The Spaces of Shapes and Geodesic Triangulations on Surfaces

By

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To my family, friends, and teachers

Contents

List of Figures

Abstract

The theory of the geometry and topology of surfaces has been well-established during the last century from various perspectives. In recent developments of discrete differential geometry, mathematicians started to think rigorously about the discrete counterparts of concepts in the smooth setting, including triangulations, metrics, curvatures. We study the geometry of discrete surfaces via discrete metrics, discrete curvatures, and discrete conformal maps, not only as approximations to the smooth counterparts, but as geometric objects in their own right. They provide fast algorithms in computer graphics, and also a parallel theory, such as discrete uniformization theorems.

This thesis is concerned with these types of problems. We study the geometry and topology of "shape" spaces of different geometric objects, including high genus surfaces in $\mathbb{R}^3$ and geodesic triangulations on surfaces with constant curvature. In Chapter 2, we study the global comparison problem for two surfaces with the same high genus, constructing a metric on the shape space and producing a correspondence between two given shapes. This leads to an algorithm to compute the distance between a pair of shapes via energy minimization. In Chapter 3, we consider the topology of the space of geodesic triangulations on a surface with a fixed combinatorial type in a fixed isotopy class, which can be regarded as a discrete version of the group of surface diffeomorphisms. We show that these spaces, after appropriate normalization, are contractible when the surface is a convex polygon. Furthermore, we provide an algorithm to generate geodesic triangulations when the surface is a star-shaped polygon.

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CHAPTER 1

Introduction

This dissertation consists of two chapters dealing with topics in the field of low dimensional geometry and topology. These two chapters are independent from each other, but consider problems about the geometry and topology of the space of "shapes" on surfaces, both in the smooth setting and in the discrete setting. Moreover, these topics lay the foundations for further applications in shape analysis and graph drawing.

In this chapter, we provide the overview of this dissertation, and explain the motivations and main results for each chapter.

1.1. Overview

In Chapter 2, we explore the mathematical foundation of the shape comparison problem for high genus surfaces. This is a continuation of the work by Hass and Koehl [**HK17**]. We define the shape space of a high genus surface and construct a metric on the shape space of a high genus surface by minimizing an energy among all the quasiconformal homeomorphisms in a fixed homotopy class of maps between a given pair of shapes. We show that the minimizer of the energy is realized by a quasiconformal homeomorphism, producing an "optimal" correspondence between two shapes. We also discuss several energies related to the Dirichlet energy of maps between surfaces. Finally, we propose an algorithm to compute the distance between two arbitrary shapes of high genus surfaces.

In Chapter 3, we study the topology of the space of geodesic triangulations of a surface. We give a new proof of the contractibility of the space of geodesic triangulations of a convex polygon with a fixed combinatorial type, using the idea of Tutte's theorem, significantly simplifying the proof by Bloch, Connelly, and Henderson [**BCH84**]. Then we prove that each component of the space of geodesic triangulations of a flat torus with a fixed combinatorial type is homotopy equivalent to a torus. We also give a constructive method to produce a geodesic triangulation of a star-shaped polygon under a mild assumption about the triangulation, and a characterization for geodesic triangulations which can not be realized as a configuration of the equilibrium state of any spring system.

1.2. Comparing shapes of high genus surfaces

How to measure the difference between two shapes is a fundamental problem in computer graphics, computer vision, and medical imaging. In Chapter 2, we investigate the shape comparison problem between two shapes of a surface of genus at least one by considering the following questions:

(1) What is the precise meaning of a "shape" of a surface?
(2) How similar are two given shapes of a surface? Or how to compare two shapes?
(3) How to construct the best global alignment of two shapes, namely an "optimal" correspondence between two shapes?

These problems have been studied extensively in the fields of surface registration, shape matching, shape morphing, and texture mapping. Effective algorithms have been developed if the topology of the surface is relatively simple, such as with the 2-dimensional disk or 2-dimensional sphere [**GGS03**, **GWC+04**, **HK17**]. Hass and Koehl [**HK17**] introduced a metric structure for smooth genus-zero surfaces by considering the *symmetric distortion energy* of conformal diffeomorphisms between two genus-zero surfaces. They showed that the infimum was achieved by a conformal diffeomorphism, and proposed an algorithm to compute the distance between two triangulated surfaces and applied it to describe shapes of proteins and generate evolutionary trees of species [**HK14**, **KH13**, **KH15**].

However, there are few results about the computation of optimal maps between high-genus surfaces [**LBG+08**, **LW14**, **WZ14**, **ZLL+12**]. On the other hand, detecting the change of the shapes of high genus surfaces is crucial to understanding various applications. For example, the vestibular system in the inner ear is modelled by a genus-three surface, and the morphometry of the vestibular system has been an active research field in the analysis of Adolescent Idiopathic Scoliosis Disease [**WWS+15**]. In the study of deformity of the vertebrae, the vertebrae bone is modeled by a genus-one surface [**LGL15**].

Comparing shapes of high genus surfaces is much more challenging than the case of the 2-sphere. Any two metrics on the 2-sphere are conformal to each other, but for high-genus surfaces, conformal maps are insufficient to measure the difference between two shapes. Algorithmically, the main difficulty is how to deal with the topology of the surfaces. One possible approach is to construct local injective maps from disk-like patches to some canonical domain and glue them to form a global map. This method requires a consistent way to cover the whole surface with patches.

An alternative method is to cut the surface using a system of disjoint loops to a disk-like surface, but boundary conditions on the loops are not natural.

The key to measuring the difference between two surfaces is finding a metric structure on the shape space of a surface. More precisely, for a metric d defined on the shape space, given shapes S_1, S_2, and S_3, we require the following properties:

(1) $d(S_1, S_2) \geq 0$;

(2) $d(S_1, S_2) = 0$ if and only if S_1 and S_2 represent the same element in the shape space;

(3) $d(S_1, S_2) = d(S_2, S_1)$;

(4) $d(S_1, S_2) + d(S_2, S_3) \geq d(S_1, S_3)$.

These properties of metric structures imply that we can distinguish two different shapes if the two shapes are not isometric, independent of the order and stable under small perturbations or noise.

The main result of Chapter 2 includes the following. We will define the shape space $\mathcal{S}(S)$ of a high genus surface as the space of equivalence classes of Riemannian metrics on a fixed smooth surface, up to isometries isotopic to the identity on the surface. We show that the shape space has a close connection with the Teichmüller space of the surface S. Then we construct a metric on the shape space by introducing an energy for quasiconformal homeomorphisms of the surface, which measures the similarity of two shapes. We show that the infimum of this energy in a fixed homotopy class is achieved by a quasiconformal homeomorphism, which produces the "optimal" correspondence between two shapes and realizes the distance between two shapes.

1.3. The space of geodesic triangulations on surfaces

A triangulation of a fixed combinatorial type of T on a surface with a Riemannian metric (S, g) is a *geodesic triangulation* if each edge in T is embedded as a geodesic arc in S. We study the space of geodesic triangulations with a fixed combinatorial type on certain surfaces, including a polygonal region in the Euclidean plane and a flat torus. We study the following two problems.

(1) The *embeddability* problem: Given a surface (S, g) with a triangulation T, can we construct a geodesic triangulation with the combinatorial type of T? In particular, if S is a 2-disk with a triangulation T and we specify the positions of the boundary vertices of T in the plane so that they form a polygon, can we find positions of the interior vertices in the plane to construct a geodesic triangulation of S with the combinatorial type of T?

(2) The *contractibility* problem: If the space of geodesic triangulations on (S, g) with a fixed combinatorial type of T is not empty, what is the topology of this space? In particular, is it a contractible space?

These two problems have been studied in [**BS78**, **BCH84**, **Cai44b**, **Ho73**], partly because they are closely related to the problem of determining the existence and uniqueness of differentiable structures on a triangulated manifolds [**CHHS83**]. They are also used to produce effective algorithms to solve graph morphing problems in [**DVPV03**, **FG99**, **SG01**, **SG03**].

In Chapter 3, we solve the contractibility problem for convex polygons, the embeddability problem for star-shaped polygons, and the contractibility problem for flat tori using the idea of Tutte's Theorem [**Tut63**]. We give a new proof of the contractibility of the space of geodesic triangulations of a fixed combinatorial type of T for the case of a convex polygon Ω in $\mathbb{R}^2$. We construct a homotopy equivalence from this space to an affine subspace in the Euclidean space, significantly simplifying the previous argument in [**BCH84**]. We then give a new constructive method to produce geodesic triangulations with a fixed combinatorial type for a star-shaped polygon under a mild assumption on the triangulation. This problem has been studied by Hong and Nagamochi [**HN08**] and Xu et al. [**XCGL11**].

These results can be regarded as discrete versions of classical results by Smale [**Sma59**] and Earle and Eells [**EE$^+$69**] about surface diffeomorphisms. The group of diffeomorphisms of the 2-disk fixing the boundary, denoted by $\mathcal{D}_0(\mathbb{D}^2)$, is contractible.

CHAPTER 2

Comparing Shapes of High Genus Surfaces

In this chapter, we study the shape comparison problem for two shapes of high genus surfaces. The goal is to construct a new metric structure on the shape space of a high genus surface. In Section 1, we provide background about the geometry and topology of surfaces and maps between surfaces. In Section 2, we define the space of shapes $\mathcal{S}(S)$ and establish its connection with Teichmüller space. In Section 4, we introduce an energy $E(f)$ for a quasiconformal homeomorphism f on a surface and prove that this energy provides a metric d on the shape space $\mathcal{S}(S)$, and an "optimal" correspondence between two shapes. In Section 5, we discuss other energies for maps to define metrics on the Teichmüller space and on a conformal class of metrics. In Section 6, we propose an algorithm to compute the distance between two shapes.

2.1. Background

The fundamental object we are going to study is a *surface* S, namely a 2-dimensional smooth manifold, possibly with boundary. In practice, most surfaces are immersed in the space $\mathbb{R}^3$. We will restrict ourselves to *surface of finite type*, namely surfaces obtained from compact, connected, orientable surfaces of genus g by removing b disjoint open disks and p points, denoted by $S_{g,b,p}$. Here we summarize related backgrounds about the basic geometry and topology of surfaces.

2.1.1. The topology and geometry of surfaces. The fundamental result in the topology of smooth surfaces is the following classification theorem.

THEOREM 2.1.1. *Every compact, connected, orientable surface without boundary is diffeomorphic to S_g for some genus g.*

The topology of a surface S is determined by the *Euler characteristic* $\chi(S)$ by the following relation with the genus

$$\chi(S_g) = 2 - 2g.$$

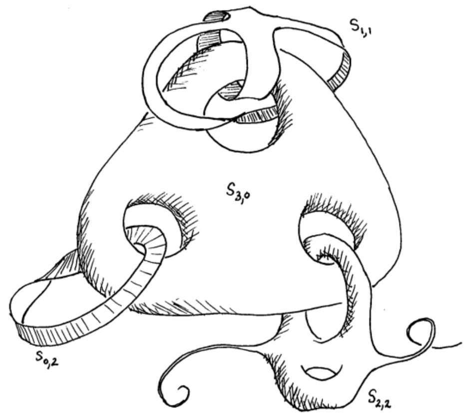


FIGURE 2.1. Surface of finite types [**Min13**]

The general formula for surfaces of finite type is given by

$$\chi(S_{g,b,p}) = 2 - 2g - b - p.$$

In applications, surfaces are represented by triangular meshes, namely a set of flat triangles glued together along edges. A triangular mesh correspond to a *triangulation* $T = (V, E, F)$ of a simplicial complex homeomorphic to S_g, where V, E, and F are the sets of vertices, edges, and faces in T. In this case we have the following two relations

$$V - E + F = \chi(S_g), \quad \text{and} \quad 2E = 3F.$$

The *fundamental group* $\pi_1(S_g)$ of S_g is given by

$$\pi_1(S_g) =< a_1, b_1 ... a_g, b_g | [a_1, b_1] ... [a_g, b_g] = 1 > .$$

Its abelianization gives the *homology group* of S_g with integer coefficients

$$H_0(S_g) = \mathbb{Z}, \qquad H_1(S_g) = \mathbb{Z}^{2g}, \quad \text{and} \quad H_2(S_g) = \mathbb{Z}.$$

A surface with finite type $S_{g,b,p}$, is homotopically equivalent to a wedge product of $2g+b+p$ copies of S^1, so $\pi_1(S_{g,b,p})$ is a free group of order $2g + b + p$.

if a surface S is immersed in $\mathbb{R}^3$, the standard Euclidean metric in $\mathbb{R}^3$ induces a Riemannian metric g on the tangent space T_pS, which defines the concepts of length of arcs, angle between two vectors in the tangent space, area, geodesic and various intrinsic curvatures.

Although there are infinitely many Riemannian metrics on a surface S, the following far-reaching result, the *Gauss-Bonnet Theorem*, establishes the connection between the topology and the geometry of S.

THEOREM 2.1.2. *Let K be the Gaussian curvature of a Riemannian metric on a surface $S_{g,b}$ and κ_g^i be the geodesic curvatures of boundary components B_i, then we have:*

$$\sum_{i=1}^{b} \int_{B_i} \kappa_g + \iint_S K = 2\pi\chi(S_{g,b}).$$

Besides Riemannian metrics, we will consider various *geometric structures* on surfaces. A geometric structure on a surface S is a maximal atlas of coordinate charts $\{U_i, \phi_i\}$ where U_i is a open subset in S and ϕ_i is a diffeomoprhism from U_i to its image in $\mathbb{R}^2$. Different geometric structures require different restrictions on the transition maps $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ of two overlapping charts. Here we consider three different types of geometric structures on a surface, including complex structure, conformal structure and hyperbolic structure.

A *complex structure* requires the transition map to be biholomorphic if we identify $\mathbb{R}^2$ with $\mathbb{C}$. A surface is a one-dimensional complex manifold referred as a Riemann surface. Similarly, a *conformal structure* requires the transition map to be a conformal map, which means the metric is stretched uniformly in all directions in the tangent space of every point in S. It turns out that holomorphic functions and conformal maps are equivalent in the case of surfaces, thus we can use complex structures and conformal structures interchangeably.

A Riemannian metric on a surface gives rise to a complex structure by the existence of the *isothermal coordinates*, in which the metric is in the form of

$$ds^2 = \rho^2(dx^2 + dy^2).$$

All the transition maps are scalar multiplications, hence we have a conformal or complex structure on a surface naturally induced by a Riemannian metric.

A *hyperbolic structure* on a surface requires the transition map to be the restriction of isometries in the hyperbolic plane, which can be identified with the group of linear fractional transformations isomorphic to $PSL_2(\mathbb{R})$. It also corresponds to a complete Riemannian metric on the surface S with constant curvature -1. Since all the linear fractional transformations are conformal, it automatically produces a conformal structure on a surface. Similarly we can define *flat structures* and *elliptic structures* on a surface when the transition maps are isometries of the Euclidean plane or the round 2-sphere.

By the Gauss-Bonnet theorem, the Euler characteristic $\chi(S)$ has to be negative for a surface with hyperbolic structures. This necessary condition turns out to be sufficient by the *Uniformization Theorem* by Poincare, Klein, and Koebe.

THEOREM 2.1.3. *Every Riemannian metric on a surface S is conformally equivalent to a complete Riemannian metric with constant curvature +1, 0, or -1, the sign depending on the sign of its Euler characteristic $\chi(S)$. Furthermore, the metric is unique if the Euler characteristic is negative.*

This theorem establishes the relation among all the geometric structures we mention above. It can be interpreted as the existence of a canonical metric with constant curvatures in each conformal class of metrics. In terms of complex structures of Riemann surfaces, it asserts that every simply connected Riemann surface is biholomorphically equivalent to one the the three Riemann surfaces: Riemann sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$ and the disk $\mathbb{D}$. They are universal coverings for all the other Riemann surfaces. Therefore, the hyperbolic structures can be constructed by the quotient of the hyperbolic plane by a discrete subgroups of $PSL_2(\mathbb{Z})$.

The formula for the Euler characteristic shows that most of the surfaces of finite type have hyperbolic structures since $\chi(S_{g,b,p}) = 2 - 2g - b - p$ is positive only for small values of g, b and p. However, the uniformization theorem does not provide a concrete method to construct a specific hyperbolic metric on a given surface. We can construct families of hyperbolic structures alternatively using the pants decomposition of surfaces with negative Euler characteristic. Every surface S with negative Euler characteristic can be decomposed into $-\chi(S)$ pairs of pants, namely 2-spheres with three boundary components or punctures. For each pair of pants, we can put a hyperbolic metric on it by gluing two identical right angle hyperbolic hexagons(possible with degeneracy) in the hyperbolic plane. Then we glue pairs of pants along the geodesic boundaries with the same lengths. The result will be a hyperbolic surfaces with finite area $-2\pi\chi(S)$. This process can be regarded as the geometrization for surfaces with negative Euler characteristics.

2.1.2. Maps between surfaces. To compare two geometric structures on a surface, one of the fundamental method is to construct maps between two geometric structures and measure the minimal "distortion" of these maps. Here we focus on three types of well-known maps between surfaces, including conformal maps, harmonic maps, and quasiconformal maps.

2.1.2.1. *Conformal maps and Harmonic maps.* A map between two surfaces $f : (S_1, g_1) \to (S_2, g_2)$ is *conformal* if there exists a positive function λ on S_1 such that

$$f^*(g_2) = \lambda^2 g_1.$$

The fundamental result is the Uniformization Theorem mentioned above.

The general theory of harmonic maps between two n dimensional manifolds was developed by Eells and Sampson [**ES64**]. We restrict our attention to the case of surfaces. The *Dirichlet energy* of a map between two surfaces $f : (S_1, g_1) \to (S_2, g_2)$ is defined by

$$E_D(f) = \int_{S_1} ||df||^2 dA$$

where df is the differential of f, considered as a section to the bundle $T^*S_1 \otimes TS_2$ with a metric induced from m_1 and m_2. It can be regarded as the measurement of total stretching of the map f. A map is *harmonic* if it is a critical point of the Dirichlet energy among maps in its homotopy class.

One of the earliest results about harmonic maps in the plane is the *Rado-Kneser-Choquet theorem* [**Dur04**].

Theorem 2.1.4. *Suppose $\phi : \mathbb{D} \to \mathbb{R}^2$ is a harmonic map sending the boundary $\partial\mathbb{D}$ homeomorphically into the boundary $\partial\Sigma$ of some convex region $\Sigma \subset \mathbb{R}^2$. Then ϕ is one to one.*

When it comes to general surfaces, a fundamental question is the existence and uniqueness of harmonic maps in a given homotopy class of maps between two surfaces. Here we summarize the result proved by Jost [**JS82**], Schoen and Yau [**SY78**], Coron and Helein [**CH89**], and Markovic and Mateljevic [**MM99**].

Theorem 2.1.5. *Given two Riemannian metrics on a surface F and a diffeomorphism f, there exists a diffeomorphism which is a critical point of the Dirichlet energy in the homotopy class of f. If the genus of F satisfies $g > 1$, then this diffeomorphism is unique.*

2.1.2.2. *Quasiconformal maps and the Teichmüller maps.* Quasiconformal maps are a generalization of conformal maps between surfaces, arising naturally when we compare two conformal structures on a surface. Let $f : D \to \mathbb{C}$ be an orientation preserving diffeomorphism from a region

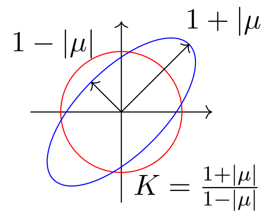


FIGURE 2.2. Quasiconformal maps

D in $\mathbb{C}$. We can consider the *Beltrami coefficient*

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z},$$

where $f_{\bar{z}}$ and f_z are the derivative of f with respect to the complex variable z and its conjugate $\bar{z}$. If f is conformal, then Cauchy-Riemann equations imply $f_{\bar{z}} = 0$, so $\mu_f = 0$. The Jacobian of f is given by $J(f) = |f_z|^2 - |f_{\bar{z}}|^2$ which is positive by assumption. Hence $|\mu_f|$ varies from 0 to 1, measuring the deviation of f from a conformal map. An alternative quantity K varying from 1 to ∞, called the *dilatation*, is defined by

$$K_f(z) = \frac{1+|\mu_f|}{1-|\mu_f|}.$$

Geometrically, at each point z in D, df maps circles in T_pD to ellipses in $T_{f(p)}\mathbb{C} = \mathbb{C}$. The dilatation $K_f(z)$ is the ratio of the major axis to the minor axis of the ellipse. Then we call the map f a *K-quasiconformal map* if there exists a $K > 0$ such that

$$\sup_{z\in D} K_f(z) = \sup_{z\in D} \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|} \leq K.$$

The composition of a K_1-quasiconformal map with a K_2-quasiconformal is a K_1K_2-quasiconformal map. Quasiconformal maps can be generalized further to non-differentiable maps using several mutually equivalent geometric and measure-theoretic definitions [**IT12**].

Every quasiconformal map $f : D \to \mathbb{C}$ gives rise to a Beltrami coefficient $\mu_f(z)$ defined on D. A remarkable theorem proved by Ahlfors and Bers (see, e.g. [**FM11**]) states the converse is also true, and can be regarded as a generalization of the Riemann mapping theorem.

