Discrete isometry groups of symmetric spaces of noncompact type

By

Subhadip Dey

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Professor Michael Kapovich, Chair

Professor Joel Hass

Professor Eugene Gorsky

Committee in Charge

2020

To my family

Contents

Abstract

This dissertation contains three main results about discrete isometry groups Γ of symmetric spaces X of noncompact type. Two of these results are about the *critical exponent*, which is a fundamental numerical invariant that one assigns to the action $\Gamma \curvearrowright X$, which quantifies the exponential growth of Γ-orbits in X. One of these two results (Theorem 2.1.1) states that if X is the complex-hyperbolic n-space and Γ is a torsion-free, *convex-cocompact* isometry group of X whose critical exponent lies below a certain optimal bound, then the quotient $\Gamma\backslash X$ is a Stein manifold. The other one (Theorem 2.2.8) states that, if X is any symmetric space of *noncompact type* and Γ is a discrete group of isometries of X satisfying the *Anosov condition* (a higher rank generalization of convex-cocompactness), then the (Finsler) critical exponent of Γ is equal to the *Hausdorff dimension* of the *flag limit set* of Γ, $\Lambda(\Gamma)$. Along the way, we also prove that $\Lambda(\Gamma)$ supports a Γ-invariant *conformal density* (called the *Patterson-Sullivan density*), and that the density is ergodic and unique conformal density. These results generalize D. Sullivan's classical results on convex-cocompact Kleinian groups.

Our last result (Theorem 2.2.4, also see Theorem 5.3.1) gives a generalization of the classical Klein combination theorem in the setting of discrete isometry groups satisfying the Anosov condition. The result states that if Γ_1 and Γ_2 are two such groups, then under suitable conditions, the isometry group that they generate is again of the same type, i.e., a discrete isometry group having the Anosov property, and isomorphic to the free product $\Gamma_1 * \Gamma_2$.

Acknowledgments

First of all, I would like to express my sincerest gratitude to my advisor Misha Kapovich, for teaching patiently since the beginning of my graduate studies, for giving me exciting problems to work on, for sharing ideas, criticisms, and advice. Thank you for being a wonderful guide. Special thanks to Misha's daughter, Esther, for suggesting that I should include more pictures; after all, the main topic here is geometry!

Many professors here in our math department have enriched my Ph.D. experience. I am particularly grateful to Eugene Gorsky, Joel Hass, Greg Kuperberg, and Jennifer Schultens, for spending their time with me in many mathematical conversations. I spent great times at Greg's weekly lunches, where discussions about everything were on the table (alongside food, of course), they will be missed! I thank Alexander Soshnikov for making the first two years of my grad study smoother by relieving me from repeating some of the compulsory courses which I took before joining UC Davis. I thank Rohit Thomas for being a great teaching mentor and writing a recommendation letter for me.

I would also like to thank some professors outside our department. I am grateful to Gerard Besson and Alex Furman for writing recommendation letters, to Alex Furman and Sara Maloni for having enriching mathematical discussions with me and providing me with various opportunities. I spent two memorable weeks in a summer school at MSRI, and I thank Alex for the opportunity; conversations with him have inspired me a lot.

I want to thank some of my former teachers, who would be very happy to see me where I am today. I am grateful to Koushik Hazra, Uttam Hazra, Susanta Roy and Parthasarathi Samanta, teachers from my school and high school days, for appreciating my fondness in mathematics and encouraging me to do more. I thank Jishnu Gupta Biswas and Aniruddha Naolekar, professors from my undergraduate and master's years, for teaching me geometry and topology and encouraging me to come to UC Davis.

After coming to Davis, I am fortunate to have made some great friends. Anubhab Gupta has stood by me during my most difficult times, and he has always been one phone call away. Sneha Chakraborty has made me smile when I felt sad. Indrajit Jana (the only mathematician among these people) has encouraged me to do non-math activities; I wish I had learned some tabla and piano from him. I miss the delicious dishes made by Abhishek Santra and his wife, Tamashree Ghosh. Thank you for filling my life at Davis with fun, laughter, smiles, and beautiful memories.

I appreciate my friends in the math department, Priya Kshirsagar, Beibei Liu, Yanwen Luo, Neetal Neel, Michael Ponce, Wencin Poh, and Maria Trnkova, for many interesting conversations about math and life. I want to thank our departmental staff, especially Tina Denena and Sarah Driver, for making my Ph.D. journey easier.

I feel lucky to have old friends like Subhajit Jana, Shubhodip Mondal, Susnata Mondal, who have visited me in Davis. Priyadip Mondal, my dear friend from undergraduate years, has visited me twice and has kept close contact with me all the time. I thank you all.

Lastly, I offer my heartfelt indebtedness to my family. My parents, Dipali De and Malay Kumar De, have given me guidance and inspiration when I am lost. My wife, Sayantani Sarkar, has given me unconditional friendship and support. I cannot thank them enough for all the endless love they have given to me and all sacrifices they've had to make for me. It is my pleasure to dedicate my thesis to them.

CHAPTER 1

Introduction

Symmetric spaces are the most beautiful examples of Riemannian manifolds with rich and fascinating geometry. The simplest examples of such spaces, which are familiar to many of us, are the euclidean plane, the sphere, and the hyperbolic plane. Each symmetric space has its unique geometry; for example, the euclidean pane is flat, and the sphere and hyperbolic plane respectively have constant positive and negative *Gaussian curvature*. There are many other (in fact, infinite number of) examples of symmetric spaces, and we will come across some of these in this thesis.

Symmetric spaces were introduced by Élie Cartan in 1926, who also, subsequently, classified all of them. We do not discuss his precise classification here, but we can get some flavor of this by putting all the *irreducible*[1] symmetric spaces into three broad categories: The first category is the simplest; it consists of the euclidean spaces. Of course, the euclidean plane lives here. The second category is called *of compact type* and consists of, well, the compact ones. The sphere belongs to this category. The third category, called *of noncompact type*, contains the rest of them, and the hyperbolic plane belongs here. The symmetric spaces which belong to the last category also have a geometric characterization: Their curvature is nonpositive but not identically zero.

In this thesis, we only consider the symmetric spaces which belong to the third category, i.e., we only discuss *symmetric spaces of noncompact type*. For convenience, we often drop the words "of noncompact type." More generally, one considers *locally* symmetric spaces[2]: These are complete Riemannian manifolds whose *local geometries* are modeled on the geometry of a symmetric space. These manifolds are also very special since many of them naturally arise, for instance, as parameter spaces (also called *moduli spaces*) of many interesting objects in mathematics. Each locally symmetric space M can be expressed as

$$M = \Gamma \backslash X,$$

[1]A symmetric space (more generally, a simply-connected Riemannian manifold) X is called *irreducible* if X cannot be written as a (Riemannian) product $X_1 \times X_2$ where both X_1 and X_2 have dimensions ≥ 1. Note that, it is enough to classify the irreducible symmetric spaces since any other symmetric space arises as products of the irreducible ones.

[2]To distinguish, the former ones are called *globally* symmetric spaces.

where X, the universal cover of M, is a globally symmetric space and $\Gamma \cong \pi_1(M)$ is a *discrete group of isometries* of X. The geometry of M strongly ties to the fundamental group, Γ. Quite often, the geometry of M is completely determined by Γ; such phenomena are commonly termed as "rigidity." A beautiful (and also the earliest) example of such is the *Mostow rigidity theorem* (1968): If M is a closed hyperbolic manifold of dimension ≥ 3 and N is another closed hyperbolic manifold such that their fundamental groups are isomorphic, then M and N are the *same*, i.e., there exists a distance preserving bijection (an *isometry*) between M and N. Therefore, the study of discrete isometry groups is very important (and fascinating in its own right), which is the main objective of this thesis.

This thesis contains three results on discrete isometry groups. In Chapter 2, we give precise formulations of these results (see Theorems 2.1.1, 2.2.4, and 2.2.8). The rest is divided as follows: In Chapter 3, we prove Theorem 2.1.1. This chapter is also independent of the other chapters. In Chapter 4, we discuss some preliminaries needed for the proofs of the other two theorems. We prove Theorem 2.2.4 in Chapter 5, and Theorem 2.2.8 in Chapter 6. These final two chapters are also mostly independent of each other and can be read in any order.

CHAPTER 2

Main results

In this chapter, we discuss our main results.

2.1. Discrete isometry groups of complex-hyperbolic spaces

The n-dimensional complex-hyperbolic space, $\mathbb{H}^n_{\mathbb{C}}$, is a complex analog of the n-dimensional real-hyperbolic space, $\mathbb{H}^n$. In the unit ball model, this geometry is described by a complete Kähler metric g, called the Bergman metric, on the unit ball $\boldsymbol{B}^n \subset \mathbb{C}^n$, which has constant negative holomorphic sectional curvature. The real part of g defines a Riemannian metric on $\mathbb{H}^n_{\mathbb{C}}$ which, after normalization, has variable sectional curvature lying in $[-4,-1]$. See Section 3.1 for details.

Let Γ be a discrete group of holomorphic isometries acting on $\mathbb{H}^n_{\mathbb{C}}$, i.e. Γ is a discrete subgroup of the Lie group $\mathrm{Isom}_0(\mathbb{H}^n_{\mathbb{C}}) = \mathrm{PU}(n,1)$. Fix a point $x \in \mathbb{H}^n_{\mathbb{C}}$. The *critical exponent* $\delta(\Gamma)$ of Γ is defined by

$$\delta(\Gamma) := \inf \left\{ s : \sum_{\gamma\in\Gamma} e^{-s\cdot d_{\mathbb{H}^n_{\mathbb{C}}}(x,\gamma x)} < \infty \right\}.$$

This number is also equal to

$$\limsup_{n\to\infty} \frac{\log N(R,x)}{R},$$

where $N(R,x)$ counts the number of points in the orbit Γx lying in the closed ball of radius R centered at x, i.e., $N(R,x) = \mathrm{card}\{y \in \Gamma x : d_{\mathbb{H}^n_{\mathbb{C}}}(x,y) \le R\}$. In other words, the critical exponent measures the rate of exponential growth of the Γ-orbit $\Gamma x \subset \mathbb{H}^n_{\mathbb{C}}$. It is a fact that the number $\delta(\Gamma)$ does not depend on x and, hence, is an invariant of the action $\Gamma \curvearrowright \mathbb{H}^n_{\mathbb{C}}$. It is also a fact that $\delta(\Gamma)$ equals the Hausdorff dimension of the *conical limit set* of Γ, see [**Cor90**] and [**CI99**].

Our result stated below demonstrates an interplay between the theory of discrete subgroups of $\mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$ and the holomorphic function theory of *complex-hyperbolic manifolds* (manifolds of the form $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$). More precisely, we prove that if Γ has critical exponent below a certain threshold, then the complex manifold $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ is rich in holomorphic functions.

THEOREM 2.1.1. *Let $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$ be a convex-cocompact, torsion-free discrete subgroup such that $\delta(\Gamma) < 2$. Then $M_\Gamma = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ is Stein.*

This theorem appears in [**DK20**]. The proof of the theorem is presented in Chapter 3. The meaning of the term *convex-cocompact* is that Γ acts cocompactly (i.e., with compact quotient) on a Γ-invariant, nonempty, closed, convex subset of $\mathbb{H}^n_{\mathbb{C}}$ or, equivalently, the *core* of M_Γ is a nonempty compact set.

The condition on the critical exponent in the above theorem is sharp since, for a *complex Fuchsian subgroup* $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$, $\delta(\Gamma) = 2$, but M_Γ is non-Stein because the convex core of M_Γ is a compact complex curve, see Example 3.1.2. On the other hand, if Γ is a torsion-free *real Fuchsian subgroup* or a small deformation of such (i.e., *real quasi-Fuchsian subgroups*, see Example 3.1.1), then Γ satisfies the condition of the above theorem.

Theorem 2.1.1 has limited intersection with [**BS76**, Prop. 6.4]: It states that, if $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$ is a discrete subgroup that stabilizes a totally-real totally-geodesic subspace of dimension n in $\mathbb{H}^n_{\mathbb{C}}$ and acts on it cocompactly (in particular, $\delta(\Gamma) = n-1$), then M_Γ is a Stein manifold. More generally, [**Che13**, Prop. 1.6] claims that same result holds if one replaces the words "acts on it cocompactly" by "injectivity radius of M_Γ is positive."

Yue [**Yue99**] made several conjectures about convex-cocompact complex-hyperbolic Kleinian groups $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$. One such conjecture states that if $\delta(\Gamma) > n-1$, then M_Γ is not Stein. Theorem 2.1.1 disproves this conjecture in dimension $n = 2$ since counter-examples are given by real quasi-Fuchsian subgroups of $\mathrm{Isom}\left(\mathbb{H}^2_{\mathbb{C}}\right)$; in this case, $\delta(\Gamma) > 1$, but M_Γ is Stein. For $n \geq 3$, the conjecture is unknown.

In Chapter 3, we also discuss a number of interesting conjectures related to Theorem 2.1.1.

2.2. Anosov subgroups of semisimple Lie groups

Convex-cocompact Kleinian groups form a rich class of discrete groups with nice geometrical, topological, and dynamical properties. Recall from the previous section that a discrete subgroup $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n) = \mathrm{PO}(n,1)$ is called convex-cocompact if Γ acts cocompactly on a nonempty Γ-invariant closed convex subset $C \subset \mathbb{H}^n$. In rank-one simple Lie groups, convex-cocompact subgroups generalize *uniform lattices*[1].

[1]A discrete subgroup Γ of a Lie group G is called a *lattice* if $\Gamma \backslash G$ supports a nontrivial G-invariant measure of finite volume. Moreover, Γ is called a *uniform lattice* if $\Gamma \backslash G$ is compact.

Lattices in semisimple Lie groups G of *noncompact type* is a widely studied topic in mathematics with rich structure theory. In contrast, infinite covolume discrete subgroups are still poorly understood. A difficulty arose from the lack of the notion of a flexible class of discrete subgroups of G that has nice geometrical and dynamical properties. In rank-one, such a class is offered by convex-cocompact (or, more generally, *geometrically finite*) subgroups. But, in higher rank, a naive generalization of "convex-cocompactness" turned out to be too restrictive: Let X be a *symmetric space* of *noncompact type*, $\operatorname{rank}(X) \geq 2$, and $G = \operatorname{Isom}(X)$ be the isometry group of X. Assume that X has no rank-one de Rham factor (e.g., X cannot be written as a Riemannian product $X_1 \times \mathbb{H}^n$). If a discrete subgroup $\Gamma < G$ acts cocompactly on a Γ-invariant, nonempty, closed, convex subset $C \subset X$, then either Γ acts cocompactly on X or Γ preserves a proper symmetric subspace of X [**KL06**]. Equivalently, such Γ can only arise as a representation of a uniform lattice into G. In particular, Γ cannot be isomorphic to, for instance, a free group F_n.

The class of *Anosov subgroups* of G gives a different higher rank generalization of convex-cocompact Kleinian groups that is flexible enough to include a wide range of discrete subgroups of G (e.g., free groups, surface groups, etc.). This notion was introduced by Labourie [**Lab06**] using *Anosov flows* in his study of the *Hitchin component* of the character variety

$$\operatorname{Rep}(\pi_1(\Sigma), G) := \operatorname{Hom}(\pi_1(\Sigma), G)/G,$$

where $G = \operatorname{PSL}(n, \mathbb{R})$, $n > 2$, and Σ is a compact surface of negative Euler characteristic. He proved that the Hitchin component consists entirely of discrete and faithful (Anosov) representations[2] $\pi_1(\Sigma) \to G$. Guichard and Weinhard [**GW12**] extended Labourie's class to include representations of word-hyperbolic groups. Later Kapovich, Leeb, and Porti [**KLP14, KLP18a**] gave several geometrical and dynamical characterizations of Anosov subgroups. Many different characterizations of Anosov subgroups are now known, e.g., see [**GGKW17**,**KLP18b**,**KL18b**,**BPS19**]. In Chapter 4, we review some of these characterizations.

For the time being, we give the following definition of Anosov subgroups of $G = \operatorname{SL}(n, \mathbb{R})$. Every element $g \in G$ has a *singular value decomposition*. The singular values of g,

$$\sigma_1(g) \geq \cdots \geq \sigma_n(g) > 0, \tag{2.1}$$

[2]Components of a character variety consisting of discrete and faithful representations are called *(higher) Teichmüller spaces*, see [**Wie18**].

can be geometrically interpreted as the lengths of the semiaxes of the ellipsoid $g(\boldsymbol{B}^n)$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map and $\boldsymbol{B}^n \subset \mathbb{R}^n$ is the closed unit ball.

DEFINITION 2.2.1 (Anosov subgroups). Let Γ be a finitely generated group and $|\cdot|_S : \Gamma \to \mathbb{N} \cup \{0\}$ be the word-length function with respect to a finite generating set S. A representation $\rho : \Gamma \to G$ is called *P_k-Anosov* ($k = 1, \ldots, n-1$) if there exists constants $L \geq 1, A \geq 0$, such that

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \geq L^{-1}|\gamma|_S - A, \quad \forall \gamma \in \Gamma. \tag{2.2}$$

The image $\rho(\Gamma)$ is called a *P_k-Anosov subgroup* of G.

REMARK. (1) The above definition[3] was introduced in [**BPS19**] where it was proved to be equivalent to Labourie's definition of Anosov representations. Note that (in contrast with Labourie's definition) Γ is not a priori assumed to be word-hyperbolic. In fact, the word-hyperbolicity of Γ is a consequence of [**KLP18b**, Thm. 4].

(2) It follows from (2.2) that, for a P_k-Anosov representation $\rho : \Gamma \to G$, every element $\gamma \in \Gamma$ of sufficiently large word-length satisfies $\sigma_k(\rho(\gamma)) > \sigma_{k+1}(\rho(\gamma))$. As a consequence, $\rho(\gamma)$ is nontrivial. In particular, Anosov representations ρ have finite kernel and discrete image.

(3) When $n = 2$, i.e. $G = \mathrm{SL}(2, \mathbb{R})$, then G acts on the hyperbolic plane $\mathbb{H}^2$ via fractional linear transformations on the upper half-plane model of $\mathbb{H}^2$. The action is transitive with point stabilizers being isomorphic to $K = \mathrm{SO}(2, \mathbb{R})$, i.e. $\mathbb{H}^2$ can be identified with G/K. The function

$$d_1(g_1K, g_2K) := \log\left(\frac{\sigma_1(g_1^{-1}g_2)}{\sigma_2(g_1^{-1}g_2)}\right), \quad g_1, g_2 \in G,$$

is the distance function of a G-invariant Riemannian metric (the hyperbolic metric) on $\mathbb{H}^2 = G/K$. Let $\Gamma < G$ be a P_1-Anosov subgroup. Then, the inequality (2.2) means that Γ is *undistorted*, i.e. the map $\Gamma \to \mathbb{H}^2$, $\gamma \mapsto \gamma K$, is a quasiisometric embedding. This is equivalent to Γ being convex-cocompact. In general, Anosov property in rank-one is equivalent to convex-cocompactness.

(4) For $n \geq 3$, $k \in \{1, \ldots, n-1\}$, the function

$$d_k(g_1K, g_2K) := \log\left(\frac{\sigma_k(g_1^{-1}g_2)}{\sigma_{k+1}(g_1^{-1}g_2)}\right), \quad g_1, g_2 \in G = \mathrm{SL}(n, \mathbb{R}),$$

[3] This definition is a special case of *URU* subgroups (Definition 4.4.11).

also defines a G-invariant (asymmetric) distance function on G/K. However, d is the distance function of a Finsler (which is not a Riemannian) metric. These metrics play central role in Chapter 6.

The theory (as well as the results below) of Anosov subgroups extends to other semisimple Lie groups of noncompact type with mild restrictions. In the sequel, G denotes a connected semisimple Lie group of noncompact type, X denotes the symmetric space of G, and P denotes a parabolic subgroup of G which is conjugate to its opposite. For such P we have the class of *P-Anosov subgroups*. See Chapter 4 for details.

A combination theorem. The combination theorems in geometric group theory provide tools to construct new groups with "nice" geometric properties out of old ones. The classical combination theorem of Klein [**Kle83**] states that under certain assumptions, the group $\langle \Gamma_1, \Gamma_2 \rangle$ generated by two Kleinian groups Γ_1 and Γ_2 is again Kleinian, and is naturally isomorphic to the free product $\Gamma_1 * \Gamma_2$. In a series of articles [**Mas65**, **Mas68**, **Mas71**, **Mas93**], Maskit generalized the Klein combination theorem to amalgamated free products and HNN extensions. See Maskit's book [**Mas88**] for a detailed account of these results.

These so called "Klein-Maskit combination theorems" have been generalized to the geometrically finite groups by several mathematicians. For instance, in [**BC08**], Baker and Cooper proved the following

THEOREM 2.2.2 (Virtual amalgam theorem, [**BC08**]). *If Γ_1 and Γ_2 are two geometrically finite subgroups of* $\operatorname{Isom}(\mathbb{H}^n)$ *which have compatible parabolic subgroups, and if $H = \Gamma_1 \cap \Gamma_2$ is separable in Γ_1 and Γ_2, then there exist finite index subgroups Γ_1' and Γ_2' of Γ_1 and Γ_2, respectively, containing H such that the group $\langle \Gamma_1', \Gamma_2' \rangle$ generated by Γ_1' and Γ_2' is geometrically finite, and is naturally isomorphic to the amalgam $\Gamma_1' *_H \Gamma_2'$.*

When Γ_1 and Γ_2 intersect trivially, the "compatibility condition" in the above theorem simply means that the limit sets of Γ_1 and Γ_2 in $\partial_\infty \mathbb{H}^n$ are disjoint. Since this case is the most relevant to us, we state it separately.

COROLLARY 2.2.3. *If Γ_1 and Γ_2 are two geometrically finite subgroups of* $\operatorname{Isom}(\mathbb{H}^n)$ *with disjoint limit sets in $\partial_\infty \mathbb{H}^n$, then there exist finite index subgroups Γ_1' and Γ_2' of Γ_1 and Γ_2, respectively, such* *that the group $\langle\Gamma_1', \Gamma_2'\rangle$ generated by Γ_1' and Γ_2' is geometrically finite and is naturally isomorphic $\Gamma_1' * \Gamma_2'$.*

The following theorem presents an analogue of this result in the setting of Anosov subgroups. Recall our convention that G is a semisimple Lie group of noncompact type and P is a parabolic subgroup.

THEOREM 2.2.4 (Combination theorem). *Let $\Gamma_1, \ldots, \Gamma_n$ be pairwise antipodal, residually finite*[4] *P-Anosov subgroups of G. Then, there exist finite index subgroups Γ_i' of Γ_i, for $i = 1, \ldots, n$, such that the subgroup $\langle\Gamma_1', \ldots, \Gamma_n'\rangle$ generated by $\Gamma_1', \ldots, \Gamma_n'$ in G is P-Anosov, and is naturally isomorphic to the free product $\Gamma_1' * \cdots * \Gamma_n'$.*

This result and its proof appears in [**DKL19**]. We present this proof in Chapter 5. The undefined term "antipodal" in the statement will be made precise later in Definition 5.2.20: This condition replaces the disjointness of the limit sets in Corollary 2.2.3. Moreover, the geometric finiteness in Corollary 2.2.3 is replaced by the Anosov condition.

Kapovich, Leeb, and Porti [**KLP14**] used the local-to-global principle for *Morse quasigeodesics* to construct (free) *Morse-Schottky subgroups* of semisimple Lie groups (cf. also [**Ben97**]):

THEOREM 2.2.5 (Morse-Schottky subgroups, [**KLP14**, Theorem 7.40]). *Suppose that $g_1, ..., g_n$ are hyperbolic isometries of a symmetric space $X = G/K$ of noncompact type, whose repelling/attracting points in the flag-manifold G/P are pairwise antipodal. Then for all sufficiently large N, the subgroup of G generated by $g_1^N, ..., g_n^N$ is P-Anosov and free of rank n.*

While Theorem 2.2.4 contains this result as a special case when the subgroups $\Gamma_1, ..., \Gamma_n$ are cyclic, our proof of the theorem involves extending their methods that work with arbitrary Anosov subgroups.

We note that the traditional statements of the Klein-Maskit combination theorems are *sharper* in the sense that under suitable assumption one does not need to pass to finite index subgroups. Therefore, it is natural to expect that Theorem 2.2.4 should have a version that is more aligned with the original form of the Klein-Maskit combination theorems. The following is a reasonable combination conjecture in the setting of Anosov subgroups:

[4] It suffices to assume that each Γ_i has trivial intersection with the center of G.

CONJECTURE 2.2.6 ([**DKL19**, Conj. 5.3]). *Let $A_1, ..., A_n \subset G/P$ be nonempty disjoint compact subsets such that any two distinct elements of $A := \bigcup_{i=1}^{n} A_i$ are antipodal. Suppose that $\Gamma_1, ..., \Gamma_n$ are P-Anosov subgroups of G such that for all $i = 1, ..., n$ and all $\gamma \in \Gamma_i - \{1\}$ we have $\gamma(A - A_i) \subset \operatorname{int}(A_i)$. Then the subgroup Γ of G generated by $\Gamma_1, ..., \Gamma_n$ is P-Anosov.*

Note that under the above assumptions, Γ is naturally isomorphic to the free product $\Gamma_1 * \cdots * \Gamma_n$, see, e.g., [**Tit72**].

Patterson-Sullivan theory. Let $\Gamma < \operatorname{Isom}(\mathbb{H}^n)$ be a Kleinian group. Recall the notion of the critical exponent $\delta(\Gamma)$ from Section 2.1. This is a fundamental numerical invariant associated with Γ that measures the asymptotic growth rate of Γ-orbits in $\mathbb{H}^n$.

In an influential paper [**Sul79**], extending pioneering work by Patterson [**Pat76**] on Fuchsian groups, Sullivan proved the following relation between the *Hausdorff dimension* of the limit set $\Lambda(\Gamma)$ of Γ and its critical exponent.

THEOREM 2.2.7 (Sullivan [**Sul79**, Thm. 8]). *Let Γ be a convex-cocompact subgroup of the isometry group of $\mathbb{H}^n$. Then the critical exponent $\delta(\Gamma)$ equals to the Hausdorff dimension of $\Lambda(\Gamma)$.*

Many generalizations of this result in different directions followed afterwards. For instance, Sullivan generalized this theorem for geometrically finite Kleinian groups [**Sul84**]. An important ingredient of Sullivan's proof of this theorem is the existence of a probability measure on $\Lambda(\Gamma)$ that changes *conformally* under the Γ-action. The construction of such measure goes back to Patterson's original idea in [**Pat76**]. Measures of this type (resp. a family of "well-behaved" measures) are commonly referred as *Patterson–Sullivan measures* (resp. *densities*). We refer to Nicholls' book [**Nic89**] for a self-contained exposition on these results.

Moreover, Corlette [**Cor90**] and Corlette-Iozzi [**CI99**] extended Theorem 2.2.7 for geometrically finite groups of isometries of rank-one symmetric spaces, and Bishop-Jones [**BJ97**] extended these results to arbitrary discrete isometry groups of rank-one symmetric spaces. Yue [**Yue96**] and Ledrappier [**Led95**] studied the case of Hadamard spaces of negative curvature.

In higher rank, there has been a considerable amount of work done by various people, starting with the early works of Bishop-Steger [**BS93**] and Burger [**Bur93**] on products of hyperbolic spaces. Later, Albuquerque [**Alb99**], Quint [**Qui02a**,**Qui02b**], and Link [**Lin04**,**Lin06**] studied Patterson-Sullivan measures associated with discrete isometry groups of general higher rank symmetric spaces. However, most of these works were limited to the case of *Zariski-dense* discrete subgroups.

In Chapter 6, we develop a Patterson-Sullivan theory for Anosov subgroups (not necessarily Zariski-dense). Using the classical construction of Patterson, we define and study a notion of Patterson-Sullivan measures in the setting of Anosov subgroups, and extend several results of [**Sul79**] in this setting. This work appears in [**DK19**]. Below we summarize these results.

Following [**KL18b**, Sec. 5], we consider an appropriate polyhedral Finsler pseudometric d_{F} on $X = G/K$ (see Section 6.1 for details). Let $\delta_{\mathrm{F}}(\Gamma)$ denote the *Finsler critical exponent* of Γ, i.e.

$$\delta_{\mathrm{F}}(\Gamma) := \inf\left\{ s : \sum_{\gamma\in\Gamma} e^{-s\cdot d_{\mathrm{F}}(x,\gamma x)} < \infty \right\}, \quad x \in X.$$

As before, this number is independent of $x \in X$.

THEOREM 2.2.8. *Let Γ be a nonelementary P-Anosov subgroup of G. Then the Patterson–Sullivan density μ on the flag limit set*[5] $\Lambda(\Gamma) \subset G/P$ *is the unique (up to rescaling) Γ-invariant conformal density. Moreover,*

(i) The density μ is non-atomic and $\delta_{\mathrm{F}}(\Gamma)$-dimensional.

(ii) The support of μ is $\Lambda(\Gamma)$ and Γ acts on $\Lambda(\Gamma)$ ergodically with respect to μ.

(iii) The critical exponent $\delta_{\mathrm{F}}(\Gamma)$ is positive and finite.

(iv) The Poincaré series of Γ diverges at $\delta_{\mathrm{F}}(\Gamma)$. Equivalently, Γ has Finsler divergence type.

(v) The $\delta_{\mathrm{F}}(\Gamma)$-dimensional Hausdorff measure on $\Lambda(\Gamma)$ with respect to a Gromov premetric is a member of a Γ-invariant conformal density. In particular, the Hausdorff dimension of $\Lambda(\Gamma)$ is $\delta_{\mathrm{F}}(\Gamma)$.

The notions of *conformal density* and its *dimension*, *divergence type*, and *Gromov premetric* are defined in Chapter 6 (see "Organization of this chapter" in the introduction of Chapter 6). Note that Theorem 2.2.8(v) generalizes Theorem 2.2.7. In Section 6.9, we also give some applications of our theorem.

[5] See Definition 4.4.2.

CHAPTER 3

Stein property of complex hyperbolic manifolds

This chapter is based on [**DK20**]. The goal of this chapter is to give a proof of

THEOREM 2.1.1. *Let* $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$ *be a convex-cocompact, torsion-free discrete subgroup such that* $\delta(\Gamma) < 2$. *Then* $M_\Gamma = \Gamma\backslash \boldsymbol{B}^n$ *is Stein.*

The main ingredients in the proof of Theorem 2.1.1 are Proposition 3.2.7 and Theorem 3.3.3. The condition "convex-cocompact" is only used in Proposition 3.2.7, whereas Theorem 3.3.3 holds for any torsion-free discrete subgroup $\Gamma < \mathrm{Isom}\,(\mathbb{H}^n_{\mathbb{C}})$ satisfying $\delta(\Gamma) < 2$.

CONJECTURE 3.0.1. *Theorem 2.1.1 holds if we omit the "convex-cocompact" assumption on* Γ.

In section 3.4 we discuss other conjectural generalizations of Theorem 2.1.1 and supporting results.

3.1. Preliminaries of complex-hyperbolic spaces

Here we recall some definitions and basic facts about the n-dimensional complex hyperbolic space, we refer to [**Gol99**] for details. See also [**Kap19**].

Consider the n-dimensional complex vector space $\mathbb{C}^{n+1}$ equipped with the pseudo-hermitian bilinear form

$$\langle z, w\rangle = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k \tag{3.1}$$

and define the quadratic form $q(z)$ of signature $(n, 1)$ by $q(z) := \langle z, z\rangle$. Then q defines the *negative light cone* $V_- := \{z : q(z) < 0\} \subset \mathbb{C}^{n+1}$. The projection of V_- in the projectivization of $\mathbb{C}^{n+1}$, $\mathbb{P}^n$, is an open ball which we denote by $\boldsymbol{B}^n$.

The tangent space $T_{[z]}\mathbb{P}^n$ is naturally identified with $z^\perp$, the orthogonal complement of $\mathbb{C}z$ in V, taken with respect to $\langle\cdot,\cdot\rangle$. If $z \in V_-$, then the restriction of q to $z^\perp$ is positive-definite, hence, $\langle\cdot,\cdot\rangle$ project to a hermitian metric h (also denoted $\langle\cdot,\cdot\rangle$) on $\boldsymbol{B}^n$. The *complex hyperbolic n-space* $\mathbb{H}^n_{\mathbb{C}}$ is $\boldsymbol{B}^n$ equipped with the hermitian metric h. The boundary $\partial\boldsymbol{B}^n$ of $\boldsymbol{B}^n$ in $\mathbb{P}^n$ gives a natural compactification of $\boldsymbol{B}^n$.

In this chapter, we usually denote the complex hyperbolic n-space by $\boldsymbol{B}^n$. The real part of the hermitian metric h defines a Riemannian metric g on $\boldsymbol{B}^n$. The sectional curvature of g varies between -4 and -1. We denote the distance function on $\boldsymbol{B}^n$ by d. The distance function satisfies

$$\cosh^2(d(0,z)) = \frac{1}{1-|z|^2}. \tag{3.2}$$

In the above formula, we are writing points in $\boldsymbol{B}^n$ in affine coordinates, i.e., a point $[(1,z)] \in \boldsymbol{B}^n$ is written as z.

A real linear subspace $W \subset \mathbb{C}^{n+1}$ is said to be *totally real* with respect to the form (3.1) if for any two vectors $z, w \in W$, $\langle z, w\rangle \in \mathbb{R}$. Such a subspace is automatically totally real in the usual sense: $JW \cap W = \{0\}$, where J is the almost complex structure on V. *(Real) geodesics* in $\boldsymbol{B}^n$ are projections of totally real indefinite (with respect to q) 2-planes in $\mathbb{C}^{n+1}$ (intersected with V_-). For instance, geodesics through the origin $0 \in \boldsymbol{B}^n$ are Euclidean line segments in $\boldsymbol{B}^n$. More generally, totally-geodesic real subspaces in $\boldsymbol{B}^n$ are projections of totally real indefinite subspaces in $\mathbb{C}^{n+1}$ (intersected with V_-). They are isometric to the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ of constant sectional curvature -1.

Complex geodesics in $\boldsymbol{B}^n$ are projections of indefinite complex 2-planes. Complex geodesics are isometric to the unit disk with the hermitian metric

$$\frac{dzd\bar{z}}{(1-|z|^2)^2},$$

which has constant sectional curvature -4. More generally, k-dimensional complex hyperbolic subspaces $\mathbb{H}^k_{\mathbb{C}}$ in $\boldsymbol{B}^n$ are projections of indefinite complex $(k+1)$-dimensional subspaces (intersected with V_-).

All complete totally-geodesic submanifolds in $\mathbb{H}^n_{\mathbb{C}}$ are either real or complex hyperbolic subspaces.

The group $\mathrm{U}(n,1) \cong \mathrm{U}(q)$ of (complex) automorphisms of the form q projects to the group $\mathrm{Aut}(\boldsymbol{B}^n) \cong \mathrm{PU}(n,1)$ of complex (biholomorphic, isometric) automorphisms of $\boldsymbol{B}^n$. The group $\mathrm{Aut}(\boldsymbol{B}^n)$ is linear, its matrix representation is given, for instance, by the adjoint representation, which is faithful since $\mathrm{Aut}(\boldsymbol{B}^n)$ has trivial center.

A discrete subgroup Γ of $\mathrm{Aut}(\boldsymbol{B}^n)$ is called a *complex-hyperbolic Kleinian group.* The accumulation set of an(y) orbit Γx in $\partial\boldsymbol{B}^n$ is called the *limit set* of Γ and denoted by $\Lambda(\Gamma)$. The complement of $\Lambda(\Gamma)$ in $\partial \boldsymbol{B}^n$ is called the *domain of discontinuity* of Γ and denoted by $\Omega(\Gamma)$. The group Γ acts properly discontinuously on $\boldsymbol{B}^n \cup \Omega(\Gamma)$.

For a (torsion-free) complex-hyperbolic Kleinian group Γ, the quotient $\Gamma \backslash \boldsymbol{B}^n$ is a Riemannian orbifold (manifold) equipped with push-forward of the Riemannian metric of $\boldsymbol{B}^n$. We reserve the notation M_Γ to denote this quotient. The *convex core* of M_Γ is defined by

$$\mathrm{core}(M) := \Gamma \backslash C,$$

where $C \subset \boldsymbol{B}^n$ is the smallest Γ-invariant, closed, convex subset. Here the word "smallest" means "intersection of all nonempty"; we allow C to be the empty set. The subgroup Γ is called *convex-cocompact* if the convex core of M_Γ is a nonempty compact subset. Equivalently (see [**Bow95**]), $\overline{M}_\Gamma = \Gamma \backslash (\boldsymbol{B}^n \cup \Omega(\Gamma))$ is compact, provided that $|\Gamma| = \infty$.

Below are two interesting examples of convex-cocompact complex-hyperbolic Kleinian groups which will also serve as illustrations our results.

Example 3.1.1 (Real Fuchsian subgroups). Let $\mathbb{H}^2_{\mathbb{R}} \subset \boldsymbol{B}^n$ be a totally real-hyperbolic plane. This inclusion is induced by an embedding $\rho : \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}}) = \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{Aut}(\boldsymbol{B}^n)$ whose image preserves $\mathbb{H}^2_{\mathbb{R}}$. Let $\Gamma' < \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$ be a cocompact subgroup. Then $\Gamma = \rho(\Gamma')$ preserves $\mathbb{H}^2_{\mathbb{R}}$ and acts on it cocompactly. Such subgroups $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ will be called *real Fuchsian subgroups.* The compact surface-orbifold $\Sigma = \Gamma \backslash \mathbb{H}^2_{\mathbb{R}}$ is the convex core, $\mathrm{core}(M_\Gamma)$. The critical exponent $\delta(\Gamma)$ is 1.

Let Γ_t, $t \geq 0$, be a continuous family of deformations of $\Gamma_0 = \Gamma$ in $\mathrm{Aut}(\boldsymbol{B}^n)$ such that Γ_t's, for $t > 0$, are convex-cocompact but not real Fuchsian. Such deformation exist as long as Γ_t is, say, torsion-free, see e.g. [**Wei64**]. The groups Γ_t, $t > 0$, are called *real quasi-Fuchsian subgroups.* The critical exponents of such subgroups are strictly greater than 1.

Example 3.1.2 (Complex Fuchsian subgroups). Let Γ' be a cocompact subgroup of $\mathrm{SU}(1,1)$, the identity component isometry group of the real-hyperbolic plane (modulo $\mathbb{Z}_2$) and let $\mathrm{SU}(1,1) \to \mathrm{SU}(n,1)$ be any embedding. Note that $\mathrm{SU}(n,1)$ modulo center (isomorphic to $\mathbb{Z}_{n+1}$) is isomorphic to $\mathrm{PU}(n,1)$. By taking compositions, we get a representation $\rho : \Gamma' \to \mathrm{PU}(n,1)$. Then $\Gamma := \rho(\Gamma')$ leaves a complex geodesic invariant in $\boldsymbol{B}^n$. Such subgroups Γ will be called *complex Fuchsian subgroups.* In this case, $\mathrm{core}(M_\Gamma) = \Gamma \backslash \mathbb{H}^1_{\mathbb{C}}$ is a compact complex curve in M_Γ where $\mathbb{H}^1_{\mathbb{C}}$ is the Γ-invariant complex geodesic. The critical exponent $\delta(\Gamma)$ is 2.

3.2. Generalities on complex manifolds

By a *complex manifold with boundary* M, we mean a smooth manifold with (possibly empty) boundary ∂M such that $\mathrm{int}(M)$ is equipped with a complex structure and that there exists a smooth embedding $f : M \to X$ to an equidimensional complex manifold X, biholomorphic on $\mathrm{int}(M)$. A holomorphic function on M is a smooth function which admits a holomorphic extension to a neighborhood of M in X.

Let X be a complex manifold and $Y \subset X$ is a codimension 0 smooth submanifold with boundary in X. The submanifold Y is said to be *strictly Levi-convex* if every boundary point of Y admits a neighborhood U in X such that the submanifold with boundary $Y \cap U$ can be written as

$$\{\phi \leq 0\},$$

for some smooth submersion $\phi : U \to \mathbb{R}$ satisfying $\mathrm{Hess}(\phi) > 0$, where $\mathrm{Hess}(\phi)$ is the holomorphic Hessian:

$$\left(\frac{\partial^2 \phi}{\partial \bar{z}_i \partial z_j}\right).$$

Definition 3.2.1. A *strongly pseudoconvex manifold* M is a complex manifold with boundary which admits a strictly Levi-convex holomorphic embedding in an equidimensional complex manifold.

Definition 3.2.2. An open complex manifold Z is called *holomorphically convex* if for every discrete closed subset $A \subset Z$ there exists a holomorphic function $Z \to \mathbb{C}$ which is proper on A.

Alternatively,[1] one can define holomorphically convex manifolds as follows: For a compact K in a complex manifold M, the *holomorphic convex hull* $\widehat{K}_M$ of K in M is

$$\widehat{K}_M = \{z \in M : |f(z)| \leq \sup_{w \in K} |f(w)|, \forall f \in \mathcal{O}_M\}.$$

See [**Hör90**, Sec. 5.1]. In the above, $\mathcal{O}_M$ denotes the ring of holomorphic functions on M. Then M is holomorphically convex iff for every compact $K \subset M$, the hull $\hat{K}_M$ is also compact.

Theorem 3.2.3 (Grauert [**Gra58**]). *The interior of every compact strongly pseudoconvex manifold M is holomorphically convex.*

[1] and this is the standard definition

Definition 3.2.4 (Stein manifolds). A complex manifold M is called *Stein* if it admits a proper holomorphic embedding in $\mathbb{C}^n$ for some n.

Equivalently, M is Stein iff it is holomorphically convex and *holomorphically separable*: That is, for every distinct points $x, y \in M$, there exists a holomorphic function $f : M \to \mathbb{C}$ such that $f(x) \neq f(y)$. See [**Hör90**, Def. 5.1.3] for this definition and [**Hör90**, Thm. 5.1.3] for one direction[2] of the equivalence.

