Stable Properties of Gapped Ground State Phases in Quantum Spin Chains

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Contents

Abstract **iii**

0 Introduction **1**

1 Mathematical Preliminaries **3**
1.1 Lattice 3
1.2 Spin 4
1.3 Algebra of Observables 4
1.4 Local Dynamics 5
1.5 Lieb-Robinson Bounds and Dynamics 8
1.6 Quantum Spin System 16
1.7 States 16

2 Summary of Results **23**
2.1 Stability of gapped ground state phases of spins and fermions in one dimension . . . 24
2.2 Automorphic equivalence preserves the split property 27
2.3 Automorphic equivalence of gapped ground state phases in the bulk 29

3 Stability of gapped ground state phases of spins and fermions in one dimension. **30**

4 Automorphic equivalence preserves the split property. **56**

5 Automorphic equivalence within gapped phases in the bulk. **76**

6 Appendix **121**

7 References for Mathematical Preliminaries and Summary **124**

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Mathematics

Stable Properties of Gapped Ground State Phases in Quantum Spin Chains

Abstract

This dissertation presents three peer-reviewed journal articles on the topic of stable properties of gapped ground state phases of quantum spin systems, primarily in one dimension. Mathematical preliminaries for these papers are found in Chapter 1. A detailed summary of results, including main results, co-author information and funding acknowledgments, are found in Chapter 2. The Appendix comments on the hypotheses of the main result of Chapter 5.

0 | Introduction

Heuristically, two quantum systems are in the same phase if there is a smooth path between the interactions of the systems which preserves characterizing properties of the material [9]. If no such interpolation exists, then the systems are separated by a phase transition. The problem of detecting quantum phase transitions has a rich and modern mathematical theory. It is a fundamental problem in the theory of quantum computing, where quantum information is stored in the ground state space of a many-body interaction.

This dissertation investigates phase transitions in *quantum spin systems* (**QSS**), which are widely applicable many-body models for quantum matter and quantum information.

A QSS associates to each site of a lattice a finite-dimensional Hilbert space representing the spin of a confined particle. The basic objects of quantum mechanics, observables and states, are built for a QSS from the lattice structure. Interactions between sites are spatially localized with respect to the lattice and generate a uniformly hyperfinite C^*-algebra of observables. Interactions define *Hamiltonian* operators, the observables of energy, and their corresponding Heisenberg dynamics by extensive summation over subregions of the lattice. States are the normalized, positive linear functionals of this algebra. *Ground states*, characterized by a positivity condition with respect to the infinitesimal generator of the Heisenberg dynamics, define expectation values for the system at its lowest energy configuration.

A ground state of a QSS is naturally associated with its Gelfand-Naimark-Segal (**GNS**) representation which contains information about the phase. In this representation, the Heisenberg dynamics of the system are implemented by a one-parameter group of unitary operators. The self-adjoint generator of this group is called the *bulk Hamiltonian*, since its properties determine bulk properties of the QSS. If there is a *ground state spectral gap* separating the ground and excited state energies of the bulk Hamiltonian and the interaction between spins is sufficiently short-range, then correlations in the ground state will decay exponentially with distance in the lattice [13, 23]. Rigorous results about invariants of ground states have evidenced the physical theory that symmetry constraints may produce nontrivial phases, even when in the absence of the symmetry there is only the trivial phase [1, 26, 27]. Accordingly, phases with symmetry constraints are called *symmetry protected topological* (**SPT**) phases. SPT phases and symmetric invariants in the GNS representation have been studied extensively in the cases of *finitely correlated* states, i.e. matrix product states [25, 28, 29], and *split* states [15] of the 1D spin chain.

In the first chapter of this dissertation, the technical preliminaries for rigorous discussion of these topics are established. The following chapters are a collection of peer-reviewed journal articles which investigate stable properties of gapped ground state phases of quantum spin chains. The results of these papers are summarized in Chapter 2. The first two papers in this collection, Chapters 3 and 4, prove results about stable properties of gapped ground state phases in one dimension − hence the title of this dissertation. It is important to note that the main result of the third paper, Theorem 1.3 of Chapter 5, applies to arbitrary integer lattice dimension, though it has applications to the theory of SPT phases in one dimension.

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1 | Mathematical Preliminaries

1.1 Lattice

In this chapter, we will define a quantum spin system with respect to the integer lattice $\mathbb{Z}^D$ equipped with the distance function $\|(x_1,\ldots,x_D)-(y_1,\ldots,y_D)\| = \max_{1\leq i\leq D}|x_i-y_i|$. In all but one of the subsequent chapters, we investigate mathematical properties of gapped ground states when $D=1$, with the exception of Chapter 5. Many condensed matter models, such as the Affleck-Kennedy-Lieb-Tasaki spin $S=3/2$ system defined on the regular hexagonal lattice, are defined on lattices which are not integer sublattices. Mathematically, many results in the theory of quantum spin systems do not depend materially on the periodicity of the lattice, and most can be developed more generally on graphs. To this point, we remark that many results of this chapter, such as the Lieb-Robinson bound in Theorem 1.5.1, will hold for countable graphs Γ for which there exist constants $c_\Gamma, d_\Gamma > 0$ such that:

$$\sup_{x\in\Gamma}|B_n(x)| \leq c_\Gamma n^{d_\Gamma}, \qquad \sup_{x\in\Gamma}|B_n(x)\setminus B_{n-1}(x)| \leq c_\Gamma n^{d_\Gamma - 1}. \tag{1.1}$$

For a general treatment of quantum spin systems on graphs which satisfy condition (1.1), see [24] or [21].

1.2 Spin

We associate to each site x of the lattice $\mathbb{Z}^D$ the finite-dimensional Hilbert space $\mathfrak{H}_x = \mathbb{C}^{d_x}$. $\mathfrak{H}_x$ has a natural interpretation as the state space of a particle with spin $S_x \in \frac{1}{2}\mathbb{N}$, in which case d_x is determined as the dimension $2S_x + 1$ of the corresponding irreducible representation of SU(2). The linear operators $B(\mathfrak{H}_x)$ on $\mathfrak{H}_x$ form the algebra of observables at the site x.

1.3 Algebra of Observables

Henceforth we consider the case of homogeneous spin S. By the previous section, this determines the dimension of the on-site algebra of observables as $d^2 = (2S+1)^2$. For a given dimension D of the integer lattice, we index a collection:

$$\left\{\mathfrak{A}_X : X \text{ is a finite subset of } \mathbb{Z}^D\right\} \tag{1.2}$$

of matrix algebras $\mathfrak{A}_X = B\big(\bigotimes_{x\in X} \mathfrak{H}_x\big)$ describing the observables for the quantum system of finite collections of spins. We require that if $Y \subset X$, then $\mathfrak{A}_Y$ is a unital subalgebra of $\mathfrak{A}_X$ under the identification:

$$\begin{aligned} \iota_{Y,X} :& \mathfrak{A}_Y \to \mathfrak{A}_X \\ A \mapsto& \mathbb{1}_{\mathfrak{A}_{X\setminus Y}} \otimes A. \end{aligned} \tag{1.3}$$

This unital mapping turns the collection in (1.2) into a net of algebras ordered by inclusion. We define the *algebra of observables* for the infinite lattice:

$$\mathfrak{A} = \overline{\bigcup_{\substack{X\subset\mathbb{Z}^D \\ X \text{is finite}}} \mathfrak{A}_X}^{\|\cdot\|}. \tag{1.4}$$

$\mathfrak{A}$ is a C^*-algebra with the norm defined by the operator norm on the dense subalgebra $\bigcup_{\substack{X\subset\mathbb{Z}^D \\ X \text{ is finite}}} \mathfrak{A}_X$, and $\mathfrak{A}$ is unique up to $*$-isomorphism independently of lattice dimension (cf. Section IV of [10]). For any $\Sigma \subset \mathbb{Z}^D$, not necessarily finite, denote by $\mathfrak{A}_\Sigma$ the closed subalgebra with dense subset $\bigcup_{\substack{Z\subset\Sigma \\ Z \text{ is finite}}} \mathfrak{A}_Z$. Further discussion of algebraic and topological properties of $\mathfrak{A}$ can be found in

Chapter 2.6 of [5]. In particular, $\mathfrak{A}$ is a simple C^*-algebra.

Elements of $\mathfrak{A}$ are known as *observables.* If Z is a finite subset such that $X \cap Z = \emptyset$, then for all $A \in \mathfrak{A}_X$ and $B \in \mathfrak{A}_Z$:

$$[A, B] = AB - BA = 0. \tag{1.5}$$

The following well-used definition makes precise the idea of locality in $\mathfrak{A}$.

Definition 1.3.1. *Let $A \in \mathfrak{A}$ be an observable. A is a **local** observable if there exists a finite subset $X \subset \mathbb{Z}^D$ such that $A \in \mathfrak{A}_X$. Denote by:*

$$\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\substack{X \subset \mathbb{Z}^D \\ X \text{ is finite}}} \mathfrak{A}_X \tag{1.6}$$

*the **algebra of local observables.***

An observable $B \in \mathfrak{A} \setminus \mathfrak{A}_{\mathrm{loc}}$ is not local, but it is *quasi-local* in the sense that there exists a sequence $B_n \in \mathfrak{A}_{\mathrm{loc}}$ such that:

$$\lim_{n \to \infty} \|B - B_n\| = 0. \tag{1.7}$$

A fundamental question which we answer in summary in Section 1.5 of this chapter is whether the convergence in (1.7) can be quantified in terms of the physical parameters of the system when $B = B(t)$ is evolved by dynamics, e.g. as in equation (1.15).

1.4 Local Dynamics

Hamiltonian operators, the observables of energy in quantum theory, are defined for quantum spin systems by extensive summation of interactions between sites. The following definitions specify these objects. Let $P_f(\Sigma)$ denote the set of finite subsets of a set Σ.

Definition 1.4.1. *An **interaction** on $\mathbb{Z}^D$ is a function $\Phi : P_f(\mathbb{Z}^D) \to \mathfrak{A}_{\mathrm{loc}}$ such that for all*

$X \in P_f(\mathbb{Z}^D)$,

$$\Phi(X) = \Phi(X)^* \in \mathfrak{A}_X. \tag{1.8}$$

Let $\mathcal{B}(\mathbb{Z}^D)$ denote the real vector space of interactions on $\mathbb{Z}^D$. On physical grounds, we may require interactions to change with respect to a parameter $s \in \mathbb{R}$. For example, adiabatic theorems for quantum spin systems use interaction parameters to introduce changes over long time scales [4]. Mathematically this dependence is represented as a curve Φ_s of interactions. In the following, we will investigate curves of interactions which are smooth in an appropriate sense. To define this rigorously, we first summarize the theory of $\mathcal{F}$-functions, which define extended norms on $\mathcal{B}(\mathbb{Z}^D)$ and provide a precise definition of differentiation of interactions. These definitions are well-known and not novel to this dissertation, and we refer to Chapter 6.2 of [6] for a comprehensive discussion of classes of interactions and the Appendix of [24] for a detailed exposition of $\mathcal{F}$-functions.

Definition 1.4.2. *A monotone decreasing function $F : [0,\infty) \to [0,\infty)$ is an **$\mathcal{F}$-function** if $\lim_{r\to\infty} F(r) = 0$ and F satisfies the following two properties:*

(i) $\sum_{x\in\mathbb{Z}^D} F(\|x\|) < \infty$,

(ii) *There exists a constant $C_F > 0$ such that for all $x, y \in \mathbb{Z}^D$,*

$$\sum_{z\in\mathbb{Z}^D} F(\|x-z\|)F(\|z-y\|) \leq C_F F(\|x-y\|).$$

$\mathcal{F}$-functions quantify the decay of an interaction in terms of the lattice distance.

Definition 1.4.3. *The $\mathcal{F}$-norm of an interaction Φ from F is:*

$$\|\Phi\|_F = \sup_{x,y\in\mathbb{Z}^D} \sum_{\substack{X\in P_f(\mathbb{Z}^D) \\ x,y\in X}} \frac{\|\Phi(X)\|}{F(\|x-y\|)}. \tag{1.9}$$

The quantity in (1.9) defines an extended norm on the space of interactions and induces a complete normed metric subspace in a natural way.

Lemma 1.4.1. *Denote by $\mathcal{B}_F(\mathbb{Z}^D)$ the set of interactions Φ such that $\|\Phi\|_F < \infty$. Then $\mathcal{B}_F(\mathbb{Z}^D)$ with the norm $\|\cdot\|_F$ is a real Banach space.*

Proof. The fact that $\mathcal{B}_F(\mathbb{Z}^D)$ is a normed vector space over $\mathbb{R}$ with $\|\cdot\|_F$ is clear. So suppose (Φ_n) is a Cauchy sequence of this space. This implies that for all $X \in P_f(\mathbb{Z}^D)$, the sequence $(\Phi_n(X))$ is uniformly Cauchy and hence convergent to some $\Phi(X) \in \mathfrak{A}_X$. Define $\Phi \in \mathcal{B}(\mathbb{Z}^D)$ by

$$\Phi(X) = \lim_{n\to\infty} \Phi_n(X).$$

Suppose $x, y \in \mathbb{Z}^D$ and denote $S_{x,y} = \{Z \in P_f(\mathbb{Z}^D) : x, y \in Z\}$. Let J be any finite subset of $S_{x,y}$. Then:

$$\sum_{Z\in J} \frac{\|\Phi(Z)\|}{F(\|x-y\|)} = \lim_{m\to\infty} \sum_{Z\in J} \frac{\|\Phi_m(Z)\|}{F(\|x-y\|)} \leq \sup_m \|\Phi_m\|_F < \infty$$

which shows

$$\|\Phi\|_F = \sup_{x,y\in\mathbb{Z}^D} \sup_{J\in P_f(S_{x,y})} \sum_{Z\in J} \frac{\|\Phi(Z)\|}{F(\|x-y\|)} < \infty$$

and $\Phi \in B_F(\mathbb{Z}^D)$. Similar computations show that $\Phi_n \to \Phi$ with respect to $\|\cdot\|_F$. □

Evidently if F, G are both $\mathcal{F}$-functions such that $\sup_r (F/G)(r) < \infty$, then $\mathcal{B}_F(\mathbb{Z}^D) \subset \mathcal{B}_G(\mathbb{Z}^D)$. In particular, the *finite-range* interactions are in the intersection of all $\mathcal{B}_F(\mathbb{Z}^D)$, showing that they are non-empty subspaces.

Definition 1.4.4. *An interaction Φ is* ***finite-range*** *if there exists $R > 0$ such that $\mathrm{diam}(X) \geq R$ implies $\Phi(X) = 0$.*

The $\mathcal{F}$-norm can be modified to also describe decay of curves of interactions. Let $\Phi(s)$ be a curve of interactions in $\mathcal{B}_F(\mathbb{Z}^D)$ parametrized by $s \in I$, where I is a subinterval of $\mathbb{R}$. Then we define:

$$\|\Phi_s\|_{F,I} = \sup_{x,y\in\mathbb{Z}^D} \sum_{\substack{Z\in P_f(\mathbb{Z}^D)\\ x,y\in Z}} \sup_{s\in I} \frac{\|\Phi_s(Z)\|}{F(\|x-y\|)}. \tag{1.10}$$

Definition 1.4.5. *Suppose $(\Phi_s)_{s\in I}$ is a curve of interactions in $\mathcal{B}_F(\mathbb{Z}^D)$. Say Φ_s is a* ***differentiable*** *interaction if the following two conditions hold:*

(i) $\forall X \in P_f(\mathbb{Z}^D)$, the map $s \mapsto \Phi_s(X)$ is continuously differentiable in $\mathfrak{A}_X$ with respect to the operator norm. Denote this derivative as $\Phi'_s(X)$.

(ii) $\|\partial\Phi_s\|_{F,I} < \infty$*, where we define:*

$$\partial\Phi_s(X) = |X|\Phi_s'(X). \tag{1.11}$$

Lastly, we define the dynamics from an interaction in terms of their generator. Let Φ_s be a differentiable interaction. Define the *local Hamiltonian* of Φ_s over $X \in P_f(\mathbb{Z}^D)$ by the extensive sum:

$$H_X(\Phi_s) = \sum_{Z \subset X} \Phi_s(Z). \tag{1.12}$$

Define $U_X(s) \in \mathfrak{A}_X$ as the unique unitary solution to the equation:

$$\frac{d}{ds}U_X(s) = -iH_X(\Phi_s)U_X(s), \quad U_X(0) = \mathbb{1}_X. \tag{1.13}$$

For each $X \in P_f(\mathbb{Z}^D)$, the associated dynamics $\beta^X : I \to \mathrm{Aut}(\mathfrak{A}_X)$ of Φ_s are defined by conjugation:

$$\beta_s^X(A) = U_X(s)^* A U_X(s). \tag{1.14}$$

For example, if Φ_s is a constant curve over $\mathbb{R}$, i.e. $\Phi_s = \Phi_t = \Phi$ for all $s, t \in \mathbb{R}$, then $U_X(t) = e^{-itH_X(\Phi)}$ is the group of unitaries defined by the generator $-iH_X(\Phi)$. In this case, we reserve the notation:

$$\tau_t^{\Phi,X}(A) = e^{itH_X(\Phi)} A e^{-itH_X(\Phi)} \tag{1.15}$$

and refer to these dynamics as the *Heisenberg* dynamics from Φ.

1.5 Lieb-Robinson Bounds and Dynamics

A C^*-dynamical system $(\mathcal{A}, \alpha)$ is particularly amenable to study when $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ is *approximately inner*, that is, the pointwise limit of strongly continuous groups of inner $*$-automorphisms of $\mathcal{A}$. The tool which will allow us to identify approximately inner Heisenberg dynamics for a QSS is the Lieb-Robinson bound. First demonstrated by Lieb and Robinson in 1972 [14], these bounds have a critical role in the analysis of the quasi-locality of the dynamics.

Theorem 1.5.1 proves a commutator bound in the case when the evolution is from the Heisenberg dynamics due to an interaction which decays by at least a power-law $\mathcal{F}$-function. The method of proof of this theorem uses the following integral inequality.

Lemma 1.5.1 (Lemma 2.3 of [24]). *Let $\mathfrak{H}$ be a Hilbert space and $I \subset \mathbb{R}$ a finite or infinite interval, and $A, B : I \to B(\mathfrak{H})$ strongly continuous mappings with $A(t) = A(t)^*$ for all $t \in I$. Then for each $t_0 \in I$ and $V_0 \in B(\mathfrak{H})$, the initial value problem:*

$$\frac{d}{dt}V(t) = -i[A(t), V(t)] + B(t), \quad V(t_0) = V_0 \tag{1.16}$$

has a unique strong solution, and:

$$\|V(t)\| \leq \|V_0\| + \int_{\min\{t,t_0\}}^{\max\{t,t_0\}} ds \ \|B(s)\|. \tag{1.17}$$

Proof. Let $W(t)$ be the unique strong solution to:

$$\frac{d}{dt}W(t) = -iA(t)W(t), \quad W(t_0) = \mathbb{1}. \tag{1.18}$$

$W(t)$ is unitary for all t. Then $V(t) = W(t)\big(V_0 + \int_{t_0}^{t} ds\ W(s)^* B(s) W(s)\big) W(t)^*$ is the unique strong solution to (1.16), and the bound follows from taking norms. □

Theorem 1.5.1. *Suppose Φ is an interaction such that $\|\Phi\|_F < \infty$, where $F(r) \leq \frac{1}{(1+r)^{D+\epsilon}}$ ($\epsilon > 0$). Then there exist constants $C, v > 0$ such that for all Λ and local observables A, B with disjoint supports S_A, S_B:*

$$\|[\tau_t^{\Lambda}(A), B]\| \leq C\|A\|\|B\|(e^{v|t|} - 1) \sum_{a \in S_A} \sum_{b \in S_B} F(\|a - b\|). \tag{1.19}$$

The constants may be taken as:

$$C = \frac{2}{C_F} \qquad\qquad v = 2C_F\|\Phi\|_F \tag{1.20}$$

Proof. We first perform a preliminary computation to which we will apply Lemma 1.5.1. This will establish the first step of an iterative procedure which will produce the desired bound. In the following, we omit dependence on Λ. Let U, V be local observables of $\mathfrak{A}_\Lambda$ with support S_U, S_V. Denote $K_U = \sum \{\Phi(X) : X \cap S_U \neq \emptyset\}$. Then:

$$\begin{aligned}\frac{d}{dt}[\tau_t(U), V] &= i[[\tau_t(H(\Phi)), \tau_t(U)], V] \\ &= i[\tau_t([H(\Phi), U]), V] \\ &= i[\tau_t(K_U), [\tau_t(U), V]] - i[\tau_t(U), [\tau_t(K_U), V]].\end{aligned} \tag{1.21}$$

where the last line follows by Jacobi's identity. Lemma 1.5.1 implies, when $\|U\| \neq 0$:

$$\frac{\|[\tau_t(U), V]\|}{\|U\|} \leq \frac{\|[U, V]\|}{\|U\|} + 2 \sum_{\substack{X : X \cap S_U \neq \emptyset \\ \|\Phi(X)\| \neq 0}} \|\Phi(X)\| \int_0^{|t|} dr \; \frac{\|[\tau_r(\Phi(X)), V]\|}{\|\Phi(X)\|}. \tag{1.22}$$

In the case when $U = A$ and $V = B$, $[A, B] = 0$, and so:

$$\frac{\|[\tau_t(A), B]\|}{\|A\|} \leq 2 \sum_{\substack{X : X \cap S_A \neq \emptyset \\ \|\Phi(X)\| \neq 0}} \|\Phi(X)\| \int_0^{|t|} dr \; \frac{\|[\tau_r(\Phi(X)), B]\|}{\|\Phi(X)\|}. \tag{1.23}$$

Applying the bound derived in (1.22) to the integrands of the righthand side of (1.23) yields:

$$\begin{aligned}\frac{\|[\tau_t(A), B]\|}{\|A\|} \leq 2 \sum_{\substack{X_1 : X_1 \cap S_A \neq \emptyset \\ \|\Phi(X_1)\| \neq 0}} \|\Phi(X_1)\| \int_0^{|t|} dr_1 \; \Bigg(&\frac{\|[\Phi(X_1), B]\|}{\|\Phi(X_1)\|} \\ &+ 2\|\Phi(X_2)\| \int_0^{r_1} dr_2 \sum_{\substack{X_2 : X_2 \cap X_1 \neq \emptyset \\ \|\Phi(X_2)\| \neq 0}} \frac{\|[\tau_{r_2}(\Phi(X_2)), B]\|}{\|\Phi(X_2)\|} \Bigg).\end{aligned} \tag{1.24}$$

Denote by $\delta_B : P_f(\Lambda) \to \{0, 1\}$ the function such that $\delta_B(X) = 1$ if and only if $S_B \cap X \neq \emptyset$. The Nth application of (1.22) yields:

$$\frac{\|[\tau_t(A), B]\|}{\|A\|} \leq \sum_{n=1}^{N} a_n + R_N \tag{1.25}$$

for coefficients defined:

$$a_n = 2^n \sum_{\substack{X_1 : X_1 \cap S_A \neq \emptyset \\ \|\Phi(X_1)\| \neq 0}} \cdots \sum_{\substack{X_n : X_n \cap X_{n-1} \neq \emptyset \\ \|\Phi(X_n)\| \neq 0}} \prod_{i=1}^{n-1} \|\Phi(X_i)\| \int_0^{|t|} dr_1 \cdots \int_0^{r_{n-1}} dr_n \, \frac{\|[\tau_{r_n}(\Phi(X_n)), B]\|}{\|\Phi(X_n)\|} \delta_B(X_n)$$

$$R_N = 2^{N+1} \sum_{\substack{X_1 : X_1 \cap S_A \neq \emptyset \\ \|\Phi(X_1)\| \neq 0}} \cdots \sum_{\substack{X_{N+1} : X_{N+1} \cap X_N \neq \emptyset \\ \|\Phi(X_{N+1})\| \neq 0}} \prod_{i=1}^{N+1} \|\Phi(X_i)\| \int_0^{|t|} dr_1 \cdots \int_0^{r_{N+1}} dr_{N+1} \, \frac{\|[\tau_{r_{N+1}}(\Phi(X_{N+1})), B]\|}{\|\Phi(X_{N+1})\|} \tag{1.26}$$

The series $\sum a_n$ can be seen to be convergent since:

$$\begin{aligned} a_n &\leq 2^{n+1} \sum_{a \in S_A} \sum_{b \in S_B} \sum_{x_1, \ldots, x_n \in \Lambda} \sum_{\substack{X_1 \subseteq \Lambda \\ a, x_1 \in X_1}} \cdots \sum_{\substack{X_{n-1} \subseteq \Lambda \\ x_{n-1}, x_n \in X_{n-1}}} \sum_{\substack{X_n \subseteq \Lambda \\ x_n, b \in X_n}} \prod_{i=1}^{n} \|\Phi(X_i)\| \\ &\leq \frac{2}{C_F} \sum_{a \in S_A} \sum_{b \in S_B} \frac{(2\|\Phi\|_F C_F |t|)^n}{n!} F(\|s - a\|) \end{aligned} \tag{1.27}$$

from which we derive the bound:

$$\sum_{n=1}^{N} a_n \leq \frac{2}{C_F} \|A\| \|B\| (e^{2\|\Phi\|_F C_F |t|} - 1) \sum_{a \in S_A} \sum_{b \in S_B} F(\|a - b\|). \tag{1.28}$$

The theorem follows from showing that the remainder R_N tends to 0. But this is evident from the same series manipulations as in (1.27):

$$R_N \leq \left(\frac{2\|F\| |S_A|}{C_F} \right) \frac{(2C_F \|\Phi\|_F |t|)^{N+1}}{(N+1)!} \|B\|. \tag{1.29}$$

□

We also record, for completeness, the existence of a family of maps $\mathbb{E}_X : \mathfrak{A} \to \mathfrak{A}_X$ which satisfy the defining features of a conditional expectation. These maps will be useful in approximating quasi-local observables, particularly time-evolutions of local observables by the Heisenberg dynamics of a rapidly decaying interaction.

Theorem 1.5.2 (Proposition 2.2 of [22]). *There exists a collection* $\{\mathbb{E}_\Lambda : \Lambda \in P_f(\mathbb{Z}^D)\}$ *of completely positive maps* $\mathbb{E}_\Lambda : \mathfrak{A} \to \mathfrak{A}_\Lambda$ *such that:*

$$\begin{aligned}&(1)\ \ \forall A \in \mathfrak{A}_\Lambda : \quad \mathbb{E}_\Lambda(A) = A \\ &(2)\ \ \forall C, D \in \bigcup_{Z: Z\cap\Lambda=\emptyset} \mathfrak{A}_Z, A \in \mathfrak{A}_\Lambda : \\ &\qquad \mathbb{E}_\Lambda(CAD) = C\mathbb{E}_\Lambda(A)D.\end{aligned} \tag{1.30}$$

Furthermore, if $A \in \mathfrak{A}$ *is such that for all* $B \in \bigcup_{Z:Z\cap\Lambda=\emptyset} \mathfrak{A}_Z$,

$$\|[A, B]\| \leq \epsilon\|A\|\|B\| \tag{1.31}$$

then $\|\mathbb{E}_\Lambda(A) - A\| \leq \epsilon\|A\|$. *Lastly, the maps are ordered by inclusion precisely in the sense that if* $\Lambda_0 \subset \Lambda$ *then:*

$$\mathbb{E}_\Lambda \circ \mathbb{E}_{\Lambda_0}(A) = \mathbb{E}_{\Lambda_0} \circ \mathbb{E}_\Lambda(A) = \mathbb{E}_{\Lambda_0}(A). \tag{1.32}$$

The proof of this theorem is detailed in Section 3 of [22], and we only record the definition of $\mathbb{E}_\Lambda$. Let ρ_{Λ^c} be the product state on $\mathfrak{A}_{\Lambda^c}$ whose tensor factors are normalized trace. Then:

$$\mathbb{E}_\Lambda = \mathrm{id}_\Lambda \otimes \rho_{\Lambda^c}. \tag{1.33}$$

One application of a Lieb-Robinson bound is proving the existence of the Heisenberg dynamics in the infinite-volume limit.

Theorem 1.5.3. *Suppose* $\|\Phi\|_F < \infty$ *for* $F(r) \leq \frac{1}{(1+r)^{D+\varepsilon}}$. *There exists a group of automorphisms* $\tau^\Phi : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A})$ *such that for all* $A \in \mathfrak{A}$*:*

$$\tau_t^\Phi(A) = \lim_{\Lambda\to\mathbb{Z}^D} \tau_t^{\Phi,\Lambda}(A). \tag{1.34}$$

For each $A \in \mathfrak{A}$, *the map* $t \mapsto \tau_t^\Phi(A)$ *is norm continuous.*

Proof. First we show the existence of the limit in (1.34). Let $\Lambda_n = [-n, n]^D$ and denote $\tau_t^{\Phi,\Lambda_n} = \tau_t^n$.

The claim is that for fixed $t \in \mathbb{R}$ and $A \in \mathfrak{A}_{\mathrm{loc}}$, the sequence $\left(\tau_t^n(A)\right)$ converges as $n \to \infty$. Suppose $n > m$ are natural numbers such that $S_A \subset \Lambda_m$. By Lemma 1.5.1,

$$\|\tau_t^n(A) - \tau_t^m(A)\| \leq \int_0^{|t|} dr \; \|[H_{\Lambda_n}(\Phi) - H_{\Lambda_m}(\Phi), \tau_r^m(A)]\|. \tag{1.35}$$

The difference of Hamiltonians in the above bound can be separated into two summands:

$$\begin{aligned} \mathcal{I}_1 &= \{X \subset \Lambda_n : X \not\subset \Lambda_m, \; X \cap S_A = \emptyset\} \\ \mathcal{I}_2 &= \{X \subset \Lambda_n : X \not\subset \Lambda_m, \; X \cap S_A \neq \emptyset\} \end{aligned} \tag{1.36}$$

$$H_{\Lambda_n}(\Phi) - H_{\Lambda_m}(\Phi) = \sum_{X \in \mathcal{I}_1} \Phi(X) + \sum_{Y \in \mathcal{I}_2} \Phi(Y).$$

The contributions to (1.35) from $\mathcal{I}_1$ can be bounded by the Lieb-Robinson bound:

$$\begin{aligned} \|[\sum_{X \in \mathcal{I}_1} \Phi(X), \tau_r^m(A)]\| &\leq \sum_{k=d(S_A,\Lambda_m)}^{2n} \sum_{\substack{X \in \mathcal{I}_1 \\ d(X,S_A)=k}} \|[\Phi(X), \tau_r^m(A)]\| \\ &\leq \sum_{k=d(S_A,\Lambda_m)}^{2n} \frac{2\|A\|}{C_F}(e^{v|r|}-1) \sum_{\substack{X \in \mathcal{I}_1 \\ d(X,S_A)=k}} \|\Phi(X)\| \sum_{a \in S_A} \sum_{y \in X} F(\|a-y\|) \\ &\leq \sum_{k=d(S_A,\Lambda_m)}^{2n} \frac{2 \cdot 3^D \|A\|}{C_F}(e^{v|r|}-1)\|\Phi\|_F |S_A| \frac{1}{k^{1+\varepsilon}} \\ &\leq \frac{2 \cdot 3^D \|A\|}{C_F}(e^{v|r|}-1)\|\Phi\|_F |S_A| \frac{1}{d(S_A,\Lambda_m)^{\varepsilon}} \end{aligned} \tag{1.37}$$

while the contributions from $\mathcal{I}_2$ can be controlled by the decay of the interaction:

$$\begin{aligned} \|[\sum_{Y \in \mathcal{I}_2} \Phi(Y), \tau_r^m(A)]\| &\leq \sum_{x \in \Lambda_n \setminus \Lambda_m} \sum_{a \in S_A} \sum_{X: x,a \in X} \|[\Phi(Y), \tau_r^m(A)\| \\ &\leq 2 \cdot 3^D \|A\| \|\Phi\|_F |S_A| \frac{1}{d(S_A,\Lambda_m)^{\epsilon}}. \end{aligned} \tag{1.38}$$

So denote the limit of this Cauchy sequence by: $\tau_t^{\Phi}(A) = \lim_{n\to\infty} \tau_t^n(A)$. The limit in (1.34) follows by fact that for any $m \in \mathbb{N}$, if $\Lambda \supset \Lambda_m$ then, by replacing $\tau_t^n(A)$ with $\tau_t^{\Phi,\Lambda}(A)$ in (1.35):

$$\begin{aligned} \|\tau_t^{\Phi,\Lambda}(A) - \tau_t^{\Phi}(A)\| &\le \|\tau_t^{\Phi,\Lambda}(A) - \tau_t^n(A)\| + \|\tau_t^n(A) - \tau_t^{\Phi}(A)\| \\ &\le 2 \cdot 3^D \|A\| \|\Phi\|_F |S_A| e^{v|t|} d(S_A, \Lambda_m)^{-\epsilon}. \end{aligned} \tag{1.39}$$

The group law $\tau_t^{\Phi} \circ \tau_s^{\Phi} = \tau_{t+s}^{\Phi}$ follows from the fact that τ^{Φ} is the pointwise limit of one-parameter groups, and that the convergence is uniform over compact intervals of time. And by similar computations as above, there exists $C > 0$ independent of Λ such that:

$$\|\tau_t^{\Lambda}(A) - \tau_{t_0}^{\Lambda}(A)\| \le \int_{\min\{t_0,t\}}^{\max\{t,t_0\}} dr \; \|[H_\Lambda(\Phi), \tau_r^{\Lambda}(A)]\| \le C|t - t_0| \|A\| \tag{1.40}$$

The continuity follows from taking the limit as $\Lambda \to \mathbb{Z}^D$. □

We end this section with a description of the infinitesimal generators of the Heisenberg dynamics.

Definition 1.5.1. *Let $\mathfrak{X}$ be a Banach space. A C_0-**group** $T : \mathbb{R} \to B(\mathfrak{X})$ is a group representation of $\mathbb{R}$ such that for all $x \in \mathfrak{X}$, $t \mapsto T(t)x$ is a continuous map.*

Definition 1.5.2. *Let $\mathfrak{X}$ be a Banach space and $T : \mathbb{R} \to B(\mathfrak{X})$ be a C_0-group. The **infinitesimal generator** A of T is defined as the linear operator:*

$$\begin{aligned} \mathrm{Dom}(A) &= \left\{ x \in \mathfrak{X} : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists.} \right\} \\ \forall x \in \mathrm{Dom}(A), &\quad Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}. \end{aligned} \tag{1.41}$$

The infinitesimal generator of a C_0-group is a closed operator. In fact, the condition in (1.42) of the theorem below characterizes generators of C_0-semigroups, although we will not use this fact.

Theorem 1.5.4 (see Theorem 2.13 of [12])**.** *Suppose A is the generator of a C_0-group T. Denote by $\rho(A)$ the resolvent set of A. Then A is closed, densely defined, and there exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\lambda > \omega$ implies $\lambda \in \rho(A)$ and*

$$\|(\lambda - \omega)^n (\lambda - A)^{-n}\| \le M, \; \forall n \in \mathbb{N}. \tag{1.42}$$

We will only consider generators arising from unitary groups or groups which are quasi-local dynamics of the form (1.15) or (1.34). When the context is clear, we will use the term *generator*. The generators δ^Φ and $\delta^{\Phi,\Lambda}$ of τ^Φ and $\tau^{\Phi,\Lambda}$, respectively, will be particularly important in the discussion of ground states.