THEOREM 2.1.6. *If $\mu \in L^\infty(\mathbb{C})$ and $||\mu||_\infty < 1$, there exists a unique quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing 0, 1, and ∞, satisfying $\mu = \mu_f$ almost everywhere.*

Since the composition of quasiconformal maps with conformal maps is again quasiconformal with the same maximal dilatation, we can define quasiconformal maps $f : S_1 \to S_2$ between Riemann surfaces using local charts. Then the Beltrami coefficient is a $(-1,1)$-form $\mu d\bar{z}/dz$ instead of a function, but $|\mu|$ is well-defined on the surface. We can define the corresponding dilatation of a map f

$$K_f = \sup_{p \in F} \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}.$$

This quantity measures the difference between two conformal structures, or equivalently, two hyperbolic structures for higher genus surfaces. We have the following extremal problem in a given homotopy class: find a map f_0 achieving this infimum of the dilatation in a homotopy class:

$$K_{f_0} = \inf\{K_f | f \text{ in a given homotopy class}\}.$$

This map is called an *extremal quasiconformal map* in the given homotopy class between two Riemann surfaces . For surfaces S_g with genus $g > 1$, the extremal quasiconformal map in certain special coordinates is locally an affine map except for some singularities, called the *Teichmüller map*. The fundamental theorem about a Teichmüller map is Teichmüller's theorem (see e.g. [**FM11**]).

THEOREM 2.1.7. *Given two conformal structures on a surface S_g with genus $g > 1$, there exists a unique Teichmüller map in every homotopy class of diffeomorphisms of S_g.*

2.1.3. Teichmüller space and its structures. To compare geometric structures of certain type on a surface, it is natural to study the space of all such geometric structures. The natural space to consider is the *moduli space*, namely the set of all the geometric structures quotient by the corresponding equivalence. For instance, if we consider the space of all conformal structures, complex structures or hyperbolic structures on a surface, the notions of equivalence should correspond to conformal diffeomorphims, biholomorphisms, and isometries respectively. It turns out that the *Teichmüller space*, instead of the moduli space, is easier to study.

We define the Teichmüller space for a surface S_g with genus g as following

$$\mathcal{T}(S) = \{\text{hyperbolic metrics } \bar{g} \text{ on surface } S_g \text{ denoted by } (S, \bar{g})\}/\mathcal{D}_0,$$

where $\mathcal{D}_0$ is the group of diffeomorphisms of S_g isotopic to identity. Equivalently, we can use conformal or complex structures.

Another formulation for the Teichmüller space involves marked surfaces $(F, f, \bar{g})$ with hyperbolic metrics $\bar{g}$ and a diffeomorphism $f : S \to F$ as a marking. Given a surface S, we consider the following equivalent class of marked surfaces

$$\mathcal{T}(S) = \{(F, f, \bar{g})\}/\sim,$$

where two marked surfaces $(F_1, f_1, \bar{g}_1)$ and $(F_2, f_2, \bar{g}_2)$ are equivalent if there exists an isometry $h : F_1 \to F_2$ isotopy to $f_2 \circ f_1^{-1}$. Similarly, we can refer to conformal and complex structures. There exist several equivalent interpretations for Teichmüller space using Riemann surfaces with marked generators for fundamental groups, the set of equivalence classes of Beltrami coefficients, the set of equivalence classes of quasiconformal maps in the universal covering, etc. We will mostly use the two definitions above and define the space of shapes in the next section in a similar fashion.

The Teichmüller space of the 2-sphere consists of just one point. The Teichmüller space $\mathcal{T}(T^2)$ of flat tori up to conformal equivalence plays a fundamental role as the analogue to its higher genus counterparts.

THEOREM 2.1.8. *The space $\mathcal{T}(T^2)$ is parametrized by the upper half plane $\mathbb{H}$, and mapping class group of T^2 acts on $\mathbb{H}$ as isometries in $PSL(2, \mathbb{Z})$.*

We have a similar identification between the Teichmüller space of surfaces with higher genus and a contractible space in the Euclidean space. However, the parametrization is more complicated than the case of tori. There are two common parametrizations: the Fenchel-Nielsen coordinates and the Fricke coordinates.

The Fenchel-Niesen coordinates stem from the pants decomposition of surfaces. Given a hyperbolic metric on surface S_g, we can cut it into $2g - 2$ hyperbolic pairs of pants by $3g - 3$ disjoint simple closed geodesics. Each hyperbolic pair of pants is determined uniquely by the lengths of the three boundary geodesics. Given two pairs of pants along two boundaries with the same length, we can glue them together to generate families of distinct hyperbolic metrics by performing twists along the boundary curves. Hence each simple closed geodesic corresponds to a length parameter and a twist parameter, leading to the following parametrization

$$\mathcal{FN} : \mathcal{T}(S_g) \to \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}, \qquad [\bar{g}] \to (l_1, l_2, ..., l_{3g-3}, \theta_1, \theta_2, ..., \theta_{3g-3}).$$

THEOREM 2.1.9. *The map* $\mathcal{FN} : \mathcal{T}(S_g) \to \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$ *is well-defined and bijective. Hence* $\mathcal{T}(S_g)$ *is contractible.*

In the contrast, the Fricke coordinates arise from the Fushsian models of Riemann surfaces $\mathbb{H}/\Gamma$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ represented by linear fractional transformations. We can make Γ canonical after the normalization below such that it is represented by a set of g pairs of generators for $\pi_1(S)$, each of which correspond to two linear fractional transformation

$$\alpha_i(z) = \frac{a_i z + b_i}{c_i z + d_i} \quad a_i d_i - b_i c_i = 1 \quad a_i, b_i, c_i, d_i \in \mathbb{R} \quad c_i > 0 \quad i = 1, 2, ..., g-1;$$

$$\beta_i(z) = \frac{a'_i z + b'_i}{c'_i z + d'_i} \quad a'_i d'_i - b'_i c'_i = 1 \quad a'_i, b'_i, c'_i, d'_i \in \mathbb{R} \quad c'_i > 0 \quad i = 1, 2, ..., g-1.$$

Note the last generator α_g and β_g is determined uniquely by the previous generators and the following normalization

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] = id$$

where $[\alpha_1, \beta_i] = \alpha_i \circ \beta_i \alpha_i^{-1} \beta_i^{-1}$. Then we can define a new parametrization

$$\mathcal{FR} : \mathcal{T}(S_g) \to \mathbb{R}^{6g-6} \quad [(S_g, f)] \to (a_1, c_1, d_1, ..., a_{g-1}, c_{g-1}, d_{g-1})$$

THEOREM 2.1.10. *The map* $\mathcal{FN} : \mathcal{T}(S_g) \to \mathbb{R}^{6g-6}$ *is well-defined and injection.*

These two coordinates introduce real analytic structures on $\mathcal{T}(S_g)$. A metric structure can be defined with the extremal quasiconformal map

$$d([F_1, f_1], [F_2, f_2]) = \inf_{f \sim f_2 \circ f_1^{-1}} \frac{1}{2} \log K(f) \quad [F_i, f_i] \in \mathcal{T}(S_g) \quad i = 1, 2.$$

By the Teichmüller existence theorem and uniqueness theorem, the metric above is defined for any pair of points in $\mathcal{T}(S_g)$, and the infimum is achieved by the Teichmüller map. This metric is a complete Finsler metric called Teichmüller metric, and its geometric properties have been studied extensively.

There exist other metrics on the Teichmüller space such as Weil-Peterson metric, which is a incomplete Riemannian metric, and Thurston's asymmetric metric, another Finsler metric which is defined with Lipschitz constant instead of dilatation. A natural complex structure is introduced by Bers embedding, then Teichmüller metric is identified with the Kobayashi metric with respect to this complex manifold. See more about properties of different metrics on Teichmuller space in [**IT12**] In the following discussion, we will mainly deal with the Teichmüller metric.

2.2. The Space of Shapes

We need to define rigorously the space of "shapes" before constructing metrics on it. Various notions of shape spaces of curves and surfaces in $\mathbb{R}^2$ or $\mathbb{R}^3$ have been formulated from different perspectives with applications in computational geometry and computer graphics. An overview of various notions about shapes is given by Bauer, Bruveris and Minchor [**BBM14**].

In this paper, we will introduce the space of shapes on surfaces from an intrinsic point of view. The idea stems from the work by Ebin [**Ebi67**], Fischer and Tromba [**FT84, Tro12**], Earle and Eells [**EE$^+$69**]. We will summarize their work, define the shape space of surfaces and complete the picture of its connection with the Teichmüller space.

2.2.1. Space of Riemannian metrics and its quotients. From the intrinsic viewpoint, the natural space to consider is the space of all possible smooth metric tensors on a given surface S, denoted by $\mathcal{M}$. Let TS and T^*S be the tangent and cotangent bundle, then a metric tensor is a section of S^2T^*S, the bundle of all symmetric (0,2)-type tensors. Since a metric tenor is positive definite, all metric tensors on F form a convex subset of the infinite-dimensional vector space of sections of symmetric 2-tensors, denoted by $\Gamma(S^2T^*S)$.

The tangent space at any element of $\mathcal{M}$, being a subset of a vector space, is naturally isomorphic to $\Gamma(S^2T^*S)$. In the tangent space at g in $\mathcal{M}$, there is a natural inner product induced by g on arbitrary tensor fields, defined as

$$(h,k)_g = \int_S tr_g(hk)dvol_g$$

where h and k are in $\Gamma(S^2T^*S)$ identified with the tangent space at g and $dvol_g$ is the volume form. In local coordinates, they are represented by

$$tr_g(hk) = g^{ij}g^{lm}h_{il}k_{jm} \quad \text{and} \quad dvol_g = \sqrt{det(g)}dx_1dx_2.$$

Clarke [**Cla10**] explored the basic properties of this metric, showing that this metric, originally defined as a weak Riemannian metric, is indeed a metric. Furthermore, it coincides with the Weil-Petersson metric when restricted to the Teichmüller space.

It is hard to compute the natural L^2 metric defined above on the space $\mathcal{M}$. Besides, $\mathcal{M}$ contains redundant information: two metric tensors h and k may describe the isometric surface with different parametrizations. Therefore we would like to simplify the definition of the space of shapes for a given surface S, as the quotient of $\mathcal{M}$ by certain groups acting on $\mathcal{M}$.

There are three topological groups acting naturally on the space of metrics: the space $\mathcal{P}$ of all smooth functions on surface S, $\mathcal{D}$ the orientation-preserving diffeomorphism group of S and its normal subgroup $\mathcal{D}_0$, the group of diffeomorphisms isotopic to the identity. The group $\mathcal{D}$ acts on $\mathcal{M}$ as isometries by pull-back

$$\mathcal{D} \times \mathcal{M} \to \mathcal{M} \quad (f, g) \to f^* g.$$

The action of $\mathcal{D}_0$ is its restriction. The action of $\mathcal{P}$ on $\mathcal{M}$ is the multiplication of positive functions with metric tensors

$$\mathcal{P} \times \mathcal{M} \to \mathcal{M} \quad (u, g) \to e^u g.$$

When we consider the two group actions above, an immediate question is whether we have a bundle structure. The natural topology for $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{P}$ is the smooth Frechet topology, which means that two metrics are close if all the coefficients and their derivatives are close under the supremum norm in every chart. The implicit function theorem and its consequences are not true in general for this topology. Hence in Ebin [**Ebi67**] and Fischer [**FM77, FT84**], $\mathcal{M}$, $\mathcal{D}$, and $\mathcal{P}$ are modelled in the corresponding Sobolev spaces. These spaces contain maps which have square integrable partial derivatives up to sufficiently large order $s > 1$ in every local charts, denoted by $\mathcal{M}^s$, $\mathcal{D}^{s+1}$, and $\mathcal{P}^s$ respectively.

The space $\mathcal{M}^s$ forms an open convex subset in the Hilbert space $\Gamma^s(S^2T^*S)$, hence a Hilbert manifold. The space $\mathcal{P}^s$ corresponds to the Sobolev space $H^s(S, \mathbf{R})$. Then the multiplication and inverse are continuous, hence $\mathcal{P}^s$ is an abelian Hilbert Lie group. Ebin [**Ebi67**] proves that $\mathcal{D}^{s+1}$ is also a Hilbert Lie group. Then we can apply the following theorem in [**FT84**] for the action of a Hilbert Lie group on an infinite-dimensional manifold, which will induce a smooth structure on the shape space, the space of pointwise conformal classes and the Teichmüller space.

THEOREM 2.2.1. *Let a smooth Hilbert Lie group $\mathcal{G}$ act on a smooth Hilbert manifold $\mathcal{N}$. If the action is smooth, proper, and free, then:*

- *For all $x \in \mathcal{N}$, the orbit of x by $\mathcal{G}$, denoted by $\mathcal{G}_x$, is a closed smooth submanifold in $\mathcal{N}$;*
- *The quotient space $\mathcal{N}/\mathcal{G}$ is a smooth manifold;*

- *The quotient map $\pi : \mathcal{N} \to \mathcal{N}/\mathcal{G}$ is a smooth submersion. It has the structure of a smooth principle fibre bundle.*

Fischer and Tromba [**FT84**] considered the action of $\mathcal{P}^s$ on $\mathcal{M}^s$, where two metrics were in the same orbit if they differed by a factor $u \in \mathcal{P}^s$, namely they were pointwise conformal to each other. The quotient manifold of this group action on $\mathcal{M}^s$ is the space of pointwise conformal structures on S, denoted by $\mathcal{C}^s$. By Theorem 3.1 above, they clarified the differential structure for $\mathcal{C}^s$ in [**FT84**].

THEOREM 2.2.2. *The group action $P : \mathcal{P}^s \times \mathcal{M}^s \to \mathcal{M}^s$ is smooth, free, and proper. The quotient space $\mathcal{C}^s = \mathcal{M}^s/\mathcal{P}^s$ by the quotient map $\pi : \mathcal{M}^s \to \mathcal{M}^s/\mathcal{P}^s = \mathcal{C}^s$, is a contractible smooth Hilbert manifold, and $(\mathcal{M}^s, \mathcal{C}^s, \pi)$ has the structure of a trivial principle fiber bundle with structure group $\mathcal{P}^s$. The orbit $\mathcal{P}^s g$ for any g is a closed smooth submanifold diffeomorphic to $\mathcal{P}^s$.*

For surfaces with genus larger than one, there is a unique hyperbolic metric in each conformal class of metrics. Let $\mathcal{M}_{-1}$ and $\mathcal{M}^s_{-1}$ be the space of all smooth hyperbolic metrics and the corresponding Hilbert manifold, then Fischer and Tromba [**FT84**] proved that $\mathcal{M}^s_{-1}$ and $\mathcal{C}^s$ were diffeomorphic, so we can use them interchangeably.

We can take further quotient of $\mathcal{C}^s$ by group action of $\mathcal{D}^{s+1}_0$. This quotient gives a trivial fibre bundle description of the Teichmüller space $\mathcal{T}^s$ in [**EE$^+$69**, **FT84**].

THEOREM 2.2.3. *Assume a surface S is of genus $g > 1$. The group action $\mathcal{D}^{s+1}_0 \times \mathcal{C}^s \to \mathcal{C}^s$ by pullback is smooth, free, and proper. The quotient space is the Teichmüller space $\mathcal{T}^s$, and the quotient map $\pi : \mathcal{C}^s \to \mathcal{C}^s/\mathcal{D}^{s+1}_0 = \mathcal{T}^s$ gives a trivial principle fibre bundle structure to $(\mathcal{C}^s, \mathcal{T}^s, \pi)$.*

The two groups can be combined to form a semidirect product $\mathcal{D}^{s+1}_0 \ltimes \mathcal{P}^s$, which is called the *conformorphism group* in Fischer [**FM77**] denoted by $\mathcal{E}^s_0$. It acts on $\mathcal{M}^s$ by

$$\mathcal{E}^s_0 \times \mathcal{M}^s \to \mathcal{M}^s \quad ((f, u), g) \to e^u \cdot f^* g;$$

$$(f_1, u_1) \cdot (f_2, u_2) = (f_2 \circ f_1, e^{u_2 + (u_1 \circ f_2)}).$$

The quotient of the group action on $\mathcal{M}^s$ gives the Teichmüller space $\mathcal{T}^s$. This follows since $\mathcal{P}^s$ is a normal subgroup of $\mathcal{E}^s_0$ hence the two-step quotient $(\mathcal{M}^s/\mathcal{P}^s)/\mathcal{D}^s_0$ is isomorphic structure to $\mathcal{M}^s/\mathcal{E}^s_0$ [**FM77**]. In summary, we have the following diagram with two trivial fibre bundle structures

$$\begin{array}{ccc} \mathcal{M}^s & & \\ {\scriptstyle \mathcal{P}^s}\downarrow & \searrow^{\mathcal{E}_0^s} & \\ \mathcal{C}^s & \xrightarrow{\mathcal{D}_0^{s+1}} & \mathcal{T}^s \end{array} .$$

Given these two trivial bundle structures, we can formally write $\mathcal{M}^s = \mathcal{P}^s \times \mathcal{D}_0^{s+1} \times \mathcal{T}^s$. It means that for any given metric $g \in \mathcal{M}^s$, there exist elements in $u \in \mathcal{P}^s$, $f \in \mathcal{D}^s$ and $[\tau] \in \mathcal{T}^s$ such that $g = e^u f^*(\sigma([\tau])) \in \mathcal{M}^s$. Here we don't have a canonical choice for a section $\sigma : \mathcal{T}^s \to \mathcal{M}^s_{-1}$, although a global section exists since the bundle is trivial.

2.2.2. The space of shapes and its quotient. Motivated by the definition of the Teichmüller space, we define the space of shapes as follows.

Definition 2.2.4. Let S be a closed orientable connected surface. The *space of shapes* of S, or the *shape space*, denoted by $\mathcal{S}(S)$, is the space of equivalence classes of metrics on the surface S, where two metrics g_1 and g_2 are equivalent if there exists an isometry $f : (S, g_1) \to (S, g_2)$ isotopic to the identity.

This space is the quotient of $\mathcal{M}$ by the action of $\mathcal{D}_0$ as pullbacks. Alternatively, we can regard the elements in the shape space as equivalence classes of marked surfaces, denoted by (F_i, ϕ_i, g_i), where F_i is a surface with metric g_i diffeomorphic to S via a marking $\phi_i : F_i \to S$. Two marked Riemannian surfaces (F_1, ϕ_1, g_1) and (F_2, ϕ_2, g_2) are equivalent if there exists an isometry $f : (F_1, g_1) \to (F_2, g_2)$ so that $f \circ \phi_1$ is isotopic to ϕ_2.

We show that $\mathcal{S}^s$, the Hilbert manifold arising as the quotient manifold of the action by $\mathcal{D}_0^{s+1}$ on $\mathcal{M}^s$, has a principal bundle structure, which defines the differential structure on the shape space $\mathcal{S}^s$.

Theorem 2.2.5. *The action by $\mathcal{D}_0^{s+1}$ on the space $\mathcal{M}^s$ is smooth, free, and proper if the surface F has genus $g > 1$. Hence the quotient space $\mathcal{S}^s = \mathcal{M}^s/\mathcal{D}_0^{s+1}$ is a smooth Hilbert manifold, and the quotient map $\pi : \mathcal{M}^s \to \mathcal{M}^s/\mathcal{D}_0^{s+1} = \mathcal{S}^s$ is smooth. $(\mathcal{M}^s, \mathcal{S}^s, \pi)$ has the structure of a principle fibre bundle with structure group $\mathcal{D}_0^{s+1}$.*

Proof. The smoothness of the action of $\mathcal{D}^{s+1}$ on $\mathcal{M}^s$ was proved in detail by Ebin [**Ebi67**]. The properness of the action of $\mathcal{D}^{s+1}$ was given by Palais and Fischer (see, e.g. [**Tro12**]) using a straightforward computation, so the same holds for the action of its normal subgroup $\mathcal{D}_0^{s+1}$. Hence we only need to prove that the action is free. We need to show that if $f^*g = g$ and $\mathcal{D}_0^{s+1}$, then f has to be the identity.

We prove it with harmonic maps. By Coron and Helein [**CH89**], any smooth harmonic diffeomorphism between two compact Riemannian surfaces is a minimizer of the Dirichlet energy in its homotopy class, and it is unique if the genus is larger than 1. Hence for any metric g_0 on a surface S, if we have an isometry $f : (S, g_0) \to (S, g_0)$ isotopic to the identity, it has to be the identity by the uniqueness of harmonic maps, since the identity is a harmonic map. □

In the previous discussions, we consider all the spaces to be in the category of Hilbert manifolds $\mathcal{M}^s$, $\mathcal{S}^s$, $\mathcal{C}^s$, and $\mathcal{T}^s$ for sufficient large $s > 0$, to guarantee the continuity of metric tensors and their derivatives. By choosing the category of the *Inverse Limit Hilbert* structure, or ILH-structure, defined by Omori [**Omo70**], the results above also hold for ILH-Lie groups $\mathcal{P}$, $\mathcal{D}_0$, and spaces $\mathcal{M}$, $\mathcal{M}_{-1}$, $\mathcal{S}$, $\mathcal{T}$(see, e.g. [**FT84**]), so we will use this category in the rest of this paper. Notice that if the genus g of S is larger than one, the corresponding spaces $\mathcal{M}$ and $\mathcal{D}_0$ are contractible in this category, so the shape space $\mathcal{S}(S)$ is a contractible space, and the bundle structure $(\mathcal{M}, \mathcal{S}, \pi)$ is trivial.

Our next goal is to understand the connection between the space of shapes $\mathcal{S}$ and the Teichmüller space $\mathcal{T}$. There is a natural projection from $\mathcal{S}$ to $\mathcal{T}$. By the Uniformization Theorem, there exists a unique hyperbolic metric $\bar{g}$ in the conformal class of g. The identity map $id : (S, g) \to (S, \bar{g})$ is conformal, so we can define the following projection

$$j : \mathcal{S} \to \mathcal{T} \quad [g] \to [\bar{g}].$$

LEMMA 2.2.6. *The projection map $j : \mathcal{S} \to \mathcal{T}$ is well-defined and smooth.*

PROOF. Given g_1 and g_2 representing one equivalent class in $\mathcal{S}$ and their conformally equivalent hyperbolic metrics $\bar{g_1}$ and $\bar{g_2}$, we have an isometry $f : (S, g_1) \to (S, g_2)$ isotopic to the identity. It induces a conformal map from $(S, \bar{g_1})$ to $(S, \bar{g_2})$ by $\bar{f} = id \circ f \circ id^{-1}$, since id^{-1}, f and id are conformal. Then $\bar{f}$ has to be an isometry since conformal diffeomorphisms between hyperbolic surfaces are isometries. Hence $\bar{g_1}$ and $\bar{g_2}$ represent the same element in the Teichmüller space.

