}$ be the category of 3-dimensional extended cobordisms. The objects are oriented compact surfaces $\Sigma$ equipped with a lagrangian subspace $L$. Morphisms are 3-dimensional cobordisms with an integer. The composition of two cobordisms*

$$(M, f_- : \partial_M \to \Sigma_1, f_+ : \partial_+ M \to \Sigma_0, m) \in Mor((\Sigma_1, L_1), (\Sigma_0, L_0))$$

*and*

$$(N, g_- : \partial_N \to \Sigma_0, g_+ : \partial_+ M \to \Sigma_2, n) \in Mor((\Sigma_0, L_0), (\Sigma_2, L_2))$$

*is defined as the cobordism:*

$$(N \sqcup_{g_- \circ f_+^{-1}} M, f_- : \partial_M \to \Sigma_1, g_+ : \partial_+ M \to \Sigma_2, m + n - \mu(K_1, L_0, K_2)) \in Mor((\Sigma_1, L_1), (\Sigma_2, L_2))$$

*where in $\mu$ the Maslov index with:*

(1) $K_1 = \{x \in H(\Sigma_0; \mathbb{R}) | (f_+^{-1})_*(x) \in (f_-^{-1})_*(L_1)\}$
(2) $K_2 = \{x \in H(\Sigma_0; \mathbb{R}) | (g_-^{-1})_*(x) \in (g_+^{-1})_*(L_2)\}$

For the extended category, we get a TQFT by the universal construction.

THEOREM 3.3.2. *[**3**] The universal construction applied to the extended RT invariants of 3-manifolds and to the category $\widetilde{Cob_3}$ yields a TQFT. Moreover, The vector spaces associated to closed surfaces is equipped with a mapping class group invariant Hermitian form $\langle \cdot, \cdot \rangle$. If $A = e^{\frac{i\pi}{2r}}$, the Hermitian form is positive definite.*

We skip the proof of the theorem here. For details of this topic, we refer readers to the original paper.

A mapping cylinder in $\widetilde{Cob_3}$ is represented by a pair $(f, n)$ where $f$ is a mapping cylinder and $n$ is an integer. That is, we get a central extension of mapping class groups which act on the vector space associated to some surface $\Sigma$. Equivalently, the mapping class group MCG($\Sigma$) acts on $Z(\Sigma)$ projectively.

CHAPTER 4

# Mapping Class Group on Surfaces

In this chapter, we revisit the definition of a mapping class group, and give a combinatorial description of the mapping class groups action on the vector spaces coming from the $SU(2)$-TQFT.

## 4.1. Basic Definitions and Properties of Mapping Class Groups

Let $S$ be the surface with or without boundary. The mapping class group of $S$ is defined to be

$$\mathrm{MCG}(S) = Diff(S;\partial S)/\{isotopy\}.$$

A example of a mapping class is Dehn twist around a simple closed curve $\gamma$. We denote this mapping class by $D_\gamma$. $D_\gamma$ equals the except on a cylinder neighborhood of $\gamma$. On the cylinder, we cut the surface along $\gamma$ and glue it back after twisting $2\pi$.

THEOREM 4.1.1. *[5] Mapping class group is generated by Dehn twist.*

MCG($\Sigma$) is embeded in Mor$(S,S)$. Thus one get mapping class group representations from TQFT, if the TQFT is defined on extended cobordism category. One get representations for the central extensions of mapping class groups. Equivalently, projective representations for mapping class groups.

THEOREM 4.1.2. *[5][The inclusion homomorphism]Let S be a closed subsurface of a surface $S'$. Assume that S is not homeomorphic to a closed annulus and no component of $S'-S$ is an open disk. Let $\eta : \mathrm{MCG}(S) \to \mathrm{MCG}(S')$ be the induced map. Let $\alpha_1,\dots\alpha_m$ donote the boundary components of S that bound once-punctured disks in $S'-S$ and let $\{\beta_1,\gamma_1\},\dots,\{\beta_n,\gamma_n\}$ denote the pairs of boundary components of S that bound annuli in $S'-S$. Then the kernel of $\eta$ is the free abelian group:*

$$ker(\eta) =< T_{\alpha_1},\dots,T_{\alpha_m}, T_{\beta_1}T_{\gamma_1}^{-1},\dots,T_{\beta_n}T_{\gamma_n}^{-1} >$$

*In particular, if no connected component of $S'-S$ is an open annulus, an open disk, or an open once marked disk, then $\eta$ is injective.*

We finish this section by the classification theorem of mapping classes.

THEOREM 4.1.3. *[5] Let $g,n \geq 0$. Each mapping class $f \in \mathrm{MCG}(\Sigma_{g,n})$ is periodic, Dehn twist, or pseudo-Anosov. Further, pseudo-Anosov mapping classes are neither periodic nor reducible.*

## 4.2. The Mapping Class Group Action

In this section, we give a combinatorial description of the $SU(2)$-TQFT representation.

By definition, MCG($\Sigma$) acting on the vector space spanned by all 3-manifold with $\partial M = \Sigma$ with some linear relation. The action is defined to be gluing mapping cylinders to $\Sigma$. To make this action more explicit, we allow links in the cobordism. We have the following proposition describing the projective action of the mapping class groups.

PROPOSITION 4.2.1. *Let Z be the TQFT comes from $\overline{\mathscr{C}_A}$. $\Sigma_g$ and $H_g$ are standard surface and handle-body with genus g respectively. $Z(\Sigma_g)$ is generated by $Z(H_g,L)$. Moreover, $D_\gamma(Z(H_g,L)) = Z(H_g, L\cup\gamma_\Omega)$.*

By considering the parallel strands as tensor product of the generating object of $\overline{\mathscr{C}_A}$, We can replace parallel strands by summation of colored strands using Equation 2.4. According to Equation 2.3, A Dehn twist along $\gamma$ bounds a disk that perpendicular to a strand colored by $i$. Then the action of the Dehn twist is by multiplying $(-1)^i A^{i(i+2)}$. As in section 2.3, we can define a set of basis for $Z(\Sigma_g)$.

PROPOSITION 4.2.2. *[3] Let $\Gamma$ be a trivalent graph that contract to $H_g$, An admissible coloring of graph $\Gamma$ gives a set of orthogonal basis of $Z(\Sigma_g)$.*

Specifically, when $A$ be a $4p$-th root of unity where $p$ is an odd prime number, we have the following straight forward proof for the proposition. We present this proof here because the distinct eigenvalue property mentioned below are crucial in our proof.

Let $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a set of curves in $\Sigma_g$ such that each $\gamma_i$ bounds a disk in $H_g$ that perpendicular to one edge of $\Gamma$. Let $H \subset \mathrm{MCG}(\Sigma_g)$ be the subgroup generated by $\{D_{\gamma_i} | 1 \leq i \leq n\}$. Notice that $(-1)^i A^{i(i+2)}$ are all distinct for $0 \leq i \leq p-2$. Thus the eigenspaces of $H$ are all one-dimensional. Each of such eigenspaces are generated by skeins represented by admissible colorings of $\Gamma$. The Hermitian structure is MCG-invariant. Thus these eigenspaces are orthogonal to each other.

The first irreducibility theorem for the TQFT representation is proved by Justin Roberts.

THEOREM 4.2.3. *[21]Let $A$ be a $4p$-th root of unity and $p \geq 3$ be a prime integer. Then $Z(\Sigma_g)$ is irreducible under the action of* $\mathrm{MCG}(\Sigma_g)$.

The following lemma utilize the prime condition.

LEMMA 4.2.4. *Let $A$ be a $4p$-th root of unity and $p \geq 3$ be a prime integer. $t_i \in Z(\Sigma_1)$ be the vector represented by $i$ parallel $(-1)$-framed $\Omega$ colored skeins along the longitude of the solid torus. Then $t_1, t_2, \ldots t_{p-2}$ generate $V(\Sigma_1)$.*

PROOF. We define the Hopf pairing

$$[\cdot,\cdot] : Z(\Sigma_1) \times Z(\Sigma_2) \to \mathbb{C}$$

by gluing two copy of solid torus along their boundary to get a sphere and then evaluating the skeins in the 3-sphere. Since the $S$-matrix of $\overline{\mathscr{C}_A}$ is non-degenerate, the Hopf pairing is non-degenerate.