We will use:

Theorem 3.2.5 (Rossi [**Ros69**], Corollary on page 20). *If a compact complex manifold M is strongly pseudoconvex and contains no compact complex subvarieties of positive dimension, then* $\mathrm{int}(M)$ *is Stein.*

We now discuss strong pseudoconvexity and Stein property in the context of complex-hyperbolic manifolds. A classical example of a complex submanifold with Levi-convex boundary is a closed round ball $\overline{\boldsymbol{B}}^n$ in $\mathbb{C}^n$. Suppose that $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ is a discrete torsion-free subgroup of the group of holomorphic automorphisms of $\boldsymbol{B}^n$ with (nonempty) domain of discontinuity $\Omega = \Omega(\Gamma) \subset \partial \boldsymbol{B}^n$. The quotient

$$\overline{M}_\Gamma = \Gamma \backslash (\boldsymbol{B}^n \cup \Omega)$$

is a smooth manifold with boundary.

Lemma 3.2.6. *$\overline{M}_\Gamma$ is strongly pseudoconvex.*

Proof. We let T_Λ denote the union of all projective hyperplanes in $P^n_{\mathbb{C}}$ tangent to $\partial \boldsymbol{B}^n$ at points of Λ, the limit set of Γ. Let $\widehat{\Omega}$ denote the connected component of $P^n_{\mathbb{C}} - T_\Lambda$ containing $\boldsymbol{B}^n$. It is clear that $\boldsymbol{B}^n \cup \Omega \subset \widehat{\Omega}$ is strictly Levi-convex. By the construction, Γ preserves $\widehat{\Omega}$. It is proven in [**CNS13**, Thm. 7.5.3] that the action of Γ on $\widehat{\Omega}$ is properly discontinuous. Hence, $X := \Gamma \backslash \widehat{\Omega}$ is a complex manifold containing $\overline{M}_\Gamma$ as a strictly Levi-convex submanifold with boundary. $\square$

Specializing to the case when $\overline{M}_\Gamma$ is compact, i.e. Γ is convex-cocompact, we obtain:

Proposition 3.2.7. *Suppose that Γ is torsion-free, convex-cocompact and $n > 1$. Then:*

(i) $\partial \overline{M}_\Gamma$ is connected.

[2] The other direction follows easily from the definitions.

(ii) If $\mathrm{int}(\overline{M}_\Gamma) = M_\Gamma$ *contains no compact complex subvarieties of positive dimension, then* M_Γ *is Stein.*

For example, as it was observed in [**BS76**], the quotient-manifold $\Gamma\backslash\boldsymbol{B}^2$ of a torsion-free real-Fuchsian subgroup $\Gamma < \mathrm{Aut}(\boldsymbol{B}^2)$ is Stein while the quotient-manifold of a complex-Fuchsian subgroup $\Gamma < \mathrm{Aut}(\boldsymbol{B}^2)$ is non-Stein.

3.3. Proof of Theorem 2.1.1

In this section, we construct certain plurisubharmonic functions on M_Γ, for each finitely generated, discrete subgroup $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ satisfying $\delta(\Gamma) < 2$. We use these functions to show that M_Γ has no compact subvarieties of positive dimension. At the end of this section, we prove Theorem 2.1.1.

Let X be a complex manifold. Recall that a continuous function $f : X \to \mathbb{R}$ is called *plurisubharmonic*[3] if for any homomorphic map $\phi : V(\subset \mathbb{C}) \to X$, the composition $f \circ \phi$ is subharmonic. Plurisubharmonic functions f satisfy the maximum principle; in particular, if f restricts to a non-constant function on a connected complex subvariety $Y \subset X$, then Y is noncompact.

Now we turn to our construction of plurisubharmonic functions. Let $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ be a discrete subgroup. Consider the Poincaré series

$$\sum_{\gamma\in\Gamma}(1 - |\gamma(z)|^2), \quad z \in \boldsymbol{B}^n. \tag{3.3}$$

Lemma 3.3.1. *Suppose that* $\delta(\Gamma) < 2$*. Then (3.3) uniformly converges on compact sets.*

Proof. Since $\delta(\Gamma) < 2$, the Poincaré series

$$\sum_{\gamma\in\Gamma} e^{-2d(0,\gamma(z))}$$

uniformly converges on compact subsets in $\boldsymbol{B}^n$. By (3.2), we get

$$e^{-2d(0,\gamma(z))} \leq (1 - |\gamma(z)|^2) \leq 4e^{-2d(0,\gamma(z))}. \tag{3.4}$$

Then, the result follows from the upper inequality. $\square$

[3]There is a more general notion of *plurisubharmonic functions*; for our purpose, we only consider this restrictive definition.

Remark. Note that when $\delta(\Gamma) > 2$, or when Γ is of *divergent type* (e.g., convex-cocompact) and $\delta(\Gamma) = 2$, then (3.3) does not converge. This follows from the lower inequality of (3.4).

Assume that $\delta(\Gamma) < 2$. Define $F : \boldsymbol{B}^n \to \mathbb{R}$,

$$F(z) = \sum_{\gamma \in \Gamma} (|\gamma(z)|^2 - 1).$$

Since F is Γ-invariant, i.e., $F(\gamma z) = F(z)$, for all $\gamma \in \Gamma$ and all $z \in \boldsymbol{B}^n$, F descends to a function

$$f : M_\Gamma \to \mathbb{R}.$$

Lemma 3.3.2. *The function $f : M_\Gamma \to \mathbb{R}$ is plurisubharmonic.*

Proof. Enumerate Γ as $\Gamma = \{\gamma_1, \gamma_2, \dots\}$. Consider the sequence of partial sums of the series F,

$$S_k(z) = \sum_{j \le k} (|\gamma_j(z)|^2 - 1).$$

Since each summand in the above is plurisubharmonic[4], S_k is plurisubharmonic for each $k \ge 1$. Moreover, the sequence of functions S_k is monotonically decreasing. Thus, the limit $F = \lim_{k\to\infty} S_k$ is also plurisubharmonic, and hence so is f. □

Note, however, that at this point we do not yet know that the function f is nonconstant.

Now we prove the main result of this section.

Theorem 3.3.3. *Let Γ be a torsion-free discrete subgroup of* $\mathrm{Aut}(\boldsymbol{B}^n)$. *If $\delta(\Gamma) < 2$, then M_Γ contains no compact complex subvarieties of positive dimension.*

Proof. Suppose that Y is a compact connected subvariety of positive dimension in M_Γ. Since $\pi_1(Y)$ is finitely generated, so is its image Γ' in $\Gamma = \pi_1(M_\Gamma)$. Since $\delta(\Gamma') \le \delta(\Gamma)$, by passing to the subgroup Γ' we can (and will) assume that the group Γ is finitely generated.

We construct a sequence of functions $F_k : \boldsymbol{B}^n \to \mathbb{R}$ as follows. For $k \in \mathbb{N}$, let $\Sigma_k \subset \Gamma - \{1\}$ denote the subset consisting of $\gamma \in \Gamma$ satisfying $d(0, \gamma(0)) \le k$. Since Γ is a finitely generated linear group, it is residually finite and, hence, there exists a finite index subgroup $\Gamma_k < \Gamma$ disjoint from

[4]This follows from the fact that the function $|z|^2$ is plurisubharmonic.

Σ_k. For each $k \in \mathbb{N}$, define $F_k : \boldsymbol{B}^n \to \mathbb{R}$ as the sum

$$F_k(z) = \sum_{\gamma \in \Gamma_k} (|\gamma(z)|^2 - 1).$$

Since

$$\bigcap_{k \in \mathbb{N}} \Gamma_k = \{1\},$$

the sequence of functions F_k converges to $(|z|^2 - 1)$ uniformly on compact subsets of $\boldsymbol{B}^n$. As before, each F_k is plurisubharmonic (cf. Lemmata 3.3.1, 3.3.2).

Let $\widetilde{Y}$ be a connected component of the preimage of Y under the projection map $\boldsymbol{B}^n \to M_\Gamma$. Since $\widetilde{Y}$ is a closed, noncompact subset of $\boldsymbol{B}^n$, the function $(|z|^2 - 1)$ is nonconstant on $\widetilde{Y}$. As the sequence (F_k) converges to $(|z|^2 - 1)$ uniformly on compacts, there exists $k \in \mathbb{N}$ such that F_k is nonconstant on $\widetilde{Y}$. Let $f_k : M_k = M_{\Gamma_k} \to \mathbb{R}$ denote the function obtained by projecting F_k to M_k, and Y_k be the image of $\widetilde{Y}$ under the projection map $\boldsymbol{B}^n \to M_k$. Since M_k is a finite covering of M_Γ, the subvariety $Y_k \subset M_k$ is compact. Moreover, f_k is a nonconstant plurisubharmonic function on Y_k since F_k is such a function on $\widetilde{Y}$. This contradicts the maximum principle. $\square$

Remark. Regarding Remark 3.3: The failure of convergence of the series (3.3) as pointed out in Remark 3.3 is not so surprising. In fact, if Γ is a complex Fuchsian group, then $\delta(\Gamma) = 2$ and the convex core of M_Γ is a compact Riemann surface, see Example 3.1.2. Thus, our construction of F must fail in this case.

We conclude this section with a

Proof of Theorem 2.1.1. By Theorem 3.3.3, M_Γ does not have compact complex subvarieties of positive dimensions. Then, by the second part of Proposition 3.2.7, M_Γ is Stein. $\square$

3.4. Further remarks

In relation to Theorem 2.1.1, it is also interesting to understand the case when $\delta(\Gamma) = 2$, that is: For which convex-cocompact, torsion-free subgroups Γ of $\mathrm{Aut}(\boldsymbol{B}^n)$ satisfying $\delta(\Gamma) = 2$, is the manifold M_Γ Stein? It has been pointed out before that a complex Fuchsian subgroup $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ satisfies $\delta(\Gamma) = 2$, but the manifold M_Γ is not Stein. In fact, the convex core of M_Γ is a complex curve, see Remark 3.3. We conjecture that complex Fuchsian subgroups are the only such non-Stein examples.

Conjecture 3.4.1. *Let $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ be a convex-cocompact, torsion-free subgroup such that $\delta(\Gamma) = 2$. Then, M_Γ is non-Stein if and only if Γ is a complex Fuchsian subgroup.*

We illustrate this conjecture in the following very special case: Let $\phi : \pi_1(\Sigma) \to \mathrm{Aut}(\boldsymbol{B}^n)$ be a faithful convex-cocompact representation where Σ is a compact Riemann surface of genus $g \geq 2$. Then ϕ induces a (unique) equivariant harmonic map

$$F : \widetilde{\Sigma} \to \boldsymbol{B}^n$$

which descends to a harmonic map $f : \Sigma \to M_\Gamma$. In the above, $\widetilde{\Sigma}$ denotes the universal cover of Σ.

Proposition 3.4.2. *Suppose that F is a holomorphic immersion. Then $\Gamma = \phi(\pi_1(\Sigma))$ satisfies $\delta(\Gamma) \geq 2$. Moreover, if $\delta(\Gamma) = 2$, then Γ preserves a complex line. In particular, Γ is a complex Fuchsian subgroup of $\mathrm{Aut}(\boldsymbol{B}^n)$.*

Proof. Noting that M_Γ contains a compact complex curve, namely $f(\Sigma)$, the first part follows directly from Theorem 2.1.1.

For the second part, we let Y denote the surface $\widetilde{\Sigma}$ equipped with the Riemannian metric obtained via pull-back of the Riemannian metric g on $\boldsymbol{B}^n$. The entropy[5] $h(Y)$ of Y is bounded above by $\delta(\Gamma)$, i.e.

$$h(Y) \leq 2. \tag{3.5}$$

This can be seen as follows: The distance function d_Y on Y satisfies

$$d_Y(y_1, y_2) \geq d(F(y_1), F(y_2)).$$

Therefore, the exponential growth-rate δ_Y of $\pi_1(\Sigma)$-orbits in Y satisfies $\delta_Y \leq \delta(\Gamma)$. On the other hand, the quantity $\delta_Y = h(Y)$ since $\pi_1(\Sigma)$ acts cocompactly on Y.

Assume that $\widetilde{\Sigma}$ is endowed with a conformal Riemannian metric of constant -4 sectional curvature. Since $\widetilde{\Sigma}$ is a symmetric space, we have

$$h^2(Y)\mathrm{Area}(\Gamma\backslash Y) \geq h^2(\widetilde{\Sigma})\mathrm{Area}(\Sigma),$$

[5] The *volume entropy* of a simply connected Riemannian manifold (X, g) is defined as $\lim_{r\to\infty} \log \mathrm{Vol}(B(r,x))/r$, where $x \in X$ is a chosen base-point and $B(r,x)$ denotes the ball of radius r centered at x. This limit exists and is independent of x, see [**Man79**].

see [**BCG96**, p. 624]. The inequality (3.5) together with the above implies that $\mathrm{Area}(\Gamma\backslash Y) \geq \mathrm{Area}(\Sigma)$.

On the other hand, since $f : \Gamma\backslash Y \to M_\Gamma$ is holomorphic, $4 \cdot \mathrm{Area}(\Gamma\backslash Y)$ equals to the *Toledo invariant* $c(\phi)$ (see [**Tol89**]) of the representation ϕ. Since $c(\phi) \leq 4\pi(g-1)$, the inequality $\mathrm{Area}(\Gamma\backslash Y) \geq \mathrm{Area}(\Sigma) = \pi(g-1)$ shows that

$$\mathrm{Area}(\Gamma\backslash Y) = \pi(g-1)$$

or, equivalently, $c(\phi) = 4\pi(g-1)$. By the main result of [**Tol89**], Γ preserves a complex-hyperbolic line in $\boldsymbol{B}^n$. $\square$

Remark. The assumption that F is an immersion can be eliminated: Instead of working with a Riemannian metric, one can work with a Riemannian metric with finitely many singularities.

Motivated by Theorem 3.3.3, we also make the following conjecture.

Conjecture 3.4.3. *If $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ is discrete, torsion-free, and $\delta(\Gamma) < 2k$, then M_Γ does not contain compact complex subvarieties of dimension $\geq k$.*

We conclude this section with a verification of this conjecture under a stronger hypothesis.

Proposition 3.4.4. *If $\Gamma < \mathrm{Aut}(\boldsymbol{B}^n)$ is discrete, torsion-free, and $\delta(\Gamma) < 2k-1$, then M_Γ does not contain compact complex subvarieties of dimension $\geq k$.*

Proof. Note that if Γ is elementary (i.e., virtually abelian), then $\delta(\Gamma) = 0$. In this case, the result follows from Theorem 3.3.3. For the rest, we assume that Γ is nonelementary.

By [**BCG08**, Sec. 4], there is a natural map $f : M_\Gamma \to M_\Gamma$ homotopic to the identity map $\mathrm{Id}_{M_\Gamma} : M_\Gamma \to M_\Gamma$ and satisfying

$$|\mathrm{Jac}_p(f)| \leq \left(\frac{\delta(\Gamma)+1}{p}\right)^p, \quad 2 \leq p \leq 2n,$$

where $\mathrm{Jac}_p(f)$ denotes the *p-Jacobian* of f. When $\delta(\Gamma) < 2k-1$, we have $|\mathrm{Jac}_p(f)| < 1$, for $p \in [2k, 2n]$. This means that f strictly contracts the volume form on each p-dimensional tangent space at every point $x \in M_\Gamma$, for $p \in [2k, 2n]$.

Let $Y \subset M_\Gamma$ be a compact complex subvariety of dimension $\geq k$ (real dimension $\geq 2k$). Then, Y is also a volume minimizer in its homology class. Since f strictly contracts volume on Y, $f(Y)$

has volume strictly lesser than that of Y. However, f being homotopic to Id_{M_Γ}, $f(Y)$ belongs to the homology class of Y. This is a contradiction to the fact that Y minimizes volume its homology class. $\square$

Remark. Note that Proposition 3.4.4 gives an alternative proof of Theorem 3.3.3 (hence Theorem 2.1.1) under a stronger hypothesis, namely $\delta(\Gamma) \in (0, 1)$. However, this method fails to verify Theorem 3.3.3 in the case when $\delta(\Gamma) \in [1, 2)$.

CHAPTER 4

Anosov subgroups

In this chapter, we present the necessary background needed for the next two chapters. Special emphasis will be put on the definition of Anosov subgroups and other characterizations of this class of subgroups given by Kapovich, Leeb, and Porti [**KLP18b**, **KL18b**]. Along the way, we also set up our notations and conventions.

4.1. Symmetric spaces

For a thorough treatment of most the material in this and the next two sections, we refer to Eberlein's book [**Ebe96**].

Semisimple Lie groups. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Throughout this chapter, G is always assumed to be a connected real Lie group. Each $Y \in \mathfrak{g}$ defines an adjoint endomorphism $\operatorname{ad} Y : \mathfrak{g} \to \mathfrak{g}$, $(\operatorname{ad} Y)(Z) = [Y, Z]$. The *Killing form* of $\mathfrak{g}$ is a symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$B(Y, Z) = \operatorname{tr}(\operatorname{ad} Y \circ \operatorname{ad} Z)\,.$$

Definition 4.1.1. A real Lie algebra $\mathfrak{g}$ is called *semisimple* if the Killing form B of $\mathfrak{g}$ is nondegenerate, i.e., if $B(Y, Z) = 0$ for some $Y \in \mathfrak{g}$ and all $Z \in \mathfrak{g}$, then $Y = 0$. A connected real Lie group is called *semisimple* if its Lie algebra is semisimple. G or $\mathfrak{g}$ is called *simple* if $\mathfrak{g}$ has no nontrivial proper ideals and $\dim(\mathfrak{g}) \geq 2$.

Each semisimple Lie algebra $\mathfrak{g}$ orthogonally decomposes into a direct sum of simple ideals,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r. \tag{4.1}$$

This decomposition is unique up to the order of the summands.

The center of a semisimple Lie algebra $\mathfrak{z}(\mathfrak{g})$ is trivial. Consequently, the center $Z(G)$ of a semisimple Lie group G is a discrete subgroup. From now on, we always assume that $Z(G)$ is finite.

Definition 4.1.2 (Noncompact type). A semisimple Lie group G is called of *compact type* if its Lie algebra $\mathfrak{g}$ is of *compact type*, i.e., the Killing form of $\mathfrak{g}$ is negative definite[1]. G is called of *noncompact type* if none of $\mathfrak{g}_i$'s in the direct sum decomposition (4.1) is of compact type.

Maximal compact subgroups. Let G be a semisimple Lie group of noncompact type. It is known that G admits a maximal compact subgroup K. Note that[2] $Z(G) < K$. On the coset space $X = G/K$, one defines a G-invariant Riemannian metric g by averaging a left-invariant Riemannian metric on G by K. As a Riemannian manifold, X has nonpositive sectional curvature, and G acts on the left by isometries. Note that the stabilizer of a point $gK \in X$ is gKg^{-1}, a conjugate of K and, hence, a maximal compact subgroup of G. In fact, by *Cartan's fixed-point theorem*, maximal compact subgroups of G represent a single conjugacy class.

Symmetric spaces. The Riemannian manifolds X arising from the above description has the following geometric characterization.

Definition 4.1.3 (Symmetric spaces of noncompact type). A simply-connected, complete Riemannian manifold X is called a *symmetric space* if every point $x \in X$ defines an isometric geodesic symmetry: An isometry $s_x : X \to X$ that fixes x such that $ds_x|_x = -\mathrm{Id}_{T_xX}$. A symmetric space is called of *noncompact type* if X has nonpositive sectional curvature and X cannot be written as a Riemannian product $\mathbb{E}^k \times X_1$ $(k \geq 1)$ where $\mathbb{E}^k$ is the k-dimensional euclidean space with flat metric.

We denote the isometry group of X by $\mathrm{Isom}\,(X)$; it is a Lie group. In fact, the identity component G of $\mathrm{Isom}\,(X)$ is a semisimple Lie group of noncompact type with trivial center so that X can be written as

$$X = G/K$$

where $K = G_x$ is the stabilizer of a point $x \in X$ that is also a maximal compact subgroup of G.

In the sequel, we fix the following notations and conventions: X denotes a symmetric space of noncompact type, G denotes a semisimple Lie group that acts on X by isometries such that

(i) Elements of G do not swap factors in the de Rham decomposition of X,

$$X = X_1 \times \dots X_r. \tag{4.2}$$

[1]This is equivalent to G being compact.

[2]Recall our assumption that G has finite center.

(ii) G is *commensurable* with $\mathrm{Isom}\,(X)$, i.e., the homomorphism $G \to \mathrm{Isom}\,(X)$ has finite kernel and cokernel.

Under the second assumption, X is a homogeneous G-space. For simplicity, in the rest of this section we take G to be the identity component of $\mathrm{Isom}\,(X)$. However, the discussion easily generalizes to the above general setting.

Let $x \in X$ be any point. We keep this choice fixed throughout this chapter. Since X is a *Hadamard manifold*[3], the *Cartan-Hadamard theorem* asserts that the exponential map

$$\exp : T_x X \to X \tag{4.3}$$

is a diffeomorphism. In other words, X is diffeomorphic to $\mathbb{R}^{\dim(X)}$.

Cartan decomposition. The choice of $x \in X$ determines a natural decomposition of the Lie algebra $\mathfrak{g}$ of G in the following way: The corresponding geodesic symmetry $s_x : X \to X$ determines an involution $G \to G$, $g \mapsto s_x g s_x$, whose differential $i : \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie algebra automorphism. This produces a decomposition of $\mathfrak{g}$, called the *Cartan decomposition* (with respect to x),

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \tag{4.4}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the ± 1-eigenspaces of i. By definition, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. In fact, $\mathfrak{k}$ is tangent to $K = G_x$ and $\mathfrak{p}$ maps isomorphically onto $T_x X$ via the differential of the projection map $\pi : G \to X = G/G_x$. We identify $\mathfrak{p}$ with $T_x X$ under $d\pi$. The Killing form B restricts to a positive (resp. negative) definite form on $\mathfrak{p}$ (resp. $\mathfrak{k}$).

The Riemannian metric of X is actually 'determined' by the Killing form of $\mathfrak{g}$: Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, the adjoint action of K on $\mathfrak{g}$ leaves $\mathfrak{p}$ invariant. Hence, we may write $\mathfrak{p}$ as an orthogonal (wrt. B) direct sum of irreducible components,

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r.$$

This decomposition relates to the de Rham decomposition (4.2) of X via the exponential map (4.3). Then,

$$g_x = c_1 B|_{\mathfrak{p}_1} + \cdots + c_r B|_{\mathfrak{p}_r},$$

[3] A *Hadamard manifold* is by definition a complete, simply-connected, nonpositively curved Riemannian manifold.

for some constants $c_1, \ldots, c_r > 0$. Therefore, we can (and will) assume that the isomorphism $d\pi : \mathfrak{p} \to T_x X$ is isometric where $\mathfrak{p}$ (resp. $T_x X$) is equipped with the Killing form (resp. Riemannian metric).

Maximal flats, Weyl groups, and Weyl chambers. The *rank* of G is the dimension of a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. The maximal abelian subalgebras of $\mathfrak{p}$ constitute a single K-orbit for the adjoint action of K on $\mathfrak{p}$. Let $K_{\mathfrak{a}} < K$ be the stabilizer of $\mathfrak{a}$. Then the action of $K_{\mathfrak{a}}$ on $\mathfrak{a}$ factors through a finite group W of automorphisms of $\mathfrak{a}$ called the *Weyl group*. The group W is generated by reflections along (a finite number of) hyperplanes in $\mathfrak{a}$. The complement of these reflecting hyperplanes in $\mathfrak{a}$ decomposes into connected components, the closures[4] of these components are called *Weyl chambers*. Each Weyl chamber is a fundamental domain for the action $W \curvearrowright \mathfrak{a}$. K acts transitively on the set of Weyl chambers in $\mathfrak{p}$. We pick a *model* maximal abelian subalgebra $\mathfrak{a}_{\mathrm{mod}}$ of $\mathfrak{p}$ and a *model* Weyl chamber $\bar{\mathfrak{a}}^+_{\mathrm{mod}} \subset \mathfrak{a}_{\mathrm{mod}}$.

The algebraic description in the previous paragraph fits into the following geometric description: Using the exponential map (4.3), $\exp : \mathfrak{p} \to X$, each maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ isometrically maps onto a *maximal flat*[5] in X passing through x. The *rank* of X is the the dimension of a maximal flat passing through x (this definition is independent of a choice of $x \in X$). The isotropy group K acts transitively on the set of all maximal flats passing through x. We call $F_{\mathrm{mod}} := \exp(\mathfrak{a}_{\mathrm{mod}})$ the *model flat* and $\Delta := \exp(\bar{\mathfrak{a}}^+_{\mathrm{mod}})$ the *model chamber*. We often think of F_{mod} as the flat euclidean space $\mathbb{E}^{\mathrm{rank}(X)}$ with x being the origin and Δ as a closed convex cone. The isometric action $K_{F_{\mathrm{mod}}} \curvearrowright F_{\mathrm{mod}}$ of the stabilizer $K_{F_{\mathrm{mod}}}$ of F_{mod} in K (modulo the kernel of the action) is identified with the Weyl group action discussed above.

Since X is a homogeneous G-space, G acts transitively on the set of all (pointed) maximal flats in X: For a given maximal flat $F' \subset X$ and a point $x' \in F'$, choose $g \in G$ such that $g(x) = x'$. Then $F'' = g(F_{\mathrm{mod}})$ passes through x'. Finally, we find a suitable element $k \in K_{x'}$ such that $k(F'') = F'$. In fact, if $F' = F_{\mathrm{mod}}$, then one can find an element $g \in G$ that preserves F_{mod} and translates (in the euclidean sense) x to x'. Consequently, the the action of stabilizer group of F_{mod} in G on F_{mod} factors through the group $\mathbb{R}^{\mathrm{rank}(X)} \rtimes W$ of affine isometries.

---

[4]Note that the standard definition of Weyl chambers does not involve taking closures.

[5]A submanifold $F \subset X$ is called a *flat* if F is the image of an totally geodesic, isometric embedding of a euclidean space. A flat F is called *maximal* if it is not properly contained in another flat.

Δ-valued distances. Using the above description, one can define a vector-valued distance function on X in the following way: Every pair of points (x', x'') of X lies in a maximal flat $F' \subset X$. Find $g \in G$ that maps F' to F_{mod}, x' to x, and x'' into Δ. The image $g(x'')$ in Δ is uniquely determined and is called the Δ-*valued distance* between x' and x'':

$$d_\Delta(x', x'') := g(x'') \in \Delta \subset \mathbb{E}^{\text{rank}(X)}.$$

Note that, tautologically,

$$d(x', x'') = \|d_\Delta(x', x'')\|,$$

where d is the Riemannian distance function on X and $\|\cdot\|$ denotes the euclidean norm inherited from the Riemannian metric of X. It is clear that d_Δ is a G-invariant pairing, and, in fact, d_Δ is a *complete G-congruence invariant* for oriented pairs: For pairs of points (x', x'') and (y', y'') in X, there exists $g \in G$ such that $g(x') = y'$ and $g(x'') = y''$ iff $d_\Delta(x', x'') = d_\Delta(y', y'')$.

The Δ-valued distance function behaves much like an ordinary distance function. Apart from the properties above, d_Δ satisfies the following generalization of the triangle inequality for ordinary distance functions:

THEOREM 4.1.4 (Triangle inequality for Δ-valued distances, [**KLM09**]). *For all triples $y, y', y'' \in X$,*

$$\|d_\Delta(y, y') - d_\Delta(y', y'')\| \leq d(y, y'').$$

The Δ-valued distances naturally relate to the *Cartan decomposition* of G: Consider the K-invariant map

$$\text{pr} : X \to \Delta, \quad x' \mapsto d_\Delta(x, x').$$

This gives an identification $K\backslash G/K = K\backslash X = \Delta = \exp(\bar{\mathfrak{a}}^+_{\text{mod}}) \implies G = K\exp(\bar{\mathfrak{a}}^+_{\text{mod}})K$. The *Cartan projection* for the choice of $x \in X$ is the induced map $G \to \bar{\mathfrak{a}}^+_{\text{mod}}$.

4.2. Boundary at infinity

Visual boundary. Since X is a Hadamard manifold, there is a notion of the *visual boundary* $\partial_\infty X$ of X: As a set,

$$\partial_\infty X = \{c : [0, \infty) \to X : c \text{ is a unit-speed geodesic ray}\}/\sim$$

where the equivalence relation $\sim$ is defined by

$$c_1 \sim c_2 \iff \limsup_{t\to\infty} d(c_1(t), c_2(t)) < \infty.$$

In words, two geodesic rays represent the same boundary point iff they are *asymptotic*. Equivalently, two geodesic rays represent the same boundary point iff they are within finite *Hausdorff distance*[6] from each other.

The exponential map (4.3) identifies $\partial_\infty X$ with the unit tangent sphere $S_xX \subset T_xX$ by sending $v \in S_xX$ to the equivalence class of the geodesic ray $\exp(tv)$, $t \geq 0$,

$$S_xX = \partial_\infty X. \tag{4.5}$$

Under this identification, $\partial_\infty X$ gets the topology of a sphere $S^{\dim(X)-1}$.

An invariant version of the identification (4.5) is useful: For $y \in Y$, $v_y \in S_yX$, and $v_x \in S_xX$, the vectors v_x, v_y represent same point at the visual boundary iff the geodesic rays starting at x and y with initial velocity vectors v_x and v_y, respectively, have finite Hausdorff distance from each other iff the geodesic flow φ_t on SX keeps v_x, v_y within bounded distance for all times $t \geq 0$.

Here and below, SX denotes the unit tangent bundle of X. The action of G on SX defines a (topological) action $G \curvearrowright \partial_\infty X$. For $y \in X$, the induced action of the maximal compact subgroup $G_y < G$ on $\partial_\infty X$ can be understood as the natural isometric action $G_y \curvearrowright S_yX$ (since G_y preserves $S_yX \subset SX$). In particular, $\partial_\infty X$ has a $K(:= G_x)$-invariant Riemannian metric inherited from S_xX. Using the identification $T_xX = \mathfrak{p}$ from the previous section, a fundamental domain for the action $K \curvearrowright \mathfrak{p}$ can be identified with the the set of unit vectors in $\bar{\mathfrak{a}}^+_{\mathrm{mod}}$. We denote this fundamental domain by σ_{mod}.This defines a K-invariant map

$$\theta : \partial_\infty X \to \sigma_{\mathrm{mod}} \tag{4.6}$$

called the *type map*. In fact, σ_{mod} can also be regarded as a fundamental domain for $G \curvearrowright SX$ and, hence, for $G \curvearrowright \partial_\infty X$. In particular, the type map θ is also G-invariant.

Tits building. The isometric actions $G_y \curvearrowright S_yX \cong \partial_\infty X$, $y \in X$, from the previous paragraph can be used to define a *spherical building structure* on $\partial_\infty X$, called the *Tits building* $\partial_{\mathrm{Tits}}X$ of G:

[6]For sets $A, B \subset X$, the *Hausdorff distance* between A and B is $d_{\mathrm{Haus}}(A, B) := \inf\{C \geq 0 : A \subset N_C(B) \text{ and } B \subset N_C(A)\}$.

The apartments are $a = \mathfrak{a}_y \cap S_y X$, for $y \in Y$ and each maximal abelian subalgebra $\mathfrak{a}_y \subset \mathfrak{p}_y$[7]. The apartment $a_{\mathrm{mod}} := \mathfrak{a} \cap S_x X$ is called the *model apartment*. The apartments are isometric copies of the $(\mathrm{rank}(X) - 1)$-dimensional sphere and they cover $\partial_\infty X$. For each apartment $a \subset \partial_\infty X$, $W(a) := G_a/\{\text{kernel of the action}\} \cong W$ and the pair $(a, W(a))$ is a Coxeter complex. Under these notations, $\sigma_{\mathrm{mod}} \subset a_{\mathrm{mod}}$ is a fundamental domain for the action $W(a_{\mathrm{mod}}) \curvearrowright a_{\mathrm{mod}}$. Moreover, the Coxeter complex structures of the apartments are compatible, i.e., the intersection of every two apartments is simplicial and the simplicial structures on these intersections agree. Therefore, all the apartments and their Coxeter complex structures can be put together to define a single large simplicial complex; the Tits boundary $\partial_{\mathrm{Tits}} X$ is this simplicial complex with its simplicial topology[8]. A maximal simplex is given by σ_{mod}, and all other maximal cells are naturally identified with σ_{mod} by the type map (4.6). These maximal cells are called (spherical) *Weyl chambers*. We reserve the notation τ_{mod} to denote faces of σ_{mod}. A simplex τ in the building $\partial_{\mathrm{Tits}} X$ is called of *type* τ_{mod} if the type map sends τ onto τ_{mod}.

For a simplex τ in $\partial_{\mathrm{Tits}} X$, the *star* $\mathrm{st}\,(\tau)$ of τ is the union of all chambers in $\partial_{\mathrm{Tits}} X$ containing τ. The *open star* $\mathrm{ost}\,(\tau)$ of τ is the union of all the open simplices whose closures contains θ. For a face τ_{mod} of σ_{mod} (viewed as a complex itself), define the open star $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ similarly. The *boundary* $\partial\mathrm{st}(\tau_{\mathrm{mod}})$ is the complement of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ in σ_{mod}.

Parabolic subgroups and generalized flag manifolds. Point stabilizers of the action $G \curvearrowright \partial_\infty X$ are called *parabolic subgroups* of G. If an element $g \in G$ fixes a point ξ in the interior of a Weyl chamber σ, then g fixes σ point-wise (since the type of points must be preserved). Hence G_ξ also fixes σ point-wise. The parabolic subgroups P fixing interior points of Weyl chambers are called *minimal parabolic subgroups*[9]. It is clear that the minimal parabolic subgroups of G form a single conjugacy class, i.e. every minimal parabolic subgroup is conjugate to $P_{\sigma_{\mathrm{mod}}}$, the stabilizer of σ_{mod} in G. The quotient space

$$\partial_{\mathrm{Fu}} X := G/P_{\sigma_{\mathrm{mod}}}$$

equipped with the quotient topology is a G-homogeneous space which is called the (general) *full flag manifold* or the *Fursternberg boundary* of X. This is the space of all simplices of type σ_{mod} in the Tits building.

[7]Here $\mathfrak{p}_y$ refers to the summand in the Cartan decomposition with respect to y, $\mathfrak{g} = \mathfrak{k}_y + \mathfrak{p}_y$, cf. (4.4).
[8]This topology is strictly finer than the topology of $\partial_\infty X$.
[9]since they do not properly contain any other parabolic subgroups

A similar discussion is true for general parabolic subgroups of G. If P stabilizes a boundary point $\xi \in \sigma$, then P also point-wise fixes the minimal face τ of σ containing ξ. Clearly, P is conjugate to $P_{\tau_{\text{mod}}}$ where $\tau_{\text{mod}} \subset \sigma_{\text{mod}}$ denotes the type of τ. The quotient space

$$\text{Flag}\,(\tau_{\text{mod}}) = G/P_{\tau_{\text{mod}}}$$

equipped with the quotient topology is called the *(partial) flag manifold*. These manifolds are compact, smooth, G-homogeneous manifolds. As in the previous paragraph, $\text{Flag}\,(\tau_{\text{mod}})$ is the space of all simplices of type τ_{mod} in the Tits building, $\partial_{\text{Tits}} X$. Note that $\text{Flag}\,(\sigma_{\text{mod}}) = \partial_{\text{Fu}} X$.

Opposition involution. The Cartan involution $i|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ induces an involutive action (called *opposition involution*) on $\partial_\infty X$,

$$\iota : \partial_\infty X \to \partial_\infty X, \quad \iota^2 = \text{Id}_{\partial_\infty X},$$

which preserves the building structure. The opposition involution also induces an involution on σ_{mod} by

$$\xi \mapsto \theta(\iota(\xi)), \quad \xi \in \sigma_{\text{mod}}.$$

We also denote this involution by $\iota : \sigma_{\text{mod}} \to \sigma_{\text{mod}}$. For every subset $\Theta \subset \sigma_{\text{mod}}$, $\iota(\Theta)$ denotes the image of Θ under $\iota : \sigma_{\text{mod}} \to \sigma_{\text{mod}}$.

Two parabolic subgroups P and P' are called *opposite* if P (resp. P') is conjugate to $P_{\tau_{\text{mod}}}$ (resp. $P_{\iota\tau_{\text{mod}}}$). In the sequel, we only consider faces $\tau_{\text{mod}} \subset \sigma_{\text{mod}}$ that are ι-invariant, i.e., $\iota\tau_{\text{mod}} = \tau_{\text{mod}}$. In other words, the parabolic subgroups $P_{\tau_{\text{mod}}}$ that we consider are conjugate to their opposite.

Examples. We describe the Tits buildings in the following two explicit examples. We will come back to further discussion of these examples in Chapter 6.

Example 4.2.1 (Product of rank-one symmetric spaces). Let X be a product of k rank-one symmetric spaces,

$$X = X_1 \times \cdots \times X_k.$$

The rank of X is k. Let G be a semisimple Lie group commensurable with the isometry group of X. For example, we may take $G = \text{Isom}\,(X_1) \times \cdots \times \text{Isom}\,(X_k)$.

The model maximal flat F_{mod} (see Figure 4.1 below) can be viewed as the product of some chosen geodesic lines (coordinate axes), one for each de Rham factor of X. The Weyl group W

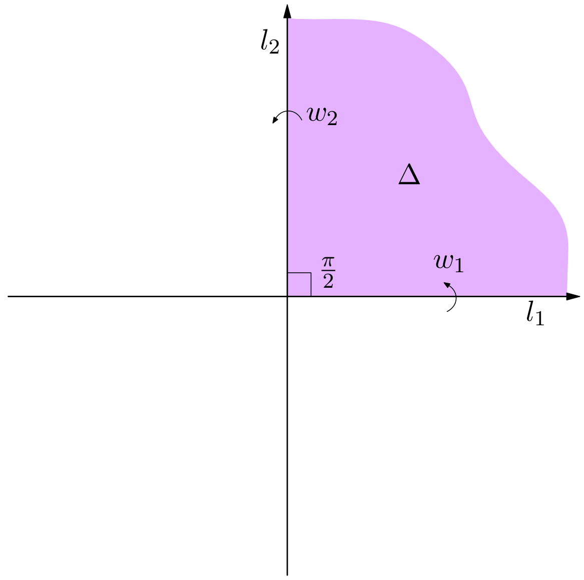

FIGURE 4.1. The model maximal flat $F_{\text{mod}} \subset X$, where $X = X_1 \times X_2$, product of two rank-one symmetric spaces. It is spanned by two geodesic lines $l_i \subset X_i$, $i = 1, 2$. The Weyl group W is generated by two elements, reflections w_i along l_i, $i = 1, 2$. The positive quadrant (shaded) is the model Weyl chamber Δ.

is generated by reflections along the coordinate hyperplanes and the longest element in it is the reflection about the origin. The model Weyl chamber Δ can be realized as the nonnegative orthant. The opposition involution ι acts on it trivially.

The Tits boundary of a product of two symmetric spaces is the simplicial join of their individual Tits buildings and, for rank-one symmetric spaces, the Tits boundary is discrete. These two facts imply that the $(p-1)$-simplices in the Tits building of X for $1 \leq p \leq k$ can be parametrized by p-tuples $(\xi_{r_1}, \ldots, \xi_{r_p}) \in \partial_\infty X_{r_1} \times \cdots \times \partial_\infty X_{r_p}$, $1 \leq r_1 < \cdots < r_p \leq k$,

$$(\xi_{r_1}, \ldots, \xi_{r_p}) \longleftrightarrow \tau = \text{span}\{\xi_{r_1}, \ldots, \xi_{r_k}\}.$$

We say that such a simplex τ has type $\tau_{\text{mod}} = (r_1, \ldots, r_p)$. The incidence structure can be understood as follows: Two simplices have a common q-face if and only if they have q equal coordinates.

The star $\text{st}(\tau)$ of $\tau = (\xi_{r_1}, \ldots, \xi_{r_p})$ is the minimal subcomplex of the Tits building containing all chambers $(\zeta_1, \ldots, \zeta_p)$ satisfying $\zeta_{r_i} = \xi_{r_i}$, for all $i \in \{1, \ldots, p\}$.

Since the opposition involution ι fixes each chamber point-wise, every face τ_{mod} of σ_{mod} and every type is ι-invariant. Every two chambers (resp. faces of the same type) in $\partial_{\text{Tits}} X$ are antipodal to each other unless they have a common face (resp. subface).

Example 4.2.2 ($X = \mathrm{SL}(k+1,\mathbb{R})/\mathrm{SO}(k+1,\mathbb{R})$). We take $G = \mathrm{SL}(k+1,\mathbb{R})$, $K = \mathrm{SL}(k+1,\mathbb{R})$; the symmetric space $X = G/K$ is identified with the set of all positive definite, symmetric matrices in $\mathrm{SL}(k+1,\mathbb{R})$. In this case $\mathrm{rank}(X) = k$ and X is irreducible. The standard choice of a model flat F_{mod} is the subset of all diagonal matrices $a = \mathrm{diag}(a_1,\ldots,a_{k+1}) \in \mathrm{SL}(k+1,\mathbb{R})$ with positive diagonal entries. We identify the model flat with $\mathfrak{a}$ via the logarithm map

$$\log : a = \mathrm{diag}(a_1,\ldots,a_{k+1}) \mapsto (\log a_1,\ldots,\log a_{k+1})$$

where $\mathfrak{a}$ is viewed as the hyperplane in $\mathbb{R}^{k+1}$ consisting of all points with zero sum of coordinates.

The Weyl group $W = \mathrm{Sym}_{k+1}$ acts on $\mathfrak{a}$ by permuting the coordinates. The standard choice for the model Weyl chamber $\Delta = \mathfrak{a}_+$ consists of all the points in $\mathfrak{a}$ with decreasing coordinate entries. The Cartan projection[10] $\rho : \mathrm{SL}(k+1,\mathbb{R}) \to \mathfrak{a}_+$ can be written as $g \mapsto \log a$ where a is associated to g via the singular value decomposition $g = uav$, $u, v \in \mathrm{SO}(k+1,\mathbb{R})$. The logarithm of i-th singular value of g, $\sigma_i(g)$ (see (2.1)), will be denoted by $\mu_i(g)$,

$$\mu_i(g) := \log \sigma_i(g).$$

The opposition involution ι sends $(\mu_1,\ldots,\mu_{k+1}) \in \mathfrak{a}_+$ to $(-\mu_{k+1},\ldots,-\mu_1)$.