This projection can be constructed explicitly using the bundle structure of $\mathcal{S}$. Since the bundle structure of $\mathcal{M}$ over $\mathcal{S}$ is trivial, there exists a smooth global section $\sigma : \mathcal{S} \to \mathcal{M}$. We can compose this section with the two smooth projections from $\mathcal{M} \to \mathcal{M}_{-1}$ and $\mathcal{M}_{-1} \to \mathcal{T}$ to construct the projection j. $\square$

Unfortunately we can't take the quotient of $\mathcal{S}$ by the group action of $\mathcal{P}$ directly to construct a well-defined group action. This is due to the fact that a function $u \in \mathcal{P}$ has a fixed value at a fixed point while every element in $\mathcal{S}$ can be represented using different metrics, which achieve possibly different values at a fixed point. It can also be seen by the fact that $\mathcal{D}_0$ is not a normal subgroup of $\mathcal{E}_0$, hence $\mathcal{E}_0/(\mathcal{D}_0, 1)$ is not isomorphic to $\mathcal{P}$ as groups.

In summary, we have the following four spaces $\mathcal{M}$, $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{T}$ in a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathcal{D}_0} & \mathcal{S} \\ {\scriptstyle \mathcal{P}}\downarrow & \searrow^{\mathcal{E}_0} & \downarrow{\scriptstyle j} \\ \mathcal{C} & \xrightarrow{\mathcal{D}_0} & \mathcal{T} \end{array}.$$

The group action $\mathcal{D} \times \mathcal{M} \to \mathcal{M}$ is more subtle since certain metric tensors have non-trivial symmetries. For example, hyperbolic surfaces with genus g may have isometry groups with order up to $84(g-1)$(see, e.g. [**FM11**]).

The diagram above holds for surfaces S with $g > 1$. For the torus, its diffeomorphism group $\mathcal{D}_0$ could contain non-trivial isometries, so the action of $\mathcal{D}_0$ on $\mathcal{M}$ may not be free. By Earle and Eells [**EE$^+$69**], $\mathcal{D}_0$ is not contractible and has the same homotopy type as the torus, so the shape space $\mathcal{S}$ is not contractible. It does not fit in the picture for higher genus cases. Nevertheless, we define a metric structure on the shape space of a surface S, including the torus in the next section.

2.3. Metrics on the Space of Shapes on Surfaces

In this section, we define a distance function between two shapes in the shape space $\mathcal{S}$ of a closed orientable surface S of genus $g \geq 1$. We first discuss how to compare shapes using diffeomorphisms, then define a metric based on two energies defined for quasiconformal homeomorphisms on S.

2.3.1. Measurement of distortion. To compare two shapes, we find an "optimal" diffeomorphism between two shapes on a surface and measure its deviation from an isometry. In general, we can measure the distortion of $f : (S, g_1) \to (S, g_2)$ by the singular values of its differential, where the differential at a point p is

$$df_p : (T_pS, g_1) \to (T_{f(p)}S, g_2).$$

After choosing appropriate orthonormal basis in each metric, it can be expressed as a matrix

$$df_p = T = \begin{bmatrix} \lambda_1(p) & 0 \\ 0 & \lambda_2(p) \end{bmatrix}.$$

where $\lambda_1(p)$ and $\lambda_2(p)$ are the singular values of df_p as a linear transformation. The area distortion of f at p is measured by the Jacobian $J_f(p) = \lambda_1(p)\lambda_2(p)$. The ratio of the two singular values at $p \in S$ corresponds to the eccentricity of the ellipse in the tangent space at $f(p)$ shown in Figure 1. To measure the angle distortion of f, we define the *dilatation* of f at p to be $K_f(p) = \lambda_1(p)/\lambda_2(p)$, assuming $\lambda_1(p) \geq \lambda_2(p)$.

Notice that we can extend these definitions from diffeomorphisms on S to quasiconformal homeomorphisms on S. For a quasiconformal homeomorphism f from a region $\Omega \subset \mathbb{C}$ into $\mathbb{C}$, f_z and $f_{\bar{z}}$ are locally square-integrable, and f is differentiable almost everywhere. The Jacobian J_f is well-defined almost everywhere and locally integrable, and the essential supremum of K_f over the surface is bounded. Then we can show that both λ_1 and λ_2 are locally square-integrable, satisfying the following relations

$$\lambda_1(p) = \sqrt{J_f(p)K_f(p)} \quad \text{and} \quad \lambda_2(p) = \sqrt{\frac{J_f(p)}{K_f(p)}} \quad \forall p \in \Omega.$$

Since the Jacobian and dilatation of f are local quantities, we can construct charts on a surface to show that λ_1 and λ_2 are well-defined and locally square-integrable for quasiconformal homeomorphisms on the surface S.

Based on the two singular values λ_1 and λ_2, we can define energies of f measuring the angle distortion and the area distortion of f respectively.

Definition 2.3.1. The *area distortion energy* of a quasiconformal homeomorphism $f : (S, g_1) \to (S, g_2)$ is

$$E_1(f) = \sqrt{\int_S (1 - \sqrt{\lambda_1(p)\lambda_2(p)})^2 dA_{g_1}}.$$

The *angle distortion energy* of f is

$$E_2(f) = \frac{1}{2}|| \log \frac{\lambda_1(p)}{\lambda_2(p)} ||_\infty.$$

where $\lambda_1(p)$ and $\lambda_2(p)$ are singular values of f at $p \in S$, and $|| \cdot ||_\infty$ is the essential supremum norm on the functions on S.

Note that if f is a pointwise area-preserving, then $E_1(f) = 0$. If f is conformal, $E_2(f) = 0$. Both of them are zero if and only if f is an isometry.

2.3.2. Metric Structure for Genus Zero Surfaces. Hass and Koehl [**HK17**] introduced a metric structure for smooth genus-zero surfaces from the intrinsic point of view. By the Uniformization Theorem in Section 2, any two metrics g_1 and g_2 on S^2 are conformally equivalent. There exists a conformal diffeomorphism $f : (S^2, g_1) \to (S^2, g_2)$ with a positive function λ_f, called the *conformal factor*, such that

$$f^*(g_2) = \lambda_f g_1 \quad \text{or} \quad g_2(f^*(v_1), f^*(v_2))_{f(p)} = \lambda_f(p) g_1(v_1, v_2)_p$$

where $v_1, v_2 \in T_p S^2$ for all $p \in S^2$. In looking for an energy minimizing map, we can restrict to the group of conformal diffeormophisms of the round 2-sphere, which coincides with the group of Mobius transformations isomorphic to $PSL(2, \mathbb{C})$. If we choose an appropriate orthonormal basis in the tangent space for each metric, the differential of a conformal diffeomorphism has a simple expression [**HK17**]

$$df_p = \begin{bmatrix} \lambda_f & 0 \\ 0 & \lambda_f \end{bmatrix}.$$

For conformal maps we have $E_2 = 0$ and E_1 simplifies to

$$E_1(f) = \sqrt{\int_{S^2} (1 - \lambda_f)^2 dA_{g_1}}.$$

This idea leads to the definition of a metric on the space of shapes of S^2 as

$$d((S^2, g_1), (S^2, g_2)) = \inf\{E_1(f) | f : (S^2, g_1) \to (S^2, g_2) \text{ a conformal diffeomorphism}\}.$$

In [**HK17**], Hass and Koehl showed this function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ gave a metric, and the infimum was achieved by a conformal diffeomorphism. In their framework, the given two surfaces are mapped to the round 2-sphere by conformal maps c_1 and c_2. They found an optimal conformal diffeomorphism $c_2^{-1} \circ m \circ c_1$ between the two surfaces by minimizing the symmetric distortion energy among the group of Mobius transformations. They proposed an algorithm to compute the distance between two triangulated surfaces and applied it to describe shapes of proteins and generate evolutionary trees of species [**HK14, KH13, KH15**].

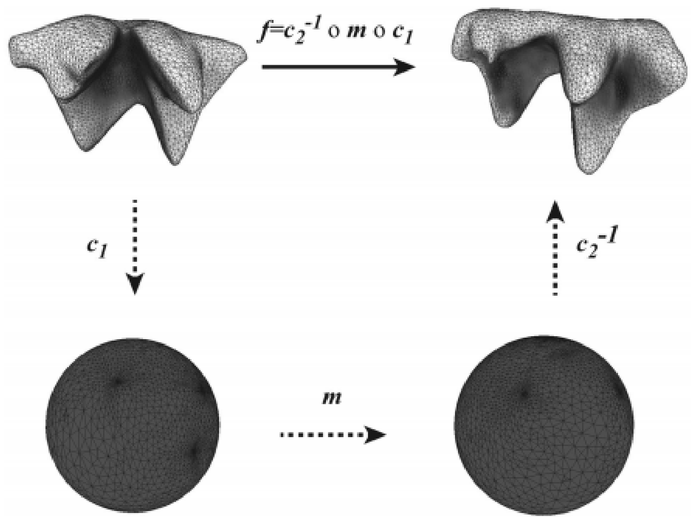


FIGURE 2.3. A framework to compare genus-zero surfaces from [**HK17**]

2.3.3. Metric Structure for High Genus Surfaces. There is a fundamental difference between the shape space of genus-zero surfaces S^2 and that of higher genus surfaces S. Any two shapes on the 2-sphere are conformal, while two shapes on a high genus surface are not necessarily conformally equivalent. We define a distance between two shapes by minimizing the sum of the energies E_1 and E_2 over the quasiconformal homeomorphisms of S isotopic to the identity. Setting $E(f) = E_1(f) + E_2(f)$, we define a distance function as follows.

DEFINITION 2.3.2. Let S be a closed connected orientable surface of genus $g \geq 1$ and $\mathcal{S}(S)$ be the shape space of S. Then we define a function $d : \mathcal{S}(S) \times \mathcal{S}(S) \to \mathbb{R}$ between two shapes in $\mathcal{S}(S)$ represented by (S, g_1) and (S, g_2) to be

$$d((S,g_1),(S,g_2)) = \inf_{f \in \mathcal{Q}_0} (E(f)) = \inf_{f \in \mathcal{Q}_0} \left(\sqrt{\int_S (1 - \sqrt{\lambda_1 \lambda_2})^2 dA_{g_1}} + \frac{1}{2} || \log \frac{\lambda_1}{\lambda_2} ||_\infty \right)$$

where $\mathcal{Q}_0$ is the space of quasiconformal homeomorphisms from (S, g_1) to (S, g_2) isotopic to the identity.

Equivalently, we can use marked surfaces to define this metric on the shape space $\mathcal{S}(S)$. Let (F_1, ϕ_1, g_1) and (F_2, ϕ_2, g_2) represent two different shapes of S, then

$$d((F_1, \phi_1, g_1), (F_2, \phi_2, g_2)) = \inf_{f \in \mathcal{Q}} (E(f))$$

where $\mathcal{Q}$ is the set of quasiconformal homeomorphisms from (F_1, g_1) to (F_2, g_2) isotopic to $\phi_2 \circ \phi_1^{-1}$.

THEOREM 2.3.3. *Let S be a closed orientable connected surface of genus $g \geq 1$. The function d induces a metric on the space of shapes $\mathcal{S}(S)$.*

PROOF. To show the function d is a metric, we need to check that for any three metrics (S, g_1), (S, g_2), and (S, g_3), we have

(1) $d((S, g_1), (S, g_2)) \geq 0$;
(2) $d((S, g_1), (S, g_2)) = 0$ if and only if g_1 and g_2 are isometric by a diffeomorphism isotopic to the identity;
(3) $d((S, g_1), (S, g_2)) = d((S, g_2), (S, g_1))$;
(4) $d((S, g_1), (S, g_3)) \leq d((S, g_1), (S, g_2)) + d((S, g_3), (S, g_2))$.

The first two properties are immediate from the definition. If $d((S, g_1), (S, g_2)) = 0$, then there exists an isometry isotopic to the identity, which means that g_1 and g_2 represent the same equivalence class in $\mathcal{S}(S)$.

The symmetry property follows from $E_1(f) = E_1(f^{-1})$ and $E_2(f) = E_2(f^{-1})$. By a similar computation in [**HK17**], we have

$$
\begin{aligned}
E_1(f^{-1}) &= \sqrt{\int_S (1 - \sqrt{\frac{1}{\lambda_1\lambda_2}})^2 dA_{g_2}} = \sqrt{\int_S (1 - \sqrt{\frac{1}{\lambda_1\lambda_2}})^2 \lambda_1\lambda_2 dA_{g_1}} \\
&= \sqrt{\int_S (1 - \sqrt{\lambda_1\lambda_2})^2 dA_{g_1}} = E_1(f).
\end{aligned}
$$

The singular values of f^{-1} are $1/\lambda_1$ and $1/\lambda_2$, so the symmetry of E_2 is immediate.

To show the triangle inequality, set $f : (S, g_1) \to (S, g_2)$ and $g : (S, g_2) \to (S, g_3)$, and we show that

$$E_1(g \circ f) \leq E_1(g) + E_1(f).$$

Let the singular values of f, g, and $g \circ f$ be λ_1 and λ_2, μ_1 and μ_2, σ_1 and σ_2 respectively. Then by a similar computation in [**HK17**], we have

$$
\begin{aligned}
(E_1(g) + E_1(f))^2 = &\int_S (1 - \sqrt{\lambda_1\lambda_2})^2 dA_{g_1} + \int_S (1 - \sqrt{\mu_1\mu_2})^2 dA_{g_2} \\
&+ 2\sqrt{\int_S (1 - \sqrt{\lambda_1\lambda_2})^2 dA_{g_1} \int_S (1 - \sqrt{\mu_1\mu_2})^2 dA_{g_2}}.
\end{aligned}
$$

Notice that $dA_{g_2} = \lambda_1\lambda_2 dA_{g_1}$, then by the Cauchy-Schwarz inequality, we have

$$\sqrt{\int_S (1-\sqrt{\lambda_1\lambda_2})^2 dA_{g_1} \int_S (1-\sqrt{\mu_1\mu_2})^2 dA_{g_2}} = \sqrt{\int_S (1-\sqrt{\lambda_1\lambda_2})^2 dA_{g_1} \int_S (1-\sqrt{\mu_1\mu_2})^2 \lambda_1\lambda_2 dA_{g_1}}$$
$$\geq \int_S (1-\sqrt{\lambda_1\lambda_2})(1-\sqrt{\mu_1\mu_2})\sqrt{\lambda_1\lambda_2} dA_{g_1}.$$

Hence

$$\begin{aligned}(E_1(g)+E_1(f))^2 &\geq \int_S (1-\sqrt{\lambda_1\lambda_2})^2 + (1-\sqrt{\mu_1\mu_2})^2\lambda_1\lambda_2 \\ &+ 2(1-\sqrt{\lambda_1\lambda_2})(1-\sqrt{\mu_1\mu_2})\sqrt{\lambda_1\lambda_2} dA_{g_1} \\ &= \int_S ((1-\sqrt{\lambda_1\lambda_2}) + \sqrt{\lambda_1\lambda_2}(1-\sqrt{\mu_1\mu_2}))^2 dA_{g_1} \\ &= \int_S (1-\sqrt{\lambda_1\lambda_2\mu_1\mu_2})^2 dA_{g_1}.\end{aligned}$$

Since $\sigma_1\sigma_2 = J_{g\circ f} = J_f J_g = \lambda_1\lambda_2\mu_1\mu_2$, it follows that

$$(E_1(g)+E_1(f))^2 \geq (E_1(g\circ f))^2.$$

To prove the second part of the inequality, namely $E_2(g\circ f) \leq E_2(g) + E_2(f)$, we assume that $\lambda_1 \geq \lambda_2$, $\mu_1 \geq \mu_2$, and $\sigma_1 \geq \sigma_2$ for simplicity. Notice that the larger singular value is the 2-norm for the differential df_p, and the smaller singular value is the reciprocal of the 2-norm of the inverse of df_p. The larger singular value of the composition $g\circ f$ is bounded by

$$\sigma_1(p) = ||d(g\circ f)_p||_2 = ||dg_{f(p)} \circ df_p||_2 \leq ||df_p||_2 ||dg_{f(p)}||_2 = \lambda_1(p)\mu_1(p).$$

Similarly for the inverse, we have

$$\frac{1}{\sigma_2(p)} = ||d(f\circ g)_p^{-1}||_2 = ||df_p^{-1} \circ dg_{f(p)}^{-1}||_2 \leq ||df_p^{-1}||_2 ||dg_{f(p)}^{-1}||_2 = \frac{1}{\lambda_2(p)\mu_2(p)}.$$

Hence we have

$$0 < \lambda_2\mu_2 \leq \sigma_2 \leq \sigma_1 \leq \lambda_1\mu_1.$$

Therefore

$$E_2(f) + E_2(g) = \frac{1}{2}||\log\frac{\lambda_1}{\lambda_2}||_\infty + \frac{1}{2}||\log\frac{\mu_1}{\mu_2}||_\infty \geq \frac{1}{2}||\log\frac{\lambda_1}{\lambda_2} + \log\frac{\mu_1}{\mu_2}||_\infty$$
$$= \frac{1}{2}||\log\frac{\lambda_1\mu_1}{\lambda_2\mu_2}||_\infty \geq \frac{1}{2}||\log\frac{\sigma_1}{\sigma_2}||_\infty = E_2(g\circ f).$$

Therefore we show that

$$E(f) + E(g) \geq E(g \circ f).$$

To pass to the infimum, we choose $f_n : (S, g_1) \to (S, g_2)$ and $g_n : (S, g_2) \to (S, g_3)$ in $\mathcal{Q}_0$ such that

$$\lim_{n\to\infty} E(f_n) = d((S, g_1), (S, g_2)) \text{ and } \lim_{n\to\infty} E(g_n) = d((S, g_2), (S, g_3)).$$

Then we have

$$E(f_n) + E(g_n) \geq E(g_n \circ f_n) \geq d((S, g_1), (S, g_3)).$$

Taking the limit as $n \to \infty$ we have

$$d((S, g_1), (S, g_2)) + d((S, g_2), (S, g_3)) \geq d((S, g_1), (S, g_3)).$$

The last thing to check is that the metric d is well-defined on the shape space. Assume g_1 and $\tilde{g_1}$ represent the same shape, and g_2 and $\tilde{g_2}$ represent another shape. Then we have an isometry $i_1 : (S, g_1) \to (S, \tilde{g_1})$ isotopic to the identity and another isometry $i_2 : (S, g_2) \to (S, \tilde{g_2})$ isotopic to the identity. Given $f : (S, g_1) \to (S, g_2)$, consider the map $\tilde{f} : (S, \tilde{g_1}) \to (S, \tilde{g_2})$ defined as

$$\tilde{f} = i_2 \circ f \circ i_1^{-1}.$$

Since i_1 and i_2 are isometries, they will not change the singular values, so the singular values of $\tilde{f}$ are given by $\tilde{\lambda}_1(p) = \lambda_1(i_1^{-1}(p))$ and $\tilde{\lambda}_2(p) = \lambda_2(i_1^{-1}(p))$. An isometry also preserves the area, so $dA_{\tilde{g_1}} = dA_{g_1}$. Hence we have

$$E_1(\tilde{f}) = \sqrt{\int_S (1 - \sqrt{\tilde{\lambda_1}(p)\tilde{\lambda_2}(p)})^2 dA_{\tilde{g_1}}} = \sqrt{\int_S (1 - \sqrt{\lambda_1(i_1^{-1}(p))\lambda_2(i_1^{-1}(p))})^2 dA_{g_1}} = E_1(f).$$

and

$$E_2(\tilde{f}) = \frac{1}{2}||\log\frac{\tilde{\lambda_1}}{\tilde{\lambda_2}}||_\infty = \frac{1}{2}||\log\frac{\lambda_1}{\lambda_2}||_\infty = E_2(f).$$

Hence we have

$$E(\tilde{f}) = E(f).$$

Since i_1 and i_2 are isotopic to the identity, $f \in \mathcal{Q}_0$ if and only if $\tilde{f} \in \mathcal{Q}_0$. Taking the infimum over $f \in \mathcal{Q}_0$, we conclude that

$$d((S, g_1), (S, g_2)) = d((S, \tilde{g_1}), (S, \tilde{g_2})).$$

Hence d is a well-defined metric on $\mathcal{S}$. □

Notice that if we restrict the metric to $\mathcal{T}$, then d will be the Teichmüller metric.

Our next goal is to show that the distance between two shapes is realized by a homeomorphism of the surface S. In general, a sequence of homeomorphisms f_n of a surface may converge to a singular map, such as a constant map. We show that singular maps will not occur for the limit of an energy-minimizing sequence.

Given two compact hyperbolic surfaces $(S, \bar{g}_1)$ and $(S, \bar{g}_2)$, all the K-quasiconformal homeomorphisms between them are D-quasi-isometries, with the constant of distortion D depending only on K(see, e.g. [**FM07**]). Hence these K-quasiconformal homeomorphisms are equicontinuous. The following lemma shows that this result also holds for K-quasiconformal homeomorphisms between two flat tori. In the proof of this lemma, we use the extremal length of curve families in the annulus (see e.g. [**FM07**]).