Let $v_i$ be the skein colored by $i$ along the longitude of the solid torus. We compute:

$$[t_i, v_j] = [v_0, D^i_\alpha v_j] = [v_0, ((-1)^j A^{j(j+2)})^i v_j] = ((-1)^j A^{j(j+2)})^i [j+1]$$

Thus the matrix $\{[t_i, v_j]\}_{ij}$ is equal to a Vandermonde matrix times a diagonal matrix. $(-1)^j A^{j(j+2)}$ are all distince for all $j$ when $p$ is prime, The first factor is invertible. The second factor is obviously invertible because the diagonal entry $[j+1] \neq 0$ when $0 \leq j \leq p-2$. This proves $\{t_i\}_{0 \leq i \leq p-2}$ generate $Z(\Sigma_g)$. □

Now we give the proof of Roberts' theorem.

PROOF. Let $v_0$ be the vector in $Z(\Sigma_g)$ representing the empty skein in $H_g$. The above lemma shows that we generate $Z(\Sigma_g)$ by applying Dehn twists on $v_0$. Let $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a set of curves in $\Sigma_g$ such that each $\gamma_i$ bounds a dick in $H_g$ that perpendicular to one edge of $\Gamma$ and $H \subset \mathrm{MCG}(\Sigma_g)$ be the subgroup generated by $\{D_{\gamma_i} | 1 \leq i \leq n\}$. Subrepresentations of $\mathrm{MCG}(\Sigma_g)$ must be summations of eigenspaces of $H$. Thus, the subrepresentation containing the eigenspace generated by $v_0$ is $Z(\Sigma_g)$. □

The action of the mapping class group naturally generalize to surfaces with boundaries. We consider the vector space generated by skein spaces in $H_g$ but with projectors on its boundary. Without special notice, we will assume working with category $\overline{\mathscr{C}_A}$, where $A$ is a $4p$-th root of unity where $p$ be a prime number.

We denote by $V_{g,b;\vec{c}}$ be the vector space generated by skein space in $H_g$ with $b$ projectors on its boundary. The projectors are $\vec{c} = (c_1, c_2, \ldots c_n)$ respectively. When $g = 0$, $V_{g,b;\vec{c}}$ can be realized as morphism space in $\overline{\mathscr{C}_A}$:

$$V_{g,b;c_1,c_2,\ldots,c_b} = \mathrm{Hom}(X_1 \otimes X_2 \otimes \ldots \otimes X_b, \mathbf{1}).$$

As in section 2.3 and the case for closed surfaces. we have the following description for some sets of orthogonal basis of $V_{g,b;\vec{c}}$: We choose a trivalent graph $\Gamma$ such that the handlebody $H_g$ retracts to $\Gamma$, and fix the colors of the edge adjacent to the uni-vertices to be $c_1, c_2, \ldots, c_b$. The set of admissible colorings of $\Gamma$ is a set of orthogonal basis for $V_{g,b;\vec{c}}$.

CHAPTER 5

# Proof of Irreducibility Theorem

In this chapter, We prove the Theorem 1.0.1. As in the setting of the theorem, we assume $A$ be a $4p$-th root of unity where $p$ is an odd prime number in this Chapter.

## 5.1. Cross-lamination Lemma for Irreducibility

The following lemma is the unitary version of the cross-lamination lemma in [**15**].

LEMMA 5.1.1. *[**15**] Consider a positive definite Hermitian vector space $X$ over some field $F$ that is a (projective) representation of some group $G$, and $G_1$, $G_2$ are two subgroups. Suppose $X = \bigoplus_I V_i$(resp. $\{X = \bigoplus_J W_j\}$) be an irreducible multiplicity free decomposition under the action of some central extension $\tilde{G}_1$(resp. $\tilde{G}_2$) of $G_1$(resp. $G_2$). Define a graph $C(X, G_1, G_2)$ on the set of irreducible summands, with an edge connecting $V_j$ and $W_k$ if there exists some element $v \in V_j$ and $w \in W_j$ such that $[v, w] \neq 0$.*

*If the graph is connected, then $X$ is irreducible as a representation of $G$.*

Note that decompositions in Lemma 5.1.1 do not depend on the choice of central extensions. So when proving irreducibility, $G$ and $\tilde{G}$ will not be distinguished. As discussed in [**21**], the central extension does not affect the irreducibility.

With the setting of Lemma 5.1.1, each $W_j$(resp. $V_i$) is connected with some $V_i$(resp. $W_j$) since $W_j$(resp. $V_i$) is not $\{0\}$ as a vector space. Thus, to prove irreducibility, we just need to prove one side of this bipartite graph is connected. We formulate it in the following corollary.

COROLLARY 5.1.2. *With the assumptions in Lemma* 5.1.1*, we can conclude irreducibility with one of the following conditions on the graph.*

(a) *For all $i, j \in I$, $V_i$, $V_j$ are connected by some path.*
(b) *For all $j \in J$, there exist an $i \in I$ such that $V_i$ connected to $W_j$.*

## 5.2. Two Base Cases

In this section, we discuss two base cases, where the surfaces are sphere with 4 punctures and one-holed torus.

### 5.2.1. Sphere with 4 Punctures.

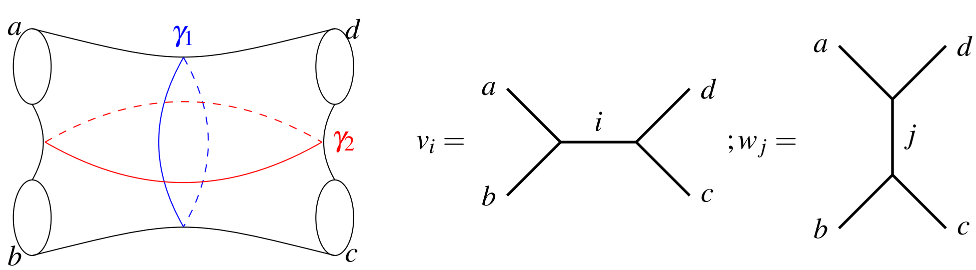
LEMMA 5.2.1. *$V_{0,4;a,b,c,d}$ is irreducible under the action of $PB_4$ for all admissible $a,b,c,d$. By admissible, we mean the vector space associated has positive dimension.*

PROOF. The Dehn twist around $\gamma_1$ and $\gamma_2$ gives 2 orthogonal decompositions:

$$V_{0,4;a,b,c,d} = \bigoplus_{i=\max\{|a-b|,|c-d|\};i\equiv a-b \mod 2}^{\min\{a+b,c+d,2p-4-a-b,2p-4-c-d\}} V_i,$$

$$V_{0,4;a,b,c,d} = \bigoplus_{j=\max\{|a-d|,|b-c|\};i\equiv a-d \mod 2}^{\min\{a+d,c+b,2p-4-a-d,2p-4-c-b\}} W_j,$$

where $V_i$ (resp. $W_j$) are spanned by the single vector $v_i$ (resp. $w_j$).

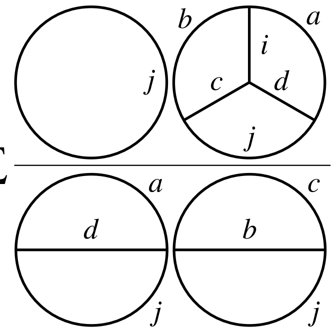



When $p$ is an odd prime, $(-1)^i A^{i(i+2)}$ are different for different $i$. Thus, both decompositions are multiplicity free because they have different eigenvalues under the action of $D_{\gamma_t}(t = 1,2)$.

By Lemma 5.1.1, the irreducibility of the representation of$PB_4$ is equivalent to the connectivity of the graph $C(V_{0,4;a,b,c,d}, < D_{\gamma_1} >, < D_{\gamma_2} >)$. Before we proceed, we give change of basis formula [**2**] for $\{v_i\}$ and

$\{w_j\}$ below.

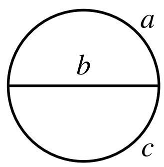
$$v_i = \sum_j \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} w_j = \sum_j$$



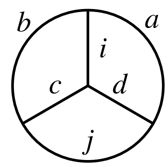

$w_j$

where we follow the conventions in [**2**]:

$< a, b, c >=$




$;\quad \left\langle \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\rangle =$




$V_i$ and $W_k$ are connected is equivalent to the span of $v_i$ in $\{w_j\}$ has non-zero coefficient on $w_k$. According to Corollary 5.1.2 case $(a)$, Our strategy is to to find some $i$ such that $V_i$ is connected to $W_j$ for all $j$.