In Theorem 1.5.3, it was proven that, assuming sufficient decay of the interaction, the map $t \mapsto \tau_t^\Phi(A)$ is continuous with respect to the norm topology. Hence $\tau^\Phi : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A})$ defines a C_0-group, and the generator of τ^Φ is a closed operator.

Because of the locality of the interaction Φ, $\delta^{\Phi,\Lambda}$ and δ^Φ have concrete formulas in terms of the lattice structure and the local Hamiltonians $\{H_\Lambda(\Phi) : \Lambda \in P_f(\mathbb{Z}^D)\}$. The former is given by a simple computation and the latter is the content of Proposition 1.5.1.

Example 1.5.1 (Generators of local dynamics)**.** For any $X \in P_f(\mathbb{Z}^D)$, the map $t \mapsto \tau_t^{\Phi,X}$ is continuous with respect to the operator norm. Hence the generator is a bounded operator with $\mathrm{Dom}(\delta^{\Phi,X}) = \mathfrak{A}$, and the generator can be computed as a derivative:

$$\delta^{\Phi,X}(A) = \frac{d}{dt} e^{itH_X(\Phi)} A e^{-itH_X(\Phi)} \Big|_{t=0} = i[H_X(\Phi), A]. \tag{1.43}$$

Proposition 1.5.1. *Suppose* $\|\Phi\|_F < \infty$ *for* $F(r) \le \frac{1}{(1+r)^{D+\varepsilon}}$*. Then* $\mathfrak{A}_{\mathrm{loc}}$ *is a core of* $\mathrm{Dom}(\delta^\Phi)$*, and for* $A \in \mathfrak{A}_{\mathrm{loc}}$*:*

$$\delta^\Phi(A) = i \sum_{\substack{X \in P_f(\mathbb{Z}^D): \\ X \cap S_A \neq \emptyset}} [\Phi(X), A]. \tag{1.44}$$

Proof. By (1.40), for fixed A, the sequence of functions $t \mapsto \tau_t^{\Phi,[-n,n]^D}$ converges uniformly on compact intervals to $t \mapsto \tau_t^\Phi(A)$. By Example 1.5.1 and an interchange of limits:

$$\lim_{t \downarrow 0} \frac{\tau_t^\Phi(A) - A}{t} = \lim_{n \to \infty} \delta^{\Phi,n}(A) = \lim_{n \to \infty} i[H_{[-n,n]^D}(\Phi), A]. \tag{1.45}$$

The bounds in (1.37) and (1.38) of Theorem 1.5.3 imply that the sequence $(\delta^{\Phi,n}(A))$ converges. Hence $A \in \mathrm{Dom}(\delta)$ and the formula in (1.44) is valid. $\square$

It follows from Proposition 1.5.1 that for all $A, B \in \mathrm{Dom}(\delta^\Phi)$,

$$\delta^\Phi(A^*) = \delta^\Phi(A)^* \text{ and } \delta^\Phi(AB) = \delta^\Phi(A)B + A\delta^\Phi(B). \tag{1.46}$$

That is to say, δ^Φ is an example of a $*$-*derivation.*

1.6 Quantum Spin System

Let $\Phi \in B_F(\mathbb{Z}^D)$ for any $\mathcal{F}$-function. In Section 1.5 we recorded the existence of an automorphism $\tau^\Phi : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A})$ such that for all $A \in \mathfrak{A}_{\mathrm{loc}}$:

$$\tau_t^\Phi(A) = \lim_{\Lambda \to \mathbb{Z}^D} \tau_t^{\Phi,\Lambda}(A)$$

where the limit is taken over the net of finite volumes Λ ordered by inclusion.

Definition 1.6.1. *A* ***quantum spin system*** *is a C^*-dynamical system of the form* $(\mathfrak{A}, \tau^\Phi)$.

1.7 States

States act on observables to produce the expected values for the system. In a quantum spin system, states are exactly the set of positive unital linear functionals of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ induces an important cyclic representation in the following way. The closed left ideal:

$$N_\varphi = \{A \in \mathfrak{A} : \varphi(A^*A) = 0\} \tag{1.47}$$

defines the zero-length vectors of the pre-Hilbert space $(\mathfrak{A}, \langle \cdot, \cdot \rangle_\varphi)$ for inner product defined by $\langle A, B \rangle_\varphi = \varphi(A^*B)$. The representation $\pi_\varphi : \mathfrak{A} \to B(\overline{\mathfrak{A}/N_\varphi})$ defined on the dense subset $\mathfrak{A}/N_\varphi$ by $\pi_\varphi(A)(B + N_\varphi) = (AB) + N_\varphi$ is called the Gelfand-Naimark-Segal (**GNS)** representation. The vector $\Omega_\varphi = 1 + N_\varphi$ is a cyclic vector for this representation, and furthermore,

$$\varphi(A) = \langle \Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle. \tag{1.48}$$

The purpose of this section is to develop properties of certain classes of states which will be the central objects of study in later chapters. We record the definitions of two important classes of states: *ground states* and *split states*. Their GNS representations of the quasi-local algebra will have critical roles in the description of ground state and SPT phases. We also discuss the quasi-adiabatic evolution of ground states.

Definition 1.7.1. *A state ω of $\mathfrak{A}$ is a* ***ground state*** *of the quantum spin system $(\mathfrak{A}, \tau^\Phi)$ if for all $A \in \mathrm{Dom}(\delta^\Phi)$,*

$$-i\omega(A^*\delta^\Phi(A)) \geq 0 \tag{1.49}$$

Let $\mathcal{E}_0(\Phi)$ denote the set of ground states of an interaction Φ. Proposition 1.7.1 shows that given a QSS $(\mathfrak{A}, \tau^\Phi)$, at least one ground state ω_Φ of the dynamics always exists.

Proposition 1.7.1 (cf. Theorem 4.2.5 of [30])**.** *Suppose $\|\Phi\|_F < \infty$ for $F(r) \leq \frac{1}{(1+r)^{D+\epsilon}}$. Then $\mathcal{E}_0(\Phi)$ is not the empty set.*

Proof. Let $\pi : \mathfrak{A} \to B(\mathfrak{H})$ be a faithful $*$-representation on a Hilbert space, guaranteed to exist by the Gelfand-Naimark theorem. Denote $H_n = H_{[-n,n]^D}(\Phi)$, and define:

$$K_n = \pi(H_n) - \inf \mathrm{spec}(H_n) \geq 0. \tag{1.50}$$

Let $\varphi_n : B(\mathfrak{H}) \to \mathbb{C}$ be a vector state from the kernel of K_n. There exists a $*$-weak limit φ of some subsequence φ_{n_k} of these states, by the $*$-weak compactness of the states of $B(\mathfrak{H})$, and so we may define:

$$\begin{aligned} &\omega : \mathfrak{A} \to \mathbb{C} \\ &\omega(A) = \varphi \circ \pi(A). \end{aligned} \tag{1.51}$$

Let $\delta^\Phi = \delta$ and $\delta^{\Phi,[-n,n]^D} = \delta_n$ denote the infinite and finite-volume generators, respectively, of the Heisenberg dynamics. By Proposition 1.5.1 and (1.45), we have that for any $A \in \mathfrak{A}_{\mathrm{loc}}$, $A \in \mathrm{Dom}(\delta)$

and:

$$\delta(A) = \lim_n \delta_n(A). \tag{1.52}$$

And so:

$$\begin{aligned} |\omega(A^*\delta(A)) - \omega_{n_k}(A^*\delta_{n_k}(A))| &\le |(\omega - \omega_{n_k})(A^*\delta(A))| + 2\|A\|\|(\delta - \delta_n)(A)\| \\ &= |(\varphi - \varphi_{n_k})(\pi(A^*\delta(A)))| + 2\|A\|\|(\delta - \delta_n)(A)\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{1.53}$$

This justifies the coupled limit in the first line of the following computation:

$$\begin{aligned} -i\omega(A^*\delta(A)) &= \lim_{k\to\infty} \varphi_{n_k}(\pi(A^*[H_{n_k}, A])) \\ &\ge \liminf_m \left[\varphi_m\big(\pi(A^*H_mA)\big) - \varphi_m\big(\pi(A^*A)\pi(H_m)\big)\right] \ge 0. \end{aligned} \tag{1.54}$$

The statement follows by fact that $\mathfrak{A}_{\mathrm{loc}}$ is dense in $\mathfrak{A}$ and the fact that δ is a closed derivation. □

Evidently $\mathcal{E}_0(\Phi)$ is a $*$-weak compact, convex subset of the states of $\mathfrak{A}$. Furthermore, by Theorem 5.3.37 of [6], $\mathcal{E}_0(\Phi)$ is a face of the states of $\mathfrak{A}$, and the extreme points of $\mathcal{E}_0(\Phi)$ are pure states. It is straightforward to show that the ground state condition implies $\omega(\delta^\Phi(A)) = 0$ for any $A \in \mathrm{Dom}(\delta^\Phi)$ and $\omega \in \mathcal{E}_0(\Phi)$, and so $\omega \circ \tau_t^\Phi = \omega$. The following well-known fact records how the Heisenberg dynamics induce a covariant representation of $\mathbb{R}$ in the GNS representation of the ground state ω.

Lemma 1.7.1. *Let $(\mathfrak{H}_\omega, \Omega_\omega, \pi_\omega)$ denote the GNS representation of ω, as constructed above. There exists a C_0-group $U : \mathbb{R} \to \mathrm{Unitaries}(\mathfrak{H}_\omega)$ such that:*

$$\pi_\omega \circ \tau_t^\Phi = \mathrm{Ad}_{U_t} \circ \pi_\omega. \tag{1.55}$$

Let $-iH_\omega$ denote the skew-adjoint generator of U. Then $H_\omega \ge 0$ and $H_\omega\Omega_\omega = 0$.

Proof. For ease of notation, we omit the dependence on ω of the representation and let A denote an arbitrary element of $\mathfrak{A}$. For fixed $t \in \mathbb{R}$, define U_t first by its adjoint on the dense subspace $\pi(\mathfrak{A})\Omega$

by:

$$U_t^* \pi(A)\Omega = \pi(\tau_t^\Phi(A))\Omega. \tag{1.56}$$

U_t^* is bounded on this dense subspace since:

$$\|U_t^* \pi(A)\Omega\|^2 = \omega(\tau_t(A^*A)) = \omega(A^*A) = \|\pi(A)\Omega\|^2. \tag{1.57}$$

Equation (1.57) implies that the extension of U_t^* to $\mathfrak{H}$ is unitary and that $U_t\Omega = \Omega$. The group law and strong continuity of $t \mapsto U_t$ follow from the same properties of the map $t \mapsto \tau_t^\Phi$, since:

$$\begin{gathered} U_s^* U_t^* \pi(A)\Omega = \pi(\tau_t \tau_s(A))\Omega = U_{s+t}^* \pi(A)\Omega \\ \|(U_t^* \pi(A) - U_s^* \pi(A))\Omega\| \leq \|\tau_{s-t}^\Phi(A) - A\| \to 0 \text{ as } s \to t. \end{gathered} \tag{1.58}$$

Hence by Stone's theorem, there exists a self adjoint operator H such that $-iH$ generates U and $\pi(\mathfrak{A})\Omega \subset \mathrm{Dom}(H)$. As it is clear that $H\Omega = 0$, we lastly prove that H is non-negative. Equation (1.56) and continuity of π imply $\delta(\pi(A)) = \pi(\delta_\Phi(A))$, where δ_Φ is the generator of τ^Φ and δ the generator of Ad_{U_t}. And so:

$$\langle \pi(A)\Omega, H\pi(A)\Omega\rangle = \langle \Omega, \pi(A^*)[H, \pi(A)]\Omega\rangle = \omega(A^*\delta_\Phi(A)) \geq 0. \tag{1.59}$$

□

The operator H_ω is called the *GNS Hamiltonian* or *bulk Hamiltonian*. It is the Hamiltonian operator of the bulk system, and its properties are closely related to those of the ground state. For example, it is known that the existence of a ground state gap in the spectrum of H_ω and $\dim\ker(H_\omega) = 1$ imply exponential decay of correlations in the ground state for a broad class of quantum spin systems [23, 13]. The precise gap condition required is recorded below.

Definition 1.7.2. *A ground state ω of $(\mathfrak{A}, \tau^\Phi)$ is **gapped** if there exists $\gamma > 0$ such that:*

$$\inf_{\substack{\psi \in \mathrm{Dom}(H_\omega): \\ \|\psi\|=1,\ \psi \perp \ker(H_\omega)}} \langle \psi, H_\omega \psi\rangle \geq \gamma. \tag{1.60}$$

A closely related but inequivalent condition is a lower bound on the spectral gaps of the local Hamiltonians $H_\Lambda(\Phi)$ which is uniform in the volume Λ.

Definition 1.7.3. *An interaction* $\Phi : P_f(\mathbb{Z}^D) \to \mathfrak{A}_{\mathrm{loc}}$ *is* ***uniformly locally gapped*** *if* (1) *there exist constants* $\gamma, R > 0$ *and a consistent partitioning:*

$$\mathrm{spec}(H_\Lambda(\Phi)) = \mathrm{spec}_0(H_\Lambda(\Phi)) \cup \mathrm{spec}_+(H_\Lambda(\Phi)) \tag{1.61}$$

such that for all connected volumes $\Lambda \in P_f(\mathbb{Z}^D)$*, if* $\mathrm{diam}(\Lambda) > R$*, then:*

$$\min\left\{\lambda - \mu : \lambda \in \mathrm{spec}_+(H_\Lambda(\Phi)), \mu \in \mathrm{spec}_0(H_\Lambda(\Phi))\right\} \geq \gamma; \tag{1.62}$$

and (2) $\lim_{\Lambda \to \mathbb{Z}^D} \mathrm{diam}(\mathrm{spec}(H_\Lambda(\Phi))) = 0$.

This definition of a uniform local gap of Φ includes the condition that the lower energies of $H_\Lambda(\Phi)$ shrink in diameter as Λ tends to the lattice volume. Mathematically, much of the analysis of finite-volume gapped ground states can be done with only condition (1). However, condition (2) is important for the analysis of infinite-volume ground states, such as in Lemma 3.14 of [19] which shows that gapped ground states of a certain class of frustration-free spin chains can be stably perturbed. And since condition (2) arises as a natural consequence of the perturbation theory of frustration free systems, we include it in Definition 1.7.3.

It is a remarkable principle for quantum spin systems that smooth, gapped curves of ground states of rapidly decaying interactions can be written in terms of quasi-local, time-dependent dynamics. This principle is often referred to as *quasi-adiabatic evolution*, and the dynamics are the *spectral flow.* In finite-volume, the quasi-adiabatic evolution provides an explicit flow of spectral projectors over smoothly evolving, isolated regions. A complete construction of the spectral flow and proof of the quasi-adiabatic principle can be found in Section VI of [24] and [3], where the following proposition is made applicable in a many-body setting for different rates of decay of interaction.

Proposition 1.7.2 (Proposition 2.4 of [3])**.** *Suppose:*

(1) *for* $0 \leq s \leq 1$, $H(s)$ *is a smooth family of self-adjoint Hamiltonians with bounded derivative* $H'(s)$ *such that* $\|H'(s)\|$ *is uniformly bounded for* $s \in [0, 1]$*;*

(2) *that the spectrum spec(s) of* $H(s)$ *can be decomposed into disjoint sets* $\mathrm{spec}_1(s), \mathrm{spec}_2(s)$ *such that:*

$$\inf_{0\leq s\leq 1} \inf\{|\lambda-\mu| : \lambda \in \mathrm{spec}_1(s), \mu \in \mathrm{spec}_2(s)\} > 0; \tag{1.63}$$

(3) *there are compact intervals* $I(s)$ *with end points depending smoothly on* s *and such that* $\mathrm{spec}_1(s) \subset I(s) \subset (\mathbb{R} \setminus \mathrm{spec}_2(s))$*, so that*

$$\inf_{0\leq s\leq 1} \inf\{|a-b| : a \in I(s), b \in \mathrm{spec}_2(s)\} > 0; \tag{1.64}$$

(4) $\omega_\gamma \in L^1(\mathbb{R})$ *is a real-valued function such that* $\int dt\omega_\gamma(t) = 1$ *and* $\mathrm{supp}(\hat{\omega}_\gamma) \subset [-\gamma, \gamma]$.

Then the spectral projector $P(s)$ *supported on* $\mathrm{spec}_1(s)$ *is given by:*

$$P(s) = U(s)P(0)U(s)^* \tag{1.65}$$

for unitaries solving:

$$\begin{aligned} -i\frac{d}{ds}U(s) &= D(s)U(s), \quad U(0) = \mathbb{1} \\ D(s) &= \int_{-\infty}^{\infty} dt\ \omega_\gamma(t) \int_0^t du\ e^{iuH(s)}H'(s)e^{-iuH(s)}. \end{aligned} \tag{1.66}$$

In the setting of QSS, the unitaries $U_\Lambda(s)$ that appear in (1.65) when $H(s)$ is a local Hamiltonian over Λ define the finite-volume spectral flow $\alpha_s^\Lambda = \mathrm{Ad}_{U_\Lambda(s)}$.

Next, we discuss an equivalence relation on factor states which is weaker than unitary equivalence of their GNS representations.

Definition 1.7.4. *Let* π_1, π_2 *be* $*$*-representations of a unital* C^**-algebra* $\mathcal{A}$*, and denote by* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *the generated von Neumann algebras from* $\pi_1(\mathcal{A})$ *and* $\pi_2(\mathcal{A})$*, respectively.* π_1 *and* π_2 *are* ***quasi-equivalent*** *representations if* $\mathcal{B}_1, \mathcal{B}_2$ *are factors and there exists a* $*$*-isomorphism* $\tau : \mathcal{B}_1 \to \mathcal{B}_2$ *such that* $\pi_1(x) = \pi_2 \circ \tau(x)$*, for all* $x \in \mathcal{B}_1$*. Say that two factor states* ω, φ *of* $\mathcal{A}$ *are* ***quasi-equivalent*** *states if their GNS representations are quasi-equivalent.*

We will express the quasi-equivalence relation between states by $\sim$. The notion of quasi-equivalence can be broadened to non-degenerate $*$-representations; however, since we consider only representations of $\mathfrak{A}$ which arise from factor states, such as pure ground states, we may freely use an asymptotic condition for quasi-equivalence (Corollary 2.6.11 in [5]): $\omega \sim \varphi$ if and only if for all $\varepsilon > 0$, there exists $X_\varepsilon \in P_f(\mathbb{Z}^D)$ such that $Y \in P_f(\mathbb{Z}^D)$ and $B \in \mathfrak{A}_Y$ with $Y \cap X_\varepsilon = \emptyset$ imply:

$$|\omega(B) - \varphi(B)| \leq \|B\|\varepsilon. \tag{1.67}$$

In Section 3, we study the consequences of the asymptotic relation in (1.67) in the context of symmetry protected topological phases when $D = 1$. A closely related feature of certain pure ground states is the *split property.*

Definition 1.7.5 (Definition 2.1 of [27]). *A pure state φ of a 1D QSS has the* ***split property*** *if $\pi_{\varphi_R}(\mathfrak{A}_{[0,\infty)})''$ is a type I factor, where π_{φ_R} is the GNS representation of the restriction φ_R of φ to the right half-infinite chain algebra $\mathfrak{A}_{[0,\infty)}$.*

The split property is naturally connected to the notion of quasi-equivalence. In [16], it is shown that a state ω of $\mathfrak{A}$ with bounded entanglement entropy is quasi-equivalent to a product state $\psi \otimes \varphi$, where ψ is a state of $\mathfrak{A}_{(-\infty,0]}$ and φ is a state of $\mathfrak{A}_{(0,\infty)}$. The following result characterizes the split property for pure states in terms of quasi-equivalence.

Proposition 1.7.3 (cf. Proposition 2.2 of [17]). *Consider $\mathfrak{A}$ as the quasi-local algebra of the integer lattice $\mathbb{Z}$, i.e. $D = 1$. Let φ be a pure state of $\mathfrak{A}$. The following conditions are equivalent:*

(i) *φ is quasi-equivalent to the state which is the tensor product of restrictions, $\varphi|_{\mathfrak{A}_L} \otimes \varphi|_{\mathfrak{A}_R}$,*

(ii) *φ has the split property.*

where $\mathfrak{A}_L = \mathfrak{A}_{(-\infty,0]}$ and $\mathfrak{A}_R = \mathfrak{A}_{(0,\infty)}$.

2 | Summary of Results

In this chapter, we summarize the results of the following three papers, which are presented in chronological order of publication.

The first paper, “Stability of gapped ground state phases of spins and fermions in one dimension," was published in the Journal of Mathematical Physics, Vol. 59 (2018). The paper is based on joint work with co-author Bruno Nachtergaele, at the Department of Mathematics and Center for Quantum Mathematics and Physics, University of California, Davis. The authors were supported by National Science Foundation Grants DMS-1207995 (A.M.), DMS-1515850 (A.M. and B.N.) and DMS-1813149 (B.N.).

The second paper, “Automorphic equivalence preserves the split property," was published in the Journal of Functional Analysis, Vol. 277 (2019). The author was supported by the National Science Foundation Grant DMS-1813149. The work was partially completed during the 2018 Thematic Semester, *Mathematical challenges in many-body physics and quantum information,* at the Centre de recherches mathématiques, where the author was supported by a Simons-CRM research grant due to Bruno Nachtergaele.

The third paper, “Automorphic equivalence within gapped phases in the bulk," was published in the Journal of Functional Analysis, Vol. 278 (2020). The paper is based on joint work with co-author Yoshiko Ogata at the Graduate School of Mathematical Sciences, University of Tokyo. The authors were supported by the National Science Foundation Grant DMS-1813149 (A.M.), and JSPS KAKENHI Grant Numbers 16K05171 and 19K03534 and JST CREST Grant Number JPMJCR19T2 (Y.O.).

2.1 Stability of gapped ground state phases of spins and fermions in one dimension

Rigorously, perturbative lower bounds for the uniform local ground state spectral gap of quantum spin system interactions have been proven for *frustration free* systems, which are defined by the requirement that a ground state of a local Hamiltonian minimizes each of its interaction terms. Pioneering work in [7, 8, 18] used periodic boundary conditions and *Local Topological Quantum Order* (**LTQO**), a "local indistinguishability" condition on the ground state space, to prove gap stability for perturbations of the form $\Phi_s = \Phi + sV$, $s \in [0, 1]$, of non-negative, frustration free interactions Φ with LTQO. The technical accomplishment of [7, 8, 18, 19] is proving a relative form bound of the perturbed Hamiltonians of local systems using constants which are independent of system size. This is accomplished by a locally block-diagonal decomposition of a unitarily equivalent system in terms of the conditional expectation map from Theorem 1.5.2 and the spectral flow automorphism defined in Proposition 1.7.2.

In lattices with open boundary conditions, boundary perturbations may violate LTQO and prevent the validity of critical estimates in [7, 8, 18]. The main result of this paper, Theorem 3.11, guarantees that phases of uniformly locally gapped, topologically ordered ground states of frustration free spin chain interactions are stable under small but extensive perturbations which may involve the boundary.

> **Theorem 3.11:** Suppose $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}_{\mathrm{loc}}$ has LTQO with $\Omega(n) \leq n^{-\nu}$ for $\nu > 4$ and there exist $K > 0, s \in (0, 1]$ such that h_Φ satisfies $h_\Phi(r) \geq Kr^s$. Then there exists $\varepsilon(\gamma_0) > 0$ such that $0 \leq \varepsilon < \varepsilon(\gamma_0)$ and $\mathrm{diam}\Lambda > \max\{2D, R\}$ imply
>
> $$\gamma(H_\Lambda(\varepsilon)) \geq \gamma_0 - (m + 2M_D)\varepsilon > 0$$

The constant $\varepsilon(\gamma_0)$ can be taken as

$$\varepsilon(\gamma_0) = \min\{1, \gamma_0/(m + 2M_D)\}\,.$$

An exact description of the mathematical assumptions for the theorem is in Section II.B. The constants m and M_D are parameters of the system, and in particular M_D depends on the strength of the interaction at the boundaries of the intervals. To illustrate this result, we present a well-known example of an interaction which satisfies the conditions of Theorem 3.11: the Affleck-Kennedy-Lieb-Tasaki (**AKLT**) spin $S = 1$ chain, which can be constructed as follows. By the Clebsch-Gordan decomposition, the state space for nearest neighbor pairs of spins $x, x+1$ can be written as a direct sum of irreducible representations of SU(2). When $S = 1$, this multiplicity-free direct sum decomposition is:

$$\mathfrak{H}_x \otimes \mathfrak{H}_{x+1} = D^{(0)}_{x,x+1} \oplus D^{(1)}_{x,x+1} \oplus D^{(2)}_{x,x+1}, \tag{2.1}$$

where the dimension of $D^{(j)}_{x,x+1}$ is $2j+1$. Let $P^{(2)}_{x,x+1}$ denote the projection onto $D^{(2)}_{x,x+1}$. Then the AKLT interaction is defined:

$$\Phi^{\mathrm{AKLT}}(X) = \begin{cases} P^{(2)}_{x,x+1} & \text{if } X = \{x, x+1\} \text{ for some } x \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \tag{2.2}$$

Evidently the interaction terms are non-negative. Results from the theory of finitely correlated states in [11] imply that for sufficiently large interval $\Lambda = [-n, n]$, the kernel of $H_\Lambda(\Phi^{\mathrm{AKLT}})$ is exactly the span of state vectors of the form:

$$\begin{gathered} \Psi_\Lambda : M_2(\mathbb{C}) \to \bigotimes_{x\in\Lambda} \mathfrak{H}_x \\ \Psi_\Lambda(B) = \sum_{(\sigma_x : x\in\Lambda)\in\{-1,0,1\}^{2n+1}} \operatorname{tr}\Big(B v_{\sigma_n} \cdots v_{\sigma_{-n}}\Big) |\sigma_{-n} \cdots \sigma_n\rangle \end{gathered} \tag{2.3}$$

where $|\iota\rangle$, $\iota \in \{-1, 0, 1\}$ denotes an element of the orthonormal eigenvector basis of the Z spin-matrix, and:

$$v_{-1} = -(2/3)^{1/2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad v_0 = (1/3)^{1/2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad v_1 = (2/3)^{1/2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.4}$$

The dimension of the kernel is 4, and the infinite-volume ground state ω^{AKLT} of the interaction is gapped and unique. Furthermore, the finite-volume ground states are uniformly locally gapped. The fact that the ground state space of the AKLT interaction satisfies the LTQO condition described in Section II.C can be derived from the following estimate: for all $k < n$ and nonzero $A \in \mathfrak{A}_{[-k,k]}$, $B, C \in M_2(\mathbb{C})$:

$$\frac{|\langle \Psi_\Lambda(B), A\Psi_\Lambda(C)\rangle - \omega^{\mathrm{AKLT}}(A)\mathrm{tr}(B^*C)|}{\|A\|_{\mathfrak{A}}\|B\|_{M_2(\mathbb{C})}\|C\|_{M_2(\mathbb{C})}} = O((1/3)^{n-k}). \tag{2.5}$$

Hence, by Theorem 3.11, small perturbations of Φ^{AKLT} remain uniformly locally gapped. The lower spectrum $\mathrm{spec}_0(H_\Lambda(\Phi^{\mathrm{AKLT}}))$ is the 0-eigenvalue group of total multiplicity 4.

We now move away from this example and discuss the results of the paper in generality. Proposition 3.12 shows that higher gaps in the spectrum of frustration free spin chain interactions with topologically ordered ground states are also stable under perturbation. Lemma 3.13 shows that if the perturbative interaction remains far from the boundary of the system, then the diameter of the lower energies of the spectrum, relative to the uniform gap, tends to 0 as the system size grows.

Generally, the conditions of the theorem allow for perturbations which include interaction terms localized at the boundaries of the interval. See equation (3.20) in Section III for the precise form of boundary perturbations we consider. The presence of these boundary perturbations raise the question of whether the spectral subspaces corresponding to the 0-eigenvalue group of the local Hamiltonians are still justifiably ground state spaces. In Section III.D, we prove that the perturbed finite-volume states which are supported over the ground state energies of the system still converge in $*$-weak limit to an infinite-volume ground state of the perturbed Heisenberg dynamics, regardless of the presence of edge perturbations $-$ the ground state condition is verified in Lemma 3.14.

Lastly, in Section IV.B, we extend our results on spin chains to even, spinless fermion interactions using the Jordan-Wigner $*$-automorphisms $\{\vartheta_\Lambda : \Lambda \text{ a finite interval}\}$ defined by:

$$\begin{aligned} &\vartheta_\Lambda : \mathfrak{A}_\Lambda^{\mathrm{CAR}} \to \mathfrak{A}_\Lambda \\ &\vartheta_\Lambda(a(x)) = \exp\Big(-i\pi \sum_{j<x} S_j^+ S_j^- \Big) S_x^- . \end{aligned} \tag{2.6}$$

In Proposition 4.3, it is proven that if Ψ is an even interaction of the CAR algebra satisfying the assumptions of the unperturbed interaction in Section II.B, modified appropriately and in the obvious ways for fermion interactions, then there exists an interaction Φ of the spin $S = 1/2$ algebra of observables such that:

$$H_\Lambda(\Phi) = \vartheta_\Lambda(H_\Lambda(\Psi)). \tag{2.7}$$

Since ϑ_Λ is an inner automorphism, Theorem 3.11 applies to even fermion interactions via the transformation in (2.7). This is the content of Theorem 4.4.

2.2 Automorphic equivalence preserves the split property

It has been remarked in the previous chapter that ground state structures of phase equivalent gapped quantum spin systems can be written in terms of a smooth path of automorphisms which uniformly satisfy a Lieb-Robinson bound. An important proposition about properties of the thermodynamic limit of the spectral flow, from [3], states:

Proposition 2.2.1 (Proposition 5.4 of [3]). *The spectral flow α_s for the infinite system has the following properties:*

(i.) $(\alpha_s)_{s\in[0,1]}$ *is a strongly continuous cocycle of automorphisms of the C^*-algebra of quasi-local observables, and it is the thermodynamic limit of the finite-volume cocycles generated by an interaction Ψ_s. The interaction Ψ_s satisfies a finite $\mathcal{F}$-norm for F_Ψ, which is superpolynomially decaying.*

(ii.) α_s *satisfies the Lieb-Robinson bound*

$$\|[\alpha_s(A), B]\| \leq 2\|A\|\|B\| \min\Big[1, g(s) \sum_{x\in S_A, y\in S_B} F_\Psi(\|x-y\|)\Big], \tag{2.8}$$

for any $A \in \mathfrak{A}_{S_A}$*,* $B \in \mathfrak{A}_{S_B}$*, and* $0 \le s \le 1$ *with* g *given by*

$$C_{F_\Psi} g(t) = \begin{cases} e^{2\|\Psi\| C_{F_\Psi}|t|} - 1 & \text{if } d(S_A, S_B) > 0, \\ e^{2\|\Psi\| C_{F_\Psi}|t|} & \text{otherwise.} \end{cases} \tag{2.9}$$

(iii.) *If* β *is a local symmetry of* Φ_s*, i.e. an automorphism such that* $\beta(\Phi_s(X)) = \Phi_s(X)$ *for all* $X \in P_f(\mathbb{Z}^D)$*, then* β *is also a symmetry of* α_s*, i.e.* $\alpha_s \circ \beta = \alpha_s$*, for all* $s \in [0,1]$*.*

Proposition 5.4 of [3] also discusses the effect of the translation symmetry on the spectral flow. We illustrate (iii) of the previous proposition with the AKLT chain defined in the previous section. It is a computation to show that:

$$P^{(2)}_{x,x+1} = \frac{1}{3}\mathbb{1} + \frac{1}{2}\vec{S}_x \cdot \vec{S}_{x+1} + \frac{1}{6}(\vec{S}_x \cdot \vec{S}_{x+1})^2 \tag{2.10}$$

where $\vec{S}_x \cdot \vec{S}_{x+1} = \sum_{j=1}^3 S_x^j S_{x+1}^j$. We denote by $\Xi_x : \mathfrak{A}_{\{x\}} \to \mathfrak{A}_{\{x\}}$ the unique antilinear $*$-automorphism which maps S_x^j to $-S_x^j$. By continuous extension, there exists an on-site symmetry Ξ determined by:

$$\forall A \in \mathfrak{A}_{\text{loc}} : \quad \Xi(A) = \lim_{\Lambda \to \mathbb{Z}} \left(\bigotimes_{x \in \Lambda} \Xi_x \right)(A). \tag{2.11}$$

Evidently, for all Λ, $\Xi(H_\Lambda(\Phi^{\text{AKLT}})) = H_\Lambda(\Phi^{\text{AKLT}})$. Furthermore, if Φ is another interaction such that $\Xi(\Phi(X)) = \Phi(X)$, for all $X \in \Phi(X)$, then by (iii) of Proposition 2.2.1 and Theorem 3.11 of [19], for sufficiently small s, if ω_s denotes the unique ground state of the Heisenberg dynamics generated by $\Phi^{\text{AKLT}} + s\Phi$,

$$\omega_s \circ \Xi = \omega_0 \circ (\alpha_s \circ \Xi) = \omega_s. \tag{2.12}$$

Thus the AKLT ground state is a stable point in a phase of ground states protected by the Ξ symmetry in one dimension. Without the additional symmetry constraint imposed by Ξ, the ground state of the AKLT chain is in the same phase as the non-interacting state. In [2], it is shown that there exists a smooth, uniformly locally gapped path between the AKLT interaction and a *Product Vacua with Boundary* model, which has a unique product state as a ground state. However, in

[27], it was proven that the AKLT interaction cannot be smoothly deformed through a path of finite-range, uniformly locally gapped, Ξ-invariant interactions to the trivial interaction. This was done by defining a $\mathbb{Z}_2$-index σ_Φ for finite-range, uniformly locally gapped interactions Φ which are Ξ-invariant.

The main result of this paper is to extend the $\mathbb{Z}_2$-index of [27] to spin chain interactions which may have an unbounded range of interaction. The mathematical obstruction to this extension was in verifying that the split property in Definition 1.7.5 holds for the ground states of these interactions.

> **Theorem 2.3:** Suppose $\tau : [0,1] \to \mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$ are quasi-local dynamics with a generating interaction $\Phi(\cdot, t)$ such that $\|\Phi\|_{0,\beta} < \infty$. If ω_0 is a split factor state and $\beta > 3$, then $\omega_t = \omega_0 \circ \tau_t$ is also a split factor state, for all $t \in [0,1]$.

Note that in the statement of this theorem, we use a definition of a split state, due to Matsui, applying to factor states which may not necessarily be pure states. In the case that the state is pure, by Proposition 1.7.3, this definition and Definition 1.7.5 coincide. In Section 3, we use this result to modify the arguments of [27] and rigorously extend the $\mathbb{Z}_2$-index to phase-equivalent interactions which decay by a stretched exponential law.

2.3 Automorphic equivalence of gapped ground state phases in the bulk

In the previous section, we discussed properties of the strongly continuous cocycle $\alpha : [0,1] \to \mathrm{Aut}(\mathfrak{A})$. It is the thermodynamic limit of finite-volume cocycles generated by a family Ψ_s of fast-decaying interactions. Conditions from [24] on the ground state gap and decay of a curve Φ_s of interactions guarantee the relation:

$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s \tag{2.13}$$

where $\mathcal{S}(s)$ is the set of $*$-weak limits of finite-volume ground states of Φ_s. These gap and decay conditions are met, in particular, when the Φ_s are uniformly locally gapped and finite-range interactions with a range which is uniformly bounded above in s.