LEMMA 2.3.4. *Let $f_n : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_2)$ be a family of K-quasiconformal homeomorphisms between two flat tori with unit area. Then the maps f_n are equicontinuous.*

PROOF. Let J be the injective radius of $(\mathbb{T}^2, g_1)$, and $d_{g_i}(x, y)$ denote the distance between x and y in the metric g_i, where $i = 1, 2$. Then for any $0 < r < J$, if $d_{g_1}(x, y) < r$, then there exists an embedded annulus A in $(\mathbb{T}^2, g_1)$ centered at the midpoint of x and y, whose inner radius is $r/2$ and outer radius is $J/2$. Moreover, it separates $\mathbb{T}^2$ into two components, one of which is a flat disk with radius $r/2$ containing x and y.

Lift A isometrically to a flat annulus $\tilde{A}$ in the universal covering $\mathbb{R}^2$, and lift x and y to $\tilde{x}$ and $\tilde{y}$ contained in the disk bounded by the inner boundary of $\tilde{A}$. We consider the extremal length $\lambda(\Gamma)$ of the family of curves Γ in $\tilde{A}$ that separate the two boundary circles of $\tilde{A}$, with the curves not leaving $\tilde{A}$. Then we have (see e.g. [**FM07**])

$$\lambda(\Gamma) = \frac{2\pi}{\log(J/r)}.$$

We also lift f_n to K-quasiconformal homeomorhisms $\tilde{f}_n : \mathbb{R}^2 \to \mathbb{R}^2$. Then by the property of K-quasiconformal homeomorphisms, $\tilde{f}_n(A)$ are annulus, and if $\Gamma_1^n = \tilde{f}_n(\Gamma)$, then the curves in Γ_1^n are contained in $\tilde{f}_n(A)$ with their extremal length bounded by

$$\lambda(\Gamma_1^n) \leq K\lambda(\Gamma).$$

By the definition of the extremal length $\lambda(\Gamma)$, notice that the area of $\tilde{f}_n(A)$ is less than one in the Euclidean metric on $\mathbb{R}^2$, so

$$\lambda(\Gamma_1^n) \geq L^2 \geq 4d^2(\tilde{f}_n(\tilde{x}), \tilde{f}_n(\tilde{y})) = 4d_{g_2}^2(f_n(x), f_n(y)) \quad \forall n,$$

where L is the length of the inner boundary curve of $\tilde{f}_n(A)$. The second inequality holds because $\tilde{f}_n$ is a homeomorphism so that $\tilde{f}_n(\tilde{x})$ and $\tilde{f}_n(\tilde{y})$ are in the disk bounded by the inner boundary curve of $\tilde{f}_n(A)$, and the last equality holds because there exists an isometric project from $\mathbb{R}^2$ to $(\mathbb{T}^2, g_2)$. Combining above relations, we have

$$d_{g_2}(f_n(x), f_n(y)) \leq \sqrt{\frac{\pi K}{2 \log \frac{J}{r}}} \quad \forall n.$$

Notice that $d_{g_2}(f(x), f(y)) \to 0$ if $r \to 0$. Hence for any $\epsilon > 0$, there exists $r > 0$ such that if $d_{g_1}(x, y) < r$, then $d_{g_2}(f(x), f(y)) < \epsilon$. Notice that r doesn't depend on n, hence the maps f_n are equicontinuous.

□

THEOREM 2.3.5. *Assume S has genus $g \geq 1$. Given two metrics (S, g_1), (S, g_2) representing two shapes in $\mathcal{S}(S)$, and an energy-minimizing sequence $f_n \in \mathcal{Q}_0(S)$ such that $E(f_n) \to d((S, g_1), (S, g_2))$ as $n \to \infty$, there is a subsequence of f_n converging to a quasiconformal homeomorphism f such that $E(f) = d((S, g_1), (S, g_2))$.*

PROOF. Since $E(f_n) \to d((S, g_1), (S, g_2))$, we assume that $E(f_n) < K$ for some $K > 0$. Then the maps f_n are K-quasiconformal homeomorphisms on a compact surface S. Then the maps f_n are equicontinuous and bounded with respect to the corresponding metrics $\bar{g}_1$ and $\bar{g}_2$ of constant curvature, and if S is the torus, we normalize $\bar{g}_1$ and $\bar{g}_2$ to be metrics with unit area. By Arzela-Ascoli, there exists a subsequence converging uniformly to a continuous map f. To show f is a homeomorphism, notice that the inverses of the maps f_n are also D-quasi-isometries where D does not depend on n. Then the equicontinuity of inverses of f_n implies that if $f_n^{-1}(x) = a$ and

$f_n^{-1}(y) = b$, then

$$d_{\bar{g}_1}(a,b) = d_{\bar{g}_1}(f_n^{-1}(x), f_n^{-1}(y)) \le C(K) d_{\bar{g}_2}(x,y) = C(K) d_{\bar{g}_2}(f_n(a), f_n(b))$$

where $d_{\bar{g}_i}(x,y)$ denotes the distance between x and y in the metric $\bar{g}_i$ for $i = 1,2$. Taking the limit $n \to \infty$, we conclude that f is injective. Then f is a continuous injection from a compact 2-manifold to a connected 2-manifold, so it is a homeomorphism by the properness of f and the theorem of invariance of domain. (see e.g. [**Lee10**]).

Replace f_n by a convergent subsequence and we have $f_n \to f$ uniformly where f is a homeomorphism. For the limit map f, notice that its energy is given by

$$E(f) = \sqrt{\int_S (1 - \sqrt{J_f})^2 dA_{g_1}} + \frac{1}{2} \log K_f.$$

where J_f is the Jacobian of f and K_f is the maximal dilatation of f. The lower semicontinuity property of the maximal dilatations for quasiconformal maps [**BGMR13**] gives

$$K_f \le \liminf_{n\to\infty} K_{f_n}.$$

Next we will to show that taking a further subsequence of f_n, we have

$$\int_F (1 - \sqrt{J_f})^2 dA_{g_1} = \lim_{n\to\infty} \int_F (1 - \sqrt{J_{f_n}})^2 dA_{g_1}.$$

Notice that this term has the following decomposition:

$$\begin{aligned} \int_F (1 - \sqrt{J_{f_n}})^2 dA_{g_1} &= \int_F dA_{g_1} + \int_F J_{f_n} dA_{g_1} - 2 \int_F \sqrt{J_{f_n}} dA_{g_1} \\ &= Area((F, g_1)) + Area((F, g_2)) - 2 \int_F \sqrt{J_{f_n}} dA_{g_1}. \end{aligned}$$

where $Area((F,g))$ is the area of the surface F with metric g. Similarly we have

$$\int_F (1 - \sqrt{J_f})^2 dA_{g_1} = Area((F, g_1)) + Area((F, g_2)) - 2 \int_F \sqrt{J_f} dA_{g_1}.$$

Consider $\sqrt{J_{f_n}}$ as functions in the function space on (F, g_1) with L^2 norm. The Area of (F, g_2) gives a uniform bound on their norms:

$$\lim_{n\to\infty} \int_F (\sqrt{J_{f_n}})^2 dA_{g_1} = Area((F, g_2)) = \int_F (\sqrt{J_f})^2 dA_{g_1}.$$

The unit closed ball in the function space on (F, g_1) with L^2 norm is weakly sequentially compact, so we have a subsequence of f_n, denoted again by f_n, such that $\sqrt{J_{f_n}}$ converges weakly to $\sqrt{J_f}$. Since (F, g_1) is compact, constant functions are in the function space on (F, g_1) with L^2 norm, hence

$$\lim_{n\to\infty} \int_F \sqrt{J_{f_n}} \cdot 1 dA_{g_1} = \int_F \sqrt{J_f} \cdot 1 dA_{g_1}.$$

Thus, we have

$$E(f) \le \liminf_{n\to\infty} E(f_n) = d((F, g_1), (F, g_2)).$$

Since f is a quasiconformal homeomorphsim, $E(f) \ge d((F, g_1), (F, g_2))$, hence

$$E(f) = d((F, g_1), (F, g_2)).$$

□

We now consider the uniqueness of the energy minimizing map in each homotopy class. In the special case where both surfaces (S, g_1) and (S, g_2) are flat tori with unit area, the minimizers are given by affine maps, because affine maps coincide with Teichmüller maps on flat tori with unit area, and the Jacobians of affine maps are constant. This forces the Jacobians to be the constant map$J \equiv 1$ on S. If we fix one point p on S, then there is a unique affine map fixing p realizing the infimum of the energy. It is not clear whether the minimizer is unique between two general surfaces.

2.4. Energies for maps between shapes

The area distortion energy E_1 is a generalization from the symmetric distortion energy for the 2-sphere in [**HK17**]. It induces a metric when restricted to a conformal class of metrics. The energy E_2 is the Teichmüller metric. Ideally, we want to find a variational framework to compute the optimal diffeomorphism, namely a convex energy for maps between surfaces with a unique minimizer in a given homotopy class of maps. Unfortunately, we don't know whether the energy E is convex, and whether its minimizer is unique in each homotopy class. So we try to construct new metrics on the shape space by considering more energies of maps between surfaces to define metrics on conformal classes of metrics and the Teichmüller space.

The Dirichlet energy is a candidate for a metric, because the critical point of the Dirichlet energy in each homotopy class of diffeomorphisms on S_g exists and is achieved by a unique diffeomorphism when $g > 1$. We will use the Dirichlet energy to define new energies of maps between surfaces, and compute various energies of maps between flat tori.

2.4.1. Energies for metrics on the Teichmüller space. There exist other metrics on the Teichmüller space, including the Teichmüller metric, the Weil-Petersson metric, and Thurston's asymmetric metrics. However, they are not easy to compute in practice. Several energies related to Dirichlet energy can be regarded as a measurement of the conformal distortion of a map f between two surfaces (S, g_1) and (S, g_2). For example, the energy $E_c(f)$ for a map $f : (S, g_1) \to (S, g_2)$ was defined in [**AL15**, **PP93**] to be

$$E_c(f) = E_D(f) - A$$

where and A is the area of the target surface. A similar energy $E_C(f)$ is defined to be

$$E_C = \log \frac{E_D(f)}{A}.$$

Notice that if f is a conformal map, $E_D(f) = A$ so $E_c(f) = E_C(f) = 0$.

We can represent these two energies using singular values λ_1 and λ_2 of the differential df of the map $f : (S, g_1) \to (S, g_2)$ as

$$E_c(f) = E_D - A = \frac{1}{2}\int_F \lambda_1^2 + \lambda_2^2 dA_{g_1} - \int_F \lambda_1\lambda_2 dA_{g_1} = \frac{1}{2}\int_S (\lambda_1 - \lambda_2)^2 dA_{g_1};$$

$$E_C(f) = \log \frac{\int_S \lambda_1^2 + \lambda_2^2 dA_{g_1}}{2\int_S \lambda_1\lambda_2 dA_{g_1}}.$$

These two energies both characterize the deviation of f from a conformal map. However, they don't induce metrics. We show this with the following counterexamples using three flat tori with unit area.

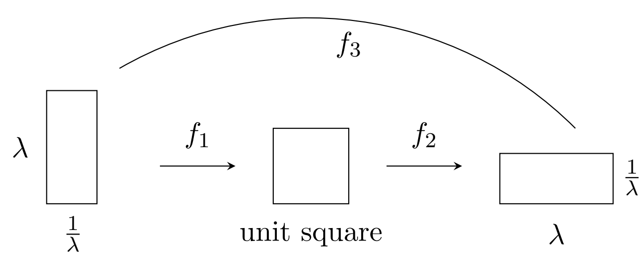


Figure 2.4. Three flat tori represented by fundamental domains

LEMMA 2.4.1. *The energies E_C and E_c do not define a metric on the Teichmüller space of flat tori with unit area.*

PROOF. We show that for each of these two energies, the triangle inequality does not hold. The Dirichlet energy of an affine map between two flat tori is given by

$$E_D(f) = \frac{1}{2}\int_S \lambda_1^2 + \lambda_2^2 dA = \frac{1}{2}(\lambda_1^2+\lambda_2^2)A = \frac{\lambda_1^2+\lambda_2^2}{2\lambda_1\lambda_2}\lambda_1\lambda_2 A = \frac{1}{2}(K+\frac{1}{K})\bar{A}$$

where λ_1 and λ_2 are the two singular values corresponding to the affine map, A and $\bar{A} = \lambda_1\lambda_2 A$ are the areas of the domain and the target surfaces, and K is the dilatation of the affine map. A similar computation can be found in [**LGY15**].

For any $\lambda \geq 1$, we have three flat tori defined in Figure 3 and three affine maps $f_1 : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_2)$, $f_2 : (\mathbb{T}^2, g_2) \to (\mathbb{T}^2, g_3)$ and $f_3 : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_3)$. The singular values for both f_1 and f_2 are λ and $1/\lambda$, and the singular values for f_3 are λ^2 and $1/\lambda^2$. We have

$$2E_c(f_1) + 2E_c(f_2) = 2(\lambda - \frac{1}{\lambda})^2 - 2 \leq (\lambda^2 - \frac{1}{\lambda^2})^2 - 1 = 2E_c(f_3).$$

This means the triangle inequality can't hold locally in the Teichmüller space for E_c. Similarly for E_C we have

$$E_C(f_3) = \log\frac{1}{2}(\lambda^4 + \frac{1}{\lambda^4}) \geq 2\log\frac{1}{2}(\lambda^2 + \frac{1}{\lambda^2}) = E_C(f_1) + E_C(f_2)$$

with equality holding if and only if $\lambda = 1$. □

Inspired by the discussion above, we can modify E_C to construct a metric on the Teichmüller space of flat tori.

LEMMA 2.4.2. *For any two flat metrics on a torus $(\mathbb{T}^2, g_1)$ and $(\mathbb{T}^2, g_2)$ representing two elements in the Teichmüller space of flat tori with unit area, define*

$$d((\mathbb{T}, g_1), (\mathbb{T}, g_2)) = \inf_{f\in\mathcal{D}_0} \sqrt{\log E_D(f)}$$

where $\mathcal{D}_0$ is the diffeomorphism of $\mathbb{T}^2$ isotopic to the identity. Then d is a metric on the Teichmüller space of flat tori with unit area.

PROOF. By the property of the Dirichlet energy, $E_D(f) \geq 1$ so $d(g_1, g_2) \geq 0$ and the equality holds if and only if two flat metrics g_1 and g_2 are conformal to each other, hence $d(g_1, g_2) = 0$ implies that g_1 and g_2 represent the same shape. The minimizer of E_D is given by an affine map, so we have the explicit formula

$$d(g_1, g_2) = \inf_{f \in \mathcal{D}_0} \sqrt{\log E_D(f)} = \sqrt{\log \frac{1}{2}(K + \frac{1}{K})}$$

where K is the dilatation of the affine map. Notice that the inverse of the affine map has the same dilatation, so $d(g_1, g_2) = d(g_2, g_1)$.

For the triangle inequality, if we have three flat metrics g_1, g_2 and g_3 with the affine maps $f_1 : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_2)$, $f_2 : (\mathbb{T}^2, g_2) \to (\mathbb{T}^2, g_3)$ and $f_3 : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_3)$, we know that $K_{f_3} \leq K_{f_2} K_{f_1}$. Hence we only need to check the following inequality for $x = K_{f_1} \geq 1$ and $y = K_{f_2} \geq 1$.

$$\begin{aligned} d(g_1, g_3) &= \sqrt{\log \frac{1}{2}(K_{f_3} + \frac{1}{K_{f_3}})} \leq \sqrt{\log \frac{1}{2}(K_{f_2} K_{f_1} + \frac{1}{K_{f_2} K_{f_1}})} = \sqrt{\log \frac{1}{2}(xy + \frac{1}{xy})} \\ &\leq \sqrt{\log \frac{1}{2}(x + \frac{1}{x})} + \sqrt{\log \frac{1}{2}(y + \frac{1}{y})} = \sqrt{\log \frac{1}{2}(K_{f_2} + \frac{1}{K_{f_2}})} + \sqrt{\log \frac{1}{2}(K_{f_1} + \frac{1}{K_{f_1}})} \\ &= d(g_1, g_2) + d(g_2, g_3). \end{aligned}$$

We need to show that the inequality below holds for $x \geq 1$ and $y \geq 1$.

$$\sqrt{\log \frac{1}{2}(xy + \frac{1}{xy})} \leq \sqrt{\log \frac{1}{2}(x + \frac{1}{x})} + \sqrt{\log \frac{1}{2}(y + \frac{1}{y})}.$$

By taking square on both sides we can show that the inequality is equivalent to

$$\log^2 \frac{2(xy + \frac{1}{xy})}{(x + \frac{1}{x})(y + \frac{1}{y})} \leq 4 \log \frac{x + \frac{1}{x}}{2} \log \frac{y + \frac{1}{y}}{2}.$$

Applying the inequality $x/(x+1) \leq \log(1+x) \leq x$ if $x \geq 0$, we deduce that

$$\log \frac{2(xy + \frac{1}{xy})}{(x + \frac{1}{x})(y + \frac{1}{y})} \leq \frac{(x - \frac{1}{x})(y - \frac{1}{y})}{(x + \frac{1}{x})(y + \frac{1}{y})};$$

$$\frac{x + \frac{1}{x} - 2}{x + \frac{1}{x}} \leq \log \frac{x + \frac{1}{x}}{2}; \quad \frac{y + \frac{1}{y} - 2}{y + \frac{1}{y}} \leq \log \frac{y + \frac{1}{y}}{2}.$$

This means that we need to check the inequality

$$\frac{(x - \frac{1}{x})^2 (y - \frac{1}{y})^2}{(x + \frac{1}{x})^2 (y + \frac{1}{y})^2} \leq 4 \frac{x + \frac{1}{x} - 2}{x + \frac{1}{x}} \frac{y + \frac{1}{y} - 2}{y + \frac{1}{y}}.$$

By cancellation and expansion of this inequality, we can see that it holds for $x \geq 1$ and $y \geq 1$. Using a similar argument as Theorem 4.3 , we can show that d is well-defined and satisfies the triangle inequality, hence d is a metric.

□

The Teichmüller metric on the Teichmüller space of flat tori with unit area is $1/2 \log K$, which coincides with the hyperbolic metric on the upper half plane model. Comparing to the Teichmüller metric, the metric defined above is more sensitive to small deformations of metrics, meaning that when K is close to one,

$$\sqrt{\log \frac{1}{2}(K + \frac{1}{K})} - \frac{1}{2} \log K > 0.$$

On the other hand, if K is large, the distance is asymptotically the same as the square root of the Teichmüller metric up to a constant multiple. The metric defined above can be regarded as a composition of the Teichmüller metric with certain function. It is not clear whether we can combine the energy E_1 with this metric to define a metric on the shape space of the torus

$$E((\mathbb{T}, g_1), (\mathbb{T}, g_2)) = \inf_{f \in \mathcal{Q}_0} \sqrt{\log \frac{E_D(f)}{A}} + \sqrt{\int_T (1 - \sqrt{\lambda_1 \lambda_2})^2 dA_{g_1}}.$$

2.4.2. Energies for metrics on a conformal class. Hass and Koehl [**HK17**] proposed a variation for the symmetric distortion energy for a surface S. For any $p \geq 1$ and a conformal map $f : (S, g_1) \to (S, g_2)$, assume the conformal factor of this map is $\lambda = e^u$ where u is a function on S, and define the L^p *energy* between the two metrics to be

$$E_{sd_p}((S, g_1), (S, g_2)) = E_p(f) + E_p(f^{-1}) = \sqrt[p]{\int_S |u|^p dA_{g_1}} + \sqrt[p]{\int_S |u|^p dA_{g_2}}.$$

LEMMA 2.4.3. *For any $1 < p < \infty$, the L^p energy between two metrics does not define a metric on a conformal class of metrics on the torus.*

PROOF. We can consider the three affine maps $f_1 : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_2)$, $f_2 : (\mathbb{T}^2, g_2) \to (\mathbb{T}^2, g_3)$, and $f_3 : (\mathbb{T}^2, g_1) \to (\mathbb{T}^2, g_3)$, with $g_2 = \lambda g_1$ and $g_3 = \lambda g_2$ where $\lambda > 0$ is a constant. Then $A_2 = \lambda^2 A_1$, $A_3 = \lambda^4 A_1$, and we have

$$\begin{aligned} E_{sd_p}(g_1, g_2) + E_{sd_p}(g_2, g_3) &= |\log \lambda| \sqrt[p]{A_1} + |\log \lambda| \sqrt[p]{A_2} + |\log \lambda| \sqrt[p]{A_2} + |\log \lambda| \sqrt[p]{A_3} \\ &= \sqrt[p]{A_1}(1 + \lambda^{\frac{2}{p}})^2 |\log \lambda|; \end{aligned}$$

$$E_{sq_p}(g_1, g_3) = |\log \lambda^2| \sqrt[p]{A_1} + |\log \lambda^2| \sqrt[p]{A_3} = \sqrt[p]{A_1}(2 + 2\lambda^{\frac{4}{p}})|\log \lambda|$$
$$\geq E_{sd_p}(g_1, g_2) + E_{sd_p}(g_2, g_3).$$

So we don't have the triangle inequality. □

2.5. Algorithms to Compute Maps and the Distance

In this section, we first summarize related works about various computational methods for maps between surfaces, including conformal maps, harmonic maps and Teichmüller maps. We assume that we have a closed connected orientable surface S, a genus-zero surface S^2 or high-genus surface S_g with genus $g \geq 1$. One comprehensive survey about surface parametrization using these maps can be found in Floater and Hormann [**FH05**]. Then, we propose an algorithm to compute the distance between two shapes of high genus surfaces.