Without loss of generality, we assume $a-b \geq |c-d|$(otherwise, we rotate the symbols), write $v_{a-b}$ as summation of $w_j$:

(5.1)
$$v_{a-b} = \Sigma_j \left\{ \begin{array}{ccc} a & b & a-b \\ c & d & j \end{array} \right\} w_j$$

If $j$ is admissible, $< b, c, j >$ and $< a, d, j >$ are nonzero, so we just need to check the tetrahedron symbols. Theorem 2 of [**2**] gives an explicit formula for the tetrahedron symbol. As in [**2**], Let $m_1 = (a+b+i)/2$, $m_2 = (a+d+j)/2$, $m_3 = (b+c+j)/2$, $m_4 = (i+d+c)/2$; $n_1 = (a+b+c+d)/2$, $n_2 = (b+i+d+j)/2$, $n_3 = (a+i+c+j)/2$.

(5.2)
$$\left\langle \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\rangle = \sum_{z=\max m_s}^{\min n_t} \frac{\prod_{s,t}[n_s - m_t]!}{[a]![b]![c]![d]![i]![j]!} \frac{(-1)^z [z+1]!}{\prod_s [n_s - z]! \prod_t [z - m_t]!}$$

In general, the tetrahedron symbol is a summation over $z$ for all $\max_t n_t \leq z \leq \min_s m_s$. in our case, we only have one summand because $\max_t n_t = z = \min_s m_s$. which is

$$\left\langle \begin{array}{ccc} a & b & a-b \\ c & d & j \end{array} \right\rangle = \frac{\prod_{s,t}[n_s - m_t]!}{[a]![b]![c]![d]![a-b]![j]!} \frac{(-1)^z [z+1]!}{\prod_s [n_s - z]! \prod_t [z - m_t]!} \tag{5.3}$$

Then we need to check the $q$-factorials in the above formula are less than $p$ to make sure all factors are nonzero. Note that $2(n_s - m_t)$ can be realized as summation of two labels of an admissible triple subtracting the other one, which is always less than $2(p-2)$. Thus $n_s - m_t \leq p-2$, which implies $[n_s - m_t]! \neq 0$; $z$ is half of the sum of labels of an admissible triple, so $z \leq p-2$, $[z+1]! \neq 0$.

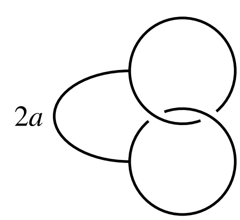
The above computation showed that $V_{a-b}$ is connected to $W_j$ for all $j$. By Corollary 5.1.2(a), $V_{0,4;a,b,c,d}$ is irreducible under the action of $PB_4$.

□

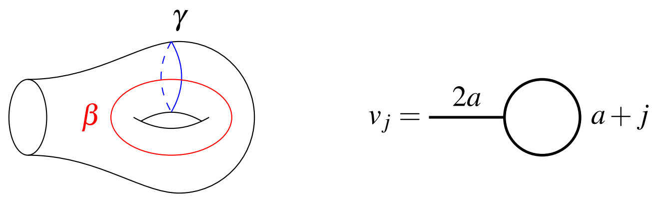
**5.2.2. One-holed Torus.** This case has been studied by G. Patrick and G. Masbaum [**18**]. They proved irreducibility of $V_{g,1;2a}$ for any $g$ when $p$ is an odd prime. In our case, we just need $g = 1$ to start the induction, and for completeness, we put a more elementary proof here, and we would like to thank Julien Korinman [**14**] for teaching us the proof.

LEMMA 5.2.2. *Let $p$ be an odd prime, $1 \leq a \leq \frac{p-3}{2}$, $V_{1,1;2a}$ is irreducible under the action of* $\mathrm{MCG}(\Sigma_{1,1})$.

PROOF. In [**18**], their computation showed that the Hopf pairing (see figure below) $\langle , \rangle$ of $V_{1,1;2a}$ is a nondegenerate bilinear form.



Let $v_j$ be the lollipop basis shown below, and $w_i = D^i_\beta v_0$ be the vector derived from applying Dehn twist along $\beta$ to $v_0$ $i$ times .




We compute the Hopf pairing of $w_i$ and $v_j$, where $0 \le i, j \le p-a-2$:

$$< w_i, v_j > = < D^i_\beta v_0, v_j > = < v_0, D^i_\gamma v_j > = (-1)^{ij} A^{i(j+a)(2+j+a)} < v_0, v_j >$$

When $p$ is an odd prime, $(-1)^j A^{(j+a)(2+j+a)}$ are distinct complex numbers, so matrix $V = \{(-1)^{ij} A^{i(j+a)(2+j+a)}\}_{ij}$ is a Vandermonde matrix, thus invertible. $\{< w_i, v_j >\}_{ij} = V \cdot diag\{< v_0, v_j >\}$. $< v_0, v_j > \neq 0$ (c. f. **[18]** Page 100), so the product of the two matrices is invertible, which implies $\{w_i\}$ spans $V_{1,1;2a}$.

Consider the bipartite graph $C(V_{1,1;2a}, < D_\beta >, < D_\gamma >)$. $< v_0 >$ is invariant under the action of $D_\gamma$, and the decomposition under the action of $< D_\gamma >$ is multiplicity free. The argument above showed that $v_0$ have component in all eigenspaces of $D_\beta$ and each eigenspaces are 1-dimensional. That is, $< v_0 >$ is connected to all eigenspaces of $< D_\beta >$, and all the eigenspaces are multiplicity free due to eigenvalue test. By Corollary 5.1.2(a), $V_{1,1;2a}$ is irreducible.

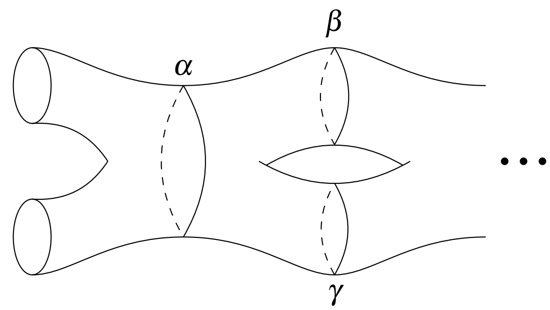
□

### 5.3. The Induction

In this section, we will develop the induction steps. The idea is the following: We can decompose the representation by restricting it to a mapping class group of a subsurface $S' \subset S$. Usually $S - S'$ is a cylinder $\alpha \times I$. The decomposition depends on the choice of $\alpha$ on $S$. Two different such decompositions will produce a bipartite graph described in Lemma 5.1.1. Thus, to prove the representation is irreducible, one just need to show the graph is connected.

The following three lemmas provide us the tools for the induction on genus $g$ and the number of boundaries $b$.

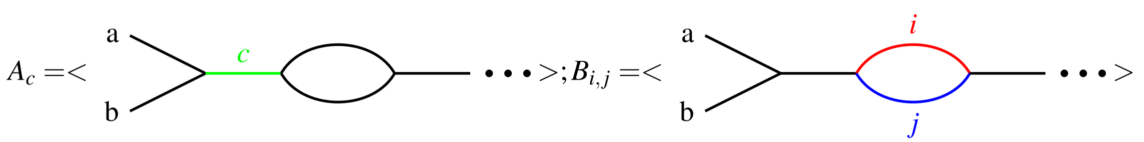
LEMMA 5.3.1. *Let $p$ be an odd prime and $g \geq 1$, Suppose $V_{g,1,c}$ and $V_{g-1,2;i,j}$ are irreducible as* $\mathrm{MCG}(\Sigma_{g,1})$ *and* $\mathrm{MCG}(\Sigma_{g-1,2})$ *representation respectively for all admissible $c, i, j$, then $V_{g,2;a,b}$ is an irreducible representation of* $\mathrm{MCG}(\Sigma_{g,2})$ *for all $a, b$.*




PROOF. Let $S$ be a representative of surface $\Sigma_{g,b}$ as above. We restrict $V_{g,2;a,b}$ to the image of $\mathrm{MCG}(S-\alpha\times I)\to \mathrm{MCG}(S)$ and the image of $\mathrm{MCG}(S-(\beta\cup\gamma)\times I)\to\mathrm{MCG}(S)$ respectively. According to Theorem 3, the kernels of group homomorphisms above factor though the corresponding representations. Notice $\mathrm{MCG}(S-\alpha\times I)\cong \mathrm{MCG}(\Sigma_{0,3})\times\mathrm{MCG}(\Sigma_{g,1})$ and $\mathrm{MCG}(S-(\beta\cup\gamma)\times I)\cong\mathrm{MCG}(\Sigma_{0,4})\times\mathrm{MCG}(\Sigma_{g-1,2})$ Thus, with the assumptions in this lemma., $V_{g,2;a,b}$ has the following two irreducible decompositions accordingly.

$$V_{g,2;a,b}=\bigoplus_c A_c=\bigoplus_c V_{g,1;c}\otimes V_{0,3;c,a,b};V_{g,2;a,b}=\bigoplus_{i,j} B_{i,j}=\bigoplus_{i,j} V_{g-1,2;i,j}\otimes V_{0,4;i,j,a,b},$$

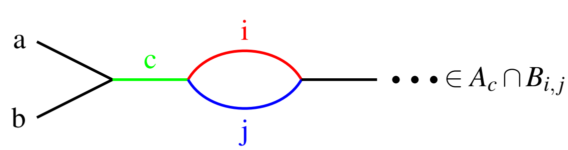
where $A_c$ and $B_{i,j}$ are invariant spaces of $D_\alpha$ and $D_\beta\times D_\gamma$ respectively. They are spanned by the graphs shown below:




When $p$ is odd prime, these invariant spaces have different eigenvalues for the Dehn twists, so the decompositions are multiplicity free for all central extensions.