The main result of this paper, Theorem 1.3, provides sufficient conditions which replaces the local gap condition with a lower bound on the bulk gaps of the associated GNS Hamiltonians. In particular, we assume that the $\Phi(s)$ have unique ground states φ_s which vary smoothly with s. The hypotheses can be found in Assumption 1.2.

Theorem 1.3: Under the Assumption 1.2, we have:

$$\varphi_s = \varphi_0 \circ \alpha_s, \quad s \in [0,1] \tag{2.14}$$

for α_s given in (1.21).

The referenced line (1.21) defines the spectral flow automorphism α_s as it is given in [3]. It is important to note that, unlike the main results of the previous two papers, Theorem 1.3 applies in arbitrary lattice dimension. Theorems 1.5 and 1.6 apply the statement of automorphic equivalence to the analysis of SPT phases by showing that the $\mathbb{Z}_2$-indices of [27] and [26], respectively, are constant within the relevant, equivalent SPT phases.

Condition (vii) of Assumption 1.2 requires that $s \mapsto \varphi_s(A)$ is differentiable with respect to s, for any A which is localized so that the tails of A decay by a stretched exponential $\zeta(r) = e^{-r^\beta}$, for some $0 < \beta < 1$. The precise formulation of this decay is in Definition 1.1, which defines an extended norm:

$$\|A\|_\zeta = \|A\| + \sup_N \frac{\|A - \mathbb{E}_{[-N,N]^D}(A)\|}{\zeta(N)} \tag{2.15}$$

and an associated $*$-algebra $\mathcal{D}_\zeta = \{A \in \mathfrak{A} : \|A\|_\zeta < \infty\}$ of subexponentially localized observables. The decay law is chosen so that $\alpha_s^{-1}(\mathfrak{A}_{\text{loc}}) \subset \mathcal{D}_\zeta$ – this is proven in Lemma 2.1. In fact, Condition (vii) is satisfied by generic paths of ground states of uniformly locally gapped, rapidly decaying QSS interactions. This is proven in the Appendix of this dissertation.

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Stability of gapped ground state phases of spins and fermions in one dimension

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We investigate the persistence of spectral gaps of one-dimensional frustration free quantum lattice systems under weak perturbations and with open boundary conditions. Assuming that the interactions of the system satisfy a form of local topological quantum order, we prove explicit lower bounds on the ground state spectral gap and higher gaps for spin and fermion chains. By adapting previous methods using the spectral flow, we analyze the bulk and edge dependence of lower bounds on spectral gaps. *Published by AIP Publishing.* https://doi.org/10.1063/1.5036751

Dedicated to the memory of Ludwig Faddeev

I. INTRODUCTION

An important result in the study of gapped ground state phases of quantum lattice systems (with or without topological order) is the stability of the spectral gap(s) under uniformly small extensive perturbations. The stability property implies that the gapped phases are full-dimensional regions in the space of Hamiltonians free of phase transitions.[1] In recent years, such results were obtained in increasing generality.[2,3,5,7,8,16–18] Our goal here is to extend the existing results applicable in one dimension to Hamiltonians with the so-called "open" boundary conditions, meaning that we consider systems defined on intervals $[a, b] \subset \mathbb{Z}$ and not on a cycle $\mathbb{Z}/(n\mathbb{Z})$. Specifically, this implies that the neighborhoods of the boundary points a and b may be treated differently than the bulk. There are physical and mathematical situations where one is naturally led to considering open boundary conditions. For example, in the series of recent studies by Ogata,[13–15] clarifying the crucial role of boundary states in the classification of quantum spin chains with matrix product ground states required the study of systems with open boundary conditions. Another situation of interest to us is the application of results for quantum spin chains to fermion models in one dimension by making use of the Jordan-Wigner transformation, which in the finite system setup only works well with open boundary conditions. In this way, we obtain explicit bounds on the spectral gaps in the spectrum of perturbed spin and even fermion chains with one or more frustration free ground states that satisfy a local topological order condition. This complements previous results that prove stability of gapped fermion systems by other approaches.[4,5,12]

II. SETTING AND MAIN RESULT

A. Notations

Denote by $(\mathbb{Z}, |\cdot|)$ the metric graph of integers. Let $P_f(X)$ denote the finite subsets of $X \subset \mathbb{Z}$. We will use Λ to refer exclusively to nonempty, finite intervals of the form

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$[a,b] = \{n \in \mathbb{N} : a \le n \le b\}$. Let $b_\Lambda(x, n) = \{m \in \Lambda\colon |x - m| \le n\}$ denote the restriction of a metric ball to the interval. For each $x \in \Lambda$, denote by r_x and R_x the following distances to the boundary:

$$r_x = \min\{x - a, b - x\}, \quad R_x = \max\{x - a, b - x\}. \tag{2.1}$$

Although r_x and R_x depend on the interval $[a, b]$, we omit this dependence from the notation since we will always fix a finite volume $[a, b]$ throughout our arguments.

In the following, we will consider both spin systems and fermion systems on the one-dimensional lattice. Without difficulty, we could also treat systems that include both types of degrees of freedom, but for simplicity of the notations, we will not consider such systems in this paper. It is also possible to consider inhomogeneous systems for which the number of spin or fermion states depends on the site. In order to present the main ideas without overly burdensome notation, we will only consider homogeneous systems in the note.

The algebra of observables of the finite system in Λ, of either spins or fermions, will be denoted by $\mathfrak{A}_\Lambda$. If we want to specify that we are specifically considering spins or fermions, we will use the notation $\mathfrak{A}^s_\Lambda$ or $\mathfrak{A}^f_\Lambda$, respectively. These algebras, and the associated Hilbert space they are represented on, are defined as follows:

For spin systems, we have

$$\mathfrak{A}^s_\Lambda = M_d(\mathbb{C})^{\otimes|\Lambda|}, \quad \mathfrak{h}_\Lambda = (\mathbb{C}^d)^{\otimes|\Lambda|},$$

where d is the dimension of the Hilbert space of a single spin, i.e., $d = 2S + 1$.

For fermions, $\mathfrak{A}^f_\Lambda$ denotes the C^*-algebra generated by $\{a(x), a^*(x)\colon x \in \Lambda\}$, the annihilation and creation operators defining a representation of the Canonical Anticommutation Relations (CAR) on the antisymmetric Fock space $\mathfrak{F}_\Lambda = \mathfrak{F}(\ell^2(\Lambda))$. The dimension of $\mathfrak{F}_\Lambda$ is $2^{|\Lambda|}$ and $\mathfrak{A}^f_\Lambda$ is $*$-isomorphic to the matrix algebra $M_{2^{|\Lambda|}}(\mathbb{C})$.

Given an exhaustive net of CAR or spin algebras $\{\mathfrak{A}_\Lambda : \Lambda \in P_f(\mathbb{Z})\}$, the inductive limit $\mathfrak{A}_\mathbb{Z}$, the d^∞ UHF algebra, is obtained by norm completion,

$$\mathfrak{A}_\mathbb{Z} = \overline{\bigcup_{\Lambda \in P_f(\mathbb{Z})} \mathfrak{A}_\Lambda}.$$

This algebra is often referred to as the *quasi-local* algebra, and $\mathfrak{A}_{\rm loc} = \bigcup \mathfrak{A}_\Lambda$ is referred as the *local* algebra.

Define by $N_X = \sum_{x\in X} a^*(x)a(x)$ the number operator for $X \in P_f(\mathbb{Z})$, and define the *parity automorphism* by

$$\rho_\Lambda(A) = \exp(i\pi N_\Lambda) A \exp(i\pi N_\Lambda). \tag{2.2}$$

Say that $A \in \mathfrak{A}^f_\Lambda$ is *even* if $\rho_\Lambda(A) = A$ and odd if $\rho_\Lambda(A) = -A$. The observable A is even if and only if it commutes with the local symmetry operator $\exp(i\pi N_\Lambda)$, which is if and only if A is the sum of even monomials in the generating set $\{a(x), a^*(x)\colon x \in \Lambda\}$. Unlike the odd observables, the even observables form a $*$-subalgebra of $\mathfrak{A}^f_\Lambda$, which we denote by $\mathfrak{A}^+_\Lambda$.

B. Assumptions

Let I be a subinterval of $\mathbb{Z}$, not necessarily finite. An *interaction on I* is a function $\Phi : P_f(I) \to \mathfrak{A}_{\rm loc}$ such that $\Phi(X) = \Phi(X)^* \in \mathfrak{A}_X$ for all $X \in P_f(I)$. The corresponding *local Hamiltonian* of the finite system on $\Lambda \subset I$ is $H_\Lambda = \sum_{X\subset\Lambda}\Phi(X)$. Say that Φ is *non-negative* if $\Phi(X) \ge 0$ for all $X \in P_f(I)$. Say that Φ is an *even* interaction of the CAR algebra if $\Phi(X) \in \mathfrak{A}^+_X$.

The interactions in our perturbative setup will satisfy the following assumptions. First, let $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}_{\rm loc}$ be a non-negative interaction with distinguished local Hamiltonians H_Λ. We will refer to η as the *unperturbed interaction*. We assume that η has the following properties:

i. **Finite range**: There exists $R > 0$ such that $\operatorname{diam}(X) > R$ implies $\eta(X) = 0$.
ii. **Uniformly bounded**: There exists $M > 0$ such that for all $X \in P_f(\mathbb{Z})$, $\|\eta(X)\| < M$.
iii. **Frustration free**: For all intervals $\Lambda \in P_f(\mathbb{Z})$, $\ker(H_\Lambda) \supsetneq \{0\}$.

iv. **Uniformly locally gapped**: There exists $\gamma_0 > 0$ such that for all intervals $[a, b] \in P_f(\mathbb{Z})$, with $b - a \geq R$, γ_0 is lower bound a for non-zero eigenvalues of $H_{[a,b]}$.
v. ***Local topological quantum order*** (**LTQO**) of the ground state projectors.

The concept of LTQO was introduced in Ref. 2. We will need to adapt the definition to take into account parity and boundary conditions, which we do in Sec. II C.

Next, we consider the perturbations. To allow edge effects, we will consider perturbations given in terms of a family of interactions on intervals. For each Λ, let $\Phi^\Lambda : P_f(\Lambda) \to \mathfrak{A}_{\text{loc}}$ be an interaction on the interval, and denote by $[\Phi]$ the collection of these perturbative interactions,

$$[\Phi] = \left\{\Phi^\Lambda : \Lambda \in P_f(\mathbb{Z})\right\}. \tag{2.3}$$

The perturbed Hamiltonians have the form

$$H_\Lambda(\varepsilon) = \sum_{X \subset \Lambda} \eta(X) + \varepsilon \Phi^\Lambda(X),\ \varepsilon \in [0, 1], \tag{2.4}$$

and while the Hamiltonians depend on the interval Λ, lower bounds on gaps in the spectrum will be uniform in the volume.

Our main assumption on the interactions Φ^Λ in $[\Phi]$ is that $\Phi^\Lambda(X)$ decays rapidly with the diameter of X. To make this precise, we use $\mathcal{F}$-functions and provide explicit bounds in terms of the $\mathcal{F}$-norm. The definition and properties of $\mathcal{F}$-functions and $\mathcal{F}$-norm can be found in the Appendix. In our argument, we will use functions of the form

$$\begin{aligned} F(x) &= e^{-h(x)} F^b(x), \\ F^b(x) &= \frac{L}{(1 + cx)^\kappa}, \end{aligned} \tag{2.5}$$

where $\kappa > 2$ and $L, c > 0$. The function h: $[0, \infty) \to [0, \infty)$ is a monotone increasing, subadditive weight function. At times, it will be necessary to precompose F with a transformation $\tau : [0, \infty) \to \mathbb{R}$, and so we will take as convention $F \circ \tau(x) = F(0)$ for $\tau(x) < 0$. We will denote by $\|\cdot\|_F$ the extended norm (A1) induced by F.

Using $\mathcal{F}$-function terminology, we assume for the perturbations:

i. **Fast decay**: There exists an $\mathcal{F}$-function $F(r) = e^{-h_\Phi(r)} \frac{L}{(1+cx)^\kappa}$, for $L, c > 0$ and $\kappa > 2$, such that $\sup_\Lambda \|\Phi^\Lambda\|_F < \infty$.
ii. **Metric ball support**: For all Λ, $\Phi^\Lambda(X) \neq 0$ implies $X = b_\Lambda(z, n)$ for some $z \in \Lambda$ and $n \in \mathbb{N}$.

The assumption that Φ^Λ is supported on metric balls is not restrictive since a finite-volume Hamiltonian of any fast-decaying interaction can be rewritten as the finite-volume Hamiltonian of a balled interaction with comparable decay (c.f. the Appendix of Ref. 18).

C. Local topological quantum order

Consider the unperturbed interaction η and its local Hamiltonians. Denote by P_X the orthogonal projection onto $\ker(H_X)$, and define the state

$$\omega_\Lambda(A) = \frac{1}{\text{tr}(P_\Lambda)} \text{tr}(P_\Lambda A),\ A \in \mathfrak{A}_\Lambda.$$

Definition. The unperturbed interaction η satisfies *local topological quantum order* if there exists a monotone function Ω: $[0, \infty) \to [0, \infty)$, decreasing to 0, such that for all $x \in \Lambda$ and $n, k \in \mathbb{N}$ satisfying $0 \leq k \leq r_x$ and $k \leq n \leq R_x$, the following bound holds:

$$\forall A \in \mathfrak{A}_{b_\Lambda(x,k)} : \|P_{b_\Lambda(x,n)}(A - \omega_\Lambda(A))P_{b_\Lambda(x,n)}\| \leq \Omega(z_x(n) - k)\|A\|, \tag{2.6}$$

where $z_x : \mathbb{N} \to \mathbb{N}$ is the cutoff function defined in terms of distance to the boundary of Λ (2.1),

$$z_x(m) = \begin{cases} m & \text{if } m \leq r_x \\ r_x & \text{else.} \end{cases} \tag{2.7}$$

If $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}^f_{\text{loc}}$ is an even interaction and (2.6) holds for the restricted class of observables $A \in \mathfrak{A}^+_{b_\Lambda(x,k)}$, then we will say that η has $\mathbb{Z}_2$-*LTQO*.

For example, the AKLT interaction with either periodic or open boundary conditions has LTQO with $\Omega(r) = (1/3)^r$. The interaction defined in (5.1) has $\mathbb{Z}_2$-LTQO with $\Omega(r) = 0$ for r greater than a cutoff $D > 0$ defined by the interaction parameters, and $\Omega(x) = 2$ otherwise (Proposition 5.4).

D. The main result

For any finite interval Λ, we consider the local Hamiltonian $H_\Lambda(\varepsilon)$ given in (2.4). There exist continuous functions $\lambda_1, \ldots, \lambda_N : [0, 1] \to \mathbb{R}$ such that for all $\varepsilon \in [0, 1]$, $\{\lambda_1(\varepsilon), \ldots, \lambda_N(\varepsilon)\}$ are the eigenvalues of $H_\Lambda(\varepsilon)$. We partition $\text{sp}(H_\Lambda(\varepsilon))$ into two disjoint regions, an upper and a lower part of the spectrum, and call the minimum distance between these two sets the *spectral gap above the ground state* or *the spectral gap*,

$$\text{sp}_{0,\Lambda}(\varepsilon) = \{\lambda_i(\varepsilon) : \lambda_i(0) = 0\}, \qquad \text{sp}_{1,\Lambda}(\varepsilon) = \left\{\lambda_j(\varepsilon) : \lambda_j(0) > 0\right\}, \tag{2.8}$$

$$\gamma(H_\Lambda(\varepsilon)) = \min\left\{\lambda - \mu : \lambda \in \text{sp}_{1,\Lambda}(\varepsilon),\ \mu \in \text{sp}_{0,\Lambda}(\varepsilon)\right\}. \tag{2.9}$$

For a class of sufficiently small perturbations, the main result of this paper establishes a lower bound for the size of the spectral gap which does not depend on Λ, under the assumptions that η has LTQO, the interactions in $[\Phi]$, from (2.3), decay sufficiently fast and, in the case of fermions, that the interactions are even. The spectrum may have other gaps which can be defined similarly in terms of eigenvalue splitting, and we also prove an estimate showing how these gaps persist under weak perturbations. To state these results, we define several constants that characterize the effect of the perturbation and the presence of edge effects.

The effect of perturbations near the boundary of Λ is, in general, different and stronger than far away from the boundary. As a consequence, our stability result for open chains features a distance parameter $D \geq 0$, in terms of which we distinguish sites near and far away from the boundary. In Sec. III, we write each Φ^Λ as the sum of an interaction $\Phi^D(\Lambda)$, with a local Hamiltonian Φ^D_Λ supported at the D-boundary, and a bulk interaction $\Phi^{\text{Int}}(\Lambda)$. Define the following two finite constants quantifying the strength of the bulk and edge perturbations, respectively:

$$M_{\text{Int}} = \sup_\Lambda\left\{\|\Phi^{\text{Int}}(\Lambda)\|_F : \text{diam}(\Lambda) > \max\{2D, R\}\right\},$$
$$M_D = \sup_\Lambda\left\{\|\Phi^D_\Lambda\| : \text{diam}(\Lambda) > \max\{2D, R\}\right\}.$$

Then, for constant,

$$m = \Big(\sum_{|n|\geq 3} 20C(3|n|+2)\Big[\Omega\Big(\frac{|n|-1}{2}\Big)^{1/2} + F_0\Big(\frac{|n|-3}{2}\Big)\Big] + C\Big(\sum_{n\in\mathbb{Z}} \Omega\Big(\frac{|n|}{2}\Big) + 2F_0\Big(\lfloor\frac{|n|}{2}\rfloor\Big) + 8\Big)\Big)(\|\eta\|_F + M_{\text{Int}}),$$

where $F_0(x) = F^b(x/18 - R - 3/2)$, we are able to prove the following theorem.

Theorem 3.11 (Ground state gap stability for spin chains). *Suppose $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}^s_{\text{loc}}$ has LTQO with $\Omega(n) \leq n^{-\nu}$, for $\nu > 4$, and there exist $K > 0$, $s \in (0, 1]$ such that h_Φ satisfies $h_\Phi(r) \geq Kr^s$. Then there exists $\varepsilon(\gamma_0) > 0$ such that $0 \leq \varepsilon < \varepsilon(\gamma_0)$ and $\text{diam}(\Lambda) > \max\{2D, R\}$ imply*

$$\gamma(H_\Lambda(\varepsilon)) \geq \gamma_0 - (m + 2M_D)\varepsilon > 0.$$

The constant $\varepsilon(\gamma_0)$ can be taken as

$$\varepsilon(\gamma_0) = \min\left\{1, \frac{\gamma_0}{m + 2M_D}\right\}.$$

As a consequence, if we assume that $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}^+_{\text{loc}}$ has $\mathbb{Z}_2$-LTQO, and Ω and $\Phi^\Lambda : P_f(\Lambda) \to \mathfrak{A}^+_\Lambda$ have the same decay assumptions as in Theorem 3.11, we are also able to prove:

Theorem 4.4. *There exist $\varepsilon'(\gamma_0) > 0$ and constant m'_D such that $0 \le \varepsilon < \varepsilon'(\gamma_0)$ and* $\mathrm{diam}(\Lambda) > \max\{2D, R\}$ *implies*

$$\gamma(H_\Lambda(\varepsilon)) \ge \gamma_0 - m'_D\varepsilon > 0.$$

The constants m'_D and $\varepsilon'(\gamma_0)$ can be explicitly determined by the constants m, M_D, and $\varepsilon(\gamma_0)$.

The proofs of Theorems 3.11 and 4.4 rely on a relative form bound argument. We remark that the proof will depend strongly on the fact that the size of the boundary of Λ can be bounded independently of the size of Λ itself. This is special about one-dimensional systems. The stability of the gap in higher dimensions requires a careful analysis of the locality of perturbations[11] and more complicated assumptions.

Additionally, due to the relative form bound, the hypotheses for a stable ground state spectral gap also imply general stability of the spectrum. Precisely, we prove the following statement about the persistence of higher spectral gaps. In the statement, J_1, J_2, J_3 refer to Eqs. (3.12) and (3.15).

Proposition 3.12. Let $T, \gamma > 0$, and denote $\mathrm{res}(H_\Lambda) = \mathbb{C} \setminus \mathrm{sp}(H_\Lambda)$. *Suppose η, $[\Phi]$ satisfy the hypotheses of Theorem 3.11. There exists $\varepsilon(\gamma, T) > 0$ such that for sufficiently large Λ and $0 \le \varepsilon < \varepsilon(\gamma, T)$, if $\nu, \mu \in \mathrm{sp}(H_\Lambda)$ with $(\nu, \mu) \subset \mathrm{res}(H_\Lambda) \cap [0, T]$ and $\mu - \nu > \gamma$, then the gap between ν and μ is stable. Precisely, if we denote*

$$\gamma(\nu, \mu, \varepsilon) = \min\{\lambda(\varepsilon) \in \mathrm{sp}(H_\Lambda(\varepsilon)) : \lambda(0) \ge \mu\} - \max\{\lambda(\varepsilon) \in \mathrm{sp}(H_\Lambda(\varepsilon)) : \lambda(0) \le \nu\},$$

then

$$\gamma(\nu, \mu, \varepsilon) \ge (1 - p\varepsilon)\gamma - 2(q + pT + M_D)\varepsilon > 0$$

for $0 \le \varepsilon < \varepsilon(\gamma, T)$ and p, q defined as

$$p = \frac{3}{\gamma_0} J_1(\|\eta\|_F + M_{\mathrm{Int}}), \qquad q = [C(J_3 + 4) + J_2](\|\eta\|_F + M_{\mathrm{Int}}).$$

III. STABILITY OF SPECTRAL GAP IN SPIN CHAINS

A. Perturbations at the boundary

Here, we make the distinction between a perturbation near the boundary and in the bulk. In this section, unless otherwise noted, we fix an interval $\Lambda = [a, b]$ and let Φ denote the interaction Φ^Λ, with a local Hamiltonian $\Phi_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$.

Let $D \in \mathbb{N}$ define a uniform distance parameter, and denote by $\mathrm{Int}_D(\Lambda)$ the relative interior $[a + D, b - D]$. The piece of the perturbation associated with $x \in \Lambda$ is $\Phi_x = \sum_{n=1}^{R_x} \Phi(b_\Lambda(x, n))$, and the whole perturbation is split by the relative interior, $\Phi_\Lambda = \Phi^D_\Lambda + \Phi^{\mathrm{Int}}_\Lambda$, where

$$\Phi^D_\Lambda = \sum_{x \in \Lambda \setminus \mathrm{Int}_D(\Lambda)} \Phi_x, \qquad \Phi^{\mathrm{Int}}_\Lambda = \sum_{x \in \mathrm{Int}_D(\Lambda)} \Phi_x$$

are the edge and bulk perturbations, respectively. Let $\Phi^D(\Lambda), \Phi^{\mathrm{Int}}(\Lambda) : P_f(\Lambda) \to \mathfrak{A}^s_\Lambda$ denote the corresponding local interactions.

If $x \in \mathrm{Int}_D(\Lambda)$, then $n \ge r_x$ implies $\|\Phi(b_\Lambda(x, n)\| \le \|\Phi\|_F F(D)$, and so even though the bulk perturbative interaction contains terms which extend to the boundary, their contribution to the total perturbation is relatively small as a function of D.

Since the Hamiltonian $H_\Lambda + \varepsilon\Phi_\Lambda$ is close in the operator norm to the bulk-perturbed Hamiltonian, it will suffice to prove ground state spectral gap stability for $H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda$. To do this, we will use a unitary decomposition method depending on *spectral flow*. First proved in Ref. 8, our present formulation of the following theorem using $\mathcal{F}$-functions comes from Ref. 18.

B. Spectral flow decomposition

Let $\Psi : P_f(I) \to \mathfrak{A}^s_{\mathrm{loc}}$ be an arbitrary interaction, $\Lambda \subset I$, and suppose $\gamma \in (0, \gamma_0)$. Let $\varepsilon_\Lambda > 0$ be such that $0 \le \varepsilon \le \varepsilon_\Lambda$ implies $\gamma(H_\Lambda(\varepsilon)) \ge \gamma$, where $H_\Lambda(\varepsilon) = H_\Lambda + \varepsilon\Psi_\Lambda$. We may take ε_Λ to be maximal. Because $\gamma(H_\Lambda(\varepsilon))$ is bounded below by γ and $\varepsilon\Psi_\Lambda$ is uniformly bounded on $[0, \varepsilon_\Lambda]$, we may construct the *spectral flow* (also known as *quasi-adiabatic evolution*) $\alpha : [0, \varepsilon_\Lambda] \to \mathfrak{A}^s_\Lambda$, whose quasi-local properties are extensively discussed in Refs. 1 and 6. Briefly summarizing, there exists a norm-continuous family $U(\varepsilon)$ of unitaries such that, if $P(\varepsilon)$ denotes the orthogonal projection onto the kernel of $H_\Lambda(\varepsilon)$,

$$\alpha_\varepsilon(A) = U(\varepsilon)^* A U(\varepsilon) \text{ and } P(\varepsilon) = U(\varepsilon)P(0)U(\varepsilon)^*. \tag{3.1}$$

The unitaries are the solution to $-i\frac{d}{d\varepsilon}U(\varepsilon) = D(\varepsilon)U(\varepsilon)$ with $U(0) = \mathbb{1}$, where the generator $D(\varepsilon)$ is given by

$$D(\varepsilon) = \int_{-\infty}^{\infty} w_\gamma(t) \int_0^t e^{isH_\Lambda(\varepsilon)} \Psi_\Lambda e^{-isH_\Lambda(\varepsilon)} ds\, dt \tag{3.2}$$

for a weight function $w_{gamma} \in L^1$ with compactly supported Fourier transform (see Lemma 2.3 in Ref. 1). Since the quasi-local properties of its generator are made clear by the expression (3.2), the spectral flow automorphism transforms the perturbed Hamiltonian $H_\Lambda(\varepsilon)$ into a unitarily equivalent finite-volume Hamiltonian of a well-behaved, local interaction. Identifying this local interaction is the content of the unitary decomposition theorem:

Theorem 3.1. *Suppose $\Psi : P_f(I) \to \mathfrak{A}^s_{\mathrm{loc}}$ satisfies a finite $\mathcal{F}$-norm for F and $h_\Psi(r) \geq \mathcal{K} r^t$ for some $\mathcal{K} > 0$ and $t \in (0, 1]$. Then for all $0 \leq \varepsilon \leq \varepsilon_\Lambda$,*

i. *there exists an interaction $\Phi^1(\varepsilon) : P_f(\Lambda) \to \mathfrak{A}^s_\Lambda$ such that $\alpha_\varepsilon(H_\Lambda(\varepsilon)) = H_\Lambda + \Phi^1(\varepsilon)$, and*
ii. *$\Phi^1(\varepsilon)$ is supported on the metric balls of Λ, that is,*

$$\Phi^1_\Lambda(\varepsilon) = \sum_{x\in\Lambda} \Phi^1_x(\varepsilon),$$

where $\Phi^1_x(\varepsilon) = \sum_{n=1}^{R_x} \Phi^1(b_\Lambda(x, n), \varepsilon)$ and each $\Phi^1(b_\Lambda(x, n), \varepsilon) \in \mathfrak{A}^s_{b_\Lambda(x,n)}$. Furthermore, for all $x \in \Lambda$, $[P(0), \Phi^1_x(\varepsilon)] = 0$.

There exists a constant $C > 0$, depending on the uniform bound M, range R, uniform gap γ_0, and decay parameters $\mathcal{K}$ and t, such that

$$\|\Phi^1(\varepsilon)\|_{F_\varphi} \leq C\varepsilon(\|\eta\|_{F_\Psi} + \|\Psi\|_{F_\Psi}),$$

where F_φ is an $\mathcal{F}$-function depending on $\mathcal{K}, t, \gamma$ such that $F_\varphi(r)$ decays faster than any polynomial in r.

Proof. This reformulated statement of the original decomposition theorem found in Ref. 8 is proved in Theorem 6.3.4 in Ref. 18, and so we record here only the precise form of F_φ. Define

$$\mu(r) = \begin{cases} (e/\kappa)^\kappa & \text{if } r \leq e^\kappa \\ r/(\log r)^\kappa & \text{else } r > e^\kappa. \end{cases} \tag{3.3}$$

Define $\mathcal{K}_0 = \min\{\mathcal{K}, 2/7\}$, and denote by v_Ψ the Lieb-Robinson velocity for the Heisenberg dynamics generated by the interaction Ψ. Denote $\widetilde{\mu}(r) = \mu(\frac{\mathcal{K}\gamma r}{2v_\Psi})$ and

$$G_\Phi(r) = e^{-\frac{\mathcal{K}_0}{\mathcal{K}}\widetilde{\mu}\circ h_\Phi(r)} F^b(r).$$

Then the $\mathcal{F}$-function in the statement of the theorem is given by

$$F_\varphi(r) = \begin{cases} G_\Psi(0) & \text{if } r \leq 18R + 27 \\ G_\Psi(r/18 - R - 3/2) & \text{else } r > 18R + 27. \end{cases} \tag{3.4}$$

$\square$

For the remainder of this section, let $U(\varepsilon)$, α_ε, and $\Phi^1(\varepsilon)$ be from an application of Theorem 3.1 when Ψ is the bulk perturbative interaction $\Phi^{\mathrm{Int}}(\Lambda)$ with a local Hamiltonian $\Phi^{\mathrm{Int}}_\Lambda$.

Lemma 3.2. The local operator $\Phi^1(\varepsilon)$ can be rewritten as

$$\Phi^1(\varepsilon) = \Phi^2(\varepsilon) + \Phi^3(\varepsilon) + \omega_\Lambda(\widetilde{\Phi^1}(\varepsilon)) + \mathcal{R}(\varepsilon) \tag{3.5}$$

for terms defined by

$$\begin{aligned}\widetilde{\Phi^1}(\varepsilon) &= \sum_{x\in \mathrm{Int}_2(\Lambda)} \Phi^1_x(\varepsilon),\\ \Phi^2(\varepsilon) &= (\mathbb{1}-P)(\widetilde{\Phi^1}(\varepsilon) - \omega_\Lambda(\widetilde{\Phi^1}(\varepsilon))\mathbb{1})(\mathbb{1}-P),\\ \Phi^3(\varepsilon) &= P(\widetilde{\Phi^1}(\varepsilon) - \omega_\Lambda(\widetilde{\Phi^1}(\varepsilon))\mathbb{1})P,\\ \mathcal{R}(\varepsilon) &= \Phi^1_a(\varepsilon) + \Phi^1_{a+1}(\varepsilon) + \Phi^1_b(\varepsilon) + \Phi^1_{b+1}(\varepsilon).\end{aligned}$$

Proof. This follows from a direct calculation using the fact that $[\Phi^1_x(\varepsilon), P] = 0$. □

The reason for separating the boundary terms $\mathcal{R}(\varepsilon)$ from the rest of the transformed perturbation is for notational convenience since the following argument will use the fact that $\lfloor r_x/2\rfloor > 0$ for $x \in \mathrm{Int}_2(\Lambda)$.

C. Relative form boundedness of perturbations

The argument for relative form boundedness of the transformed perturbation $\Phi^1(\varepsilon)$ will depend on the following two elementary lemmas.

Lemma 3.3. Suppose $x \in \Lambda$. For any $1 \le m \le r_x$,

$$\|P(\Phi^1_x(\varepsilon) - \omega_\Lambda(\Phi^1_x(\varepsilon)))P\| \le \|\Phi^1(\varepsilon)\|\Big(\Omega(r_x - m) + 2F_\varphi(m)\Big).$$

Proof. Denote $A - \omega_\Lambda(A) = A_0$ and $b_\Lambda(x, n) = b_x(n)$, for brevity. For $0 \le m \le r_x$, by linearity of ω_Λ,

$$P\Big(\Phi^1_x(\varepsilon)\Big)_0 P = \sum_{k=1}^{R_x} P\Phi^1(b_x(k),\varepsilon)_0 P = \sum_{k=1}^{m} P\Phi^1(b_x(k),\varepsilon)_0 P + \sum_{k=m+1}^{R_x} P\Phi^1(b_x(k),\varepsilon)_0 P.$$

We bound the two summands separately. The right summand is bounded by Proposition A.1,

$$\sum_{k=m+1}^{R_x} \|P\Phi^1(b_x(k),\varepsilon)_0 P\| \le 2\|\Phi^1(\varepsilon)\|_{F_\varphi} F_\varphi(m).$$

The left summand is bounded by local topological quantum order and the $\mathcal{F}$-norm,

$$\begin{aligned}\sum_{k=1}^{m} \|P\Phi^1(b_x(k),\varepsilon)_0 P\| &\le \sum_{k=1}^{m} \Omega(r_x - k)\|\Phi^1(b_x(k),\varepsilon)\|\\ &\le \Omega(r_x - m)\|\Phi^1(\varepsilon)\|_{F_\varphi}.\end{aligned}$$

Combining these bounds proves the lemma. □

The next lemma uses the cutoff function z_x defined in Sec. II C, Eq. (2.7).

Lemma 3.4. Suppose $x \in \mathrm{Int}_2(\Lambda)$. If $1 \le m \le r_x$ and $m \le n \le R_x$, then

$$\Big\|\sum_{k=1}^{m}\Big(\Phi^1(b_x(k),\varepsilon)\Big)_0 P_{b_x(n)}\Big\| \le \|\Phi^1(\varepsilon)\|_{F_\varphi}\Big[5\Omega(z_x(n) - m)^{1/2} + 4F_\varphi(m)\Big].$$

Proof. Suppose $A \in \mathfrak{A}^s_{b_x(k)}$. The C^*-identity and LTQO imply

$$\Big|\|AP_{b_x(n)}\| - \|AP\|\Big|^2 \le \Big|\|AP_{b_x(n)}\|^2 - \|AP\|^2\Big| \le 2\|A\|^2\Omega(z_x(n) - m).$$

In the case $A = \sum_{k=1}^m \Phi^1(b_x(k),\varepsilon)_0$, the above bound and Proposition A.1 imply

$$\|AP_{b_x(n)}\| \le 4\|\Phi^1(\varepsilon)\|_{F_\varphi}\Omega(z_x(n) - m)^{1/2} + \|AP\|$$

By Theorem 3.1, $\Phi^1_x(\varepsilon)$ commutes with P. So, using Lemma 3.3, we get

$$\begin{aligned}\|AP\| &\leq \|P\Phi^1_x(\varepsilon)_0 P\| + 2\sum_{k=m+1}^{R_x} \|\Phi^1(b_x(k),\varepsilon)\| \\ &\leq \|\Phi^1(\varepsilon)\|_{F_\varphi}\Big[\Omega(r_x - m) + 4F_\varphi(m)\Big].\end{aligned}$$

□

Proposition 3.6 uses a finite resolution of identity $\{E^x_n\}$ defined at each site $x \in \mathrm{Int}_2(\Lambda)$ by

$$E^x_n = \begin{cases} \mathbb{1} - P_{b_x(1)} & \text{if } n = 1 \\ P_{b_x(n-1)} - P_{b_x(n)} & \text{if } 1 < n \leq r_x \\ P_{b_x(r_x)} - P & \text{if } n = r_x + 1 \\ P & \text{else } n = r_x + 2. \end{cases}$$

Lemma 3.5. The family $\{E^x_n\}$ has the properties

1. $\displaystyle\sum_{k=1}^{r_x+2} E^x_k = \mathbb{1}$ *and* $\displaystyle\sum_{k=1}^{m} E^x_k = \begin{cases} \mathbb{1} - P_{b_x(m)} & \text{if } 1 \leq m \leq r_x \\ \mathbb{1} - P & \text{if } m = r_x + 1, \end{cases}$
2. $P_{b_x(k)}E^x_k = 0$ *for* $k \leq r_x$.