2.5.1. Discretization and Computation of Maps. One of the natural ideas to compute these maps reduces to solving partial differential equations using finite elements methods. This method usually gives fast and robust algorithms, while the geometric properties of the maps are not preserved in an intuitive fashion. The recent development of discrete differential geometry provides structure-preserving methods to compute these maps. It asks not only the algorithms but a parallel discrete theory, such as discrete metric, discrete curvature, and discrete uniformization theorem. The key questions to ask in this field includes the following

(1) What should be the definition for the discrete version of smooth maps, such as discrete conformal maps? Does this **discretization of smooth maps converge** to the smooth map in weak or strong sense when we subdivide the meshes?
(2) What is the algorithm to compute the map? Does this **algorithm converge**? In particular, can we transform this problem to a convex optimization problem?

Here we summarize the ideas of the previous works on the discretization of conformal maps, harmonic maps and Teichmüller maps and their computational methods.

2.5.1.1. *Conformal maps.* Conformal maps have been applied to various problems such as constructing maps from brain to the round 2-sphere. Here we only mention part of the literature related to high genus surfaces.

Gu and Yau [**GY02, GY03**] proposed a computional method for the conformal structures on general Riemann surfaces with non-trivial topologies. Zeng et al. [**ZLY+07**] computed spherical parametrizations for high genus surfaces by introducing branch points. Thurston [**Thu79**] pointed out that conformal maps could be approximated numerically by circle packings. This idea was implemented by Hurdal and Stephenson [**HS09**]. See Stephenson [**Ste05**] for a detailed treatment with circle packings. It also led to the definition of a discrete conformal structure and discrete Ricci flow for triangulated surfaces proposed by Chow and Luo [**CL+03**]. Similarly, Luo [**Luo04**] proposed another discrete conformal structure and discrete Yamabe flow, which was transformed to a variational framework for computation by Bobenko, Pinkall, and Springborn [**BPS15**] and Springborn, Schroeder, and Pinkall [**SSP08**]. These two notions of discrete conformal equivalence provide not only fast algorithms but also analogous theorems in the discrete setting, including a discrete uniformization theorem by Thurston [**Thu79**] and Gu et al. [**GLS+18**]. The convergence of discrete conformal mappings to smooth mappings under subdivisions was proved by Rodin and Sullivan [**RS+87**] and Gu, Luo and Wu [**GLW19**]. Glickenstein and Thomas [**GT17**] showed that the two notions fall into a unified framework, providing a family of discretizations for conformal mappings. The algorithms of the discrete flows can be formulated into a convex optimization problem.

2.5.1.2. *Harmonic map.* Discrete versions of harmonic maps and Dirichlet energy have been studied in the theory of finite elements as a numerical solution to second-order elliptic PDEs. Pinkall and Polthier [**PP93**] proposed a discrete energy for triangular meshes to compute the harmonic map as an approximation to the minimal surface with a prescribed boundary in $\mathbb{R}^3$. For a smooth genus zero surface, harmonic maps coincide with conformal maps, providing a variational method implemented by Gu et al. [**GWC+04**] with applications to brian mappings. For high-genus surfaces, Li et al. [**LBG+08**] computed harmonic maps by directly minimizing Dirichlet energy. They need to first compute the hyperbolic metric for the target surface. Then the gradient flow of the Dirichlet energy from the domain surface to the target surface with a hyperbolic metric converges due to the convexity of the energy.

Another natural approach to discretize harmonic maps is through convex combination mappings. This idea emerges from Tutte's theorem [**Tut63**] for planar graphs, which states that we can embed a 3-vertex-connected planar graph in the plane with a prescribed convex boundary by solving a sparse linear system, producing a piecewise linear map as an approximation to the harmonic map. This result was further developed for triangulations of the disk by Floater [**Flo03a**], and can be regarded as a discrete version of the Rado-Kneser-Choquet theorem. Dym, Slutsky, and

Lipman [**DSL19**] showed that the piecewise linear maps generated by Tutte's theorem converge to the smooth harmonic map under subdivisions of the triangulations. Aigerman and Lipman [**AL15**] generalized Tutte's embedding to orbifolds.

There is currently no well-developed theory for discrete harmonic maps between general triangulated surfaces with non-trivial topologies.

2.5.1.3. *Quasiconformal map and Teichmüller map.* Quasiconformal maps and Teichmüller maps provide a method to compare two surfaces by measuring the deviation from conformality. Various methods have been proposed to compute quasiconformal maps and the Teichmüller map. Lui et al. [**LGY15**, **LLYG12**] studied numerical methods for computing quasiconformal maps and the Teichmüller maps extensively with applications in medical image and surface matching. Zeng et al. [**ZLL$^+$12**] computed quasiconformal maps with prescribed Beltrami coefficients μ by modifying the original metric such that the quasiconformal map corresponds to a conformal map in the new metric, then computing conformal maps using Ricci flow or Yamabe flow. Lam, Gu and Lui [**LGL15**] computed the Teichmüller map with constraints to analyze the shape of vertebrae bones. Weber, Myles, and Zorin [**WMZ12**] implemented a non-convex energy to compute Teichmuler maps. Wong and Zhao [**WZ14**] proposed a method to compute quasiconformal maps using discrete Beltrami flow. Most of the current algorithms deal with triangulated disks or 2-spheres. There is no well-developed notion for discrete quasiconformal maps between general surfaces.

2.5.2. An Algorithm to Compute the Distance. In this section, we propose an algorithm to compute the distance on the shape space of a high genus surface. The framework of this algorithm stems from the work in [**LBG$^+$08**, **LW14**, **WZ14**].

In practice, a surface is described by a triangular mesh $T = (V, E, F)$ with the set of vertices V, the set of edges E, and the set of faces F. Each vertex has coordinates in $\mathbb{R}^3$ for its position, and the length of an edge connecting two vertices is computed in $\mathbb{R}^3$ by their coordinates. Then a *discrete metric* is determined by the set of edge lengths, namely a function l from E to $\mathbb{R}^+$ so that triangle inequalities hold for all triangles in F.

To compare two shapes, we are given three lists for each mesh T_1 and T_2, a list of vertices v_i, a list of edges $e_{ij} = [v_i, v_j]$ and a list of faces $f_{ijk} = [v_i, v_j, v_k]$ with orientation given by the order of the three vertices. We have all coordinates of vertices in T_1 and T_2 in $\mathbb{R}^3$ and the two sets of edge lengths l_1 and l_2 of all the edges in T_1 and T_2. The combinatorial information and the discrete metrics for two shapes are contained in (T_1, l_1) and (T_2, l_2).

A map f between (T_1, l_1) and (T_2, l_2) is determined by the images in (T_2, l_2) of the vertices in T_1. Each map f restricted to a face of T_1 sends three vertices of the domain triangle with edge lengths L_1, L_2, and L_3 to three points on the target, and the corresponding distances l_1, l_2, and l_3 between each pair of vertices are computed using the discrete metric on the target, forming a triangle with potential cone singularities. Here we approximate the singular values in the smooth setting by the two singular values $\lambda_1(f_{ijk})$ and $\lambda_2(f_{ijk})$ of the linear map from each triangle f_{ijk} in T_1 to the Euclidean triangle with edge lengths l_1, l_2, and l_3, ignoring the potential singularities. Then the discrete energy of f between (T_1, l_1) and (T_2, l_2) is given by

$$E(f) = \sqrt{\sum_{f_{ijk}} (1 - \sqrt{\lambda_1(f_{ijk})\lambda_2(f_{ijk})})^2 A(f_{ijk})} + \frac{1}{2} \max_{f_{ijk}} \left| \log \frac{\lambda_1(f_{ijk})}{\lambda_2(f_{ijk})} \right|$$

where the integral is replaced by a sum over all the faces weighted with their areas $A(f_{ijk})$, the supremum is replaced by the maximum over all the faces.

To compute the two singular values $\lambda_1(f_{ijk})$ and $\lambda_2(f_{ijk})$ for each face, set L_1, L_2, L_3 to be the lengths of the domain triangle with opposite angles α, β, γ, and l_1, l_2, l_3 to be the corresponding lengths of the target triangle. Set λ_1 and λ_2 to be the two singular values and assume $\lambda_1 \geq \lambda_2$. Then set $D = \lambda_1^2 + \lambda_2^2$, $R = \frac{\lambda_1}{\lambda_2}$ and $P = \lambda_1 \lambda_2$. We have

$$D = \frac{1}{2A}(\cot\alpha |l_1|^2 + \cot\beta |l_2|^2 + \cot\gamma |l_3|^2)$$

and

$$P = \lambda_1 \lambda_2 = \sqrt{\frac{c(c - l_1)(c - l_2)(c - l_3)}{C(C - L_1)(C - L_2)(C - L_3)}}$$

where A is the area of the domain triangle, C is its semi-perimeter, and c is the semi-perimeter of the image triangle. The formula for D can be found in [**PP93**], and P is the ratio of the areas of two triangles where we can apply Heron's formula. Then we have the relation

$$(R + \frac{1}{R})P = D.$$

We can solve R by

$$R = \frac{\frac{D}{P} + \sqrt{(\frac{D}{P})^2 - 4}}{2}.$$

A similar computation of the singular values using coordinates of the vertices in $\mathbb{R}^3$ can be found in [**LBG+08**].

2.5.2.1. *Overview of the Algorithm.* We describe an algorithm to compute the distance between two shapes (T_1, l_1) and (T_2, l_2) as follows.

(1) Check that T_1 and T_2 represent closed orientable connected surfaces with the same genus, and normalize both areas to be one.
(2) Fix a vertex v_0 in T_1 and find a system of disjoint loops based on v_0 using the greedy algorithm proposed by Erickson and Whittlesey [**EW05**] and a local refinement algorithm of the triangulation in [**LBG+08**] to construct new triangulations T_1' with a new discrete metric l_1'. Similarly compute a system of disjoint loops for T_2 with a new triangulation T_2' and a new discrete metric l_2'.
(3) Slice the two triangulations T_1' and T_2' along the corresponding systems of loops computed in Step 2 into two "$4g$-gon" G_1 and G_2 by the procedure in Gu and Yau [**GY08**].
(4) Compute the initial map using Tutte's theorem as follows. First construct maps from G_i to a regular $4g$-gon G by Tutte's embedding using the degree of the vertex as weights for the linear system, and imposing the following boundary conditions:
    (a) Map the $4g$ vertices from the base point to the $4g$ vertices of the regular $4g$-gon in order.
    (b) Distribute the boundary vertices between any two vertices from the base point evenly to the boundary of $4g$-gon.

    Then compose these maps $G_1 \to G$ and $G \to G_2$ to generate the initial map. This initial map is described by the barycentric coordinates (a^i, b^i, c^i) of the image of each vertex v_i in T_1' contained in some face of T_2'.
(5) Compute the uniformization metric from the original metric (T_2', l_2') using discrete Ricci flow [**JKLG08**] or the variational method corresponding to discrete Yamabe flow [**SSP08**].
(6) Compute the harmonic map from (T_1', l_1') to T_2' with the uniformization metric from the previous step using *local charts* construction in [**LBG+08**]. It produces a new face and new barycentric coordinates for each vertex in T_1'.
(7) Minimize the distance energy E using the map in Step 6 as an initial guess and the local charts in the previous step.

The output of the algorithm contains a number E and a correspondence between two surfaces, described by barycentric coordinates of the image of each vertex of T_1' in a triangular face of T_2'. This algorithm computes a local minimum of the discrete energy. Unfortunately we can't show that this energy is convex so we can not guarantee the local minimum we find is the global minimum. The implementation of this algorithm is not complete, and we will focus on it in the further studies.

CHAPTER 3

The Spaces of Geodesic Triangulations on Surfaces

In this chapter, we study the topology of the space of geodesic triangulations on a surface. In Section 1, we provide the background for the basic property of surface diffeomorphisms and the history of the contractibility and embeddability problems for polygons. In Section 2, we recall various method to construct geodesic triangulations on different surfaces, especially Tutte's theorem and its generalizations. In Section 3, we give a new proof of the contractibility of $\mathcal{GT}(\Omega, T)$ if Ω is a convex polygon using Tutte's method. In Section 4, we give an explicit construction of a geodesic triangulation in $\mathcal{GT}(\Omega, T)$ if Ω is a strictly star-shaped polygon, assuming the triangulation does not contain any dividing edge. In Section 5, we give a characterization of a special class of geodesic triangulations corresponding to the minimizers of weighted length energies.

3.1. Background

We will first review basic facts about diffeomorphism groups of surfaces with different topologies. Then we briefly summarize the history of these two problems, especially the Bloch-Connelly-Henderson Theorem.

3.1.1. Surface Diffeomorphism. We consider the diffeomorphism group $\mathcal{D}(S)$, consisting of all orientation-preserving self-diffeomorphisms of a smooth surface, and its subspace $\mathcal{D}_0(S)$, the diffeomorphisms isotopic to identity. For the 2-disk we require the diffeomorphisms to fix the boundary $\partial\mathbb{D}^2$ pointwise. Then we summarize the following well-known results [**Sma59**, **EE$^+$69**].

THEOREM 3.1.1. *The diffeomorphism group for the disk $\mathcal{D}(\mathbb{D}^2) = \mathcal{D}_0(\mathbb{D}^2)$ and it is contractible. The diffeomorphism group for the sphere $\mathcal{D}(\mathbb{S}^2) = \mathcal{D}_0(\mathbb{S}^2)$ and the inclusion from orientation preserving rotation $SO(3) \to \mathcal{D}(\mathbb{S}^2)$ is a homotopy equivalence. It admits a product structure $\mathcal{D}_0(\mathbb{S}^2) = PSL(2,\mathbb{C}) \times \mathcal{D}_0(\mathbb{S}^2; 0, 1, \infty)$, where $\mathcal{D}_0(\mathbb{S}^2; 0, 1, \infty)$ is a contractible subgroup of diffeomorphisms fixing $0, 1, \infty$. The diffeomorhism group of tori also admits a similar product $\mathcal{D}_0(\mathbb{T}^2) = \mathcal{D}_0(\mathbb{T}^2; x) \times \mathbb{T}^2$ where $\mathcal{D}_0(\mathbb{T}^2; x)$ is a contractible subgroup of diffeomorphisms fixing $x \in \mathbb{T}^2$. Finally, $\mathcal{D}_0(S_g)$ is contractible for all hyperbolic surfaces $g \geq 2$.*

Smale proved the sphere case. Earle and Eells proved the other cases using a fiber bundle description for Teichmuller space. The fact that $\mathcal{D}_0(S_g)$ is contractible can also be deduced from the fiber bundle description of Teichmuller space with shape space. The product structures for the sphere and tori give immediately the topology of their respective diffeomorphism groups. Our goal is to show that exactly the same results hold for the space of geodesic triangulations as finite dimensional manifolds.

3.1.2. Prior work. The embeddability problem and the contractibility problem have been studied in [**BS78**, **BCH84**, **Cai44b**, **Ho73**], partly because they are closely related to the problem of determining the existence and uniqueness of differentiable structures on a triangulated manifolds [**CHHS83**]. They are also used to produce effective algorithms to solve graph morphing problems in [**DVPV03**, **FG99**, **SG01**, **SG03**].

In the general setting, we can consider a finite n-dimensional simplicial complex T, whose polyhedron $|T|$ is homeomorphic to the n-dimensional disk $\mathbb{D}^n$. A geodesic triangulation of $\mathbb{D}^n$ with the combinatorial type of T is determined by the positions of vertices of T in $\mathbb{R}^n$. The space of all such geodesic triangulations is denoted by $\mathcal{GT}(\mathbb{D}^n, T)$.

We can also interpret this space in terms of homeomorphisms. First assume there exists an initial geodesic triangulation of $\mathbb{D}^n$. Then all the other geodesic triangulations are the images of the initial triangulation under simplexwise linear homeomorphisms fixing the boundary vertices of T, determined by the images of the interior vertices of T in $\mathbb{R}^n$. The space of all such simplexwise linear homeomorphisms is denoted by $L(\mathbb{D}^n, T)$. Ho showed in [**Ho73**] that it was homeomorphic to $\mathcal{GT}(\mathbb{D}^n, T)$.

When we restrict to the 2-dimensional case, Cairns [**Cai44a**, **Cai44b**] initiated an investigation of the topology of the space of geodesic triangulations of a geometric triangle in the Euclidean plane and the round 2-sphere.

THEOREM 3.1.2. *If Ω is a geometric triangle with a triangulation T in the plane, then $\mathcal{GT}(\Omega, T)$ is path-connected.*

Ho [**Ho73**] proved that this space was simply-connected.

THEOREM 3.1.3. *If Ω is a geometric triangle with a triangulation T in the plane, then $\mathcal{GT}(\Omega, T)$ is simply-connected.*

A *dividing edge* in a triangulation T is an interior edge connecting two boundary vertices. Using an induction argument, Bing and Starbird [**BS78**] considered the general case of star-shaped polygons.

THEOREM 3.1.4. *If Ω is a star-shaped polygon with a triangulation T in the plane, and T does not contain any dividing edge, then $\mathcal{GT}(\Omega, T)$ is non-empty and path-connected.*

By the following pictures, Bing and Starbird [**BS78**] showed that $\mathcal{GT}(\Omega, T)$ was not necessarily path-connected if we didn't assume star-shaped boundary.

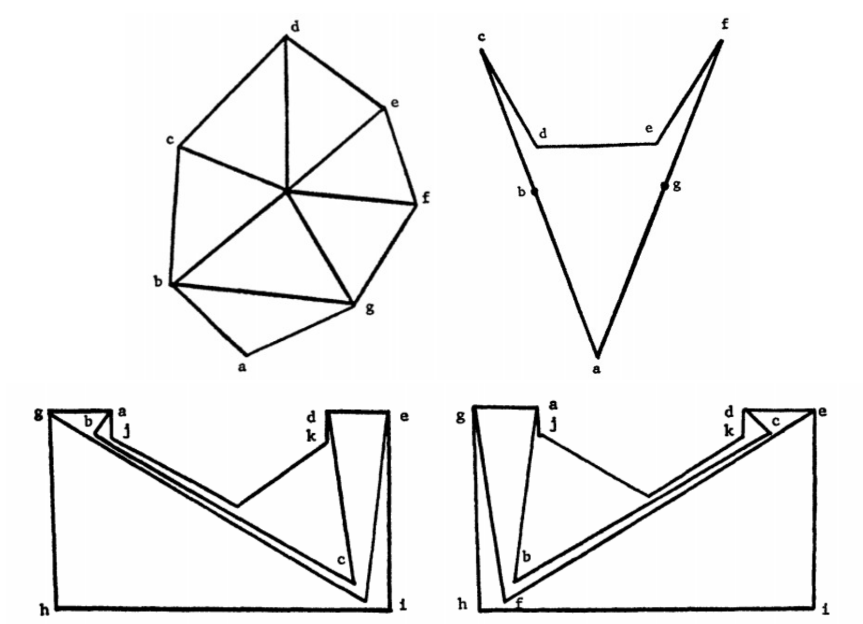


FIGURE 3.1. Counterexamples from [**BS78**]

Bloch, Connelly, and Henderson [**BCH84**] proved the contractibility of the space of simplexwise linear homeomorphisms of a convex 2-disk. In a very recent paper, Cerf [**Cer19**] improved the original argument in [**BCH84**] to give a new proof of the Bloch-Connelly-Henderson theorem.

THEOREM 3.1.5. *If Ω is a convex polygon with a triangulation T in the plane, and T does not contain any dividing edge, then $\mathcal{GT}(\Omega, T)$ is homeomorphic to $\mathbb{R}^{2k}$, where k is the number of interior vertices of T.*

We will give a new short proof of this theorem using the idea of Tutte's theorem [**Tut63**].

3.2. Construction of geodesic triangulations on surfaces

In this section we summarize well-established results about the constructions of geodesic triangulations for different surfaces. The basic result stems from Tutte's idea of embedding a 3-vertex-connected graph on a convex polygon and its generalizations. This construction is effective to solve problems in practice including surface parametrizations and planar graph morphing.

3.2.1. Tutte's embedding for the disk. Given a triangulation $T = (V, E, F)$ of the 2-disk with the sets of vertices V, edges E and faces F, the 1-skeleton of T is a planar graph. There is no canonical method to embed this graph in the plane. Tutte [**Tut63**] provided an efficient method to construct a straight-line embedding of a 3-vertex-connected planar graph by specifiying the coordinates of vertices of one face as a convex polygon and solving for the coordinates of other vertices with a linear system of equations. Using a discrete maximal principle, Floater [**Flo03a**] proved the same result for triangulations of the 2-disk. Gortler, Gotsman, and Thurston [**GGT06**] reproved Tutte's theorem with *discrete one forms* and generalized this results to the case of multiple-connected polygonal regions with appropriate assumptions on the boundaries. Since we are dealing with triangulations, we use the formulation given by Floater [**Flo03a**].

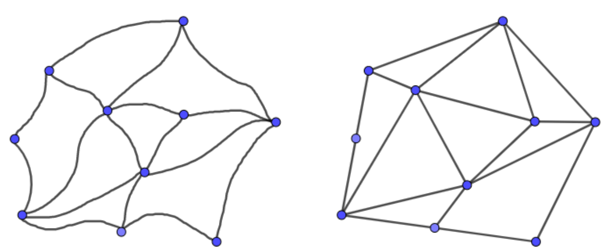


FIGURE 3.2. Tutte's embedding

THEOREM 3.2.1. *Assume $T = (V, E, F)$ is a triangulation of a convex polygon Ω, and ϕ is a simplexwise linear homeomorphism from T to $\mathbb{R}^2$. If ϕ maps every interior vertex in T into the convex hull of the images of its neighbors, and maps the cyclically ordered boundary vertices of T to the cyclically ordered boundary vertices of Ω, then ϕ is one to one.*

As Floater pointed out, this theorem gave a discrete version of the Rado-Kneser-Choquet theorem about harmonic maps from the disk to a convex polygon. Moreover, it gives a constructive method to produce geodesic triangulations of a convex polygon with the combinatorial type of T as follows.