According to Lemma 5.1.1, we just need to prove the bipartite graph $C(V_{g,2;a,b},\mathrm{MCG}(S-\alpha\times I),\mathrm{MCG}(S-(\beta\cup\gamma)\times I))$ is connected.

Note that if $(c,i,j)$ is an admissible triple, $A_c$ and $B_{i,j}$ are connected because the following element will be in the intersection of $A_c$ and $B_{i,j}$.

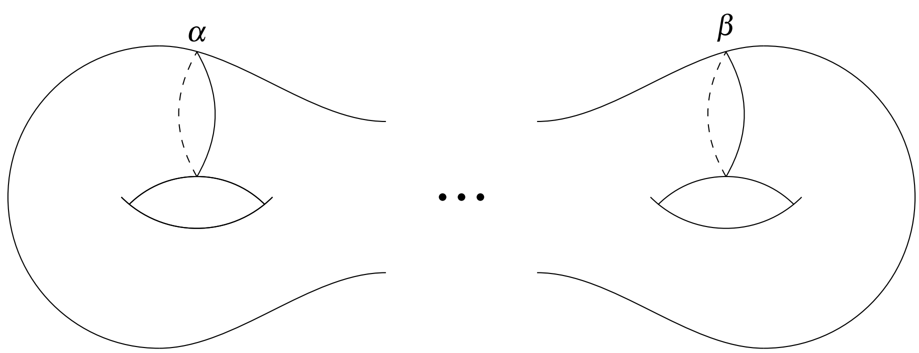


Then notice for all $c$, $(c, \frac{p-3}{2}, \frac{p-3}{2})$ will be an admissible triple, so all $A_c$ are connected to $B_{\frac{p-3}{2},\frac{p-3}{2}}$. $B_{\frac{p-3}{2},\frac{p-3}{2}}$ is of positive dimension for all $g$. Thus, $V_{g,2;a,b}$ is an irreducible $\mathrm{MCG}(\Sigma_{g,2})$ representation by Corollary 5.1.2.

□

If all boundaries are colored by 0, we should consider it as the closed surface. This case not only shows up in the question itself, but also contribute to the induction. Although closed surface case have been proved by Roberts [**21**], we still put this case in our induction for completeness.

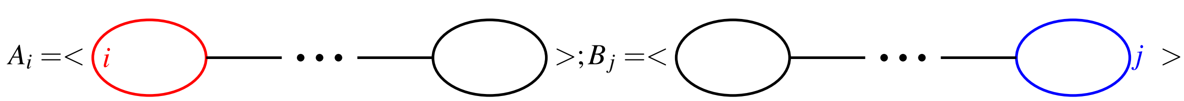
LEMMA 5.3.2. *Let $p$ be an odd prime and $g \geq 2$. Assume $V_{g-1,2;i,i}$ is irreducible under the action of* $\mathrm{MCG}(\Sigma_{g-1,2})$ *for all $i$. Then $V_{g,0}$ is irreducible under the action of* $\mathrm{MCG}(\Sigma_g)$.



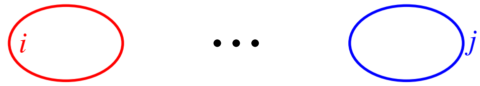

PROOF. Let $S$ be a representative of $\Sigma_{g,0}$ as above. We restrict $V_{g,0}$ to the representation of subgroups of the image of $\mathrm{MCG}(S-\alpha\times I)$ and the image of $\mathrm{MCG}(S-\beta\times I)$ in $\mathrm{MCG}(S)$ respectively. By Theorem 4.1.2, the kernels factor though the corresponding representations. Note that both $\mathrm{MCG}(S-\alpha\times I)$ and $\mathrm{MCG}(S-\beta\times I)$ are isomorphic to $\mathrm{MCG}(\Sigma_{g-1,2})$. Thus, with the assumptions in this lemma, $V_{g,0}$ has the following two irreducible decompositions accordingly.

$$V_{g,0} \cong \oplus_i A_i = V_{g-1,2;i,i}; V_{g,0} \cong \oplus_j B_j = V_{g-1,2;j,j}$$

Where $A_i$ and $B_j$ are invariant spaces of $D_\alpha$ and $D_\beta$ respectively. They are spanned by the graphs shown below, where uncolored edges run over all admissible colors.




When $p$ is odd prime, these invariant spaces have different eigenvalues, so the decompositions are multiplicity free for all central extension of $\mathrm{MCG}(\Sigma_g)$. Notice the following element is in the intersection of $A_i$ and $B_j$:

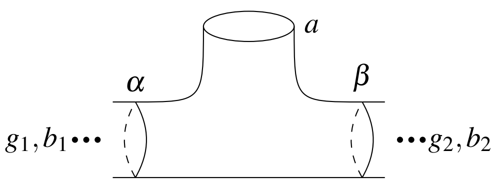



$\in A_i \cap B_j$.

Argument above showed that the bipartite graph $< V_{g,0}; \mathrm{MCG}(S-\alpha\times I), \mathrm{MCG}(S-\beta\times I) >$ is complete. Thus, the representation is irreducible by Lemma 5.1.1. $\square$

Before we introduce the next lemma, we define $\prec$ to be the lexicographical order on pair $(g,b)$.

LEMMA 5.3.3. *Let $p$ be an odd prime and $(g,b) \notin X = \{(0,1),(0,2),(0,3),(0,4),(1,1),(1,2)\}$ and $b \geq 0$. Consider vector space $V_{g,b;\vec{c}}$ with $b \geq 1$ and one of the boundary is colored by $a \neq p-2$. If $V_{g',b';\vec{c}'}$ is irreducible under the action of* $\mathrm{MCG}(\Sigma_{g',b'})$ *for all $(g',b') \prec (g,b)$ and any color $\vec{c}'$, $V_{g,b;\vec{c}}$ is irreducible under the action of* $\mathrm{MCG}(\Sigma_{g,b})$.

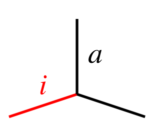


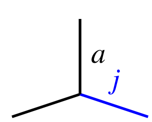

PROOF. Since $(g,b) \notin X$, we can find $(g_1,b_1) \prec (g,b)$ and $(g_2,b_2) \prec (g,b)$ satisfying $g = g_1+g_2$ and $b = b_1+b_2-1$. By restricting to the group actions of $\mathrm{MCG}(S-\alpha\times I)$ and $\mathrm{MCG}(S-\beta\times I)$, we can decompose the representation $V_{g,b,\vec{c}} = V_{g_1+g_2,b_1+b_2+1;\vec{c}_1,\vec{c}_2,a}$ in following two ways, where the indexes $i$ and $j$ run over all colors such that the corresponding vector spaces are of positive dimension.

$$A_i = V_{g_1,b_1+1;\vec{c}_1,i} \otimes V_{g_2,b_2+2;\vec{c}_i,i,a}$$

$$B_j = V_{g_1,b_1+2;\vec{c}_1 a,j} \otimes V_{g_2,b_2+1;\vec{c}_2,j}$$

Where $A_i$ and $B_j$ are invariant spaces of $D_\alpha$ and $D_\beta$ respectively. They are spanned by the graphs shown below, where the uncolored edges run over all admissible colors.

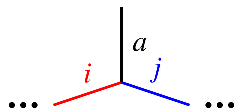
$A_i =<\cdots$ 

$\cdots>; B_j =<\cdots$ 

$\cdots>$.

$i$(resp. $j$) run through all colors such that both of their tensor factor have positive dimension. For instance, if both $g_1$ and $g_2$ are positive, $i$ and $j$ run through all colors satisfying parity condition. If $g_1$ or $g_2$ is zero, then we have extra inequality constraints for $i$ and $j$, but these constraints always reduce to an interval.