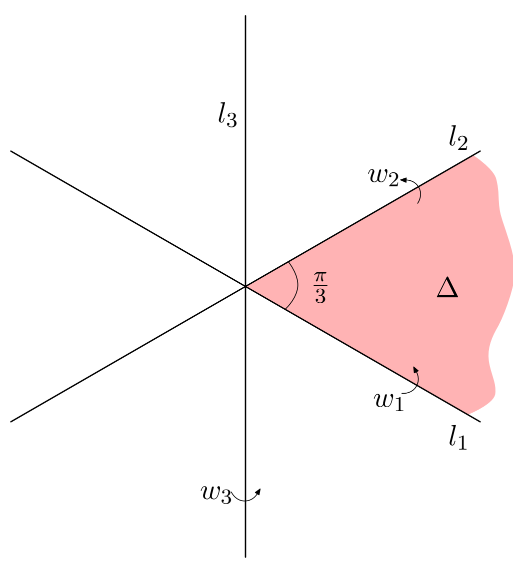

Figure 4.2. The model maximal flat in $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$ ($\mathrm{rank}(X) = 2$). The Weyl group W is generated by three reflections, w_1, w_2, w_3. A chosen model Weyl chamber Δ is shaded.

[10]Or the Δ-valued distance in the sense that $d_\Delta(x, gx) = \rho(g)$.

The Tits building of X can be identified with the incidence geometry of flags in $\mathbb{R}^{k+1}$. The Fursternberg boundary consists of full flags

$$V_1 \subset \cdots \subset V_{k+1} = \mathbb{R}^{k+1}, \quad \dim(V_i) = i.$$

The partial flags are

$$V: \ V_{r_1} \subset \cdots \subset V_{r_p} \subset V_{r_{p+1}} = \mathbb{R}^{k+1}, \quad \dim(V_{r_i}) = r_i,$$

$1 \leq r_1 < \cdots < r_p < r_{p+1} = k+1$, which are elements of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ where $\tau_{\mathrm{mod}} = (r_1, \ldots, r_p)$. The opposition involution sends τ_{mod} to $\iota\tau_{\mathrm{mod}} = (k+1-r_p, \ldots, k+1-r_1)$. It follows that τ_{mod} is ι-invariant if and only if $r_i + r_{p+1-i} = k+1$, for each $i = 1, \ldots, p$. The partial flag manifold $\mathrm{Flag}(\tau_{\mathrm{mod}})$ consisting of all partial flags V of type $\tau_{\mathrm{mod}} = (r_1, \ldots, r_p)$ naturally embeds into the product of Grassmanians $\mathrm{Gr}_{r_1}(\mathbb{R}^{k+1}) \times \cdots \times \mathrm{Gr}_{r_p}(\mathbb{R}^{k+1})$.

Suppose that $\tau_{\mathrm{mod}} = (r_1, \ldots, r_p)$ is ι-invariant. A pair $V^{\pm} \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ is antipodal (see definition in page 33) if and only if $V^+_{r_i} + V^-_{r_{p+1-i}} = \mathbb{R}^{k+1}$ for each $i = 1, \ldots, p$.

4.3. Parallel sets, cones, and diamonds

In this section, we describe certain convex subsets of X that generalize geodesics in the hyperbolic spaces in a meaningful way.

Regularity. Let $I \subset \mathbb{R}$ be a an interval containing zero. Recall that each (unit-speed parametrized) geodesic $c : I \to X$ satisfying $c(0) = y$ and $c'(0) = v \in S_yX$ can be written as $c(t) = \exp_y(tv)$ where $\exp_y : T_yX \to X$ is the exponential map. We often identify a geodesic c with its image and write it as yz where y is the initial point and z is the final point. One or both of the endpoints y, z may lie in $\partial_\infty X$: If $y, z \in X$, then we call yz a (geodesic) segment. If $y \in X$ and $z \in \partial_\infty X$, then we call yz a (geodesic) ray starting at y and asymptotic to z. If $y, z \in \partial_\infty X$, then we call yz a (geodesic) line forward (resp. backward) asymptotic to z (resp. y). We use the word "geodesic" to mean any of these. It is a fact that geodesics in a Hadamard manifolds are uniquely determined (up to reparameterizations) by their end points.

Let c be a geodesic in X and $v(t) = c'(t) \in S_{c(t)}X$. Under the identifications $S_{c(t)}X = \partial_\infty X$ from the previous section, for all times t, $v(t)$ represent the same boundary point[11] ξ_+ in $\partial_\infty X$.

[11] This follows since for any time t_1, t_2, the geodesic flow moves $v(t_1)$ and $v(t_2)$ along the same orbit keeping the distance fixed.

We say that c is *forward asymptotic* to ξ_+. Similarly, we say that c is *backward asymptotic* to ξ_- if ξ_- is represented by $c'(-t)$, for any t. It is easy to see that $c \subset \xi_-\xi_+$ and

$$\theta(\xi_+) = \iota\theta(\xi_-)$$

where θ is the type map defined in the previous section.

Let $\tau_{\text{mod}} \subset \sigma_{\text{mod}}$ be a face. We recall our convention that τ_{mod} is ι-invariant.

DEFINITION 4.3.1 (Regularity). A geodesic $c \subset X$ is called τ_{mod}*-regular* if $\theta(\xi_+) \in \operatorname{ost}(\tau_{\text{mod}})$ where $\xi_+ \in \partial_\infty X$ is the point such that c is forward asymptotic to ξ_+. Equivalently, c is τ_{mod}-regular if $\xi_- \in \operatorname{ost}(\tau_{\text{mod}})$ where $\xi_- \in \partial_\infty X$ is the point such that c is backward asymptotic to ξ_-. A pair of points (y, z) in X is called τ_{mod}*-regular* if the segment yz is τ_{mod}-regular.

Parallel sets. Two points $\tau_\pm \in \operatorname{Flag}(\tau_{\text{mod}})$ are called *antipodal* if there exists a τ_{mod}-regular line $c \subset X$ such that c is *forward* (resp. *backward*) *asymptotic* to τ_+ (resp. τ_-): I.e., there exist $\xi_\pm \in \operatorname{st}(\tau_\pm)$ such that c is forward asymptotic to ξ_+ and backward asymptotic to ξ_-. In general, any two points $\tau_\pm \in \operatorname{Flag}(\tau_{\text{mod}})$ are not[12] always antipodal. Given $\tau \in \operatorname{Flag}(\tau_{\text{mod}})$, the set $C(\tau) = \{\tau_- \in \operatorname{Flag}(\tau_{\text{mod}}) : \tau_- \text{ is antipodal to } \tau\}$ is an open Schubert stratum in the flag variety $\operatorname{Flag}(\tau_{\text{mod}}) = G/P_{\tau_{\text{mod}}}$.

DEFINITION 4.3.2 (Parallel sets). Given antipodal points $\tau_\pm \in \operatorname{Flag}(\tau_{\text{mod}})$, the *parallel set* $P(\tau_-, \tau_+)$ is the union of all lines $c \subset X$ such that c is forward asymptotic to τ_+ and backward asymptotic to τ_-.

Parallel sets are totally geodesic (in particular, convex) submanifolds of X. For antipodal points $\sigma_\pm \in \operatorname{Flag}(\sigma_{\text{mod}})$, $P(\sigma_-, \sigma_+)$ is a maximal flat in X.

Cones and diamonds. The following convex sets were introduced in [**KLP17**].

DEFINITION 4.3.3 (Cones). Given $y \in X$ and $\tau \in \operatorname{Flag}(\tau_{\text{mod}})$, the *cone* $V(y, \operatorname{st}(\tau))$ is the union of all rays starting at y and forward asymptotic to τ.

DEFINITION 4.3.4 (Diamonds). Let (y, z) be a τ_{mod}-regular pair in X. The *diamond* $\diamondsuit_{\tau_{\text{mod}}}(y, z)$ is the set $V(y, \operatorname{st}(\tau_+)) \cap V(z, \operatorname{st}(\tau_-))$ where $\tau_\pm \in \operatorname{Flag}(\tau_{\text{mod}})$ are the (unique) points such that yz is forward (resp. backward) asymptotic to τ_+ (resp. τ_-).

[12]For instance, if $\tau_\pm$ share a common face, then they are not antipodal.

Cones and (hence) diamonds are closed, convex subsets of X. In fact, each lives in a unique[13] parallel set. The convexity property implies that cones are *nested*: $V(z, \mathrm{st}(\tau)) \subset V(y, \mathrm{st}(\tau))$ for every $z \in V(y, \mathrm{st}(\tau))$.

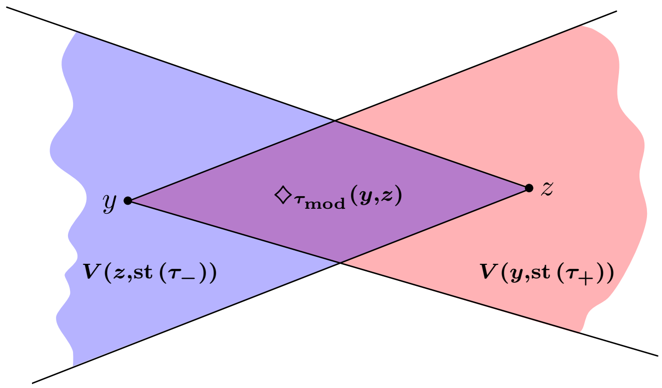

FIGURE 4.3. Two opposite cones intersecting on a diamond.

Quantified regularity. The following *more regular* version of the above definitions will be frequently used: Let $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be a face and let $\Theta \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ be a nonempty ι-invariant, compact, τ_{mod}-*Weyl convex*[14] subset. A geodesic $c \subset X$ is called Θ-*regular* if $\theta(\xi_+) \in \Theta$ (or, equivalently, $\theta(\xi_-) \in \Theta$) where $\xi_\pm \in \partial_\infty X$ are the points such that c is forward and backward asymptotic to $\xi_\pm$. A pair of points (y, z) in X is called Θ-*regular* if the segment yz is Θ-regular. Given $y \in X$ and $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, the Θ-*cone* $V(y, \mathrm{st}_\Theta(\tau))$ is the union of all Θ-regular rays in X starting at y and forward asymptotic to τ. Given a Θ-regular pair (y, z) in X, the Θ-*diamond* $\diamondsuit_\Theta(y, z)$ is the set $V(y, \mathrm{st}_\Theta(\tau_+)) \cap V(z, \mathrm{st}_\Theta(\tau_-))$ where $\tau_\pm \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ are the (unique) points such that yz is forward and backward asymptotic to $\tau_\pm$. The Θ-cones/diamonds are closed, convex subset of X.

As a final remark, we note that parallel sets, cones, and diamonds in a rank-one symmetric space are precisely the lines, rays, and segments, respectively.

[13]same type τ_{mod} is understood

[14]A subset Θ of $\mathrm{ost}(\tau_{\mathrm{mod}})$ is called τ_{mod}-*Weyl convex* if its symmetrization $W_{\tau_{\mathrm{mod}}}\Theta$ is convex in a_{mod}. Here $W_{\tau_{\mathrm{mod}}}$ is the stabilizer of τ_{mod} in W for $W \curvearrowright a_{\mathrm{mod}}$. This condition is needed for the desired convexity of Θ-cones/diamonds defined next, see [**KLP17**, Prop. 2.10].

4.4. Definition and different characterizations of Anosov subgroups

In this section, we describe the notion of Anosov subgroups in detail. First, we recall Labourie's original definition. Although we never use this formulation in our thesis, we include this definition as a historical motivation.

Original definition. In [**Lab06**], Labourie introduced the notion of Anosov representations in terms of certain contraction/expansion properties of a flow.

A *flow* on a manifold N is a continuous action of $\mathbb{R}$ on N by homeomorphisms, $t \mapsto \varphi_t : N \to N$, such that $\varphi_0 = \mathrm{Id}_N$ and $\varphi_{s+t} = \varphi_s \varphi_t$ for all $s, t \in \mathbb{R}$. In other words, $t \mapsto \varphi_t$ is a continuous homomorphism from $\mathbb{R}$ to the group of homeomorphisms of N. A C^∞ flow φ_t on a smooth Riemannian manifold N is called *Anosov* if the following conditions are satisfied: The flow φ_t does not have a fixed point in N, and there is a φ_t-invariant splitting of TN,

$$TN = E^s \oplus E^u \oplus E^0,$$

where E^0 is is spanned by the vector field on N defined by the flow, and constants $A, B > 0$ such that

(i) E^s is *uniformly contracting* (stable subbundle): For any $v \in E_s$, $t \in \mathbb{R}$,

$$\|d\varphi_t(v)\| \leq Ae^{-Bt}\|v\|.$$

(ii) E^u is *uniformly expanding* (unstable subbundle): For any $v \in E_s$, $t \in \mathbb{R}$,

$$\|d\varphi_t(v)\| \geq Ae^{Bt}\|v\|.$$

By a celebrated theorem of Anosov, if N is a closed, negatively-curved manifold, then the geodesic flow $\varphi_t : SN \to SN$ is a topologically transitive Anosov flow. In fact, among closed, (locally) symmetric spaces M, the geodesic flow is Anosov iff $\mathrm{rank}(M) = 1$ [**Ebe73**]. It is noted that if N is compact, the Riemannian metric plays no special role in the Anosov condition, i.e., one may replace the Riemannian metric with any continuous family of norms on TN and the above contraction/expansion properties are still satisfied with respect to these norms. In particular, the Anosov property of the geodesic flow is a property for the flow itself.

In higher rank, one needs to replace the unit tangent bundle with an appropriate bundle to define the notion of Anosov flow. This was Labourie's original viewpoint. Let $\Gamma = \pi_1(N)$ where

N is a closed, negatively-curved Riemannian manifold, and let $\rho : \Gamma \to G$ be a representation. Let $M := SN$ and φ_t denotes the geodesic flow on M. The fundamental group Γ of N acts on the universal cover $\widetilde{N}$ by Deck transformations which lifts to an action $\Gamma \curvearrowright \widehat{M} := S\widetilde{N}$. Now, fix a pair $P_\pm$ of opposite parabolic[15] subgroups of G. With these data, we define an associated bundle as follows:

Let $F^\pm := G/P_\pm$, and $G \curvearrowright F^+ \times F^-$ be the action by left multiplication. This action has a unique open G-orbit $\mathcal{X} \subset F^+ \times F^-$ which can be identified with $G/(P_+ \cap P_-)$ (the space of opposite flags). The associated bundle we are considering is

$$\mathcal{X}_\rho := \Gamma \backslash (\widehat{M} \times \mathcal{X}) \longrightarrow M,$$

where the action $\Gamma \curvearrowright \widehat{M} \times \mathcal{X}$ is understood as the diagonal action such that Γ acts on $\widehat{M}$ by Deck transformations and Γ acts on $\mathcal{X}$ via the representation $\rho : \Gamma \to G$. Moreover, the projection map

$$\widehat{M} \times \mathcal{X} \longrightarrow \widehat{M}$$

gives $\mathcal{X}_\rho$ a structure of a *flat bundle* over M with fibers $\mathcal{X}$.

Next, we discuss the Anosov condition: The geodesic flow φ_t on M lifts to a geodesic flow $\hat{\varphi}_t$ on $\widehat{M}$ which commutes with the action $\Gamma \curvearrowright \widehat{M}$. From this, we obtain a flow $\hat{\psi}_t$ on $\widehat{M} \times \mathcal{X}$, $\hat{\psi}_t(m, x) := (\hat{\varphi}_t(m), x)$. Using Γ-invariance, $\hat{\psi}_t$ induces a flow ψ_t on $\mathcal{X}_\rho$ and $(\mathcal{X}_\rho, \psi_t)$ is a replacement for the geodesic flow. Now, using the local product structure of $\mathcal{X}$, $T\mathcal{X}$ splits as a pair of Γ-invariant subbundles, $T\mathcal{X} = E^+ \oplus E^-$. Therefore, $\mathcal{X}_\rho$ comes equipped with two distributions,

$$\mathcal{E}^\pm \to \mathcal{X}_\rho,$$

which are preserved by the flow ψ_t.

Definition 4.4.1 (Anosov representations). The representation $\rho : \Gamma \to G$ is called a (P_+, P_-)-*Anosov representation* if the bundle $\mathcal{X}_\rho \to M$ admits a section $\sigma : M \to \mathcal{X}_\rho$ which is *flat* along the flow lines of φ_t such that the (lifted) action of φ_t on $\sigma^* \mathcal{E}^\pm$ is uniformly expanding/contracting.

A section $\sigma : M \to \mathcal{X}_\rho$ is completely determined by a ρ-equivariant map $\hat{\sigma} : \widehat{M} \to \mathcal{X}$. The flatness condition in the definition then simply means that $\hat{\sigma}(\varphi_t m) = \hat{\sigma}(m)$, for all $m \in M$ and

[15] For the moment, we do not impose the parabolic subgroups to be conjugate to its opposite.

$t \in \mathbb{R}$. This, in fact, gives us Γ-equivariant topological embedding

$$\partial_\infty^{(2)} \widetilde{N} \cong \mathbb{R} \backslash \widehat{N} \longrightarrow \mathcal{X} \subset F^+ \times F^-,$$

where $\partial_\infty^{(2)} \widetilde{N}$ is $\partial_\infty \widetilde{N} \times \partial_\infty \widetilde{N}$ minus the diagonal. The contraction of the flow on the bundle $\sigma^* \mathcal{E}^+$ implies that the projection map $\partial_\infty^{(2)} \widetilde{N} \to F^+$ factors through the projection $\partial_\infty^{(2)} \widetilde{N} \to \partial_\infty \widetilde{N}$ onto the first factor. Similar is also true for the expanding bundle $\sigma^* \mathcal{E}^-$. This determines a pair of Γ-equivariant Hölder maps (*boundary embedding*)

$$\xi^\pm : \partial_\infty N \to F^\pm = G/P_\pm. \tag{4.7}$$

Moreover, the uniform expansion/contraction in the above is measured with respect to a chosen continuous family of norms on the fibers of the bundles $\sigma^* \mathcal{E}^\pm$. Again, since N is compact, this choice plays no role.

In [**GW12**], Guichard and Weinhard extended Definition 4.4.1 for representation of any word-hyperbolic group into G. In this setting, one needs to replace the geodesic flow (M, ϕ_t) in a suitable way, and such a replacement is offered by the *Gromov geodesic flow* $\widehat{\Gamma}$ of Γ. See [**GW12**, Sec. 2] for details. Apart from many other things discussed in this paper, they proved that Anosov condition is an *open* condition: I.e., a small perturbation of an Anosov representation $\rho : \Gamma \to G$ is again Anosov.

P-Anosov representations. In Definition 4.4.1, when both P_+ and P_- are conjugate to $P < G$, then we call a (P_+, P_-)-Anosov representation simply by a *P-Anosov representation.* It turns out that every (P_+, P_-)-Anosov representation is a P-Anosov representation for an appropriate[16] parabolic subgroup $P < G$ which is conjugate to its opposite. Compare with [**GW12**, Lem. 3.18]. Moreover, if $\rho : \Gamma \to G$ is a P-Anosov representation (where P is conjugate to its opposite), then the boundary embeddings $\xi^\pm : \partial_\infty \Gamma \to G/P$ are equal[17], i.e., $\xi^+ = \xi^-$. Hence, we obtain a single map

$$\xi : \partial_\infty \Gamma \to G/P.$$

[16]For instance, one can take $P = P_{\tau_{\mathrm{mod}}} \cap P_{\iota\tau_{\mathrm{mod}}}$ where $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ is a face such that P_+ is conjugate to $P_{\tau_{\mathrm{mod}}}$ and P_- is conjugate to $P_{\iota\tau_{\mathrm{mod}}}$. See Section 4.2.

[17]In fact, a stronger result holds for Zariski-dense Anosov representations, see [**KLP17**, Thm. 5.14].

Definition 4.4.2 (Limit set). For a P-Anosov subgroup $\Gamma < G$, the *(flag) limit set* $\Lambda(\Gamma)$ is the image of ξ,

$$\Lambda(\Gamma) := \xi(\partial_\infty \Gamma) \subset G/P.$$

The notion of "flag limit set" makes sense for more general class of discrete subgroups of G. See Definition 4.4.5 below.

From now on, we abandon the discussion of Anosov subgroups *à la* Labourie since this will play no role in the next two chapters. Instead, we use the following characterizations due to Kapovich, Leeb, and Porti [**KLP17**, **KLP18a**, **KL18b**]. See [**KL18a**] for a survey.

Morse subgroups. The *Morse property* in higher rank was introduced by Kapovich-Leeb-Porti [**KLP14**]. Recall that a *quasigeodesic* in X is a quasiisometric embedding $\phi : I \to X$ of an interval $I \subset \mathbb{R}$. We say that ϕ is τ_{mod}*-regular quasigeodesic* if for all sufficiently separated points $t_1, t_2 \in I$, the segment $\phi(t_1)\phi(t_2)$ is τ_{mod}-regular. We say that ϕ is a τ_{mod}*-Morse quasigeodesic* if it is τ_{mod}-regular and for all sufficiently separated points $t_1, t_2 \in I$, the image $\phi([t_1, t_2])$ is uniformly close to $\diamondsuit_{\tau_{\text{mod}}}(\phi(t_1), \phi(t_2))$.

Let Z be a geodesic Gromov-hyperbolic metric space[18]. A quasiisometric map $\phi : Z \to X$ is called a τ_{mod}*-Morse embedding* if the image of every geodesic in Z is a τ_{mod}-Morse quasigeodesic with uniformly controlled coarse-geometric quantifiers: There exists a constant $D > 0$ and a ι-invariant, compact, τ_{mod}-Weyl convex subset $\Theta \subset \text{ost}(\tau_{\text{mod}})$ such that if $z_1 z_2$ is a geodesic segment in Z of length $\geq D$, then $\phi(z_1)\phi(z_2)$ is a Θ-regular geodesic in X and the image $\phi(z_1 z_2)$ is D-close to $\diamondsuit_{\tau_{\text{mod}}}(\phi(z_1), \phi(z_2))$. See Footnote 14 in page 34 for the definition of τ_{mod}-Weyl convexity.

Definition 4.4.3 (τ_{mod}-Morse subgroups). Let $\Gamma < G$ be a discrete, finitely-generated subgroup equipped with a word metric. Then, Γ is called τ_{mod}*-Morse subgroup* if Γ is word-hyperbolic and, for an(y) $x \in X$, the orbit map $\Gamma \to \Gamma x$ is a τ_{mod}-Morse embedding.

RCA subgroups. We first define the notion of *regular* sequences in X. Let τ_{mod} be an ι-invariant face of σ_{mod}. Let $V(0, \partial\text{st}(\tau_{\text{mod}}))$ denote the union of all rays in Δ emanating from 0 asymptotic to points $\xi \in \partial\text{st}(\tau_{\text{mod}})$. A sequence (x_n) on X diverging to infinity is τ_{mod}*-regular* if for all $x \in X$, the sequence $(d_\Delta(x, x_n))_{n\in\mathbb{N}}$ in Δ diverges away from $V(0, \partial\text{st}(\tau_{\text{mod}}))$. A sequence (g_n) in G is τ_{mod}-regular if for some (equivalently, every) $x \in X$, the sequence $(g_n(x))$ is τ_{mod}-regular.

[18]See Definition 6.3.2.

Definition 4.4.4 (τ_{mod}-regular subgroups). A discrete subgroup $\Gamma < G$ is τ_{mod}*-regular* if for some (equivalently, all) $x \in X$ and all sequences of distinct elements (γ_n) in Γ, the sequence $(\gamma_n x)$ is τ_{mod}-regular.

For every τ_{mod}-regular subgroup $\Gamma < G$, one can define the notion of *flag limit set* in the following way: For $x \in X$ and $A \subset X$, define the *shadow of* A *on* $\text{Flag}(\tau_{\text{mod}})$ *from* x as

$$S(x : A) = \{\tau \in \text{Flag}(\tau_{\text{mod}}) \mid A \cap V(x, \text{st}(\tau)) \neq \emptyset\}. \tag{4.8}$$

Let (g_n) be a τ_{mod}-regular sequence in G. A sequence (τ_n) in $\text{Flag}(\tau_{\text{mod}})$ is called a *shadow sequence* for (g_n) if there exists $x \in X$ such that, for every $n \in \mathbb{N}$, $\tau_n = S(x : \{g_n x\})$. A τ_{mod}-regular sequence (g_n) is said to be τ_{mod}*-flag-convergent* to $\tau \in \text{Flag}(\tau_{\text{mod}})$ if a(ny) shadow sequence (τ_n) of (g_n) converges to τ.

Definition 4.4.5 (Limit set). Let Γ be a τ_{mod}-regular subgroup of G. The τ_{mod}*-flag limit set* of Γ denoted by $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is the subset of $\text{Flag}(\tau_{\text{mod}})$ which consists of all limit points of τ_{mod}-flag-convergent sequences on Γ.

The flag limit set $\Lambda_{\tau_{\text{mod}}}$ is Γ-invariant, closed subset of $\text{Flag}(\tau_{\text{mod}})$. Moreover, it provides a compactification of the orbit $\Gamma x \subset X$, i.e., $\Gamma x \sqcup \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is compact in the shadow topology. More generally, one defines τ_{mod}-flag limit sets of a subset $Z \subset X$ as the accumulation set in $\text{Flag}(\tau_{\text{mod}})$ of τ_{mod}-regular sequences in Z with respect to the topology of flag-convergence.

Definition 4.4.6 (τ_{mod}-RA subgroups). A τ_{mod}-regular subgroup Γ is τ_{mod}*-RA*[19] if its limit set $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is *antipodal*: Every two distinct elements of $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ are antipodal to each other.

For τ_{mod}-RA subgroups Γ, the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is a convergence action (see [**KLP14**, Prop. 5.38]). A τ_{mod}-RA subgroup Γ is called *nonelementary* if $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ consists of at least three (hence infinitely many) points; otherwise Γ is called *elementary*. If Γ is nonelementary then the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is *minimal*: Every orbit of Γ is dense, and $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is perfect.[20]

We note that when $\text{rank}(G)$ is one, then the above condition of being τ_{mod}-RA is vacuously satisfied by all discrete subgroups of G.

[19]RA stands for regular and antipodal

[20]This follows from a general result for convergence actions by Gehring-Martin [**GM87**] and Tukia [**Tuk94**]. See also [**KLP14**, Subsec. 3.2] or [**KLP17**, Subsec. 3.3].

Finally, we define the notion of *conicality*: For a τ_{mod}-regular subgroup Γ, a limit point $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is a *conical limit point* if there exists $x \in X$, $c > 0$ and a sequence (γ_n) of pairwise distinct elements of Γ such that

$$d(\gamma_n x, V(x, \text{st}(\tau))) \leq c.$$

The set of all conical limit points is denoted by $\Lambda^{\text{con}}_{\tau_{\text{mod}}}(\Gamma)$.

DEFINITION 4.4.7 (τ_{mod}-RC subgroups). A τ_{mod}-regular subgroup $\Gamma < G$ is called τ_{mod}-*RC*[21] if $\Lambda_{\tau_{\text{mod}}}(\Gamma) = \Lambda^{\text{con}}_{\tau_{\text{mod}}}(\Gamma)$, i.e., every limit point of Γ is a conical limit point.

When G has rank one, then the conicality condition is precisely satisfied by the convex-cocompact subgroups of G.

Finally,

DEFINITION 4.4.8 (τ_{mod}-RCA subgroups). A discrete subgroup $\Gamma < G$ is called τ_{mod}-RCA if Γ is τ_{mod}-RA and τ_{mod}-RC.

URU subgroups. A τ_{mod}-regular sequence (x_n) is called *uniformly* τ_{mod}*-regular* if the following holds: Consider the sequence $(d_\Delta(x, x_n))_{n \in \mathbb{N}}$ in Δ. Then, the sequence (x_n) is called *uniformly* τ_{mod}*-regular* if $(d_\Delta(x, x_n))_{n \in \mathbb{N}}$ diverges away from $V(0, \partial\text{st}(\tau_{\text{mod}}))$ with a linear rate, i.e.,

$$\liminf_{n \to \infty} \frac{d_{\text{E}}\left(d_\Delta(x, x_n), V(0, \partial\text{st}(\tau_{\text{mod}}))\right)}{d(0, d_\Delta(x, x_n))} > 0.$$

In the above, d_{E} denotes the euclidean distance on Δ. Accordingly, a sequence (g_n) in G is uniformly τ_{mod}-regular if for some (equivalently, every) $x \in X$, the sequence $(g_n(x))$ is uniformly τ_{mod}-regular.

DEFINITION 4.4.9 (Uniformly τ_{mod}-regular subgroups). A discrete subgroup $\Gamma < G$ is *uniformly* τ_{mod}*-regular* if for some (equivalently, all) $x \in X$ and all sequences of distinct elements (γ_n) in Γ, the sequence $(\gamma_n x)$ is uniformly τ_{mod}-regular.

We note that again when $\text{rank}(G)$ is one, then the above definition is vacuously satisfied by all discrete subgroups of G.

DEFINITION 4.4.10 (Undistorted). A finitely generated subgroup $\Gamma < G$ (equipped with the word metric) is said to be *undistorted* if one (equivalently, every) orbit map $\Gamma \to \Gamma x \subset X$ is a quasiisometric embedding.

[21]RC stands for regular and conical

Again, when $\mathrm{rank}(G)$ is one, then the discrete subgroups which are undistorted are precisely the convex-cocompact subgroups of G.

Finally,

Definition 4.4.11 (τ_{mod}-URU subgroups). A subgroup $\Gamma < G$ is said to be τ_{mod}-URU if it is both τ_{mod}-uniformly regular and undistorted.

Equivalent characterizations of Anosov subgroups. Let τ_{mod} be a ι-invariant face of σ_{mod} and let P be a parabolic subgroup of G conjugate to $P_{\tau_{\mathrm{mod}}}$.

Equivalence Theorem 4.4.12 (Kapovich-Leeb-Porti [**KLP17**, Thm. 1.1]). *The following classes of nonelementary discrete subgroups of G are equal:*

(i) *P-Anosov,*

(ii) *τ_{mod}-RCA,*

(iii) *τ_{mod}-Morse,*

(iv) *τ_{mod}-URU.*

As a final remark, we note that, for an Anosov subgroup Γ, the two definitions of limit sets (Definitions 4.4.2 & 4.4.5) agree with each other.

Examples. We discuss some examples of Anosov subgroups in the higher rank. Note that, few rank-one examples were discussed in Chapter 3.

The following are two interesting classes of examples of Anosov subgroups; both of these arise as representations of surface groups Γ into some Lie groups G:

(1) *Hitchin representations*: In his seminal paper [**Hit92**], Hitchin studied the character variety

$$\mathrm{Rep}(\Gamma, \mathrm{PSL}(n,\mathbb{R})) := \mathrm{Hom}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R}), \quad n > 2,$$

where Σ_g is a surface of genus $g \geq 2$. Using *Higgs bundle* techniques, he proved that a distinguished component (or two isomorphic ones, when n is even) of $\mathrm{Rep}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ is homeomorphic to the euclidean space of dimension $(2g-2)(n^2-1)$. Labourie [**Lab06**] introduced the term *Hitchin component* for this special component. He proved that each element of this component is P_{min}-Anosov, where P_{min} is a minimal parabolic subgroup of $\mathrm{PSL}(n,\mathbb{R})$.

(2) *Maximal representations*: These representations were introduced in [**BIW10**]. Let G be a semisimple Lie group of *Hermitian type*, i.e., the symmetric space $X = G/K$ admits a

G-invariant complex structure. In Chapter 3, we encountered such examples, namely, the complex-hyperbolic n-space, the symmetric space of $\mathrm{SU}(n, 1)$. Some other examples are given by symmetric spaces of $G = \mathrm{SU}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$, and the products of such spaces, etc.

Now, let $\rho : \Gamma \to G$ be a representation, where Γ is again the fundamental group of a compact surface Σ. When $G = \mathrm{SU}(n, 1)$, Toledo [**Tol89**] assigned a number $c(\rho)$ to ρ which is an invariant, meaning, that it does not change under continuous deformations of ρ. In fact, we also briefly encountered this number in Chapter 3 (see Proposition 3.4.2). The paper [**BIW10**] generalized the notion of Toledo invariant for representations $\rho : \Gamma \to G$ into arbitrary Lie groups G of Hermitian type. They proved that $|c(\rho)| \leq |\chi(\Sigma)| \cdot \mathrm{rank}(G)$. A representation $\rho : \Gamma \to G$ is called *maximal* if it has the maximal value for the Toledo invariant, i.e., $|c(\rho)| = |\chi(\Sigma)| \cdot \mathrm{rank}(G)$. The authors further gave a characterization of maximal representations in terms of equivariant boundary maps.

In [**BILW05**], it was proven that, when $G = \mathrm{Sp}(2n, \mathbb{R})$, then maximal representations $\rho : \Gamma \to G$ are P-Anosov for a specific maximal parabolic subgroup $P < G$.

Another interesting class of examples of Anosov subgroups are given by *Morse-Schottky subgroups*, see Theorem 2.2.5. Furthermore, in Section 6.9, we discuss other examples of Anosov subgroup of $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$.

In fact, one obtains many examples of Anosov representations from the known ones: For example, if G has rank-one and $\varphi : G \to G'$ is a monomorphism induced by a totally geodesic embedding of symmetric spaces $X \to X'$, where X' is the symmetric space of G', and $\Gamma < G$ is an Anosov subgroup of G, then $\varphi(\Gamma)$ is a P'-Anosov subgroup of G', for a suitable parabolic subgroup $P' < G'$. Our combination theorem (Theorem 2.2.4) provides another way to construct Anosov subgroups using known ones.

CHAPTER 5

A combination theorem for Anosov subgroups

This chapter is based on [**DKL19**]. We continue with the terminology introduced in Section 2.2.

The goal of this chapter is to prove

THEOREM 2.2.4 (Combination theorem). *Let $\Gamma_1, \ldots, \Gamma_n$ be pairwise antipodal, residually finite*[1] *P-Anosov subgroups of G. Then there exist finite index subgroups Γ_i' of Γ_i, for $i = 1, \ldots, n$, such that the subgroup $\langle \Gamma_1', \ldots, \Gamma_n' \rangle$ generated by $\Gamma_1', \ldots, \Gamma_n'$ in G is P-Anosov, and is naturally isomorphic to the free product $\Gamma_1' * \cdots * \Gamma_n'$.*

Although we have stated this theorem in the language of Anosov representations, we do not use it in our proof. Instead, we use the language of Morse subgroups to prove the following statement.

THEOREM 5.0.1. *Let $\Gamma_1, \ldots, \Gamma_n$ be pairwise antipodal, residually finite τ_{mod}-Morse subgroups of G. Then, there exist finite index subgroups $\Gamma_i' < \Gamma_i$, for $i = 1, \ldots, n$, such that $\langle \Gamma_1', \ldots, \Gamma_n' \rangle$ is τ_{mod}-Morse, and is naturally isomorphic to $\Gamma_1' * \cdots * \Gamma_n'$*

Organization of this chapter. In Section 5.1, we prove several estimates on ξ-angles which will provide crucial ingredients for construction of Morse embeddings in the proof of Theorem 5.0.1. In Section 5.2, we discuss more about the Morse condition. In this section, we introduce the replacement lemma (Theorem 5.2.11 and a generalized version Theorem 5.2.13) which is another important ingredient in the proof of our main result. Finally, in Section 5.3, we prove Theorem 5.0.1.

5.1. Visual angle estimates

The key result in this section is Proposition 5.1.9 which will be used in the proof of Theorem 5.0.1 to construct *Morse quasigeodesics* (see Definition 5.2.3). In the first section, we first obtain some weaker results which would lead to the estimates in Proposition 5.1.9 in the later section.

[1]It suffices to assume that each Γ_i has trivial intersection with the center of G. See also the remark following Theorem 5.0.1.

In what follows, we always denote by τ_{mod} an ι-invariant face of the model chamber σ_{mod}. The sets denoted by Θ, Θ' etc. will always be ι-invariant, compact, τ_{mod}-Weyl convex subset of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$. By ξ_{mod} we denote an ι-invariant point in the interior of τ_{mod}.

Small visual angles I. Define the *space of antipodal simplices*

$$\mathcal{X} = \left(\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}})\right)^{\mathrm{opp}} \underset{\mathrm{open}}{\subset} \mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}})\,,$$

which consists of all pairs of antipodal simplices of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. This space has a transitive G-action which makes it a homogeneous G-space. The point stabilizer H of this action is the intersection of two opposite parabolic subgroups of G.

Throughout in this section x will be a fixed point of X. For a point $\omega = (\tau_+, \tau_-) \in \mathcal{X}$, let $P(\omega)$ denote the parallel set $P(\tau_+, \tau_-)$. We define a function $d_x^{\mathrm{opp}} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ by

$$d_x^{\mathrm{opp}}\,(\omega) = d\,(x, P(\omega))\,.$$

Proposition 5.1.1. *The function d_x^{opp} is continuous.*

Proof. The proof is the same as of Lemma 2.21 of [**KLP17**]. Fix a point $\omega_0 \in \mathcal{X}$. From the fiber bundle theory, we have a fibration

$$H \longrightarrow G \xrightarrow{\mathrm{ev}_{\omega_0}} \mathcal{X},$$

where H denotes the point stabilizer of the transitive G action, and ev_{ω_0} denotes the evaluation map $\mathrm{ev}_{\omega_0}(g) = g \cdot \omega_0$. See [**Ste99**, Sections 7.4, 7.5]. For any $\omega \in \mathcal{X}$, there exists a neighborhood U such that ev_{ω_0} has a local section σ over U,

$$\sigma : U \to G, \quad \mathrm{ev}_{\omega_0} \circ \sigma = \mathrm{Id}_U.$$

It suffices to show that d_x^{opp} is continuous on such neighborhoods U.

Define a function $d' : X \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ by $d'(x, \omega) = d_x^{\mathrm{opp}}(\omega)$. Note that the action of G on $X \times \mathcal{X}$ given by $g(x, \omega) = (gx, g\omega)$ leaves d' invariant. Therefore, on U,

$$\begin{aligned} d_x^{\mathrm{opp}}(\omega) &= d'(x, \omega) \\ &= d'(x, \sigma(\omega)\omega_0) = d'(\sigma(\omega)^{-1}x, \omega_0) \\ &= d^{\mathrm{opp}}_{\sigma(\omega)^{-1}x}(\omega_0) = d(\sigma(\omega)^{-1}x, P(\omega_0)), \end{aligned}$$

where the last function is continuous on U. Therefore, d_x^{opp} is continuous on U. $\square$

Definition 5.1.2 (Antipodal subsets). A pair of subsets Λ_1, Λ_2 of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ is called *antipodal*, if any simplex $\tau_1 \in \Lambda_1$ is antipodal[2] to any simplex $\tau_2 \in \Lambda_2$ and vice versa.

Let Λ_1 and Λ_2 be a pair of compact, antipodal subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. Then, $\Lambda_1 \times \Lambda_2$ is a compact subset of $\mathcal{X}$.

Proposition 5.1.1 implies:

Corollary 5.1.3. *Let Λ_1 and Λ_2 be compact, antipodal subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. *If Λ_1 and Λ_2 are antipodal, then, for any point $x \in X$, there is a number $D = D(\Lambda_1, \Lambda_2, x)$ such that*

$$d(x, P(\tau_1, \tau_2)) \leq D, \quad \forall \tau_1 \in \Lambda_1, \forall \tau_2 \in \Lambda_2.$$

Proposition 5.1.4. *Let $\Lambda_1, \Lambda_2 \subset \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be compact, antipodal subsets. There exists a function $f = f(\Lambda_1, \Lambda_2, x) : [0, \infty) \to [0, \pi]$ satisfying $f(R) \to 0$ as $R \to \infty$ such that for any $\tau_1 \in \Lambda_1$, $\tau_2 \in \Lambda_2$, and for any $z_1 \in x\xi_{\tau_1}$, $z_2 \in x\xi_{\tau_2}$ satisfying $d(z_1, x), d(z_2, x) \geq R$, we have*

$$\alpha_1 = \angle_{z_1}(x, z_2) \leq f(R), \quad \alpha_2 = \angle_{z_2}(x, z_1) \leq f(R).$$

Notation. Throughout, we use the following notation for angles: For $x \in X$ and $y, z \in X \cup \partial_\infty X$,

$$\angle_x(y, z) := \text{angle formed at } x \text{ between the geodesics } xy \text{ and } xz \in [0, \pi].$$

Moreover, when $\eta, \zeta \in \partial_\infty X$, then the *Tits angle* is defined by

$$\angle_{\mathrm{Tits}}(\eta, \zeta) := \sup_{x \in X} \angle_x(\eta, \zeta). \tag{5.1}$$

[2]See Section 4.3 for the definition of *antipodal* simplices.

PROOF OF PROPOSITION 5.1.4. Let $\bar{x} \in P(\tau_1, \tau_2)$ be the point closest to x. Recall that we denote the Cartan involution about a point $y \in X$ by s_y. Note that $s_{\bar{x}}$ preserves $P(\tau_1, \tau_2)$. Since τ_1 and τ_2 are antipodal, $s_{\bar{x}}(\tau_1) = \tau_2$. Hence $\angle^{\xi}_{\bar{x}}(\tau_1, \tau_2) = \pi$, i.e. $s_{\bar{x}}(\xi_{\tau_1}) = \xi_{\tau_2}$. Let $c : (-\infty, \infty) \to P(\tau_1, \tau_2)$, $c(0) = \bar{x}$, be the biinfinite geodesic passing through $\bar{x}$ and asymptotic to $c(+\infty) = \xi_{\tau_1}$ and $c(-\infty) = \xi_{\tau_2}$. For $i = 1, 2$, let $c_i : [0, \infty) \to X$ be the geodesic ray $x\xi_{\tau_i}$ (see Figure 5.1(a)). Since the functions $d(c(t), c_1(t))$ and $d(c(-t), c_2(t))$ are bounded convex functions, they are decreasing with maximum at $t = 0$. Therefore,

$$d(c(t), c_1(t)) \leq D,\ d(c(-t), c_2(t)) \leq D, \quad \forall t \in [0, \infty), \tag{5.2}$$

where $D > 0$ is a number as in Corollary 5.1.3.