First assign a positive weight c_{ij} to a directed edge $(i,j) \in \bar{E}$, where $\bar{E}$ is the set of directed edges of T. We normalize the weights by

$$w_{ij} = \frac{c_{ij}}{\sum_{j \in N(v_i)} c_{ij}}$$

where the set $N(v_i)$ consists of all the vertices that are neighbors of v_i, so that $\Sigma_{j \in N(v_i)} w_{ij} = 1$ for all $i = 1, 2, ..., N_I$. Notice that we don't impose symmetry condition $w_{ij} = w_{ji}$. We are given the coordinates $\{(b_i^x, b_i^y)\}_{i=N_I+1}^{|V|}$ for all the boundary vertices such that they form a convex polygon Ω in $\mathbb{R}^2$. Then we can solve the following linear system

$$\sum_{j \in N(v_i)} w_{ij} x_j = x_i \quad i = 1, 2, ...N_I;$$

$$\sum_{j \in N(v_i)} w_{ij} y_j = y_i \quad i = 1, 2, ...N_I;$$

$$x_i = b_i^x \quad i = N_I + 1, N_I + 2, ...N_I + N_B = |V|;$$

$$y_i = b_i^y \quad i = N_I + 1, N_I + 2, ...N_I + N_B = |V|$$

where $N_I = |V_I|$ is the size of the set of interior vertices V_I, and $N_B = |V_B|$ is the size of the set of boundary vertices V_B. The solution to this linear system produces the coordinates of all the interior vertices in $\mathbb{R}^2$. We put the vertices in the positions given by their coordinates, and connect the vertices based on the combinatorics of the triangulation T. Tutte's theorem claims that the result is a geodesic triangulation of Ω with the combinatorial type of T.

The linear system above implies that the x-coordinate(or y-coordinate) of one interior vertex is a convex combination of the x-coordinates(or y-coordinates) of its neighbors. Notice that the coefficient matrix of this system is not necessarily symmetric but it is diagonally dominant, so the solution exists uniquely.

Tutte's theorem solves the embeddability problem for a triangulation of a convex polygon. We can vary the coefficients w_{ij} to construct families of geodesic triangulations of a convex polygon. We will see that this idea will lead to a simple proof of the contractibility of the space of geodesic triangulations.

Bing and Starbird also shows that we can embed an geodesic triangulation of T when the boundary is a star-shaped polygon, as long as there is no dividing edge. However, we can not choose weights arbitrarily to generate families of geodesic triangulations. The embeddability problem of star-shaped polygon has been also studied by Hong and Nagamochi [**HN08**] and Xu et al. [**XCGL11**] using iterative methods.

3.2.2. Circle packing for the 2-sphere. Given a triangulation of 2-sphere, we can regard it as the tangency graph of a circle packing on 2-sphere. Then the Koebe-Andreev-Thurston theorem ensures the existence and uniqueness of this geodesic triangulation.

THEOREM 3.2.2. *Given any triangulation T of the 2-sphere, the circle packing whose tangency graph is isomorphic to T is unique up to Mobius transformations.*

Based on the circle packing, we can construct a geodesic triangulation of the 2-sphere by connecting the centers of two tangent circles using geodesic arc. Notice that the Mobius group does not map geodesic triangulations to geodesic triangulations.

3.2.3. Tutte's embedding for flat tori. In the case of a flat torus $(\mathbb{T}^2, g)$ with a triangulation T, the situation is similar to the disk case, because we can lift a geodesic triangulation of $(\mathbb{T}^2, g)$ to the universal covering $\mathbb{R}^2$. Using the method in Gu and Yau [**GY03**] and Gortler, Gotsman, and Thurston [**GGT06**], we can compute the *harmonic one form* to produce geodesic triangulations on $\mathbb{T}^2$ with a fixed combinatorial type of T.

Specifically, we first assign a positive weight c_{ij} to each directed edge in T and normalize the weights as in the case of the 2-disk to produce positive weights w_{ij} satisfying $\Sigma_{j\in N(v_i)} w_{ij} = 1$ for all $i = 1, 2, ..., N_I$. Instead of computing the coordinates for vertices in T directly, we compute the harmonic one forms $\Delta z : \bar{E} \to \mathbb{R}$ by solving the following system of equations

$$\begin{aligned}
\Delta z_{ij} &= -\Delta z_{ji} \quad \text{for all directed edges } (i,j) \in \bar{E}; \\
\sum_{v_j \in N(v_i)} w_{ij} \Delta z_{ij} &= 0 \quad \text{for all vertices } v_i \in V; \\
\Delta z_{ij} + \Delta z_{jk} + \Delta z_{ki} &= 0 \quad \text{for all faces } f_{ijk} \in F.
\end{aligned} \tag{3.2.1}$$

Gortler, Gotsman, and Thurston [**GGT06**] showed that this linear system had exactly two independent solutions, denoted by Δx and Δy. Then we can assign a vertex v_0 in V to the origin in $\mathbb{R}^2$ and compute the coordinates for other vertices v by summing the entries of the discrete one forms along a path p consisting a sequence of directed edges in T from v_0 to v

$$(3.2.2) \qquad (x_0, y_0) = (0,0) \text{ and } (x_i, y_i) = \Big(\sum_{(i,j)\in p} \Delta x_{ij}, \sum_{(i,j)\in p} \Delta y_{ij}\Big) \text{ for other vertices.}$$

Since the discrete form is closed, the coordinates for (x_i, y_i) are independent of the choice of the paths.

THEOREM 3.2.3. *Given a triangulation T of $(\mathbb{T}^2, g)$ whose 1-skeleton is a 3-vertex-connected graph, the two linearly independent solutions of the system above produce embeddings of any sub-triangulations T' of T with the topology of a disk.*

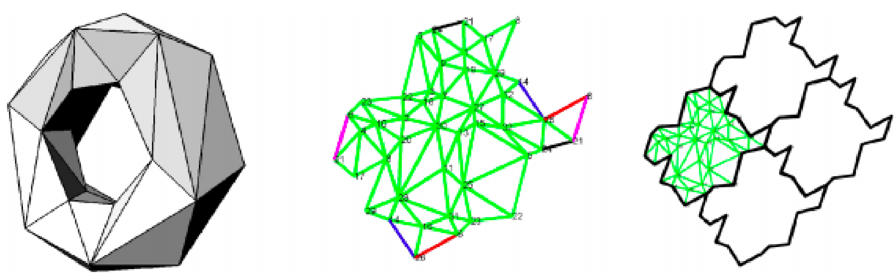


FIGURE 3.3. Tutte's embedding for flat tori from [**GGT06**]

Gortler, Gotsman, and Thurston pointed out that this statement of local injectivity produced a globally injective map from the universal cover of the torus to the Euclidean plane. We can generate families of equivariant geodesic triangulations in $\mathbb{R}^2$ projecting to geodesic triangulations on $(\mathbb{T}^2, g)$ by varying the weights w_{ij} in the linear system. If we choose a different pair of harmonic one forms $\Delta x'$ and $\Delta y'$, then the resulting geodesic triangulation in $\mathbb{R}^2$ is the image of the original geodesic triangulation under an affine transformation. This method was extended by Aigerman and Lipman [**AL15**] to Euclidean orbifolds with spherical topology.

3.2.4. Energy minimization on hyperbolic surfaces. For a hyperbolic surface, Colin de Verdier [**dV91**] proved a similar result as the disk case using the *discrete Dirichlet energy* of a given triangulation and showed that the minimizer exists uniquely and it is a geodesic triangulation in the hyperbolic surface. The energy of an embedding of one-skeleton of a triangulation T, denoted by $\psi : T^1 \to S_g$ is given by

$$E(\psi) = \frac{1}{2} \sum_{(i,j)\in E} c_{ij} \Big|\Big| \frac{d\psi_{ij}}{dt} \Big|\Big|^2 dt,$$

where ψ_{ij} as a map from $[0,1]$ to S_g is the restriction of ψ on the edge $(i,j) \in E$. This energy can be regarded as the energy of a system of ideal springs, where , each edge is a spring with Hook constant c_{ij}. The minimizer of this energy corresponds to exactly the equilibrium state of this system. Each point is in the convex hull of its neighbors. Colin de Verdiere's result can be summarized as following.

THEOREM 3.2.4. *Fix a triangulation $T = (V, E, F)$ of a surface S_g, the minimizer of $E(\psi)$ in its isotopy class exists uniquely and the image of ψ is a geodesic triangulation of surface S_g.*

Physically this means that we can vary the Hook constants for each ideal spring to generate families of geodesic triangulation in the same isotopy class. Hass and Scott [**HS12**] prove similar results for more general triangulations with new definition of combinatorial area and energy. Using this energy, they showed that the space of geodesic triangulation of a hyperbolic surface with one-vertex triangulation is contractible.

3.3. Geodesic Triangulations of the 2-Disk with Convex Boundary

In this section, we define the space of geodesic triangulations for the disk, and give a new proof of the contractibility of $\mathcal{GT}(\Omega, T)$ if Ω is a convex polygon.

DEFINITION 3.3.1. Given a triangulation $T = (V, E, F)$ of the 2-disk, fix the boundary vertices $\{v_i\}_{i=N_I+1}^{|V|}$ of T in $\mathbb{R}^2$ with coordinates $\{(b_i^x, b_i^y)\}_{i=N_I+1}^{|V|}$ and connect them based on T such that they form a convex polygon Ω in $\mathbb{R}^2$. The space of geodesic triangulations $\mathcal{GT}(\Omega, T)$ is defined as the set of all the geodesic triangulations of Ω with the combinatorial type of T whose boundary vertices $\{v_i\}_{i=N_I+1}^{|V|}$ have the corresponding coordinates $\{(b_i^x, b_i^y)\}_{i=N_I+1}^{|V|}$.

Every geodesic triangulation is uniquely determined by the positions of the interior vertices in V_I, so its topology is the subspace topology induced by $\Omega^{|V_I|} \subset \mathbb{R}^{2|V_I|}$. Notice that this space could be empty if the boundary is complicated. For instance, if the polygon is not star-shaped, then there doesn't exist any geodesic embedding of a triangulation with only one interior vertex. Nevertheless, Tutte's theorem shows that this space is not empty if the polygonal region Ω is convex.

Let us consider the topology of the space $\mathcal{GT}(\Omega, T)$ where Ω is a fixed convex polygon in $\mathbb{R}^2$. Let E_I be the set of interior edges in T and E_B be the set of boundary edges in T.

Definition 3.3.2. Given a triangulation T of Ω with coordinates of the boundary vertices $\{(b_i^x, b_i^y)\}_{i=N_I+1}^{|V|}$, define W to be the space of positive weights $(w_{ij}) \in \mathbb{R}^{2|E_I|}$ on the set of directed edges of T satisfying the normalization condition $\sum_{j \in N(v_i)} w_{ij} = 1$ for all $v_i \in V_I$. The *Tutte map* Ψ sends the weights in W to the solution to the linear system in Tutte's theorem with coefficients (w_{ij}) and $\{(b_i^x, b_i^y)\}$.

The weight space W is a $2|E_I| - |V_I|$ dimensional affine manifold in $\mathbb{R}^{2E_I}$. The image $\mathcal{GT}(\Omega, T)$ is a $2|V_I|$ dimensional manifold. By Euler characteristic $\chi(\Omega) = |V| - |E| + |F| = 1$ and the requirement of simplicial complex $3|F| = 2|E_I| + |E_B|$, we can deduce that $|E_I| - 3|V_I| = |E_B| - 3$. Hence the dimension of the space of weights W is not lower than the dimension of $\mathcal{GT}(\Omega, T)$.

Lemma 3.3.3. *The Tutte map Ψ is continuous and surjective from the space of weights W to $\mathcal{GT}(\Omega, T)$.*

Proof. By Tutte's theorem, for any $(w_{ij}) \in W$, the solution to the linear system generates a geodesic triangulation of T. The continuity follows from the continuous dependence of the solutions on the coefficients in the linear system. To show surjectivity, given a geodesic triangulation τ, any interior vertex v_i in τ is in the convex hull of its neighbors. Then we can construct the weights (w_{ij}) for a geodesic triangulation τ using the *mean value coordinates* defined in [**Flo03b**] below.

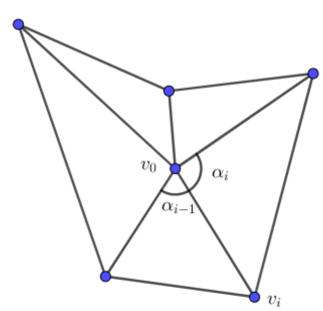


Figure 3.4. The mean value coordinate at v_0

The mean value coordinates on the directed edges of a geodesic triangulation are given by

$$w_{ij} = \frac{c_{ij}}{\sum_{j \in N(v_i)} c_{ij}} \quad \text{and} \quad c_{ij} = \frac{\tan(\alpha_{i-1}^j/2) + \tan(\alpha_i^j/2)}{||v_i - v_j||}$$

where the two angles α_{i-1}^j and α_i^j at v_i sharing the edge $(i,j) \in E_I$ in the Figure 3. The mean value coordinates provide a smooth map from $\mathcal{GT}(\Omega, T)$ to W. □

There are various ways to construct the weights from a given geodesic triangulation other than the mean value coordinates. Floater proposed another construction by taking the average of barycentric coordinates [**FG99**]. An alternative method to construct weights from a geodesic triangulation τ is to take the center of mass of the space of weights $(w_{ij}) \in W$ such that $\Psi((w_{ij})) = \tau$. This subspace is a convex subspace of W and the center of mass is well-defined. All three methods agree with the barycentric coordinates of a vertex when the star of this vertex is a triangle. Notice that we can not use the well-known cotangent weights, which are symmetric, but not necessarily positive to guarantee the condition for the embedding.

Definition 3.3.4. The map $\sigma : \mathcal{GT}(\Omega, T) \to W$ sends a geodesic triangulation τ to weights (w_{ij}) in W determined by the mean value coordinates.

Theorem 3.3.5. *If Ω is a convex polygon in $\mathbb{R}^2$ with a triangulation T, the space of geodesic triangulations $\mathcal{GT}(\Omega, T)$ is contractible.*

Proof. The map σ is continuous. By Tutte's theorem, $\Psi(\sigma(\tau)) = \tau$ for any $\tau \in \mathcal{GT}(\mathbb{D}^2, T)$, so the map σ is a global section of Ψ from $\mathcal{GT}(\Omega, T)$ to W. We need to show $\sigma \circ \Psi$ is homotopic to the identity map on W. From the previous discussion, we know that W is an affine manifold in $\mathbb{R}^{2|E_I|}$, so we can use the isotopy $(1-t)\sigma \circ \Psi + t\mathbf{1}$ where $\mathbf{1}$ is the identity map on W. Since W is a contractible space, $\mathcal{GT}(\Omega, T)$ is contractible by this homotopy equivalence. □

Although we mainly consider triangulations in this paper, this argument can be generalized to the case of the *convex geodesic embedding* of a 3-vertex-connected graph G, which is defined to be a geodesic embedding of G in the plane such that all its faces are convex. Then using the same idea of Tutte's theorem, we can show the contractiblity of the space of convex geodesic triangulations of G with the prescribed convex boundary Ω.

We can extend this result to convex polygons in other geometries of constant curvature. More precisely, if we have a convex polygon in the hyperbolic plane or a convex polygon in the round 2-sphere contained in a hemisphere, we can reduce it to the case of convex polygon in the Euclidean plane.

For a hyperbolic convex polygon Ω_H, we embed it in the Klein model of the hyperbolic plane so that all the edges of Ω_H are straight arcs in the Euclidean metric, inducing a convex polygon Ω in the Euclidean plane. Given a triangulation T of Ω_H , there is a bijection between the space of all hyperbolic geodesic triangulations of Ω_H represented in the Klein model and $\mathcal{GT}(\Omega, T)$, induced by the identity map on $\Omega^{|V_I|}$. Hence the space of hyperbolic geodesic triangulations $\mathcal{GT}(\Omega_H, T)$ is also contractible.

Similarly, if Ω_S is a spherical convex polygon contained in a hemisphere with a triangulation T, we can apply the *gnomonic transformation* from the center of the 2-sphere to the plane P tangent to the center of the hemisphere containing Ω_S. Then Ω_S is mapped to a convex polygon Ω in the plane P under the gnomonic transformation. This projective transformation keeps the incidence and maps geodesic arcs in hemisphere to the straight arcs in P. Hence it induces a bijection between the space of spherical geodesic triangulations of Ω_S with combinatorial type of T and $\mathcal{GT}(\Omega, T)$ in P.

Corollary 3.3.1. *Assume Ω is a hyperbolic convex polygon, or a spherical convex polygon contained in a hemisphere, and T is a triangulation of Ω. Then the space of geodesic triangulations $\mathcal{GT}(\Omega, T)$ is contractible.*

3.4. Geodesic Triangulations of the 2-Disk with star-shaped Boundary

In this section, we consider a star-shaped subset Ω of $\mathbb{R}^2$. An *eye* of a star-shaped region Ω is a point p in Ω such that for any other point q in Ω the line segment $l(t) = tp + (1-t)q$ lies inside Ω. The set of eyes of Ω is called the *kernel* of Ω. A set is called *strictly star-shaped* if the interior of the kernel is not empty.

In the case of polygons in $\mathbb{R}^2$, the kernel is the intersection of a family of closed half-spaces, each defined by the line passing one boundary edge of Ω. Every closed half space contains a half disk in Ω centered at one point on its corresponding boundary edge. If the star-shaped polygon is strict, the intersection of the *open* half-spaces is not empty. This means that we can pick an eye e with a neighborhood U of e such that if $q \in U$, then q is also an eye of Ω.

The first question to address is how to construct a geodesic triangulation of a strictly star-shaped polygon Ω with a combinatorial type of T. As Bing and Starbird [**BS78**] pointed out, it was not always possible if there was a dividing edge. Assuming there was no dividing edge in T, they proved that such geodesic triangulations existed by induction.

We give an explicit method to produce a geodesic triangulation for a strictly star-shaped polygon. We can regard all the edges e_{ij} in T as ideal springs with Hook constants w_{ij}. Fixing the boundary vertices, the equilibrium state corresponds to the critical point of the *weighted length energy* defined as

$$\mathcal{E} = \frac{1}{2}\sum_{e_{ij}\in E_I} w_{ij}L_{ij}^2$$

where L_{ij} is the length of the edge connecting v_i and v_j. This energy can be regarded as a discrete version of the Dirichlet energy [**dV91**, **HS12**], and it has a unique minimizer corresponding to the equilibrium state. Tutte's theorem guarantees that the equilibrium state is a geodesic embedding of T if the boundary is a convex polygon.

Given a triangulation T of a fixed strictly star-shaped polygon Ω, assume that the weighted length energy $\mathcal{E}$ satisfies $\sum_{e_{ij}\in E_I} w_{ij} = 1$. Notice that if the polygon is star-shaped but not convex, we can't choose arbitrary weights to generate a geodesic embedding of T. Hence we need to assign weights carefully to avoid singularities such as intersections of edges and degenerate triangles.

The idea is to distribute more and more weights to the interior edges connecting two interior vertices. As the weights for interior edges connecting two interior vertices tend to 1, all the interior vertices will concentrate at a certain point. If we can choose this point to be an eye of the polygon, we will produce an geodesic embedding of T of Ω.

Fix a polygon Ω with a triangulation T and the coordinates $\{(b_j^x, b_j^y)\}_{i=N_I+1}^{|V|}$ for its boundary vertices. Given a set of coordinates in $\mathbb{R}^2$ for all the interior vertices $\{(x_i, y_i)\}_{i=1}^{N_I}$, we define a family of weighted length energies with a parameter $0 < \epsilon < 1$ as

$$\mathcal{E}(\epsilon) = \frac{1-\epsilon}{2M_I}\sum_{e_{ij}\in E_I^I} L_{ij}^2 + \frac{\epsilon}{2M_B}\sum_{e_{ij}\in E_I^B} L_{ij}^2$$

where E_I^B is the set of all the interior edges connecting an interior vertex to a boundary vertex and E_I^I is the set of all the interior edges connecting two interior vertices. Let $M_B = |E_I^B|$ and $M_I = |E_I^I|$. The edge lengths L_{ij} are determined by the coordinates of the vertices

$$L_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2.$$

As $\epsilon \to 0$, most weights are assigned to interior edges in E_I^I, forcing all the interior vertices of the minimizer of $\mathcal{E}(\epsilon)$ to concentrate to one point.

Theorem 3.4.1. *Let Ω be a polygonal region with a triangulation T of Ω. Let $v_j^B = (x_j^B, y_j^B) = (b_j^x, b_j^y)$ for $j = 1, ..., N_B$ be the coordinates of the boundary vertices of Ω and $v_i^I(\epsilon) = (x_i^I(\epsilon), y_i^I(\epsilon))$ for $i = 1, ..., N_I$ be the coordinates of the interior vertices of the minimizer of the energy $\mathcal{E}(\epsilon)$. Then for all $i = 1, 2, ..., N_I$,*

$$\lim_{\epsilon \to 0} v_i^I = \lim_{\epsilon \to 0} (x_i^I(\epsilon), y_i^I(\epsilon)) = (x_0, y_0) = v_0$$

where

$$v_0 = \sum_{j=1}^{N_B} \lambda_j v_j^B \quad and \quad \lambda_j = \frac{deg(v_j^B) - 2}{\sum_j deg(v_j^B - 2)} = \frac{deg(v_j^B) - 2}{M_B},$$

assuming $deg(v)$ is the degree of the vertex v in T.