Proof. We only comment that the second property follows from the frustration free assumption on η. □

Proposition 3.6. Let $x \in \mathrm{Int}_2(\Lambda)$ and $0 \leq \varepsilon \leq \varepsilon_\Lambda$. There exist local operators $\Theta^x_\beta(n,\varepsilon)$, for $3 \leq n \leq r_x$, and operator $\Theta^x_\alpha(\varepsilon)$ such that

$$\Phi^1_x(\varepsilon)_0 = \sum_{n=3}^{r_x} \Theta^x_\beta(n,\varepsilon) + \Theta^x_\alpha(\varepsilon).$$

Furthermore, $P_{b_x(n)}\Theta^x_\beta(n,\varepsilon) = 0$, and $\Theta^x_\beta(n,\varepsilon)$ and $\Theta^x_\alpha(\varepsilon)$ decay rapidly,

$$\|\Theta^x_\beta(n,\varepsilon)\| \leq 20\|\Phi^1(\varepsilon)\|_{F_\varphi}\left[\Omega\Big(\frac{n-1}{2}\Big)^{\frac{1}{2}} + F_\varphi\Big(\frac{n-3}{2}\Big)\right],$$

$$\|\Theta^x_\alpha(\varepsilon)\| \leq 20\|\Phi^1(\varepsilon)\|_{F_\varphi}\left[\Omega\Big(\frac{r_x-1}{2}\Big)^{\frac{1}{2}} + F_\varphi\Big(\frac{r_x-3}{2}\Big)\right].$$

Proof. Fix $x \in (a,b)$ and $\varepsilon \in [0,\varepsilon_\Lambda]$. Abbreviate $Q = \mathbb{1} - P$ and $\Phi^1_k = \Phi^1(b_x(k),\varepsilon)_0$, i.e.,

$$\Phi^1_x(\varepsilon)_0 = \sum_{k=1}^{R_x} Q\Phi^1_k Q.$$

Define a "cutoff" parameter $n_x = \lfloor \frac{r_x}{2} \rfloor$ and split $\Phi^1_x(\varepsilon)_0$ into two sums,

$$\Phi^1_x(\varepsilon)_0 = \sum_{k=1}^{n_x} Q\Phi^1_k Q + \sum_{k=n_x+1}^{R_x} Q\Phi^1_x Q. \tag{3.6}$$

The tail $\mu^x_\alpha = \sum_{k=n_x+1}^{R_x} Q\Phi^1_k Q$ can be bounded above in operator norm by using LTQO, so we turn our attention to the other summand. Denote by $Q_{b_x(l)}$ the complement projection $\mathbb{1} - P_{b_x(l)}$. Using the resolution $\{E_n\}$ at x, we rewrite $Q\Phi^1_k Q$ for all $1 \leq k \leq n_x$ as

$$Q\Phi^1_k Q = Q_{b_x(2k)}\Phi^1_k Q_{b_x(2k)} + \sum_{n=2k+1}^{r_x+1}\left[E_n\Phi^1_k\Big(\sum_{m=1}^{n-1} E_m\Big) + \Big(\sum_{m=1}^{n} E_m\Big)\Phi^1_k E_n\right]. \tag{3.7}$$

Define the following terms to organize the summands in (3.7):

$$\nu^x_\alpha(k) = E_{r_x+1}\Phi^1_k Q_{b_x(r_x)} + Q\Phi^1_k E_{r_x+1}, \quad \theta^x_\beta(n,k) = E_n\Phi^1_k Q_{b_x(n-1)} + Q_{b_x(n)}\Phi^1_k E_n,$$
$$\tau^x_\beta(2k) = Q_{b_x(2k)}\Phi^1_k Q_{b_x(2k)}$$

so that

$$Q\Phi^1_k Q = \nu^x_\alpha(k) + \tau^x_\beta(2k) + \sum_{n=2k+1}^{r_x} \theta^x_\beta(n,k).$$

For convenience, extend $\tau^x_\beta(m)$ to previously undefined m by declaring $\tau^x_\beta(m) = 0$. The derivation of the $\Theta^x_\beta(\varepsilon,n), \Theta^x_\alpha(\varepsilon)$ operators will result from an interchange of order for the summation of terms in (3.3) over n and k. The following definition for $\Theta^x_\beta(n,\varepsilon)$ accounts for the parity of r_x,

$$\forall 3 \le n < r_x : \ \Theta^x_\beta(n,\varepsilon) = \left[\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \theta^x_\beta(n,k)\right] + \tau^x_\beta(n),$$
$$\Theta^x_\beta(r_x,\varepsilon) = \sum_{k=1}^{\lfloor\frac{r_x}{2}\rfloor} \theta^x_\beta(r_x,k) + \tau^x_\beta(r_x).$$

Then

$$Q\Phi^1_x(\varepsilon)_0 Q = \sum_{k=1}^{R_x} Q\Phi^1_k Q = \sum_{n=3}^{r_x} \Theta^x_\beta(n,\varepsilon) + \Theta^x_\alpha(\varepsilon),$$

where $\Theta^x_\alpha(\varepsilon) = \mu^x_\alpha + \sum_{k=1}^{n_x} \nu^x_\alpha(k)$. Next, the frustration free property of H_Λ implies that $\ker(H_{b_x(n)}) \subset \ker(H_{b_x(n-1)})$, and so

$$\forall 3 \le n \le r_x : P_{b_x(n)}\Theta^x_\beta(n,\varepsilon) = \Theta^x_\beta(n,\varepsilon)P_{b_x(n)} = 0. \tag{3.8}$$

Furthermore, we have the following bounds on operator norm, for all $x \in \mathrm{Int}_2(\Lambda)$ and $3 \le n < r_x$, by Lemma 3.4 and Proposition A.1,

$$\begin{aligned}\|\Theta^x_\beta(n,\varepsilon)\| &\le \|(\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \Phi^1_k)^* E_n\| + \|\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \Phi^1_k E_n\| + \|\tau^x_\beta(n)\| \\ &\le 20\|\Phi^1(\varepsilon)\|_{F_\varphi}\left[\Omega\left(\frac{n-1}{2}\right)^{\frac12} + F_\varphi\left(\frac{n-3}{2}\right)\right], \\ \max\left\{\|\Theta^x_\beta(r_x,\varepsilon)\|, \|\Theta^x_\alpha(\varepsilon)\|\right\} &\le 20\|\Phi^1(\varepsilon)\|_{F_\varphi}\left[\Omega\left(\frac{r_x-1}{2}\right)^{\frac12} + F_\varphi\left(\frac{r_x-3}{2}\right)\right].\end{aligned} \tag{3.9}$$

□

Now, we define several quantities which will appear in the derivation of the form bound. Note that the weight function $e^{-h_\varphi(x)}$ of F_φ is bounded above by 1 on its domain. So any expression in F_φ is bounded above by the corresponding sum using the shifted base $\mathcal{F}$-function

$$F_0(r) = F^b(r/18 - R - 3/2) \tag{3.10}$$

from (2.5) and (3.4). Define

$$\kappa(n,\varepsilon) = 20C\varepsilon(\|\eta\|_F + \|\Phi^{\mathrm{Int}}(\Lambda)\|_F)\left[\Omega\left(\frac{n-1}{2}\right)^{\frac12} + F_0\left(\frac{n-3}{2}\right)\right]. \tag{3.11}$$

$\kappa(n,\varepsilon)$ does not depend on either Λ or the lower bound γ on the instantaneous gap, and the inequalities from (3.9) are rewritten as

$$\|\Theta^x_\beta(n,\varepsilon)\| \le \kappa(n,\varepsilon), \quad \|\Theta^x_\alpha(\varepsilon)\| \le \kappa(r_x,\varepsilon).$$

Last, we see by the assumed decay of Ω that the following sums are finite:

$$\begin{aligned}J_1 &= \sum_{n\in\mathbb{Z}} 20C|n|[\Omega((|n|-1)/2)^{1/2} + F_0((|n|-3)/2)], \\ J_2 &= \sum_{n\in\mathbb{Z}} 20C[\Omega((|n|-1)/2)^{1/2} + F_0((|n|-3)/2)].\end{aligned} \tag{3.12}$$

The following argument for concluding form boundedness is essentially due to Ref. 8, modified to work with the boundary terms introduced by Proposition 3.6. We divide a large part of the Hamiltonian with respect to a convenient partition of $\mathrm{Int}_2(\Lambda)$. For $n \in \mathbb{N}$, define the relation $x \sim_n y$ if and only if $x - y \in (2n+1)\mathbb{Z}$. Index each of the parts Λ_n^i of $\mathrm{Int}_2(\Lambda)/\sim_n$ by a representative $i \in I(n) \subset \mathrm{Int}_2(\Lambda)$. Note that the cardinality of $I(n)$ is roughly bounded above by $3n$. The corresponding parts of the Hamiltonian are defined by

$$H_n^i = \sum_{x \in \Lambda_n^i} H_{b_x(n)}, \quad \Phi_n^i = \sum_{x \in \Lambda_n^i} \Theta_\beta^x(n, \varepsilon).$$

By definition of the $\Theta_\beta^x(n,\varepsilon)$ operators, $\Phi^2(\varepsilon) = \sum_{n,i} \Phi_n^i$. In order to compare H_n^i to Φ_n^i, we use a resolution of identity from Ref. 8, whose properties we record here.

Lemma 3.7. For a configuration $\sigma : \Lambda_n^i \to \{0,1\}$, define the projection $S_n^i(\sigma) = \prod_{x \in \Lambda_n^i} \sigma_x Q_{b_x(n)} + (1-\sigma_x) P_{b_x(n)}$. Then

$$1. \sum_{\sigma : \Lambda_n^i \to \{0,1\}} S_n^i(\sigma) = \mathbb{1},$$

$$2. \; S_n^i(\sigma) S_n^i(\sigma') = \delta_{\sigma,\sigma'} S_n^i(\sigma),$$

$$3. \; \textit{for all } x \in \Lambda_n^i, \; [\Theta_\beta^x(n,\varepsilon), S_n^i(\sigma)] = 0.$$

Proof. These properties follow immediately from the fact that $P_{b_x(n)} \Theta_\beta^x(n,\varepsilon) = 0$ and that $x \sim_n y$ implies $b_x(n) \cap b_y(n) = \emptyset$. □

Proposition 3.8. Suppose $\mathrm{diam}(\Lambda) > \max\{4, R\}$. *There exist constants δ, $\beta > 0$, dependent on $\|\Phi^{\mathrm{Int}}(\Lambda)\|_F$ such that $0 \le \varepsilon \le \varepsilon_\Lambda$ implies, for all $v \in \mathfrak{H}_\Lambda$,*

$$|\langle v, \Phi^2(\varepsilon) v\rangle| \le \delta \varepsilon \|v\|^2 + \beta \varepsilon \langle v, H_\Lambda v\rangle. \tag{3.13}$$

Precisely, we may choose

$$\delta = J_2(\|\eta\|_F + \|\Phi^{\mathrm{Int}}(\Lambda)\|_F) \textit{ and } \beta = \frac{3}{\gamma_0} J_1(\|\eta\|_F + \|\Phi^{\mathrm{Int}}(\Lambda)\|_F).$$

Proof. Denote $d_\Lambda = \mathrm{diam}(\Lambda)$. For any $x \in \mathrm{Int}_2(\Lambda)$, if $n > r_x$, say that $\Theta_\beta^x(n,\varepsilon) = 0$. Suppose $u \in \mathfrak{H}_\Lambda$. Then by Proposition 3.6,

$$|\langle u, \Phi^2(\varepsilon) u\rangle| \le |\langle u, \sum_{n=3}^{d_\Lambda} \sum_{i \in I(n)} \Phi_n^i u\rangle| + \sum_{x \in \mathrm{Int}_2(\Lambda)} \kappa(r_x, \varepsilon) \|u\|^2.$$

The second term $\sum_{x \in \mathrm{Int}_2(\Lambda)} \kappa(r_x,\varepsilon)$ is bounded above by the constants in (3.12), so we focus on the first summand. Since $[\Phi_n^i, S_n^i(\sigma)] = 0$,

$$\begin{aligned}
|\langle u, \sum_{n=3}^{d_\Lambda} \sum_{i \in I(n)} \Phi_n^i u\rangle| &\le \sum_{n=3}^{d_\Lambda} |\langle u, \sum_{i \in I(n)} \Phi_n^i \Big[\sum_\sigma S_n^i(\sigma) \Big] u\rangle| \\
&\le \sum_{n=3}^{d_\Lambda} \sum_{i \in I(n)} \sum_{\sigma : \Lambda_n^i \to \{0,1\}} \sum_{x \in \Lambda_n^i} \|S_n^i(\sigma) \Theta_\beta^x(n,\varepsilon)\| \langle u, S_n^i(\sigma) u\rangle \\
&\le \sum_{n=3}^{d_\Lambda} \sum_{i \in I(n)} \frac{\kappa(n,\varepsilon)}{\gamma_0} \sum_{x \in \Lambda_n^i} \sum_{\substack{\sigma : \Lambda_n^i \to \{0,1\} \\ \sigma_x = 1}} \gamma_0 \langle u, S_n^i(\sigma) u\rangle \\
&= \sum_{n=3}^{d_\Lambda} \sum_{i \in I(n)} \frac{\kappa(n,\varepsilon)}{\gamma_0} \sum_{x \in \Lambda_n^i} \langle u, \gamma_0 Q_{b_x(n)} u\rangle \\
&\le \sum_{n=3}^{d_\Lambda} \frac{3n\kappa(n,\varepsilon)}{\gamma_0} \langle u, H_\Lambda u\rangle.
\end{aligned} \tag{3.14}$$

Hence

$$|\langle u, \Phi^2(\varepsilon)u\rangle| \le \Big[\sum_{x\in \mathrm{Int}_2(\Lambda)} \kappa(r_x,\varepsilon)\Big]\|u\|^2 + \Big[\sum_{n=3}^{d_\Lambda} \frac{3n\kappa(n,\varepsilon)}{\gamma_0}\Big]\langle u, H_\Lambda u\rangle$$
$$\le J_2(\|\eta\|_F + \|\Phi^{\mathrm{Int}}(\Lambda)\|_F)\|u\|^2 + \frac{3}{\gamma_0}J_1(\|\eta\|_F + \|\Phi^{\mathrm{Int}}(\Lambda)\|_F)\varepsilon\langle u, H_\Lambda u\rangle.$$

□

Corollary 3.9. There exists a constant α, dependent on $\|\Phi^{\mathrm{Int}}(\Lambda)\|_F$, such that $0 \le \varepsilon \le \varepsilon_\Lambda$ and $\mathrm{diam}(\Lambda) > \max\{4, R\}$ *imply*

$$\forall u \in \mathfrak{H}_\Lambda : \ |\langle u, (\Phi^2(\varepsilon) + \Phi^3(\varepsilon) + \mathcal{R}(\varepsilon))u\rangle| \le \alpha\varepsilon\|u\|^2 + \beta\varepsilon\langle u, H_\Lambda u\rangle.$$

Precisely, we may take $\alpha = C(\|\eta\|_F + \|\Phi^{\mathrm{Int}}(\Lambda)\|_F)[J_3 + 4] + \delta$.

Proof. Suppose $x \in \mathrm{Int}_2(\Lambda)$. Set $m = \lfloor \frac{r_x}{2} \rfloor$ in an application of Lemma 3.3 to show

$$\|P(\Phi^1_x(\varepsilon))_0 P\| \le \|\Phi^1(\varepsilon)\|_{F_\varphi}[\Omega(r_x/2) + 2F_\varphi(\lfloor r_x/2\rfloor)].$$

But by the decay of Ω and F_0, we have that the following sum is finite:

$$J_3 = \sum_{z\in\mathbb{Z}} \Omega(|z|/2) + 2F_0(\lfloor |z|/2\rfloor). \tag{3.15}$$

And, summing over $x \in \mathrm{Int}_2(\Lambda)$,

$$\|\Phi^3(\varepsilon)\| \le \sum_{x\in\mathrm{Int}_2(\Lambda)} \|P(\Phi^1_x(\varepsilon))_0 P\| \le \|\Phi^1(\varepsilon)\|_{F_\varphi} J_3.$$

Next, it is straightforward to apply Proposition A.1 to $\mathcal{R}(\varepsilon)$ to get an upper bound on the norm,

$$\|\mathcal{R}(\varepsilon)\| \le 4\|\Phi^1(\varepsilon)\|_{F_\varphi}.$$

□

Until now, all estimates have been expressed using a local bound $\|\Phi^{\mathrm{Int}}(\Lambda)\|_F$ on the strength of the bulk perturbation for fixed Λ. In order to obtain volume independent lower bounds on the spectral gap, we use the following uniform quantity:

$$M_{\mathrm{Int}} = \sup_\Lambda \Big\{\|\Phi^{\mathrm{Int}}(\Lambda)\|_F : \mathrm{diam}(\Lambda) > \max\{2D, R\}\Big\}.$$

Proposition 3.10. There exist $\varepsilon_{\mathrm{Int}} > 0$ and constant $m > 0$ such that $0 \le \varepsilon < \varepsilon_{\mathrm{Int}}$ and $\mathrm{diam}(\Lambda) > \{4, R\}$ *imply*

$$\gamma(H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda) \ge \gamma_0 - m\varepsilon > 0.$$

The constants $\varepsilon_{\mathrm{Int}}$ and m can be taken as the following expressions:

$$m = \Big(3J_1 + 2J_2 + C(J_3 + 8)\Big)(\|\eta\|_F + M_{\mathrm{Int}}),$$
$$\varepsilon_{\mathrm{Int}} = \min\Big\{1, \tfrac{\gamma_0}{m}\Big\}.$$

Proof. Let $\gamma \in (0, \gamma_0)$. For fixed Λ with $\mathrm{diam}(\Lambda) > \max\{4, R\}$, there exists $\varepsilon_\Lambda > 0$ such that for all $0 \le \varepsilon \le \varepsilon_\Lambda$, $\gamma(H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda) \ge \gamma$. By continuity of the eigenvalue functions, we may assume that ε_Λ is maximal, i.e., either $\varepsilon_\Lambda = 1$ or there exists $c > 0$ such that for all $\mu \in (\varepsilon_\Lambda, \varepsilon_\Lambda + c)$, $\gamma(H_\Lambda + \mu\Phi^{\mathrm{Int}}_\Lambda) < \gamma$.

Since the gap does not close on $[0, \varepsilon_\Lambda]$, we use the spectral flow decomposition (3.5) to transform $H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda$ by unitaries and a shift in the spectrum,

$$\alpha_\varepsilon(H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda) - \omega_\Lambda(\widetilde{\Phi^1(\varepsilon)}) = H_\Lambda + \Phi^2(\varepsilon) + \Phi^3(\varepsilon) + \mathcal{R}(\varepsilon).$$

But by Corollary 3.9, if $\varepsilon \le \varepsilon_\Lambda$, then $\Phi(\varepsilon) = \Phi^2(\varepsilon) + \Phi^3(\varepsilon) + \mathcal{R}(\varepsilon)$ is H_Λ-bounded. Now, by the relation $P(\varepsilon) = U(\varepsilon)P(0)U(\varepsilon)^*$ in (3.1), the span of the eigenvectors to the 0-group of $H_\Lambda + \Phi(\varepsilon)$ is exactly $\ker(H_\Lambda)$. So, if λ is in the 0-group, which we will denote by $\mathrm{sp}(0,\varepsilon)$, then there exists a unit norm $u \in \ker(H_\Lambda)$ such that

$$|\lambda| = |\langle u, (H_\Lambda + \Phi(\varepsilon))u\rangle| \le \alpha\varepsilon. \tag{3.16}$$

Next, define $\varepsilon_1 > 0$ as the solution to $h(\varepsilon) = \gamma$, where h is defined as

$$h(\varepsilon) = (1 - \beta\varepsilon)\gamma_0 - \delta\varepsilon - 4C\varepsilon(\|\eta\|_F + M_{\mathrm{Int}}) - \alpha\varepsilon.$$

Set $\varepsilon_\gamma = \min\{\varepsilon_1, 1\}$. Combining (3.16) and Corollary 3.9, we see that if $0 \le \varepsilon < \min\{\varepsilon_\gamma, \varepsilon_\Lambda\}$, then

$$\begin{aligned}\gamma(H_\Lambda(\varepsilon)) &= \min_{v\in\ker(H_\Lambda)^\perp:\|v\|=1} \langle v, [H_\Lambda + \Phi^2(\varepsilon) + \mathcal{R}(\varepsilon)]v\rangle - \max \mathrm{sp}(0,\varepsilon)\\ &\ge h(\varepsilon)\\ &> \gamma.\end{aligned}$$

By maximality, either $\varepsilon_\Lambda = 1$ or $\gamma(H_\Lambda + \varepsilon_\Lambda\Phi^{\mathrm{Int}}_\Lambda) = \gamma$. Hence $\varepsilon_\gamma \le \varepsilon_\Lambda$ necessarily and $\gamma(H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda) \ge h(\varepsilon) > \gamma$ for all $\varepsilon < \varepsilon_\gamma$. But now, γ was arbitrarily smaller than γ_0. Set

$$\varepsilon_{\mathrm{Int}} = \sup\{\varepsilon_\gamma : \gamma \in (0, \gamma_0)\}.$$

Evidently $\varepsilon_{\mathrm{Int}}$ does not depend on Λ, and if $0 \le \varepsilon < \varepsilon_{\mathrm{Int}}$, then

$$\gamma(H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda) \ge h(\varepsilon) = \gamma_0 - m\varepsilon > 0,$$

where the constant

$$m = (3J_1 + 2J_2 + C(J_3 + 8))(\|\eta\|_F + M_{\mathrm{Int}})$$

comes from rewriting the lower bound $h(\varepsilon)$ as a linear equation of ε. ◻

Denote by M_D the following finite uniform bound on the strength of the edge perturbations:

$$M_D = \sup_\Lambda \left\{\|\Phi^D_\Lambda\| : \mathrm{diam}(\Lambda) > \max\{2D, R\}\right\}.$$

We remark that M_{Int} and m are defined in terms of $\mathcal{F}$-function decay, while M_D is defined in terms of the operator norm.

Theorem 3.11 (Ground state gap stability for spin chains). *Suppose $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}^s_{\mathrm{loc}}$ has LTQO with $\Omega(n) \le n^{-\nu}$, for $\nu > 4$, and there exist $K > 0$, $s \in (0, 1]$ such that h_Φ satisfies $h_\Phi(r) \ge Kr^s$. Then there exists $\varepsilon(\gamma_0) > 0$ such that $0 \le \varepsilon < \varepsilon(\gamma_0)$ and $\mathrm{diam}(\Lambda) > \max\{2D, R\}$ imply*

$$\gamma(H_\Lambda(\varepsilon)) \ge \gamma_0 - (m + 2M_D)\varepsilon > 0.$$

The constant $\varepsilon(\gamma_0)$ can be taken as

$$\varepsilon(\gamma_0) = \min\left\{1, \frac{\gamma_0}{m + 2M_D}\right\}. \tag{3.17}$$

Proof. Considering $\varepsilon\Phi^D_\Lambda$ as a perturbation of $H + \varepsilon\Phi^{\mathrm{Int}}_\Lambda$, the spectrum of $H + \varepsilon\Phi^D_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda$ must be contained in the compact neighborhood,

$$\mathcal{O}_\Lambda(\varepsilon) = \left\{r \in \mathbb{R} : d(r, \mathrm{sp}(H + \varepsilon\Phi^{\mathrm{Int}}_\Lambda)) \le \|\varepsilon\Phi^D_\Lambda\|\right\}.$$

That is,

$$\gamma(H_\Lambda(\varepsilon)) \ge \gamma(H_\Lambda + \varepsilon\Phi^{\mathrm{Int}}_\Lambda) - 2\|\varepsilon\Phi^D_\Lambda\| \ge \gamma_0 - (m + 2M_D)\varepsilon.$$

◻

Since the stability theorem guarantees a Λ-independent neighborhood of 0 where a relative form bound of the perturbation will hold, we can also conclude the stability of spectral gaps which are located higher in the spectrum.

Proposition 3.12. Let T, $\gamma > 0$, and denote $\mathrm{res}(H_\Lambda) = \mathbb{C} \setminus \mathrm{sp}(H_\Lambda)$. *Suppose η, $[\Phi]$ satisfy the hypotheses of Theorem 3.11. There exists $\varepsilon(\gamma, T) > 0$ such that for sufficiently large Λ and $0 \le \varepsilon < \varepsilon(\gamma, T)$, if ν, $\mu \in sp(H_\Lambda)$ with $(\nu, \mu) \subset res(H_\Lambda) \cap [0, T]$ and $\mu - \nu > \gamma$, then the gap between ν and μ is stable. Precisely, if we denote*

$$\gamma(\nu, \mu, \varepsilon) = \min\{\lambda(\varepsilon) \in \mathrm{sp}(H_\Lambda(\varepsilon)) : \lambda(0) \ge \mu\} - \max\{\lambda(\varepsilon) \in \mathrm{sp}(H_\Lambda(\varepsilon)) : \lambda(0) \le \nu\},$$

then

$$\gamma(\nu, \mu, \varepsilon) \ge (1 - p\varepsilon)\gamma - 2(q + pT + M_D)\varepsilon > 0$$

for p, q defined by

$$p = \frac{3}{\gamma_0} J_1(\|\eta\|_F + M_{\mathrm{Int}}), \quad q = (\|\eta\|_F + M_{\mathrm{Int}})[C(J_3 + 4) + J_2]. \tag{3.18}$$

Proof. Let $\Phi(\varepsilon)$ be defined as in Proposition 3.10, for $0 \le \varepsilon < \varepsilon_{\mathrm{Int}}$. By Proposition 3.9, for all $u \in \mathfrak{H}_\Lambda$,

$$|\langle u, \Phi(\varepsilon)u\rangle| \le p\varepsilon\langle u, H_\Lambda u\rangle + q\varepsilon\|u\|^2.$$

Let $z = \frac{\nu+\mu}{2}$ and denote $R_\zeta(\varepsilon') = (\zeta - H_\Lambda - \Phi(\varepsilon'))^{-1}$, with $R_\zeta = R_\zeta(0)$. Let U denote the polar unitary such that $R_z = U|R_z|$. Since R_z is self-adjoint, $|R_z|U^* = U|R_z|$, and so for unit norm u,

$$\begin{aligned}\sup_{\|w\|=1} |\langle w, |R_z|^{1/2} U^*\Phi(\varepsilon)|R_z|^{1/2}u\rangle| &\le \||R_z|^{1/2}\Phi(\varepsilon)|R_z|^{1/2}\| \\ &\le \sup_{\|v\|=1}\left[q\varepsilon\||R_z|^{1/2}v\|^2 + p\varepsilon\langle v, H_\Lambda|R_z|v\rangle\right].\end{aligned} \tag{3.19}$$

That is, for sufficiently small ε,

$$\||R_z|^{1/2}U^*\Phi(\varepsilon)|R_z|^{1/2}\| \le q\varepsilon\|R_z\| + p\varepsilon(1 + |z|\|R_z\|) < 1,$$

and by the expansion

$$R_z(\varepsilon)^{-1} = U|z - H|^{1/2}(\mathbb{1} - |R_z|^{1/2}U^*\Phi(\varepsilon)|R_z|^{1/2})|z - H|^{1/2},$$

we derive the lower bound

$$d(z, \mathrm{sp}(H_\Lambda + \Phi(\varepsilon))) \ge (1 - p\varepsilon)\gamma - 2(q + pT)\varepsilon.$$

Hence for sufficiently small ε, independently of sufficiently large Λ,

$$\gamma(\nu, \mu, \varepsilon) \ge (1 - p\varepsilon)\gamma - 2(q + pT + M_D)\varepsilon > 0.$$

□

D. The thermodynamic limit

So far, we have studied finite spin chains and shown that, under a set of general assumptions, the group of eigenvalues continuously connected to the ground state energy of a finite frustration-free Hamiltonian remains separated by a gap from the rest of the spectrum, uniformly in the length of the chain and as long as the perturbations are not too large. We now want to show that the states associated with this group of eigenvalues all converge to a ground state of the model in the thermodynamic limit. The lower bound for the gap of finite chains is then also a lower bound for the gap above those ground states of the infinite chain.

For concreteness, we consider Hamiltonians of the form (2.4), where η satisfies the assumption set out in Sec. II B, and $[\Phi] = \{\Phi^\Lambda \mid \Lambda \in \mathcal{P}_f(\mathbb{Z})\}$ is a family of perturbations given in terms of interactions $\Phi, \Phi^b \in \mathcal{B}_F$ and a few parameters that define the boundary conditions. Specifically, consider intervals $\Lambda \subset \mathbb{Z}$ of the form $[-a, b]$, $a, b \ge 0$, and for any $D \ge 0$, let $\mathrm{Int}_D(\Lambda) = [-a + D, b - D]$. Let ∂ denote the triple of parameters (D_1, D_2, s), $D_1, D_2 \ge 0$, $s \in [0, 1]$, and consider

$$H_\Lambda^\partial(\epsilon) = \sum_{X\subset\Lambda}\eta(X) + \epsilon\left(\sum_{X\subset\Lambda_{D_1}}\Phi(X) + s\sum_{X\subset(\Lambda\setminus\Lambda_{D_2})}\Phi^b(X)\right). \tag{3.20}$$

This form of the Hamiltonian covers a broad range of perturbations and boundary conditions. The dynamics generated by $H^{\partial}_{\Lambda}(\epsilon)$ is the one-parameter group of automorphism $\tau_t^{H^{\partial}_{\Lambda}(\epsilon)}$.

As explained in Subsection 2 of the Appendix, if we take, for example, $\Lambda_n = [-a_n, b_n]$, $s_n \in [0, 1]$ arbitrary, and $D_{1,n}$, $D_{2,n}$ such that $\min(a_n, b_n) - \max(D_{1,n}, D_{2,n}) \to \infty$, then there is a strongly continuous group of automorphisms $\tau_t^{\epsilon}, t \in \mathbb{R}$ on $\mathfrak{A}_{\mathbb{Z}}$ such that

$$\lim_{n\to\infty} \|\tau_t^{H^{\partial_n}_{\Lambda_n}(\epsilon)}(A) - \tau_t^{\epsilon}(A)\| = 0, \text{ for all } A \in \mathfrak{A}_{\mathbb{Z}}^{\mathrm{loc}}. \tag{3.21}$$

If we take $\epsilon \in [0, \epsilon(\gamma_0))$, with $\epsilon(\gamma_0)$ as in Theorem 3.11, we have a uniform gap separating the lower portion of the spectrum of $H^{\partial_n}_{\Lambda_n}(\epsilon)$, denoted by $\mathrm{sp}_{0,\Lambda_n}(\varepsilon)$ in (2.8), and the rest of the spectrum. The following results provide an estimate of $\mathrm{diam}(\mathrm{sp}_{0,\Lambda_n}(\varepsilon))$. For simplicity, let $\Lambda_n = [-n, n]$ for the remainder of the section.

Lemma 3.13. In the assumptions of above, choose $s_n = 0$ and put $D_{1,n} = D_n$. Then, there exists a function $\mathcal{G}: [0,\infty) \to [0,\infty)$ which decreases to 0 as n tends to infinity and, for large enough n,

$$\mathrm{diam}(\mathrm{sp}_{0,\Lambda_n}(\varepsilon)) \le \varepsilon \mathcal{G}(D_n).$$

Precisely, we may take

$$\mathcal{G}(r) = 8 \sum_{k=\lfloor r/2 \rfloor}^{\infty} \tilde{F}(k/2) + C(M_{\mathrm{Int}} + \|\eta\|_F)[\Omega(k/2) + F_0(\lfloor k/2 \rfloor)],$$

where $\tilde{F}$ is an $\mathcal{F}$-function depending on $\|\eta\|_F$ and M_{Int}.