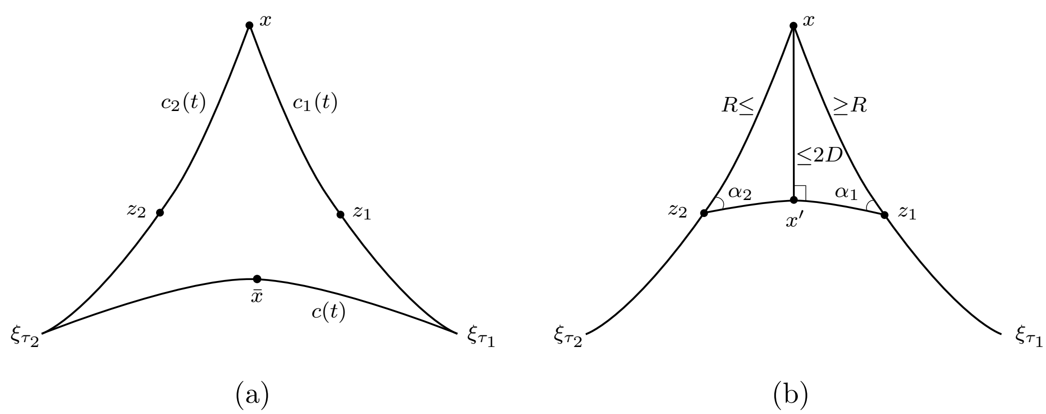

FIGURE 5.1

For $R \geq 0$, let $c_1(t_1) = z_1 \in x\xi_{\tau_1}$ and $c_2(t_2) = z_2 \in x\xi_{\tau_2}$ be any points satisfying $t_1 = d(z_1, x) \geq R$ and $t_2 = d(z_2, x) \geq R$. By (5.2), the Hausdorff distance between the segments z_1z_2 and $c(t_1)c(-t_2)$ is bounded above by D. Combining with $d(x, \bar{x}) \leq D$ we obtain

$$d(x, z_1z_2) \leq 2D. \tag{5.3}$$

Let x' be the point on z_1z_2 nearest to x. When $R \geq 2D + 1$, x' is in the interior of z_1z_2. Consider geodesic triangles $\triangle_1 = \triangle(x, x', z_1)$ and $\triangle_2 = \triangle(x, x', z_2)$; the angle of $\triangle_1$ and $\triangle_2$ at the vertex x' is $\pi/2$. Let $\alpha_1 = \angle_{z_1}(x, x') = \angle_{z_1}(x, z_2)$ and $\alpha_2 = \angle_{z_2}(x, x') = \angle_{z_2}(x, z_1)$ (see Figure 5.1(b)). Let $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$ be the Euclidean comparison triangles of $\triangle_1$ and $\triangle_2$, respectively; we denote the corresponding vertices of $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$ by the same symbols. In the triangles $\tilde{\triangle}_1$, $\tilde{\triangle}_2$, since the angles at the vertex x' are at least $\pi/2$, we have

$$\tilde{\alpha}_i \leq \sin^{-1}\left(\frac{xx'}{xz_i}\right) \leq \sin^{-1}\left(\frac{2D}{R}\right), \qquad i = 1, 2,$$

where $\tilde{\alpha}_i$ denotes the angle corresponding to α_i. The second inequality in above comes from (5.3). Since the triangles $\triangle_1$ and $\triangle_2$ are thinner than the triangles $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$, respectively, we have $\alpha_i \leq \tilde{\alpha}_i$. Therefore, when $R \geq 2D+1$, $f(R)$ can be given by the following formula:

$$f(R) = \sin^{-1}\left(\frac{2D}{R}\right).$$

The domain of f can be extended to $R < 2D+1$ continuously. However, the continuity of f is irrelevant; we can simply set $f(R) = \pi$ for $R < 2D+1$. $\square$

For the next result, we first need to review the notion of ξ-*angles*. Let τ_{mod} be a ι-invariant face of σ_{mod}. For every such face we pick one and for all a fixed point $\xi = \xi_{\mathrm{mod}}$ of ι in the interior of τ_{mod}. Then, for every simplex τ in $\partial_{\mathrm{Tits}}X$ of type τ_{mod}, we define a point $\xi_\tau \in \tau$ by

$$\{\xi_\tau\} = \theta^{-1}(\xi_{\mathrm{mod}}) \cap \tau.$$

For a type (face) τ_{mod} of σ_{mod} and a point $x \in X$, we define the ξ-*angle* between two simplices τ_1 and τ_2 of type τ_{mod} with respect to x by

$$\angle_x^\xi(\tau_1, \tau_2) := \angle_x(\xi_{\tau_1}, \xi_{\tau_2}).$$

Given τ_{mod}-regular segments xy_1, xy_2 in X, we define the ξ-*angle*

$$\angle_x^\xi(y_1, y_2) := \angle_x^\xi(\tau_1, \tau_2),$$

where $y_i \in V(x, \mathrm{st}(\tau_i)), i = 1, 2$.

Now we give a ξ-angle version of the proposition above which will be useful in the next section.

Proposition 5.1.5. *Let $\Lambda_1, \Lambda_2 \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ be compact antipodal subsets. Given $\Theta \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ containing ξ_{mod} in its interior, there exists $R_0 = R_0(x, \Lambda_1, \Lambda_2, \Theta, \xi)$ such that for any $\tau_1 \in \Lambda_1$, $\tau_2 \in \Lambda_2$, and for any $z_1 \in x\xi_{\tau_1}$, $z_2 \in x\xi_{\tau_2}$ satisfying $d(z_1, x), d(z_2, x) \geq R_0$, the segment z_1z_2 is Θ-regular.*

Moreover, there exists a function $f_0 = f_0(x, \Lambda_1, \Lambda_2, \xi) : [0, \infty) \to [0, \pi]$ satisfying $f_0(R) \to 0$ as $R \to \infty$ such that for any $\tau_1 \in \Lambda_1$, $\tau_2 \in \Lambda_2$, and for any $z_1 \in x\xi_{\tau_1}$, $z_2 \in x\xi_{\tau_2}$ satisfying

$d(z_1, x), d(z_2, x) \geq R \geq R_0$, *we have*

$$\angle^{\xi}_{z_1}(x, z_2), \angle^{\xi}_{z_2}(x, z_1) \leq f_0(R).$$

PROOF. Let $\alpha = \min\{\angle_{\mathrm{Tits}}(\xi, \zeta) : \zeta \in \partial\Theta\} > 0$. Here $\angle_{\mathrm{Tits}}$ denotes the Tits angle, see (5.1). Using the triangle inequality for the Δ-valued distances (Theorem 4.1.4), we get

$$\|d_\Delta(x, z_1) - d_\Delta(x_1, z_1)\| \leq d(x, x_1),$$

for any point $x_1 \in X$. Specializing to $x_1 = x'$, the point on $z_1 z_2$ closest to x, we obtain

$$\left\|d_\Delta(x, z_1) - d_\Delta(x', z_1)\right\| \leq 2D.$$

Then $x'z_1$ is Θ-regular when xz_1 has length $\geq 2D/\sin\alpha$. Therefore, the constant R_0 can be given by

$$R_0 = \frac{2D}{\sin\alpha}. \tag{5.4}$$

This proves first part of the proposition.

For the second part, let $(\Theta_n)_{n\in\mathbb{N}}$ be a nested sequence of ι-invariant, compact, τ_{mod}-Weyl convex subsets of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ such that ξ is an interior point of each Θ_n, and $\bigcap_{n=1}^{\infty} \Theta_n = \{\xi\}$. Let α_n be the Tits-distance from ξ to the boundary of Θ_n,

$$\alpha_n = \min\{\angle_{\mathrm{Tits}}(\xi, \zeta) : \zeta \in \partial\Theta_n\} > 0.$$

Clearly, $(\alpha_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence converging to zero. This implies that $R_0(\Theta_n)$ is strictly increasing which diverges to infinity, where R_0 is as in (5.4). If $R_0(\Theta_n) \leq d(x, z_1) < R_0(\Theta_{n+1})$, then the first part of the proposition implies that $z_2 z_1$ is Θ_n-regular, which then implies

$$\begin{aligned}\angle^{\xi}_{z_2}(x, z_1) &\leq \angle_{z_2}(x, z_1) + \angle_0(\xi, d_\Delta(z_2, z_1)) \\ &\leq f(R) + \alpha_n,\end{aligned}$$

where the function f is as in Proposition 5.1.4. Therefore, when $R_0(\Theta_n) \leq R < R_0(\Theta_{n+1})$, we may define

$$f_0(R) = f(R) + \alpha_n.$$

As in the case of f in Proposition 5.1.4, continuity of f_0 is irrelevant. $\square$

Small visual angles II. The Θ-cones (over a fixed simplex $\tau \in \mathrm{Flag}\,(\tau_{\mathrm{mod}})$) vary continuously with their tips. Here, the topology on the set of Θ-cones over a fixed simplex τ is given by their Hausdorff distances. Precisely, we have,

THEOREM 5.1.6 (Uniform continuity of Θ-cones, [**KLP14**]). *The Hausdorff distance between two Θ-cones over a fixed $\tau \in \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ is bounded by the distance between their tips,*

$$d_{\mathrm{Haus}}\left(V(x, \mathrm{st}_{\Theta}(\tau)), V(\bar{x}, \mathrm{st}_{\Theta}(\tau))\right) \leq d(x, \bar{x}).$$

Moreover, for diamonds, one also has the following form of uniform continuity. This will be useful here, especially in the discussion of replacements in Section 5.2.

THEOREM 5.1.7 (Uniform continuity of diamonds). *Given any Θ' with $\mathrm{int}\,(\Theta') \supset \Theta$, and any $\delta > 0$, there exists $c = c(\Theta, \Theta', \delta)$ such that for all Θ-regular segments xy and $x'y'$ with $d(x, x') \leq \delta$, $d(y, y') \leq \delta$, we have*

$$\diamondsuit_{\Theta}(x, y) \subset N_c\left(\diamondsuit_{\Theta'}(x', y')\right).$$

PROOF. We will prove this theorem as a corollary of [**KLP18b**, Theorem 5.16]: For every (Θ, B)-regular (L, A)-quasigeodesic $q : [a_-, a_+] \to X$ and points $x_{\pm} \in X$ within distance $\leq B$ from $q(a_{\pm})$, the image of q is contained in the $D(L, A, \Theta, B)$-neighborhood of the diamond $\diamondsuit_{\tau_{\mathrm{mod}}}(x_-, x_+)$.

REMARK. Using the hard theorem [**KLP18b**, Theorem 5.16] in order to prove Theorem 5.1.7 is an overkill, but it is quicker than a direct argument. We refer the reader to Section 5.2 for the definition of (Θ, B)-regular quasigeodesics.

By appealing to the triangle inequality for the Δ-valued distances (Theorem 4.1.4), one gets a slightly more precise statement, namely, there exists $D(L, A, \Theta, \Theta', B)$ such that the image of q is contained in the $D(L, A, \Theta, \Theta', B)$-neighborhood of the diamond $\diamondsuit_{\Theta'}(x_-, x_+)$.

We observe that for every point $z \in \diamondsuit_{\Theta}(x, y)$ the broken geodesic segment

$$xz \star zy$$

is $(L,0)$-quasigeodesic for some $L = L(\Theta)$, and is $(\Theta,0)$-regular. Hence, according to the above sharpening of [**KLP18b**, Theorem 5.16], the point z belongs to the $c = D(L,0,\Theta,\Theta',\delta)$-neighborhood of the diamond $\diamondsuit_{\Theta'}(x',y')$, provided that

$$d(x,x') \leq \delta, d(y,y') \leq \delta.$$

Thus,

$$\diamondsuit_{\Theta}(x,y) \subset N_c\left(\diamondsuit_{\Theta'}(x',y')\right). \qquad \square$$

The following lemma will be useful.

LEMMA 5.1.8. *Let $\Theta, \Theta' \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ be compact subsets such that Θ is contained in the interior of Θ', and let $xy \subset X$ be a Θ-regular geodesic. Also, let $x', y' \in X$ points which satisfy $d(x,x'), d(y,y') \leq D$ for some $D > 0$. If $d(x,y) \geq 2D/\sin\alpha$, where $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta') < \pi/2$, then $x'y'$ is Θ'-regular.*

PROOF. Let $xy \subset x\xi$ where ξ is a Θ-regular ideal point. Let y'' be a point on $x'\xi$ which satisfies $d(x,y) = d(x',y'')$. Then, $d(y,y'') \leq d(x,x') \leq D$, where the first inequality comes from the fact that $d(x(t),x'(t))$ is a non-increasing function, $x(t), x'(t)$ being unit speed parameterizations of $x\xi, x'\xi$, respectively. Hence, $d(y',y'') \leq d(y,y') + d(y,y'') \leq 2D$.

The triangle inequality for the Δ-valued distances (Theorem 4.1.4) implies

$$\|d_\Delta(x',y') - d_\Delta(x',y'')\| \leq d(y',y'') \leq 2D.$$

Since $d_\Delta(x',y'')$ is Θ-regular, $d_\Delta(x',y')$ is Θ'-regular whenever

$$\angle\left(d_\Delta(x',y'), d_\Delta(x',y'')\right) \leq \alpha$$

which happens whenever $d(x',y'') \geq 2D/\sin\alpha$. $\square$

Now we turn to the main estimate in this section.

PROPOSITION 5.1.9 (Uniformly small visual angles). *Let $\Lambda_1, \Lambda_2 \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ be compact, antipodal sets, and let Θ' be a subset of $\mathrm{ost}(\tau_{\mathrm{mod}})$ containing Θ in its interior. Let $y_1 \in V(x, \mathrm{st}_\Theta(\tau_1))$ and $y_2 \in V(x, \mathrm{st}_\Theta(\tau_2))$ be any points, where $\tau_1 \in \Lambda_1$ and $\tau_2 \in \Lambda_2$ are any simplices. Then,*

(1) There exists a constant $R_1 = R_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta)$ such that y_1y_2 is Θ'-regular if $d(x,y_i) \geq R_1$.

(2) There exists a function $f_1 = f_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta, \xi) : [0, \infty) \to [0, \pi]$ satisfying

$$\lim_{R\to\infty} f_1(R) = 0$$

such that if $d(x, y_i) \geq R \geq R_1$, then

$$\angle^{\xi}_{y_1}(x, y_2), \angle^{\xi}_{y_2}(x, y_1) \leq f_1(R). \tag{5.5}$$

Proof. For part 1, we take an approach similar to the one given in the proof of Proposition 5.1.5. Let $\bar{x}$ be the nearest point projection of x into the parallel set $P(\tau_1, \tau_2)$, and for each $i = 1, 2$ let $\bar{y}_i$ denote the nearest point projection of y_i into $V(\bar{x}, \mathrm{st}_\Theta(\tau_i)) \subset P(\tau_1, \tau_2)$. Let $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta') > 0$, and $\alpha' = \angle_{\mathrm{Tits}}(\Theta, \partial\mathrm{st}(\tau_{\mathrm{mod}})) \geq \alpha$. Finally, let $D = D(\Lambda_1, \Lambda_2, x)$ be the constant given by Corollary 5.1.3.

Since $d(x, \bar{x}) \leq D$, we combine this with Theorem 5.1.6 to get

$$d(y_i, \bar{y}_i) \leq D, \quad i = 1, 2. \tag{5.6}$$

Then, using the triangle inequality for Δ-valued distances (Theorem 4.1.4), we deduce

$$\begin{aligned} \|d_\Delta(y_1, y_2) - d_\Delta(\bar{y}_1, \bar{y}_2)\| &\leq \|d_\Delta(y_1, y_2) - d_\Delta(\bar{y}_1, y_2)\| + \|d_\Delta(\bar{y}_1, y_2) - d_\Delta(\bar{y}_1, \bar{y}_2)\| \\ &\leq d(y_1, \bar{y}_1) + d(y_2, \bar{y}_2) \leq 2D. \end{aligned}$$

Since $\bar{y}_1\bar{y}_2$ is Θ-regular, y_1y_2 is Θ'-regular whenever y_1y_2 has length $\geq 2D/\sin\alpha$. See Lemma 5.1.8. Moreover,

$$\begin{aligned} d(y_1, y_2) &\geq d(\bar{y}_1, \bar{y}_2) - 2D \\ &\geq d(\bar{y}_i, \bar{x}) \sin\alpha' - 2D \\ &\geq (d(y_i, x) - 2D) \sin\alpha - 2D \\ &= d(y_i, x) \sin\alpha - 2D(1 + \sin\alpha), \end{aligned} \tag{5.7}$$

where the second inequality comes from triangle comparisons (note that $\angle_{\bar{x}}(\bar{y}_1, \bar{y}_2) \geq \alpha'$ because $\bar{y}_1$, $\bar{y}_2$ are in different Θ-cones with tip $\bar{x}$), and the third inequality follows from (5.6), $d(x, \bar{x}) \leq D$, $\pi/2 > \alpha' \geq \alpha$ and the polygon inequality. Using (5.7), we obtain: $d(y_1, y_2) \geq 2D/\sin\alpha$ whenever

$d(x, y_1)$ or $d(x, y_2)$ is greater than $2D(1/\sin^2\alpha + 1/\sin\alpha + 1)$. We may set

$$R_1 = 2D\left(1 + \frac{1}{\sin\alpha} + \frac{1}{\sin^2\alpha}\right).$$

This proves part 1.

To prove part 2 we need the following lemmas.

Recall that $s_x : X \to X$ denotes the Cartan involution of X fixing x.

LEMMA 5.1.10. *Let $\tau, \tau' \in \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be a pair of simplices, let $x \in X$ be any point, and let $y \in V(x, \mathrm{st}_\Theta\,(s_x\tau))$ be a point satisfying $d(x, y) \geq l$. For sufficiently small ϵ, $\epsilon \leq \epsilon_0(\xi_{\mathrm{mod}})$, we have: If $\angle_x^\xi(\tau, \tau') \leq \epsilon$, then*

$$\angle_y^\xi(\tau, \tau') \leq \epsilon'(\Theta, l)$$

with $\epsilon'(\Theta, l) \to 0$ as $l \to \infty$.

PROOF. Let $\xi_+ \in \tau$, $\xi_- \in s_x\tau$, $\xi' \in \tau'$ be ξ_{mod}-regular points. Then

$$\angle_x^\xi(\tau, \tau') \leq \epsilon \implies \angle_x(\xi_+, \xi') \leq \epsilon \implies \angle_x(\xi_-, \xi') \geq \pi - \epsilon.$$

Using [**KLP14**, Lemma 2.44(ii)], there exists a function $\epsilon'(\Theta, l)$ satisfying $\lim_{l\to\infty} \epsilon'(\Theta, l) = 0$ such that

$$\angle_y(\xi_-, \xi') \geq \pi - \epsilon'(\Theta, l).$$

Then,

$$\angle_y^\xi(\tau, \tau') = \angle_y(\xi_+, \xi') = \pi - \angle_y(\xi_-, \xi') \leq \epsilon'(\Theta, l). \qquad \square$$

In the following, Θ'' will denote an auxiliary subset of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ such that $\mathrm{int}\,(\Theta'') \supset \Theta$. Let $\alpha'' = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta'')$.

LEMMA 5.1.11. *Let $\tau \in \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be any simplex, and $y \in V(x, \mathrm{st}_\Theta\,(\tau))$ be any point. If $z \in x\xi_\tau$ is any point such that $d(x, y)\sin\alpha'' \geq d(x, z)$, then*

$$y \in V(z, \mathrm{st}_{\Theta''}\,(\tau)).$$

PROOF. Let F be a maximal flat asymptotic to τ containing x and y, and let y' be the nearest point projection of y into $x\xi_\tau$. Since $\xi \in \tau_{\mathrm{mod}}$, the Tits distance from ξ to any point in Θ is bounded above by $\pi/2 - \epsilon(\Theta)$ where $\epsilon(\Theta) > 0$. Then, the distance from x to y' is comparable to $d(x, y)$, i.e.

$$d(x, y)\cos(\theta) = d(x, y'), \quad \theta = \angle_x(y, y') \leq \pi/2 - \epsilon(\Theta).$$

Notice that $0 < \alpha'' \leq \theta + \alpha'' \leq \pi/2 - \epsilon(\Theta'')$. For any point $z' \in xy'$, $y \in V(z', \mathrm{st}_{\Theta''}(\tau))$ whenever $d(y, y') \leq d(z', y')\tan(\theta + \alpha'')$.

Let $z' \in xy'$ be a point which satisfies $d(y, y') = d(z', y')\tan(\theta + \alpha'')$. In that case,

$$\begin{aligned} d(x, z') &= d(x, y') - d(z', y') \\ &= d(x, y)\cos\theta - d(y, y')\cot(\theta + \alpha'') \\ &= d(x, y)\cos\theta - d(x, y)\sin\theta\cot(\theta + \alpha'') \\ &= d(x, y)\frac{\sin\alpha''}{\sin(\theta + \alpha'')} \\ &\geq d(x, y)\sin\alpha''. \end{aligned}$$

Moreover, if $z \in x\xi_\tau$ is the point satisfying $d(x, z) = d(x, y)\sin\alpha''$, then $z' \in V(z, \mathrm{st}_{\Theta''}(\tau))$, and from convexity of cones, $y \in V(z, \mathrm{st}_{\Theta''}(\tau))$. □

LEMMA 5.1.12. *There exists a function $f_1'(x, \Lambda_1, \Lambda_2, \Theta, \xi) : [0, \infty) \to [0, \pi]$ satisfying $f_1'(R) \to 0$ as $R \to \infty$ such that the following holds: For $\tau_1 \in \Lambda_1$, let $y_1 \in V(x, \mathrm{st}_\Theta(\tau_1))$ be any point. If $d(x, y_1) \geq R$, then*

$$\max_{\tau_2 \in \Lambda_2} \angle^\xi_{y_1}(x, \tau_2) \leq f_1'(R).$$

PROOF. Let $\alpha'' = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta'')$ as defined above, where Θ'' is some auxiliary subset of $\mathrm{ost}(\tau_{\mathrm{mod}})$. Using Lemma 5.1.11, if $z_1 \in x\xi_{\tau_1}$ satisfies $d(x, z_1) = d(x, y_1)\sin\alpha''$, then we have $y_1 \in V(z_1, \mathrm{st}_{\Theta''}(\tau_1))$. See Figure 5.2. Letting $d(x, z_2) \to \infty$ in Proposition 5.1.5, we get

$$\angle^\xi_{z_1}(s_x(\tau_1), \tau_2) = \angle^\xi_{z_1}(x, \tau_2) \leq f_0(R\sin\alpha''), \quad \forall \tau_2 \in \Lambda_2.$$

When R is sufficiently large, $R \geq R_2(x, \Lambda_1, \Lambda_2, \xi)$, then $f_0(R\sin\alpha) \leq \epsilon_0(\xi_{\mathrm{mod}})$, where $\epsilon_0(\xi_{\mathrm{mod}})$ is as in Lemma 5.1.10. Moreover, since $d(y_1, z_1) \geq R(1 - \sin\alpha'')$, Lemma 5.1.10 implies that

$$\angle^\xi_{y_1}(x, \tau_2) = \angle^\xi_{y_1}(s_x(\tau_1), \tau_2) \leq \epsilon'(\Theta'', R(1 - \sin\alpha'')), \quad \forall \tau_2 \in \Lambda_2.$$

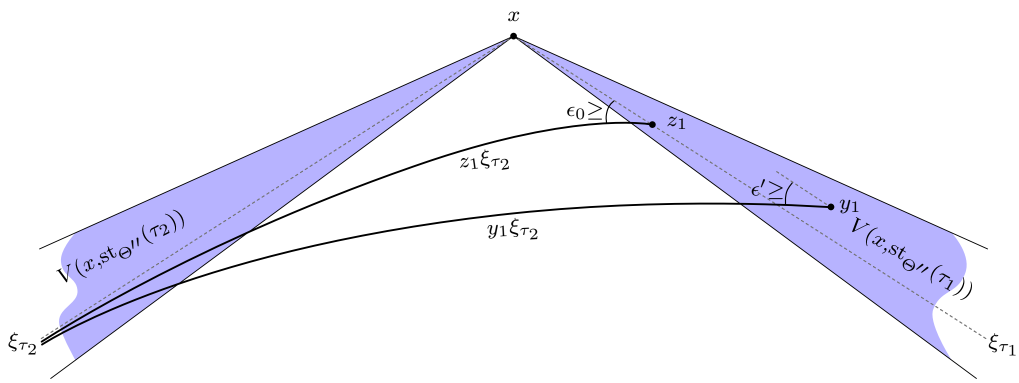

FIGURE 5.2

So, we may define

$$f_1'(R) = \begin{cases} \epsilon'(\Theta'', R(1-\sin\alpha'')), & \text{if } R \geq R_2 \\ \pi, & \text{otherwise} \end{cases}. \qquad \square$$

Now we are ready to prove the estimate (5.5). We first observe that the only property of the point $x \in X$ we have used to estimate the functions f in Proposition 5.1.4, R_0 and f_0 in Proposition 5.1.5 and subsequently f_1' in Lemma 5.1.12 is that

$$d(x, P(\tau_1, \tau_2)) \leq D, \quad \forall \tau_1 \in \Lambda_1, \forall \tau_2 \in \Lambda_2.$$

All these estimates for x work for any other point x_1 satisfying this inequality with the same number D. Moreover, all these estimates work if we replace Λ_1 or Λ_2 by their proper subsets. In particular, we may replace Λ_2 by any of its singleton subsets $\{\tau_2\}$, or replace x by a point $y_2 \in V(x, \mathrm{st}_\Theta(\tau_2))$. Here we use the fact that for a fixed τ_2 and a point y_2 in $V(x, \mathrm{st}_\Theta(\tau_2))$,

$$d(y_2, P(\tau_1, \tau_2)) \leq d(x, P(\tau_1, \tau_2)) \leq D, \quad \forall \tau_1 \in \Lambda_1.$$

Therefore, the same estimate $f_1'(x, \Lambda_1, \Lambda_2, \xi)$ works when x and Λ_2 (elsewhere) in Lemma 5.1.12 are replaced by y_2 and $\{\tau_2\}$, respectively. Precisely, whenever y_1y_2 is a Θ'-regular,

$$\angle_{y_1}^{\xi}(y_2, \tau_2) \leq f_1'(d(y_1, y_2)), \tag{5.8}$$

where $f_1' = f_1'(x, \Lambda_1, \Lambda_2, \Theta', \xi)$. Θ'-regularity of y_1y_2 is also guaranteed whenever, for $i = 1, 2$, $d(x, y_i) \geq R_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta)$.

Therefore, if $R \geq R_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta)$ and $d(x, y_1), d(x, y_2) \geq R$, we can use Lemma 5.1.12, (5.7) and (5.8) to get

$$\begin{aligned}\angle^{\xi}_{y_1}(x, y_2) &\leq \angle^{\xi}_{y_1}(x, \tau_2) + \angle^{\xi}_{y_1}(y_2, \tau_2)\\ &\leq f_1'(R) + f_1'(R \sin\alpha - 2D(1 + \sin\alpha))\\ &\leq 2f_1'(R\sin\alpha - 4D),\end{aligned}$$

where $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta')$.

This completes the proof of part 2. $\square$

5.2. Morse condition

In this section, we discuss Morse quasigeodesics, Morse embeddings and Morse subgroups and their various properties in more detail (cf. page 38). These notions were introduced in [**KLP14**], and it was proved that the notions of Morse subgroups and Anosov subgroups are equivalent (Equivalence Theorem 4.4.12).

The main technical result in this section is the *replacement lemma* (see Theorems 5.2.11 and 5.2.13). This will be an important ingredient in the proof of Theorem 5.0.1.

Stability of quasigeodesics. Recall that a *quasigeodesic* in a metric space (Y, d_Y) is a quasi-isometric embedding of an interval $I \subset \mathbb{R}$ into Y. Quantitively speaking, an (L, A)-quasigeodesic in Y is a map, not necessarily continuous, $\gamma : I \to Y$ which satisfies

$$L^{-1}|a - b| - A \leq d_Y(\gamma(a), \gamma(b)) \leq L|a - b| + A,$$

where d_Y is the metric of Y. When Y is assumed to be a geodesic δ-hyperbolic space, the Morse lemma, proven for these spaces by Gromov [**Gro87**, Proposition 7.2.A], establishes stability of quasigeodesics. Precisely, an (L, A)-quasigeodesic in a δ-hyperbolic space stays within a uniform neighborhood of a geodesic; the radius H of this neighborhood solely depends on the given parameters, namely L, A and δ,

$$H = L^2(A_1 A + A_2\delta),$$

where A_1 and A_2 are universal constants, see [**Shc13**, **GS19**]. The stability of quasigeodesics can also be stated without referring to geodesics: An (L, A)-quasigeodesic path is *stable* if the image of any (L', A')-quasigeodesic with the same endpoints is uniformly close to the original path. Thus, any uniform quasigeodesic in a δ-hyperbolic space is stable. Morse lemma is a vital ingredient to prove the invariance of hyperbolicity under quasiisometries, see [**DK18**, Corollary 11.43].

One of the major differences between the coarse geometric nature of CAT(0) (or *non-positively curved*) and δ-hyperbolic spaces is that the quasigeodesics in CAT(0) spaces can be unstable, and thus the most naive generalization of the Morse lemma fails in the CAT(0) settings, already for the Euclidean plane. Some versions of the Morse lemma are known for CAT(0) spaces; in [**Sul14**] it has been shown that a quasigeodesic is stable if and only if it is *strongly contracting*. However, this class of quasigeodesics is too restrictive in the context of symmetric spaces.

Nevertheless, according to the main theorem of [**KLP18b**], an analogue of the Morse lemma holds for τ_{mod}-*regular quasigeodesics*, with diamonds (or, cones, or parallel sets) replacing geodesic segments (rays, complete geodesics).

The letters B, D which appear bellow are non-negative numbers.

Definition 5.2.1 (Regular quasigeodesics). A pair of points in X is called Θ-*regular* if the geodesic segment connecting them is Θ-regular. An (L, A)-quasigeodesic $\gamma : I \to X$ is called (Θ, B)-*regular* if for all $t_1, t_2 \in I$, $|t_1 - t_2| \geq B$ implies that $(\gamma(t_1), \gamma(t_2))$ is Θ-regular.

In [**KLP18b**, Theorem 5.17], it is shown that (finite) regular quasigeodesics are special in the sense that they live close to the diamonds. This gives a higher rank generalization of the Morse lemma for hyperbolic spaces. We state this result in the next theorem.

Morse Lemma 5.2.2. *[Kapovich-Leeb-Porti [**KLP18b**]] Let $\gamma : [a, b] \to X$ be a (Θ, B)-regular (L, A)-quasigeodesic. There exists a constant $D = D(L, A, \Theta, \Theta', B, X) > 0$ such that the image of γ is contained in the D-neighborhood of a diamond $\diamondsuit_{\Theta'}(x_1, x_2)$ with tips satisfying $d(\gamma(a), x_1) \leq D$, $d(\gamma(b), x_2) \leq D$.*

Now we review the notion of Morse quasigeodesics.

Definition 5.2.3 (Morse quasigeodesics, [**KLP14**]). A (finite, semiinfinite, or biinfinite) (L, A)-quasigeodesic $\gamma : I \to X$ is called a (L, A, Θ, D)-*Morse quasigeodesic* if for all $t_1, t_2 \in I$, the image $\gamma([t_1, t_2])$ is D-close to a Θ-diamond $\diamondsuit_\Theta(x_1, x_2)$ with tips satisfying $d(x_i, \gamma(t_i)) \leq D$, for $i = 1, 2$.

Remark. (1) In light of this definition, the Morse lemma 5.2.2 is equivalent to saying that *the uniformly regular uniform quasigeodesics are uniformly Morse.* Conversely, uniformly Morse quasigeodesics are also uniformly regular.

(2) Note that it is not in general true that for an (L, A, Θ, D)-Morse quasigeodesic γ, the segment $\gamma(t_1)\gamma(t_2)$ is τ_{mod}-regular. However, when $t_2 - t_1$ is uniformly large, $\gamma(t_1)\gamma(t_2)$ becomes uniformly τ_{mod}-regular, and in this case one can say that $\gamma([t_1, t_2])$ lies in a uniform neighborhood of $\diamondsuit_{\Theta'}(\gamma(t_1), \gamma(t_2))$ for any subset $\Theta' \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ containing Θ in its interior (cf. Theorem 5.1.7).

Straight sequences. We review some important tools for constructing Morse quasigeodesics. Let Θ, Θ' be ι-invariant, compact, τ_{mod}-Weyl convex subsets of $\mathrm{ost}(\tau_{\mathrm{mod}})$ such that

$$\mathrm{int}\left(\Theta'\right) \supset \Theta.$$

Definition 5.2.4 (Straight-spaced sequences, [**KLP14**]). Let $\epsilon \geq 0$ be a number. A (finite, infinite, or biinfinite) sequence (x_n) is called (Θ, ϵ)-*straight* if, for each n, the segments $x_n x_{n+1}$ are Θ-regular and

$$\angle^{\xi}_{x_n}(x_{n-1}, x_{n+1}) \geq \pi - \epsilon.$$

Moreover, (x_n) is called l-*spaced* if $d(x_n, x_{n+1}) \geq l$ for all n.

Definition 5.2.5 (Morse sequence). A sequence (x_n) is called (Θ, D)-*Morse* if the piecewise geodesic path formed by connecting consecutive points by geodesic segments is a (Θ, D)-Morse quasigeodesic.

Theorem 5.2.6 (Morse lemma for straight spaced sequences, [**KLP14**]). *For Θ, Θ', D, there exists l, ϵ such that any (Θ, ϵ)-straight l-spaced sequence (x_n) in X is D-close to a parallel set $P(\tau_+, \tau_-)$ of type τ_{mod}. Moreover, the nearest point projection $\bar{x}_n$ of x_n on $P(\tau_+, \tau_-)$ satisfies*

$$\bar{x}_{n \pm m} \in V(\bar{x}_n, \mathrm{st}_{\Theta'}(\tau_\pm)), \quad \forall m \in \mathbb{N}.$$

Furthermore, the sequence (x_n) is a uniform Morse sequence with parameters depending only on the given data Θ, Θ', D.

Replacements. Here we define an alternative notion of stability of quasigeodesics, namely that the *Morse property is stable under replacements*. See Theorem 5.2.11, and its generalized version Theorem 5.2.13.

We first develop an important tool which will be needed in the proof of these results.

Definition 5.2.7 (Longitudinal segments). Let y_1, y_2 be any points in $P(\tau_-, \tau_+)$. The (oriented) segment y_1y_2 is called Θ-*longitudinal* if $y_2 \in V(y_1, \mathrm{st}_\Theta(\tau_+))$. Moreover, y_1y_2 is called *(τ_{mod}-)longitudinal* if $y_2 \in V(y_1, \mathrm{ost}(\tau_+))$

Convexity of Θ-cones implies:

Lemma 5.2.8 (Concatenation of longitudinal segments). *Let $x_1, x_2, x_3 \in P(\tau_-, \tau_+)$ be points such that x_1x_2 and x_2x_3 are Θ-longitudinal. Then x_1x_3 is also Θ-longitudinal.*

Proposition 5.2.9 (Nearby diamonds). *Let $\gamma : [a, b] \to X$ be an (L, A, Θ, D)-Morse qusaigeodesic, and let $\delta > 0$ be any number. Let $P(\tau_-, \tau_+)$ be a parallel set such that the image of γ is contained in $N_\delta(P(\tau_-, \tau_+))$. Denote the nearest point projection of $\gamma(t)$ into $P(\tau_-, \tau_+)$ by $\bar{\gamma}(t)$. Suppose that $\bar{\gamma}(a)\bar{\gamma}(b)$ is longitudinal. Then, there exist $R' = R'(L, A, \Theta, \Theta', D, \delta)$, $D' = D'(L, A, \Theta, \Theta', D, \delta)$ such that the following holds: For any $t_1, t_2 \in [a, b]$, if $(t_2 - t_1) \geq R'$, then $\bar{\gamma}(t_1)\bar{\gamma}(t_2)$ is Θ'-longitudinal and $\gamma([t_1, t_2]) \subset N_{D'}(\diamondsuit_{\Theta'}(\bar{\gamma}(t_1), \bar{\gamma}(t_2)))$.*

Proof. Let $\Theta'', \Theta''' \subset \tau_{\mathrm{mod}}$ be auxiliary subsets such that $\mathrm{int}(\Theta') \supset \Theta'''$, $\mathrm{int}(\Theta''') \supset \Theta''$, and $\mathrm{int}(\Theta'') \supset \Theta$. Note that when $(b - a)$ is sufficiently large, the triangle inequality for the Δ-valued distances (Theorem 4.1.4) asserts that $\bar{\gamma}(a)\bar{\gamma}(b)$ is Θ''-regular, which in turn makes $\bar{\gamma}(a)\bar{\gamma}(b)$ Θ''-longitudinal. Therefore, it follows that

$$\bar{\gamma}([a, b]) \subset N_{c+\delta}(\diamondsuit_{\Theta'''}(\bar{\gamma}(a), \bar{\gamma}(b))) \subset N_{c+\delta}(V(\bar{\gamma}(a), \mathrm{st}_{\Theta'''}(\tau_+))),$$

where $c = c(\Theta'', \Theta''', D + \delta)$ is the constant as in Theorem 5.1.7.

Let $t \in [a, b]$ be any point. From above, we get $d(\bar{\gamma}(t), V(\bar{\gamma}(a), \mathrm{st}_{\Theta'''}(\tau_+))) \leq c + \delta$. Using the triangle inequality for the Δ-valued distances (Theorem 4.1.4) again, we obtain that when $t - a \geq R \gg L(c + \delta)$, then

$$\bar{\gamma}(t) \in V(\bar{\gamma}(a), \mathrm{st}_{\Theta'}(\tau_+)),$$

i.e. $\bar{\gamma}(a)\bar{\gamma}(t)$ is Θ'-longitudinal. By reversing the direction of γ, we also get that when $b - t \geq R$, then $\bar{\gamma}(t)\bar{\gamma}(b)$ is Θ'-longitudinal.

For arbitrary $t_1, t_2 \in [a, b]$, we let $t = (t_2 - t_1)/2$. The same argument applied to the paths $\gamma([a, t])$, $\gamma([t, b])$ implies that when $t - t_1 \geq R$, and $t_2 - t \geq R$, then $\bar{\gamma}(t_1)\bar{\gamma}(t)$ and $\bar{\gamma}(t)\bar{\gamma}(t_2)$ are Θ'-longitudinal segments. Using Lemma 5.2.8, we get that $\bar{\gamma}(t_1)\bar{\gamma}(t_2)$ is Θ'-longitudinal.

Therefore, $\bar{\gamma}(t_1)\bar{\gamma}(t_2)$ is Θ'-longitudinal whenever $t_2 - t_1 \geq 2R$.

After enlarging Θ', the second part follows from Theorem 5.1.7. $\square$

We now turn to the discussion of replacements.

DEFINITION 5.2.10 (Morse quasigeodesic replacements). Let $\gamma : I \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $[t_1, t_2]$ be a subinterval of I. Let $\gamma' : [t_1, t_2] \to X$ be another (L', A', Θ', D')-Morse quasigeodesic s.t. $\gamma|_{\{t_1,t_2\}} = \gamma'|_{\{t_1,t_2\}}$ (see Figure 5.3). An (L', A', Θ', D')-*Morse quasigeodesic replacement* of $\gamma|_{[t_1,t_2]}$ is the concatenation of $\gamma|_{I-(t_1,t_2)}$ with $\gamma'|_{[t_1,t_2]}$.

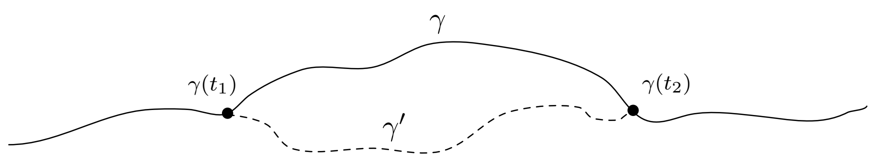

FIGURE 5.3. Replacement: The original path γ is depicted as a solid line, and the path γ' is depicted as a dashed line.

THEOREM 5.2.11 (Replacement lemma). *Uniform Morse quasigeodesic replacements are uniformly Morse.*

PROOF. Suppose that $I = [a, b]$ is some interval. Let $\gamma : I \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $\rho : I \to X$ be obtained by replacing $\gamma|_{[t_1,t_2]}$ by a (L', A', Θ', D')-Morse quasigeodesic $\gamma' : [t_1, t_2] \to X$. Let Θ'' be subset of $\operatorname{ost}(\tau_{\mathrm{mod}})$ which contains Θ and Θ'. Replacing the parameters (L, A, Θ, D) and (L', A', Θ', D') by $(L'', A'', \Theta'', D'')$, where $L'' = \max\{L, L'\}$, $A'' = \max\{A, A'\}$, $D'' = \max\{D, D'\}$, and some $\Theta'' \supset \Theta \cup \Theta'$, we may assume that $(L, A, \Theta, D) = (L', A', \Theta', D')$ to begin with.

By definition, there exists a diamond $\diamondsuit_\Theta(x_1, x_2)$ with $d(x_1, \gamma(a)) \le D$, $d(x_2, \gamma(b)) \le D$ such that $\gamma([a, b]) \subset N_D(\diamondsuit_\Theta(x_1, x_2))$. Without loss of generality, we may assume that $x_1 \ne x_2$. The diamond $\diamondsuit_\Theta(x_1, x_2)$ spans a unique parallel set $P(\tau_-, \tau_+)$ such that $x_2 \in V(x_1, \operatorname{st}_\Theta(\tau_{\mathrm{mod}}))\, \tau_+$. We denote the nearest point projections of $\gamma(t)$ and $\gamma'(t)$ to $P(\tau_-, \tau_+)$ by $\bar\gamma(t)$ and $\bar\gamma'(t)$, respectively.

By the triangle inequality for the Δ-valued distances (Theorem 4.1.4) we get that when $(b-a)$ is sufficiently large, $(b-a) \ge C(\Theta, \Theta', D)$, then $\bar\gamma(a)\bar\gamma(b)$ is Θ'-longitudinal[3]. Using Proposition

[3]the nearest point projection might not send $\gamma(a)$ (resp. $\gamma(b)$) to x_1 (resp. x_2), but sends into a D-neighborhood of x_1 (resp. x_2).