By the assumption, $V = \oplus_i A_i$(resp.$V = \oplus_j B_j$) is an irreducible and multiplicity free (by checking eigenvalues of the Dehn twist around the circle in the graph) decomposition under the group action of $\mathrm{MCG}(\Sigma_{g_1,b_1+1}) \times \mathrm{MCG}(\Sigma_{g_2,b_2+2})$ (resp. $\mathrm{MCG}(\Sigma_{g_1,b_1+2}) \times \mathrm{MCG}(\Sigma_{g_2,b_2+1})$), and we noticed that if $(i,j,a)$ is an admissible triple, $A_i$ and $B_j$ are connected because the following element is in the intersection:



$\in A_i \cap B_j$

Now we prove the following claim: Suppose $i \geq i'$ and there exist $j, j'$ such that $(i,j,a)$ and $(i',j',a)$ are admissible triples, then there exist some $j''$ such that $(i,j'',a)$ and $(i-2,j'',a)$ are admissible triples.

If $(i-2,j,a)$ is admissible, we are done. Otherwise, one of the inequality conditions must fail, so we have either $i+j=a$ or $j-i=a$:

Case 1 $i+j=a$: $i'+j' \geq a$ and $i' < i$ implies $j' > j$, so $B_{j+2}$ is not $\{0\}$. Let $j'' = j+2$, we check $(i-2)+j''=a$, $|(i-2)-j''| = |i-4+j| \leq |i-4|+j$. Since $i > i'$, $i \geq 2$, so $|i-4|+j \leq i+j = a$. Thus $(i-2,j+2,a)$ is an admissible triple.

Case 2 $j-i=a$: $j'-i' \leq a$ and $i' < i$ implies $j' < j$, so $B_{j-2}$ is not $\{0\}$. Let $j'' = j-2$, we check $|(i-2)-j''| = a$, $(i-2)+j'' = j-i+2i-4 \geq j-i = a$. Thus $(i-2,j-2,a)$ is an admissible triple.

We take $i'$ to be the least $i$ such that $A_i \neq 0$. The claim said $B_{j''}$ is connected to $A_i$ and $A_{i-2}$. This gives us paths connect all pairs of adjacent $A_i's$. Together with Corollary 5.1.2(b), we proved $V_{g_1+g_2,b_1+b_2+1;\vec{c}_1,\vec{c}_2,a}$ is irreducible.

□

### 5.4. Proof of Theorem 1.0.1

PROOF OF THEOREM 1.0.1. We are going to do induction on the lexicographical order $\prec$ of pairs (g, b). We just need to show that for all $V_{g,b;\vec{c}}$ where $(g,b) \notin \{(0,1),(0,2),(0,3),(0,4),(1,1),(1,2)\}$, The irreducibility of $V_{g,b;\vec{c}}$ is implied by some $V_{g',b';\vec{c}'}$ satisfying $(g',b') \prec (g,b)$.

We first consider the case that one of the boundary is colored by $p-2$. Because summation on colors on the boundary should be an even number, $b \geq 2$. Pick another boundary that is colored by $i$. We have the following isomorphism of $\mathrm{MCG}(\Sigma_{g,b-1})$ representations:

$$V_{g,b;\vec{c}} \cong V_{0,3;p-2,i,p-2-i} \otimes V_{g,b-1;\vec{c},p-2-i \setminus \{p-2,i\}}$$

By this, we lower the number of boundary components by 1.

If $b \geq 1$ and none of the boundary is colored by $p-2$, and $(g,b) \notin \{(0,1),(0,2),(0,3),(0,4),(1,1),(1,2)\}$, Lemma 5.3.3 implies it is enough to show $V_{g',b';\vec{c}'}$ are irreducible for all $(g',b') \prec (g,b)$.

If $b = 0$ and $g \leq 2$, Lemma 5.3.2 implies it is enough to show $V_{g,2;i,i}$ is irreducible.

If $(g,b) = (1,2)$, Lemma 5.3.1 implies it is enough to show $V_{1,1;a}$ is irreducible.

For Base cases:

- $(g,b) = (0,1),(0,2),(0,3)$. The vector spaces are either 0 or 1-dimensional.
- $(g,b) = (1,0)$. This is Weil representation, and it is irreducible.
- $(g,b) = (0,4)$ and $(1,1)$. We proved them in section 4.

The above induction proves $V_{g,b;\vec{c}}$ is irreducible for all $g,b$ and all coloring $\vec{c}$. □

CHAPTER 6

# Proof of Denseness Theorem

## 6.1. Background for denseness theorem

**6.1.1. Zariski Denseness and Analytic Denseness.** Let $K$ be a infinite field, the Zariski topology on $K^n$, or a subset of $K^n$, is the coarsest topology that solution set of a polynomial equation is closed. We interpret Lie group $SL(n,\mathbb{C})$ as a subset of $\mathbb{C}^{n^2}$. For projective special linear group $PSL(n,\mathbb{C})$, We consider the topology induced by the map $SL(n,\mathbb{C}) \to PSL(n,\mathbb{C})$. For $SL(n,\mathbb{R})$ and $SU(n,\mathbb{C})$, we consider the subset topology of the Zariski topology of $SL(n,\mathbb{C})$.

The analytic topology is the topology defined by the Euclidean metric.

It is obvious that the Zariski topology is much coarser than the analytic topology. However, The following theorem by Chevallay states that these two topology are equivalent when we consider the closed subgroups of $SU(n,\mathbb{C})$.

THEOREM 6.1.1. *(Chevallay)Every analytically closed subgroup of $SU(n,\mathbb{C})$ is Zariski closed.*

The case we studied in this chapter is the Zariski image of the braid group in $PSL(n,\mathbb{C})$. The Analytic closed subgroup of $PSL(n,\mathbb{C})$ are difficult to work with. Working with the Zariski topology can help us rule out many nasty possibilities.

**6.1.2. Lemmas for Induction.** In this subsection, we introduce some tools to study image of group homomorphisms that were developed by Ribet [**15**,**20**] and others and that originate with Goursat's Lemma.

In generic $q$ case, we no longer have the Hermitian form on the skein spaces. Thus, we rephrase Lemma 5.1.1.

LEMMA 6.1.2. *[**15**] Consider a vector space $X$ over some field $F$ is a (projective) representation of some group $G$, and $G_1$, $G_2$ are two subgroups. Let $X = \bigoplus_I V_i$(resp. $\{X = \bigoplus_J W_j\}$) be an irreducible multiplicity free decomposition under the action of some central extension $\tilde{G}_1$(resp. $\tilde{G}_2$) of $G_1$(resp. $G_2$). Define a directed graph $C(X, G_1, G_2)$ on the set of irreducible summands, with an edge from $V_j$ to $W_k$ if there exists some element $v \in V_j$ has nonzero component in $W_k$.*

*If the graph is strongly connected, then $X$ is irreducible as a representation of $G$.*

The following theorem is often called non-commutative Chinese remainder theorem. That locally surjectivity implies global surjectivity.

THEOREM 6.1.3. *Suppose that each of $G_1, G_2, \ldots, G_l$ is a minimal simple Lie group or a non-abelian finite simple group, and suppose that*

$$H \subset G = G_1 \times G_2 \times \ldots \times G_l$$

*is a closed subgroup that surjects onto each factor $G_k$. Then $H$ is a diagonal subgroup of $G$.*

In our case, The Gamma is the Zariski closure of the mapping class group and $G_i$ are complex simple Lie groups. Thus, we have a corollary for our case.

COROLLARY 6.1.4. *If*

$$f : \Gamma \to PSL(W_1) \times PSL(W_2) \times \ldots \times PSL(W_n)$$

*is surjective when restrict to $\Gamma \to G_i$ for all i, and non of pairs $W_i$ and $W_j$ are isomorphic or dual to each other, then $f$ is surjective.*

Lastly, we need the following surjectivity theorem for the induction purpose.

THEOREM 6.1.5. *Let $V$ be a finite-dimensional complex representation of a connected Lie group G, Let $H \subset G$ be a closed, connected subgroup, and let*

$$V|_H \cong \bigoplus_{k=1}^{n} W_k$$

*be the decomposition of the restricted representation. Suppose that:*

(1) *$V$ is G-irreducible.*

(2) *At most one of $W_i$ is of dimension 2 and at most one of $W_i$ is of dimension 1.*

(3) *For every $j \neq k$, the summands $W_j$ and $W_k$ are neither isomorphic nor dual as projective representation of $H$.*

(4) *For each $j$, $H$ surjects onto $PSL(W_j)$.*

*Then $G$ suject to $PSL(V)$.*

PROOF. Let $L$ be the Lie algebra of $G$, $K \subset L$ be the corresponding Lie algebra of $H$. Then we have the splitting

$$\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \mathbb{C}$$

. The non-commutative Chinese remainder theorem for Lie algebra gives us that $H$ is jointly surjective:

$$H \twoheadrightarrow \bigoplus_k \mathfrak{sl}(W_k)$$

.