Proof. Suppose n is sufficiently large so that $\frac{1}{2}\lfloor D_n/2 \rfloor > R$, the range of the interaction η. By the spectral flow decomposition in (3.5),

$$\begin{aligned} \mathrm{diam}(\mathrm{sp}_{0,\Lambda_n}(\varepsilon)) &\le 2\|P\Phi^1(\varepsilon)_0 P\| \\ &= 2\|\sum \{P\Phi^1_x(\varepsilon)_0 P : x \in \Lambda_n\}\| \\ &\le 2(A + B) \end{aligned}$$

for A, B defined by complementary regions of the interval $\Lambda_n = [-n, n]$,

$$\begin{aligned} A &= \sum \Big\{ \|P\Phi^1_x(\varepsilon)_0 P\| \mid \forall x \in \Lambda_n : -n + \lfloor D_n/2 \rfloor \le x \le n - \lfloor D_n/2 \rfloor \Big\}, \\ B &= \sum \Big\{ \|P\Phi^1_x(\varepsilon)_0 P\| \mid \forall x \in \Lambda_n : -n \le x < -n + \lfloor D_n/2 \rfloor \text{ or } n - \lfloor D_n/2 \rfloor < x \le n \Big\}. \end{aligned}$$

By applying LTQO and $\mathcal{F}$-norm bounds,

$$\|A\| \le 4C(M_{\mathrm{Int}} + \|\eta\|_F)\varepsilon \sum_{k=\lfloor D_n/2 \rfloor}^{\infty} [\Omega(k/2) + F_0(\lfloor k/2 \rfloor)],$$

where F_0 is the shifted base $\mathcal{F}$-function from (3.10). For the norm bound on B, let $\Delta_{X(k)}$ denote the partial trace difference operators from the Proof of Theorem 3.1 (c.f. Theorem 6.3.4 in Ref. 18), defined with respect to an enlargement of $X \subset \Lambda_n$. Suppose $-n \le x < -n + \lfloor D_n/2 \rfloor$or $n - \lfloor D_n/2 \rfloor <$ $x \le n$. Denote $d_x(n) = d(x, \mathrm{Int}_{D_n}(\Lambda_n))$. By the locality assumption on Φ^{Λ_n} and the fact that $d_x(n)/2 >$ R, if $k \le \lfloor d_x(n)/2 \rfloor$, then, in the notation of the Proof of Theorem 3.1,

$$\Phi^1(b_x(k), \varepsilon) = \Delta_{b_x(k)}((\alpha_\varepsilon - \mathrm{id}) \circ \mathcal{F}_{w_{\gamma_0,\varepsilon}}(h_x)),$$

and so

$$\begin{aligned} \|P\Phi^1_x(\varepsilon)_0 P\| &\le \|\sum_{k=1}^{\lfloor d_x(n)/2 \rfloor} P\Phi^1(b_x(k), \varepsilon)_0 P\| + \sum_{k=\lfloor d_x(n)/2 \rfloor + 1}^{R_x} \|P\Phi^1(b_x(k), \varepsilon)_0 P\| \\ &\le 4\|(\alpha_\varepsilon - \mathrm{id}) \circ \mathcal{F}_{w_{\gamma_0,\varepsilon}}(h_x)\| + 2C(M_{\mathrm{Int}} + \|\eta\|_F)\varepsilon F_0(\lfloor d_x(n)/2 \rfloor). \end{aligned}$$

Using the quasi-locality of the generator $iD(\varepsilon)$ of the spectral flow unitaries,

$$(\alpha_\varepsilon^{\Lambda_n} - \mathrm{id}) \circ \mathcal{F}_{w_{\gamma_0},\varepsilon}(h_x) = \int_0^\varepsilon i\alpha_s^{\Lambda_n}\Big([D(s), \sum_{k=1}^{R_x} \Delta_{b_x(k)}(\mathcal{F}_{w_{\gamma_0,\varepsilon}}(h_x))]\Big)\, \mathrm{d}s,$$

and there exists an $\mathcal{F}$-function $\tilde{F}$, independent of Λ_n, such that

$$2\|(\alpha^{\Lambda_n} - \mathrm{id}) \circ \mathcal{F}_{w_{\gamma_0,\varepsilon}}(h_x)\| \leq \varepsilon \tilde{F}(\lfloor d_x(n)/2 \rfloor).$$

Hence

$$\|B\| \leq \sum_{k=\lfloor D_n/2 \rfloor}^{\infty} \varepsilon \tilde{F}(k/2) + 2C(M_{\mathrm{Int}} + \|\eta\|_F)\varepsilon F_0(k/2).$$

Let $\mathcal{G}(r) = 8\sum_{k=\lfloor r/2 \rfloor}^{\infty} \tilde{F}(k/2) + C(M_{\mathrm{Int}} + \|\eta\|_F)[\Omega(k/2) + F_0(\lfloor k/2 \rfloor)]$. Then

$$\mathrm{diam}(\mathrm{sp}_{0,\Lambda_n}(\varepsilon)) \leq \varepsilon \mathcal{G}(D_n).$$

□

Let $P_n(\varepsilon)$ denote the spectral projection of $H^{\partial_n}_{\Lambda_n}(\epsilon)$ associated with the isolated portion of the spectrum $\mathrm{sp}_{0,\Lambda_n}(\varepsilon)$ and define the set of states of $\mathfrak{A}^s_{\Lambda_n}$, $\mathcal{S}_n(\varepsilon)$, with support in the range of $P_n(\varepsilon)$,

$$\mathcal{S}_n(\varepsilon) = \{\omega \mid \omega \text{ is a state on } \mathfrak{A}^s_{\Lambda_n} \text{ with } \omega(P_n(\varepsilon)) = 1\}.$$

We now consider the thermodynamic limits of these states,

$$\mathcal{S}(\varepsilon) = \{\omega \text{ state on } \mathfrak{A}^s_{\mathbb{Z}} \mid \exists (n_k) \text{ increasing and } \omega_k \in \mathcal{S}_{n_k}(\varepsilon) \text{ s.t. } \lim_k \omega_k(A) = \omega(A), \forall A \in \mathfrak{A}^{\mathrm{loc}}_{\mathbb{Z}}\}.$$

Lemma 3.14. Let $c_n(\varepsilon) = \mathrm{diam}(\mathrm{sp}_{0,\Lambda_n}(\varepsilon))$. *Then*

(i) *for all* $\omega \in \mathcal{S}_n(\varepsilon)$ *and* $A \in \mathfrak{A}^s_{\Lambda_n}$, *we have*

$$\mathrm{Re}\,\omega(A^*[H^{\partial_n}_{\Lambda_n}(\epsilon), A]) \geq -c_n(\varepsilon)\|A\|^2, \textit{and } \left|\mathrm{Im}\,\omega(A^*[H^{\partial_n}_{\Lambda_n}(\epsilon), A])\right| \leq c_n(\varepsilon)\|A\|^2.$$

(ii) *If* $s_n = 0$ *and* $D_{1,n}$ *is such that* $\lim_n[n - D_{1,n}] = \lim_n D_{1,n} = \infty$, *then, for all* $\omega \in \mathcal{S}(\varepsilon)$ *and* $A \in \mathfrak{A}^{\mathrm{loc}}_{\mathbb{Z}}$, *we have*

$$\lim_{n\to\infty} \omega(A^*[H^{\partial_n}_{\Lambda_n}(\epsilon), A]) \geq 0.$$

Proof. The proof of (i) is elementary and the proof of (ii) follows by noting that the additional assumptions imply that the sequence $[H^{\partial_n}_{\Lambda_n}(\epsilon), A]$ converges in norm and that $\lim c_n(\varepsilon) = 0$ by Lemma 3.13. □

In other words, the conditions of part (ii) of the lemma imply that the states in $\mathcal{S}_n(\epsilon)$ converge to ground states of the infinite system. In Subsection 2 of the Appendix, it is explained that the spectral flow automorphisms, like the time evolution of the system, converge to the same limit regardless of the choice of boundary condition ∂_n. Since we have the relation $P_n(0) = \alpha^{\Lambda_n,\partial_n}_{\varepsilon}(P_n(\varepsilon))$, we also have

$$\mathcal{S}_n(\varepsilon) = \mathcal{S}_n(0) \circ \alpha^{\Lambda_n,\partial_n}_{\varepsilon},$$

and as an easy consequence of the convergence (see Ref. 1, Lemma 5.6), we then also have

$$\mathcal{S}(\varepsilon) = \mathcal{S}(0) \circ \alpha_{\varepsilon}.$$

Since the same α_ϵ relates limiting states regardless of the boundary conditions, for example, with a constant sequence $\partial_n = \partial$, for any n, these limiting states must be the same and, hence, also ground states of the infinite systems defined by the dynamics τ_t. The same conclusion then holds for the lower bound on the spectral gap above these ground states (see Ref. 10 for the details).

IV. STABILITY OF SPECTRAL GAP IN FERMION CHAINS

A. Quasi-local maps

Suppose $\mathfrak{A}_\Lambda$ is a local algebra of observables which is $*$-isomorphic to $\mathfrak{A}^s_\Lambda$. Let $\phi : \mathfrak{A}_\Lambda \to \mathfrak{A}^s_\Lambda$ denote a possible $*$-isomorphism. Given a local Hamiltonian H_Λ in $\mathfrak{A}_\Lambda$, ϕ unitarily transforms H_Λ into a Hamiltonian $H^s_\Lambda = \phi(H_\Lambda)$ of the spin algebra. Using an exhaustive family of conditional expectations $\{\theta_{X_i} : X_i \subset X_{i+1}\}$, H^s_Λ can again be realized as the sum of local operators through a telescoping sum,

$$\forall B \in \mathfrak{A}_\Lambda : \quad \phi(B) = \theta_{X_1}(\phi(B)) + \sum_{j=1}^{N-1} \theta_{X_{j+1}}(\phi(B)) - \theta_{X_j}(\phi(B)).$$

The Proof of Theorem 3.1 uses this method of decomposition in the setting where ϕ is a *quasi-local* $*$-automorphism, and the θ_{X_j} are normalized partial trace over increasing metric balls $X_j = b_x(j)$. The quasi-locality property, defined below, guarantees that the transformed local interaction will have decay comparable to that of the original interaction.

In this section, we prove the stability of the spectral gap for even Hamiltonians in the CAR algebra of fermions satisfying $\mathbb{Z}_2$-LTQO. To do this, we will use the Jordan-Wigner isomorphism to transform even fermion interactions into spin interactions in a way that respects the parity symmetry.

Definition. Let $\Lambda \in P_f(\mathbb{Z})$ be a nonempty interval. A linear map $\alpha : \mathfrak{A}^s_\Lambda \to \mathfrak{A}^s_\Lambda$ is *quasi-local* if there exist constants $C > 0, p \in \mathbb{N}$ and a decay function g: $[0, \infty) \to [0, \infty)$ such that if $X, Y \subset \Lambda$ are disjoint subsets, then for all $A \in \mathfrak{A}^s_X$ and $B \in \mathfrak{A}^s_Y$, the following bounds hold:

$$\|\alpha(A)\| \le C|X|^p\|A\| \quad \|[\alpha(A), B]\| \le C\|A\|\|B\||X|^p g(d(X, Y)). \tag{4.1}$$

Example. The local Heisenberg dynamics $\tau^\Lambda : U \subseteq \mathbb{R} \to \mathrm{Aut}(\mathfrak{A}^s_\Lambda)$ generated by an interaction Ψ with a finite $\mathcal{F}$-norm is a collection of quasi-local maps parametrized by t. Let F be an $\mathcal{F}$-function such that $\|\Psi\|_F < \infty$, and denote by v_Ψ the Lieb-Robinson velocity. There exists a constant $C_\Psi > 0$ such that for $X, Y \in P_f(\Lambda)$ disjoint sets and $A \in \mathfrak{A}^s_X, B \in \mathfrak{A}^s_Y$, the following Lieb-Robinson bound holds:

$$\|[\tau^\Lambda_t(A), B]\| \le C_\Psi(e^{v_\Psi|t|} - 1)\|A\|\|B\| \sum_{x\in X, y\in Y} F(|x - y|).$$

But by properties of the $\mathcal{F}$-function,

$$\sum_{x\in X, y\in Y} F(|x - y|) \le |X| \sup\left\{ \sum_{\substack{y\in\mathbb{Z}\\ |x-y|\ge d(X,Y)}} F(|x - y|) : x \in \mathbb{Z} \right\} < \infty.$$

So take $C_t = C_\Psi(e^{v_\Psi|t|} - 1)$, $p_t = 1$, and

$$g_t(n) = \sup\left\{ \sum_{\substack{y\in\mathbb{Z}\\ |x-y|\ge n}} F(|x - y|) : x \in \mathbb{Z} \right\}.$$

In particular, the spectral flow automorphism $\alpha^\Lambda : [0, \varepsilon_\Lambda] \to \mathrm{Aut}(\mathfrak{A}^s_\Lambda)$ is quasi-local.[1]

Last, we specify the normalized partial trace maps. Let

$$X(n) = \{z \in \Lambda : \exists x \in X,\ |z - x| \le n\}$$

denote an enlargement of $X \in P_f(\Lambda)$. Denote the normalized partial trace of the state space over $\Lambda\backslash X(n)$ by

$$\theta_{X(n)} = \frac{1}{\dim \mathfrak{H}_{\Lambda\backslash X(n)}} \mathrm{tr}_{\mathfrak{H}_{\Lambda\backslash X(n)}}.$$

For convention, we will take the trace over $\mathfrak{H}_\emptyset$ as the identity map. Then define, for all $A \in \mathfrak{A}^s_\Lambda$,

$$\Delta_{X(0)}(A) = \theta_{X(0)}(A), \quad \Delta_{X(n)}(A) = \theta_{X(n)}(A) - \theta_{X(n-1)}(A).$$

B. Transformation of even fermion interactions

Recall, we denote by $\mathfrak{A}^+_\Lambda \subset \mathfrak{A}^f_\Lambda$ the even operators of the CAR algebra over Λ. We say that $\beta \in \mathrm{Aut}(\mathfrak{A}_\Lambda)$ is even if it preserves the parity. Even interactions are defined similarly. We also denote $S^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$. The following definition is the well-known Jordan-Wigner transformation, which gives a C^*-isomorphism of CAR and spin algebras.

Definition. Consider the case $\mathfrak{A}^s_{\{x\}} = M_2(\mathbb{C})$. Let $\vartheta_\Lambda : \mathfrak{A}^f_\Lambda \to \mathfrak{A}^s_\Lambda$ denote the *Jordan-Wigner map* defined by

$$a(x) \mapsto \exp\left(-i\pi \sum_{j<x} S^+_j S^-_j \right) S^-_x, \quad a^*(x) \mapsto \exp\left(i\pi \sum_{j<x} S^+_j S^-_j \right) S^+_x.$$

The Jordan-Wigner transformation extends the notion of parity to the spin 1/2 algebra. We say that $A \in \mathfrak{A}^s_\Lambda$ is even if $\vartheta^{-1}_\Lambda(A) \in \mathfrak{A}^+_\Lambda$.

Proposition 4.1. Let $X \subset \Lambda$ be any subinterval.

1. *If $A \in \mathfrak{A}^+_X$, then $\vartheta_\Lambda(A) \in \mathfrak{A}^s_X$.*
2. *If $\alpha : \mathfrak{A}^s_\Lambda \to \mathfrak{A}^s_\Lambda$ is an even quasi − local map, then $\Delta_{X(n)} \circ \alpha$ is also even.*

Proof. Suppose A is a monomial $ca^\#(x_1)\cdots a^\#(x_{2n})$. By the CAR, we may assume $x_j \leq x_{j+1}$. A direct computation shows that the first part of the lemma holds for the even monomials which generate $\mathfrak{A}^+_X$,

$$\vartheta_\Lambda(A) = c \prod_{k=2}^{2n} S^\flat_{x_k} S^\flat_{x_{k-1}} \exp\left(\pm i\pi \sum_{j=x_{k-1}}^{x_k-1} S^+_j S^-_j \right) \in \mathfrak{A}^s_X.$$

Next, we show that the partial trace is an even map. For any $x \in \Lambda$, define the following four unitary operators:

$$u^{(0)}_x = \mathbb{1},\ u^{(1)}_x = \sigma^1_x,\ u^{(2)}_x = i\sigma^2_x,\ u^{(3)}_x = \sigma^3_x.$$

Now, let $Z \subset \Lambda$ and $B \otimes C \in \mathfrak{A}^s_Z \otimes \mathfrak{A}^s_{\Lambda\setminus Z}$. Denote by $I_{\Lambda\setminus Z}$ the set of finite sequences ι: $\Lambda\setminus Z \to \{0, 1, 2, 3\}$. Define

$$u(\iota) = \prod_{z\in Z} u^{(\iota_z)}_z.$$

Using elementary properties of trace and locality in the spin algebra,

$$\frac{1}{\dim(\mathfrak{H}_{\Lambda\setminus Z})} B \otimes \mathrm{tr}(C)\mathbb{1} = \frac{1}{4^{|\Lambda\setminus Z|}} \sum_{\iota\in I_{\Lambda\setminus Z}} u(\iota)^*[B \otimes C]u(\iota) \in \mathfrak{A}^s_Z. \tag{4.2}$$

The relation in (4.2) uniquely defines the partial trace; hence,

$$\theta_Z(\cdot) = \frac{1}{4^{|\Lambda\setminus Z|}} \sum_{\iota\in I_{\Lambda\setminus Z}} u(\iota)^*[\cdot]u(\iota).$$

The second part of the lemma follows from this formula.

□

In the following, we will assume the interactions are supported on intervals:

Definition. An interaction Φ is *supported on intervals* if $\Phi(X) \neq 0$ only if $X = [a, b]$ for some $a, b \in \mathbb{Z}$.

Any interaction can be "regrouped" into one with interval support, and while the methods to do this are neither new nor canonical, we record here a simple way without changing the local Hamiltonians, at the expense of rate of decay.

Proposition 4.2. Suppose $I \subset \mathbb{Z}$ is an interval and $\Psi : P_f(I) \to \mathfrak{A}_{\mathrm{loc}}$ is an interaction. Then there exists an interaction $\Phi : P_f(I) \to \mathfrak{A}_{\mathrm{loc}}$, supported on intervals, such that for all finite intervals $\Lambda \subset I$, the local Hamiltonians are equal,

$$\Phi_\Lambda = \sum_{X\subset\Lambda} \Phi(X) = \Psi_\Lambda.$$

If Ψ is an unperturbed interaction with uniform bound M, range R, and local gap γ_0, then so is Φ, with uniform bound $2^R M$ and the same range and local gap.

Furthermore, if $\|\Psi\|_F < \infty$*, where* F *is the* $\mathcal{F}$*-function in (A2), and* $h(r) \geq Kr^s$ *for some* $K > 0$ *and* $s \in (0, 1]$*, then* $\|\Phi\|_G \leq \|\Psi\|_F$ *for the* $\mathcal{F}$*-function defined by*

$$G(r) = e^{-\frac{1}{2}h(r)} \frac{C_\Phi}{(1+cr)^\kappa},$$
$$C_\Phi = L \sum_{n=1}^{\infty} n e^{-\frac{1}{2}h(n)}.$$

Proof. If $I \subsetneq \mathbb{Z}$, then we may extend Ψ to $\mathbb{Z}$ by $\Psi(Z) = 0$ for $Z \not\subset I$, and by construction, Φ defined in terms of the extension will restrict to an interaction on I. So we may assume $I = \mathbb{Z}$. We will define Φ by induction on the diameter n of intervals $[k, k+n]$. When $n = 0, 1$, define

$$\Phi(\{x\}) = \Psi(\{x\}) \text{ and } \Phi(\{x, x+1\}) = \Psi(\{x, x+1\}).$$

For larger n, define

$$\Phi([k, k+n]) = \sum \{\Psi(X) : X \subset [k, k+n], \operatorname{diam}(X) = n\}.$$

By construction, $\Phi_\Lambda = \Psi_\Lambda$. Now, suppose Φ is an unperturbed interaction with constants M, R, γ_0. Since $\Phi_{b_\Lambda(x,n)} = \Psi_{b_\Lambda(x,n)}$ for all x and n, Φ and Ψ have the same local gap. Similarly, it is clear that Φ and Ψ have the same range R, and if $\operatorname{diam}([a, b]) \leq R$,

$$\|\Phi([a, b])\| \leq 2^R M.$$

Now, suppose Φ is some interaction, not necessarily finite range, with $\|\Phi\|_F$. For fixed $k \in \mathbb{Z}$ and $n \geq 0$, by Proposition A.1,

$$\|\Phi([k, k+n])\| \leq \sum_{\substack{X \in P_f(\mathbb{Z}) \\ k, k+n \in X}} \|\Psi(X)\| \leq \|\Psi\|_F F(n).$$

So for $x, y \in \mathbb{Z}$,

$$\begin{aligned}
\sum_{\substack{k,n \\ x,y \in [k,k+n]}} \|\Phi([k, k+n])\| &= \sum_{n \geq |x-y|} \sum_{\substack{k \\ x,y \in [k,k+n]}} \|\Phi([k, k+n])\| \\
&\leq \sum_{n \geq |x-y|} \sum_{\substack{k \\ x,y \in [k,k+n]}} \|\Psi\|_F F(n) \\
&\leq \|\Psi\|_F \sum_{n \geq |x-y|} (n + 1 - |x-y|) e^{-h(n)} \frac{L}{(1+cn)^\kappa} \\
&\leq \|\Psi\|_F \left(\sum_{n=1}^{\infty} n e^{-\frac{1}{2}h(n)} \right) e^{-\frac{1}{2}h(|x-y|)} \frac{L}{(1+c|x-y|)^\kappa}.
\end{aligned}$$

That is,

$$\|\Phi\|_G = \sup_{x,y \in \mathbb{Z}} \left\{ \sum_{\substack{X \in P_f(\mathbb{Z}) \\ x,y \in X}} \frac{\|\Phi(X)\|}{G(|x-y|)} \right\} \leq \|\Psi\|_F.$$

□

Proposition 4.3. Suppose $\Psi : P_f(I) \to \mathfrak{A}^f_{\text{loc}}$ *is an even interaction supported on intervals. Then there exists an even interaction* $\Phi : P_f(I) \to \mathfrak{A}^s_{\text{loc}}$ *such that for any* $\Lambda \subset I$,

$$\vartheta_\Lambda(\Psi_\Lambda) = \Phi_\Lambda.$$

If Ψ *satisfies a finite* $\mathcal{F}$*-norm for some* F *of the form (A2), then so does* Φ*. If* Ψ *is an unperturbed interaction, then so is* Φ *for the same constants.*

Proof. For $\Lambda_0 \subset \Lambda$, let $\iota_{\Lambda_0,\Lambda}$ denote the inclusion $\mathfrak{A}^f_{\Lambda_0} \hookrightarrow \mathfrak{A}^f_{\Lambda}$. If $A \in \mathfrak{A}^+_{\Lambda_0}$, then by expanding in an even generating set of monomials, we see

$$\iota_{\Lambda_0,\Lambda} \circ \vartheta_{\Lambda_0}(A) = \vartheta_\Lambda \circ \iota_{\Lambda_0,\Lambda}(A).$$

So there exists an injective $*$-morphism $\vartheta : \bigcup \mathfrak{A}^+_\Lambda \to \mathfrak{A}^s_{\mathrm{loc}}$ which extends every ϑ_Λ, from which we define $\Phi(X) = \vartheta\,(\Psi(X))$. By Proposition 4.1, this is a well-defined interaction which is also supported on intervals. Evidently Φ is an even interaction; i.e., $\vartheta^{-1}(\Phi(X))$ is even for any X.

ϑ is isometric, and for the $\mathcal{F}$-function F,

$$\|\Psi\|_F = \sup_{x,y} \sum_{\substack{X \in P_f(\mathbb{Z}) \\ x,y \in X}} \frac{\|\vartheta(\Psi(X))\|}{F(|x-y|)} = \|\Phi\|_F.$$

Now suppose Ψ is an unperturbed interaction. Then evidently Φ is uniformly bounded. Φ is frustration free and uniformly locally gapped since, for any Λ, there exists a unitary $Q_\Lambda : \mathfrak{H}_\Lambda \to \mathfrak{F}_\Lambda$ such that for $A \in \mathfrak{A}^f_\Lambda$,

$$\vartheta(A) = \vartheta_\Lambda(A) = Q^*_\Lambda A Q_\Lambda.$$

Since ϑ is an isometry which preserves support for even observables, and Q_Λ is unitary, Φ has the same uniform bound, range, and local gap as Ψ. □

Theorem 4.4 (Ground state gap stability for fermion chains). *There exist $\varepsilon'_{\gamma_0} > 0$ and constant m'_D such that $0 \le \varepsilon < \varepsilon'_{\gamma_0}$ and* $\mathrm{diam}(\Lambda) > \max\{2D, R\}$ *imply*

$$\gamma(H_\Lambda(\varepsilon)) \ge \gamma_0 - m'_D \varepsilon > 0.$$

The constants m'_D and ε'_{γ_0} can be explicitly determined by the expressions in (3.17).

Proof. By Proposition 4.2, we assume that η and $\Phi^\Lambda = \Phi$ are supported on intervals. Proposition 4.3 implies the existence of spin interactions η^S and Φ^S with the same uniform bound, range, local gap γ_0, and decay.

Let $\gamma \in (0, \gamma_0)$ and $D \in \mathbb{N}$ be a chosen distance from the boundary, uniform in the volume, and consider fixed Λ with sufficiently large diameter. By Theorem 3.1, the spectral flow decomposes the local Hamiltonian $H_\Lambda(\varepsilon)$ of $\eta^S + \varepsilon\Phi^S$,

$$\alpha^\Lambda_\varepsilon(H_\Lambda + \varepsilon\Phi_\Lambda) = H_\Lambda + \sum_{x\in\Lambda} \Phi^1_x(\varepsilon) = H_\Lambda + \Phi^2(\varepsilon) + \Phi^3(\varepsilon) + \mathcal{R}(\varepsilon) + \omega_\Lambda(\widetilde{\Phi^1(\varepsilon)}).$$

Since ϑ_Λ is implemented by some unitary, η^S has $\mathbb{Z}_2$-LTQO for the same decay function Ω. So to apply the norm boundedness argument in Sec. III, it suffices to argue that $\Phi^1(b_\Lambda(x, n), \varepsilon)$ is even.

But the Proof of Theorem 3.1 in Ref. 18 guarantees the existence of even interactions $\Psi_i : P_f(\Lambda) \to \mathfrak{A}^s_\Lambda$, $i = 1, 2, 3$, and quasi-local maps $\mathcal{K}^{(\varepsilon)}_i : \mathfrak{A}^s_\Lambda \to \mathfrak{A}^s_\Lambda$ such that

$$\begin{aligned}\Phi^1(b_\Lambda(x,n),\varepsilon) = \Delta_{b_\Lambda(x,n)} \circ \mathcal{K}^{(\varepsilon)}_1(\Psi_1(\{x\})) + \sum_{k=1}^{n} \varepsilon\Delta_{b_\Lambda(x,n)} \circ \mathcal{K}^{(\varepsilon)}_2(\Psi_2(b_\Lambda(x,k))) \\ + \Delta_{b_\Lambda(x,n)} \circ \mathcal{K}^{(\varepsilon)}_3(\Psi_3(b_\Lambda(x,k))).\end{aligned}$$

The $K^{(\varepsilon)}_i$ are defined in terms of the spectral flow automorphism and are also even maps. Hence, by Lemma 4.1, $\Phi^1(b_\Lambda(x, n), \varepsilon)$ must also be even since the even observables form a subalgebra. □

V. EXAMPLE OF EVEN HAMILTONIAN SATISFYING STABILITY HYPOTHESES

Here we describe an example of an interaction of the CAR algebra which satisfies the stability hypotheses of Theorem 3.11. Let $\mathcal{X} = \{f_i : i \in \mathcal{B}\}$ and $\mathcal{Y} = \{g_j : j \in \mathcal{B}\}$ be two collections of vectors in $\ell^2(\mathbb{Z})$ such that

(i) $\mathcal{X} \cup \mathcal{Y}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$;
(ii) there exist $R \geq 0$ and collections $\{x_i : i \in \mathcal{B}\}$, $\{y_j : j \in \mathcal{B}\}$ such that for all i, j,

$$\mathrm{supp}(f_i) \subset b(x_i, R), \quad \mathrm{supp}(g_j) \subset b(y_j, R),$$
$$i \neq j \text{ implies } b(x_i, R) \cap b(x_j, R) = \emptyset = b(y_i, R) \cap b(y_j, R).$$

Furthermore, denote $\mathcal{X}_W = \{f_i : \mathrm{supp}(f_i) \subset W\}$ and $\mathcal{Y}_W = \{g_j : \mathrm{supp}(g_j) \subset W\}$. We will also assume the following:
(iii) There exists a diameter N_0 such that for all intervals Λ, $\mathrm{diam}(\Lambda) > N_0$ implies $\mathcal{X}_\Lambda \neq \emptyset$ and $\mathcal{Y}_\Lambda \neq \emptyset$.

Definition. Let $\eta : P_f(\mathbb{Z}) \to \mathfrak{A}^f_{\mathrm{loc}}$ be the finite-range interaction defined by

$$\eta(b(x_i, R)) = \mathbb{1} - a^*(f_i)a(f_i), \quad \eta(b(y_j, R)) = a^*(g_j)a(g_j). \tag{5.1}$$

Lemma 5.1. Suppose Λ is an interval such that diam(Λ) $> N_0$. Then H_Λ is non-negative, uniformly gapped, and frustration free.

Proof. Let $(f_{n_1}, \ldots, f_{n_\Lambda})$ and $(g_{m_1}, \ldots, g_{m_\Lambda})$ be the collections of vectors whose support is contained in Λ. If necessary, complete the list to an orthonormal basis of $\ell^2(\Lambda)$ with $(h_1, \ldots, h_p)$, $p = |\Lambda| - n_\Lambda - m_\Lambda$. Evidently H_Λ is uniformly gapped and non-negative. So we prove that

$$\ker(H_\Lambda) = \mathrm{span}(\psi_X : X \subset [1, p]),$$

where we define

$$\phi_\Lambda = f_{n_1} \wedge \cdots f_{n_\Lambda}, \quad \zeta_X = \bigwedge \{h_{i_k} : i_k \in X \subset [1, p]\}, \quad \psi_X = \phi_\Lambda \wedge \zeta_X.$$

By calculation, $\psi_X \in \ker(H_\Lambda)$ for any $X \subset [1, p]$. But each term of the interaction H_Λ is a projection, the complement projection of some $a^*(f_q)a(f_q)$. So $H_\Lambda \psi = 0$ implies $\psi \in \mathrm{ran}(a^*(f_i)a(f_i))$ for each $i = n_1, \ldots, n_\Lambda$. □

Next, we show that the number of auxiliary orthonormal basis vectors h_i needed to complete $\mathcal{X}_\Lambda$ and $\mathcal{Y}_\Lambda$ to a basis of $\ell^2(\Lambda)$ is uniformly bounded in Λ, and that each h_i has support contained toward the edge of Λ.

Lemma 5.2. Suppose $\mathrm{diam}(\Lambda) > N_0$. *Let $\mathcal{Z}(\Lambda) = \{h_1^{(\Lambda)}, \ldots, h_n^{(\Lambda)}\}$, $n = n(\Lambda)$, be a basis for the complement of span($\mathcal{X}_\Lambda \cup \mathcal{Y}_\Lambda$) in $\ell^2(\Lambda)$. Then*

i. *for each $i \in$ [1, n] ,* $\mathrm{supp}(h_i^{(\Lambda)}) \subset \Lambda \setminus \mathrm{Int}_{3R}(\Lambda)$;
ii. $|\mathcal{Z}(\Lambda)| \leq 6R$

Proof. Let (ξ_k) denote the orthonormal basis from $\mathcal{X} \cup \mathcal{Y}$. Suppose $\mathrm{supp}(f) \subset \mathrm{Int}_{3R}(\Lambda)$. Then $x_i \notin \Lambda$ implies $\mathrm{supp}(f_i) \cap \mathrm{supp}(f) = \emptyset$, that is, $\langle f_i, f\rangle = 0$ (respectively, y_j and $\langle g_j, f\rangle = 0$). Hence

$$f = \sum \langle \xi_k, f\rangle \xi_k = \sum_{\xi \in \mathcal{X}_\Lambda \cup \mathcal{Y}_\Lambda} \langle \xi, f\rangle \xi.$$

Hence $f \in \mathrm{span}(\mathcal{X}_\Lambda \cup \mathcal{Y}_\Lambda)$. Now, a basis of the orthogonal complement of $\ell^2(\mathrm{Int}_{3R}(\Lambda))$ in $\ell^2(\Lambda)$ is necessarily supported on $\Lambda \setminus \mathrm{Int}_{3R}(\Lambda)$, proving (1). Additionally, the dimension of $\ell^2(\Lambda \setminus \mathrm{Int}_{3R}(\Lambda))$ is an upper bound for $|Z(\Lambda)|$, which proves (2). □

This lemma has an immediate corollary:

Corollary 5.3. Let $\mathfrak{A}(\mathcal{W})$ denote the C^*-subalgebra of $\mathfrak{A}^f_{\mathbb{Z}}$ generated by the operators $a^*(f)$, $a(f)$ such that $f \in \mathcal{W} \subset \ell^2(W)$. Then for all intervals Λ with diameter larger than $6R$,

$$\mathfrak{A}_{\mathrm{Int}_{3R}(\Lambda)} \subset \mathfrak{A}(\mathcal{X}_\Lambda \cup \mathcal{Y}_\Lambda).$$

To conclude this section, we prove that the interaction defined in (5.1) satisfies $\mathbb{Z}_2$-LTQO. Denote $D = \max\{N_0, 3R\}$. Recall that if $n \geq D$ then $H_{b_\Lambda(x,n)}$ is non-negative and frustration free with kernel indexed by $\mathcal{Z}(b_\Lambda(x, n))$.

Define the step-function Ω: $[0, \infty) \to [0, \infty)$ by

$$\Omega(x) = \begin{cases} 0 & \text{if } x \geq D \\ 2 & \text{otherwise.} \end{cases}$$

Proposition 5.4. Suppose $\operatorname{diam}(\Lambda) > 2D$*, and let* $x \in \Lambda$*, and* $(n, k) \in \mathbb{N}^2$ *be such that* $0 \leq k \leq r_x$, $k \leq n \leq R_x$. *Let* P_n *denote the projection onto* $H_{b_\Lambda(x,n)}$. *Then for all* $A \in \mathfrak{A}^+_{b_\Lambda(x,k)}$,

$$\|P_n(A - \omega_\Lambda(A))P_n\| \leq \Omega(z_x(n) - k)\|A\|.$$

Proof. We handle the two cases of n when $\operatorname{diam}(b_\Lambda(x, n)) \geq N_0$ or $\operatorname{diam}(b_\Lambda(x, n)) < N_0$. Suppose the former. Now, there are two subcases for k: either $b_\Lambda(x, k) \not\subset \operatorname{Int}_D(b_\Lambda(x, n))$ or $b_\Lambda(x, k)$ is contained in that interior.

Suppose $b_\Lambda(x, k) \subset \operatorname{Int}_D(b_\Lambda(x, n))$. Then $z_x(n) - k \geq D$, necessarily. Denote

$$\mathcal{X}_{b_\Lambda(x,n)} = \mathcal{X}_n = \{f_{i_1} \ldots, f_{i_M}\}, \quad \mathcal{Z}(b_\Lambda(x, n)) = \mathcal{Z}(n) = \left\{h_1, \ldots, h_p\right\}$$

Let $\psi^n_X = f_{i_1} \wedge \cdots f_{i_M} \wedge h_{n_1} \wedge \cdots h_{n_{|X|}}$ be a generic unit norm basis vector of the kernel, indexed by $X \subset \mathcal{Z}(n)$. A calculation shows

$$\|P_n(A - \omega_\Lambda(A))P_n\| \leq 6R \sup_{X\subset\mathcal{Z}(n)} |\langle\psi^n_X, A\psi^n_X\rangle - \omega_\Lambda(A)| + 2^{6R} \sup_{X\neq Y} |\langle\psi^n_X, A\psi^n_Y\rangle|.$$

But by the theory of quasi-free states and Corollary 5.3,

$$\sup_{X\subset\mathcal{Z}(n)} |\langle\psi^n_X, A\psi^n_X\rangle - \omega_\Lambda(A)| = \sup_{X\neq Y} |\langle\psi^n_X, A\psi^n_Y\rangle| = 0.$$

Now suppose $b_\Lambda(x, k)$ is not contained in the D-interior of $b_\Lambda(x, n)$. This implies $z_x(n) - k < D$. And by the trivial commutator bound,

$$\|P_n(A - \omega_\Lambda(A))P_n\| \leq 2\|A\| = \Omega(z_x(n) - k)\|A\|.$$

Last, suppose $\operatorname{diam}(b_\Lambda(x, n)) < N_0$. Then $n - k \leq n < N_0 \leq D$. Hence $z_x(n) - k < D$ as well, and the trivial bound agrees with Ω. Conclude that H_Λ satisfies LTQO for Ω. ◻

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APPENDIX: DETAILS ABOUT $\mathcal{F}$-FUNCTIONS AND SPECTRAL FLOW

1. $\mathcal{F}$-functions and decay of interactions

In addition to LTQO, a critical assumption for our spectral gap stability argument is rapid decay of the perturbations in [Φ]. We choose to describe this decay through $\mathcal{F}$-functions, which have several useful properties, one of which define an extended norm on the real vector space of interactions.

Definition. A function F: $[0, \infty) \to (0, \infty)$ is an *$\mathcal{F}$-function for* $(\mathbb{Z}, |\cdot|)$ if

i. $\|F\| = \sum_{x\in\mathbb{Z}} F(x) < \infty$;
ii. there exists $C_F > 0$ such that for all $x, y \in \mathbb{Z}$,

$$\sum_{z\in\mathbb{Z}} F(|x - z|)F(|z - y|) \leq C_F F(|x - y|).$$

Furthermore, if Φ is an interaction, then the $\mathcal{F}$*-norm* of Φ (with respect to F) is defined as

$$\|\Phi\|_F = \sup\left\{\sum_{\substack{Z\in P_f(\mathbb{Z})\\ x,y\in Z}} \frac{\|\Phi(Z)\|}{F(|x-y|)}\right\} \in [0,\infty]. \tag{A1}$$

Example. Suppose h: $[0, \infty) \to [0, \infty)$ is a monotone increasing, subadditive function and $\kappa > 2$. The following function F defines an $\mathcal{F}$-function,

$$F(r) = e^{-h(r)}\frac{L}{(1+cr)^{\kappa}},\ L, c > 0. \tag{A2}$$

The $\mathcal{F}$-function in (A2) and following properties will be used extensively in the proof of spectral gap stability.