5.2.9, when $(t_2 - t_1) \geq R'$, then $\bar{\gamma}(t_1)\bar{\gamma}(t_2) = \bar{\gamma}'(t_1)\bar{\gamma}'(t_2)$ is also Θ'-longitudinal. From Theorem 5.1.7 we get a constant D' such that $\gamma'([t_1, t_2]) \subset N_{D'}(P(\tau_-, \tau_+))$.

We prove that any subpath $\rho|_{[r_1,r_2]}$ is uniformly close to a diamond. From above, if $(r_2 - r_1) \geq R'$, for $r_1, r_2 \in I$, then $\bar{\gamma}(r_2) \in V(\bar{\gamma}(r_1), \mathrm{st}_{\Theta'}(\tau_+))$. This also holds for γ' and $r_1, r_2 \in [t_1, t_2]$ for a bigger R' because $\gamma'([t_1, t_1])$ is D'-close to $P(\tau_-, \tau_+)$, and $\bar{\gamma}'(t_1)\bar{\gamma}'(t_2)$ is longitudinal. Also, note that in this case, possibly after enlarging Θ', $\gamma|_{[r_1,r_2]}$ and $\gamma'|_{[r_1,r_2]}$ become uniformly close to $\diamondsuit_{\Theta'}(\bar{\gamma}(r_1), \bar{\gamma}(r_2))$ and $\diamondsuit_{\Theta'}(\bar{\gamma}'(r_1), \bar{\gamma}'(r_2))$, respectively (Theorem 5.1.7).

Clearly, when both r_1, r_2 belong to one of the sets $[a, t_1]$, $[t_1, t_2]$, $[t_2, b]$, then $\rho([r_1, r_2])$ is uniformly close to a diamond. Therefore, the following are the only nontrivial cases.

Case 1. $r_1 \in [a, t_1]$ *and* $r_2 \in [t_1, t_2]$.

In this case, if $(t_1 - r_1) \geq R'$ and $(r_2 - t_1) \geq R'$, then from the discussion above we get

$$\bar{\gamma}(t_1) \in V(\bar{\gamma}(r_1), \mathrm{st}_{\Theta'}(\tau_+)), \quad \bar{\gamma}'(r_2) \in V(\bar{\gamma}(t_1), \mathrm{st}_{\Theta'}(\tau_+)).$$

From convexity of cones, it follows that

$$\bar{\gamma}'(r_2) \in V(\bar{\gamma}(r_1), \mathrm{st}_{\Theta'}(\tau_+)).$$

Since $\diamondsuit_{\Theta'}(\bar{\gamma}(r_1), \bar{\gamma}(t_1))$ and $\diamondsuit_{\Theta'}(\bar{\gamma}'(t_1), \bar{\gamma}'(r_2))$ are subsets of $\diamondsuit_{\Theta'}(\bar{\gamma}(r_1), \bar{\gamma}'(r_2))$, $\rho|_{[r_1,r_2]}$ is uniformly close to $\diamondsuit_{\Theta'}(\bar{\gamma}(r_1), \bar{\gamma}'(r_2))$.

Now we prove the quasiisometric inequality for $\rho(r_1)$ and $\rho(r_2)$. Since the points $\bar{\gamma}(r_1)$ and $\bar{\gamma}'(r_2)$ belong to two opposite cones with tip $\bar{\gamma}(t_1) = \bar{\gamma}'(t_1)$,

$$\angle_{\bar{\gamma}(t_1)}\left(\bar{\gamma}(r_1), \bar{\gamma}'(r_2)\right) \geq \alpha',$$

where $\alpha' = \angle_{\mathrm{Tits}}(\Theta', \partial \mathrm{st}(\tau_{\mathrm{mod}}))$. Comparing the geodesic triangle $\triangle(\bar{\gamma}(r_1), \bar{\gamma}(t_1), \bar{\gamma}'(r_2))$ with a Euclidean one, we get

$$d\left(\bar{\gamma}(r_1), \bar{\gamma}'(r_2)\right) \geq \frac{\sin \alpha'}{2}\left(d\left(\bar{\gamma}(r_1), \bar{\gamma}(t_1)\right) + d\left(\bar{\gamma}'(t_1), \bar{\gamma}'(r_2)\right)\right).$$

This, together with standard polygon inequality, implies

$$
\begin{aligned}
d(\rho(r_1), \rho(r_2)) = d\left(\gamma(r_1), \gamma'(r_2)\right) &\geq \frac{\sin\alpha'}{2}\left(d\left(\gamma(r_1), \gamma(t_1)\right) + d\left(\gamma'(t_1), \gamma'(r_2)\right)\right) - 2D'(1+\sin\alpha') \\
&\geq \frac{\sin\alpha'}{2L}|r_1 - r_2| - \left(4D' + A\right).
\end{aligned}
$$

In the last inequality, we are using the quasigeodesic data for the paths γ and γ'.

Case 2. $r_1 \in [t_1, t_2]$ *and* $r_2 \in [t_2, b]$.

This case is settled by reversing the direction of γ in the case 1.

Case 3. $r_1 \in [a, t_1]$ *and* $r_2 \in [t_2, b]$.

The quasiisometric inequality for $\rho(r_1)$ and $\rho(r_2)$ is clear, since

$$
d\left(\rho(r_1), \rho(r_2)\right) = d\left(\gamma(r_1), \gamma(r_2)\right) \geq \frac{|r_1 - r_2|}{L} - A.
$$

It remains only to show that the image $\rho([r_1, r_2])$ is uniformly close to a Θ'-diamond.

We know that $\gamma([r_1, r_2])$ is D-close to a diamond $\diamondsuit_\Theta(x_1, x_2)$ satisfying $d(x_i, \gamma(r_i)) \leq D$, and that $\gamma'([t_1, t_2])$ is D-close to a diamond $\diamondsuit_\Theta(y_1, y_2)$ satisfying $d(y_i, \gamma'(t_i)) \leq D$, for $i = 1, 2$. Since $\gamma(t_i) = \gamma'(t_i)$, it follows that the points y_1 and y_2 are $2D$-close to $\diamondsuit_\Theta(x_1, x_2)$. Let $P(\tau_-, \tau_+)$ be the unique parallel set spanned by $\diamondsuit_\Theta(x_1, x_2)$ satisfying $x_2 \in V\left(x_1, \mathrm{st}_\Theta(\tau_{\mathrm{mod}})\right)\tau_+$. Then,

$$
y_1y_2 \subset N_{2D}\left(P(\tau_-, \tau_+)\right).
$$

Let $\bar{y}_1$, $\bar{y}_2$ denote the projections of y_1, y_2, respectively, in $P(\tau_-, \tau_+)$. Note that the points y_1, y_2 are D-close to $\gamma([r_1, r_2])$. Using Proposition 5.2.9, it follows that when $d(y_1, y_2)$, or equivalently $(t_2 - t_1)$, is sufficiently large, then $\bar{y}_1\bar{y}_2$ is Θ'-longitudinal. In addition, note that the points $\bar{y}_1, \bar{y}_2$ are $4D$-close to the cones $V(x_1, \mathrm{st}_\Theta(\theta_+)), V(x_2, \mathrm{st}_\Theta(\theta_-))$, respectively. Using the triangle inequality for the Δ-valued distances (Theorem 4.1.4) it follows that when $d(x_1, \bar{y}_1)$ and $d(x_2, \bar{y}_2)$, or equivalently $(t_1 - r_1)$ and $(r_2 - t_2)$, are large enough, then $x_1\bar{y}_1$ and $\bar{y}_2x_2$ are Θ'-longitudinal. Therefore,

$$
\bar{y}_1\bar{y}_2 \subset \diamondsuit_{\Theta'}(x_1, x_2).
$$

Using Theorem 5.1.7, there is a constant c which depends only on D, Θ', Θ'' such that

$$
\diamondsuit_{\Theta''}(y_1, y_2) \subset N_c\left(\diamondsuit_{\Theta'}(\bar{y}_1, \bar{y}_2)\right) \subset N_c\left(\diamondsuit_{\Theta'}(x_1, x_2)\right).
$$

Therefore, $\rho([r_1, r_2])$ is $(c + D)$-close to $\diamondsuit_{\Theta'}(x_1, x_2)$. $\square$

Remark. The replacement lemma is false if we relax the Morse condition. It is not generally true that a *uniform quasigeodesic replacement* of an (ordinary) quasigeodesic in a CAT(k) space, $k \geq 0$, is a uniform quasigeodesic. See the example below. However, if $k < 0$, then the ordinary quasigeodesics are Morse quasigeodesics, so the replacement lemma for ordinary quasigeodesics holds.

Example 5.2.12. Let $Y = \mathbb{R}^2$, γ be the x-axis. For $r \geq 0$, define a constant speed piecewise path $\gamma'_r : [-r, r] \to \mathbb{R}^2$ with endpoints at $(-r, 0)$ and $(r, 0)$ as in Figure 5.4 (red path). An easy calculation shows that γ' is a $(4, 0)$-quasigeodesic. Let ρ_r denote the replacements, for $r \geq 0$. Then $\rho_r(2r) = \rho_r(r - k_r)$, for some number $k_r > 0$ (observe the point $(2r, 0)$). However, if ρ_r is an (l, a)-quasigeodesic, then $d(\rho_r(2r), \rho_r(r - k_r)) \geq r/l - a$, which is false for large r's.

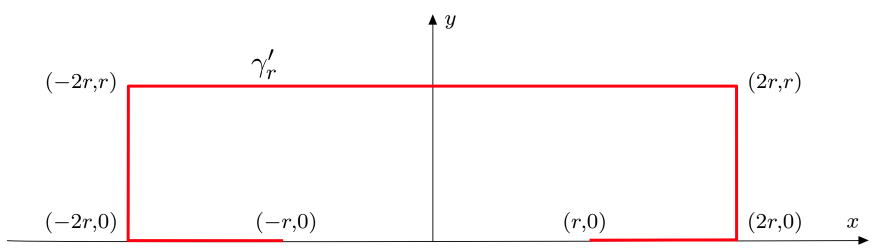

Figure 5.4

We can also replace a Morse quasigeodesic at multiple segments.

Theorem 5.2.13 (Generalized replacement lemma). *Let $\gamma : [a, b] \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $a = t_0 \leq t_1 \leq \cdots \leq t_{r_0 - 1} \leq t_{r_0} = b$. For $r = 1, \ldots, r_0$, let $\gamma_r : [t_{r-1}, t_r] \to X$ be an (L', A', Θ', D')-Morse quasigeodesic with $\gamma_r|_{\{t_{r-1}, t_r\}} = \gamma|_{\{t_{r-1}, t_r\}}$. Then the concatenation of γ_r's is an $(L'', A'', \Theta'', D'')$-Morse quasigeodesic where $(L'', A'', \Theta'', D'')$ depends only on (L, A, Θ, D) and (L', A', Θ', D').*

The proof of this theorem closely follows the proof of the previous one and, we are omitting the details.

Residual finiteness. An important feature shared by many finitely generated subgroups of G is the *residual finiteness* property which enables us to obtain finite index subgroups which avoid a given finite set of elements.

DEFINITION 5.2.14 (Residual finiteness). A group H is called *residually finite (RF)* if it satisfies one of the following equivalent conditions: (1) Given a finite subset $S \subset H \setminus \{1_H\}$, there exists a finite index subgroup $F < H$ such that $F \cap S = \emptyset$. (2) Given an element $h \in H \setminus \{1_H\}$, there exits a finite group Φ and a homomorphism $\phi : H \to \Phi$ such that $\phi(h) \neq 1_\Phi$. (3) The intersection of finite index subgroups of H is trivial.

Residual finiteness of Morse subgroups is a corollary to the following celebrated theorem.

Let R be a commutative ring with unity, and let $GL(N, R)$ denote the group (with multiplication) of non-singular $N \times N$ matrices with entries in R. Then,

THEOREM 5.2.15 (A. I. Mal'cev, [**Mal40**]). *Finitely generated subgroups of $GL(N, R)$ are RF.*

As an application to this theorem, one obtains,

COROLLARY 5.2.16. *Each finitely generated subgroup $\Gamma < G$ which intersects the center of G trivially is RF.*

PROOF. Under our assumptions, the adjoint representation $\Gamma \to GL(\mathfrak{g})$ is faithful. □

For a subgroup $\Gamma < G$, we define the *norm* of Γ with respect to $x \in X$ as

$$\|\Gamma\|_x = \inf\{d(x, \gamma x) : 1_\Gamma \neq \gamma \in \Gamma\}.$$

Note that when $\|\Gamma\|_x > 0$, Γ is discrete. Residual finiteness implies the following useful lemma which we use to obtain subgroups whose nontrivial elements send x arbitrarily far:

LEMMA 5.2.17. *Let Γ be a RF discrete subgroup of G. For any $R \in \mathbb{R}$, there exist a finite index subgroup $\Gamma' < \Gamma$ such that $\|\Gamma'\|_x \geq R$.*

PROOF. Since Γ is discrete, the set $\Phi = \{\gamma : d(x, \gamma x) < R\}$ is finite. The assertion follows from the residual finiteness property. □

Morse subgroups. Now we briefly turn to the discussion of Morse subgroups before proving our main theorem in the next section.

Recall the notion of Morse subgroups (Definition 4.4.3). By the Equivalence Theorem 4.4.12, τ_{mod}-Morse subgroups are uniformly τ_{mod}-regular and, hence, the accumulation set in $\partial_\infty X$ of any orbit Γx contains only points whose types are elements of Θ, for some ι-invariant, compact, τ_{mod}-Weyl convex subset $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$. The smallest such Θ will be denoted by Θ_Γ.

PROPOSITION 5.2.18. *Let Γ be a τ_{mod}-Morse subgroup, let Λ' be any compact set in* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ *whose interior contains $\Lambda = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, the flag limit set of Γ (see Definition 4.4.5), and let Θ' be any ι-invariant, compact, τ_{mod}-Weyl convex subset of* $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ *containing $\Theta = \Theta_\Gamma(x)$ in its interior. There exists a number $S > 0$ such that any $\gamma \in \Gamma$ satisfying $d(x, \gamma x) > S$ also satisfies $\gamma x \in V\,(x, \mathrm{st}_{\Theta'}(\tau))$, for some $\tau \in \Lambda'$.*

PROOF. We first prove that there exists $S' > 0$ such that $d(x, \gamma x) > S'$ implies that $(x, \gamma x)$ is Θ'-regular. Suppose that S' does not exist; then, there is an unbounded sequence $(\gamma_i)_{i\in\mathbb{N}}$ in Γ such that $(x, \gamma_i x)$ is not Θ'-regular for all i. Then, $(\gamma_i x)_{i\in\mathbb{N}}$ subconverges to an ideal point whose type $\notin \mathrm{int}\,(\Theta')$. This can not happen because the interior of Θ' contains Θ.

Next we prove that S exists. Assuming that it does not exist, we get an unbounded sequence $(\gamma_i')_{i\in\mathbb{N}}$ in Γ such that $\gamma_i' x \notin V\,(x, \mathrm{st}_{\Theta'}(\tau))$, for all $i \in \mathbb{N}$ and $\tau \in \Lambda'$. After extraction we may assume that $(x, \gamma_i' x)$ is Θ'-regular, for all i. But then, $(\gamma_i' x)_{i\in\mathbb{N}}$ does not accumulate in any simplex in the interior of Λ' i.e. Γ has a limit simplex outside Λ, but this gives a contradiction. ☐

Combining Lemma 5.2.17 and Proposition 5.2.18, we get the following:

COROLLARY 5.2.19. *Let $\Gamma < G$ be a RF τ_{mod}-Morse subgroup, let Λ' be any compact set in* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ *whose interior contains $\Lambda = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, and let Θ' be any compact set containing $\Theta = \Theta_\Gamma$ in its interior. There exists $S_1 > 0$ such that for any $S \geq S_1$ there exists a finite index subgroup Γ' of Γ satisfying $\|\Gamma'\|_x > S$ which also satisfies the following: For any $\gamma' \in \Gamma'$ exists $\tau \in \Lambda'$ for which $\gamma' x \in V\,(x, \mathrm{st}_{\Theta'}(\tau))$.*

DEFINITION 5.2.20 (Antipodal Morse subgroups). A pair of τ_{mod}-Morse subgroups $\Gamma_1, \Gamma_2 < G$ is called *antipodal* if their flag limit sets in $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ are antipodal.

PROPOSITION 5.2.21. *Let $\Gamma_1, \dots, \Gamma_n$ be pairwise antipodal, RF τ_{mod}-Morse subgroups of G. Let $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$ be a subset which contains the sets Θ_{Γ_i}, for $i = 1, \dots, n$, in its interior. Then, there* *exists a collection* $\{\Lambda'_1, \ldots, \Lambda'_n\}$ *of pairwise antipodal, compact subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$*, and a number* $S_2 > 0$ *such that for any* $S \geq S_2$ *there exists a collection of finite index subgroups* $\Gamma'_1, \ldots, \Gamma'_n$ *of* $\Gamma_1, \ldots, \Gamma_n$*, respectively, satisfying* $\|\Gamma'_1\|_x \geq S, \ldots, \|\Gamma'_n\|_x \geq S$ *which also satisfies the following: For each* $i = 1, \ldots, n$*, and for each* $\gamma_i \in \Gamma'_i$*, there exists* $\tau_i \in \Lambda'_i$ *such that*

$$\gamma_i x \in V(x, \mathrm{st}_\Theta\,(\tau_i)).$$

Proof. Once we show that there exists a collection $\{\Lambda'_1, \ldots, \Lambda'_n\}$ such that, for each i, the interior of Λ'_i contains the flag limit set Λ_i of Γ_i, the first part of the proposition follows from the Corollary 5.2.19. We may construct $\Lambda'_1, \ldots, \Lambda'_n$ as follows:

Lemma 5.2.22. *Let* $\{\Lambda_1, \ldots, \Lambda_n\}$ *be a set of pairwise antipodal, compact subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$*. Then, there exists a set* $\{\Lambda'_1, \ldots, \Lambda'_n\}$ *of pairwise antipodal, compact subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ *such that each* Λ_i *is contained in the interior of* Λ'_i*.*

Proof. The case $n = 2$ can be proved as follows. Let Λ_1, Λ_2 be a pair of antipodal, compact subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. Then,

$$\Lambda_1 \times \Lambda_2 \underset{\text{compact}}{\subset} (\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}}))^{\mathrm{opp}} \underset{\text{open}}{\subset} \mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}})\,.$$

There is a open neighborhood of $\Lambda_1 \times \Lambda_2$ in $(\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}}))^{\mathrm{opp}}$ of the form $U_1 \times U_2$. In particular, the subsets U_1 and U_2 are antipodal. Then any pair of compact subsets Λ'_1 and Λ'_2 of U_1 and U_2, respectively, containing Λ_1 and Λ_2 in their respective interiors, does the job.

We consider now the general case $n \geq 3$ and let $\{\Lambda_1, \ldots, \Lambda_n\}$ be a collection of subsets as in proposition. For Λ_1, using the lemma, we find a compact neighborhood Λ'_1 of Λ_1 which is antipodal to the compact

$$\bigcup_{k=2}^{n} \Lambda_k.$$

Then, $\{\Lambda'_1, \Lambda_2, \ldots, \Lambda_n\}$ is new collection pairwise antipodal, compact subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. The same argument yields a compact neighborhood Λ'_2 of Λ_2 antipodal to $\Lambda'_1, \Lambda_3, ..., \Lambda_k$. We continue inductively. □

This completes the proof of the proposition. □

5.3. Proof of Theorem 5.0.1

In this section, we prove

Theorem 5.0.1. *Let $\Gamma_1, \dots, \Gamma_n$ be pairwise antipodal, RF τ_{mod}-Morse subgroups of G. Then, there exist finite index subgroups $\Gamma'_i < \Gamma_i$, for $i = 1, \dots, n$, such that $\langle \Gamma'_1, \dots, \Gamma'_n \rangle$ is τ_{mod}-Morse, and is naturally isomorphic to $\Gamma'_1 * \cdots * \Gamma'_n$*

Proof. We first fix our notations. We denote the τ_{mod}-flag limit sets of $\Gamma_1, \dots, \Gamma_n$ by $\Lambda_1, \dots, \Lambda_n$, respectively. Let $\Theta \subset \text{ost}\,(\tau_{\text{mod}})$, let $\{\Lambda'_1, \dots, \Lambda'_n\}$ be a collection of compact, pairwise antipodal subsets in $\text{Flag}\,(\tau_{\text{mod}})$, and let $S_2 > 0$ be as in Proposition 5.2.21. As always, the point x will be treated as a fixed base point in X. Finally, $\Theta \subset \Theta' \subset \Theta''$ are ι-invariant, compact, τ_{mod}-Weyl convex subsets of $\text{ost}\,(\tau_{\text{mod}})$ such that

$$\text{int}\left(\Theta''\right) \supset \Theta', \quad \text{int}\left(\Theta'\right) \supset \Theta.$$

By Proposition 5.2.21, for each $S > S_2$ there exist finite index subgroups $\Gamma'_1, \dots, \Gamma'_n$ of $\Gamma_1, \dots, \Gamma_n$, respectively, of norms $\|\Gamma'_i\|_x \geq S$, such that for each $i = 1, \dots, n$, and each $\gamma_i \in \Gamma'_i$,

$$\gamma_i x \in V(x, \text{st}_\Theta\,(\tau_i)), \tag{5.9}$$

for some $\tau_i \in \Lambda'_i$. Let A_i be a finite generating set of Γ'_i, for each $i = 1, \dots, n$. This choice endows each Γ'_i with a word metric, and thus yields a Θ-Morse embedding $o^i_x : \text{Cay}\,(\Gamma'_i, A_i) \to X$ induced by the orbit map $\Gamma'_i \to \Gamma'_i x$. We take the standard generating set $A = A_1 \cup \cdots \cup A_n$ of the abstract free product $\Gamma' = \Gamma'_1 * \cdots * \Gamma'_n$. We obtain a natural homomorphism $\varphi : \Gamma' \to G$. When S sufficiently large we prove that $o_x : \text{Cay}\,(\Gamma', A) \to X$ is a Θ'-Morse embedding, i.e. we prove that the geodesics of $\text{Cay}\,(\Gamma', A)$ are mapped to uniform Morse quasigeodesics in X. This not only will prove that φ is injective, but also will show that the subgroup $\langle \Gamma'_1, \dots, \Gamma'_n \rangle$ of G generated by $\Gamma'_1, \dots, \Gamma'_n$ is τ_{mod}-Morse.

Claim. *There exists $S_0 > 0$ such that if $S \geq S_0$, then the map $o_x : \text{Cay}\,(\Gamma', A) \to X$ sends (finite) geodesics to uniform Morse quasigeodesics.*

Proof. Given any $\gamma \in \Gamma'$, there is a natural embedding of $\text{Cay}\,(\Gamma'_i)$ into $\text{Cay}\,(\Gamma')$ given by the right multiplication map $\gamma_i \mapsto \gamma_i \gamma$. Any geodesic in $\text{Cay}\,(\Gamma')$ is a concatenation of paths which are images of the geodesics under the maps above. By equivariance, it suffices to study the geodesics in Γ' starting at $1_{\Gamma'}$. Any geodesic ψ with starting point $1_{\Gamma'}$ in $\mathrm{Cay}\,(\Gamma')$ can be written as

$$\psi \ : \ 1_{\Gamma'},\ \gamma_{k_1},\ \gamma_{k_2}\gamma_{k_1},\ldots, \tag{5.10}$$

where the path joining $\gamma_{k_r}\gamma_{k_{r-1}}\ldots\gamma_{k_1}$ and $\gamma_{k_{r-1}}\ldots\gamma_{k_1}$ in $\mathrm{Cay}\,(\Gamma')$ is the image of a geodesic segment in $\mathrm{Cay}\,(\Gamma'_i)$ connecting the identity to γ_{k_r} under the map $(\cdot)\mapsto(\cdot)(\gamma_{k_{r-1}}\ldots\gamma_{k_1})$, assuming that $\gamma_{k_r}\in\Gamma'_i$. We group together γ_{k_r}'s in above to avoid two consecutive ones coming from same Γ_i's.

The (finite) sequence (5.10) is mapped to $x,\ \gamma_{k_1}x,\ \gamma_{k_2}\gamma_{k_1}x,\ldots$ under the map o_x; to avoid cumbersome notations, we denote $\gamma_{k_r}\gamma_{k_{r-1}}\ldots\gamma_{k_1}$ by g_r, denote $\gamma_{k_r}\gamma_{k_{r-1}}\ldots\gamma_{k_1}x$ by p_r and assume that the index r of this sequence varies between 0 and r_0. Using these notations, we have

$$g_r=\gamma_{k_r}g_{r-1},\quad g_rx=p_r. \tag{5.11}$$

Let $m_1=p_0$, $m_{r_0}=p_{r_0}$, and, for $2\le r\le r_0-1$, let m_r denote the midpoint of p_{r-1} and p_r (see Figure 5.5).

It follows from (5.9) that all the segments $p_{r-1}p_r$ in X are Θ-regular and have length at least S. Moreover, it follows from (5.11) that, for any $1\le r\le r_0-1$, precomposing the right multiplication action $g_r^{-1}\curvearrowright\Gamma$ with o_x maps the hinge $p_{r-1}p_rp_{r+1}$ to $(\gamma_{k_r}^{-1}x)(x)(\gamma_{k_{r+1}}x)$ which is of the form $(\gamma_ix)(x)(\gamma_jx)$, for some $\gamma_i\in\Gamma'_i$, $\gamma_j\in\Gamma'_j$, $i\neq j$. From (5.9), we get that $\gamma_ix\in V(x,\mathrm{st}_\Theta\,(\tau_i))$ and $\gamma_jx\in V(x,\mathrm{st}_\Theta\,(\tau_j))$, for some $\tau_i\in\Lambda'_i$ and $\tau_j\in\Lambda'_j$. To simplify our notation, the corresponding images of m_r and m_{r+1} are denoted by m'_i and m'_j, respectively.

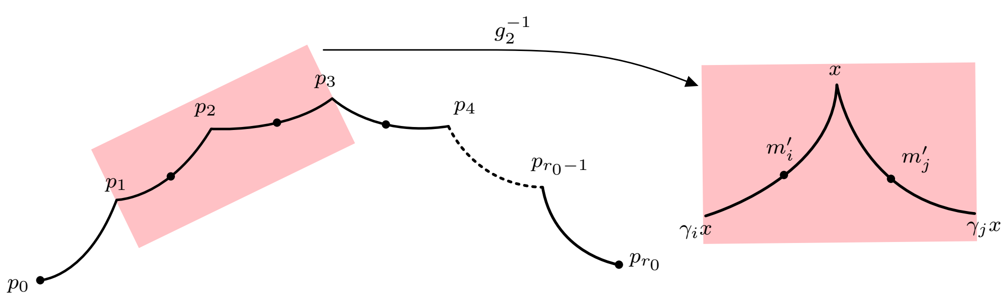

FIGURE 5.5. Morse embedding of quasigeodesics. The points represent the midpoint sequence (m_i).

Let $D = \max\{D(\Lambda_i', \Lambda_j', x) : 1 \leq i < j \leq n\}$, where $D(\Lambda_i', \Lambda_j', x)$ is the constant given by Corollary 5.1.3. Moreover, let $R_1(x, \Lambda_i', \Lambda_j', \Theta', \Theta)$ and $f_1(x, \Lambda_i', \Lambda_j', \Theta', \Theta, \xi)$ be the quantities as in Proposition 5.1.9. Define

$$R_1 = \max_{i,j,\ i \neq j} \left\{ R_1(x, \Lambda_i', \Lambda_j', \Theta', \Theta) \right\},$$

and

$$f_1 = \max_{i,j,\ i \neq j} \left\{ f_1(x, \Lambda_i', \Lambda_j', \Theta', \Theta, \xi) \right\}.$$

Note that $d(x, m_i'), d(x, m_j') \geq S/2$. Using part 1 of Proposition 5.1.9, when $S/2 \geq R_1$, then $m_i'm_j'$ is Θ'-regular. Using part 2 of the same proposition we get

$$\angle^{\xi}_{m_i'}(x, m_j'), \angle^{\xi}_{m_j}(x, m_i') \leq f_1(S/2).$$

Moreover, using and (5.7), we obtain

$$d(m_i', m_j') \geq \frac{S \sin \alpha}{2} - 4D,$$

where $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta')$. Therefore, when $S \geq 2R_1$, the sequence (m_r) is $(\Theta', 2f_1(S/2))$-straight and $((S \sin \alpha)/2 - 4D)$-spaced.

For any $\delta' > 0$, Theorem 5.2.6 applied to Θ', Θ'' and δ' concludes that there exists $S_0 \gg R_1$ such that when $S \geq S_0$, then the sequence (m_r) is δ'-close to a parallel set $P(\tau_-, \tau_+)$ such that the nearest point projection map sends $m_r m_{r+1}$ to a Θ''-longitudinal segment. This proves that the piecewise geodesic path $p_0 p_1 \dots p_{r_0}$ is a uniform Morse quasigeodesic for sufficiently small δ'.

Finally, we prove that $o_x \circ \psi$ is uniformly Morse. By invoking the Morse property of Γ_i''s, we get that each segment of $o_x \circ \psi$ connecting a consecutive pair p_r and p_{r+1} is a uniform Morse quasigeodesic. Therefore, $o_x \circ \psi$ is obtained by replacing the geodesic segments $p_r p_{r+1}$ of the path $p_0 p_1 \dots p_{r_0}$ by uniform Morse quasigeodesics. From the generalized replacement lemma (Theorem 5.2.13), it follows that $o_x \circ \psi$ is also a uniform Morse quasigeodesic. $\square$

This completes the proof of the theorem. $\square$

Remark. The RF condition in the above theorem can be relaxed by integrating the content of Corollary 5.2.16 into the hypothesis. Precisely, instead of requiring Γ_i's to be RF one may require that Γ_i's intersect the center of G trivially. When $G \cong \mathrm{Isom}_0(X)$, this happens automatically, because $\mathrm{Isom}_0(X)$ is centerless.

Below is a more general form of Theorem 5.0.1 which does not involve passing to finite index subgroups, but instead requires "sufficient antipodality and sparseness" of the subgroups Γ_i. Let

$$(\underbrace{\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times ... \times \mathrm{Flag}\,(\tau_{\mathrm{mod}})}_{n \text{ times}})^{\mathrm{opp}}$$

denote the subset of $(\mathrm{Flag}\,(\tau_{\mathrm{mod}}))^n$ consisting of n tuples of pairwise antipodal flags. For a subset

$$A \subset (\underbrace{\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times ... \times \mathrm{Flag}\,(\tau_{\mathrm{mod}})}_{n \text{ times}})^{\mathrm{opp}}$$

and for $x \in X$ define the subset $O_{A,x} \subset X^n$ consisting of n-tuples $(x_1, ..., x_n)$ such that for some $(\tau_1, ..., \tau_n) \in A$, we have $x_i \in V(x, \mathrm{st}\,(\tau_i))$, $i = 1, ..., n$.

Theorem 5.3.1. *For each compact*

$$A \subset (\underbrace{\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times ... \times \mathrm{Flag}\,(\tau_{\mathrm{mod}})}_{n \text{ times}})^{\mathrm{opp}},$$

and $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$*, there exists a constant* $S = S(A, \Theta, x)$ *such that the following holds. Let* $\Gamma_1, ..., \Gamma_n$ *be* P*-Anosov subgroups of* G *such that:*

a. $\|\Gamma_i\|_x \geq S$, $i = 1, ..., n$.

b. For each $\gamma_i \in \Gamma_i, i = 1, ..., n$*, the segment* $x\gamma_i(x)$ *is* Θ*-regular.*

c. For each n*-tuple of nontrivial elements* $\gamma_i \in \Gamma_i - \{1\}$, $i = 1, ..., n$*, we have*

$$(\gamma_1(x), ..., \gamma_n(x)) \in O_{A,x}.$$

Then the subgroup of G *generated by* $\Gamma_1, ..., \Gamma_n$ *is* P*-Anosov, and is naturally isomorphic to the free product* $\Gamma_1 * ... * \Gamma_n$.

Proof. The proof is very similar to the one of Theorem 5.0.1. The conclusion of Proposition 5.2.21 now becomes a hypothesis on the subgroups Γ_i, so no passage to finite index subgroups is required. Hence, the rest of the proof of Theorem 5.0.1 goes through. $\square$

Remark. We should note that this theorem is in the spirit of the "quantitative ping-pong lemma" of Breuillard and Gelander, see [**BG08**, Lemma 2.3].

CHAPTER 6

Patterson-Sullivan theory for Anosov subgroups

This chapter is based on [**DK19**]. We continue with the notations from Section 2.2.

In this chapter, we prove the following

THEOREM 2.2.8. *Let Γ be a nonelementary P-Anosov subgroup of G. Then the Patterson–Sullivan density μ on the flag limit set $\Lambda(\Gamma) \subset G/P$ is the unique (up to rescaling) Γ-invariant conformal density. Moreover,*

(i) The density μ is non-atomic and $\delta_{\mathrm{F}}(\Gamma)$-dimensional.

(ii) The support of μ is $\Lambda(\Gamma)$ and Γ acts on $\Lambda(\Gamma)$ ergodically with respect to μ.

(iii) The critical exponent $\delta_{\mathrm{F}}(\Gamma)$ is positive and finite.

(iv) The Poincaré series of Γ diverges at $\delta_{\mathrm{F}}(\Gamma)$. Equivalently, Γ has Finsler divergence type.

(v) The $\delta_{\mathrm{F}}(\Gamma)$-dimensional Hausdorff measure on $\Lambda(\Gamma)$ with respect to a Gromov premetric is a member of a Γ-invariant conformal density. In particular, the Hausdorff dimension of $\Lambda(\Gamma)$ is $\delta_{\mathrm{F}}(\Gamma)$.

Proofs of different parts of this theorem can be found at different places in this chapter. Along the way, we introduce definitions of various notions appearing in the statement, such as *conformal density* and its *dimension*, *divergence type*, *Gromov premetric*, etc.

The uniqueness of conformal density in Theorem 2.2.8 is proven in Corollary 6.7.4. The main ingredients in the proof are a generalization of Sullivan's *shadow lemma* proven in Theorem 6.5.1, and an ergodicity argument (see Theorem 6.7.1) due to Sullivan. The proof of part (i) of the theorem is given in Corollaries 6.5.2 and 6.6.5. The second half of part (ii) follows from Theorem 6.7.3 while the first half follows from the facts that the support of μ is a closed Γ-invariant subset of $\Lambda(\Gamma)$ and the action $\Gamma \curvearrowright \Lambda(\Gamma)$ is minimal. The part (iii) is proven in Propositions 6.1.3 and 6.2.1. This also implies that the *Riemannian* critical exponent $\delta_{\mathrm{R}}(\Gamma)$ positive, see the remarks following Propositions 6.1.3 and 6.2.1. The part (iv) follows from Corollary 6.5.5. The Hausdorff density in part (v) is studied in Section 6.8 (cf. Theorem 6.8.3). The background Gromov (pre)metric is introduced in Section 6.4 where we also prove that the action $\Gamma \curvearrowright \Lambda(\Gamma)$ with respect to this metric is *conformal* (see Corollary 6.4.6).

Organization of this chapter. In Section 6.1, we introduce the Finsler metric d_{F} on X which plays a central role in what follows. Also, the notions of *critical exponent*, *convergence/divergence type* are defined in this section. In Section 6.2, we introduce the notion of *conformal densities* and the *Patterson-Sullivan density* on the limit set of Γ. In Section 6.4, we define *Gromov premetric* on the limit set which is used as a background metric to compute the Hausdorff dimension of the limit set. We also show that this is an actual metric under a suitable assumption that is enough for our purpose. In Section 6.5, we generalize Sullivan's shadow lemma to our setting. This lemma plays a vital role in the proof several parts of Theorem 2.2.8 such as in the proof of ergodicity of the action $\Gamma \curvearrowright \Lambda(\Gamma)$ with respect to any conformal density in Section 6.7. In Sections 6.6 and 6.7, we prove that the Patterson-Sullivan density is the unique conformal density. In Section 6.8, we introduce the *Hausdorff density* and prove that it is a conformal density. As a corollary, we obtain Theorem 2.2.8(v). Finally, in Section 6.9, we give some applications of Theorem 2.2.8.

6.1. Critical exponent

On the symmetric space $X = G/K$, we consider two natural (pseudo-)metrics. Let $d_{\mathrm{R}}(\cdot,\cdot)$ denote the distance function on X of the (fixed) G-invariant Riemannian metric on X. Furthermore, for a fixed ι-invariant face τ_{mod} of σ_{mod} and a fixed ι-invariant type $\bar\theta$ in the interior of τ_{mod}, we let d_{F} denote the polyhedral Finsler (pseudo-)metric on X:

$$d_{\mathrm{F}}(x,y) = \langle d_{\Delta}(x,y) | \bar\theta \rangle \tag{6.1}$$

(cf. [**KL18b**, Subsec. 5.1]). The inner product above is the euclidean inner product on F_{mod} coming from the Riemannian metric on X. These two metrics are related by the inequality

$$d_{\mathrm{F}}(x,y) \le d_{\mathrm{R}}(x,y). \tag{6.2}$$

Since the Finsler metric d_{F} inherently depends on the choice of τ_{mod} and $\bar\theta$, from now on we fix $\bar\theta$ and use the notation d_{F} to denote the corresponding Finslder metric.

The metric space (X, d_{R}) is a complete Riemannian manifold and, in particular, it is *geodesic*, i.e., any two points in X can be connected by a geodesic segment. The (pseudo-)metric space (X, d_{F})

is also a geodesic space. The geodesics in (X, d_{F}) are called *Finsler geodesics*. All the Riemannian geodesics are also Finsler, however, there are other Finsler geodesics when $\mathrm{rank}(X) \geq 2$. The precise description of all Finsler geodesics is given in [**KL18b**, Subsec. 5.1.3]. We merely use this description as a definition of Finsler geodesics.

Definition 6.1.1 (Finsler geodesics). Let $I \subset \mathbb{R}$ be an interval. A path $\ell : I \to X$ is called a *Finsler geodesic* if there exists a pair of antipodal flags $\tau_{\pm} \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ such that $\ell(I) \subset P(\tau_+, \tau_-)$ and

$$\ell(t_2) \in V(\ell(t_1), \mathrm{st}(\tau_+)), \quad \forall t_1 \leq t_2.$$

Moreover, given an ι-invariant, compact, τ_{mod}-Weyl convex subset $\Theta \subset \mathrm{ost}(\tau_{\mathrm{mod}})$, a Finsler geodesic $\ell : I \to X$ is called a Θ-*Finsler geodesic* if, in addition to the above, it satisfies the following stronger condition:

$$\ell(t_2) \in V(\ell(t_1), \mathrm{st}_{\Theta}(\tau_+)), \quad \forall t_1 \leq t_2.$$

Remark. Finsler geodesics give alternative description of diamonds, namely, $\diamondsuit_{\tau_{\mathrm{mod}}}(x, y)$ (resp. $\diamondsuit_{\Theta}(x, y)$) is the union of all Finsler (resp. Θ-Finsler) geodesics connecting the endpoints x and y (cf. Figure 6.1). See [**KL18b**, Subsec. 5.1.3].

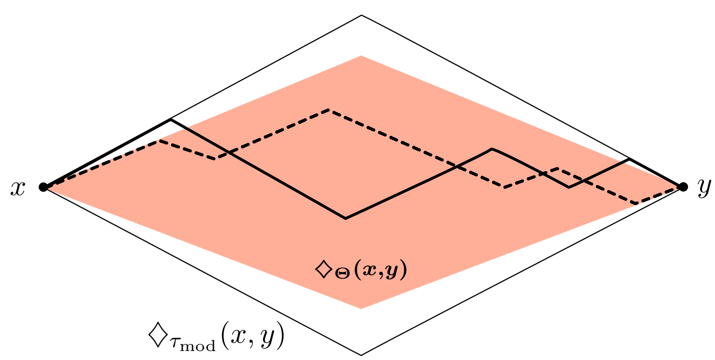

Figure 6.1. Finsler (solid) and Θ-Finsler (dashed) geodesics.

Notation. In this chapter, we use the notation $\overline{xy}$ to denote the Riemannian geodesic segment connecting a pair of points $x, y \in X$. To denote a Finsler geodesic segment connecting x and y, we use the notation $\widehat{xy}$.

Below we let $*$ be either R or F. Let $\Gamma < G$ be a subgroup, and $x, x_0 \in X$. Define the *orbital counting function* $N_*(r, x, x_0) : [0, \infty) \to [0, \infty]$,

$$N_*(r) = N_*(r, x, x_0) = \operatorname{card}\{\gamma \in \Gamma : d_*(x, \gamma x_0) < r\}.$$

Using $N_*(r)$, following [**Alb99**] and [**Qui02b**], we define the *critical exponent* δ_* of Γ by

$$\delta_* = \limsup_{r \to \infty} \frac{\log N_*(r)}{r} \in [0, \infty]. \tag{6.3}$$

The critical exponents δ_{F} and δ_{R} will be called the *Finsler critical exponent* and *Riemannian critical exponent*, respectively.

Remark. The discussion in [**Alb99**] and [**Qui02b**] is mostly limited to the case when $\bar{\theta}$ is regular, i.e., belongs to the interior of σ_{mod}.

We note that the critical exponent is independent of the chosen points x and x_0. This can be proved as follows: Consider the Poincaré series

$$g_s^*(x, x_0) = \sum_{\gamma \in \Gamma} \exp(-s d_*(x, \gamma x_0)). \tag{6.4}$$

It is a standard fact that $g_s^*(x, x_0)$ converges if $s > \delta_*(x, x_0)$ and diverges if $s < \delta_*(x, x_0)$ where $\delta_*(x, x_0)$ denotes the right side of (6.3). Using the triangle inequality, we obtain

$$\exp\left(-s d_*(x, x_0)\right) g_s^*(x_0, x_0) \le g_s^*(x, x_0) \le \exp\left(s d_*(x, x_0)\right) g_s^*(x_0, x_0).$$

Hence, convergence or divergence of $g_s^*(x, x_0)$ is independent of the choice of x and so is $\delta_*(x, x_0)$. For a similar reason, it is also independent of the choice of x_0.