Meanwhile we have the partial decomposition

$$\mathfrak{sl}(V) = \bigoplus_{j \neq k} W_j \otimes W_k^* \oplus \bigoplus_k \mathfrak{sl}(W_k) \oplus \mathbb{C}^{n-1}$$

.

Notice that if for some $i$, $\dim W_i > 2$, then $W_i \ncong W_i^*$ as $\mathfrak{sl}(W_i)$ representation. Together with the 3rd assumption in the theorem, $W_i \otimes W_j^*$ and $W_j \otimes W_i^*$ are both unique in the decomposition in the sense that they are not isomorphic to each other and all other summands as a representation of $H$. That is, The off-diagonal blocks are multiplicity free if and only if at most one of $W_i$ satisfying $\dim W_i \leq 2$. In the rest of the proof, we discuss it case by case. Without lose of generality, we assume the $\dim W_i \leq \dim W_j$ if $i \leq j$.

Case 1 $\dim W_2 \geq 3$.

As we discussed above, the off diagonal summands are multiplicity free in this case. so the image of $L$ in $\mathfrak{sl}(V)$ contains a subset of the off-diagonal blocks $W_j \otimes W_k^*$ and some subspace of $\mathbb{C}^{n-1}$. We can make a directed graph $\Gamma$ with a directed edge $k \to j$ for every off-diagonal block $V_j \otimes V_k^*$ which is in the image of $L$. We claim $\Gamma$ is complete.

First, we prove $\Gamma$ is strongly connected. Assuming otherwise, $\Gamma$ would have a strongly connected component $C$ with no outward edges. In this case $\bigoplus_{k \in C} W_k$ would be a non-trivial subrepresentation of $L$. $G$ is connected, so $\bigoplus_{k \in C} W_k$ is also a subrepresentation of $G$, contradicting the hypothesis that $V$ is irreducible.

Second, we prove that $\Gamma$ is transitively closed. Suppose that $i \to k \to j$ is a path of length two using three distinct vertices. Choose two operators

$$Y \in \mathrm{Hom}(W_l, W_k) \cong W_k \otimes W_l^*$$

$$X \in \mathrm{Hom}(W_k, W_j) \cong W_k \otimes W_l^*$$

whose product

$$XY \in \mathrm{Hom}(W_l, W_j) \cong W_j \otimes W_l^*$$

is non-zero. Then $[X,Y]$ is a non-zero element in $W_j \otimes W_l^*$. The image of $L$ thus contains some elements of $W_j \otimes W_l^*$, and therefore it contains all of them and $\Gamma$ is transitively closed.

If $\Gamma$ is both strongly connected and transitively close, then it is the complete directed graph. In the final step, choose some basis of $V$ that refines the decomposition of $V|_H$. In this basis, the image of $L$ contains all off-diagonal elementary matrices, so there commutators gives us all of $\mathfrak{sl}(V)$

Case 2 $n \geq 3$ and $\dim W_2 = 2$, $\dim W_1 = 1$.

Define $V' = \bigoplus_{k=2}^n W_k$. The identical argument as in case 1 shows that $\mathfrak{sl}(V')$ is in the image of $L$. Thus we have decomposition

$$\mathfrak{sl}(V) = \mathfrak{sl}(V') \oplus V' \oplus V'^* \oplus \mathbb{C}$$

as representation of $\mathfrak{sl}(V')$. The decomposition is multiplicity free since $\dim V' \geq 3$. $V$ is irreducible as $L$ representation, so The image of $L$ contains both $V$ and $V^*$. The commutators will give us all of $\mathfrak{sl}(V)$.

Case 3 $n = 2$ and $\dim W_2 = 2$, $\dim W_1 = 1$.

In this case, the $V$ is a 3-dimensional representation. we list all the Lie subalgebra of $\mathfrak{sl}(V)$. Only $\mathfrak{sl}(V)$ itself makes $V$ irreducible.

□

**6.1.3. Simple Lie Algebras and Their Representation Theory.** In this subsection, we review some basic concept of root spaces and representation theories of simple Lie algebras which can be found in standard Lie algebra textbooks. Then we introduce a lemma that is conjectured by the Greg Kuperberg and supported by David Speyer.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ be one of its Cartan subalgebra. $\mathfrak{h}$ is a maximal commutative Lie subalgebra in a sense that there is no commutative Lie subalgebra strictly contains $\mathfrak{h}$. Since the representation theory of $\mathfrak{g}$ is semisimple. Without loss of generality, Let $X$ be a simple representation of $\mathfrak{g}$. We restrict $X$ as a representation of $\mathfrak{h}$. $X$ decomposes to 1-dimensional sub-representations due to the following elementary lemma from linear algebra.

LEMMA 6.1.6. *Let $S$ be a set of $n \times n$ matrices. If the matrices are pairwise commutative then they are simultaneously diagonalizable.*

Notice that each eigenspaces $W$ of $\mathfrak{h}$ defines a functional $\lambda$ on $\mathfrak{h}$ by $h(w) = \lambda(h)\dot{w}$ for all $h \in \mathfrak{h}$ and $w \in W$. Functionals comes from the eigenspaces of $\mathfrak{h}$ are called weights of $X$. Weights of the adjoint representation are called roots. the set of roots is symmetric respect to the origin, that is, if $a \in \mathfrak{h}^*$ is a root, then $-a$ is also a root. Moreover, all eigenspaces corresponding to the nonzero roots are 1-dimensional. We denote the eigenspace correspond to root $\alpha$ by $\mathfrak{g}_\alpha$. The set of roots of $\mathfrak{g}$ in $\mathfrak{h}^*$ are called root system of $\mathfrak{g}$. Notice that all Cartan subalgebras of simple Lie algebra $\mathfrak{g}$ are conjugate to each other. The root system is independent of the choice of $\mathfrak{h}$.

Roots span $\mathfrak{h}^*$. There is a inner product structure $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ defined by extending a inner product on the set of roots. The inner product in defined as following: Let $\alpha$ and $\beta$ be two roots, $x_\alpha \in \mathfrak{g}$ be an element in $\mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}$ be the unique element in $\mathfrak{g}_{-\alpha}$ such that

$$K(x_\alpha, y_\alpha) := \mathrm{Tr}(\mathrm{ad}(x_\alpha)\mathrm{ad}(y_\alpha)) = 1.$$

The inner product $\langle \beta, \alpha \rangle$ is defined to be $\beta([x_\alpha, y_\alpha])$. We remark that $\{x_\alpha, y_\alpha, [x_\alpha, y_\alpha]\}$ is a standard $\mathfrak{sl}_2$ triple. The isometry group of the root system is called Weyl group.

Let $\Phi$ be the set of roots of $\mathfrak{g}$. We pick a generic vector $t$ in the $\mathbb{R}$-span $\Phi$. Denote the set of positive roots by

$$\Phi_+ = \{\alpha \in \Phi | \langle \alpha, t \rangle \geq 0\}$$

and denote the set of negative roots $\Phi_-$ by the other half. For different choices of $t$, the sets of positive roots are equivalent up to the action of the Weyl group. We denote nilpotent subalgebras

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha; \qquad \mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha.$$

Root systems of Lie algebras are used to study the representation theories of Lie algebras. Before we proceed, we first remind the reader the universal enveloping algebra and the PBW theorem.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Let $U(\mathfrak{g})$ be its universal enveloping algebra. That is, $U(\mathfrak{g})$ is the universal object in the category of associative unital algebras that contains $\mathfrak{g}$. It is a easy consequence from the defintion that the representation theory of $U(\mathfrak{g})$ is equivalent to the representation theory of $\mathfrak{g}$. A explicit representation of $U(\mathfrak{g})$ is $T(\mathfrak{g})/I$, where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and $I$ is the ideal generated by $a \otimes b - b \otimes a - [a,b]$ for all $a, b \in \mathfrak{g}$. In the rest of the paper, we omit $\otimes$ symbol without confusion.

The following theorem named after Henri Poincaré, Garrett Birkhoff, and Ernst Witt discusses the basis of universal enveloping algebra.