Proposition A.1. Suppose Φ *is an interaction with finite* $\mathcal{F}$*-norm for some* F*. Then*

1. *for any collection* $Z_1 \subset Z_2 \subset \cdots \subset Z_N$, $\sum_{k=1}^{N} \|\Phi(Z_k)\| \le \|\Phi\|_F F(\mathrm{diam}(Z_1))$;
2. *if* η *is a uniformly bounded, finite range interaction, then* $\|\eta\|_F < \infty$ *and* $\|\eta + \Phi\|_F < \infty$.

Proof. Let $\mathrm{diam}(Z_1) = n$, and choose $x, y \in Z_1$ such that $|x - y| = n$. Then

$$\sum_{k=1}^{N} \|\Phi(Z_k)\| \le \sum_{\substack{X\in P_f(\mathbb{Z})\\ x,y\in X}} \|\Phi(X)\| \le \|\Phi\|_F F(n).$$

Now, denote the range of η by R and uniform bound by M. Suppose $x, y \in \mathbb{Z}$. If $Z \in P_f(\mathbb{Z})$ contains x, y, and $\Phi(Z) \neq 0$, then $Z \subset b(x, R) \cap b(y, R)$. Hence

$$\sum_{\substack{X\in P_f(\mathbb{Z})\\ x,y\in X}} \|\eta(X)\| \le 2^{3R} M.$$

Then $\|\eta + \Phi\|_F < \infty$ by the triangle inequality. □

2. Thermodynamic limit of the spectral flow

There are standard results giving conditions on an interaction Φ under which the finite-volume dynamics defined by

$$\tau_{s,t}^{\Phi,\Lambda}(A) = U(s,t)^* A U(s,t), A \in \mathfrak{A}_\Lambda,$$

with

$$H_\Lambda(s) = \sum_{X\subset\Lambda} \Phi(X,s)$$

and

$$\frac{d}{ds}U(s,t) = iH_\Lambda(s)U(s,t), U(t,t) = \mathbb{1}, s,t \in I \subset \mathbb{R},$$

converges to a strongly continuous co-cycle of automorphism, $\tau_{s,t}^{\Phi}$, of the algebra of quasi-local observables $\mathfrak{A}_\Gamma$, which is defined as the norm completion of the (strictly) local observables given by

$$\mathfrak{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\in\mathcal{P}_f(\Gamma)} \mathfrak{A}_\Lambda.$$

One sufficient condition is that $\Phi(\cdot, t)$ is a continuous curve taking values in the space $\mathcal{B}_F$, where F defines an F-norm, $\|\cdot\|_F$ on interactions as in Subsection 1 of the Appendix. For any compact interval I, we can define the space $\mathcal{B}_F(I)$ as the set of all such continuous curves, and for $\Phi \in \mathcal{B}_F(I)$, the function

$$\|\Phi\|_F(t) = \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{X\in\mathcal{P}_f(\Gamma):\\ x,y\in X}} \|\Phi(X,\cdot)\|(t) \tag{A3}$$

is continuous and bounded. Strong convergence of the automorphisms here means that

$$\lim_{\Lambda\uparrow\Gamma} \|\tau_{s,t}^{\Phi,\Lambda}(A) - \tau_{s,t}^{\Phi}(A)\| = 0, \text{ for all } A \in \mathfrak{A}_\Gamma^{\mathrm{loc}}.$$

The limit is taken over any sequence of $\Lambda_n \in \mathcal{P}(\Gamma)$ increasing to Γ and the limiting dynamics is independent of the choice of sequence.

One can also show that the dynamics depends continuously on the interaction Φ in the following sense:[9]

$$\|\tau_{t,s}^{\Phi}(A) - \tau_{t,s}^{\Psi}(A)\| \le \frac{2\|A\|}{C_F}\|F\|\,|X| e^{2\min(I_{t,s}(\Phi),I_{t,s}(\Psi))} I_{t,s}(\Phi-\Psi). \tag{A4}$$

which holds for all $A \in \mathcal{A}_X$ and $s, t \in I$, and where, for $\Phi \in \mathcal{B}_F(I)$, and $s, t \in I$, the quantity $I_{t,s}(\Phi)$ is defined by

$$I_{t,s}(\Phi) = C_F \int_{\min(t,s)}^{\max(t,s)} \|\Phi\|_F(r)\, dr. \tag{A5}$$

It is often important to include in the definition of the finite-volume Hamiltonian H_Λ terms that correspond to a particular boundary condition. Such terms affect the ground states and equilibrium states of the system, including in the thermodynamic limit but, in general, do not affect the infinite-volume dynamics. In order to express this freedom in the interactions defining the finite-volume dynamics that lead to the same thermodynamic limit, we use another, weaker, notion of convergence of interactions interaction Φ introduced in Ref. 9, where it is called *local convergence in F-norm*.

Definition. Let $(\Gamma, d, \mathcal{A}_\Gamma)$ be a quantum lattice system, F be an F-function for (Γ, d), and $I \subset \mathbb{R}$ be an interval. We say that a sequence of interactions $\{\Phi_n\}_{n\ge1}$ *converges locally in F-norm* to Φ such that

(i) $\Phi_n \in \mathcal{B}_F(I)$ for all $n \ge 1$,
(ii) $\Phi \in \mathcal{B}_F(I)$, and
(iii) for any $\Lambda \in \mathcal{P}_0(\Gamma)$ and each $[a, b] \subset I$, one has

$$\lim_{n\to\infty} \int_a^b \|(\Phi_n - \Phi)\restriction_\Lambda\|_F(t)\, dt = 0\,. \tag{A6}$$

In this appendix, we want to apply this notion to the spectral flow generated by perturbations of the form (2.4) and its thermodynamic limit.

The spectral flow $\alpha_\epsilon^{\Lambda,\partial}$ for the curve of Hamiltonians $H_\Lambda^\partial(\epsilon)$, $\epsilon \in [0, \epsilon_0)$, defined in (3.20), also depends on a parameter $\gamma > 0$. This parameter is assumed to be a lower bound for the gap of interest in the spectrum of $H_\Lambda^\partial(\epsilon)$ in the stability argument, but this assumption is not needed for the construction of $\alpha_\epsilon^{\Lambda,\partial}$. The automorphisms $\alpha_\epsilon^{\Lambda,\partial}$ are generated by the self-adjoint operators $D_\Lambda^\partial(\epsilon)$, defined by

$$D_\Lambda^\partial(\epsilon) = \mathcal{K}_\epsilon^{\Lambda,\partial}\left(\sum_{X\subset\Lambda_{D_1}} \Phi(X) + s \sum_{X\subset(\Lambda\setminus\Lambda_{D_2})} \Phi(X)\right), \tag{A7}$$

where the map $\mathcal{K}_\epsilon^{\Lambda,\partial} : \mathfrak{A}_\Lambda \to \mathfrak{A}_\Lambda$ is given by

$$\mathcal{K}_\epsilon^{\Lambda,\partial}(A) = \int_{-\infty}^{\infty} \tau^{H_\Lambda^\partial(\epsilon)}(A) W_\gamma(t) dt. \tag{A8}$$

Note that $\mathcal{K}_\epsilon^{\Lambda,\partial}$ is defined as a linear map, but it depends itself on $H_\Lambda^\partial(\epsilon)$ and therefore $D_\Lambda^\partial(\epsilon)$ depends non-linearly on the perturbation. In Ref. 9 [Sec. V D], a detailed study of transformations of the form $\mathcal{K}_\epsilon^{\Lambda,\partial}$ is performed. The following proposition follows directly from applying the more general results in that work to the situation here.

Proposition A.2 (Ref. 9). *There exists an F-function $\tilde{F}$ of the form (A2) and interactions $\Psi^{\Lambda,\partial} \in \mathcal{B}_{\tilde{F}}([0,\epsilon_0])$ such that*

$$D_\Lambda^\partial(\epsilon) = \sum_{X\subset\Lambda} \Psi^{\Lambda,\partial}(X,\epsilon).$$

Furthermore, there exists an interaction $\Psi \in \mathcal{B}_{\tilde{F}}([0,\epsilon_0])$ such that for any sequences $\Lambda_n = [a_n, b_n] \subset \mathbb{Z}$, $\partial_n = (D_{1,n}, D_{2,n}, s_n)$, the interactions $\Psi^{\Lambda_n,\partial_n}$ have uniformly bounded $\tilde{F}$-norm and converge locally in $\tilde{F}$-norm to Ψ.

As a consequence, we can apply the following theorem from Ref. 9 to the sequences of interactions $\Psi^{\Lambda_n,\partial_n}$.

Theorem A.3 (Ref. 9, Theorem 3.8). *Let $(\Phi_n)_{n\geq 1}$ be a sequence of time-dependent interactions on Γ with Φ_n converging locally in the F-norm to Φ with respect to F. Suppose that for every* [a, b] $\subset I$,

$$\sup_{n\geq 1} \int_a^b \|\Phi_n\|_F(t)\,dt < \infty\,. \tag{A9}$$

Then for any $X \in \mathcal{P}_0(\Gamma)$,

$$\lim_{n\to\infty} \|\tau_{t,s}^{\Phi_n}(A) - \tau_{t,s}^{\Phi}(A)\| = 0 \tag{A10}$$

for all $A \in \mathcal{A}_X$ and each s, $t \in I$. Moreover, the convergence is uniform for s, t in compact intervals.

Now, consider a sequence $\Lambda_n = [a_n, b_n] \subset \mathbb{Z}$, $\partial_n = (D_{1,n}, D_{2,n}, s_n)$. As the result of applying Proposition A.2 and Theorem A.3, we obtain the strong convergence of the finite-volume spectral flow automorphism generated by $\Psi^{\Lambda_n,\partial_n}$ to one and the same spectral flow for the infinite chain: for $\epsilon \in [0, 1]$,

$$\lim_{n\to\infty} \alpha_\epsilon^{\Lambda_n,\partial_n}(A) = \alpha_\epsilon(A), \qquad \text{for all } A \in \mathfrak{A}_\Gamma^{\rm loc}. \tag{A11}$$

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Automorphic equivalence preserves the split property

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We prove that the split property is a stable feature for spin chain states which are related by composition with $*$-automorphisms generated by power-law decaying interactions. We apply this to the theory of the $\mathbb{Z}_2$-index for gapped ground states of symmetry protected topological phases to show that the $\mathbb{Z}_2$-index is an invariant of gapped classification of phases containing fast-decaying interactions.

1. Introduction

Recent studies have rigorously proven the existence of symmetry protected topological (SPT) phase transitions in one dimension using an invariant of smooth, gapped classification known as the $\mathbb{Z}_2$-index [13,17]. The $\mathbb{Z}_2$-index follows a line of investigation of the invariants which arise from symmetries of a quantum spin chain [1,12,15,16].

In [13], it is proven that this index is a well-defined invariant for finite-range interactions, regardless of boundary conditions, and that the index agrees with the matrix product state index defined in [15]. Thus it is concluded that the AKLT interaction belongs to a non-trivial topological phase of finite-range interactions protected by time

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reversal symmetry. It is also known that extensive but sufficiently small and fast-decaying perturbations of the AKLT interaction on the chain will not move the system out of the phase (e.g. see [8]). In particular, an SPT phase may contain interactions which are not finite range.

The objective of this note is to investigate when the $\mathbb{Z}_2$-index is a stable invariant of an SPT phase in one dimension. We prove that under certain hypotheses, including superpolynomial but still subexponential decay of interactions and uniqueness of the gapped ground state, that if an SPT phase contains an interaction with a well-defined $\mathbb{Z}_2$-index, then all interactions in the phase have a well-defined index, and that the index is an invariant of the phase (see Section 3 for the hypotheses). For this, we follow the proof of Ogata for the finite-range case closely, making the necessary and material modifications to handle an unbounded range of interaction. This stability provides further evidence that the $\mathbb{Z}_2$-index detects a true phase transition between interactions in distinct symmetry protected topological phases.

A significant mathematical obstruction to assuming weaker decay conditions is in proving that certain gapped ground states of interactions satisfy the *split property*. So far, general results on sufficient conditions for the split property to hold critically use characteristics of finite-range or exponentially decaying one-dimensional interactions, such as boundedness of the entanglement entropy or the validity of Haag duality for the spin chain interactions [6,7]. We comment on the relationship between split property for translation invariant ground states and Haag duality in Section 2.

Our main result is that quasi-local deformations of split states preserve the split property. Our proofs make use of Lieb-Robinson bounds on the speed of propagation of time-evolved observables which do not depend on the sizes of support. To the best of our knowledge, the results of this note are the first which generally guarantee the split property for ground states of interactions which do not necessarily decay exponentially.

1.1. Notations and assumptions

We consider the one-dimensional lattice $(\mathbb{Z}, |\cdot|)$. Let $P_f(\Sigma)$ denote the finite subsets of $\Sigma \subset \mathbb{Z}$. The onsite Hilbert space at $x \in \mathbb{Z}$ is $\mathfrak{H}_x = \mathbb{C}^d$, where $d \geq 2$ is taken to be independent of x for simplicity. Let $\mathfrak{A}_{\{x\}} = M_d(\mathbb{C})$ denote the onsite algebra of observables. Local algebras of observables for $X \in P_f(\mathbb{Z})$ are defined by tensor product:

$$\mathfrak{A}_X = \bigotimes_{x \in X} \mathfrak{A}_{\{x\}}. \tag{1.1}$$

We reserve Λ as notation for a finite interval of the form $[a, b] \cap \mathbb{Z}$. Let $\mathfrak{A}_{loc}$ denote the maximal algebra obtained by inclusion of local algebras, and $\mathfrak{A}_{\mathbb{Z}}$ its closure with respect to the operator norm:

$$\mathfrak{A}_{loc} = \bigcup_{X \in P_f(\mathbb{Z})} \mathfrak{A}_X, \qquad \mathfrak{A}_{\mathbb{Z}} = \overline{\mathfrak{A}_{loc}}^{\|\cdot\|}. \tag{1.2}$$

Similarly, let $\mathfrak{A}_L$ and $\mathfrak{A}_R$ denote the C^*-algebras obtained from the local algebras of the left and right complementary half-infinite chains, respectively:

$$\mathfrak{A}_L = \overline{\bigcup_{X \in P_f((-\infty,0])} \mathfrak{A}_X}^{\|\cdot\|} \quad (\text{resp. } \mathfrak{A}_R). \tag{1.3}$$

We model the interactions between sites of the lattice with *interaction functions* parametrized by a dependence $t \in [0,1]$:

$$\Phi(\cdot, t) : P_f(\mathbb{Z}) \to \mathfrak{A}_{\mathbb{Z}} \tag{1.4}$$

$$X \mapsto \Phi(X,t) = \Phi(X,t)^* \in \mathfrak{A}_X. \tag{1.5}$$

For regularity, we assume for each $X \in P_f(\mathbb{Z})$ that the dependence $t \mapsto \Phi(X,t)$ is continuously differentiable. The dynamics $\tau^\Lambda : [0,1] \to \mathrm{Aut}(\mathfrak{A}_\Lambda)$ of the model are generated by the family of Hamiltonians:

$$H_\Lambda(\Phi, t) = \sum_{Z \subset \Lambda} \Phi(Z,t), \tag{1.6}$$

are continuous in t and satisfy $\tau_0^\Lambda = \mathrm{id}$. For a thorough investigation of properties of $\Phi(\cdot, t)$ and the limit of the family (τ^Λ), the curve $\tau : [0,1] \to \mathrm{Aut}(\mathfrak{A}_{\mathbb{Z}})$ of $*$-automorphisms, we refer to Section 3 of [11]. In this case, we say τ are the *quasi-local dynamics generated by* $\Phi(\cdot, t)$.

In this note, we study antilinear symmetries of the spin chain. Precisely, if $\theta : \mathfrak{A}_{\mathbb{Z}} \to \mathfrak{A}_{\mathbb{Z}}$ is an antilinear $*$-automorphism, we say that τ is θ-invariant if the generating interactions are fixed by θ:

$$\forall X \in P_f(\mathbb{Z}) : \theta\big(\Phi(X,t)\big) = \Phi(X,t). \tag{1.7}$$

Physical considerations require decay of the interaction. To account for the t-dependence, we quantify the decay using $\mathcal{F}$-functions and $\mathcal{F}$-norms. Precisely, denote:

$$F_\beta(x) = e^{-h(x)} \frac{1}{(1+x)^\beta}, \ \beta > 0, \tag{1.8}$$

where $h : [0,\infty) \to [0,\infty)$ is a non-negative, non-decreasing, subadditive function. We observe that there exists a constant C_β such that for any $x, y \in \mathbb{Z}$: $\sum_{z \in \mathbb{Z}} F_\beta(|x - z|) F_\beta(|z-y|) \le C_\beta F_\beta(|x-y|)$. We refer to C_β as the *convolution constant* of F_β.

The function F_β depends on h, but we will suppress the h-dependence in notation, and when the choice of $\beta > 0$ is immaterial, we will suppress the β-dependence as $F = F_\beta$. Then $\|\cdot\|_{h,\beta}$ is defined for the family $\Phi(\cdot, t)$ as:

$$\|\Phi\|_{h,\beta} = \sup_{x,y\in\mathbb{Z}} \sum_{\substack{Z\in P_f(\mathbb{Z})\\ x,y\in Z}} \sup_{t\in[0,1]} \left(\frac{\|\Phi(Z,t)\|}{F_\beta(|x-y|)} \right). \tag{1.9}$$

We note that if $\|\Phi\|_{h,\beta} < \infty$, then the interaction decays uniformly as a function of the diameter:

$$\|\Phi(X,t)\| \leq \|\Phi\|_{h,\beta} F_\beta(\mathrm{diam}(X)). \tag{1.10}$$

We will denote:

$$\|\Phi(Z)\|_{[0,1]} = \sup_{t\in[0,1]} \|\Phi(Z,t)\|. \tag{1.11}$$

Lastly, we state the split property from [6,7] which will be best suited for our analysis.

Definition 1.1. A state ω of $\mathfrak{A}_\mathbb{Z}$ **satisfies the split property** if there exist states ω_L and ω_R of the left and right algebras $\mathfrak{A}_L, \mathfrak{A}_R$, respectively, such that ω is quasi-equivalent to $\omega_L \otimes \omega_R$.

For brevity, we will refer to states which satisfy Definition 1.1 as *split* states. The formulation of the split property in Definition 1.1 agrees with that of e.g. [5,13] when ω is pure and ω_L and ω_R are the restrictions of ω to the left and right algebras, respectively. There are higher-dimensional generalizations of the split property, such as the distal or approximate split property of [9]; however, we do not comment on whether these are stable.

We will express the quasi-equivalence relation between states by $\sim$. We consider only factor states, so we recall an asymptotic condition for quasi-equivalence of factor states ω and φ of the quasi-local algebra $\mathfrak{A}_\mathbb{Z}$ (cf. Corollary 2.6.11 in [3]): $\omega \sim \varphi$ if and only if for all $\varepsilon > 0$, there exists $X_\varepsilon \in P_f(\mathbb{Z})$ such that $Y \in P_f(\mathbb{Z})$ and $B \in \mathfrak{A}_Y$ with $Y \cap X_\varepsilon = \emptyset$ imply:

$$|\omega(B) - \varphi(B)| \leq \|B\| \varepsilon. \tag{1.12}$$

2. Split states

2.1. Support-independent Lieb-Robinson bounds

In the following, we prove special cases of Lieb-Robinson bounds for the integer lattice and certain configurations of supports. These bounds will be useful in proving Theorem 2.3.

Let $n, m \in \mathbb{N}_+$ such that $m < n$. For ease of notation, we define the following family of sets:

$$\mathrm{An}(m,n) = [-n,-m] \cup [m,n] \tag{2.1}$$

Lemma 2.1. *Suppose $\|\Phi\|_{h,\beta} < \infty$ for $\beta > 2$. There exists a constant $\kappa(\beta) > 0$ such that for all choices $n, m, c, p \in \mathbb{N}$ with $c < m < n$, if $[-n-p, n+p] \subset \Lambda$, the following inequality holds for all $A \in \mathrm{An}(m,n)$ and $B \in \Lambda \setminus \mathrm{An}(m-c, n+p)$:*

$$\left\|[\tau_t^\Lambda(A), B]\right\| \le \kappa(\beta) \|A\| \|B\| (e^{\nu|t|} - 1) F_{\beta-2}(\min\{p,c\}) \tag{2.2}$$

where $\kappa(\beta)$ and ν can be taken as:

$$\begin{gathered} \kappa(\beta) = \frac{16}{C_\beta}(\beta/2-1)^{-2} \\ \nu = 2\|\Phi\|_{h,\beta} C_\beta. \end{gathered} \tag{2.3}$$

In particular, $\kappa(\beta)$ does not depend on n, m, c, p or the function $h(x)$.

Proof. Denote $\mathrm{Cr}(a,b) = \Lambda \setminus \mathrm{An}(a,b)$. By iterative arguments (cf. [10]), it can be shown that the commutator in (2.2) is bounded above by the series:

$$\sup_{\substack{A \in \mathfrak{A}_{\mathrm{An}(m,n)} \setminus \{0\} \\ \|A\|=1}} \left\|[\tau_t^\Lambda(A), B]\right\| \le 2\|B\| \sum_{k=1}^{\infty} \frac{(2|t|)^k}{k!} a_k \tag{2.4}$$

where the right-hand side is convergent for a_k defined:

$$a_k = \sum_{Z_1 \in S_\Lambda(\mathrm{An}(m,n))} \sum_{Z_2 \in S_\Lambda(Z_1)} \cdots \sum_{Z_k \in S_\Lambda(Z_{k-1})} \delta_Y(Z_k) \|\Phi(Z_1)\|_{[0,1]} \cdots \|\Phi(Z_k)\|_{[0,1]} \tag{2.5}$$

$$\delta_Y(W) = \begin{cases} 1 & \text{if } W \cap Y \neq \emptyset \\ 0 & \text{else} \end{cases}. \tag{2.6}$$

Here, $S_\Lambda(W) = \{Z \subset \Lambda : Z \cap W \neq \emptyset, Z \cap W^c \neq \emptyset\}$ denotes the boundary sets of W. Let C_β denote the convolution constant of F_β. For any k:

$$\begin{aligned} a_k &\le \sum_{\substack{x \in \mathrm{An}(m,n) \\ y \in \mathrm{Cr}(m-c,n+p)}} \sum_{z_1,\dots,z_{k-1} \in \Lambda} \sum_{\substack{Z_1 \in S_\Lambda(\mathrm{An}(n,m)) \\ x, z_1 \in Z_1}} \sum_{\substack{Z_2 \in S_\Lambda(Z_1) \\ z_1, z_2 \in Z_2}} \\ &\qquad \cdots \sum_{\substack{Z_k \in S_\Lambda(Z_{k-1}) \\ z_{k-1}, y \in Z_k}} \|\Phi(Z_1)\|_{[0,1]} \cdots \|\Phi(Z_k)\|_{[0,1]} \\ &\le \sum_{\substack{x \in \mathrm{An}(m,n) \\ y \in \mathrm{Cr}(m-c,n+p)}} \sum_{z_1 \in \Lambda} \sum_{\substack{Z_1 \in S_\Lambda(\mathrm{An}(m,n)) \\ x, z_1 \in Z_1}} \|\Phi(Z_1)\|_{[0,1]} (C_\beta^{k-2} \|\Phi\|_{h,\beta}^{k-1}) F_\beta(|z_1 - y|) \\ &\le (\|\Phi\|_{h,\beta}^k C_\beta^{k-1}) e^{-h(\min\{p,c\})} \sum_{x \in \mathrm{An}(m,n)} \sum_{y \in \mathrm{Cr}(m-c,n+p)} \frac{1}{(1+|x-y|)^\beta}. \end{aligned} \tag{2.7}$$

Since:

$$\begin{aligned}
&\sum_{m\le x\le n}\ \sum_{y\in \mathrm{Cr}(m-c,n+p)} \frac{1}{(1+|x-y|)^{\beta}} \\
&\le 2\sum_{m\le x\le n} \frac{1}{(1+d(x,\mathrm{Cr}(m-c,n+p)))^{\beta/2}} \sum_{r\ge \min\{p,c\}} \frac{1}{(1+r)^{\beta/2}} \\
&\le 4\Big(\sum_{r\ge \min\{p,c\}} \frac{1}{(1+r)^{\beta/2}}\Big)^2
\end{aligned} \tag{2.8}$$

the symmetry in the sum of the last inequality of (2.7) implies:

$$\begin{aligned}
a_k &\le 8\,\|\Phi\|_{h,\beta}^k\, C_\beta^{k-1} e^{-h(\min\{p,c\})}\Big(\sum_{r\ge \min\{p,c\}} \frac{1}{(1+r)^{\beta/2}}\Big)^2 \\
&\le 8\,\|\Phi\|_{h,\beta}^k\, C_\beta^{k-1}(\beta/2-1)^{-2} e^{-h(\min\{p,c\})} \frac{1}{(1+\min\{p,c\})^{\beta-2}} \\
&\le 8\,\|\Phi\|_{h,\beta}^k\, C_\beta^{k-1}(\beta/2-1)^{-2} F_{\beta-2}(\min\{p,c\}).
\end{aligned} \tag{2.9}$$

Hence the inequality (2.2) holds with the choices:

$$\begin{aligned}
\kappa(\beta) &= \frac{16}{C_\beta}(\beta/2-1)^{-2} \\
\nu &= 2\,\|\Phi\|_{h,\beta}\, C_\beta. \quad \square
\end{aligned} \tag{2.10}$$

We also record for completeness the following useful bound.

Corollary 2.2. *Suppose* $\|\Phi\|_{h,\beta} < \infty$ *for* $\beta > 2$*. If* $X, Y \subset \Lambda$ *with* $\max X < \min Y$*, then for all* $A \in \mathfrak{A}_X$*,* $B \in \mathfrak{A}_Y$*,*

$$\big\|[\tau_t^\Lambda(A), B]\big\| \le \kappa(\beta)\,\|A\|\,\|B\|\,(e^{\nu|t|}-1)F_{\beta-2}\big(d(X,Y)\big). \tag{2.11}$$

Proof. The conclusion follows from observing that the origin has no distinguished role in the proof of Lemma 2.1. $\square$

We remark that taking the $\Lambda \to \mathbb{Z}$ limit in Lemma 2.1 and Corollary 2.2 shows that the infinite volume dynamics τ also satisfies the corresponding support-independent Lieb-Robinson bound.

2.2. Automorphic equivalence and the split property

We say states ω and φ of $\mathfrak{A}_\mathbb{Z}$ are *automorphically equivalent* if there exist quasi-local dynamics $\tau : [0,1] \to \mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$ such that:

$$\omega = \varphi \circ \tau_1. \tag{2.12}$$

In this section we prove that the split property is stable under automorphic equivalence. To proceed, we remark that if ω is a split factor state, $\omega \sim \omega_L \otimes \omega_R$, then for any $\beta \in \mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$, the following states are also factor: $\omega_L \otimes \omega_R$, $\omega_L \otimes \omega_R \circ \beta$ and $\omega \circ \beta$. Next, let $\Phi^L(\cdot,t) : P_f((-\infty,0]) \to \bigcup_{Z\subset(-\infty,0]} \mathfrak{A}_Z$ denote the restriction of $\Phi(\cdot,t)$ to the left half-infinite chain. Define $\Phi^R(\cdot,t)$ the same way using the complementary right half-infinite chain. $\Phi^L(\cdot,t)$ generates quasi-local dynamics $\tau^L : [0,1] \to \mathrm{Aut}(\mathfrak{A}_L)$ (resp. τ^R). Likewise, the interaction $\Phi^\cup(\cdot,t) : P_f(\mathbb{Z}) \to \mathfrak{A}_{loc}$ defined by:

$$\Phi^\cup(X,t) = \begin{cases} \Phi(X,t) & \text{if } X \subset (-\infty,0] \text{ or } X \subset [1,\infty) \\ 0 & \text{else} \end{cases} \tag{2.13}$$

generates quasi-local dynamics $\tau^\cup : [0,1] \to \mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$. Then in the notation:

$$(\omega_L \circ \tau_t^L) \otimes (\omega_R \circ \tau_t^R) = (\omega_L \otimes \omega_R) \circ \tau_t^\cup. \tag{2.14}$$

In the following theorem, we consider interactions which decay by at least a power law, setting h in (1.8) to be the zero function.

Theorem 2.3. *Suppose $\tau : [0,1] \to \mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$ are quasi-local dynamics with a generating interaction $\Phi(\cdot,t)$ such that $\|\Phi\|_{0,\beta} < \infty$. If ω_0 is a split factor state and $\beta > 3$, then $\omega_t = \omega_0 \circ \tau_t$ is also a split factor state, for all $t \in [0,1]$.*

Proof. Denote by $\omega_{L,t} = \omega_L \circ \tau_t^L$ (resp. $\omega_{R,t}$) and $\omega_0 = \omega$. Suppose $\varepsilon > 0$ and $n, r \in \mathbb{N}$ such that $r > n$. Recalling the sets $\mathrm{An}(a,b)$ in (2.1), let $\mathbb{E}_{n,r} : \mathfrak{A}_\mathbb{Z} \to \mathfrak{A}_{\mathrm{An}(n,2(n+r))}$ denote the conditional expectation with respect to the product trace state. Since ω is split and factor, there exists $N_\omega(\varepsilon) \in \mathbb{N}$ such that $n > N_\omega(\varepsilon)$ implies:

$$|\omega \circ \mathbb{E}_{n,r}(\tau_t(A)) - \omega_L \otimes \omega_R \circ \mathbb{E}_{n,r}(\tau_t(A))| \le \varepsilon \,\|\mathbb{E}_{n,r}(\tau_t(A))\| \le \varepsilon \,\|A\|\,. \tag{2.15}$$

The following bounds will be derived independently of r, and so we will be able to let r tend to infinity. Evidently for $A \in \mathfrak{A}_{loc}$:

$$\begin{aligned} &|\omega_t(A) - \omega_{L,t} \otimes \omega_{R,t}(A)| \\ &\quad \le |(\omega - \omega_L \otimes \omega_R) \circ \tau_t(A)| + |\omega_L \otimes \omega_R\big(\tau_t(A) - \tau_t^\cup(A)\big)| \\ &\quad \le \Big(|\omega \circ \mathbb{E}_{n,r}(\tau_t(A)) - \omega_L \otimes \omega_R \circ \mathbb{E}_{n,r}(\tau_t(A))| + 2\,\|\tau_t(A) - \mathbb{E}_{n,r}(\tau_t(A))\|\Big) \\ &\qquad\qquad + \|\tau_t(A) - \tau_t^\cup(A)\|\,. \end{aligned} \tag{2.16}$$

Lemma 2.1 implies that if $\mathrm{supp}(A) \subset \mathrm{An}(2n, 2n+r)$:

$$\|\tau_t(A) - \mathbb{E}_{n,r}(\tau_t(A))\| \leq 2\kappa(\beta)\,\|A\|\,(e^{\nu|t|}-1)F_{\beta-2}(n). \tag{2.17}$$

To conclude the proof, it is left to show that for fixed $t \in \mathbb{R}$, the quantity $\|\tau_t(A) - \tau_t^{\cup}(A)\|$ decays as a function of n, uniformly in the norm of A. This will follow from a Gronwall-type inequality. Let Λ be any interval containing $[-2(n+r), 2(n+r)]$. Define:

$$f_\Lambda(t) = \tau_t^{\Lambda}(A) - \tau_t^{\cup,\Lambda}(A) \tag{2.18}$$

where τ^{Λ} and $\tau^{\cup,\Lambda}$ are the corresponding finite-volume dynamics. Since $f_\Lambda(t)$ satisfies the ODE and initial value problem:

$$\begin{aligned} \frac{d}{dt} f_\Lambda(t) &= i[H_\Lambda(\Phi^{\cup},t), f_\Lambda(t)] + i[H_\Lambda(\Phi,t) - H_\Lambda(\Phi^{\cup},t), \tau_t^{\Lambda}(A)] \\ f_\Lambda(0) &= 0 \end{aligned} \tag{2.19}$$

the following bound is valid:

$$\begin{aligned} \|f_\Lambda(t)\| &\leq \|f_\Lambda(0)\| + \int_0^{|t|} ds\ \left\|[H_\Lambda(\Phi,s) - H_\Lambda(\Phi^{\cup},s), \tau_s^{\Lambda}(A)]\right\| \\ &= \int_0^{|t|} ds \left\| \sum_{\substack{Z\subset\Lambda: \\ Z\cap(-\infty,0]\neq\emptyset \\ Z\cap[1,\infty)\neq\emptyset}} [\Phi(Z,s), \tau_s^{\Lambda}(A)] \right\|. \end{aligned} \tag{2.20}$$

We can further divide the admissible Z in the sum of the last line of (2.20) into:

$$\begin{aligned} C_I &= \{Z \subset \Lambda : Z \cap (-\infty,0] \neq \emptyset, Z \cap [1,\infty) \neq \emptyset, Z \subseteq [-n,n]\} \\ C_{II} &= \{Z \subset \Lambda : Z \cap (-\infty,0] \neq \emptyset, Z \cap [1,\infty) \neq \emptyset, Z \nsubseteq [-n,n]\}\,. \end{aligned} \tag{2.21}$$

The contribution of the C_{II} terms to the upper bound in (2.20) are majorized using decay of the interaction. Let $\delta > 0$ such that $\beta > 2+\delta$. Then:

$$\begin{aligned} \left\| \sum_{Z\in C_{II}} [\Phi(Z,s), \tau_s^{\Lambda}(A)] \right\| &\leq \sum_{\substack{x\in(-\infty,-n] \\ y\in[1,\infty)}} 2\,\|A\| \left(\sum \{\|\Phi(Z,s)\| : x,y \in Z\} \right) \\ &\quad + \sum_{\substack{x\in[n,\infty) \\ y\in(-\infty,0]}} 2\,\|A\| \left(\sum \{\|\Phi(Z,s)\| : x,y \in Z\} \right) \\ &\leq 4\,\|A\|\,\|\Phi\|_{0,\beta} \sum_{x=n}^{\infty}\sum_{y=0}^{\infty} F_\beta(x+y) \end{aligned}$$

$$\leq 4\,\|A\|\,\|\Phi\|_{0,\beta}\sum_{x=n}^{\infty}F_{\beta-1-\delta/2}(x+1)\sum_{y=1}^{\infty}\frac{1}{(1+x+y)^{1+\delta/2}}$$

$$\leq\left[\frac{8}{(\beta-2-\delta/2)\delta}\,\|A\|\,\|\Phi\|_{0,\beta}\right]F_{\beta-2-\delta/2}(n). \tag{2.22}$$

And an application of Lemma 2.1 majorizes the contribution from C_I. Note we have the simple bound $\left\|\sum_{Z\in C_I}\Phi(Z,s)\right\|\leq 3\,\|\Phi\|_{0,\beta}\,n$. And so:

$$\left\|\sum_{Z\in C_I}\left[\Phi(Z,s),\tau_s^{\Lambda}(A)\right]\right\|\leq[3\kappa(\beta)\,\|\Phi\|_{0,\beta}\,\|A\|](e^{\nu|s|}-1)nF_{\beta-2}(n). \tag{2.23}$$

The conclusion follows from the fact that the upper bounds in (2.22) and (2.23) are independent of the sufficiently large, finite interval Λ and r. $\square$

Lastly, we remark on when the left and right states in Theorem 2.3 can be taken to be restrictions.