Definition 6.1.2. A discrete subgroup Γ of G is of *(Finsler) convergence type* if the Poincaré series $g_s^{\mathrm{F}}(x, x_0)$ converges at the critical exponent δ_{F}. Otherwise, we say that Γ has *(Finsler) divergence type.*

Since the action $\Gamma \curvearrowright X$ is properly discontinuous, δ_{R} is bounded above by the *volume entropy* of X which is finite.[1] For the Finsler critical exponent, (6.2) implies the following lower bound,

$$\delta_{\mathrm{R}} \le \delta_{\mathrm{F}}. \tag{6.5}$$

[1]Finiteness of the volume entropy of a symmetric space follows, for instance, from the fact that X has curvature bounded below combined with the Bishop-Günter volume comparison theorem, see e.g. [**BC01**, Sec. 11.10, Cor. 4].

Finiteness of δ_{F} is more subtle because, in general, d_{F} is only a pseudo-metric and therefore, the orbital counting function N_{F} may take infinity as a value. However, if the angular radius of the model Weyl chamber σ_{mod} with respect to $\bar{\theta}$ is $< \pi/2$, then d_{F} is a metric equivalent to d_{R} and, consequently, δ_{F} is finite in this case. In particular, when G is simple, then diameter of σ_{mod} is $< \pi/2$ and therefore, δ_{F} is finite.

The following finiteness result holds in the general pseudo-metric case.

Proposition 6.1.3. *For a uniformly τ_{mod}-regular subgroup $\Gamma < G$, the Finsler critical exponent δ_{F} is finite.*

Proof. When Γ is uniformly τ_{mod}-regular, the Riemannian and Finsler (pseudo-)metrics restricted to an orbit Γx are coarsely equivalent: There exist $L \geq 1, A \geq 0$ such that, for all $x_1, x_2 \in \Gamma x$,

$$L^{-1} d_{\mathrm{R}}(x_1, x_2) - A \leq d_{\mathrm{F}}(x_1, x_2) \leq d_{\mathrm{R}}(x_1, x_2). \tag{6.6}$$

The right side of this inequality comes from (6.2). From this we get $\delta_{\mathrm{R}} \leq \delta_{\mathrm{F}} \leq L\delta_{\mathrm{R}}$. Since δ_{R} is finite, δ_{F} is also finite. □

Remark. (1) It is clear from the proof of Proposition 6.1.3 that when Γ is uniformly τ_{mod}-regular, then δ_{F} is positive if and only if δ_{R} is positive.
(2) As Anosov subgroups are uniformly regular (see Equivalence Theorem 4.4.12), the above proposition applies to the class of Anosov subgroups.

Before closing this section, we compute Finsler metrics in two examples.

Example 6.1.4 (Product of rank-one symmetric spaces). We continue with the discussion from Example 4.2.1. The Finsler metric can be described as follows. Let $\tau_{\mathrm{mod}} = (r_1, \ldots, r_p)$ be a face of the model chamber, let $\bar{\theta} = (1/\sqrt{p}, \ldots, 1/\sqrt{p})$ be its barycenter, and let d_{F} be the corresponding metric on X. Given $x = (x_1, \ldots, x_k),\ y = (y_1, \ldots, y_k) \in X$, the Δ-valued distance is

$$d_\Delta(x, y) = (d_{X_1}(x_1, y_1), \ldots, d_{X_k}(x_k, y_k))$$

where d_{X_i} denotes the Riemannian distance function on X_i. Then

$$d_{\mathrm{F}}(x, y) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} d_{X_{r_j}}(x_{r_j}, y_{r_j}). \tag{6.7}$$

Example 6.1.5 ($X = \mathrm{SL}(k{+}1,\mathbb{R})/\mathrm{SO}(k{+}1,\mathbb{R})$). We continue with the discussion from Example 4.2.2. The Riemannian metric on X is given by the restriction of the Killing form B of $\mathfrak{g} = \mathfrak{sl}(k+1,\mathbb{R})$ to $\mathfrak{p}$,

$$B(P,Q) = 2(k+1)\,\mathrm{tr}(PQ^T), \quad P,Q \in \mathfrak{g}. \tag{6.8}$$

Note that the inner product B on $\mathfrak{a}$ (which we identify with F_{mod}) can be written as

$$\langle(\mu_1,\ldots,\mu_{k+1})|(\mu_1',\ldots,\mu_{k+1}')\rangle = 2(k+1)\sum_{i=1}^{k+1}\mu_i\mu_i'. \tag{6.9}$$

Let $\tau_{\mathrm{mod}} = (r_1,\ldots,r_p)$ be an ι-invariant face of the model chamber σ_{mod} and let $\Delta_{\tau_{\mathrm{mod}}}$ be the corresponding face of the model euclidean Weyl chamber Δ,

$$\Delta_{\tau_{\mathrm{mod}}} = \big\{\boldsymbol{\mu} \in \mathfrak{a}_+ : \boldsymbol{\mu} = (\underbrace{\mu_1,\ldots,\mu_1}_{r_1\text{-times}},\ldots,\underbrace{\mu_i,\ldots,\mu_i}_{(r_i-r_{i-1})\text{-times}},\ldots,\underbrace{\mu_{p+1}\ldots,\mu_{p+1}}_{(k+1-r_p)\text{-times}})\big\}.$$

For notational convenience we denote $\boldsymbol{\mu}$ in the above expression simply by the $(p+1)$-vector $(\mu_1,\ldots,\mu_{p+1})$ (by identifying the repeated entries). With this convention, the opposition involution acts by

$$\iota(\mu_1,\ldots,\mu_{p+1}) = (-\mu_{p+1},\ldots,-\mu_1).$$

We identify τ_{mod} with the unit sphere (w.r.t. the metric in (6.9)) in $\Delta_{\tau_{\mathrm{mod}}}$ centered at the origin, i.e., τ_{mod} consists of all elements $(\mu_1,\ldots,\mu_{p+1}) \in \Delta_{\tau_{\mathrm{mod}}}$ satisfying $2(k+1)\sum_{i=1}^{p+1}(r_i - r_{i-1})\mu_i^2 = 1$. An element $\bar\theta = (\mu_1,\ldots,\mu_{p+1}) \in \tau_{\mathrm{mod}}$ lies in the interior of τ_{mod} if and only if $\mu_1 > \cdots > \mu_{p+1}$. Moreover, $\bar\theta$ is ι-invariant if and only if $\mu_i + \mu_{p+2-i} = 0$ for all $i = 1,\ldots,p+1$.

The Finsler metric corresponding to $\bar\theta$ can be calculated explicitly in terms of the above formulas. In the special case when $\tau_{\mathrm{mod}} = (1,k)$ and $\bar\theta = (1/2\sqrt{k+1}, 0, -1/2\sqrt{k+1})$, for all $g \in \mathrm{SL}(k{+}1,\mathbb{R})$,

$$d_{\mathrm{F}}(x,gx) = \sqrt{k+1}\,(\mu_1(g) - \mu_{k+1}(g))\,, \tag{6.10}$$

where $x \in X$ is the point whose stabilizer is $\mathrm{SO}(k+1,\mathbb{R})$.

6.2. Conformal densities

Recall that the Busemann functions define the notion of "distance from infinity." For $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, let $b_\tau : X \to \mathbb{R}$ denote the Busemann function based at the ideal point $\bar\theta(\tau) \in \partial_\infty X$

normalized at x_0 (i.e., $b_\tau(x_0)$ is set to be zero). Using Busemann functions, one defines the *horo-spherical signed distance functions* as

$$d_\tau^{\mathrm{hor}}(x,y) = b_\tau(x) - b_\tau(y). \tag{6.11}$$

(Note that these functions can take negative values. However, their absolute values $|d_\tau^{\mathrm{hor}}(x,y)|$ satisfy the triangle inequality and, hence, are pseudo-metrics on X.) These functions are related by Finsler distance functions by

$$d_\tau^{\mathrm{hor}}(x,y) = \lim_{n\to\infty} \left(d_{\mathrm{F}}(x,z_n) - d_{\mathrm{F}}(y,z_n)\right) \tag{6.12}$$

whenever (z_n) is a sequence in X flag-converging to τ, cf. [**KL18b**, Prop. 5.43].

We define conformal densities on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ using these horospherical distance functions.

For a topological space S, we let $M_+(S)$ be the set of positive, totally finite, regular Borel measures on S. Recall that a group H of self-homeomorphisms of S acts on $M_+(S)$ by *pull-back*: For every $B \in \mathfrak{B}(S)$, $h \in H$,

$$\mu \mapsto h^*\mu, \quad h^*\mu(B) = \mu(h^{-1}(B)).$$

Notation. For a topological space S, we use the notation $\mathfrak{B}(S)$ to denote the σ-algebra of Borel sets of S.

Let $\Gamma < G$ be a discrete subgroup and let $A \subset X$ be a nonempty Γ-invariant subset. By a *Γ-invariant conformal A-density μ of dimension $\beta \geq 0$* (or "conformal A-density" in short) on $\mathrm{Flag}(\tau_{\mathrm{mod}})$, we mean a continuous Γ-equivariant map

$$\mu : A \to M^+(\mathrm{Flag}(\tau_{\mathrm{mod}})), \quad a \mapsto \mu_a,$$

satisfying the following properties:

(i) For each $a \in A$, $\mathrm{supp}(\mu_a) \subset \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.

(ii) (Invariance) μ is Γ-invariant, i.e., $\gamma^*\mu_a = \mu_{\gamma a}$ for each $\gamma \in \Gamma$ and each $a \in A$.

(iii) (Conformality) For every pair $a, b \in A$, $\mu_a \ll \mu_b$, i.e., μ_a is absolutely continuous with respect to μ_b, and the Radon Nikodym derivative $d\mu_a/d\mu_a$ can be expressed as

$$\frac{d\mu_a}{d\mu_b}(\tau) = \exp\left(-\beta d_\tau^{\mathrm{hor}}(a,b)\right), \quad \forall \tau \in \mathrm{Flag}(\tau_{\mathrm{mod}}). \tag{6.13}$$

Remark. Though we define conformal densities for general discrete subgroups of G, for our purpose, we restrict the discussion only to τ_{mod}-regular subgroups.

A conformal X-density μ is simply called a *conformal density*. Note that conformal X-densities and conformal A-densities are in a one-to-one correspondence:

$$\{\text{conformal } X\text{-densities}\} \longleftrightarrow \{\text{conformal } A\text{-densities}\}. \tag{6.14}$$

From an X-density, define an A-density by restricting the family. On the other hand, given an A-density μ, extend it to an X-density $\{\mu_x\}_{x\in X}$ by

$$d\mu_x(B) = \int_B \exp\left(-\beta d_\tau^{\mathrm{hor}}(x,a)\right) d\mu_a(\tau), \quad B \in \mathfrak{B}(\mathrm{Flag}(\tau_{\mathrm{mod}}))$$

where μ_a is a density in the family μ. Note that this extension is unique because μ_x and μ_a are absolutely continuous with respect to each other. To check Γ-invariance, note that

$$\begin{aligned}\gamma^*\mu_x(B) &= \int_{\gamma^{-1}B} \frac{d\mu_x}{d\mu_a}(\tau) d\mu_a(\tau) = \int_B \exp\left(-\beta d_{\gamma^{-1}\tau}^{\mathrm{hor}}(x,a)\right) d\mu_a(\gamma^{-1}\tau) \\ &= \int_B \exp\left(-\beta d_\tau^{\mathrm{hor}}(\gamma x,\gamma a)\right) d\mu_{\gamma a}(\tau) = \int_B \frac{d\mu_{\gamma x}}{d\mu_{\gamma a}}(\tau) d\mu_{\gamma a}(\tau) = \mu_{\gamma x}(B),\end{aligned}$$

for every $B \in \mathfrak{B}(\mathrm{Flag}(\tau_{\mathrm{mod}}))$. The other two defining properties are also satisfied.

Next we construct a conformal density using the Patterson–Sullivan construction. This definition is standard and already appeared in the work of Albuquerque and Quint, although only in the setting of Zariski dense subgroups $\Gamma < G$ and regular vectors $\bar{\theta}$; we present it here for the sake of completeness. We let $\Gamma < G$ be a τ_{mod}-regular subgroup and let Z denote the Γ-orbit of a point $x_0 \in X$. The union

$$\bar{Z} = Z \cup \Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \subset \bar{X}^{\tau_{\mathrm{mod}}},$$

equipped with the topology of flag-convergence, is a compactification of Z.

For $s > \delta_{\mathrm{F}}$ we define a family of positive measures $\mu_s = \{\mu_{x,s}\}_{x\in X}$ on $\bar{Z}$ by

$$\mu_{x,s} = \frac{1}{g_s^{\mathrm{F}}(x_0,x_0)} \sum_{\gamma\in\Gamma} \exp\left(-s d_{\mathrm{F}}(x,\gamma x_0)\right) D(\gamma x_0), \tag{6.15}$$

where $D(\gamma x_0)$ denotes the Dirac point mass of weight one at γx_0. Note that $\mu_{x,s}$ is a probability measure when $x \in Z$. Also, note that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is a null set for these measures. For $\gamma \in \Gamma$ a straightforward computation shows that

$$\gamma^* \mu_{x,s} = \mu_{\gamma x,s}. \tag{6.16}$$

Moreover, it is easy to see that the measures in the family μ_s are absolutely continuous with respect to each other. Using (6.15) we compute the Radon-Nikodym derivatives $d\mu_{x,s}/d\mu_{x_0,s}$,

$$\psi_s(z) = \frac{d\mu_{x,s}}{d\mu_{x_0,s}}(z), \tag{6.17}$$

where for $s \geq 0$,

$$\psi_s(z) := \exp\left(-s\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right).$$

The formula for ψ_s above only makes sense when $z \in Z$. Since $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is a null set, we extend ψ_s continuously to $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ by setting

$$\psi_s(\tau) = \exp\left(-s d_\tau^{\mathrm{hor}}(x,x_0)\right).$$

The continuity of this function can be verified using properties of Finsler distances (e.g., see [**KL18b**, Sec. 5.1.2] and (6.12)).

Next we prove that $\psi_s \to \psi_{\delta_{\mathrm{F}}}$ uniformly as $s \to \delta_{\mathrm{F}}$. For $S \geq s, s' > \delta_{\mathrm{F}}$ and $z \in Z$,

$$\begin{aligned}
&|\psi_{s'}(z) - \psi_s(z)| \\
&\quad = |\exp\left(-s'\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right) - \exp\left(-s\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right)| \\
&\quad = \exp\left(-s\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right)|\exp\left((s-s')\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right) - 1| \\
&\quad \leq \exp\left(S d_{\mathrm{R}}(x,x_0)\right)|\exp\left((s-s')\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right) - 1|.
\end{aligned}$$

Switching s and s' in the above, we also get

$$|\psi_{s'}(z) - \psi_s(z)| \leq \exp\left(S d_{\mathrm{R}}(x,x_0)\right)|\exp\left((s'-s)\left(d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)\right)\right) - 1|.$$

Combining the above two inequalities, we get

$$\begin{aligned}
|\psi_{s'}(z) - \psi_s(z)| &\leq \exp\left(S d_{\mathrm{R}}(x,x_0)\right)\left(\exp\left(|s'-s| \cdot |d_{\mathrm{F}}(z,x) - d_{\mathrm{F}}(z,x_0)|\right) - 1\right) \\
&\leq \exp\left(S d_{\mathrm{R}}(x,x_0)\right)\left(\exp\left(|s'-s| d_{\mathrm{R}}(x,x_0)\right) - 1\right).
\end{aligned}$$

Since Z is dense in $\bar{Z}$, the above yields

$$\|\psi_s - \psi_{s'}\|_\infty \leq \exp\left(S d_{\mathrm{R}}(x, x_0)\right)\big(\exp\big(|s' - s| d_{\mathrm{R}}(x, x_0)\big) - 1\big)$$

Therefore, $\psi_s \to \psi_{\delta_{\mathrm{F}}}$ uniformly as $s \to \delta_{\mathrm{F}}$.

Now we construct a conformal density as a limit of the family of densities $\{\mu_s\}_{s>\delta_{\mathrm{F}}}$. We first assume that Γ has divergence type.[2] Then, as s decreases to δ_{F}, the family $\mu_s = \{\mu_{x,s}\}_{x\in X}$ weakly accumulates to a density μ supported on some subset of $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.

By (6.16) we have the Γ-invariance of μ, namely, for $\gamma \in \Gamma$,

$$\gamma^* \mu_x = \mu_{\gamma x}. \tag{6.18}$$

Any such limit density is called a *Patterson–Sullivan density.*

Since μ_x is obtained as a weak limit of the measures $\mu_{x,s}$ and the derivatives $\psi_s = d\mu_{x,s}/d\mu_{x_0,s}$ converge uniformly to $\psi_{\delta_{\mathrm{F}}}$, it follows that the Radon-Nikodym derivative $d\mu_x/d\mu_{x_0}$ exists and equals to the limit

$$\lim_{s\to\delta_{\mathrm{F}}} \frac{d\mu_{x,s}}{d\mu_{x_0,s}} = \psi_{\delta_{\mathrm{F}}},$$

or more explicitly,

$$\frac{d\mu_x}{d\mu_{x_0}}(\tau) = \exp\Big(-\delta_{\mathrm{F}} d_\tau^{\mathrm{hor}}(x, x_0)\Big). \tag{6.19}$$

Note that in general weak limits are not unique. In Corollary 6.7.4 we will prove that for Anosov subgroups Γ we get a unique density in this limiting process.

When Γ has convergence type, we change weights of the Dirac masses by a small amount (as in [**Nic89**, Sec. 3.1]) in the definition (6.15) to force the Poincaré series to diverge. Define

$$\mu_{x,s} = \frac{1}{\bar{g}_s^{\mathrm{F}}(x_0, x_0)} \sum_{\gamma\in\Gamma} \exp\left(-s d_{\mathrm{F}}(x, \gamma x_0)\right) h\left(d_{\mathrm{F}}(x, \gamma x_0)\right) D(\gamma x_0)$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a subexponential function such that the following modified Poincaré series

$$\bar{g}_s^{\mathrm{F}}(x, x_0) = \sum_{\gamma\in\Gamma} \exp\left(-s d_{\mathrm{F}}(x, \gamma x_0)\right) h\left(d_{\mathrm{F}}(\gamma x, x_0)\right)$$

diverges at the critical exponent $s = \delta_{\mathrm{F}}$. In this case also, limit density μ has the properties (6.18) and (6.19).

[2] This will be the case for Anosov subgroups. See Corollary 6.5.5.

The existence of a conformal density implies that the Finsler critical exponent of Γ is positive.

PROPOSITION 6.2.1. *Suppose that Γ is a nonelementary τ_{mod}-regular antipodal subgroup. Then, the critical exponent δ_{F} is positive.*

PROOF. Suppose to the contrary that $\delta_{\text{F}} = 0$. Let μ be a Patterson–Sullivan density constructed above. It follows from the Γ-invariance and conformality that for all $\gamma \in \Gamma$,

$$\mu_x(\gamma A) = \mu_{\gamma^{-1}x}(A) = \mu_x(A), \quad \forall A \in \mathfrak{B}(\Lambda_{\tau_{\text{mod}}}(\Gamma)). \tag{6.20}$$

Note that this implies that μ is atom-free. For if $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ were an atom, then, by the minimality of the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ and (6.20), $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ would have infinite μ_x-mass,

Let (γ_n) be a sequence on Γ such that $\gamma_n^{\pm 1} \to \tau_\pm \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$. Let (U_n), $U_n \subset \text{Flag}(\tau_{\text{mod}})$, be a *contraction sequence*[3] for (γ_n). By the definition,

(i) (U_n) *exhausts* $\text{Flag}(\tau_{\text{mod}})$ in the sense that every compact set antipodal to τ_- is contained in U_n for all sufficiently large n.
(ii) The sequence $\gamma_n U_n$ Hausdorff-converges to τ_+.

Let $A \subset \Lambda_{\tau_{\text{mod}}}(\Gamma) - \{\tau_+\}$ be a compact set of positive mass (this exists because τ_+ has zero mass). Therefore, by property (1), there exists $n_0 \in \mathbb{N}$ such that $\mu_x(U_n) \geq \mu_x(A) > 0$, for all $n \geq n_0$, and together with property (2) above, we get

$$\mu_x(\tau_+) \geq \lim_{n\to\infty} \mu_x(\gamma_n U_n) \geq \mu_x(A) > 0$$

Hence τ_+ is an atom which gives a contradiction. $\square$

REMARK. As a corollary to the above proposition, the Riemannian critical exponent δ_{R} of a nonelementary uniformly τ_{mod}-regular antipodal subgroup is also positive. See the remark after Proposition 6.1.3.

6.3. Hyperbolicity of the Morse image

In this section we prove that the image of a Morse map is Gromov-hyperbolic with respect to the Finsler pseudo-metric d_{F}. As a corollary, we prove that each orbit of an Anosov subgroup is also Gromov-hyperbolic with respect to the Finsler metric.

We first recall two notions of hyperbolicity.

[3]See [**KLP14**, Def. 5.9, Prop. 5.14] or [**KLP17**, Defn. 4.1, Prop. 4.13].

Definition 6.3.1 (Rips hyperbolic). Let (Z, d) be a geodesic metric space. Then, (Z, d) is called *$\delta(\geq 0)$-hyperbolic in the sense of Rips* (or *Rips hyperbolic*) if every geodesic triangle $\triangle$ is δ-thin, i.e., each side of $\triangle$ lies in the δ-neighborhood of the union of the other two sides.

Definition 6.3.2 (Gromov hyperbolic). Let (Z, d) be a metric space. For any three points $z, z_1, z_2 \in Z$, the *Gromov product* is defined as

$$\langle z_1 | z_2 \rangle_z = \frac{1}{2}[d(z, z_1) + d(z, z_2) - d(z_1, z_2)].$$

Then (Z, d) is called *$\delta(\geq 0)$-hyperbolic in the sense of Gromov* (or *Gromov hyperbolic*) if the Gromov product satisfies the following *ultrametric inequality*: For all $z, z_1, z_2, z_3 \in Z$,

$$\langle z_1 | z_2 \rangle_z \geq \min\{\langle z_1 | z_3 \rangle_z, \langle z_2 | z_3 \rangle_z\} - \delta.$$

It should be noted that Gromov's definition applies to all metric spaces whereas Rips' definition works only for geodesic metric spaces. Moreover, Gromov hyperbolicity is not quasiisometric invariant whereas Rips hyperbolicity is (as a consequence of Morse lemma, cf. [**DK18**, Cor. 11.43])). For geodesic metric spaces, these two notions of hyperbolicity are equivalent (e.g., see [**DK18**, Lemma 11.27]).

Let (Z', d') be Rips hyperbolic and $f : (Z', d') \to (X, d_{\mathrm{R}})$ be a τ_{mod}-Morse map. We denote the image $f(Z')$ by Z. Recall that the Finsler metric is coarsely equivalent to the Riemannian metric on Z.[4] Therefore, since f is a quasiisometric embedding with respect to d_{R}, it is also a quasiisometric embedding with respect to d_{F}. Moreover, the image of a geodesic (of length bounded below by a constant) in Z' stays within a uniformly bounded Riemannian distance, say $\lambda_0 \geq 0$, from a τ_{mod}-regular Finsler geodesic connecting the images of the endpoints. This is a consequence of the Morse property [**KLP18b**, Thm. 1.1], see also [**KL18b**, Prop. 12.2]. A consequence of this is that Z is λ_0-quasiconvex in X with respect to the Finsler metric (or *Finsler quasiconvex*).

For $\lambda \geq \lambda_0$, let $Y = Y_\lambda$ be the Riemannian λ-neighborhood of Z in X. From the discussion above, it is clear that any two points $z_1, z_2 \in Z$ (with $d_{\mathrm{R}}(z_1, z_2)$ sufficiently large) can be connected by a Finsler geodesic $\widehat{z_1 z_2}$ in Y.

[4] This is also true for any finite Riemannian tubular neighborhood of Z.

Proposition 6.3.3. *Let c and c' be two Finsler geodesics in Y connecting two points z_1, z_2. Then they are uniformly Hausdorff close. Here the Hausdorff distance is induced by either of Riemannian or Finsler metric.*

Proof. Since Riemannian and Finsler metrics are comparable on Y, it is enough to prove the proposition for the Riemannian metric.

Let $\bar{c}$ and $\bar{c}'$ be the respective nearest point projections of c and c' to Z. Applying the coarse inverse of f, $\bar{c}$ and $\bar{c}'$ map to uniform quasigeodesics $\tilde{c}$ and $\tilde{c}'$, respectively, in Z'. Since Z' is Rips hyperbolic, $\tilde{c}$ and $\tilde{c}'$ are uniformly close. Applying f to $\tilde{c}$ and $\tilde{c}'$, we see that $\bar{c}$ and $\bar{c}'$ are uniformly close. Hence c and c' are also uniformly close. □

Next we observe that geodesic triangles in (Y, d_{F}) with vertices on Z are uniformly thin.

Proposition 6.3.4. *There exists $\delta \geq 0$ such that all Finsler geodesic triangles $\triangle = \triangle(z_1, z_2, z_3)$ in Y is δ-thin both in Riemannian and Finsler sense.*

Proof. Since Z' is Rips hyperbolic, geodesic triangles in Z' are δ'-thin, for some $\delta' \geq 0$. We map $\triangle$ to a uniformly quasigeodesic triangle $\triangle' \subset Z'$ via the coarse inverse map $Y \to Z'$ of the map f. Since Z' is Rips-hyperbolic, the Morse quasigeodesic triangle $\triangle'$ is uniformly thin. Therefore, $\triangle$ is also uniformly thin as well. □

Imitating the proof of [**DK18**, Lem. 11.27], we prove that (Z, d_{F}) is Gromov-hyperbolic

Theorem 6.3.5 (Hyperbolicity of Morse maps). *Let $Z \subset X$ be the image of a τ_{mod}-Morse map $f : (Z', d') \to (X, d_{\mathrm{R}})$. Then (Z, d_{F}) is Gromov-hyperbolic.*

Proof. Let δ be as in Proposition 6.3.4. Then the following holds.

Lemma 6.3.6. *Let $z, z_1, z_2 \in Z$, and let $\widehat{z_1 z_2}$ be any Finsler geodesic in Y connecting z_1 and z_2. Then,*

$$\langle z_1 | z_2 \rangle_z \leq d_{\mathrm{F}}(z, \widehat{z_1 z_2}) \leq \langle z_1 | z_2 \rangle_z + 2\delta.$$

Proof. The proof is exactly same as [**DK18**, Lem. 11.22]. Note that the proof uses δ-thinness of a triangle with vertices z, z_1, z_2. □

Let z, z_1, z_2, z_3 be any four points in Z, and let $\triangle$ be a Finsler geodesic triangle in Y with the vertices z_1, z_2, z_3. Let m be a point on the side $\widehat{z_1 z_2}$ nearest to z. By Proposition 6.3.4, since $\triangle$ is

δ-thin, $d_{\mathrm{F}}(m, \widehat{z_2 z_3} \cup \widehat{z_1 z_3}) \leq \delta$. Without loss of generality, assume that there is a point n on $\widehat{z_2 z_3}$ which is δ-close to m. Then, using the above lemma, we get

$$\langle z_2 | z_3 \rangle_z \leq d_{\mathrm{F}}(z, \widehat{z_2 z_3}) \leq d_{\mathrm{F}}(z, \widehat{z_1 z_2}) + \delta,$$

and

$$d_{\mathrm{F}}(z, \widehat{z_1 z_2}) \leq \langle z_1 | z_2 \rangle_z + 2\delta.$$

The theorem follows from this. $\square$

Quasiisometry of hyperbolic metric spaces extends to a homeomorphism of their Gromov boundaries. At the same time, it is proven in [**KLP18b**] that each τ_{mod}-Morse map

$$f : Z' \to Z = f(Z') \subset X$$

extends continuously (with respect to the topology of flag-convergence) to a homeomorphism

$$\partial_\infty f : \partial_\infty Z' \to \Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}}).$$

Thus, we obtain

Corollary 6.3.7. *The Gromov boundary $\partial_\infty Z$ of (Z, d_{F}) is naturally identified with the flag-limit set $\Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ of Z: A sequence (z_n) in Z converges to a point in $\partial_\infty Z$ if and only if (z_n) flag-converges to some $\tau \in \Lambda$.*

For a τ_{mod}-Anosov subgroup Γ we know that the orbit map $\Gamma \to \Gamma x_0$ is a τ_{mod}-Morse embedding. Then, using Theorem 6.3.5 we obtain:

Corollary 6.3.8 (Hyperbolicity of Anosov orbits). *For $x_0 \in X$, let $Z = \Gamma x_0$ where Γ is a τ_{mod}-Anosov subgroup. Then (Z, d_{F}) is Gromov hyperbolic. The Gromov boundary of (Z, d_{F}) is naturally identified with the τ_{mod}-limit set $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.*

6.4. Gromov distance at infinity

The definition of horospherical signed distances given in (6.11) is free of choice of any particular normalization for the Busemann functions. Note that

$$-d_{\mathrm{F}}(x_1, x_2) \leq d_\tau^{\mathrm{hor}}(x_1, x_2) \leq d_{\mathrm{F}}(x_1, x_2).$$

Furthermore, d_τ^{hor} satisfies the *cocycle condition:* For each triple $x_1, x_2, x_3 \in X$,

$$d_\tau^{\mathrm{hor}}(x_1,x_2) + d_\tau^{\mathrm{hor}}(x_2,x_3) = d_\tau^{\mathrm{hor}}(x_1,x_3). \tag{6.21}$$

For a pair of antipodal simplices $\tau_\pm \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, the *Gromov product* with respect to a base point $x \in X$ is defined as

$$\langle \tau_+ | \tau_- \rangle_x = \frac{1}{2}\left(d_{\tau_+}^{\mathrm{hor}}(x,z) + d_{\tau_-}^{\mathrm{hor}}(x,z) \right), \tag{6.22}$$

where z is some point on the parallel set $P(\tau_+, \tau_-)$ spanned by $\tau_\pm$.

The following lemma proves that the Gromov products do not depend on the chosen $z \in P(\tau_+, \tau_-)$.

Lemma 6.4.1. *For $z_1, z_2 \in P(\tau_+, \tau_-)$, one has $b_{\tau_+}(z_1) + b_{\tau_-}(z_1) = b_{\tau_+}(z_2) + b_{\tau_-}(z_2)$.*

Proof. Let z be the midpoint of $\overline{z_1 z_2}$ and let $s_z : X \to X$ be the point reflection about z. Assuming that Busemann functions are normalized at z, s_z transforms $b_{\tau_+}(z_1) + b_{\tau_-}(z_1)$ into $b_{\tau_-}(z_2) + b_{\tau_+}(z_2)$. Hence the quantities are equal. $\square$

Using the Gromov product, we define a *premetric*[5] on $\mathrm{Flag}(\tau_{\mathrm{mod}})$.

Definition 6.4.2 (Gromov premetric). Given fixed $x \in X$, $\epsilon > 0$, define the *Gromov premetric* $d_{\mathrm{G}}^{x,\epsilon}$ on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ as

$$d_{\mathrm{G}}^{x,\epsilon}(\tau_1,\tau_2) = \begin{cases} \exp\left(-\epsilon \langle \tau_1 | \tau_2 \rangle_x\right), & \text{if } \tau_1, \tau_2 \text{ are antipodal,} \\ 0, & \text{otherwise.} \end{cases}$$

Note that a pair of points $\tau_\pm \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ is antipodal if and only if $d_{\mathrm{G}}^{x,\epsilon}(\tau_+,\tau_-) \neq 0$.

Lemma 6.4.3. *$d_{\mathrm{G}}^{x,\epsilon}$ is a continuous function.*

Proof. The claim follows from [**Bey**, Lem. 3.8]. $\square$

Lemma 6.4.4. *Let $\gamma \in G$ and $\Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ be a γ-invariant antipodal subset. Then the map $\gamma : \Lambda \to \Lambda$ is conformal with respect to the premetric $d_{\mathrm{G}}^{x,\epsilon}$.*

[5]A *premetric* on X is a symmetric, continuous function $d : X \times X \to [0,\infty)$ such that $d(x,x) = 0$ for all $x \in X$.

Proof. Given distinct points $\tau_\pm \in \Lambda$,

$$\begin{aligned}
d_{\mathrm{G}}^{x,\epsilon}(\gamma\tau_+, \gamma\tau_-) &= \exp\left(-\epsilon\langle\gamma\tau_+|\gamma\tau_-\rangle_x\right)\\
&= \exp\left(-\frac{\epsilon}{2}\left(d^{\mathrm{hor}}_{\gamma\tau_+}(x,z) + d^{\mathrm{hor}}_{\gamma\tau_-}(x,z)\right)\right)\\
&= \exp\left(-\frac{\epsilon}{2}\left(d^{\mathrm{hor}}_{\tau_+}(\gamma^{-1}x,\gamma^{-1}z) + d^{\mathrm{hor}}_{\tau_-}(\gamma^{-1}x,\gamma^{-1}z)\right)\right)\\
&= \exp\left(-\frac{\epsilon}{2}\left(d^{\mathrm{hor}}_{\tau_+}(\gamma^{-1}x,x) + d^{\mathrm{hor}}_{\tau_-}(\gamma^{-1}x,x)\right)\right) d_{\mathrm{G}}^{x,\epsilon}(\tau_+,\tau_-),
\end{aligned}$$

where the last equality follows from the cocycle condition (6.21). Moreover, the continuity of Busemann functions b_τ as a function of τ implies that

$$\lim_{\tau_-\to\tau_+} d^{\mathrm{hor}}_{\tau_-}(\gamma^{-1}x,x) = d^{\mathrm{hor}}_{\tau_+}(\gamma^{-1}x,x).$$

Therefore,

$$\lim_{\tau_-\to\tau_+} \frac{d_{\mathrm{G}}^{x,\epsilon}(\gamma\tau_+,\gamma\tau_-)}{d_{\mathrm{G}}^{x,\epsilon}(\tau_+,\tau_-)} = E(\gamma,\tau_+) := \exp\left(-\epsilon d^{\mathrm{hor}}_{\tau_+}(\gamma^{-1}x,x)\right). \tag{6.23}$$

The lemma follows from this. ☐

The premetric $d_{\mathrm{G}}^{x,\epsilon}$ is not a metric in general since:

(i) Pairs of distinct non-antipodal points have zero distance.

(ii) The triangle inequality may fail.

However, as we shall see below, $d_{\mathrm{G}}^{x,\epsilon}$ defines a metric when restricted to "nice" antipodal subsets $\Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ for sufficiently small $\epsilon > 0$.

Theorem 6.4.5. *Let $Z \subset X$ be the image of a τ_{mod}-Morse map $f : (Z', d') \to (X, d)$, and $\Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ be the flag limit set of Z. There exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \le \epsilon_0$ and all $x \in Z$, the premetric $d_{\mathrm{G}}^{x,\epsilon}$ restricts to a metric on Λ. Moreover, the topology induced by $d_{\mathrm{G}}^{x,\epsilon}$ on Λ coincides with the subspace topology of $\Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$.*

Proof. For the first part of the theorem we only need to check that $d_{\mathrm{G}}^{x,\epsilon}$ satisfies the triangle inequality for sufficiently small $\epsilon > 0$. The idea of the proof is due to Gromov [**Gro87**]: We show that the Gromov product defined in (6.22) restricted to Λ satisfies an ultrametric inequality (see (6.27)).

Let $Y \subset X$ be a Riemannian λ-neighborhood of Z. We assume that λ here is so large such that $x \in Y$ and the image of any complete geodesic l in Z' lies within distance λ from the parallel set spanned by the images of the ideal endpoints of l under $\bar{f} : \partial_\infty Z' \to \mathrm{Flag}(\tau_{\mathrm{mod}})$. Note that λ satisfying the last condition exists as a consequence of the Morse property.

Observe that (Y, d_{F}) is a Gromov δ-hyperbolic metric space for some $\delta \geq 0$. This follows from the Gromov hyperbolicity of (Z, d_{F}) (cf. Theorem 6.3.5) and the fact that Z and Y are (Hausdorff) λ-close to each other.

We recall from Väisälä [**Väi05**, Sec. 5] that there are multiple ways to define Gromov products on Λ viewed as the Gromov boundary of (Z, d_{F}) and, hence, of (Y, d_{F}). For a distinct pair $\tau_\pm \in \Lambda$, define using the Gromov product $\langle \cdot | \cdot \rangle_x$ on (Y, d_{F}) the following two products:

$$\langle \tau_+ | \tau_- \rangle_x^{\inf} = \inf \left\{ \liminf_{i,j \to \infty} \langle y_i^+ | y_j^- \rangle_x : (y_n^\pm) \subset Y, y_n^\pm \to \tau_\pm \right\}$$

and

$$\langle \tau_+ | \tau_- \rangle_x^{\sup} = \sup \left\{ \limsup_{i,j \to \infty} \langle y_i^+ | y_j^- \rangle_x : (y_n^\pm) \subset Y, y_n^\pm \to \tau_\pm \right\}.$$

Then the difference of the above two quantities is uniformly bounded (see [**Väi05**, 5.7]), namely, for all distinct pairs $\tau_\pm \in \Lambda$,

$$0 \leq \langle \tau_+ | \tau_- \rangle_x^{\sup} - \langle \tau_+ | \tau_- \rangle_x^{\inf} \leq 2\delta. \tag{6.24}$$

Finally, $\langle \cdot | \cdot \rangle_x^{\inf}$ satisfies the ultrametric inequality (see [**Väi05**, 5.12]), i.e., for distinct triples $\tau_1, \tau_2, \tau_3 \in \Lambda$,

$$\langle \tau_1 | \tau_2 \rangle_x^{\inf} \geq \min \left\{ \langle \tau_1 | \tau_3 \rangle_x^{\inf}, \langle \tau_2 | \tau_3 \rangle_x^{\inf} \right\} - \delta. \tag{6.25}$$

By (6.24), $\langle \cdot | \cdot \rangle_x^{\sup}$ also satisfies the ultrametric inequality but with a different constant, 5δ.

Next we compare Väisälä's Gromov products with ours (see (6.22)). Let $\tau_\pm \in \Lambda$ be a pair of antipodal points and let $P = P(\tau_+, \tau_-)$. Note that our assumption on largeness of λ implies that there exist uniformly τ_{mod}-regular sequences (y_n^+) and (y_n^-) on $Y \cap P$ such that $y_n^\pm \to \tau_\pm$ as $n \to \infty$. Let $p \in P(\tau_+, \tau_-)$. Then, the additivity of Finsler distances on τ_{mod}-cones (cf. [**KL18b**, Lem. 5.10]) yields, for large n, $\langle y_n^+ | y_n^- \rangle_p = 0$. By definition,

$$\langle y_n^+ | y_n^- \rangle_x = \langle y_n^+, y_n^- \rangle_z + \frac{1}{2} \left[\left(d_{\mathrm{F}}(y_n^+, x) - d_{\mathrm{F}}(y_n^+, p) \right) + \left(d_{\mathrm{F}}(y_n^-, x) - d_{\mathrm{F}}(y_n^-, p) \right) \right],$$

and for large n,

$$\langle y_n^+ | y_n^- \rangle_x = \frac{1}{2} \left[\left(d_{\mathrm{F}}(y_n^+, x) - d_{\mathrm{F}}(y_n^+, p) \right) + \left(d_{\mathrm{F}}(y_n^-, x) - d_{\mathrm{F}}(y_n^-, p) \right) \right].$$

The limit, as $n \to \infty$, of the right side of this equation equals $\langle \tau_+ | \tau_- \rangle_x$ (cf. (6.12)). Therefore,

$$\langle \tau_+ | \tau_- \rangle_x^{\inf} \leq \langle \tau_+, \tau_- \rangle_x \leq \langle \tau_+ | \tau_- \rangle_x^{\sup}. \tag{6.26}$$

Hence, by (6.24) and (6.25), $\langle \cdot | \cdot \rangle_x$ satisfies the ultrametric inequality with constant 5δ, i.e., for distinct points $\tau_1, \tau_2, \tau_3 \in \Lambda$,

$$\langle \tau_1 | \tau_2 \rangle_x \geq \min \{ \langle \tau_1 | \tau_3 \rangle_x, \langle \tau_2 | \tau_3 \rangle_x \} - 5\delta. \tag{6.27}$$

This completes the proof of the first part of the theorem.

For the second part, note that the inequality (6.26) implies that $d_{\mathrm{G}}^{x,\epsilon}$ induces the standard topology on Λ as the Gromov boundary of (Y, d_{F}) (see [**Väi05**, 5.29]). Since, as we noted earlier, this topology is the same as the subspace topology of the flag-manifold $\mathrm{Flag}(\tau_{\mathrm{mod}})$, the second claim of the theorem follows as well. □

Corollary 6.4.6 (Conformal metric on Anosov limit set). *Let Γ be a τ_{mod}-Anosov subgroup, $x \in X$. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ and all $z \in \Gamma x$, $d_{\mathrm{G}}^{z,\epsilon}$ is a metric on $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. Moreover, the action $\Gamma \curvearrowright \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is conformal with respect to $d_{\mathrm{G}}^{z,\epsilon}$.*

Proof. Since Anosov subgroups satisfy the Morse property, corollary follows from Theorem 6.4.5 combined with Lemma 6.4.4. □

Example 6.4.7 (Product of rank-one symmetric spaces). We continue with Example 6.1.4. Let $\tau = (\xi_{r_1}, \ldots, \xi_{r_p})$ be a simplex in the Tits building of type $\tau_{\mathrm{mod}} = (r_1, \ldots, r_p)$ and $\bar\theta = (1/\sqrt{p}, \ldots, 1/\sqrt{p}) \in \tau_{\mathrm{mod}}$. We compute the horospherical distance, Gromov distance associated with τ_{mod} and type $\bar\theta$.