THEOREM 6.1.7. *Let $\mathfrak{g}$ be a lie algebra over field K, and I is a total ordered set such that $\{x_i\}_{i\in I}$ is a set of basis of $\mathfrak{g}$. Then*

$$\{x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_t}^{n_t} | i_1 < i_2 < \dots < i_t \in I; n_1, n_2 \dots n_t \in \mathbb{N}\}$$

*is a set of basis of $U(\mathfrak{g})$.*

COROLLARY 6.1.8. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and $\mathfrak{n}_-, \mathfrak{n}_+, \mathfrak{h}$ defined as above. Then we have*

$$U(\mathfrak{g}) = U(\mathfrak{n}_-)U(\mathfrak{h})U(\mathfrak{n}_+).$$

Let $X$ be a finite dimensional irreducible representation of the simple Lie algebra $\mathfrak{g}$. We denote $X_\mu$ be the eigenspace corresponding to weight $\mu$. A elementary computation shows that $\mathfrak{g}_\alpha X_\mu \subset X_{\mu+\alpha}$. Since $X$ is finite dimensional, there exist a weight $\lambda$ such that $X_\lambda \neq 0$ and $U(n_+)X_\lambda = 0$. Because $X$ is irreducible, $X_\lambda$ is of dimension one. Moreover, we have $U_{n_-}X_\lambda = X$. The following lemma is useful for treating the denseness problem for base cases. Where we prove denseness only by looking at the eigenvalues of some mapping class.

LEMMA 6.1.9. *[22]Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and fix a Cartan subalgebra $\mathfrak{h}$. Let $D(\lambda)$ be the set of weights in the weight diagram of the irreducible representation V with highest weight $\lambda$. Consider $S(\lambda) = \lambda - D(\lambda)$ as a set of vectors in $\mathfrak{h}^*$. The elements of $S(\lambda)$ which can not be written as sum of other elements are roots of $\mathfrak{g}$.*

PROOF. Let $\Phi_+$ be the set of positive roots of $\mathfrak{g}$. $C = S(\lambda) \cap \Phi_+$, $C' = \Phi_+ - C$ and $v_\lambda$ be a vector in $V_\lambda$. By PBW theorem, $U(\mathfrak{n}_-)$ is spanned by monomials of form $f_{c_1} f_{c_2} \dots f_{c_m} f_{c'_1} f_{c'_2} \dots f_{c'_n}$ where $f_c \in \mathfrak{g}_{-c}$ and $c_i \in C$, $c'_j \in C'$. Notice $U(\mathfrak{n}_-)(v_\lambda) = V$, then $V$ is generated by monomials of form $f_{c_1} f_{c_2} \dots f_{c_m} f_{c'_1} f_{c'_2} \dots f_{c'_n} v_\lambda$.

According to the definition of set $C'$ and $D(\lambda)$, $\lambda - c'_j \notin D(\lambda)$. Thus $f_{c'_j} v_\lambda = 0$ for all $c'_j \in C'$, $V(\lambda)$ is spanned by $f_{c_1} f_{c_2} \dots f_{c_m} v_\lambda$. If weight $\lambda' \in D(\lambda)$ such that $\lambda - \lambda'$ indecomposable in $S(\lambda)$, then $\lambda'$ is either $\lambda$ or $v_{\lambda'} = f_{c_i} v_\lambda$ for some $f_{c_i} \in \mathfrak{g}_{-c_i}$ and $c_i \in C$. This proves $\lambda - \lambda_i = c_i$ is a positive root. □

### 6.2. Proof of Theorem 1.0.2

The proof is by induction on number of boundaries. The induction begins at $b = 4$.

LEMMA 6.2.1. *Let $A$ be a transcendental complex number. $\widetilde{PB_4}$ is dense in the algebraic group $SL(V_{0,4;a,b,c,d})$ if* $\dim V_{0,4;a,b,c,d} \geq 2$.

PROOF.

- The representation is irreducible.

  We adapt the notations of section 4.1 and claim the graph $C(V_{0,4;a,b,c,d}, < D_{\gamma_1} >, < D_{\gamma_2} >)$ is complete.

  We still have the change of basis formula (1) in section 4.1. To prove the $v_i$ have a non-zero component in $W_j$, need to show coefficient

$$\left\langle \begin{matrix} a & b & i \\ c & d & j \end{matrix} \right\rangle = \sum_{z=\max m_s}^{\min n_t} \frac{\prod_{s,t}[n_s - m_t]!}{[a]![b]![c]![d]![i]![j]!} \frac{(-1)^z [z+1]!}{\prod_s [n_s - z]! \prod_t [z - m_t]!} \neq 0. \tag{6.1}$$

  Notice that when consider it as a rational function of $A$, the degree is different among its summands. Thus, the leading term only appeared in one of the summands. Since $A$ transcendental, it is not a root of any rational function, so the coefficient is not zero. We proved the graph is complete and the representation is irreducible.

- The image is infinite.

  Consider two consecutive colors $v_i$ and $v_{i+2}$. Let $\lambda_i$ and $\lambda_{i+1}$ be the eigenvalues of $D_{\gamma_1}$ respectively. $\lambda_{i+2}/\lambda_i = -A^{4i+8}$. $A$ is not a root of unity. Thus, the image is infinite.

- The closure of the image is $SL(V_{0,4;a,b,c,d})$

  Assume the Zariski closure of the image is a semisimple Lie group $G$. the action of $D_{\gamma_1}$ have the following eigenvalue set under some central extension:

$$X = \{A^{i^2} | i \in [\max\{|a-b|, |c-d|\}, \min\{|a+b|, |c+d|\}], i \equiv a - b \mod 2\}$$

  Let's assume $i$ start at $i_0$ and end at $i_n$, There exist an element $a \in \mathfrak{g} = Lie(G)$, the eigenvalues for $a$ acting on $V_{0,4;a,b,c,d}$ is $\lambda_0 = C + i_0^2, \lambda_1 = C + (i_0+2)^2, \ldots, \lambda_n = C + (i_n)^2$ for some constant $C$. We claim $\mathfrak{g}$ is $\mathfrak{sl}_{n+1}$.

  Notice in the weight space $\mathfrak{h}^*$, $\lambda_0$ is the highest weight under some choice of simple roots. $\lambda_0 - \lambda_1$ satisfying the following properties:

(1) $\lambda_0 - \lambda_1$ is not repeated, that is, $\lambda_0 - \lambda_1 \neq \lambda_k - \lambda_j$ for any $(k, j) \neq (0, 1)$.

(2) $\lambda_0 - \lambda_1$ is indecomposable, that is $\lambda_0 - \lambda_1$ can not be written as positive integral linear combination of $\lambda_0 - \lambda_k$ for $k \neq 1$.

By Speyer's lemma, $\lambda_0 - \lambda_1$ is a root. Consider the action of $W_{\lambda_0 - \lambda_1}$ on the weight diagram. It send $\lambda_k$ to $\lambda_k$ plus (or minus) some copy of $\lambda_0 - \lambda_1$. Because $\lambda_0 - \lambda_1$ is not repeated, $W_{\lambda_0 - \lambda_1}$ interchanges $\lambda_0$ and $\lambda_1$ and fix all other $\lambda_k$.

Then we prove for all $k$, $\lambda_0 - \lambda_k$ is indecomposable. Assume otherwise, say $\lambda_0 - \lambda_k = \sum_t \alpha_t (\lambda_0 - \lambda_t)$. the decomposition is preserved by the Weyl group action. We have $\lambda_1 - \lambda_k = \sum_t \alpha_t (\lambda_1 - \lambda_t)$. These two equation together implies $\sum_t \alpha_t = 1$. Thus $\lambda_0 - \lambda_k$ is indecomposable for all $k$.

Use Speyer's lemma again, we know all $\lambda_0 - \lambda_k$ are roots. Consider the action of $W_{\lambda_0 - \lambda_k}$. It interchanges $\lambda_0$ and $\lambda_k$. Otherwise, $\lambda_0 - W_{\lambda_0 - \lambda_k}(\lambda_0)$ will be multiple of $\lambda_0 - \lambda_k$. The action also have to fix all other $\lambda_i$. Assume otherwise, it send $\lambda_i$ to $\lambda_j$. then $\lambda_0 - \lambda_j = \lambda_0 - \lambda_i + (\lambda_i - \lambda_j)$ which is equal to $\lambda_0 - \lambda_i$ plus multiple of $\lambda_0 - \lambda_k$, so the indecomposable condition is contradicted.

$W_{\lambda_0 - \lambda_k}$ generate $S_{n+1}$. The Lie algebra $\mathfrak{g}$ have an $(n+1)$-dimensional irreducible representation and has $S_{n+1}$ as a sub-quotient group of its Weyl group. $\mathfrak{g}$ have to be $\mathfrak{sl}_{n+1}$.