Corollary 2.4. *Suppose ω_0 is a factor state such that $\omega_0\sim\omega_0|_{\mathfrak{A}_L}\otimes\omega_0|_{\mathfrak{A}_R}$, and τ satisfies the hypotheses of Theorem 2.3. Then $\omega_t=\omega_0\circ\tau_t\sim\omega_t|_{\mathfrak{A}_L}\otimes\omega_t|_{\mathfrak{A}_R}$ for all $t\in[0,1]$.*

Proof. It suffices to show that $\omega_t|_{\mathfrak{A}_L}\sim\omega_0|_{\mathfrak{A}_L}\circ\tau_t^L$ (resp. for the right algebra). This will follow by methods used in the proof of Theorem 2.3, and so we will be brief. By a familiar asymptotic condition of being a factor state (cf. Theorem 2.6.10 of [3]), the assumptions that ω_0 is factor and τ is a quasi-local map imply $\omega_t|_{\mathfrak{A}_L}$ is also a factor state. Then Gronwall-type inequalities on $f_\Lambda(t)=(\tau_t^{\Lambda}-\tau_t^{\cup,\Lambda})(A)$, $A\in\mathfrak{A}_L\cap\mathfrak{A}_{loc}$, show that $\omega_t|_{\mathfrak{A}_L}$ and $\omega_0|_{\mathfrak{A}_L}\circ\tau_t^L$ are quasi-equivalent. $\square$

2.3. Comment on Haag duality and translation invariant states

Now, we consider the split property for translation invariant pure states. A result of Matsui [6] shows that uniform decay of correlations in a translation invariant pure state φ of $\mathfrak{A}_{\mathbb{Z}}$ which satisfies Haag duality, implies φ is split.

It is also proven in [6] that if Φ is a translation invariant, finite-range interaction whose local Hamiltonians have a unique ground state and uniform spectral gap, φ is a translation invariant, pure ground state of Φ, and the GNS Hamiltonian $H_\varphi\geq 0$ has a nondegenerate eigenvalue at 0, then φ satisfies Haag duality. The conclusion is then φ is necessarily split.

In the following, we remark a sufficient condition on the decay of an interaction to guarantee uniform decay of correlations, i.e. in terms of bounds which do not depend on the support size of the observables. We leave open the question of sufficient conditions for Haag duality to hold for a general translation invariant state.

Corollary 2.5 *(Uniform correlation decay). Suppose ω is a gapped ground state of Φ, with $\|\Phi\|_{h,\beta} < \infty$, and the GNS Hamiltonian $H_\omega \geq 0$ has a nondegenerate ground state, i.e.:*

$$(i)\ \mathrm{sp}(H_\omega) \setminus \{0\} \subset [\gamma, \infty) \quad \text{and} \quad (ii)\ ker(H_\omega) = \mathbb{C}\Omega \tag{2.24}$$

There exists a constant $\mu(F) > 0$ such that for all X, Y finite with $\max X < \min Y$,

$$|\omega(AB) - \omega(A)\omega(B)| \leq \mu(F)\, \|A\|\, \|B\|\, e^{-uh(d(X,Y))} \tag{2.25}$$

We may take:

$$\begin{aligned} \mu(F) &= \left(1 + \frac{\kappa(\beta)}{\pi} + \sqrt{\frac{2\nu+\gamma}{\pi\gamma h(d(X,Y))}}\right) \\ u &= \frac{\gamma}{2\nu+\gamma} \end{aligned} \tag{2.26}$$

Proof. The proof is essentially the same as the one given in [10] changed only to use the Lieb-Robinson bound from Corollary 2.2, and so we will be brief. We suppress in notation the dependence on the representation. We may assume $\langle \Omega, B\Omega \rangle = 0$. For free parameters α, s, taking b sufficiently small, the method of proof in [10] gives:

$$|\omega(A\tau_{ib}(B))| = |\langle \Omega, A\tau_{ib}(B)\Omega \rangle| \leq \|A\|\, \|B\| \left(e^{-\frac{\gamma^2}{4\alpha}} + \frac{\kappa(\beta)}{\pi} e^{\nu s - h(d(X,Y))} + \frac{1}{s\sqrt{\pi\alpha}} e^{-\alpha s^2}\right). \tag{2.27}$$

Setting $\alpha = \gamma/2s$ and s such that:

$$s(\nu + \gamma/2) = h(d(X,Y)) \tag{2.28}$$

and taking the limit $b \to 0$ yields the bound. $\square$

Proposition 2.6. *Let Φ be a translation invariant interaction on a quantum spin chain such that $\|\Phi\|_{h,\beta} < \infty$. Suppose ω is a pure, translation invariant, gapped ground state of Φ, and that the normalized GNS Hamiltonian H_ω has a nondegenerate eigenvalue at 0.*

If ω satisfies Haag duality, then ω is quasi-equivalent to $\omega|_{\mathfrak{A}_L} \otimes \omega|_{\mathfrak{A}_R}$.

Proof. This follows immediately from Corollary 3.2 of [6] and the uniform decay of correlations guaranteed by Corollary 2.5. $\square$

3. Application to SPT phases

We recall the heuristic notion of a topological phase as an equivalence class of uniformly gapped interactions, where two such interactions Φ_0, Φ_1 are related if and only if there exists a sufficiently smooth interpolating family of interactions $\Phi(s)$, $0 \leq s \leq 1$, such that $\Phi(0) = \Phi_0$ and $\Phi(1) = \Phi_1$, and $\Phi(s)$ is gapped above the ground state, uniformly in s. It is known that in this case, the infinite-volume ground states of Φ and Ψ obtained through weak$-*$ limits of finite-volume ground states are automorphically equivalent (cf. Theorem 5.5 of [2]). The equivalence relation for a symmetry protected topological phase has the additional requirement that the $\Phi(s)$ are fixed by the given symmetry. The hypothesis of a uniform gap is essential, and we formulate this condition as the following working definition: Say that Φ has a **uniform gap** if there exist $\gamma > 0$ and minimum interval length $R_\gamma > 0$ such that if Λ is a finite interval, $\operatorname{diam}(\Lambda) \geq R_\gamma$ implies:

$$\operatorname{sp}(H_\Lambda(\Phi)) = \operatorname{sp}_-(H_\Lambda(\Phi)) \cup \operatorname{sp}_+(H_\Lambda(\Phi)) \tag{3.1}$$

with:

$$\min\left\{\lambda - \mu : \lambda \in \operatorname{sp}_+(H_\Lambda(\Phi)), \mu \in \operatorname{sp}_-(H_\Lambda(\Phi))\right\} \geq \gamma \tag{3.2}$$

and $\operatorname{diam}(\operatorname{sp}_-(H_\Lambda(\Phi))) \to 0$ as $\operatorname{diam}(\Lambda) \to \infty$. Let $\Gamma(\mathbb{Z})$ denote the uniformly gapped interactions on $\mathbb{Z}$.

In the following, we also work with a familiar formulation of equivalence in a gapped SPT phase [2]. While we note that more general symmetries may be handled in this framework, we restrict our discussion to the antilinear $*$-automorphism Ξ of time reversal since it is one of three symmetries which protect the Haldane phase in odd-spin quantum spin chains [4,13–17]. We do not claim that these are necessary conditions for being in the same topological phase.

Our application is showing that the $\mathbb{Z}_2$-index is a well-defined invariant for a Ξ-protected topological phase which contains at least one interaction with a well-defined $\mathbb{Z}_2$ index (e.g. a finite-range interaction), provided the decay F_β is sufficiently strong.

Assumption on decay: Suppose $\beta > 0$. Let F_β be determined by $h(x) = Rx^b$ for any $R > 0$ and $b \in (0, 1]$, so that (1.8) becomes:

$$F_\beta(x) = e^{-Rx^b} \frac{1}{(1+x)^\beta}. \tag{3.3}$$

We may assume, without loss of generality, that $\beta > 6$. We will suppress the dependence of the $\mathcal{F}$-norm on the variables:

$$\|\cdot\|_{Rx^b,\beta} = \|\cdot\|_F\,. \tag{3.4}$$

Definition 3.1 *(Equivalence in an SPT phase).* Define:

$$\mathfrak{B}(F) = \Big\{ \Phi \in \Gamma(\mathbb{Z}) : \ (i)\ \|\Phi\|_F < \infty,\ (ii)\ \Phi \text{ has a unique ground state}, \\ (iii)\ \forall X \in P_f(\mathbb{Z}),\ \Xi(\Phi(X)) = \Phi(X) \Big\}. \tag{3.5}$$

Define an equivalence relation $\approx$ on $\mathfrak{B}(F)$ in the following way: $\Phi_0 \approx \Phi_1$ if there exists an interpolating path $s \mapsto \Phi(\cdot, s) \in \mathfrak{B}(F)$ such that:

$$\begin{aligned} &(iv) \quad \text{for each } X \in P_f(\mathbb{Z}),\ s \mapsto \Phi(X,s) \text{ is continuously differentiable} \\ &(v) \quad \sup_{x,y\in\mathbb{Z}} \sum_{\substack{X \in P_f(\mathbb{Z}) \\ x,y \in X}} \sup_{s\in[0,1]} \left(\frac{\|\Phi(X,s)\| + |X|\,\|\Phi'(X,s)\|}{F(|x-y|)} \right) < \infty \\ &(vi) \quad \text{the } \gamma > 0 \text{ and } R_\gamma \text{ in the uniform gap condition (3.2) for } \Phi(\cdot,s) \\ &\qquad \text{can be taken independent of } s. \end{aligned} \tag{3.6}$$

Assumption (iii) of Definition 3.1 implies $\omega_\Phi(\Xi(A^*)) = \omega_\Phi(A)$, where ω_Φ is the unique ground state of some representative Φ. Condition (iv) of Definition 3.1 specifies the smoothness of the local Hamiltonians, and (v) is an assumption on the uniform spatial decay of the interactions. Precisely, (v) is sufficient decay to guarantee that the generated spectral flow will be a quasi-local map.

3.1. Extension of the $\mathbb{Z}_2$ index

We first describe the $\mathbb{Z}_2$-index defined by Ogata and defer to [13] for the details. Suppose $\Psi \in \mathfrak{B}(F)$ is finite-range with pure gapped ground state φ. Since the entanglement entropy of φ is bounded, it follows by [7] that $\varphi \sim \varphi|_{\mathfrak{A}_L} \otimes \varphi|_{\mathfrak{A}_R}$; and if $(\pi_R, \mathfrak{H}_R, \Omega_R)$ is the associated cyclic representation of $\varphi|_{\mathfrak{A}_R}$, then $\pi_R(\mathfrak{A}_R)''$ is a Type I factor. Hence we may assume there is an isomorphism $\iota : \pi_R(\mathfrak{A}_R)'' \to B(\mathfrak{K})$ for some Hilbert space $\mathfrak{K}$. Since $\varphi|_{\mathfrak{A}_R}$ is Ξ-invariant, Ξ defines a unique antilinear $*$-automorphism $\hat{\Xi}$ of $B(\mathfrak{K})$ satisfying:

$$\forall A \in \mathfrak{A}_R : \ \hat{\Xi} \circ \iota(\pi_R(A)) = \iota\Big(\pi_R \circ \Xi(A)\Big), \text{ and } \hat{\Xi}^2 = \mathrm{id}. \tag{3.7}$$

By Wigner's theorem for antilinear $*$-automorphisms, there exists an antiunitary J_{π_R} on $\mathfrak{K}$, unique up to phase, such that $\hat{\Xi}(T) = J_{\pi_R}^* T J_{\pi_R}$. Evidently $J_{\pi_R}^2 \in \{-1, 1\}$, and Theorems 2.2 and 2.6 of [13] show that the quantity $J_{\pi_R}^2$ does not depend on $\mathfrak{K}$ and is an invariant of the $\approx$ relation restricted to finite-range interactions. The $\mathbb{Z}_2$-index is thus defined by Ogata as $\hat{\sigma}_\Psi = J_{\pi_R}^2$. The extension is straightforward to define.

Definition 3.2 *(cf. Definition 3.3 of [13]).* For $\Phi \in \mathfrak{B}(F)$ with pure ground state ω such that $\omega \sim \omega|_{\mathfrak{A}_L} \otimes \omega|_{\mathfrak{A}_R}$, define:

$$\hat{\sigma}_\Phi = J_\Phi^2 \in \{-1, 1\} \tag{3.8}$$

where J_Φ is an antiunitary implementing the extension $\hat{\Xi}$ of time reversal to the von Neumann algebra generated by the associated cyclic representation of $\omega|_{\mathfrak{A}_R}$.

Lemma 3.3. *Suppose there exists $\Phi_0 \in \mathfrak{B}(F)$ such that the unique ground state ω_0 of Φ_0 is quasi-equivalent to $\omega_0|_{\mathfrak{A}_L} \otimes \omega_0|_{\mathfrak{A}_R}$. Then $\hat{\sigma}_\Phi$ is well-defined for all $\Phi \in \mathfrak{B}(F)$ such that $\Phi \approx \Phi_0$.*

Proof. Let $\Phi(\cdot, s)$, $0 \leq s \leq 1$, be an interpolating path in $\mathfrak{B}(F)$ between $\Phi_0 = \Phi(\cdot, 0)$ and $\Phi_1 = \Phi(\cdot, 1)$. By Theorem 2.2 of [13], it suffices to show that the GNS representation of the right chain restriction of ω, the pure ground state of Φ, generates a Type I factor. But Theorem 6.14 of [11], and the assumptions in (3.5) and (3.6) imply that the interaction $\Psi(s)$ which generates the spectral flow $\alpha_s \in \mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$ of the family $\Phi(\cdot, s)$ satisfies the hypotheses of Theorem 2.3; we may take $h(x) = O\big(x^b / \log^2(x^b)\big)$. $\square$

Proposition 3.4. *If $\Phi \in \mathfrak{B}(F)$ and $\Phi \approx \Phi_0$, then $\hat{\sigma}_{\Phi_0} = \hat{\sigma}_\Phi$.*

Proof. The proof direction is essentially due to Ogata in [13], and so we prove in detail only the necessary modifications to handle unbounded range of interaction. It is sufficient to show that the composition $\alpha_s \circ [(\alpha_s^L)^{-1} \otimes (\alpha_s^R)^{-1}]$ is an inner automorphism, for all $s \in [0, 1]$. Here we take the spectral flow maps to be generated by an interpolating curve $\Phi(\cdot, s)$ as in Lemma 3.3.

Let γ denote the uniform gap of the $\Phi(s)$. We show that there exists a continuous family $V(s) = V(s)^* \in \mathfrak{A}_\mathbb{Z}$ such that in the uniform topology:

$$\lim_{n\to\infty} D_{[-n,n]}(s) - D^{\cup}_{[-n,n]}(s) = V(s). \tag{3.9}$$

$D_{[-n,n]}(s)$ denotes the Hastings generator defined in (4.2) of the Appendix. This implies the composition $\alpha_s \circ [(\alpha_s^L)^{-1} \otimes (\alpha_s^R)^{-1}]$ is inner. To do this, define $g_n \in C([0, 1], \mathfrak{A}_\mathbb{Z})$ by:

$$g_n(s) = D_{[-n,n]}(s) - D^{\cup}_{[-n,n]}(s). \tag{3.10}$$

We will prove that the sequence $g_n(s)$ is uniformly Cauchy. Fix $N_0 \in \mathbb{N}$, and let $m, n \in \mathbb{N}$ be such that $4N_0 < m \leq n$. Then:

$$g_n(s) - g_m(s) = \Bigg[\int_{-\infty}^{\infty} dt\; W_\gamma(t)(\tau_t^{n,s} - \tau_t^{m,s})\Bigg(\sum_{X\subset[-N_0,N_0]} \Phi'(X, s)\Bigg)$$

$$- \int_{-\infty}^{\infty} W_\gamma(t)(\tau_t^{\cup,n,s} - \tau_t^{\cup,m,s})\Big(\sum_{X \subset [-N_0,N_0]} (\Phi^\cup)'(X,s) \Big) \Big]$$
$$+ \mathcal{R}(n,m,N_0,s) \tag{3.11}$$

where $\mathcal{R}(n,m,N_0,s)$ is defined to be the remainder difference between $g_n(s) - g_m(s)$ and the bracketed quantity in (3.11). Using Lemmas 3.5 and 3.6 and the simple bound $\left\| H_{[-N_0,N_0]}(\Phi'(\cdot,s)) \right\| \leq 3N_0 \left\| \Phi' \right\|_F$:

$$\|g_n(s) - g_m(s)\| \leq 2\big(3N_0\Omega_1(N_0) + \Omega_2(N_0)\big) \left\|\Phi'\right\|_F \tag{3.12}$$

which tends to 0 uniformly in s as $N_0 \to \infty$. □

We conclude this section with the necessary technical lemmas used in the proof of Proposition 3.4, which prove bounds analogous to those in the proof of Lemma 5.1 of [13] but remain valid for interactions which are not finite-range but decay by (3.6). We freely use the function I_γ defined in Lemma 3.7 of the Appendix.

Lemma 3.5. *Let $\gamma > 0$ and W_γ be the weight function in (4.2). Let $\Psi : P_f(\mathbb{Z}) \to \mathfrak{A}_{loc}$ be an interaction such that $\|\Psi\|_F < \infty$ with generated time-independent dynamics $\tau : \mathbb{R} \to$ $\mathrm{Aut}(\mathfrak{A}_\mathbb{Z})$. Let τ^n denote the finite-volume time-independent dynamics generated by Ψ in the interval $[-n,n]$.*

If $N, K \in \mathbb{N}$ and $N \leq K$, then for all $A \in \mathfrak{A}_{[-N,N]}$ and $n \geq m > 2K$:

$$\left\| \int_{-\infty}^{\infty} dt\ W_\gamma(t)(\tau_t^n - \tau_t^m)(A) \right\| \leq \Omega_1\big(K-N\big) \|A\| \tag{3.13}$$

for the decaying function:

$$\Omega_1(x) = 4I_\gamma(Rx^b/2\nu) + (\pi^2/6)^2 \left(10 \left\|W_\gamma(t)t\right\|_{L^1} + 2\frac{\kappa(\beta)\left\|W_\gamma\right\|_{L^\infty}}{\nu} \right) \|\Psi\|_F \, e^{-\frac{Rx^b}{2\nu}} \tag{3.14}$$

Proof. Let $T > 0$ be a positive parameter. We can find a bound for the integral:

$$\left\| \int_{-T}^{T} dt\ W_\gamma(t)(\tau_t^n - \tau_t^m)(A) \right\| \leq \int_{-T}^{T} dt\ |W_\gamma(t)| \int_0^{|t|} dr\ \left\| [H_{[-n,n]}(\Psi) - H_{[-m,m]}(\Psi), \tau_r^m(A)] \right\| \tag{3.15}$$

by further dividing the difference of the local Hamiltonians as:

$$H_{[-n,n]}(\Psi) - H_{[-m,m]}(\Psi) = \sum_{X \in \mathcal{L}} \Psi(X) + \sum_{Y \in \mathcal{R}} \Psi(Y) + \sum_{Z \in \mathcal{C}} \Psi(Z) \tag{3.16}$$

for index sets defined:

$$\mathcal{L} = \{X \subset [-n,0] : X \cap [-n,-m-1] \neq \emptyset\} \quad \mathcal{R} = \{Y \subset [0,n] : \; Y \cap [m+1,n] \neq \emptyset\} \tag{3.17}$$

$$\mathcal{C} = \{Z \subset [-n,n] : Z \cap A(m,n) \neq \emptyset, \; Z \cap (-\infty,0] \neq \emptyset, \; Z \cap (0,\infty) \neq \emptyset\}. \tag{3.18}$$

First we bound the contribution from $\mathcal{L}$. For $a, b \in \mathbb{Z}$ such that $-n \leq a \leq -m-1$ and $a \leq b \leq 0$, denote:

$$\Psi(a;b) = \sum \{X \in \mathcal{L} : \min X = a, \;\; \max X = b\}. \tag{3.19}$$

Then:

$$\begin{aligned} &\int_{-T}^{T} dt \; |W_\gamma(t)| \int_0^{|t|} dr \; \left\| \Big[\sum_{X \in \mathcal{L}} \Psi(X), \tau_r^m(A) \Big] \right\| \\ &\quad \leq \int_{-T}^{T} dt \; |W_\gamma(t)| \int_0^{|t|} dr \sum_{-n \leq a \leq -m-1} \; \sum_{a \leq b \leq 0} \| [\Psi(a;b), \tau_r^m(A)] \|. \end{aligned} \tag{3.20}$$

Using Lemma 2.2,

$$\begin{aligned} &\sum_{-n \leq a \leq -m-1} \; \sum_{a \leq b \leq 0} \| [\Psi(a;b), \tau_r^m(A)] \| \\ &\quad \leq \sum_{-n \leq a \leq -m-1} \left(\sum_{a \leq b \leq a/2} \| [\Psi(a;b), \tau_r^m(A)] \| + \sum_{a/2 < b \leq 0} \| [\Psi(a;b), \tau_r^m(A)] \| \right) \\ &\quad \leq (\pi^2/6)^2 \, \|\Psi\|_F \, \|A\| \left(\kappa(\beta)(e^{\nu|r|} - 1) F_{\beta-4}(K-N) + 2F_{\beta-4}(K) \right). \end{aligned} \tag{3.21}$$

Denote $\mathcal{I}_L = \int_{-T}^{T} dt \; |W_\gamma(t)| \int_0^{|t|} dr \; \sum_{-n \leq a \leq -m-1} \; \sum_{a \leq b \leq 0} \| [\Psi(a;b), \tau_r^m(A)] \|$. Substituting (3.21) into (3.20) yields:

$$\mathcal{I}_L \leq (\pi^2/6)^2 \, \|\Psi\|_F \, \|A\| \left(2 \, \|W_\gamma(t)t\|_{L^1} \, F_{\beta-4}(K) + \frac{\kappa(\beta) \, \|W_\gamma\|_{L^\infty}}{\nu} e^{\nu T} F_{\beta-4}(K-N) \right). \tag{3.22}$$

By symmetry, if $\mathcal{I}_R$ is the corresponding integral using the interaction on $\mathcal{R}$, then (3.22) holds with $\mathcal{I}_R$ in place of $\mathcal{I}_L$. Next we bound the contribution from $\mathcal{C}$. But since these sets in $\mathcal{C}$ have diameter of at least $2K$,

$$\left\| \sum_{Z \in \mathcal{C}} \Psi(Z) \right\| \leq 3(\pi^2/6)^2 \, \|\Psi\|_F \, F_{\beta-4}(2K). \tag{3.23}$$

Hence we have the inequality:

$$\begin{aligned}
&\left\| \int_{-\infty}^{\infty} dt\ W_\gamma(t)(\tau_t^n - \tau_t^m)(A) \right\| \\
&\quad \le 4\|A\| I_\gamma(T) + [10(\pi^2/6)^2 \|W_\gamma(t)t\|_{L^1}] \|\Psi\|_F \|A\| F_{\beta-4}(K) \\
&\qquad + \left[2(\pi^2/6)^2 \frac{\kappa(\beta)\|W_\gamma\|_{L^\infty}}{\nu} \right] \|\Psi\|_F \|A\| e^{\nu T} F_{\beta-4}(K-N).
\end{aligned} \tag{3.24}$$

Setting $T = \frac{R(K-N)^b}{2\nu}$ yields (3.13). $\square$

It can be shown that $\lim_{n\to\infty} \int dt\ W_\gamma(t)\tau_t^n(A) = \int dt\ W_\gamma(t)\tau_t(A)$, although we do not use this fact here.

Lemma 3.6. *Let γ, W_γ, and Ψ be the same as in Lemma 3.5. Suppose $K \in \mathbb{N}$ and $K < n$. Then:*

$$\left\| \int_{-\infty}^{\infty} dt\ W_\gamma(t)(\tau_t^n - \tau_t^{\cup,n}) \Bigg(\sum_{\substack{Z\subset[-n,n] \\ Z\not\subset[-K,K]}} \Psi(Z) \Bigg) \right\| \le \Omega_2(K)\|\Psi\|_F \tag{3.25}$$

where $\tau^n, \tau^{\cup,n}$ are generated by Ψ and $\Psi^\cup$, respectively, and Ω_2 is the decaying function:

$$\begin{aligned}
\Omega_2(x) &= 6x \sum_{\substack{m\in\mathbb{N} \\ m\ge x}} I_\gamma\left(\frac{R}{2}(m/4)^b\right) + \sum_{\substack{m\in\mathbb{N} \\ m\ge x}} Q(m) \\
Q(y) &= (\pi^2/6)^4 \left(\frac{12\kappa(\beta)}{\nu} \|\Psi\|_F \|W_\gamma\|_\infty + 10 \max\left\{ \|\Psi\|_F^1, \|\Psi\|_F^2 \right\} \|W_\gamma(t)|t|\|_{L^1} \right) \\
&\quad \times e^{-\frac{R}{2}(y/4)^b}.
\end{aligned} \tag{3.26}$$

Proof. First, let $J, m \in \mathbb{N}$ be natural numbers such that $J < m \le n$. Denote:

$$\mathcal{I}_{m,J} = \left\| \int_{-\infty}^{\infty} dt\ W_\gamma(t)(\tau_t^n - \tau_t^{\cup,n}) \Bigg(\sum_{\substack{Z\subset[-m,m] \\ Z\not\subset[-J,J]}} \Psi(Z) \Bigg) \right\|.$$

Furthermore, denote:

$$\mathcal{B} = \{X \subset [-n,n] : X \cap [-n,0] \neq \emptyset, X \cap (0,n] \neq \emptyset\} \tag{3.27}$$

$$\mathcal{D} = \{Z \subset [-m,m] : Z \cap \mathrm{An}(J+1,m) \neq \emptyset\}. \tag{3.28}$$

Then for $T > 0$, as in Lemma 3.5,

$$\mathcal{I}_{m,J} \leq 6(m-J)\,\|\Psi\|_F\, I_\gamma(T) + \int_{-T}^{T} dt\ |W_\gamma(t)| \int_0^{|t|} dr\ \left\| \Big[\sum_{X\in\mathcal{B}} \Psi(X), \tau_r^n\Big(\sum_{Z\in\mathcal{D}} \Psi(Z)\Big)\Big]\right\|. \tag{3.29}$$

As before, we separate the sum $\sum_{Z\in\mathcal{D}} \Psi(Z)$ into left, right and centrally localized terms of the interaction:

$$\sum_{Z\in\mathcal{D}} \Psi(Z) = \sum_{X\in\mathcal{L}} \Psi(X) + \sum_{Y\in\mathcal{R}} \Psi(Y) + \sum_{Z\in\mathcal{C}} \Psi(Z) \tag{3.30}$$

$$\mathcal{L} = \{X \in \mathcal{D} : X \subset [-m,0]\}\,, \quad \mathcal{R} = \{Y \in \mathcal{D} : Y \subset (0,m]\} \tag{3.31}$$

$$\mathcal{C} = \{Z \in \mathcal{D} : Z \cap [-m,0] \neq \emptyset,\ Z \cap (0,m] \neq \emptyset\}\,. \tag{3.32}$$

We first control the contribution to the integral from $\mathcal{L}$. We start this by gathering the interactions of $\mathcal{L}$ by intervals into $\Psi_{\mathcal{L}}(a;b) = \sum\{\Psi(X) : X \in \mathcal{L},\ \min X = a,\ \max X = b\}$:

$$\sum_{W\in\mathcal{L}} \Psi(W) = \sum_{\substack{-m\leq a\leq -J-1\\ a\leq b\leq a/2}} \Psi_{\mathcal{L}}(a;b) + \sum_{\substack{-m\leq a\leq -J-1\\ a/2<b\leq 0}} \Psi_{\mathcal{L}}(a;b) := \Psi_{\mathcal{L}}^1 + \Psi_{\mathcal{L}}^2. \tag{3.33}$$

Let $I_a = [-|a/4|, |a/4|]$. Then:

$$\begin{aligned}\left\| \Big[\tau_r^n(\Psi_{\mathcal{L}}^1), \sum_{X\in\mathcal{B}} \Psi(X)\Big]\right\| \leq \sum_{-m\leq a\leq -J-1} \Bigg(&\left\| \Big[\sum_{a\leq b\leq a/2} \tau_r^n(\Psi_{\mathcal{L}}(a;b)), \sum_{X\in\mathcal{B}:X\subset I_a} \Psi(X)\Big]\right\| \\ &+ \left\| \Big[\sum_{a\leq b\leq a/2} \tau_r^n(\Psi_{\mathcal{L}}(a;b)), \sum_{X\in\mathcal{B}:X\not\subset I_a} \Psi(X)\Big]\right\| \Bigg).\end{aligned} \tag{3.34}$$

By applying Lieb-Robinson bounds, the following inequality is valid:

$$\begin{aligned}&\left\| \Big[\sum_{a\leq b\leq a/2} \tau_r^n(\Psi_{\mathcal{L}}(a;b)), \sum_{X\in\mathcal{B}:X\subset I_a} \Psi(X)\Big]\right\| \\ &\leq \sum_{a\leq b\leq a/2} \kappa(\beta)\,\|\Psi\|_F^2\, F_\beta(|b-a|)|a|(e^{\nu|r|}-1)F_{\beta-2}(|a/4|) \\ &\leq \frac{\pi^2}{6}\kappa(\beta)\,\|\Psi\|_F^2\,(e^{\nu|r|}-1)\frac{|a|}{(1+|a/4|)^3} F_{\beta-5}(|J/4|)\end{aligned} \tag{3.35}$$

and the right-hand side is summable in $|a|$. And by both decay of the interaction and application of Lieb-Robinson bounds,

$$\begin{aligned}
&\left\| \Big[\sum_{a\le b\le a/2} \tau_r^n(\Psi_{\mathcal{L}}(a;b)), \sum_{X\in\mathcal{B}:X\not\subset I_a} \Psi(X) \Big] \right\| \\
&\le \left\| \Big[\sum_{a\le b\le a/2} \tau_r^n(\Psi_{\mathcal{L}}(a;b)), \sum_{\substack{-n\le c<a/4\\ 0\le d\le n}} \Psi_{\mathcal{B}}(c;d) \Big] \right\| \\
&\qquad + \left\| \Big[\sum_{a\le b\le a/2} \tau_r^n(\Psi_{\mathcal{L}}(a;b)), \sum_{\substack{a/4\le c\le 0\\ |a/4|<d\le n}} \Psi_{\mathcal{B}}(c;d) \Big] \right\| \\
&\le 2\kappa(\beta)(\pi^2/6)^3 \left\|\Psi\right\|_F^2 \frac{1}{(1+|a/4|)^2}\left(e^{\nu|r|} F_{\beta-2}(J/4)\right)
\end{aligned} \tag{3.36}$$

where $\Psi_{\mathcal{B}}(c;d)$ is defined as $\Psi_{\mathcal{L}}(a;b)$ only with respect to the index set $\mathcal{B}$. Hence:

$$\left\| \Big[\tau_r^n(\Psi_{\mathcal{L}}^1), \sum_{X\in\mathcal{B}} \Psi(X) \Big] \right\| \le 6\kappa(\beta)(\pi^2/6)^4 \left\|\Psi\right\|_F^2 e^{\nu|r|} F_{\beta-5}(J/4). \tag{3.37}$$

And again by decay of the interaction:

$$\left\| \Big[\tau_r^n(\Psi_{\mathcal{L}}^2), \sum_{X\in\mathcal{B}} \Psi(X) \Big] \right\| \le 2 \left\|\tau_r^n(\Psi_{\mathcal{L}}^2)\right\| \left\| \sum_{X\in\mathcal{B}} \Psi(X) \right\| \le 2(\pi^2/6)^4 \left\|\Psi\right\|_F^3 F_{\beta-4}(J/2). \tag{3.38}$$

By symmetry on the chain about 0, this majorizes the contribution from $\mathcal{R}$ as well. And decay of the interaction also yields a bound on the contribution from $\mathcal{C}$ in the same manner as in (3.38):

$$\left\| \Big[\sum_{X\in\mathcal{B}} \Psi(X), \tau_r^n\Big(\sum_{Z\in\mathcal{C}} \Psi(Z)\Big) \Big] \right\| \le 3(\pi^2/6)^4 \left\|\Psi\right\|_F^2 F_{\beta-4}(J). \tag{3.39}$$

Hence if we set $T = \frac{R}{2\nu}(J/4)^b$, the integral expression

$$\mathcal{J} = \int_{-T}^{T} dt\, |W_\gamma(t)| \int_0^{|t|} dr \left\| \Big[\sum_{X\in\mathcal{B}} \Psi(X), \tau_r^n\Big(\sum_{Z\in\mathcal{D}} \Psi(Z)\Big) \Big] \right\|$$

of the right-hand side of the inequality (3.29) is bounded:

$$\begin{aligned}
\mathcal{J} \le \frac{12\kappa(\beta)(\pi^2/6)^4}{\nu} \left\|\Psi\right\|_F^2 \left\|W_\gamma\right\|_\infty e^{-\frac{R}{2}(J/4)^b} \\
+ 10(\pi^2/6)^4 \max\left\{ \left\|\Psi\right\|_F^2, \left\|\Psi\right\|_F^3 \right\} \left\|W_\gamma(t)|t|\right\|_{L^1} F_{\beta-4}(J/2).
\end{aligned} \tag{3.40}$$

The right-hand side of the inequality (3.40) is bounded above by $Q(J)$, as defined in (3.26). Then (3.29) continues as:

$$\mathcal{I}_{m,J} \leq 6(m-J)\,\|\Psi\|_F\, I_\gamma\left(\frac{R}{2}(J/4)^b\right) + Q(J). \tag{3.41}$$

Now we prove the inequality (3.25). There exists a maximal $M_0 \in \mathbb{N}$ such that $n > M_0 K$, and so:

$$\begin{aligned} \mathcal{I}_{n,K} &= \mathcal{I}_{2K,K} + \mathcal{I}_{3K,2K} + \ldots + \mathcal{I}_{M_0K,(M_0-1)K} + \mathcal{I}_{n,M_0K} \\ &\leq 6\,\|\Psi\|_F\, K \sum_{j=1}^{M_0} I_\gamma\left(\frac{R}{2}(jK/4)^b\right) + \sum_{j=1}^{M_0} Q(jK). \end{aligned} \tag{3.42}$$

Set $H_\gamma(x) = 6\,\|\Psi\|_F\, x \sum_{\substack{m\in\mathbb{N}\\ m\geq x}} I_\gamma\left(\frac{R}{2}(m/4)^b\right) + \sum_{\substack{m\in\mathbb{N}\\ m\geq x}} Q(m)$. $\square$

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Appendix A. Generator of the spectral flow

In this appendix we briefly recall notations and properties of the spectral flow. For a more detailed analysis of quasi-locality and symmetries of the spectral flow, see e.g. Sections 6 and 7 of [11] and Proposition 5.4 of [2]. In finite volume Λ, the spectral flow is implemented for gapped, continuously differentiable families of Hamiltonians $H_\Lambda(s)$ by unitiaries solving:

$$\frac{d}{ds}U_\Lambda(s) = iD_\Lambda(s)U_\Lambda(s),\ U_\Lambda(0) = \mathbb{1} \tag{4.1}$$

for the Hastings generator:

$$D_\Lambda(s) = \int_{-\infty}^{\infty} dt\ W_\gamma(t)\tau_t^{\Lambda,s}\left(\frac{d}{ds}H_\Lambda(s)\right). \tag{4.2}$$

Here $\gamma > 0$ refers to the uniform gap of the $H_\Lambda(s)$, and $W_\gamma \in L^1 \cap L^\infty$ is chosen as the odd function, positive on $(0, \infty)$ from Equation (2.12) of [2]. Explicit estimates on the integral $I_\gamma(t) = \int_t^\infty dr\ W_\gamma(r) \geq 0$ are known:

Lemma 3.7 *(Lemma 2.6 of [2]). For $t > 36058$,*

$$I_\gamma(t) \leq [130e^2\gamma^9]t^{10} \exp\left(-\frac{2}{7}\frac{\gamma t}{(\ln(\gamma t))^2}\right). \tag{4.3}$$

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Automorphic equivalence within gapped phases in the bulk

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We develop a new adiabatic theorem for unique gapped ground states which does not require the gap for local Hamiltonians. We instead require a gap in the bulk and a smoothness of expectation values of sub-exponentially localized observables in the unique gapped ground state $\varphi_s(A)$. This requirement is weaker than the requirement of the gap of the local Hamiltonians, since a uniform spectral gap for finite dimensional ground states implies a gap in the bulk for unique gapped ground states, as well as the smoothness.