Let $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in X$. Then

$$d_\tau^{\mathrm{hor}}(x, y) = \lim_{t \to \infty} \left(d_X(\ell(t), x) - t \right)$$

where $\ell(t)$ is a geodesic ray emanating from y and asymptotic to $\bar\theta(\tau)$. A direct computation yields

$$d_\tau^{\mathrm{hor}}(x, y) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left(b_{\xi_{r_j}}(x_{r_j}) - b_{\xi_{r_j}}(y_{r_j}) \right) = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} d_{\xi_{r_j}}^{\mathrm{hor}} \left(x_{r_j}, y_{r_j} \right).$$

Hence the Gromov product can be written as

$$\langle \tau_+ | \tau_- \rangle_x = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \langle \xi_{r_j}^+ | \xi_{r_j}^- \rangle_{x_{r_j}}, \quad \forall \tau_\pm = (\xi_{r_1}^\pm, \dots, \xi_{r_p}^\pm) \in \mathrm{Flag}(\tau_{\mathrm{mod}})$$

and, finally the Gromov predistance is

$$d_{\mathrm{G}}^{x,1/\sqrt{p}}(\tau_+, \tau_-) = \prod_{j=1}^{p} d_{\mathrm{G}}^{x_{r_j},1/p} \left(\xi_{r_j}^+, \xi_{r_j}^- \right). \tag{6.28}$$

Example 6.4.8 ($X = \mathrm{SL}(k+1, \mathbb{R})/\mathrm{SO}(k+1, \mathbb{R})$). In this case the computations of Busemann functions (see [**Hat95**]) and Gromov products (see [**Bey**]) are explicitly known, and therefore, the Gromov distance can also be computed explicitly. We only give a formula for the Gromov distance in the special case when $\tau_{\mathrm{mod}} = (1, k)$ that corresponds to the partial flags $\{$*line* $\subset$ *hyperplane*$\}$ of $\mathbb{R}^{k+1}$.

We continue with the notations from Example 6.1.5. The unique ι-invariant type is

$$\bar{\theta} = (1/2\sqrt{k+1}, 0, -1/2\sqrt{k+1}).$$

After equipping $\mathbb{R}^{k+1}$ with the inner product induced by the choice of $x \in X$, the Gromov product (with respect to $x = I_{k+1}$, the identity matrix) can be written as

$$\langle (l_1, h_1) \mid (l_2, h_2) \rangle_x = -\frac{\sqrt{k+1}}{2} \log \left(\sin \angle(l_1, h_2) \cdot \sin \angle(l_2, h_1) \right)$$

where $\angle(l, h)$ denotes the angle between the line l and the hyperplane h. Thus, the Gromov predistance can be written as

$$d_{\mathrm{G}}^{x,1/\sqrt{k+1}}((l_1, h_1), (l_2, h_2)) = \sin \angle(l_1, h_2)^{\frac{1}{2}} \cdot \sin \angle(l_2, h_1)^{\frac{1}{2}}. \tag{6.29}$$

6.5. Shadow lemma

In this section we prove a generalization Sullivan's shadow lemma in higher rank. The proof we present here is inspired by that of Albuquerque's [**Alb99**, Thm. 3.3] who treated the case of full flag manifold and Quint [**Qui02b**] who treated general flag-manifolds but only in the case of regular vectors $\bar{\theta}$.

Recall the notion of *shadow* from (4.8). We mainly consider shadows of closed balls in X from a fixed base point $x \in X$ (see Figure 6.2). The topology generated by these shadows is the topology of flag convergence.

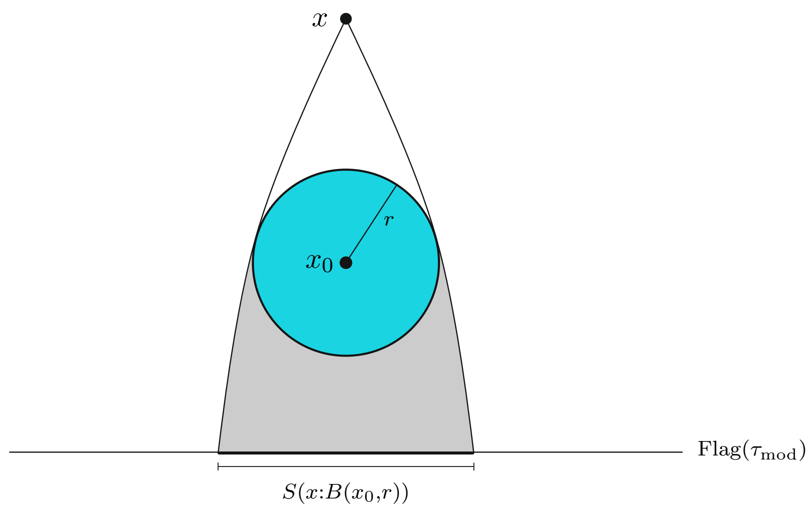

FIGURE 6.2. Shadow of a ball.

The main result in this section is the following.

THEOREM 6.5.1 (Shadow lemma). *Let Γ be a nonelementary τ_{mod}-RA subgroup, $x \in X$, and μ be a Γ-invariant conformal density of dimension β. There exists $r_0 > 0$ such that for all $r \geq r_0$ and all $\gamma \in \Gamma$ satisfying $d_{\mathrm{R}}(x, \gamma x_0) > r$,*

$$C^{-1} \exp\left(-\beta d_{\mathrm{F}}(x, \gamma x_0)\right) \ \leq \ \mu_x(S(x : B(\gamma x_0, r))) \ \leq \ C \exp\left(-\beta d_{\mathrm{F}}(x, \gamma x_0)\right),$$

for some constant $C \geq 1$.

Before presenting the proof, we note two consequences of this theorem.

COROLLARY 6.5.2. *Let Γ be a nonelementary uniformly τ_{mod}-RA subgroup. Then any conformal density μ does not have conical limit points as atoms.*

PROOF. Any conical limit point $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ lies in infinitely many shadows $S(x, B(\gamma x_0, r))$ for sufficiently large $r > 0$ (depending on τ). If τ is an atom, then (by Theorem 6.5.1) the Poincaré series

$$g^{\mathrm{F}}_{\beta}(x, x_0) = \sum_{\gamma \in \Gamma} \exp\left(-\beta d_{\mathrm{F}}(x, \gamma x_0)\right) \tag{6.30}$$

diverges for every $\beta \geq 0$. Hence δ_{F} must be infinite. But this contradicts Proposition 6.1.3. $\square$

The second application of shadow lemma will be given for the following class of subgroups.

Definition 6.5.3 (Uniform conicality). A τ_{mod}-RA subgroup is called *uniformly conical* if for a given pair of points $x, x_0 \in X$, there is a constant $r > 0$ such that for each conical limit point $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, there exists a sequence (γ_k) on Γ flag-converging to τ satisfying $d_{\mathrm{R}}(\gamma_k x_0, V(x, \mathrm{st}(\tau))) < r$, $\forall k \in \mathbb{N}$.

We observe that Anosov subgroups satisfy the uniform conicality condition:

Proposition 6.5.4. *Anosov subgroups are uniformly conical.*

Proof. This follows from the fact that the orbit map $\Gamma \to \Gamma x_0 \subset X$ is a Morse embedding. Let $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ be any point and $\xi \in \partial_\infty \Gamma$ be the preimage of τ under the boundary map. Let (γ_k), $\gamma_1 = 1_\Gamma$ be a geodesic sequence in Γ asymptotic to ξ. Then the sequence $(\gamma_k x_0)$ is a Morse quasigeodesic in X that is uniformly close to $V(x, \mathrm{st}(\tau))$ (by definition of a Morse embedding). $\square$

Corollary 6.5.5. *Let Γ be a nonelementary uniformly conical τ_{mod}-RA subgroup and μ be a conformal density of dimension β. If the conical limit set $\Lambda^{\mathrm{con}}_{\tau_{\mathrm{mod}}}(\Gamma)$ is non-null, then the Poincaré series $g^{\mathrm{F}}_\beta(x, x_0)$ (see (6.30)) diverges.*

Proof. Writing the elements of Γ in a sequence (γ_n), define

$$S_N = \sum_{n \geq N} \exp(-\beta d_{\mathrm{F}}(\gamma_n x_0, x)).$$

Convergence of the series (6.30) asserts that $\lim_{N\to\infty} S_N = 0$. Since Γ is uniformly conical, there exists $r > 0$ such that for all $N \in \mathbb{N}$,

$$\Lambda^{\mathrm{con}}_{\tau_{\mathrm{mod}}}(\Gamma) \subset \bigcup_{n \geq N} S(x : B(\gamma_n x_0, r)).$$

Applying Theorem 6.5.1, we get

$$\mu_x(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)) \leq \sum_{n \geq N} \mu_x \left(S(x : B(\gamma_n x_0, r))\right) \leq \mathrm{const} \cdot S_N$$

and, the bound above approaches to zero as $N \to \infty$. Hence we must have $\mu_x(\Lambda^{\mathrm{con}}_{\tau_{\mathrm{mod}}}(\Gamma)) = 0$. $\square$

The proof of shadow lemma occupies the rest of the section.

Proof of Theorem 6.5.1. In this proof, we equip $\text{Flag}(\tau_{\text{mod}})$ with a G_x-invariant Riemannian metric. We use the notation $L(\tau)$ to denote the set of all $\tau' \in \text{Flag}(\tau_{\text{mod}})$ which are not antipodal to τ. The complement of $L(\tau)$ in $\text{Flag}(\tau_{\text{mod}})$ is denoted by $C(\tau)$. Note that $L(\tau)$ is closed and hence, compact. Moreover, if $\tau_n \to \tau_0$, then the sequence of sets $(L(\tau_n))$ Hausdorff-converges to $L(\tau_0)$.

Lemma 6.5.6. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $\tau_0 \in \text{Flag}(\tau_{\text{mod}})$ and every $\tau \in B(\tau_0, \delta)$,*

$$N_{\varepsilon/2}(L(\tau)) \subset N_\varepsilon(L(\tau_0)).$$

Proof. We equip the set

$$Y = \{L(\tau) : \tau \in \text{Flag}(\tau_{\text{mod}})\}$$

with the Hausdorff distance $d_{\text{Haus}}(,)$. Then, as we noted above, the function $f : \text{Flag}(\tau_{\text{mod}}) \to Y$, $\tau \mapsto L(\tau)$, is continuous and, hence, uniformly continuous. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\tau, \tau_0) < \delta$ implies $d_{\text{Haus}}(L(\tau), L(\tau_0)) < \varepsilon/2$, which then implies $L(\tau) \subset N_{\epsilon/2}(L(\tau_0))$. The lemma follows from this. $\square$

Let $\mathfrak{m} = \mu_x(\Lambda_{\tau_{\text{mod}}}(\Gamma))$ denote the total mass of μ_x, and $\mathfrak{l} = \sup\{\mu_x(\tau) : \tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)\}$. Since μ_x is a regular measure and $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ is compact, $\mathfrak{l}$ is realized, i.e., if μ_x has an atomic part, then it has a largest atom. Moreover, since Γ is nonelementary, $\text{supp}(\mu_x)$ is not singleton. In fact, if τ is an atom, then the every point in the orbit $\Gamma\tau$ (which has infinite number of points) is an atom. In particular, $\mathfrak{l} < \mathfrak{m}$.

Lemma 6.5.7. *Given $\mathfrak{l} < q < \mathfrak{m}$, there exists an $\varepsilon_0 > 0$ such that for all $\tau \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ and all $B \in \mathfrak{B}(\text{Flag}(\tau_{\text{mod}}))$ contained in $N_{\varepsilon_0}(L(\tau))$, $\mu_x(B) \le q$.*

Proof. If this were false, then we would get a sequence (B_n) of Borel sets, a sequence (ε_n) positive numbers converging to zero, and a sequence (τ_n) on $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ converging to a point τ_0 such that for every $n \in \mathbb{N}$,

$$B_n \subset N_{\varepsilon_n}(L(\tau_n)), \quad \mu_x(B_n) > q.$$

To get a contradiction, we will show that $\mu_x(\tau_0) \ge q$. Let U be an open neighborhood of $L(\tau_0)$. As $L(\tau_0)$ is compact, there exists $\varepsilon > 0$ such that $N_\varepsilon(L(\tau_0)) \subset U$. Let $\delta > 0$ be a number that corresponds to this ε as in Lemma 6.5.6. Choose n so large such that $\tau_n \in B(\tau_0, \delta)$ and $\varepsilon_n \le \varepsilon/2$.

By Lemma 6.5.6, we get $N_{\varepsilon_n}(L(\tau_n)) \subset N_\varepsilon(L(\tau_0))$ and, consequently, $B_n \subset U$. This shows that every open set U containing $L(\tau_0)$ has mass $\mu_x(U) > q$. Therefore, $\mu_x(L(\tau_0)) = \mu_x(\tau_0) \geq q$. $\square$

LEMMA 6.5.8. *Given $\varepsilon > 0$ there exists $r_1 > 0$ such that for all $r \geq r_1$, the complement of $S(x : B(x_0, r))$ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is contained in $N_\varepsilon(L(\tau))$, for some $\tau \in S(x_0 : \{x\})$.*

PROOF. For $r > 0$ and $\tau_0 \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, $\tau' \in C(\tau_0)$, consider

$$U(\tau_0, x_0, r) = \{\tau' \in \mathrm{Flag}(\tau_{\mathrm{mod}}) : P(\tau_0, \tau') \cap B(x_0, r) \neq \emptyset\}.$$

This is an analogue of shadows (4.8) as viewed from the infinity (see Figure 6.3). It is easy to verify that

$$\bigcup_{r \geq 0} U(\tau_0, x_0, r) = C(\tau_0).$$

Moreover, for $g \in G$, these shadows from infinity transform as $gU(\tau_0, x_0, r) = U(g\tau_0, gx_0, r)$.

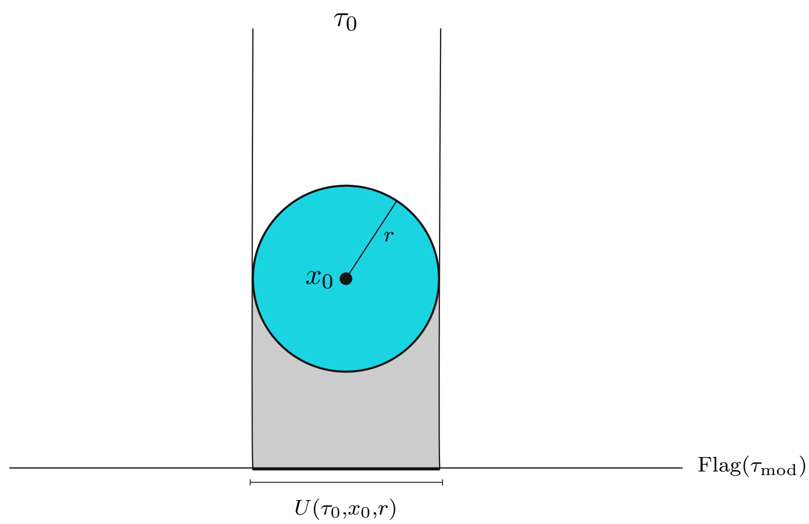

FIGURE 6.3. Shadow of a ball from infinity.

If $k \in K = G_{x_0}$, the stabilizer of x_0, then $kU(\tau_0, x_0, r) = U(k\tau_0, x_0, r)$. Since K is compact, there exists $M \geq 1$ such that the action $k \curvearrowright \mathrm{Flag}(\tau_{\mathrm{mod}})$ is M-Lipschitz for all $k \in K$. Let $r_1 > 0$ be such that $U(\tau_0, x_0, r_1/2)^c \subset N_{\varepsilon/M}(L(\tau_0))$. Here and below, for $A \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$, $A^c = \mathrm{Flag}(\tau_{\mathrm{mod}}) - A$. Then, for any $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$,

$$U(\tau, x_0, r/2)^c \subset N_\varepsilon(L(\tau)), \quad \forall r \geq r_1. \tag{6.31}$$

For $x \in X$, let $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ be a simplex such that $x \in V(x_0, \mathrm{st}(\tau))$. Then there exists a parameterized geodesic ray x_t starting from x_0, passing through x and asymptotic to some $\xi \in \mathrm{st}(\tau)$.

Claim. *For all $r > 0$, $S(x : B(x_0, 2r)) \supset U(\tau, x_0, r)$.*

Proof of claim. Pick $\tau' \in U(\tau, x_0, r)$ and let $\bar{x}_0 \in P(\tau, \tau')$ denote the nearest point projection of x_0. In addition to the ray x_t, we define another parameterized geodesic ray $\bar{x}_t$, starting at $\bar{x}_0$ and asymptotic to ξ. Due to the convexity of the Riemannian distance function on X, the distance $d_{\mathrm{R}}(x_t, \bar{x}_t)$ monotonically decreases with t. Moreover, the cones $V(\bar{x}_t, \mathrm{st}(\tau'))$ are nested as t decreases. Then,

$$\begin{aligned} d_{\mathrm{R}}(x_0, V(x_t, \mathrm{st}(\tau'))) &\leq d_{\mathrm{R}}(x_0, V(\bar{x}_t, \mathrm{st}(\tau'))) + d_{\mathrm{R}}(x_t, \bar{x}_t) \\ &\leq d_{\mathrm{R}}(x_0, V(\bar{x}_0, \mathrm{st}(\tau'))) + r \leq d_{\mathrm{R}}(x_0, \bar{x}_0) + r \leq 2r. \end{aligned}$$

Therefore, $\tau' \in S(x : B(x_0, 2r))$. □

Using (6.31) it follows from the above claim that whenever $r \geq r_1$, the complement of the shadow $S(x : B(x_0, r))$ is contained in $N_\varepsilon(L(\tau))$ for some τ satisfying $x \in V(x_0, \mathrm{st}(\tau))$. □

Lemma 6.5.9. *For all $r > 0$ and all $\tau \in S(x : B(x_0, r))$,*

$$|d_{\mathrm{F}}(x, x_0) - d_\tau^{\mathrm{hor}}(x, x_0)| \leq 2r.$$

Proof. We recall that the Finsler distance can alternatively be defined as

$$d_{\mathrm{F}}(y, z) = \max_{\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})} d_\tau^{\mathrm{hor}}(y, z),$$

where the maximum above occurs at any point in $S(y : \{z\})$ (see [**KL18b**, Sec. 5.1.2]). Fix some $\tau_0 \in S(x, \{x_0\})$. Then for any $\tau_1 \in S(x : B(x_0, r))$,

$$\begin{aligned} |d_{\mathrm{F}}(x, x_0) - d_{\tau_1}^{\mathrm{hor}}(x, x_0)| &= |b_{\tau_0}(x_0) - b_{\tau_1}(x_0)| \\ &= |b_{\tau_0}(x_0) - b_{\tau_0}(k^{-1}x_0)| \\ &\leq d_{\mathrm{R}}(x_0, k^{-1}x_0) = d_{\mathrm{R}}(kx_0, x_0), \end{aligned}$$

where $k \in K$, stabilizer of x, is some isometry satisfying $\tau_1 = k\tau_0$. In the above we chose the normalizations of the Busemann functions at x.

Let $y \in V(x, \mathrm{st}(\tau)) \cap B(x_0, r)$. Then $y \in V(x, \sigma)$ for some chamber σ in $\mathrm{st}(\tau)$. We identify $V(x, \sigma)$ with the model Weyl chamber Δ. Let $k_1 \in K$ such that $k_1 x \in V(x, \sigma)$. Then $k_1 x_0 = d_\Delta(x, x_0)$ via the identification above. Moreover, since the map

$$X \to \Delta, \quad z \mapsto d_\Delta(x, z)$$

is 1-Lipschitz (by the triangle inequality for Δ-valued distances, Theorem 4.1.4) and $d_\Delta(x, y) = y$, we obtain,

$$d_{\mathrm{R}}(y, k_1 x_0) \leq d(y, x_0) < r$$

and, in particular, $d(x_0, k_1 x_0) < 2r$. ◻

Using the above lemmata, we now complete the proof of Theorem 6.5.1. We first fix some auxiliary quantities. Let $q \in (\mathfrak{l}, \mathfrak{m})$ and ε_0 be corresponding constant as given in Lemma 6.5.7. Let δ be a constant given by Lemma 6.5.6 which corresponds to $\varepsilon = \varepsilon_0$. By Λ we denote the δ-neighborhood of $\Lambda_{\tau_{\mathrm{mod}}}$ and let

$$V = \bigcup_{\tau \in \Lambda} V(x, \mathrm{st}(\tau)) \ \subset X.$$

Since Γ is discrete, the elements of Γ which send x_0 outside V form a finite set Φ. Let

$$r_0 = \max\{r_1, d_{\mathrm{R}}(x, \gamma x_0) : \gamma \in \Phi\}$$

where r_1 is a constant that corresponds to $\varepsilon_0/2$ as in Lemma 6.5.8.

For every $\gamma \in \Gamma$ satisfying $d_{\mathrm{R}}(x, \gamma x_0) > r \geq r_0$, we assign an element $\tau_\gamma \in S(x : \{\gamma x_0\}) \cap \Lambda$ (the intersection is nonempty by above). Using Lemma 6.5.6, for every such τ_γ there exists $\tau_0 \in \Lambda_{\tau_{\mathrm{mod}}}$ so that

$$N_{\varepsilon_0/2}(L(\tau_\gamma)) \subset N_{\varepsilon_0}(L(\tau_0)).$$

By Lemmata 6.5.7 and 6.5.8, $\mu_x(S(\gamma^{-1} x : B(x_0, r))) \geq \mathfrak{m} - q$ and by properties of conformal measures,

$$\begin{aligned} \mu_x(S(x : B(\gamma x_0, r))) &= \mu_{\gamma^{-1}x}(S(\gamma^{-1} x : B(x_0, r))) \\ &= \int_{S(\gamma^{-1}x : B(x_0, r))} \exp\left(-\beta d_\tau^{\mathrm{hor}}(\gamma^{-1} x, x)\right) d\mu_x \\ &\asymp \exp\left(-\beta d_{\mathrm{F}}(x, \gamma x_0)\right) \end{aligned}$$

where in the last step we have additionally used Lemma 6.5.9. This completes the proof. $\square$

6.6. Dimension of a conformal density

In this section, we establish a lower bound for the dimension of a conformal density. For Anosov subgroups, we prove that the dimension equals the Finsler critical exponent (see Corollary 6.6.5).

THEOREM 6.6.1. *Suppose that Γ is a nonelementary τ_{mod}-RA subgroup. Let μ be a Γ-invariant conformal density of dimension β. Then β has the following lower bound:*

$$\beta \geq \delta_{\mathrm{F}} - \delta_{\mathrm{F}}^{\mathrm{c}}. \tag{6.32}$$

The proof of this theorem is given at the end of this section. The number $\delta_{\mathrm{F}}^{\mathrm{c}}$ above quantifies the maximal exponential growth rate of the orbit Γx_0 in a conical direction. The precise definition is given below.

DEFINITION 6.6.2 (Critical exponent in conical directions). Suppose that Γ is a τ_{mod}-regular subgroup. For $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, define

$$N_{\mathrm{F}}^{\mathrm{c}}(r, c, x, x_0, \tau) = \mathrm{card}\{\gamma \in \Gamma : d_{\mathrm{F}}(x, \gamma x_0) < r,\ d_{\mathrm{R}}(\gamma x_0, V(x, \mathrm{st}(\tau))) < c\}$$

and

$$\delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma) = \sup_{\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)} \left(\lim_{c \to \infty} \left(\limsup_{r \to \infty} \frac{\log N_{\mathrm{F}}^{\mathrm{c}}(r, c, x, x_0, \tau)}{r} \right) \right).$$

Note that it is sufficient to take the supremum in the definition of $\delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma)$ over the conical limit set $\Lambda_{\tau_{\mathrm{mod}}}^{\mathrm{con}}(\Gamma)$. For rank-one symmetric spaces, and, more generally, for σ_{mod}-regular subgroups, this number is zero. Below we see that for τ_{mod}-Anosov subgroups also, $\delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma) = 0$. It should be noted that, however, for general discrete subgroups, $\delta_{\mathrm{F}}^{\mathrm{c}}$ could be ∞.

PROPOSITION 6.6.3. *Suppose that Γ is a nonelementary τ_{mod}-Anosov subgroup. Then the function $N(r) = N_{\mathrm{F}}^{\mathrm{c}}(r, c, x, x_0, \tau)$ grows linearly with r. In particular, $\delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma) = 0$.*

PROOF. Without loss of generality, we can assume that $x = x_0$.[6]

LEMMA 6.6.4. *Fix $c > 0$. For any $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, the set*

$$\{\gamma x_0 : \gamma \in \Gamma,\ d_{\mathrm{R}}(\gamma x_0, V(x, \mathrm{st}(\tau))) < c\}$$

[6]Note that the number $\delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma)$ does not depend on x and x_0 as we have seen in the case of δ_{F} in Sec. 6.1.

is within a uniformly bounded distance from a uniform $\tau_{\rm mod}$-Morse quasiray α emanating from x_0 and asymptotic to τ.

PROOF. Let $\tau \in \Lambda_{\tau_{\rm mod}}(\Gamma)$ be arbitrary. Denote the preimage of τ in $\partial_\infty\Gamma$ under the boundary homeomorphism $\partial_\infty\Gamma \to \Lambda_{\tau_{\rm mod}}(\Gamma)$ by ζ. Since Γ is discrete, we can arrange the elements of $\{\gamma \in \Gamma : d_{\rm R}(\gamma x_0, V(x, {\rm st}(\tau))) < c\}$ in a sequence (γ_n). The sequence $x_n = \gamma_n x_0$ converges conically to τ. Let $\alpha : \mathbb{Z}_{\geq 0} \to X$ be the image (under the orbit map $\Gamma \to \Gamma x$) of a parametrized geodesic ray $\mathbb{Z}_{\geq 0} \to \Gamma$ starting at 1_Γ and asymptotic to ζ. Then α is a uniform $\tau_{\rm mod}$-Morse quasiray starting at x_0 and asymptotic to τ. Hence α is uniformly close to $V(x_0, {\rm st}(\tau))$. Since both sequence (x_n) and $(\alpha(n))$ are uniformly close to $V(x_0, {\rm st}(\tau))$, it is enough to understand the simpler case when $\alpha(n),\ x_n \in V(x_0, {\rm st}(\tau))$, for all $n \in \mathbb{N}$.

We claim that the sequence (x_n) is uniformly close to α. Otherwise, after extraction, (x_n) would diverge away from α. Since α is a Morse quasiray, α eventually enters each cone $V(x_n, {\rm st}(\tau))$, but further and further away from the tip x_n as n grows. Since the separation between two successive points on α (being a quasigeodesic) is uniformly bounded, we could find arbitrarily large m's such that $\alpha(m)$ is uniformly close to the boundary of a cone $V(x_n, {\rm st}(\tau))$ and is arbitrarily far away from its tip x_n. But this would contradict the $\tau_{\rm mod}$-regularity of the group Γ. □

We continue with the notations from the proof of the lemma. Since any $\tau_{\rm mod}$-Anosov subgroup $\Gamma < G$ is uniformly $\tau_{\rm mod}$-regular (cf. Equivalence Theorem 4.4.12), we may work with the Riemannian metric in place of the Finsler metric. Moreover, we may assume that the sequence (x_n) is sufficiently spaced. Let $\bar{x}_n$ denote the nearest-point projection of x_n to the image of α. The above lemma implies that $d(x_n, \bar{x}_n)$ is uniformly bounded. Since x_n's are sufficiently spaced, $\bar{x}_n$'s are also sufficiently spaced which guarantees that $d_{\rm R}(\bar{x}_n, x_0) \geq {\rm const} \cdot n$, for all large n, which in turn implies that $d_{\rm R}(x_n, x_0) \geq {\rm const} \cdot n$. The proposition follows from this. □

As a corollary of the above results, we obtain that any Γ-invariant conformal density must have dimension $\delta_{\rm F}$ when Γ is $\tau_{\rm mod}$-Anosov. The Patterson–Sullivan densities constructed in Section 6.2 also had this dimension.

COROLLARY 6.6.5. *Suppose that Γ is a nonelementary $\tau_{\rm mod}$-Anosov subgroup. Let μ be a Γ-invariant conformal density of dimension β. Then $\beta = \delta_{\rm F}$.*

PROOF. By Corollary 6.5.5 we know that the Poincaré series $g_\beta^{\mathrm{F}}(x, x_0)$ diverges and, consequently, $\beta \leq \delta_{\mathrm{F}}$. The reverse inequality is obtained in combination of Theorem 6.6.1 and Proposition 6.6.3. $\square$

To close this section, we prove Theorem 6.6.1.

PROOF OF THEOREM 6.6.1. We fix some $r \geq r_0$ where r_0 is given by Theorem 6.5.1. Assume that the stabilizer of x_0 in Γ is trivial in which case the function $N(R) = N_{\mathrm{F}}(R, x, x_0)$ counts the number of orbit points (in Γx_0) within the Finsler r-ball centered at x. The general case follows immediately.

We place a Riemannian ball of radius r at each point in the orbit. In this proof, we reserve the word *ball* to specify these balls. Let

$$c = \min_{1_\Gamma \neq \gamma \in \Gamma} \{d_{\mathrm{R}}(x_0, \gamma x_0)\}.$$

There exists a number $N \in \mathbb{N}$ that depends only on r, c, and X such that any ball intersect at most N other balls (including itself). Note that the shadows in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ (from x) of two distinct balls are disjoint unless they intersect some common τ_{mod}-cone with tip at x. Also note that, at large distances from x, the balls do not intersect the boundaries of the τ_{mod}-cones because of the τ_{mod}-regularity of the orbit.

Let n_R denote the maximal number of balls in $B_{\mathrm{F}}(x, R)$ that intersect a particular τ_{mod}-cone $V(x, \mathrm{st}(\tau))$. It follows from the definition of $\delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma)$ that

$$\limsup_{R \to \infty} \frac{\log n_R}{R} \leq \delta_{\mathrm{F}}^{\mathrm{c}}(\Gamma). \tag{6.33}$$

On the other hand, for each $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$, the maximal number of balls in $B_{\mathrm{F}}(x, R)$ whose shadows intersect τ is n_R. Therefore,

$$\frac{N_{\mathrm{F}}(R, x, x_0)}{N \cdot n_R} s(R) \leq \mathfrak{m} = \text{total mass of } \mu_x, \tag{6.34}$$

where $s(R)$ is any lower bound for the measures of the shadows of balls in $B_{\mathrm{F}}(x, R)$. We note that the shadow lemma (Theorem 6.5.1) produces such a positive lower bound[7], namely, we may take

[7]We may need to disregard a finite number of balls from the picture.

$s(R) = \mathrm{const} \cdot e^{-\beta R}$. Then (6.34) yields

$$N_{\mathrm{F}}(R, x, x_0) \leq \frac{\mathfrak{m}N \cdot n_R}{\mathrm{const}} e^{\beta R}.$$

Together with (6.33), the above results in (6.32). $\square$

6.7. Uniqueness of conformal density

Recall that an action of a group H on a measure space (S, σ) is said to be *ergodic* if each H-invariant measurable set $B \subset S$ is either null or co-null. In [**Sul79**], Sullivan proved that for a discrete group Γ of Möbius transformations of the Poincare ball $\mathbb{B}^3$, a Γ-invariant conformal density μ of non-zero dimension is unique (here and henceforth, by "unique" we mean unique up-to a constant factor) in the class of all conformal densities of same dimension if and only if the action Γ on the limit set $\Lambda(\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$. See also [**Nic89**, Thm. 4.2.1]. Generalizing this statement in our setting, we obtain the following result. The proof is essentially same of Sullivan's theorem, hence we omit the details.

Theorem 6.7.1. *Suppose that Γ is a nonelementary τ_{mod}-RA subgroup. A Γ-invariant conformal density μ of dimension $\beta > 0$ is unique in the class of all Γ-invariant conformal densities of dimension β if and only if the action $\Gamma \curvearrowright \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$.*

It is then natural to ask

Question 6.7.2. *For which τ_{mod}-regular subgroups Γ, the action $\Gamma \curvearrowright \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is ergodic with respect to a conformal measure?*

In this section we prove that the Anosov property is a sufficient condition:

Theorem 6.7.3 (Anosov implies ergodic). *Suppose that Γ is a nonelementary τ_{mod}-Anosov subgroup and μ be a Γ-invariant conformal density. Then the action $\Gamma \curvearrowright \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is ergodic with respect to any $\mu_x \in \mu$.*

As a corollary, we obtain that when Γ is τ_{mod}-Anosov, then, up to a constant factor, there is exactly one Γ-invariant conformal density, namely, the Patterson–Sullivan density.

Corollary 6.7.4 (Existence and uniqueness of conformal density). *Suppose that Γ is a nonelementary τ_{mod}-Anosov subgroup. Then, up to a constant factor, there exists a unique Γ-invariant conformal density μ, namely, the Patterson–Sullivan density.*

Proof. First of all, by Proposition 6.2.1, any such density must have a positive dimension. Secondly, by Corollary 6.6.5 this dimension equals to the critical exponent δ_{F}. Then the uniqueness follows from the combination of Theorems 6.7.1 and 6.7.3. $\square$

Now we return to the proof of Theorem 6.7.3.

Proof of Theorem 6.7.3. Let μ be a Γ-invariant conformal density. Note that the dimension β of μ must be positive (by Proposition 6.2.1 and Corollary 6.6.5).

Let B be a Γ-invariant Borel subset of $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. We need to prove that if B is not a null set, then it is co-null. From now on, we assume that B is not a null set, i.e., $\mu_x(B) > 0$.

We need the following lemmata.

Lemma 6.7.5. *There exists $r_1 > 0$ such that for every $r \geq r_1$ and every $\gamma \in \Gamma$, the shadow $S(x, B(\gamma x_0, r))$ intersects $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.*

Proof. The proof simply follows from the Morse property of the Anosov subgroup Γ. $\square$

We assume that the r_1 in the lemma also satisfies the “uniform conicality” property for Γ (cf. Proposition 6.5.4).

Lemma 6.7.6. *Let $r \geq \max\{r_0, r_1\}$ where r_0 is as in Theorem 6.5.1. For μ_x-a.e. $\tau \in B$ and every sequence (γ_n) on Γ, $\gamma_n \to \tau$, satisfying $\tau \in S_n := S(x : B(\gamma_n x_0, r))$, we have*

$$\lim_{n\to\infty} \frac{\mu_x(S_n \cap B)}{\mu_x(S_n)} = 1. \tag{6.35}$$

Assuming this lemma for a moment, we complete the proof of the theorem. The proof of this lemma is given at the end of this section. Note that, Lemma 6.7.5 is used to ensure that the ratios in the above lemma are not degenerate.

Let $\tau \in B$ be a *density point*, i.e., τ satisfies (6.35). Such point exist by Lemma 6.7.6 because B has positive mass. Note that, Γ-invariance of B and μ implies that

$$\begin{aligned}
\frac{\mu_x(S(\gamma_n^{-1}x : B(x_0, r)) \cap B)}{\mu_x(S(\gamma_n^{-1}x : B(x_0, r)))} &= \frac{\mu_{\gamma_n x}(S_n \cap B)}{\mu_{\gamma_n x}(S_n)} = 1 - \frac{\mu_{\gamma_n x}(S_n - B)}{\mu_{\gamma_n x}(S_n)} \\
&= 1 - \frac{\int_{S_n - B} \exp\left(-\beta d_\tau^{\mathrm{hor}}(\gamma_n x, x)\right) d\mu_x}{\int_{S_n} \exp\left(-\beta d_\tau^{\mathrm{hor}}(\gamma_n x, x)\right) d\mu_x} \\
&\geq 1 - \mathrm{const} \cdot \frac{\mu_x(S_n - B)}{\mu_x(S_n)},
\end{aligned}$$

where the inequality follows by Lemma 6.5.9. Together with (6.35), we get

$$\lim_{n\to\infty} \frac{\mu_x(S(\gamma_n^{-1}x : B(x_0,r)) \cap B)}{\mu_x(S(\gamma_n^{-1}x : B(x_0,r)))} = 1. \tag{6.36}$$

Note that by Corollary 6.5.2, μ is atom-free. Therefore, for every $\varepsilon > 0$ there exists $r > r_1$ such that

$$\mu_x(S(\gamma_n^{-1}x : B(x_0,r))) \geq \mathfrak{m} - \varepsilon,$$

for all large n, where $\mathfrak{m}$ denotes the total mass of μ_x. The above follows from the combination of Lemmata 6.5.7 and 6.5.8. Therefore, by (6.36),

$$\mu_x(B) \geq \lim_{n\to\infty} \mu_x(S(\gamma_n^{-1}x : B(x_0,r)) \cap B) \geq \mathfrak{m} - \varepsilon,$$

which holds for every $\varepsilon > 0$. Hence $\mu_x(B) = \mathfrak{m}$. This completes the proof of the theorem. $\square$

Now we prove Lemma 6.7.6. The lemma would have followed from a generalization of the Lebesgue density theorem (cf. [**Fed69**, Subsec. 2.9.11, 2.9.12]) if we knew that μ_x is, e.g., a *doubling measure*. Since this property is unclear, we adopt a more direct approach. The idea of the proof follows [**Rob03**, Subsec. 1E] (see also [**Lin06**, Sec. 3]).

Proof of Lemma 6.7.6. The proof requires a version of the Lebesgue differentiation theorem.

Sublemma 6.7.7. *For every bounded measurable function $\Phi : \mathrm{Flag}(\tau_{\mathrm{mod}}) \to \mathbb{R}_{\geq 0}$,*

$$\Phi(\tau) = \lim_{n\to\infty} \frac{1}{\mu_x(S(x : B(\gamma_n x_0, r)))} \int_{S(x:B(\gamma_n x_0,r))} \Phi d\mu_x.$$

for μ_x-a.e. $\tau \in \Lambda_{\tau_{\mathrm{mod}}}$ and all $\gamma_n \in \Gamma$ satisfying $\tau \in S(x : B(\gamma_n x_0, r))$.

Proof. For every bounded measurable function $\Psi : \mathrm{Flag}(\tau_{\mathrm{mod}}) \to \mathbb{R}_{\geq 0}$, define a function Ψ^* on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ which is zero outside $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ and on $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ it is defined by

$$\Psi^*(\tau) = \limsup_{N\to\infty} \frac{1}{\mu_x(S(x : B(\gamma x_0, r)))} \int_{S(x:B(\gamma x_0,r))} \Psi d\mu_x, \tag{6.37}$$

Here and in the following the limit superior is taken over all $\gamma \in \Gamma$ that satisfy $d_{\mathrm{R}}(x, \gamma x_0) \geq N$ and $\tau \in S(x : B(\gamma x_0, r))$.

Let Φ_k be a sequence of continuous functions converging to Φ μ_x-almost surely such that

$$\int_{\mathrm{Flag}(\tau_{\mathrm{mod}})} |\Phi_k - \Phi| d\mu_x < \frac{1}{k}, \quad \forall k \in \mathbb{N}.$$

Then for every $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}\limsup_{N\to\infty}&\left|\frac{1}{\mu_x(S(x:B(\gamma x_0,r)))}\int_{S(x:B(\gamma x_0,r))}\Phi d\mu_x - \Phi(\tau)\right| \\ &\le |\Phi-\Phi_k|^*(\tau) + |\Phi_k(\tau)-\Phi(\tau)| \\ &\quad + \limsup_{N\to\infty}\left|\frac{1}{\mu_x(S(x:B(\gamma x_0,r)))}\int_{S(x:B(\gamma x_0,r))}\Phi_k d\mu_x - \Phi_k(\tau)\right|. \end{aligned} \tag{6.38}$$

Since Φ_n are continuous, the last quantity in the right side of the above vanishes. Moreover, the limit of $|\Phi_k(\tau)-\Phi(\tau)|$ as $k\to\infty$ vanishes at μ_x-a.e. $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$. Therefore, we only need to control the first term of the right side of (6.38): We show that, for all bounded nonnegative measurable functions Ψ on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ and all $\varepsilon > 0$,

$$\mu_x\left(\{\Psi^* > \varepsilon\}\right) \le \frac{\mathrm{const}}{\varepsilon}\int_{\mathrm{Flag}(\tau_{\mathrm{mod}})}\Psi d\mu_x \tag{6.39}$$

where the constant does not depend on ε or Ψ. The sublemma follows from this as follows: Setting $\Psi = |\Phi-\Phi_k|$ and taking limit as $k\to\infty$ in (6.39), we see that $|\Phi-\Phi_k|^*$ μ_x-a.s. converges to zero. Hence left-hand side of (6.38) also converges to zero for μ_x-a.e. $\tau \in \Lambda_{\tau_{\mathrm{mod}}}$.

Now we verify (6.39). Let $\varepsilon > 0$ be arbitrary. For $d \ge 0$, let Γ_d be the set of all elements $\gamma \in \Gamma$ such that $d_{\mathrm{F}}(x,\gamma x_0) \ge d$ and

$$\int_{S(x:B(\gamma x_0,r))}\Psi d\mu_x \ge \frac{\varepsilon}{2}\mu_x(S(x:B(\gamma x_0,r))). \tag{6.40}$$

Claim 1. *The union of all shadows $S(x:B(\gamma x_0,r))$ over $\gamma\in\Gamma_d$ covers $\{\Psi^*>\varepsilon\}$.*

Proof of claim. The proof is straightforward. □

We recursively construct a sequence of subsets, $(\Gamma_{d,N})$, of Γ_d in the following way: Let $\Gamma_{d,1} = \{\gamma\in\Gamma_d : 0 \le d_{\mathrm{F}}(x,\gamma x_0) < 1\}$, and, for $N\ge 2$, define

$$\Gamma_{d,N} = \left\{\gamma\in\Gamma_d \;\middle|\; \begin{array}{l} N-1 \le d_{\mathrm{F}}(x,\gamma x_0) < N \text{ and } S(x:B(\gamma x_0,r))\cap \\ S(x:B(\phi x_0,r)) = \emptyset, \forall\phi\in\Gamma_{d,1}\cup\cdots\cup\Gamma_{d,N-1}\end{array}\right\}.$$

Set $\Gamma_d^* = \bigcup_{N\ge1}\Gamma_{d,N}$.

Claim 2. *There exists a constant $R \geq r$ such that, for every $d \geq 0$,*

$$\{\Psi^* > \varepsilon\} \subset \bigcup_{\phi \in \Gamma_d^*} S(x : B(\phi x_0, R)).$$

Proof of claim. It is enough to prove the claim for very large d. In fact, we assume that d is so large such that $\overline{x(\gamma x_0)}$ is uniformly τ_{mod}-regular for all $\gamma \in \Gamma_d$.