Proof. The minimizer of $\mathcal{E}(\epsilon)$ satisfies the following linear system formed by taking derivatives with respect to x_i and y_i for all $i = 1, 2, ..., N_I$

$$\frac{1-\epsilon}{M_I} \sum_{i \in N(v_k^I)} (v_k^I - v_i^I) + \frac{\epsilon}{M_B} \sum_{j \in N(v_k^I)} (v_k^I - v_j^B) = 0 \quad \text{for } k = 1, 2, \cdots N_I.$$

Notice that we separate the interior vertices $v_i^I \in V_I$ and the boundary vertices $v_j^B \in V_B$ in the summation. This system can be represented as

$$M(\epsilon)x = b_x \quad M(\epsilon)y = b_y$$

where the variables are

$$x = (x_1^I, x_2^I, ..., x_{N_I}^I, x_1^B, ..., x_{N_B}^B)^T$$

and

$$y = (y_1^I, y_2^I, ..., y_{N_I}^I, y_1^B, ..., y_{N_B}^B)^T.$$

The boundary conditions are

$$b_x = (0, 0, ..., 0, x_1^B, ..., x_{N_B}^B)^T$$

and

$$b_y = (0, 0, ..., 0, y_1^B, ..., y_{N_B}^B)^T.$$

The coefficient matrix $M(\epsilon)$ is an $(N_I + N_B) \times (N_I + N_B)$ matrix, and it can be decomposed as

$$M(\epsilon) = \left(\begin{array}{c|c} S(\epsilon) & -\epsilon W \\ \hline 0 & Id \end{array}\right)$$

where W is an $N_I \times N_B$ matrix, $S(\epsilon)$ is a square matrix of size N_I, and Id is the identity matrix of size N_B. The matrix W is defined as

$$W(i,j) = \begin{cases} \frac{1}{M_B} & \text{if } v_i^I \text{ is connected to } v_j^B; \\ 0 & \text{if } v_i^I \text{ is not connected to } v_j^B. \end{cases}$$

The matrix S is defined as

$$S(i,j)(\epsilon) = \begin{cases} -\sum_{i\neq k} S(i,k) + \epsilon \sum_{k=1}^{N_B} W(i,k) & \text{if } i = j; \\ -\frac{1-\epsilon}{M_I} & \text{if } v_i^I \text{ is connected to } v_j^I; \\ 0 & \text{if } v_i^I \text{ is not connected to } v_j^I. \end{cases}$$

Notice that for the first N_I rows in $M(\epsilon)$, the sums of their respective entries are zero, and all the off-diagonal terms are non-positive. The matrix W represents the relations of the boundary vertices with the interior vertices, and the sum of all its entries equals one. The matrix $S(\epsilon)$ is symmetric, strictly diagonally-dominant, and the sum of all its entries equals ϵ.

To show the limiting behavior of the solution to the system as $\epsilon \to 0$, we need the lemma below.

LEMMA 3.4.2. *Given the notations above, we have*

$$\lim_{\epsilon\to 0} \epsilon S(\epsilon)^{-1} = \mathbb{1}$$

where the matrix $\mathbb{1}$ *is the* $N_I \times N_I$ *matrix with all entries equal to* 1.

PROOF. Notice that $S(\epsilon)$ is symmetric and strictly diagonally dominant, so it is invertible. Let $S = S(0)$ and $M = M(0)$, then S has an eigenvalue $\lambda = 0$ with the normalized eigenvector $v = (1/\sqrt{N_I}, 1/\sqrt{N_I}, ..., 1/\sqrt{N_I})^T$.

First, we show that $\lambda = 0$ is a simple eigenvalue for S. If S has another eigenvector $u = (u_1, u_2, ..., u_{N_I})^T$ corresponding to $\lambda = 0$ not parallel to v, then it is orthogonal to v so $\sum_i u_i = 0$. Without loss of generality, we assume that $u_1 > 0$ achieves the maximal absolute value among u_i. Then we have

$$Su = 0 \quad \Rightarrow \quad \sum_{i=1}^{N_I} S(1,i)u_i = 0 \quad \Rightarrow \quad S(1,1)u_1 = -\sum_{i=2}^{N_I} S(1,i)u_i.$$

Notice that S is weakly diagonally dominant, $S(1,1) > 0$, and $S(1,i) \leq 0$, so we can deduce that

$$S(1,1)u_1 \geq -\sum_{i=2}^{N_I} S(1,i)u_1 \quad \Rightarrow \quad -\sum_{i=2}^{N_I} S(1,i)(u_i - u_1) \geq 0.$$

By our assumption, $u_i - u_1 \leq 0$ for all $i = 1, ..., N_I$, so the only possibility is $u_i = u_1$ for all i, which contradicts to the fact that u is orthogonal to v. Hence all the other eigenvalues of S are positive by Gershgorin circle theorem. (See, e.g. [**GL13**])

Second, we show that the eigenvalue $\lambda(\epsilon)$ of $S(\epsilon)$ approaching to 0 satisfies

$$\lim_{\epsilon \to 0} \frac{\lambda(\epsilon)}{\epsilon} = \frac{1}{N_I}.$$

This means that the derivative $(d\lambda/d\epsilon)(0) = 1/N_I$. To compute the derivative, notice that the sum of all the entries of $S(\epsilon)$ is ϵ, hence we have

$$v^T S(\epsilon) v = \frac{1}{N_I}(1,1,...,1)S(\epsilon)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{\epsilon}{N_I}.$$

The derivative of a simple eigenvalue of a symmetric matrix is given in [**PP$^+$08**] by

$$\frac{d\lambda}{d\epsilon}(0) = \frac{d(v^T S(\epsilon) v)}{d\epsilon} = \frac{d(\epsilon/N_I)}{\epsilon} = \frac{1}{N_I}.$$

Finally, we are ready to prove the lemma. Since $S(\epsilon)$ is symmetric, we have the diagonalization with an orthonormal matrix $P(\epsilon)$

$$\epsilon S^{-1}(\epsilon) = P(\epsilon)\begin{pmatrix} \epsilon\lambda_1^{-1}(\epsilon) & & & \\ & \epsilon\lambda_2^{-1}(\epsilon) & & \\ & & \ddots & \\ & & & \epsilon\lambda_{N_I}^{-1}(\epsilon) \end{pmatrix} P^T(\epsilon).$$

Without loss of generality, we assume the first eigenvalue $\lim_{\epsilon \to 0} \lambda_1(\epsilon) = 0$. Given any $0 < \delta < 1$, we can choose small $\epsilon > 0$ such that the following three inequality holds

$$\lambda_i(\epsilon) > C > 0 \text{ for } i = 2, 3, ..., N_i;$$

$$||P(\epsilon)\begin{pmatrix}\epsilon\lambda_1^{-1}(\epsilon) & & & \\ & \epsilon\lambda_2^{-1}(\epsilon) & & \\ & & \ddots & \\ & & & \epsilon\lambda_{N_I}^{-1}(\epsilon)\end{pmatrix}P^T(\epsilon)-P(\epsilon)\begin{pmatrix}N_I & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}P^T(\epsilon)||_2<\delta;$$

and the eigenvector $v_1(\epsilon)$ of $S(\epsilon)$ corresponding to the eigenvector $\lambda_1(\epsilon)$ satisfies

$$||v_1(\epsilon)-\frac{1}{\sqrt{N_I}}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}||_\infty<\delta.$$

Notice that the columns of $P(\epsilon)=(v_1,v_2,...,v_{N_I})$ form a set of the orthonormal basis formed by eigenvectors v_i, where the first eigenvector $v_1(\epsilon)$ approaches $v=(1/\sqrt{N_I},...,1/\sqrt{N_I})$. Then we have

$$||\epsilon S^{-1}(\epsilon)-\mathbb{1}||_2\leq||P(\epsilon)\begin{pmatrix}\epsilon\lambda_1^{-1}(\epsilon) & & \\ & \ddots & \\ & & \epsilon\lambda_{N_I}^{-1}(\epsilon)\end{pmatrix}P^T(\epsilon)-P(\epsilon)\begin{pmatrix}N_I & & \\ & \ddots & \\ & & 0\end{pmatrix}P^T(\epsilon)||_2$$

$$+||P(\epsilon)\begin{pmatrix}N_I & & \\ & \ddots & \\ & & 0\end{pmatrix}P^T(\epsilon)-\mathbb{1}||_2\leq\delta+||N_Iv_1^T(\epsilon)v_1(\epsilon)-\mathbb{1}||_2.$$

Notice that

$$||N_Iv_1^T(\epsilon)v_1(\epsilon)-\mathbb{1}||_2\leq 2N_I^2\delta.$$

Hence

$$||\epsilon S^{-1}(\epsilon)-\mathbb{1}||_2\leq(1+2N_I^2)\delta.$$

□

The inverse of the matrix $M(\epsilon)$ can be represented as

$$M^{-1}(\epsilon) = \left(\begin{array}{c|c} S^{-1}(\epsilon) & \epsilon S^{-1}(\epsilon)W \\ \hline 0 & I \end{array}\right).$$

Then the solution of the linear system $M(\epsilon)x = b_x$ is $x = M^{-1}(\epsilon)b_x$, whose first N_I entries are given by

$$\begin{pmatrix} x_1^I(\epsilon) \\ x_2^I(\epsilon) \\ \vdots \\ x_{N_I}^I(\epsilon) \end{pmatrix} = \epsilon S^{-1}(\epsilon)W \begin{pmatrix} x_1^B \\ x_2^B \\ \vdots \\ x_{N_B}^B \end{pmatrix}.$$

As $\epsilon \to 0$, the solution approaches $\mathbb{1}Wx^B$. All the x_i^I approach the same point

$$\lim_{\epsilon\to 0} x_i^I = (1, ..., 1)W \begin{pmatrix} x_1^B \\ x_2^B \\ \vdots \\ x_{N_B}^B \end{pmatrix} = \sum_{i=1}^{N_B} \frac{deg(v_i^B) - 2}{N_B} x_i^B.$$

A similar result holds for y-coordinates of the interior vertices. Hence we conclude the limit of the solutions $\lim_{\epsilon\to 0} v_i^I = v_0$. □

Notice that the matrix W can be replaced with more general matrices. The original energy $\mathcal{E}(\epsilon)$ distributes ϵ percentage of weights evenly to all the edges in E_I^B. We can define new energies by redistributing the weights

$$\mathcal{E}^W(\epsilon) = \frac{1-\epsilon}{2M_I} \sum_{e_{ij}\in E_I^I} L_{ij}^2 + \frac{\epsilon}{2} \sum_{e_{ij}\in E_I^B} w_{ij} L_{ij}^2$$

with $w_{ij} > 0$ and $\sum_{(i,j)\in E_I^B} w_{ij} = 1$. The matrix W is defined as

$$W(i,j) = \begin{cases} w_{ij} & \text{if } v_i^I \text{ is connected to } v_j^B; \\ 0 & \text{if } v_i^I \text{ is not connected to } v_j^B. \end{cases}$$

The limit of the solution is

$$v_0 = \sum_{j=1}^{N_B} \lambda_j v_j^B \quad \text{where} \quad \lambda_j = \sum_{i=1}^{N_I} w_{ij}.$$

To construct a geodesic triangulation, pick an eye e of Ω such that $e = \sum_{i=1}^{N_B} \lambda_i v_i^B$ where $\lambda_i > 0$ and $\sum_{i=1}^{N_B} \lambda_i = 1$, then define

$$W(i,j) = \begin{cases} w_{ij} = \frac{\lambda_i}{deg(v_j^B)-2} & \text{if } v_i^I \text{ is connected to } v_j^B; \\ 0 & \text{if } v_i^I \text{ is not connected to } v_j^B. \end{cases}$$

and the corresponding energy $\mathcal{E}^W(\epsilon)$. The remaining task is to show that the critical point of $\mathcal{E}^W(\epsilon)$ is a geodesic embedding of T for small ϵ.

If Ω is not convex, there exists a *reflex vertice*, defined as a boundary vertice of Ω where the turning angle is negative. We use the result by Gortler, Gotsman and Thurston [**GGT06**] to show that the minimizer of $\mathcal{E}^W(\epsilon)$ constructed above is an embedding for some $\epsilon > 0$.

THEOREM 3.4.3. *Given a strictly star-shaped polygon Ω with a triangulation T without dividing edges, if the reflex vertices of Ω are in the convex hull of their respective neighbors, then the solution to the linear system generates a straight-line embedding of T.*

THEOREM 3.4.4. *Given a strictly star-shaped polygon Ω with a triangulation T without dividing edges, and an eye e in Ω with coefficients W, there exists an $\epsilon > 0$ such that the critical point of the energy $\mathcal{E}^W(\epsilon)$ generates a geodesic embedding of T.*

PROOF. Theorem 4.3 implies that we only need to check that the reflex vertices v_r are in the convex hulls of their respective neighbors.

Choose an ϵ small enough such that the vertices of the critical point of $\mathcal{E}^W(\epsilon)$ defined above are eyes of Ω. Assume v_r is a reflexive point on the boundary of Ω. Let v be an interior vertex of the geodesic triangulation in the star of v_r, and let v_1 and v_2 be the two boundary vertices connecting to v_r. Since there is no dividing edge in T, v_1 and v_2 are the only boundary vertices connecting to v_r. We want to show that v_r is in the convex hull of its neighbors.

Assume the opposite, then all the edges connecting to v_r lie in a closed half plane, so the inner product of any pair of three vectors $\overrightarrow{v_r v_1}$, $\overrightarrow{v_r v_2}$ and $\overrightarrow{v_r v}$ is non-negative. But the inner angle at v_r is larger than π, then either angle $\angle v_1 v_r v$ or $\angle v v_r v_2$ is strictly larger than than $\frac{\pi}{2}$, which means one inner product is negative. This leads to a contradiction. □

This result solves the embeddability problem for strictly star-shaped polygons Ω with a triangulation T. We can construct a geodesic triangulation of Ω as follows. Pick an eye e of Ω with the coefficients W defined above. Then choose $\epsilon = 1/2$ and solve the linear system corresponding to the critical point of $\mathcal{E}^W(1/2)$. If the solution is not an embedding, replace ϵ by $\epsilon/2$ and continue.

We conjecture that the space of geodesic triangulations for strictly star-shaped polygon with a fixed combinatorial type is contractible.

3.5. A Characterization of Geodesic Triangulations From Energies

We use the weighted length energy to generate families of geodesic triangulations for both convex polygons and strictly star-shaped polygons in the previous sections. One interesting question is whether we can realize any given geodesic triangulation in $\mathcal{GT}(\Omega, T)$ as the critical point of certain weighted length energy by choosing appropriate weights. Unfortunately, this is not the case, given the example in Eades, Healy, and Nikolov [**EHN18**].

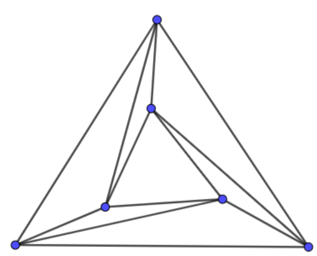


FIGURE 3.5. A geodesic triangulation can not be the minimizer of any energy

We have two equilateral triangles with different sizes determined by the vertices below

$$v_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -\sin\epsilon \\ \cos\epsilon \end{bmatrix},$$

$$v_5 = \begin{bmatrix} -\frac{\sqrt{3}}{2}\cos\epsilon + \frac{1}{2}\sin\epsilon \\ -\frac{\sqrt{3}}{2}\sin\epsilon - \frac{1}{2}\cos\epsilon \end{bmatrix}, \text{ and } v_6 = \begin{bmatrix} \frac{\sqrt{3}}{2}\cos\epsilon + \frac{1}{2}\sin\epsilon \\ \frac{\sqrt{3}}{2}\sin\epsilon - \frac{1}{2}\cos\epsilon \end{bmatrix},$$

and the triangulation given in Figure 4. The weighted length energy is given by

$$\mathcal{E}(\epsilon) = 3((2-\cos\epsilon)^2 + \sin^2\epsilon + (2 + \frac{\sqrt{3}}{2}\sin\epsilon + \frac{1}{2}\cos\epsilon)^2 + (-\frac{\sqrt{3}}{2}\cos\epsilon + \frac{1}{2}\sin\epsilon)^2)$$

$$= 30 - 6\cos\epsilon + 6\sqrt{3}\sin\epsilon.$$

Notice that when ϵ is close to zero, $\mathcal{E}(\epsilon)$ is a monotonic increasing function with respect to ϵ. Moreover, the length of *every* interior edge decreases or at least stays with the same length when $\epsilon \to 0^+$. Then it can't be a critical point of any energy in the form of $\mathcal{E} = \frac{1}{2}\sum w_{ij}L_{ij}^2$.

The triangulation in Figure 4 is not a critical point of any energy, because we can construct a vector field to move the interior vertices of the triangulation so that *no* edge is lengthened. We can show that this condition leads to a necessary and sufficient condition for a geodesic triangulation to be realized as the minimizer of a weighted length energy. Eades, Healy, and Nikolov [**EHN18**] gave another characterization for this class of geodesic triangulations.

LEMMA 3.5.1. *A geodesic triangulation τ of a polygon Ω can be realized by the critical point of a weighted length energy if and only if any vector field at the set of interior vertices of τ will shorten at least one edge and lengthen at least one edge.*

PROOF. Let (x_i, y_i) be the coordinate for vertex v_i of a given geodesic triangulation in $\mathbb{R}^2$. If there exists a vector field not increasing any edge length, then all the edge lengths will decrease or at least stay with the same length as we move the vertices of the geodesic triangulation along the vector field. Then it can't be a critical point of $\mathcal{E}$ for any choice of w_{ij}.

Conversely, assume that we are given a geodesic triangulation τ such that any vector field at interior vertices of τ will increase the length of some edge and decrease the length of another edge. We want to show that we can find some positive weights w_{ij} for every edge in E_I such that τ is the critical point of the weighted length energy

$$\mathcal{E} = \frac{1}{2}\sum_{e_{ij}\in E_I} w_{ij}L_{ij}^2.$$

To find these weights, consider the linear system corresponding to the critical point of weighted length energy, denoted by $Vw = 0$,

$$\sum_{j\in N(v_i)} v_{ij}^T w_{ij} = 0 \quad i = 1, ..., N_I$$

where we regard w_{ij} as the unknowns for the system and $v_{ij} = -v_{ji} = (x_i - x_j, y_i - y_j)$ are determined by τ. For each interior vertex v_i, we have two equations corresponding to the x coordinate and the y coordinate of v_i, so V is a $2N_I \times |E_I|$ matrix. If w_{ij} is the weight of an interior edge connecting two interior vertices, then the column c_{ij} of V corresponding to w_{ij} is

$$(0, ..., 0, v_{ij}, 0, ..., 0, v_{ji}, 0, ..., 0)^T.$$

If w_{ij} is the weight of an interior edge connecting one interior vertex v_i with a boundary vertex v_j, then the column c_{ij} of V corresponding to w_{ij} is

$$(0, ..., 0, v_{ij}, 0, ..., 0)^T.$$

To show the existence of a positive solution, consider an arbitrary vector field X defined on the set of interior vertices of τ. It can be represented as $(\alpha_1, \alpha_2, ..., \alpha_{N_I})^T$ where α_i is a row vector in $\mathbb{R}^2$. Then consider the derivative of the length of an interior edge connecting two interior vertices under X

$$\frac{dL_{ij}^2}{dt}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} (x_i + \alpha_i^x t - x_j - \alpha_j^x t)^2 + (y_i + \alpha_i^y t - y_j - \alpha_j^y t)^2 = v_{ij} \cdot (\alpha_i - \alpha_j) = 2X \cdot c_{ij}.$$

Similarly for an interior edge connecting one interior vertex v_i with one boundary vertex v_j, we have

$$\frac{dL_{ij}^2}{dt}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} (x_i + \alpha_i^x t - x_j)^2 + (y_i + \alpha_i^y t - y_j)^2 = v_{ij} \cdot \alpha_i = 2X \cdot c_{ij}.$$

By assumption, we know that X shortens one edge with weight w_{ij} and lengthens another with weight w'_{ij}. Hence the corresponding columns c_{ij} and c'_{ij} produce different signs, namely $X \cdot c_{ij}$ and $X \cdot c'_{ij}$ has different signs. This means that all the entries of $X^T V$ can't have the same sign. Since X is arbitrary, by Farkas's alternative [**RAG05**], $Vw = 0$ has a positive solution (w_{ij}).

□

From this characterization, we show that all the triangulations can be realized by the critical points of some weighted length energies if we only have a few interior edges in E_I^I.

Corollary 3.5.1. *If a triangulation T of a polygon Ω has at most one interior edges connecting two interior vertices, namely $|E_I^I| < 2$, then all the geodesic triangulations in $\mathcal{GT}(\Omega, T)$ can be realized by the critical points of some energies.*

Proof. If $|E_I^I| = 0$, the equations for each vertex are decoupled from equations for other vertices since the interior vertices are not connected to each other. The energy $\mathcal{E}$ can also be decomposed to separate terms for each interior vertex. Hence we can directly use mean value coordinates proposed by Floater to define the weights w_{ij}.

If $|E_I^I| = 1$, given a geodesic triangulation $\tau \in \mathcal{GT}(\Omega, T)$, let e_0 be the unique interior edge with two interior vertices v_1 and v_2. Assume that we have a vector field $X = (\alpha_1, \alpha_2)$ at v_1 and v_2 with $\alpha_1, \alpha_2 \in \mathbb{R}^2$ such that it will not increase the length of any edge. Then it will not increase the lengths of the edges connecting v_1 with boundary vertices. Then α_1 increases the lengths of e_0, otherwise α_1 determines a closed half space containing all the edges connecting v_1 with other vertices. Then v_1 is not a convex combination of its neighbors, which is impossible for a geodesic triangulation. Similarly α_2 increases the length of e_0. Hence the length of e_0 is increasing under X, which contradicts to the assumption of X. $\square$

CHAPTER 4

Further Directions

In this chapter, we summarize the open problems in the further study.