□

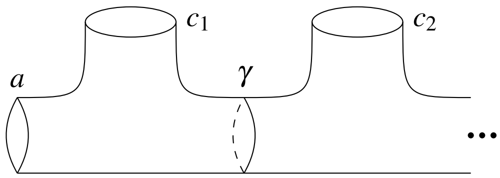
To apply Theorem 6.1.5, we need the following proposition about irreducibility.

PROPOSITION 6.2.2. *For A any transcendental number, $V_{0,b;\vec{c}}$ is irreducible as a representation of $PB_b$.*

PROOF. The proof is identical to Lemma 5.3.3. Notice that we find a element in the intersection of $V_i$ and $W_j$. In transcendental case, this means we have directed edges of both direction. □

The following lemma gives a criterion that for some specific decomposition, the components is not isomorphic nor dual to each other.

LEMMA 6.2.3. *Let A be a transcendental number, and $c_1, c_2 \ldots c_n$ be a sequence of fixed non-decreasing natural numbers and $n \geq 3$. Then there is no pair of elements in $\{V_{0,n+1;a,c_1,..c_n} | a \leq 2c_1\}$ are isomorphic or dual to each other.*




PROOF. Consider the Dehn twist around the curve $\gamma$ that bounds the first two boundaries. The eigenvalue set will be $E_a = \{A^{i^2} | \max\{c_n - \sum_{k=2}^{n-1} c_k, |a - c_1|\} \leq i \leq \min\{a + c_1, \sum_{k=2}^{n} c_k\}\}$ up to some central extension. For different $a$, we claim either the cardinality of the eigenvalue set will be different or the set of ratios of eigenvalues are different which is an invariant under central extensions.

Suppose $a, a' \leq c_1$ such that $E_a$ and $E'_a$ have the same cardinality. Then the interval for $i$ will be of the same length. To make the set of ratios of eigenvalues the same, the interval should start and end at the same place. Since we assumed $a, a' \leq 2c_1 \leq 2c_2$, so at least one of the boundary of the interval is determined by $a$ or $a'$. Thus we proved the claim. □

Now we can give the proof of Theorem 1.0.2 by induction:

PROOF. The case number of boundary components $b = 4$ is proved in Lemma 6.2.1.

Suppose the Zariski closure of the image of $PB_i$ in $PSL(V_{0,i;\vec{c}})$ is surjective for all $i \leq n$ and coloring $\vec{c}$. we prove $PB_{n+1}$ has a dense image in $PSL(V_{0,n+1;\vec{c}'})$ for any $\vec{c}'$.

Let $G$ be the Zariski closure of $PB_{n+1}$. $G$ contains elements of infinite order, so $G$ must be of positive dimension. $G$ is generated by 1-dimensional subgroups that densely generated by Dehn twists, so $G$ is connected.

Without loss of generality, assume $\vec{c}' = (c_0, c_1, c_2 \ldots c_n)$ such that $c_i \leq c_j$ if $i \leq j$. Then we can have a decomposition of $V_{0,n+1;\vec{c}'}$ by restricting to a subgroup $H = \mathrm{MCG}(S') \cong PB_n$.

$$V_{0,n+1;\vec{c}'} = \bigoplus_{i=c_1-c_0}^{c_0+c_1} V_{0,n;i,c_2,c_3,\ldots c_n}$$

According to Lemma 6.2.3, all summands are not isomorphic nor dual to each other.

To apply Theorem 6.1.5, we prove the dimension of the summands satisfying (2) of Theorem 6.1.5.

If the number of boundary component $b \geq 4$, the dimension of the vector spaces $V_{0,b;\vec{c}}$ have dimension 1 if and only if one of the colors is equal to the summation of colors on all other boundaries. In our case, $b \geq 5$, $n \geq 4$, $i \leq c_0 + c_1$. $\dim V_{0,n;i,c_2,c_3,\ldots c_n} = 1$ if and only if $i = c_n - c_{n-1} - \ldots - c_2$, so it happens at most once.

Next, we consider the 2-dimensional summands. We fix a uni-trivalent tree $\Gamma$ with $n$ boundary vertex colored by $\vec{c}$. If $\dim V_{0,n;\vec{c}} = 2$, then we have 2 admissible coloring for $\Gamma$. Fix an edge $e$ that have different colors $a$ and $a+2$ in the two different admissible colorings. Cut the tree at $e$, the tree $\Gamma$ split to $\Gamma_1$ and $\Gamma_2$, and coloring $e$ by $a$ and $a+2$ both give unique admissible coloring for $\Gamma_1$ and $\Gamma_2$. This implies $\Gamma_1$ and $\Gamma_2$ can only have 3 boundary vertices and $n = 4$. When $n = 4$, we can check by hand that at most for only one $i \leq c_0 + c_1$, $\dim V_{0,n;i,c_2,c_3,\ldots c_n} = 2$.

By Theorem 6.1.5, $G$ surjects onto $PSL(V_{0,n+1;\vec{c}'})$.

□

CHAPTER 7

# Open Problems and Future Work

In this section, we present some possible future projects and the relation between our work with other works. There are two interest directions for the future work.

(1) The denseness property for $SO(3)$-TQFT representations.

(2) The irreducibilty property for TQFT-representations comes from other modular tensor categories.

## 7.1. Denseness Problem at Root of Unity

The denseness problem at root of unity have been studied by Freedman, Larsen, Wang [**6**] [**16**] and Kuperberg [**15**]. In [**6**] and [**15**]. They studied the denseness property of the Jones representation. That is, the natural braid group $B_n$ action on the vector space $\mathrm{Hom}(X_1^{\otimes n}, X_c)$. In [**6**], they also proved the denseness of $SO(3)-TQFT$ representations when $p=5$ and $g+b\neq 1$. The reason that the denseness problem for those cases are tractable is because:

(1) The starting case is two dimensional. One can prove the surjectivity using the ADE classification.

(2) The eigenvalue set of certain Dehn twist contains only two elements.

In our case, the starting case is $g=0, b=4$ and $g=1, b=1$. As $p$ grow, the dimension of $V_{0,4;a,b,c,d}$ and $V_{1,1;a}$ will increase. Thus it is hopeless for us to rule out all possible subgroups. Another method is by looking at eigenvalue sets of some certain mapping classes. Since the eigenvalue of the Dehn twists are all powers of $A$, thus of finite order. According to the classification Theorem 4.1.3 of mapping classes, we should look at the pseudo-Anosov mapping classes. Computing the eigenvalues of these operators are still open. Jorgen Andersen, Gregor Masbaum and Kenji Ueno suggested the following conjecture.

CONJECTURE 1. *[1] A Pseudo-Anosov mapping class is represented in TQFT by matrices of infinite order, except for finitely many values of level r.*

So far, Gregor Masbaum computed the eigenvalue of $D_{\gamma_1}^{-1} D_{\gamma_2}$ acting on $V_{0,4;1,1,1,3}$. The eigenvalues are not root of unity, which proves the denseness of the representation in this case.

In [**14**], Julien Korinman have the following theorem about the finiteness of the image of the mapping class group of one-holed torus.

THEOREM 7.1.1. *Assume $A$ is a $4r$-th root of unity and the boundary is colored $2c$. For $r \geq 4$, we have:*

(1) *If $2c = r-2$ or $2c = r-3$ and $r$ is odd, then the representation has finite image.*
(2) *If $2c \leq r-3$ and $r$ is an odd prime, then the representation has infinite image.*

Korinman's method relies on the irreducibility of the representation. That is, one may verify the finiteness condition for more representations with our irreducibility result. However, his method does not provide information of eigenvalues of any mapping classes.

## 7.2. TQFT from Other MTCs

Another direction of this program is to look at the image of the mapping class group representation where the TQFT comes from other MTCs.

Our method may partly generalize to those MTCs come from other quantum groups. Especially for quantum groups of rank 2, Greg Kuperberg invented a graphical calculus called spiders plays the same role as the skeins in this paper.

Although the first few examples of TQFTs comes from Lie algebras. There are other interesting MTCs that are not from quantum groups.

One can construct a MTC from a spherical fusion category by taking its quantum double(see chapter 7 and 8 of [**4**]). The braided structure of the MTC comes from the associator of the fusion category. Paul Gustafson [**10**] studied the TQFT representation comes from the quantum double of $\mathbf{Vec}_G^{\omega}$, where $G$ is a finite group and $\omega$ is a 3-cocycle of $G$. $\mathbf{Vec}_G^{\omega}$ is the category of $G$-graded vector space with a twisted associative isomorphism. After taking the skeleton, the associative natural isomorphism is a 3-cocycle.

Gustafson proved the following theorem regarding the image of the TQFT representation:

THEOREM 7.2.1. *[**10**] The image of any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, compact surface with or without boundary is finite. In particular, the image of any such braid group representation is finite.*

# References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