Let $\tau \in \{\Psi^* > \varepsilon\}$ be arbitrary. Then there exists $\gamma \in \Gamma_d$ such that $\tau \in S(x : B(\gamma x_0, r))$. Assume that $\gamma \notin \Gamma_d^*$. By construction of Γ_d^*, there exists $\phi \in \Gamma_d^*$ such that $S(x : B(\gamma x_0, r)) \cap S(x : B(\phi x_0, r)) \neq \emptyset$ and $d_{\mathrm{F}}(x, \phi x_0) < d_{\mathrm{F}}(x, \gamma x_0)$.

By Lemma 6.6.4, both γx_0 and ϕx_0 stay uniformly close to a τ_{mod}-uniform Morse quasigeodesic α with one endpoint at x. Since $d_{\mathrm{F}}(x, \phi x_0) < d_{\mathrm{F}}(x, \gamma x_0)$, we may assume that the other endpoint of α is uniformly close to γx_0. It follows that ϕx_0 is uniformly close to the diamond $\diamondsuit_\Theta(x, \gamma x_0)$, since α is, for some ι-invariant, compact, τ_{mod}-Weyl convex subset $\Theta \subset \mathrm{ost}(\tau_{\mathrm{mod}})$. Pick $y \in B(\gamma x_0, r) \cap V(x, \mathrm{st}(\tau))$. Then, by uniform continuity of diamonds (cf. Theorem 5.1.7), for some $\Theta' \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ bigger than Θ, $\diamondsuit_\Theta(x, \gamma x_0)$ is contained in a uniform neighborhood of $\diamondsuit_{\Theta'}(x, y)$. Therefore, ϕx_0 is uniformly close to $\diamondsuit_{\Theta'}(x, y)$ and, in particular, to $V(x, \mathrm{st}(\tau))$. We may choose R to be this upper bound. $\square$

In particular, we get

$$\mu_x\left(\{\Psi^* > \varepsilon\}\right) \leq \sum_{\phi \in \Gamma_R^*} \mu_x\left(S(x : B(\phi x_0, R))\right). \tag{6.41}$$

Claim 3. *If $S(x : B(\gamma x_0, r)) \cap S(x : B(\phi x_0, r)) \neq \emptyset$, for $\gamma, \phi \in \Gamma_d^*$, then $d_{\mathrm{F}}(\gamma x_0, \phi x_0)$ is uniformly bounded.*

Proof of claim. This follows from the Gromov hyperbolicity of $(\Gamma x_0, d_{\mathrm{F}})$ (see Corollary 6.3.8) and the fact that both γx_0 and ϕx_0 lie in an annulus $\{x' \in X : N - 1 \leq d_{\mathrm{F}}(x', x) < N\}$ in the following way: Let $\tau \in S(x : B(\gamma x_0, r)) \cap S(x : B(\phi x_0, r))$. Let $z \in V(x, \mathrm{st}(\tau))$ be a point uniformly close to Γx_0. By δ-hyperbolicity,

$$\langle \gamma x_0 | \phi x_0 \rangle_x + \delta \geq \min\left\{\langle \gamma x_0 | z \rangle_x, \langle \phi x_0 | z \rangle_x\right\}. \tag{6.42}$$

Expanding the left side, we get

$$\begin{aligned}\langle\gamma x_0|\phi x_0\rangle_x+\delta&=\frac{1}{2}\left(d_{\mathrm{F}}(\gamma x_0,x)+d_{\mathrm{F}}(\phi x_0,x)-d_{\mathrm{F}}(\gamma x_0,\phi x_0)\right)+\delta\\&\leq\left(d_{\mathrm{F}}(\phi x_0,x)-\frac{1}{2}d_{\mathrm{F}}(\gamma x_0,\phi x_0)\right)+\delta+\frac{1}{2},\end{aligned}\tag{6.43}$$

and expanding the right side, we get

$$\min\left\{\langle\gamma x_0|z\rangle_x,\langle\phi x_0|z\rangle_x\right\}=\min\left\{\begin{array}{l}\frac{1}{2}\left(d_{\mathrm{F}}(\gamma x_0,x)+d_{\mathrm{F}}(x,z)-d_{\mathrm{F}}(\gamma x_0,z)\right),\\\frac{1}{2}\left(d_{\mathrm{F}}(\phi x_0,z)+d_{\mathrm{F}}(x,z)-d_{\mathrm{F}}(\phi x_0,z)\right)\end{array}\right\}.$$

Taking $z\to\tau$ in the right side of the last one and using (6.12), we get

$$\min\left\{\frac{1}{2}\left(d_{\mathrm{F}}(\gamma x_0,x)+d_\tau^{\mathrm{hor}}(x,\gamma x_0)\right),\frac{1}{2}\left(d_{\mathrm{F}}(\phi x_0,x)+d_\tau^{\mathrm{hor}}(x,\phi x_0)\right)\right\}.$$

which, by Lemma 6.5.9, is at least

$$\min\left\{d_{\mathrm{F}}(\gamma x_0,x),d_{\mathrm{F}}(\phi x_0,x)\right\}-r\geq d_{\mathrm{F}}(\gamma x_0,x)-r-1.$$

Combining this with (6.42) and (6.43), we get

$$d_{\mathrm{F}}(\gamma x_0,\phi x_0)\leq 2r+2\delta+3.\qquad\square$$

In particular, for each $\tau\in\mu_x\left(\{\Psi^*>\varepsilon\}\right)$, $|\{\phi\in\Gamma_d^*:\tau\in S(x:B(\phi x_0,r))\}|$ is uniformly bounded, say, by $D>0$. Therefore,

$$\sum_{\phi\in\Gamma_R^*}\mu_x\left(S(x:B(\phi x_0,r))\right)\leq D\mu_x\left(\bigcup_{\phi\in\Gamma_R^*}S(x:B(\phi x_0,r))\right).\tag{6.44}$$

We would like to use the shadow lemma (Theorem 6.5.1). To this end, we have

$$\mu_x\left(\{\Psi^*>\varepsilon\}\right)\leq\sum_{\phi\in\Gamma_R^*}\mu_x\left(S(x:B(\phi x_0,R))\right)\leq C'\sum_{\phi\in\Gamma_R^*}\exp\left(-\beta d_{\mathrm{F}}(x,\phi x_0)\right)\tag{6.45}$$

where the first inequality is given by (6.41) and the last inequality is given by the shadow lemma with $r_0\leq r=R$. Note that the necessary condition $d_{\mathrm{F}}(x,\phi x_0)\geq R$ which we needed to apply the shadow lemma in the above follows from the definition of Γ_R^*. Moreover, applying shadow lemma again with $r_0 \leq r = r$, we get another constant $C > 0$ such that

$$C^{-1} \sum_{\phi \in \Gamma_R^*} \exp\left(-\beta d_{\mathrm{F}}(x, \phi x_0)\right) \leq \sum_{\phi \in \Gamma_R^*} \mu_x\left(S(x : B(\phi x_0, r))\right). \tag{6.46}$$

Combined with (6.44), the inequalities in (6.45) and (6.46) give

$$\mu_x\left(\{\Psi^* > \varepsilon\}\right) \leq DC'C\mu_x\left(\bigcup_{\phi \in \Gamma_R^*} S(x : B(\phi x_0, r))\right).$$

Finally, the above and (6.40) yield

$$\mu_x\left(\{\Psi^* > \varepsilon\}\right) \leq \frac{2DC'C}{\varepsilon} \int_{\mathrm{Flag}(\tau_{\mathrm{mod}})} \Psi d\mu_x.$$

This proves (6.39). □

The proof of the lemma follows from the sublemma by taking Φ in the sublemma to be the indicator function for B. □

6.8. Hausdorff density

In this section, we restrict our attention to Anosov subgroups. Usually, one defines Hausdorff measures and Hausdorff dimension for metric spaces. In Appendix A, we verify that the theory goes through for premetrics as well. The reader who prefers to work with metrics can assume that $\epsilon > 0$ is chosen so that $d_{\mathrm{G}}^{x,\epsilon}$ defines a metric on $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ (cf. Corollary 6.4.6).

For $\beta \geq 0$ we let $\mathcal{H}_x^\beta$ denote the *β-dimensional Hausdorff measure* on $(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma), d_{\mathrm{G}}^{x,\epsilon})$ (defined with respect to the premetric $d_{\mathrm{G}}^{x,\epsilon}$ as in the appendix). The *Hausdorff dimension* of a Borel subset $B \subset \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is then defined as

$$\mathrm{Hd}(B) = \inf\{\beta : \mathcal{H}_x^\beta(B) = 0\} = \sup\{\beta : \mathcal{H}_x^\beta(B) = \infty\}.$$

Note that if for some $\beta \geq 0$, $\mathcal{H}_x^\beta(B) \in (0, \infty)$, then $\mathrm{Hd}(B) = \beta$.

Proposition 6.8.1. *Suppose that for some $\beta \geq 0$*

$$\mathcal{H}_x^\beta(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)) \in (0, \infty). \tag{6.47}$$

Let $Z = \Gamma x$. Then $\mathcal{H}^\beta = \{\mathcal{H}_z^\beta\}_{z \in Z}$ is a $\beta\epsilon$-dimensional Γ-invariant conformal Z-density.

Proof. Let $y, z \in Z$. Define a function $f : \Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \times \Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \to \mathbb{R}_{\geq 0}$ by

$$f(\tau_1, \tau_2) = \begin{cases} d_{\mathrm{G}}^{y,\epsilon}(\tau_1, \tau_2)/d_{\mathrm{G}}^{z,\epsilon}(\tau_1, \tau_2), & \tau_1 \neq \tau_2, \\ \exp\left(-\epsilon d_{\tau}^{\mathrm{hor}}(y, z)\right), & \tau_1 = \tau_2 = \tau. \end{cases}$$

By a calculation similar to the proof of Proposition 6.4.4, we obtain

$$\lim_{\tau_1, \tau_2 \to \tau} \frac{d_{\mathrm{G}}^{y,\epsilon}(\tau_1, \tau_2)}{d_{\mathrm{G}}^{z,\epsilon}(\tau_1, \tau_2)} = \exp\left(-\epsilon d_{\tau}^{\mathrm{hor}}(y, z)\right)$$

which shows that f is continuous. For $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ and small $\eta > 0$, let U_η be a neighborhood of τ in $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ such that $\forall \tau_1, \tau_2 \in U$,

$$d_{\mathrm{G}}^{y,\epsilon}(\tau_1, \tau_2) \leq \left(\exp\left(-\epsilon d_{\tau}^{\mathrm{hor}}(y, z)\right) + \eta\right) d_{\mathrm{G}}^{z,\epsilon}(\tau_1, \tau_2).$$

Hence the identity map $\mathrm{Id} : (\Lambda_{\tau_{\mathrm{mod}}}(\Gamma), d_{\mathrm{G}}^{z,\epsilon}) \to (\Lambda_{\tau_{\mathrm{mod}}}(\Gamma), d_{\mathrm{G}}^{y,\epsilon})$ on U_η is L_ε-Lipschitz, where $L_\eta = \exp\left(-\epsilon d_{\tau}^{\mathrm{hor}}(y, z)\right) + \eta$. In particular, the map Id is locally Lipschitz. Therefore, for any $B \in \mathfrak{B}(U)$, $\mathcal{H}_y^\beta(B) \leq L_\eta^\beta \mathcal{H}_z^\beta(B)$. This also shows that $\mathcal{H}_y^\beta \ll \mathcal{H}_z^\beta$. Taking limit as $\eta \to 0$, we obtain

$$\frac{d\mathcal{H}_y^\beta}{d\mathcal{H}_z^\beta}(\tau) \leq \exp\left(-\beta\epsilon d_{\tau}^{\mathrm{hor}}(y, z)\right)$$

and switching the role of y and z in the above we also obtain the reverse inequality. Hence

$$\frac{d\mathcal{H}_y^\beta}{d\mathcal{H}_z^\beta}(\tau) = \exp\left(-\beta\epsilon d_{\tau}^{\mathrm{hor}}(y, z)\right)$$

which proves conformality. Suppose that $y = \gamma z$ for some $\gamma \in \Gamma$. Then for any $B \in \mathfrak{B}(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma))$,

$$\mathcal{H}_{\gamma z}^\beta(B) = \int_B \exp\left(-\beta\epsilon d_{\tau}^{\mathrm{hor}}(\gamma z, z)\right) d\mathcal{H}_z^\beta = \int_B d\left(\gamma^* \mathcal{H}_z^\beta\right) = \gamma^* \mathcal{H}_z^\beta(B)$$

and Γ-invariance also follows. Therefore, $\mathcal{H}^\beta$ is a conformal Z-density of dimension $\beta\epsilon$. $\square$

Remark. (1) Note that if such a family $\{\mathcal{H}_z^\beta : z \in Z\}$ exists, then it may be extended to a *full* conformal density via the correspondence in (6.14).

(2) By the uniqueness of conformal density (Theorem 6.7.4), the number β in Proposition 6.8.1 equals to $\delta_{\mathrm{F}}/\epsilon$.

(3) In the following we shall see that, indeed, the $\delta_{\mathrm{F}}/\epsilon$-dimensional Hausdorff measure $\mathcal{H}_x^{\delta_{\mathrm{F}}/\epsilon}$ is finite and non-null (i.e., it satisfies (6.47)).

Next we show that if $\beta = \delta_{\mathrm{F}}/\epsilon$, then the β-dimensional Hausdorff measure $\mathcal{H}_x^\beta$ satisfies (6.47). Let us first discuss the simpler case, namely, when the Finsler pseudo-metric d_{F} is a metric. There is an abundance of examples when this occurs, e.g., in the case when $X = G/K$ is an irreducible symmetric space.

Let (Y, d) be a proper, geodesic, Gromov hyperbolic metric space and Γ be a nonelementary discrete group of isometries acting properly discontinuously on Y. Let Λ be the limit set of Γ in $\partial_\infty Y$. Further, assume that Γ is *quasiconvex-cocompact*, i.e., the quasiconvex hull $\mathrm{QCH}(\Lambda)$ is nonempty and the quotient $\Gamma \backslash \mathrm{QCH}(\Lambda)$ is compact. In [**Coo93**], Coornaert proved the following result.

THEOREM 6.8.2 (Coornaert [**Coo93**, Cor. 7.6]). *Suppose that the critical exponent δ of Γ is finite. Then the δ-dimensional Hausdorff measure on Λ with respect to a Gromov metric d_{G} is finite and non-null.*

To apply this theorem to our case, we need an appropriate setting. In Section 6.3, we proved that the orbit $Z = \Gamma x$ is a Gromov hyperbolic space with respect to the Finsler metric (cf. Corollary 6.3.8) and it is also proper. But Z fails to be geodesic. This problem can be remedied by taking a uniform neighborhood Y of Z in X such that Z is quasiconvex in Y, and then putting the intrinsic path-metric d on Y induced by d_{F} (this requires positivity of d_{F}), and finally by completing Y in this metric. Then (Y, d) is proper, geodesic and Gromov hyperbolic. Moreover, (Y, d) and the isometrically embedded (Z, d_{F}) are Hausdorff-close and, in particular, (Y, d) is quasiisometric to (Z, d_{F}) by a $(1, A)$-quasiisometry. This implies that there is a bi-Lipschitz homeomorphism from $\partial_\infty Y$ (equipped with the metric d_{G}^ϵ defined by $d_{\mathrm{G}}^\epsilon(\xi_1, \xi_2) = d_{\mathrm{G}}(\xi_1, \xi_2)^\epsilon$ where d_{G} is a Gromov metric on $\partial_\infty Y$) to $(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma), d_{\mathrm{G}}^{x,\epsilon})$. Note that the action $\Gamma \curvearrowright (Y, d)$ satisfies all the properties needed to apply Theorem 6.8.2. Therefore, by this theorem the $\delta_{\mathrm{F}}/\epsilon$-dimensional Hausdorff measure on $\partial_\infty Y$ (and, consequently, also on $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$) is finite and non-null.

In the general case where the positivity of d_{F} is unknown, the above argument still works after some modifications. Let us go back to our construction in the above paragraph. Let Y be a uniform Riemannian neighborhood of Z in which Z is Finsler quasiconvex. Define a new Γ-invariant metric $\bar{d_{\mathrm{F}}}$ on Y by

$$\bar{d_{\mathrm{F}}}(y, z) = \max\left\{d_{\mathrm{F}}(y, z),\ \varepsilon d_{\mathrm{R}}(y, z)\right\}, \quad \forall y, z \in Y$$

where $\varepsilon > 0$ is some number that is strictly lesser than L^{-1} given in (6.6). Note that for $y, z \in Z$, if $d_{\mathrm{F}}(y,z)$ is sufficiently large, then $\bar{d_{\mathrm{F}}}(y,z) = d_{\mathrm{F}}(y,z)$. Moreover, for a given ι-invariant, compact, τ_{mod}-Weyl convex subset $\Theta \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ and a possibly smaller ε (depending on the choice of Θ), any Θ-Finsler geodesic (see Definition 6.1.1) connecting these two points remains a geodesic in this new metric. In other words, Z remains quasiconvex in Y with respect to $\bar{d_{\mathrm{F}}}$.

Observe that the identity embedding $(Z, d_{\mathrm{F}}) \to (Y, \bar{d_{\mathrm{F}}})$ is a $(1, A)$-quasiisometric embedding for some large enough A and the image is Hausdorff-close to Y. Therefore, in this case also we get a natural identification of the Gromov boundaries of (Z, d_{F}) and $(Y, \bar{d_{\mathrm{F}}})$. Next, considering intrinsic metrics, we complete Y as before to get a proper, geodesic, Gromov hyperbolic space (Y, d). The rest of the argument works as before.

Using Proposition 6.8.1, we obtain the following result.

Theorem 6.8.3. *Suppose that Γ is a nonelementary τ_{mod}-Anosov subgroup. If $\beta = \delta_{\mathrm{F}}/\epsilon$, then the β-dimensional Hausdorff density $\mathcal{H}^\beta = \{\mathcal{H}^\beta_z\}_{z \in \Gamma x}$ is a Γ-invariant conformal density of dimension δ_{F}. In particular, the Hausdorff dimension with respect to the metric $d^{x,\epsilon}_{\mathrm{G}}$ satisfies*

$$\mathrm{Hd}(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)) = \delta_{\mathrm{F}}/\epsilon.$$

Moreover, $\mathcal{H}^\beta$ equals to a non-zero multiple of the Patterson–Sullivan density.

We have mostly completed the proof of this theorem. The remaining "moreover" part follows from the uniqueness of Γ-invariant conformal densitiy (Theorem 6.7.4).

Corollary 6.8.4. *With respect to the Gromov premetric $d^x_{\mathrm{G}} := d^{x,1}_{\mathrm{G}}$ the Hausdorff dimension satisfies*

$$\mathrm{Hd}(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)) = \delta_{\mathrm{F}}.$$

6.9. Applications

6.9.1. Product of hyperbolic spaces. Let Γ_1, Γ_2 be isomorphic discrete cocompact subgroups of $\mathrm{PSL}(2,\mathbb{R})$ where the isomorphism is given by $\phi : \Gamma_1 \to \Gamma_2$. We let $f : S^1 \to S^1$ be the equivariant homeomorphism of ideal boundaries of hyperbolic planes determined by ϕ.

The discrete subgroup

$$\Gamma = \{(\gamma_1, \phi\gamma_1) : \gamma_1 \in \Gamma_1\} < G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$$

acts on $X = \mathbb{H}^2 \times \mathbb{H}^2$ as a σ_{mod}-Anosov subgroup. (This follows, for instance, from the fact that Γ is an URU subgroup of G.) The σ_{mod}-limit set of Γ (in the full flag-manifold $S^1 \times S^1$) equals the graph of the map f.

We denote d_1 (resp. d_2) the distance functions of the constant -1 curvature Riemannian metrics on the first (resp. second) factor of the product $\mathbb{H}^2 \times \mathbb{H}^2$.

Unlike in section 6.4, we work with the Finsler metric on $\mathbb{H}^2 \times \mathbb{H}^2$ given by

$$d_{\mathrm{F}}((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2}. \tag{6.48}$$

(We multiply the distance function (6.7), for $p = 2$, by a factor $1/\sqrt{2}$ in order to avoid cumbersome radical constants.)

By the formula of the Gromov predistance (6.28), for $\epsilon = 1$ and $x = (x_1, x_2)$, $d_{\mathrm{G}}^{x,1}(\tau_+, \tau_-)$ is bi-Lipschitz equivalent to the product

$$\sqrt{\alpha_1 \alpha_2},$$

where $\tau_\pm = (\xi_1^\pm, \xi_2^\pm)$ and α_i is the angle between ξ_i^+, ξ_i^- as measured from x_i, $i = 1, 2$.

By [**BS93**, Thm. 2 & 3] we note that the Finsler critical exponent δ_{F} of Γ is at most 1. This can also be obtained by comparing the Hausdorff dimensions as follows. Note that by the formula of the Gromov predistance, the identity map

$$(S^1 \times S^1, \rho) \to (\mathrm{Flag}(\sigma_{\mathrm{mod}}), d_{\mathrm{G}}^{x,1})$$

is Lipschitz, where ρ is a Riemannian distance function on $S^1 \times S^1 = \partial_\infty \mathbb{H}^2 \times \partial_\infty \mathbb{H}^2$. Moreover, the limit set of Γ in $S^1 \times S^1$ is the graph of a BV function, hence, is a rectifiable curve, and, thus, has Hausdorff dimension 1 with respect to ρ. Consequently, with respect to $d_{\mathrm{G}}^{x,1}$, $\mathrm{Hd}(\Lambda_{\sigma_{\mathrm{mod}}}(\Gamma)) \leq 1$. By Theorem 6.8.3, $\delta_{\mathrm{F}} \leq 1$ as well.

Moreover, by [**BS93**, Thm. 2], $\delta_{\mathrm{F}} = 1$ if and only if ϕ is induced by an isometry of $\mathbb{H}^2$, equivalently, f is a Möbius transformation.

We further note that one can use [**Bur93**] as an alternative argument for both inequality and the equality case.

6.9.2. Projective Anosov representations. Recall that a representation $\rho : \Gamma \to \mathrm{SL}(k+1, \mathbb{R})$, $k \geq 2$, is called *projective Anosov* if it is τ_{mod}-Anosov for $\tau_{\mathrm{mod}} = (1, k)$ (see Examples 4.2.2, 6.1.5, and 6.4.8 for notations). Equivalently, ρ is P_1-Anosov, see Definition 2.2.1. The Finsler critical exponent associated to the ι-invariant type

$$\bar{\theta} = (1/2\sqrt{k+1}, 0, -1/2\sqrt{k+1})$$

will be denoted by δ_{F}.

Let $\rho : \Gamma \to \mathrm{SL}(k+1, \mathbb{R})$ be a projective Anosov representation. In [**GMT19**], the authors defined the following two critical exponents of Γ, namely, the *Hilbert critical exponent*

$$\delta_{1,k+1} = \limsup_{r \to \infty} \frac{\log \operatorname{card}\{\gamma \in \Gamma : \mu_1(\gamma) - \mu_{k+1}(\gamma) < r\}}{r}$$

and the *simple root critical exponent*

$$\delta_{1,2} = \limsup_{r \to \infty} \frac{\log \operatorname{card}\{\gamma \in \Gamma : \mu_1(\gamma) - \mu_2(\gamma) < r\}}{r}.$$

A direct computation yields

$$\sqrt{k+1}\delta_{\mathrm{F}} = \delta_{1,k+1} \leq \delta_{1,2}/2,$$

where the left equality follows from the formula for the Finsler metric given by (6.10) and the right inequality follows from $2(\mu_1 - \mu_2) \leq \mu_1 - \mu_{k+1}$. Also note that (by (6.29)) for a pair of partial flags $(l_1, h_1), (l_2, h_2) \in \mathrm{Flag}(\tau_{\mathrm{mod}})$,

$$d_{\mathrm{G}}^{x, 1/\sqrt{k+1}}((l_1, h_1), (l_2, h_2)) \leq \sin \angle(l_1, l_2)$$

where the right side equals the distance (with respect to the constant curvature Riemannian metric on $\mathbb{R}P^k$ determined by $x \in X$) between the lines l_1, l_2 in $\mathbb{R}P^k$. This together with Theorem 6.8.3 implies that

$$\delta_{1,k+1} = \delta_{\mathrm{F}}\sqrt{k+1} = \mathrm{Hd}(\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)) \leq \mathrm{Hd}_{\mathrm{R}}(\xi^1(\partial_\infty \Gamma)) \tag{6.49}$$

where $\xi^1 : \partial_\infty \Gamma \to \mathbb{R}P^k$ is the Γ-equivariant embedding[8] of $\partial_\infty \Gamma$ into $\mathbb{R}P^k$ and Hd_{R} denotes the Hausdorff dimension with respect to the Riemannian metric. Together with a recently obtained upper-bound for $\mathrm{Hd}_{\mathrm{R}}(\xi^1(\partial_\infty \Gamma))$ (see [**PSW19**, Prop. 4.1] or [**GMT19**, Thm. 4.1]), we obtain the following result.

[8]Composition of the Γ-equivariant boundary embedding $\partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ and the projection map $\mathrm{Flag}(\tau_{\mathrm{mod}}) \to \mathbb{R}P^k = \mathrm{Gr}_1(\mathbb{R}^{k+1})$.

THEOREM 6.9.1. *Let* $\Gamma \to \mathrm{SL}(k+1, \mathbb{R})$ *be a projective Anosov representation. Then*

$$\delta_{1,k+1} \leq \mathrm{Hd}_{\mathrm{R}}(\xi^1(\partial_\infty \Gamma)) \leq \delta_{1,2}.$$

Also compare [**GMT19**, Cor. 1.2] where the authors obtain identical bounds for the Hausdorff dimension of the flag limit set equipped with a *certain* Gromov metric.

APPENDIX A

Hausdorff measures on premetric spaces

Let X be a metrizable topological space. Recall that an *outer measure* is a function $\mu : \mathcal{P}(X) \to [0, \infty]$ that satisfies

(i) $\mu(\emptyset) = 0$,

(ii) for all $A, B \in \mathcal{P}(X)$ with $A \subset B$, $\mu(A) \leq \mu(B)$, and

(iii) for all countable collection $\{A_k \mid k \in \mathbb{N}\}$ of subsets of X,

$$\mu\left(\bigcup_{k\in\mathbb{N}} A_k\right) \leq \sum_{k\in\mathbb{N}} \mu(A_k).$$

A set $A \subset X$ is called *μ-measurable* if for every $E \in \mathcal{P}(X)$, $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$. By Carathéodory's theorem (cf. [**Fol99**, Thm. 1.11]), μ-measurable sets form a σ-algebra to which μ restricts as a complete measure.

Assume now that X is compact. The outer measure μ is called *good* if additionally,

(iv) for all $A, B \subset X$ with $\bar{A} \cap \bar{B} = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

The next lemma asserts that, for outer measures μ on compact metrizable spaces, the σ-algebra of Borel sets is a subalgebra of the σ-algebra of μ-measurable sets.

Lemma A.0.1. *Let X be a compact metrizable space. If μ is a good outer measure on X, then every Borel set $B \in \mathfrak{B}(X)$ is measurable.*

Proof. Let d be a metric on X. Then the condition (iv) above implies that

(iv') for all $A, B \subset X$ with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Therefore, μ is a metric outer measure on (X, d). By [**Fol99**, Prop. 11.16], Borel subsets of X are measurable. $\square$

Definition A.0.2 (Premetric space). Let X be a topological space. A symmetric continuous function $d : X \times X \to [0, \infty]$ is called a *premetric* on X. A pair (X, d) consisting of a metrizable topological space X and a premetric d on X is called a *premetric* space.

In what follows, we consider only *positive* premetrics, i.e.,

$$d(x,y) > 0 \iff x \neq y, \quad \forall x, y \in X$$

Let (X, d) be a compact positive premetric space. Then d satisfies the following *separation property*:

$$d(A,B) > 0 \iff \bar{A} \cap \bar{B} = \emptyset, \quad \forall A, B \subset X. \tag{A.1}$$

Let $\varepsilon > 0$, $\beta > 0$. For every $A \subset X$, define

$$\mathcal{H}_\varepsilon^\beta(A) = \inf_{\mathcal{U}} \left\{ \sum_{k\in\mathbb{N}} \operatorname{diam}_d(U_k)^\beta \;\middle|\; \mathcal{U} = \{U_k \mid k \in \mathbb{N}\} \text{ covers } A,\ \operatorname{mesh}(\mathcal{U}) \leq \varepsilon \right\}.$$

In the above, $\operatorname{mesh}(\mathcal{U})$ is the supremum of the d-diameters of the members of $\mathcal{U}$. Then

$$\mathcal{H}_\varepsilon^\beta : \mathcal{P}(X) \to [0, \infty]$$

is an outer measure on X (cf. [**Fol99**, Prop. 1.10]). Define the β-dimensional *Hausdorff measure* $\mathcal{H}^\beta$ by

$$\mathcal{H}^\beta(A) = \lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon^\beta(A).$$

Theorem A.0.3. *The Hausdorff measure $\mathcal{H}^\beta$ is a good outer measure.*

Proof. We need to check the properties (i)-(iv) above. Since, for all $\varepsilon > 0$, $\mathcal{H}_\varepsilon^\beta$ is an outer measure, taking limit $\varepsilon \to 0$, properties (i)-(iii) are easily verified. Therefore, we only need to check that $\mathcal{H}^\beta$ satisfies property (iv).

Let $A, B \subset X$ such that $\bar{A} \cap \bar{B} = \emptyset$. By (A.1), $d(A,B) = d_0 > 0$. Let $\varepsilon < d_0$ be a positive number and $\mathcal{U}$ be a countable open cover of $A \cup B$ with $\operatorname{mesh}(\mathcal{U}) \leq \varepsilon$. If such open cover does not exist, then $\mathcal{H}_\varepsilon^\beta(A \cup B)$ (and hence, $\mathcal{H}^\beta(A\cup B)$) is infinity. Otherwise, $\mathcal{U}$ can be written as a disjoint union $\mathcal{U}_A \sqcup \mathcal{U}_B$ where $\mathcal{U}_A$ consists of all open sets in $\mathcal{U}$ that intersect A and $\mathcal{U}_B$ consists of the rest. Clearly, $\mathcal{U}_A$ and $\mathcal{U}_B$ are open covers of A and B, respectively. Therefore,

$$\sum_{E\in\mathcal{U}} \operatorname{diam}_d(E)^\beta = \sum_{E\in\mathcal{U}_A} \operatorname{diam}_d(E)^\beta + \sum_{E\in\mathcal{U}_B} \operatorname{diam}_d(E)^\beta \geq \mathcal{H}_\varepsilon^\beta(A) + \mathcal{H}_\varepsilon^\beta(B).$$

Since the above holds for any cover $\mathcal{U}$ with mesh $\leq \varepsilon$, we have

$$\mathcal{H}_\varepsilon^\beta(A \cup B) \geq \mathcal{H}_\varepsilon^\beta(A) + \mathcal{H}_\varepsilon^\beta(B).$$

Taking limit $\varepsilon \to 0$, we get $\mathcal{H}^\beta(A \cup B) \geq \mathcal{H}^\beta(A) + \mathcal{H}^\beta(B)$. The reverse inequality follows from property (iii). Therefore, $\mathcal{H}^\beta(A \cup B) = \mathcal{H}^\beta(A) + \mathcal{H}^\beta(B)$. This completes the proof. $\square$

By Lemma A.0.1 and the above theorem, we obtain the following result.

COROLLARY A.0.4. *Every Borel subset of X is $\mathcal{H}^\beta$-measurable.*

The *Hausdorff dimension* of a Borel subset $B \subset (X, d)$ is then defined as

$$\mathrm{Hd}(B) = \inf\{\beta \mid \mathcal{H}^\beta(B) = 0\} = \sup\{\beta \mid \mathcal{H}^\beta(B) = \infty\}.$$

Bibliography

[Alb99] P. Albuquerque, *Patterson-Sullivan theory in higher rank symmetric spaces*, Geom. Funct. Anal. **9** (1999), no. 1, 1–28.

[BC01] R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1964 original.

[BC08] M. Baker and D. Cooper, *A combination theorem for convex hyperbolic manifolds, with applications to surfaces in 3-manifolds*, J. Topol. **1** (2008), no. 3, 603–642.

[BCG96] G. Besson, G. Courtois, and S. Gallot, *Minimal entropy and Mostow's rigidity theorems*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 623–649.

[BCG08] G. Besson, G. Courtois, and S. Gallot, *Rigidity of amalgamated products in negative curvature*, J. Differential Geom. **79** (2008), no. 3, 335–387.

[Ben97] Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. **7** (1997), no. 1, 1–47.

[Bey] J. Beyrer, *Cross ratios on boundaries of symmetric spaces and euclidean buildings*, To appear in Transform. Groups.

[BG08] E. Breuillard and T. Gelander, *Uniform independence in linear groups*, Invent. Math. **173** (2008), no. 2, 225–263.

[BILW05] M. Burger, A. Iozzi, F. Labourie, and A. Wienhard, *Maximal representations of surface groups: symplectic Anosov structures*, Pure Appl. Math. Q. **1** (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543–590.

[BIW10] M. Burger, A. Iozzi, and A. Wienhard, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), no. 1, 517–566.

[BJ97] C. J. Bishop and P. W. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. **179** (1997), no. 1, 1–39.

[Bow95] B. H. Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Math. J. **77** (1995), no. 1, 229–274.

[BPS19] J. Bochi, R. Potrie, and A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 11, 3343–3414.

[BS76] D. Burns, Jr. and S. Shnider, *Spherical hypersurfaces in complex manifolds*, Invent. Math. **33** (1976), no. 3, 223–246.

[BS93] C. Bishop and T. Steger, *Representation-theoretic rigidity in* $\mathrm{PSL}(2,\mathbf{R})$, Acta Math. **170** (1993), no. 1, 121–149.

[Bur93] M. Burger, *Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank* 2, Internat. Math. Res. Notices (1993), no. 7, 217–225.

[Che13] B.-Y. Chen, *Discrete groups and holomorphic functions*, Math. Ann. **355** (2013), no. 3, 1025–1047.

[CI99] K. Corlette and A. Iozzi, *Limit sets of discrete groups of isometries of exotic hyperbolic spaces*, Trans. Amer. Math. Soc. **351** (1999), no. 4, 1507–1530.

[CNS13] A. Cano, J. P. Navarrete, and J. Seade, *Complex Kleinian groups*, Progress in Mathematics, vol. 303, Birkhäuser/Springer Basel AG, Basel, 2013.

[Coo93] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. **159** (1993), no. 2, 241–270.

[Cor90] K. Corlette, *Hausdorff dimensions of limit sets. I*, Invent. Math. **102** (1990), no. 3, 521–541.

[DK18] C. Druţu and M. Kapovich, *Geometric group theory*, American Mathematical Society Colloquium Publications, vol. 63, American Mathematical Society, Providence, RI, 2018, With an appendix by Bogdan Nica.

[DK19] S. Dey and M. Kapovich, *Patterson-Sullivan theory for Anosov subgroups*, arXiv:1904.10196 (2019).

[DK20] ______, *A note on complex-hyperbolic Kleinian groups*, arXiv:2001.04012 (2020).

[DKL19] S. Dey, M. Kapovich, and B. Leeb, *A combination theorem for Anosov subgroups*, Math. Z. **293** (2019), no. 1-2, 551–578.

[Ebe73] P. Eberlein, *When is a geodesic flow of Anosov type? I,II*, J. Differential Geometry **8** (1973), 437–463; ibid. 8 (1973), 565–577.

[Ebe96] P. B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.

[Fed69] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[Fol99] G. B. Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999, Modern techniques and their applications, A Wiley-Interscience Publication.

[GGKW17] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, *Anosov representations and proper actions*, Geom. Topol. **21** (2017), no. 1, 485–584.

[GM87] F. Gehring and G. Martin, *Discrete quasiconformal groups. I*, Proc. London Math. Soc. (3) **55** (1987), no. 2, 331–358.

[GMT19] O. Glorieux, D. Monclair, and N. Tholozan, *Hausdorff dimension of limit sets for projective anosov representations*, arXiv:1902.01844 (2019).

[Gol99] W. M. Goldman, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1999, Oxford Science Publications.

[Gra58] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472.

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[GS19] S. Gouëzel and V. Shchur, *Corrigendum: A corrected quantitative version of the Morse lemma*, J. Funct. Anal. **277** (2019), no. 4, 1258–1268.

[GW12] O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math. **190** (2012), no. 2, 357–438.

[Hat95] T. Hattori, *Collapsing of quotient spaces of* $\mathrm{SO}(n)\backslash\mathrm{SL}(n,\mathbf{R})$ *at infinity*, J. Math. Soc. Japan **47** (1995), no. 2, 193–225.

[Hit92] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473.

[Hör90] L. Hörmander, *An introduction to complex analysis in several variables*, third ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990.

[Kap19] M. Kapovich, *Lectures on complex hyperbolic kleinian groups*, arxiv:1911.12806 (2019).

[KL06] B. Kleiner and B. Leeb, *Rigidity of invariant convex sets in symmetric spaces*, Invent. Math. **163** (2006), no. 3, 657–676.

[KL18a] M. Kapovich and B. Leeb, *Discrete isometry groups of symmetric spaces*, Handbook of group actions. Vol. IV, Adv. Lect. Math. (ALM), vol. 41, Int. Press, Somerville, MA, 2018, pp. 191–290.

[KL18b] ———, *Finsler bordifications of symmetric and certain locally symmetric spaces*, Geom. Topol. **22** (2018), no. 5, 2533–2646.

[Kle83] F. Klein, *Neue Beiträge zur Riemann'schen Functionentheorie*, Math. Ann. **21** (1883), no. 2, 141–218.

[KLM09] M. Kapovich, B. Leeb, and J. J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, Journal of Differential Geometry **81** (2009), no. 2, 297–354.

[KLP14] M. Kapovich, B. Leeb, and J. Porti, *Morse actions of discrete groups on symmetric space*, arXiv:1403.7671 (2014).

[KLP17] ———, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math. **3** (2017), no. 4, 808–898.

[KLP18a] ———, *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*, Geom. Topol. **22** (2018), no. 1, 157–234.

[KLP18b] ———, *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings*, Geom. Topol. **22** (2018), no. 7, 3827–3923.

[Lab06] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

[Led95] F. Ledrappier, *Structure au bord des variétés à courbure négative*, Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, Sémin. Théor. Spectr. Géom., vol. 13, Univ. Grenoble I, Saint-Martin-d'Hères, 1995, pp. 97–122.

[Lin04] G. Link, *Hausdorff dimension of limit sets of discrete subgroups of higher rank Lie groups*, Geom. Funct. Anal. **14** (2004), no. 2, 400–432.

[Lin06] ———, *Ergodicity of generalised Patterson-Sullivan measures in higher rank symmetric spaces*, Math. Z. **254** (2006), no. 3, 611–625.

[Mal40] A. I. Mal'cev, *On isomorphic matrix representations of infinite groups*, Rec. Math. [Mat. Sbornik] N.S. **8 (50)** (1940), 405–422.

[Man79] A. Manning, *Topological entropy for geodesic flows*, Ann. of Math. (2) **110** (1979), no. 3, 567–573.

[Mas65] B. Maskit, *On Klein's combination theorem*, Trans. Amer. Math. Soc. **120** (1965), 499–509.

[Mas68] ———, *On Klein's combination theorem. II*, Trans. Amer. Math. Soc. **131** (1968), 32–39.

[Mas71] ———, *On Klein's combination theorem. III*, Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, No. 66, vol. 66, Princeton Univ. Press, Princeton, N.J., 1971, pp. 297–316.

[Mas88] ———, *Kleinian groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 287, Springer-Verlag, Berlin, 1988.

[Mas93] ———, *On Klein's combination theorem. IV*, Trans. Amer. Math. Soc. **336** (1993), no. 1, 265–294.

[Nic89] P. J. Nicholls, *The ergodic theory of discrete groups*, vol. 143, Cambridge University Press, 1989.

[Pat76] S. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976), no. 3-4, 241–273.

[PSW19] B. Pozzetti, A. Sambarino, and A. Wienhard, *Conformality for a robust class of non-conformal attractors*, arXiv:1902.01303 (2019).

[Qui02a] J.-F. Quint, *Divergence exponentielle des sous-groupes discrets en rang supérieur*, Comment. Math. Helv. **77** (2002), no. 3, 563–608.

[Qui02b] ———, *Mesures de Patterson-Sullivan en rang supérieur*, Geom. Funct. Anal. **12** (2002), no. 4, 776–809.

[Rob03] T. Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. (N.S.) (2003), no. 95, vi+96.

[Ros69] H. Rossi, *Strongly pseudoconvex manifolds*, Lectures in Modern Analysis and Applications, I, Springer, Berlin, 1969, pp. 10–29.

[Shc13] V. Shchur, *A quantitative version of the Morse lemma and quasi-isometries fixing the ideal boundary*, J. Funct. Anal. **264** (2013), no. 3, 815–836.

[Ste99] N. Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, Reprint of the 1957 edition, Princeton Paperbacks.

[Sul79] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171–202.

[Sul84] ———, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. **153** (1984), no. 3-4, 259–277.

[Sul14] H. Sultan, *Hyperbolic quasi-geodesics in CAT(0) spaces*, Geom. Dedicata **169** (2014), 209–224.

[Tit72] J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.

[Tol89] D. Toledo, *Representations of surface groups in complex hyperbolic space*, J. Differential Geom. **29** (1989), no. 1, 125–133.

[Tuk94] P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. **23** (1994), no. 2, 157–187.

[Väi05] J. Väisälä, *Gromov hyperbolic spaces*, Expo. Math. **23** (2005), no. 3, 187–231.

[Wei64] A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. (2) **80** (1964), 149–157.

[Wie18] A. Wienhard, *An invitation to higher Teichmüller theory*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 1013–1039.

[Yue96] C. Yue, *The ergodic theory of discrete isometry groups on manifolds of variable negative curvature*, Trans. Amer. Math. Soc. **348** (1996), no. 12, 4965–5005.

[Yue99] ———, *Webster curvature and Hausdorff dimension of complex hyperbolic Kleinian groups*, Dynamical Systems–Proceedings Of The International Conference In Honor Of Professor Liao Shantao (Y. Jiang and L. Wen, eds.), World Sci. Publ., Singapore, 1999, pp. 319–328.