The further direction for the shape comparison problem in Chapter 2.

- The implementation of the algorithm mentioned previously. The high genus surface is more difficult to work with in general. Due to its non-trivial topology, we open needs to cut it to a disk for computation. Moreover, since the canonical metrics on most high genus surfaces have constant curvature -1, we need to construct charts to the hyperbolic disk, where the metric tensor is approximated by Euclidean metric only in small scales.
- The uniqueness of the minimizer of the energy in a fixed homotopy class of maps between two surfaces.

The further direction for the space of geodesic triangulations in Chapter 3.

- The topology of the space of geodesic triangulations of star-shaped polygons. Evidence shows that it is plausible to show that the spaces for star-shaped polygons are also contractible, so that we might be able to generalize the Bloch-Henderson-Connelly theorem.
- Construction of counterexamples other than the example given by Bing and Starbird. Their example shows that for general polygons, the space could be not path-connected. Can we show that the fundamental group and higher homotopy groups are also not trivial?
- The topology of the space of geodesic triangulations of other surfaces, including 2-sphere, tori, and hyperbolic surfaces. The space of geodesic triangulations on the 2-sphere was studies by Awartani-Henderson [**AH87**]. The conjecture is that $\mathcal{GT}(S^2, T)$ is homotopic to $SO(3)$. For hyperbolic surfaces S, Hass and Scott [**HS12**] showed that $\mathcal{GT}(S, T)$ was contractible if T is an 1-vertex triangulation. It is conjectured that $\mathcal{GT}(S, T)$ is contractible.

CHAPTER 5

Appendix A: Details about Algorithms

We propose an algorithm to compute the distance between surfaces with genus $g \geq 2$ and produce the correspondence.

Input: (S_1, T_1, l_1) and (S_2, T_2, l_2).

- **Intrinsic Information**: These means we have three lists for each mesh, a list of vertices v_i, a list of (half)-edges $e_{ij} = [v_i, v_j]$ and a list of faces $f_{ijk} = [v_i, v_j, v_k]$ with orientation encoded. And for each edge e_{ij} we have its length restored in a matrix.
- **Extrinsic Information**: All coordinates of vertices in S_1 and S_2. We only need to use extrinsic information in the last step.

The algorithm consists of the following 7 steps.

(1) Setup: check S_1 and S_2 are closed surfaces with the same genus, then normalized the both areas to be a constant C, given by the scale of the problem. C may be the average of two areas or their scale. C should not be too small to avoid numerical issues.

Input: (S_1, T_1, l_1) and (S_2, T_2, l_2)

Output: the constant C for area.

(2) Fix a base vertex v_0 and find system of loops with subdivition to determine homotopy class.

Input: (S_1, T_1, l_1) and (S_2, T_2, l_2), a base vertex in S_1

Output: (S_1, T_1', l_1'), a new triangulation given by some subdivision, and disjoint loops $\alpha_1, \alpha_2, ..., \alpha_{2g}$, each of which is described by a list of consecutive half-edges:

$$\{[v_0, v_1], [v_1, v_2], [v_2, v_3], ..., [v_n, v_0]\}$$

We also have similar output for S_2, namely (S_2, T_2', l_2') and loops $\beta_1, \beta_2, ..., \beta_{2g}$.

(3) Slice the surfaces S_1 and S_2 along the corresponding systems of loops.

Input: the output from step 2, including system of loops and new mesh T'

Output: two "$4g$-gon" G_1 and G_2, each with flags of three types of vertices: interior vertex, boundary vertex and the base point. G_1 and G_2 are described by meshes.

(4) Compute initial map with Tutte embedding:

Input: the output in step 3

Output: a face in S_2 and a barycentric coordinate (a^i, b^i, c^i) for each vertex v_i in S_1

(a) Construct maps from G_i to a regular $4g$-gon G by Tutte embedding, with specific boundary conditions.

(b) Then compose these maps $G_1 \to G$ and $G \to G_2$ to get the initial map. This initial map is described by barycentric coordinates of the image of each vertex of S_1 in some face of S_2.

(5) Compute the uniformization metric $\bar{l}_2$ for S_2 using Ricci flow or Bobenko's method.

Input: the output in step 2, only the metric part (S_2, T_2', l_2')

Output: the uniformization metric $\bar{l}_2$ of S_2.

(6) Compute the harmonic map from S_1 to S_2 with the uniformization metric using **local charts** construction in [**LBG**$^+$**08**].

Input: output of step 5, the uniformization metric $\bar{l}_2$; output of step 4, a face and barycentric coordinates for each vertex in S_1

Output: new face and new barycentric coordinates for each vertex in S_1

(a) for each vertex v in S_1, find all vertices in its neighborhood $v_1, v_2, ..., v_n$ where $n = \deg(v)$

(b) for each v_i, find the face of S_2 containing the image of v_i

(c) embed the face containing image of v in the center of Poincare disk model

(d) iteratively embed faces next to the existing faces in the Poincare model until it contains all the faces containing v_i, $v_2 ... v_n$

(e) move the image of v to the center of Poincare disk model by some Mobius transformation.

(f) for each v_i, use barycentric coordinates of v_i to compute its distance from the center in Euclidean metric d_i

(g) compute Laplaican, ω_{ij} is the cotangent weight from the metric l_1' in S_1 in the output of step 2.

$$\Delta f = \sum_{j \in N_i} \omega_{ij}(f(v_i) - f(v_j)) = \sum_{j \in N_i} \omega_{ij} d_j$$

(h) Δf is a point in Poincare disk, it gives the new face and new barycentric coordinate for the vertex v.

(i) iterate above procedure until Δf is small enough. Then for each vertex v in S_1 we will have a new face and new barycentric coordinates.

(7) Minimize the distance energy E using the result in (5) as an initial guess and local charts. **Step(a)-(e) is exactly the same as in step 6, which is the construction of local charts**. The only difference is the energy. Please note the the constant C in the energy is from step 1.

$$E(f) = \sqrt{\sum_{f_i}(1 - \sqrt{\lambda_1(f_i)\lambda_2(f_i)})^2 A(f_i)} + \frac{C}{2} \max_{f_i} \left| \log \frac{\lambda_1(f_i)}{\lambda_2(f_i)} \right|$$

Input: output of step 6

Output: distance $d(S_1, S_2)$; new face and new barycentric coordinates for each vertex in S_1 representing the map.

(a) for each vertex v in S_1, find all vertices in its neighborhood $v_1, v_2, ..., v_n$ where $n = \deg(v)$

(b) for each v_i, find the face of S_2 containing the image of v_i

(c) embed the face containing image of v in the center of Poincare disk model

(d) iteratively embed faces next to the existing faces in the Poincare model until it contains all the faces containing v_i, $v_2 ... v_n$

(e) move the image of v to the center of Poincare disk model by some Mobius transformation.

(f) for each face f_i in the star of image of v, use barycentric coordinates of v_i to compute its three edge lengths in Euclidean geometry.

(g) for each face f_i in the star of image of v, compute two singular values $\lambda_1(f_i)$ and $\lambda_2(f_i)$. Then we can compute the energy E_{old}.

(h) decrease E by moving v around the center of the disk, then we will have a new face and new barycentric coordinate for v and a smaller energy E_{new}

(i) iterate above procedure until $|E_{new} - E_{old}|$ is small enough. Then for each vertex v in S_1 we will have a new face and new barycentric coordinates, and the distance $d(S_1, S_2) = E_{new}$. This will be the output of the algorithm.

Output: a nonegative real number $d(S_1, S_2)$ which is distance and the corresponding map, described by barycentric coordinates of the image of each vertex of S_1 in certain triangular face of S_2.

CHAPTER 6

Appendix B: Examples of Geodesic Triangulations for Star-Shaped Polygons

The following gives four examples of geodesic triangulation of a star-shaped region. The first star-shaped region is

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

The second star-shaped region is

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

The third star-shaped region is

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1.8 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

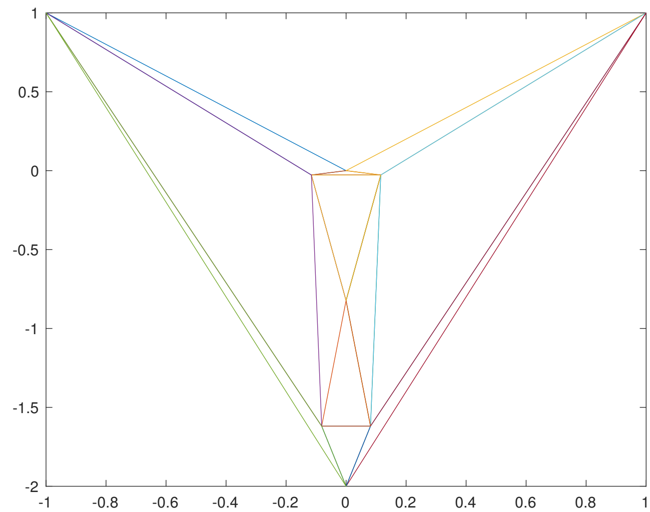
The last star-shaped region is

$$v_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The eyes for the four regions are chosen to be

$$e_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ -1.5 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ -1.9 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0.9875 \\ 0.15 \end{bmatrix}.$$

In all the examples, the parameter in the energy is $0.9, 0.5, 0.1$ respectively. For the first example, all the three parameters produce geodesic triangulations. For the rest of three examples, when the parameter is 0.9, it is not a embedding; when the parameter is 0.5 and 0.1, the result is an embedding.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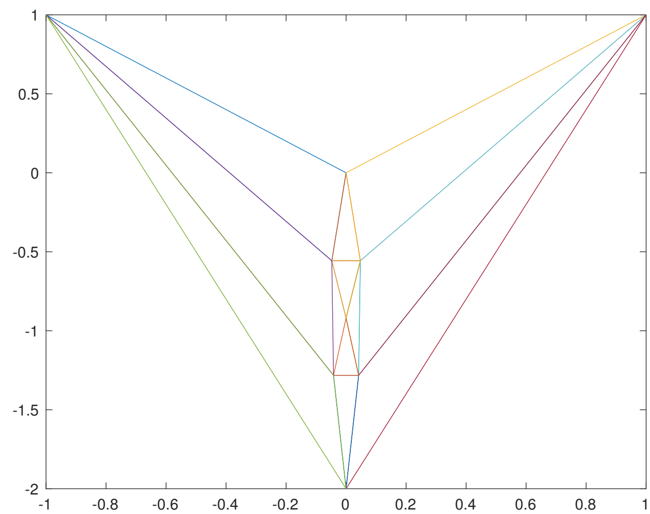
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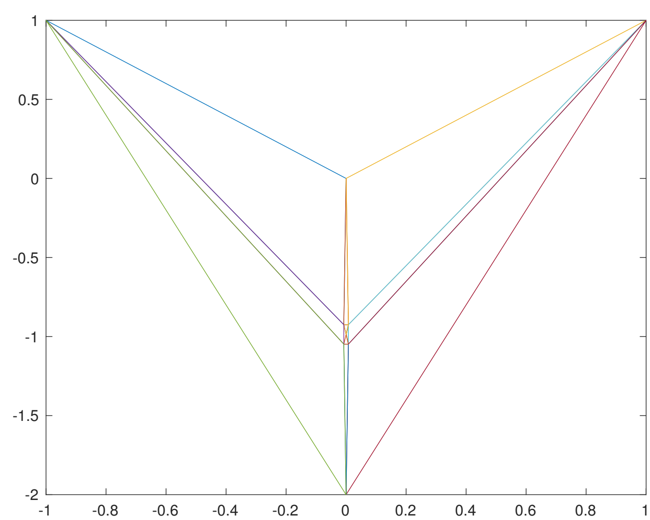
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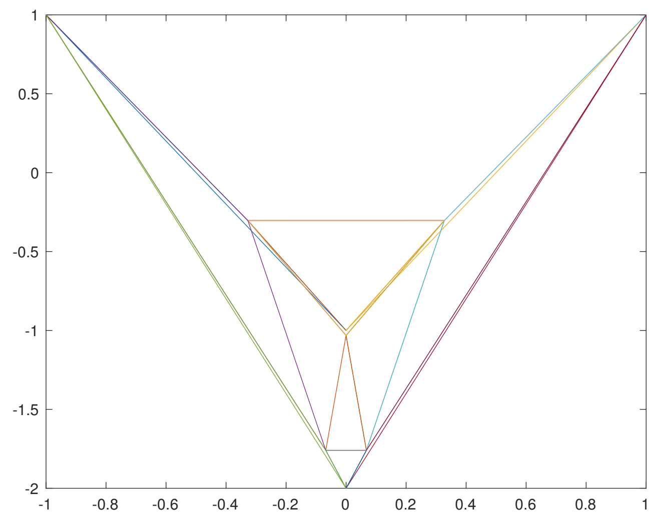
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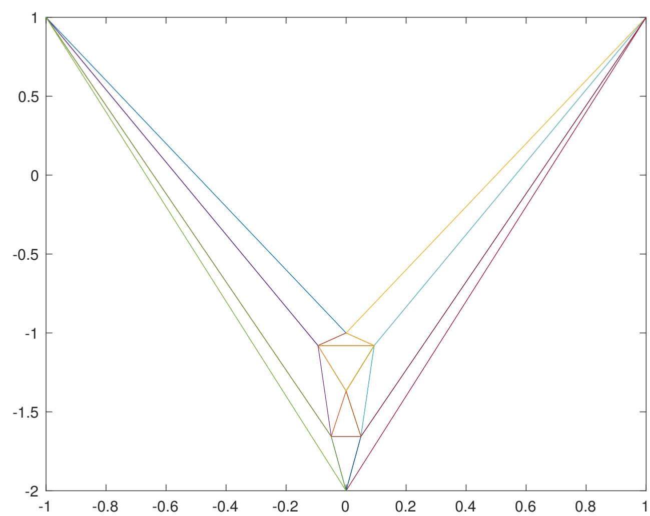
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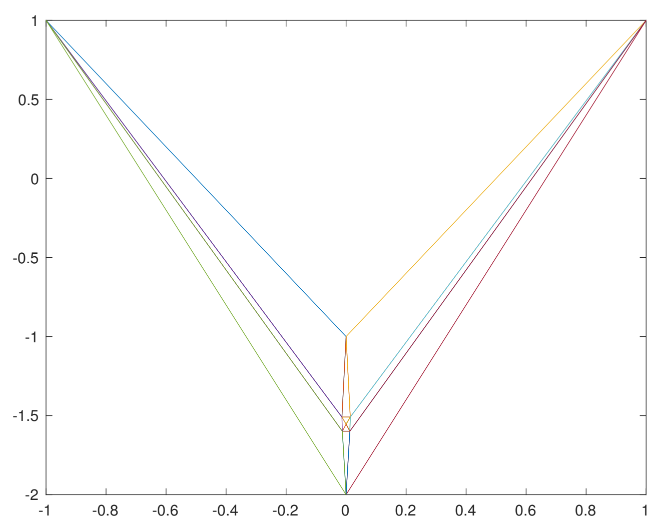
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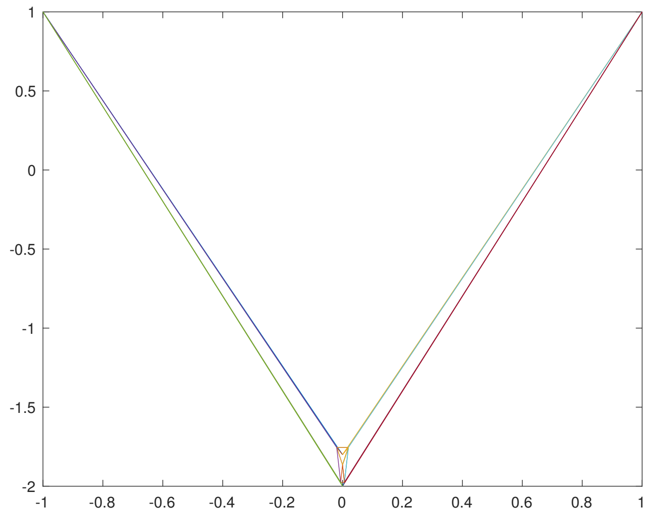
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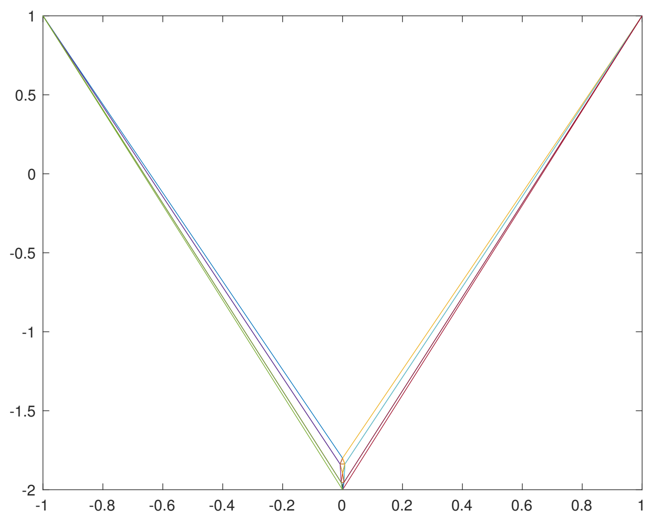
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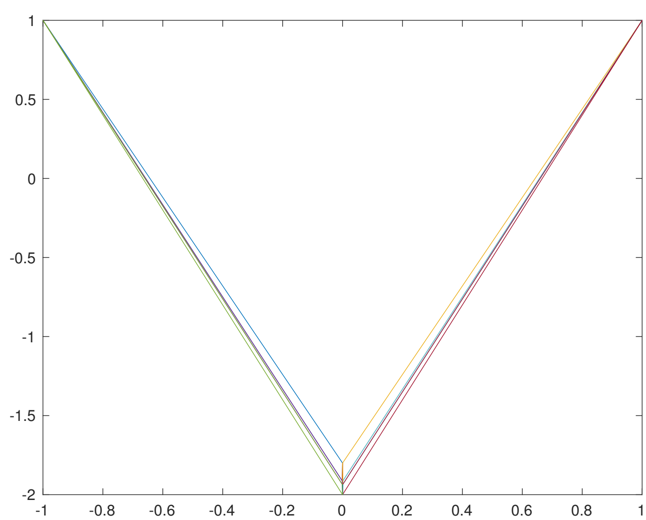
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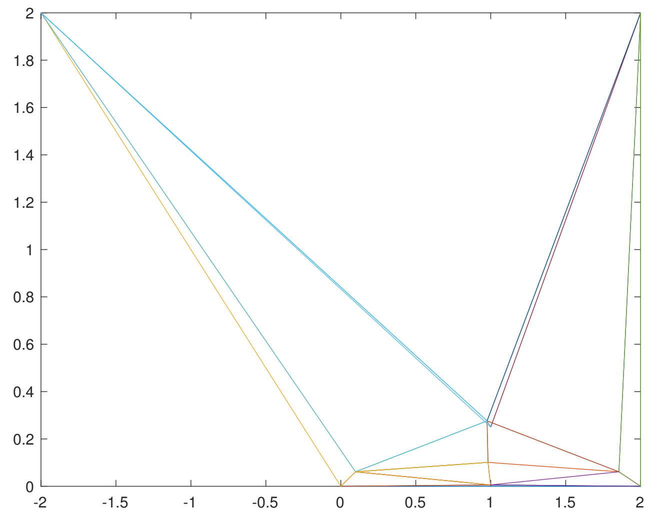
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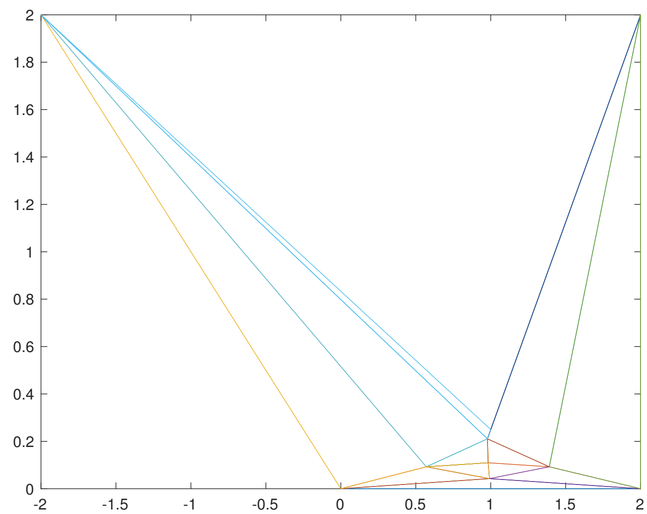
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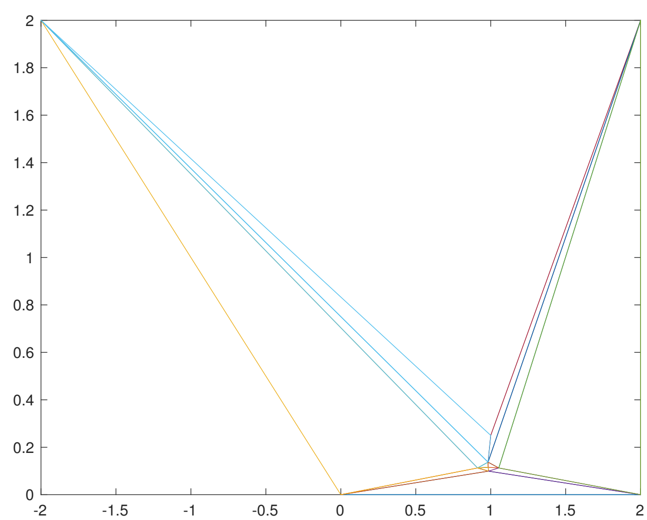
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