1. Introduction

Hastings's [6] [8] adiabatic method is a powerful tool in the analysis of gapped Hamiltonians in quantum many-body systems. Seminal mathematical developments from [1], [11], [17] and onwards have established a strong mathematical framework of adiabatic

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theory for quantum many-body systems. The adiabatic theorems from these works state that for a smooth path of gapped Hamiltonians, there is an automorphic equivalence between ground state spaces along the path. Furthermore, these automorphisms are quasi-local.

This framework has proven to be broadly applicable to many situations. In [7], the long standing problem of explaining the quantization of the Hall conductance was finally solved with this method. The Kubo formula was derived in [2] using the method.

Another use of the adiabatic theorem is the analysis of symmetry protected topological (SPT) phase, in [13] and [14]. In [13] and [14], indices for SPT phases which extend the indices by Pollmann et al. [15], [16] were introduced. The adiabatic theorem was used to show the stability of these indices. See [10] for the extension of [13] to interactions with unbounded interaction range with fast decay.

All of the adiabatic theorems developed so far require a uniform spectral gap for local Hamiltonians. Therefore, even if what we are interested in is the bulk, the use of known adiabatic theorems requires conditions on the gap in finite boxes. This is conceptually unsatisfactory because bulk-classification of gapped Hamiltonians can be coarser than the classification in finite volume [12]. For this reason, many works have been carried on torus. In this paper, we develop a new adiabatic theorem for unique gapped ground states which does not require the gap for local Hamiltonians. We instead require a gap in the bulk and a smoothness of expectation values of sub-exponentially localized observables in the unique gapped ground state $\varphi_s(A)$. This requirement is weaker than the requirement of the gap of the local Hamiltonians, since a uniform spectral gap for finite dimensional ground states implies a gap in the bulk for unique gapped ground states, as well as the smoothness. (See Remark 4.15.) Under such conditions, we show that there is a smooth path of quasi-local automorphisms α_s, such that $\omega_s = \omega_0 \circ \alpha_s$. This α_s is the same as the one given in the literatures [1], [11].

Although the result is analogous to those of finite systems, there is a crucial difference for the proof. For the finite system $\mathcal{A}_\Lambda$, there is a Hamiltonian $H_s(\Lambda)$ in the C^*-algebra $\mathcal{A}_\Lambda$. By considering a differential equation satisfied by the spectral projection $P_s(\Lambda)$ of the Hamiltonian $H_s(\Lambda)$ corresponding to the lowest eigenvalue, we may explicitly define in this case the automorphisms connecting the ground state spaces. In contrast, for infinite systems, we do not have a Hamiltonian H_s in the C^*-algebra of quantum spin systems. Of course we can consider the bulk Hamiltonian H_s, but H_s depends on the GNS representation, and the meaning of $\frac{d}{ds}H_s$ is ambiguous. Therefore, we have to find an alternative way to prove our adiabatic theorem.

In particular, for finite systems, the parallel transport condition $P_s(\Lambda)\dot{P}_s(\Lambda)P_s(\Lambda) = 0$ plays a crucial role. In infinite systems, this condition is replaced by Proposition 2.2.

Let us now give a more precise description of our result. We start by summarizing the standard setup of quantum spin systems [4,5]. Let $\nu \in \mathbb{N}$ and $d \in \mathbb{N}$. Throughout this article, we fix these numbers. We denote the algebra of $d \times d$ matrices by M_d.

We denote the set of all finite subsets in $\mathbb{Z}^\nu$ by $\mathfrak{S}_{\mathbb{Z}^\nu}$. For each $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$, $\mathrm{diam}(X)$ denotes the diameter of X. For $X, Y \subset \mathbb{Z}^\nu$, we denote by $d(X, Y)$ the distance between them. The number of elements in a finite set $\Lambda \subset \mathbb{Z}^\nu$ is denoted by $|\Lambda|$. For each $n \in \mathbb{N}$, we denote $[-n, n]^\nu \cap \mathbb{Z}^\nu$ by Λ_n. The complement of $\Lambda \subset \mathbb{Z}^\nu$ in $\mathbb{Z}^\nu$ is denoted by Λ^c.

For each $z \in \mathbb{Z}^\nu$, let $\mathcal{A}_{\{z\}}$ be an isomorphic copy of M_d, and for any finite subset $\Lambda \subset \mathbb{Z}^\nu$, let $\mathcal{A}_\Lambda = \otimes_{z\in\Lambda}\mathcal{A}_{\{z\}}$, which is the local algebra of observables in Λ. For finite Λ, the algebra $\mathcal{A}_\Lambda$ can be regarded as the set of all bounded operators acting on the Hilbert space $\otimes_{z\in\Lambda}\mathbb{C}^d$. We use this identification freely. If $\Lambda_1 \subset \Lambda_2$, the algebra $\mathcal{A}_{\Lambda_1}$ is naturally embedded in $\mathcal{A}_{\Lambda_2}$ by tensoring its elements with the identity. The algebra $\mathcal{A}$, representing the quantum spin system on $\mathbb{Z}^\nu$ is given as the inductive limit of the algebras $\mathcal{A}_\Lambda$ with $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$. Note that $\mathcal{A}_\Lambda$ for $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$ can be regarded naturally as a subalgebra of $\mathcal{A}$. We denote the set of local observables by $\mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda\in\mathfrak{S}_{\mathbb{Z}^\nu}} \mathcal{A}_\Lambda$.

A uniformly bounded interaction on $\mathcal{A}$ is a map $\Psi : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ such that

$$\Psi(X) = \Psi(X)^* \in \mathcal{A}_X, \quad X \in \mathfrak{S}_{\mathbb{Z}^\nu}, \tag{1.1}$$

and

$$\sup_{X\in\mathfrak{S}_{\mathbb{Z}^\nu}} \|\Psi(X)\| < \infty. \tag{1.2}$$

It is of finite range with interaction length less than or equal to $R \in \mathbb{N}$ if $\Psi(X) = 0$ for any $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$ whose diameter is larger than R. We denote by Ψ_n for each $n \in \mathbb{N}$ the interaction given by

$$\Psi_n(X) := \begin{cases} \Psi(X), & \text{if} \quad X \subset \Lambda_n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

For a uniformly bounded and finite range interaction Ψ and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$ define the local Hamiltonian

$$(H_\Psi)_\Lambda := \sum_{X\subset\Lambda} \Psi(X), \tag{1.4}$$

and denote the dynamics

$$\tau^t_{\Psi,\Lambda}(A) := e^{it(H_\Psi)_\Lambda} A e^{-it(H_\Psi)_\Lambda}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}. \tag{1.5}$$

By the uniform boundedness and finite rangeness of Ψ, for each $A \in \mathcal{A}$, the following limit exists:

$$\lim_{\Lambda\to\mathbb{Z}^\nu} \tau^t_{\Psi,\Lambda}(A) =: \tau^t_\Psi(A), \quad t \in \mathbb{R}, \tag{1.6}$$

and defines the dynamics τ_Ψ on $\mathcal{A}$. Note that $\tau_{\Psi_n} = \tau_{\Psi,\Lambda_n}$. We denote by δ_Ψ the generator of τ_Ψ.

For a uniformly bounded and finite range interaction Ψ, a state φ on $\mathcal{A}$ is called a τ_Ψ-ground state if the inequality $-i\,\varphi(A^*\delta_\Psi(A)) \geq 0$ holds for any element A in the domain $\mathcal{D}(\delta_\Psi)$ of δ_Ψ. Let φ be a τ_Ψ-ground state, with the GNS triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$. Then there exists a unique positive operator $H_{\varphi,\Psi}$ on $\mathcal{H}_\varphi$ such that $e^{itH_{\varphi,\Psi}}\pi_\varphi(A)\Omega_\varphi = \pi_\varphi(\tau^t_\Psi(A))\Omega_\varphi$, for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{\varphi,\Psi}$ the bulk Hamiltonian associated with φ. Note that Ω_φ is an eigenvector of $H_{\varphi,\Psi}$ with eigenvalue 0. See [5] for the general theory.

Let $\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N}$ be the conditional expectation with respect to the trace state. Let us consider the following subset of $\mathcal{A}$. (See [3] and [9] for analogous definitions.)

Definition 1.1. Let $f : (0,\infty) \to (0,\infty)$ be a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. For each $A \in \mathcal{A}$, let

$$\|A\|_f := \|A\| + \sup_{N\in\mathbb{N}} \left(\frac{\|A - \mathbb{E}_N(A)\|}{f(N)} \right). \tag{1.7}$$

We denote by $\mathcal{D}_f$ the set of all $A \in \mathcal{A}$ such that $\|A\|_f < \infty$.

Properties of $\mathcal{D}_f$ are collected in Appendix B. The set $\mathcal{D}_f$ is a $*$-algebra which is a Banach space with respect to the norm $\|\cdot\|_f$ (see Lemma B.1).

Assumption 1.2. Let $\Phi(\cdot\,;s) : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ be a family of uniformly bounded, finite range interactions parameterized by $s \in [0,1]$. We assume the following:

- **(i)** For each $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$, the map $[0,1] \ni s \to \Phi(X;s) \in \mathcal{A}_X$ is continuous and piecewise C^1. We denote by $\dot{\Phi}(X;s)$ the corresponding derivatives. The interaction obtained by differentiation is denoted by $\dot{\Phi}(s)$, for each $s \in [0,1]$.
- **(ii)** There is a number $R \in \mathbb{N}$ such that $X \in \mathfrak{S}_{\mathbb{Z}^\nu}$ and $\mathrm{diam}(X) \geq R$ imply $\Phi(X;s) = 0$, for all $s \in [0,1]$.
- **(iii)** Interactions are bounded as follows

$$\sup_{s\in[0,1]} \sup_{X\in\mathfrak{S}_{\mathbb{Z}^\nu}} \left(\|\Phi(X;s)\| + |X| \left\|\dot{\Phi}(X;s)\right\| \right) < \infty. \tag{1.8}$$

- **(iv)** Setting

$$b(\varepsilon) := \sup_{Z\in\mathfrak{S}_{\mathbb{Z}^\nu}} \sup_{s,s_0\in[0,1],0<|s-s_0|<\varepsilon} \left\| \frac{\Phi(Z;s) - \Phi(Z;s_0)}{s - s_0} - \dot{\Phi}(Z;s_0) \right\| \tag{1.9}$$

  for each $\varepsilon > 0$, we have $\lim_{\varepsilon\to 0} b(\varepsilon) = 0$.
- **(v)** For each $s \in [0,1]$, there exists a unique $\tau_{\Phi(s)}$-ground state φ_s.
- **(vi)** There exists a $\gamma > 0$ such that $\sigma(H_{\varphi_s,\Phi(s)}) \setminus \{0\} \subset [2\gamma, \infty)$ for all $s \in [0,1]$, where $\sigma(H_{\varphi_s,\Phi(s)})$ is the spectrum of $H_{\varphi_s,\Phi(s)}$.

(vii) There exists $0 < \beta < 1$ satisfying the following: Set $\zeta(t) := e^{-t^\beta}$. Then for each $A \in D_\zeta$, $\varphi_s(A)$ is differentiable with respect to s, and there is a constant C_ζ such that:

$$|\dot{\varphi}_s(A)| \le C_\zeta \|A\|_\zeta, \tag{1.10}$$

for any $A \in D_\zeta$.

The main theorem of this paper is that under the Assumption 1.2, there is a strongly continuous path of automorphisms $[0,1] \ni s \mapsto \alpha_s$ such that $\varphi_s = \varphi_0 \circ \alpha_s$, $s \in [0,1]$.

In fact, this α_s is the same one as in [1] and [11], which is given through some differential equation. Let us recall it.

We use the function ω_1 introduced in [11]. Set

$$a_n := \frac{a_1}{n \ln(n)^2}, \quad n \ge 2, \tag{1.11}$$

and choose a_1 so that $\sum_{n=1}^\infty a_n = \frac{1}{2}$. Let $\omega_1(t) \in L^1(\mathbb{R})$ be the function on $\mathbb{R}$ defined by

$$\omega_1(t) := \begin{cases} c, \quad t = 0, \\ c \prod_{n=1}^\infty \left(\frac{\sin(a_n t)}{a_n t}\right)^2, \quad t \neq 0 \end{cases} \tag{1.12}$$

with normalization factor $c > 0$ such that

$$\int dt \omega_1(t) = 1. \tag{1.13}$$

As shown in [1] and [11], ω_1 is indeed an even nonnegative L^1-function and

$$\omega_1(t) \le c_1 \frac{t}{\ln(t)^2} e^{-\frac{\eta t}{\ln(t)^2}}, \quad t > e, \tag{1.14}$$

$$W_1(x) := \int_x^\infty dt \omega_1(t) \le \begin{cases} c_1 \left(\frac{x}{\ln(x)^2}\right)^2 e^{-\frac{\eta x}{\ln(x)^2}}, \quad x > e^9, \\ 1, \quad x \le e^9 \end{cases} \tag{1.15}$$

for constants $\eta = 2a_1 \in (\frac{2}{7}, 1)$ and $c_1 = (27/14)ce^4$. We set $\omega_\gamma(t) := \gamma\omega_1(\gamma t)$, where $\gamma > 0$ is from Assumption 1.2, and $W_\gamma(x) := W_1(\gamma x)$, for $x \in \mathbb{R}_+$. The function ω_γ is an even nonnegative L^1-function with

$$\int dt \omega_\gamma(t) = 1. \tag{1.16}$$

We also have

$$W_\gamma(x) = \int_x^\infty dt\omega_\gamma(t), \quad x \in \mathbb{R}_+. \tag{1.17}$$

Furthermore, the Fourier transform of ω_γ is supported in the interval $[-\gamma, \gamma]$. (See [11].)

For each $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$, let U_Λ be the solution of the differential equation

$$-i\frac{d}{ds}U_\Lambda(s) = D_\Lambda(s)U_\Lambda(s), \quad U_\Lambda(0) = \mathbb{I}. \tag{1.18}$$

Here, $D_\Lambda(s)$ is defined by

$$D_\Lambda(s) := \int_{-\infty}^{\infty} dt\ \omega_\gamma(t) \int_0^t du\tau^u_{\Phi(s),\Lambda}\left(\frac{d}{ds}\left(H_{\Phi(s)}\right)_\Lambda\right), \quad s \in [0,1]. \tag{1.19}$$

We set

$$\alpha_{s,\Lambda}(A) := U_\Lambda(s)^* A U_\Lambda(s), \quad A \in \mathcal{A}, \quad s \in [0,1]. \tag{1.20}$$

By the results of [1] and [11], conditions (i), (ii) and (iii) of Assumption 1.2 imply that the thermodynamic limit

$$\alpha_s(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} \alpha_{s,\Lambda}(A), \quad A \in \mathcal{A}, \quad s \in [0,1], \tag{1.21}$$

exists and defines a strongly continuous path of automorphisms $[0,1] \ni s \mapsto \alpha_s$. We also have the limit of the inverse

$$\alpha_s^{-1}(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} \alpha^{-1}_{s,\Lambda}(A), \quad A \in \mathcal{A}, \quad s \in [0,1]. \tag{1.22}$$

See [11]. Our main theorem is as follows.

Theorem 1.3. *Under the Assumption 1.2, we have*

$$\varphi_s = \varphi_0 \circ \alpha_s, \quad s \in [0,1], \tag{1.23}$$

for α_s given in (1.21).

Remark 1.4. In fact the conditions (v), (vi), (vii) in Assumption 1.2 can be relaxed as follows. Suppose that there is a path of pure states $[0,1] \ni s \mapsto \varphi_s$ such that

- **(v)** for each $s \in [0,1]$, φ_s is a $\tau_{\Phi(s)}$-ground state.
- **(vi)** There exists a $\gamma > 0$ such that $\sigma(H_{\varphi_s,\Phi(s)}) \setminus \{0\} \subset [2\gamma, \infty)$ for all $s \in [0,1]$, where $\sigma(H_{\varphi_s,\Phi(s)})$ is the spectrum of $H_{\varphi_s,\Phi(s)}$. The eigenvalue 0 of $H_{\varphi_s,\Phi(s)}$ is non-degenerate.

(vii) The condition (vii) of Assumption 1.2 holds for the path.

Then we have (1.23) for α_s given in (1.21).

Our motivation to develop this bulk version of automorphic equivalence was the index theorems for SPT-phases [13] and [14]. In [13] and [14], the path of interactions was required to have a uniform spectral gap for corresponding local Hamiltonians. It is a bit unpleasant that we have to ask for the existence of the gap for local Hamiltonians while what we really would like to investigate is the bulk. From our Theorem 1.3, combined with Theorem 2.6, and the proof of Proposition 3.5 of [13], we obtain the following version of the index theorem for the time reversal symmetry.

Theorem 1.5. *Let $\Phi(\cdot\ ; s) : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ be a path of time-reversal interactions satisfying Assumption 1.2. Then $\mathbb{Z}_2$-index defined in Definition 3.3 of [13] is constant along the path.*

From our Theorem 1.3, combined with Theorem 2.9 of [14], and the proof of Proposition 3.5 of [13], we obtain the following version of the index theorem for the reflection symmetry.

Theorem 1.6. *Let $\Phi(\cdot\ ; s) : \mathfrak{S}_{\mathbb{Z}^\nu} \to \mathcal{A}_{\mathrm{loc}}$ be a path of reflection invariant interactions satisfying Assumption 1.2. Then $\mathbb{Z}_2$-index defined in Definition 3.3 of [14] is constant along the path.*

The rest of the paper is devoted to the proof of Theorem 1.3.

2. Proof of the Theorem 1.3

Throughout this Section, we will always assume Assumption 1.2. For $s \in [0, 1]$ and $A \in \mathcal{A}$, we set

$$I_s(A) := \int dt\ \omega_\gamma(t)\tau^t_{\Phi(s)}(A). \tag{2.1}$$

The integral can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$.

We need the following Lemma for the proof.

Lemma 2.1. *Fix $0 < \beta = \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and set $f(t) := t^{-1}\exp(-t^{\beta_1})$, $f_0(t) := \exp(-t^{\beta_1})$, $f_1(t) := \exp(-t^{\beta_2})$, $f_2(t) := t^{-2(\nu+2)}\exp(-t^{\beta_3})$, $g(t) := \exp(-t^{\beta_4})$, $\zeta(t) := \exp(-t^{\beta_5})$. (Here β is the one in (vii) of Assumption 1.2.) Then we have the following.*

1. *For any* $s \in [0,1]$*, we have*

$$\alpha_s^{-1}(\mathcal{A}_{\rm loc}) \subset \mathcal{D}_f \subset \mathcal{D}_{f_0} \subset \mathcal{D}_{f_1} \subset \mathcal{D}_{f_2} \subset \mathcal{D}_g \subset \mathcal{D}_\zeta.$$

2. *We have* $\tau^t_{\Phi(s)}(\mathcal{D}_f) \subset \mathcal{D}_{f_1}$ *and there is a non-negative non-decreasing function on* $\mathbb{R}_{\geq 0}$, $b_{f,f_1}(t)$ *such that*

$$\int dt\, \omega_\gamma(t)\, |t| \cdot b_{f,f_1}(|t|) < \infty, \tag{2.2}$$

$$\sup_{s\in[0,1]} \left\| \tau^t_{\Phi(s)}(A) \right\|_{f_1} \leq b_{f,f_1}(|t|)\, \|A\|_f\,, \quad A \in \mathcal{D}_f. \tag{2.3}$$

3. *We have* $\mathcal{D}_\zeta \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)})$ *for any* $s \in [0,1]$.
4. *There is a constant* $C^{(1)}_{f_2,\zeta} > 0$ *such that*

$$\sup_{s\in[0,1]} \left\| \delta_{\Phi(s)}(A) \right\|_\zeta,\ \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \delta_{\Phi_N(s)}(A) \right\|_\zeta \leq C^{(1)}_{f_2,\zeta}\, \|A\|_{f_2} \tag{2.4}$$

$$\sup_{s\in[0,1]} \left\| \delta_{\dot{\Phi}(s)}(A) \right\|_\zeta,\ \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \delta_{\dot{\Phi}_N(s)}(A) \right\|_\zeta \leq C^{(1)}_{f_2,\zeta}\, \|A\|_{f_2} \tag{2.5}$$

$$\begin{aligned}
&\sup_{s,s_0\in[0,1],0<|s-s_0|\leq\varepsilon} \left\| \delta_{\frac{\Phi(s)-\Phi(s_0)}{s-s_0} - \dot{\Phi}(s_0)}(A) \right\|_\zeta, \\
&\sup_{N\in\mathbb{N}} \sup_{s,s_0\in[0,1],0<|s-s_0|\leq\varepsilon} \left\| \delta_{\frac{\Phi_N(s)-\Phi_N(s_0)}{s-s_0} - \dot{\Phi}_N(s_0)}(A) \right\|_\zeta \\
&\leq b(\varepsilon) C^{(1)}_{f_2,\zeta}\, \|A\|_{f_2}
\end{aligned} \tag{2.6}$$

for all $A \in \mathcal{D}_{f_2}$*. (Here the meaning of the inequality is that each term on the left hand side is bounded by the right hand side. We use this notation throughout this article.) In particular,* $\delta_{\Phi(s)}(\mathcal{D}_{f_2}) \subset \mathcal{D}_\zeta$*, for any* $s \in [0,1]$*. (Recall* $b(\varepsilon)$ *in Assumption 1.2 (iv).)*

5. *For any* $A \in \mathcal{D}_f$*, and* $(s', u', s'', s''') \in [0,1] \times \mathbb{R} \times [0,1] \times [0,1]$*, we have* $\tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_{f_2} \subset \mathcal{D}_\zeta \subset D(\delta_{\Phi(s')}) \cap D(\delta_{\dot{\Phi}(s')})$ *and* $\delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A), \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_\zeta$*. For any compact intervals* $[a,b]$*,* $[c,d]$ *of* $\mathbb{R}$ *and* $A \in \mathcal{D}_f$*, the maps:*

$$\begin{aligned}
&[a,b] \times [0,1] \times [0,1] \times [c,d] \times [0,1] \times [0,1] \ni (u, s, s', u', s'', s''') \\
&\mapsto \tau^u_{\Phi(s)} \circ \delta_{\Phi(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A},
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&[a,b] \times [0,1] \times [0,1] \times [c,d] \times [0,1] \times [0,1] \ni (u, s, s', u', s'', s''') \\
&\mapsto \tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A}
\end{aligned} \tag{2.8}$$

are uniformly continuous with respect to $\|\cdot\|$, *and maps*

$$[0,1]\times[c,d]\times[0,1]\times[0,1] \ni (s',u',s'',s''') \mapsto \delta_{\Phi(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A) \in \mathcal{D}_\zeta \tag{2.9}$$

$$[0,1]\times[c,d]\times[0,1]\times[0,1] \ni (s',u',s'',s''') \mapsto \delta_{\dot{\Phi}(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A) \in \mathcal{D}_\zeta \tag{2.10}$$

are uniformly continuous with respect to $\|\cdot\|_\zeta$.

6. *For any* $A \in \mathcal{D}_f$, $\alpha_s^{-1}(A)$ *is differentiable with respect to* $\|\cdot\|$ *and*

$$\frac{d}{ds}\alpha_s^{-1}(A) = \int dt\omega_\gamma(t)\int_0^t du\tau_{\Phi(s)}^u\circ\delta_{\dot{\Phi}(s)}\left(\tau_{\Phi(s)}^{-u}\left(\alpha_s^{-1}(A)\right)\right) \tag{2.11}$$

The right hand side can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$.

7. *For any* $A \in \mathcal{D}_f$, *the integral*

$$\int dt\omega_\gamma(t)\int_0^t du\tau_{\Phi(s)}^u\circ\delta_{\dot{\Phi}(s)}\left(\tau_{\Phi(s)}^{-u}(A)\right) \tag{2.12}$$

$$\int dt\,\omega_\gamma(t)\int_0^t du\,\tau_{\Phi(s)}^{t-u}\circ\left(\delta_{\dot{\Phi}(s)}\right)\circ\tau_{\Phi(s)}^u(A) \tag{2.13}$$

are well-defined as a Bochner integral with respect to $(\mathcal{A}, \|\cdot\|)$.

8. *For any* $A \in \mathcal{D}_f$ *and* $s \in [0,1]$, *we have* $I_s(A) \in \mathcal{D}_{f_1}$.
9. *For each* $A \in \mathcal{A}$, $\mathbb{R}\times[0,1] \ni (u,s) \to \tau_{\Phi(s)}^u(A) \in \mathcal{A}$ *is continuous with respect to the norm* $\|\cdot\|$.
10. *For any* $A \in \mathcal{D}_f$, *the integrals*

$$\int dt\omega_\gamma(t)\int_0^t du\delta_{\Phi(s)}\circ\tau_{\Phi(s)}^u(A), \quad \int_0^t du\delta_{\Phi(s)}\circ\tau_{\Phi(s)}^u(A), \tag{2.14}$$

are well-defined as Bochner integrals with respect to $(\mathcal{D}_\zeta, \|\cdot\|_\zeta)$.

The proof of Lemma 2.1 is given in Section 4. Throughout Section 2 and Section 3 (but not in Section 4), we fix $0 < \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and set $f, f_0, f_1, f_2, g, \zeta$, given in Lemma 2.1, and apply Lemma 2.1.

In Section 3, we prove the following:

Proposition 2.2. *For any* $A \in \mathcal{D}_f$, *we have*

$$\dot{\varphi}_s\left(I_s(A)\right) = 0, \quad s \in [0,1]. \tag{2.15}$$

Note that by *8.* of Lemma 2.1, $I_s(A)$ belongs to $\mathcal{D}_{f_1} \subset \mathcal{D}_\zeta$, and that $\dot{\varphi}_s(I_s(A))$ in Proposition 2.2 is well-defined by (vii) of Assumption 1.2. This corresponds to the parallel transport condition $P_s(\Lambda)\dot{P}_s(\Lambda)P_s(\Lambda) = 0$ in finite systems. Note that from its definition, $I_s(A)$ does not have "off-diagonal parts," which holds for finite systems as well by the equation

$$\forall A \in \mathcal{A}_\Lambda, \ \left[\int dt\ \omega_\gamma(t)\tau^t_{\Phi(s),\Lambda}(A), P_s(\Lambda)\right] = 0.$$

We now prove Theorem 1.3 using this proposition. In order to prove the Theorem, it suffices to show

$$\frac{d}{ds}\left(\varphi_s \circ \alpha_s^{-1}(X)\right) = 0, \tag{2.16}$$

for any $X \in \mathcal{A}_{\mathrm{loc}}$. Note that from Assumption 1.2 (vii), and *1.* of Lemma 2.1, the function $[0,1] \ni s \to \varphi_s \circ \alpha_{s_0}^{-1}(X)$ is differentiable for any $X \in \mathcal{A}_{\mathrm{loc}}$ and $s_0 \in [0,1]$. Furthermore, from *6.* of Lemma 2.1, $[0,1] \ni s \mapsto \alpha_s^{-1}(X) \in \mathcal{A}$ is differentiable with respect to the norm for any $X \in \mathcal{A}_{\mathrm{loc}} \subset \mathcal{D}_f$. Therefore, for any $X \in \mathcal{A}_{\mathrm{loc}}$, $[0,1] \ni s \to \varphi_s \circ \alpha_s^{-1}(X)$ is differentiable, the left hand side of (2.16) makes sense, and we have

$$\frac{d}{ds}\left(\varphi_s \circ \alpha_s^{-1}(X)\right) = \dot{\varphi}_s \circ \alpha_s^{-1}(X) + \varphi_s \circ \frac{d}{ds}\alpha_s^{-1}(X), \quad X \in \mathcal{A}_{\mathrm{loc}}. \tag{2.17}$$

For the proof of (2.16), we use the following Lemma.

Lemma 2.3. *For any $A \in \mathcal{D}_f$,*

$$A - I_s(A) = -\int dt\omega_\gamma(t)\int_0^t du\delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A). \tag{2.18}$$

The integrand of the right hand side is continuous with respect to $\|\cdot\|_\zeta$ and the integral can be understood as the Bochner integral of $(\mathcal{D}_\zeta, \|\cdot\|_\zeta)$.

Proof. The latter part is *5., 10.* of Lemma 2.1. To show (2.18), recall the Duhamel formula

$$A - \tau^t_{\Phi(s)}(A) = \int_0^t du\ \left(-\delta_{\Phi(s)}\right) \circ \tau^u_{\Phi(s)}(A), \quad A \in \mathcal{D}_f. \tag{2.19}$$

Here we used the fact that $\tau^u_{\Phi(s_0)}(\mathcal{D}_f) \subset \mathcal{D}_{f_1} \subset \mathcal{D}_\zeta \subset D\left(\delta_{\Phi(s)}\right)$, which follows from *2.,1.,3.* of Lemma 2.1.

We multiply (2.19) by $\omega_\gamma(t)$ and integrate over $t \in \mathbb{R}$. Then recalling (1.16), we obtain

$$\begin{aligned} A - I_s(A) &= \int dt\, \omega_\gamma(t) A - \int dt\, \omega_\gamma(t) \tau^t_{\Phi(s)}(A) \\ &= \int dt\, \omega_\gamma(t) \int_0^t du\, \left(-\delta_{\Phi(s)}\right) \circ \tau^u_{\Phi(s)}(A), \quad A \in \mathcal{D}_f. \quad \square \end{aligned} \tag{2.20}$$

In order to show (2.16), we need to know $\dot{\varphi}_s$ on $\mathcal{D}_f$. From Proposition 2.2 and Lemma 2.3, for any $A \in \mathcal{D}_f$, we have

$$(\dot{\varphi}_s)(A) = (\dot{\varphi}_s)(A) - (\dot{\varphi}_s)(I_s(A)) = -\int dt \omega_\gamma(t) \int_0^t du \dot{\varphi}_s \left(\delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A)\right). \tag{2.21}$$

Here we used the Bochner integrability of the right hand side of (2.18) with respect to $\|\cdot\|_\zeta$, and the continuity of $\dot{\varphi}_s$ (1.10) with respect to $\|\cdot\|_\zeta$.

As φ_s is the $\tau_{\Phi(s)}$-ground state, we have

$$\varphi_s \circ \delta_{\Phi(s)}(B) = 0, \quad B \in \mathcal{D}_{f_1}, \quad s \in [0,1]. \tag{2.22}$$

(Recall that $\mathcal{D}_{f_1} \subset \mathcal{D}_\zeta \subset D(\delta_{\Phi(s)})$, from *1.*, *3.* of Lemma 2.1.) Differentiating this by s, we obtain

$$\dot{\varphi}_s \circ \delta_{\Phi(s)}(B) + \varphi_s \circ \delta_{\dot{\Phi}(s)}(B) = 0, \quad B \in \mathcal{D}_{f_1}, \quad s \in [0,1]. \tag{2.23}$$

More precisely, note that

$$\delta_{\Phi(s)}\left(\mathcal{D}_{f_1}\right) \subset \delta_{\Phi(s)}\left(\mathcal{D}_{f_2}\right) \subset \mathcal{D}_\zeta, \quad s \in [0,1], \tag{2.24}$$

by Lemma 2.1, *1.*, *4.* Therefore, for $B \in \mathcal{D}_{f_1}$, we have $\delta_{\Phi(s)}(B) \in \mathcal{D}_\zeta$, $s \in [0,1]$, and for any $s, s_0 \in [0,1]$ with $s \neq s_0$, we have

$$\begin{aligned} &\left| -\left(\dot{\varphi_{s_0}} \circ \delta_{\Phi(s_0)}(B) + \varphi_{s_0} \circ \delta_{\dot{\Phi}(s_0)}(B)\right) \right| \\ &= \left| \frac{\varphi_s \circ \delta_{\Phi(s)}(B) - \varphi_{s_0} \circ \delta_{\Phi(s_0)}(B)}{s - s_0} - \left(\dot{\varphi_{s_0}} \circ \delta_{\Phi(s_0)}(B) + \varphi_{s_0} \circ \delta_{\dot{\Phi}(s_0)}(B)\right) \right| \\ &\leq \left| \varphi_s \left(\frac{\delta_{\Phi(s)}(B) - \delta_{\Phi(s_0)}(B)}{s - s_0} - \delta_{\dot{\Phi}(s_0)}(B) \right) \right| \\ &+ \left| \frac{\varphi_s \circ \delta_{\Phi(s_0)}(B) - \varphi_{s_0} \circ \delta_{\Phi(s_0)}(B)}{s - s_0} - \left(\dot{\varphi_{s_0}} \circ \delta_{\Phi(s_0)}(B)\right) \right| \\ &+ \left| \left(\varphi_s - \varphi_{s_0}\right) \left(\delta_{\dot{\Phi}(s_0)}(B)\right) \right|. \end{aligned} \tag{2.25}$$

As $\delta_{\Phi(s_0)}(B) \in \mathcal{D}_\zeta$, the second and the third terms of the last line converge to 0 as $s \to s_0$. The first term of the last line can be bounded as

$$\left|\varphi_s\left(\frac{\delta_{\Phi(s)}(B) - \delta_{\Phi(s_0)}(B)}{s - s_0} - \delta_{\dot{\Phi}(s_0)}(B)\right)\right| \le \left\|\frac{\delta_{\Phi(s)}(B) - \delta_{\Phi(s_0)}(B)}{s - s_0} - \delta_{\dot{\Phi}(s_0)}(B)\right\|$$
$$\le b(|s - s_0|)C^{(1)}_{f_2,\zeta}\left\|B\right\|_{f_2} \to 0, \quad s \to s_0, \tag{2.26}$$

and goes to 0 as $s \to s_0$. Here, in the last line, we used *4.* of Lemma 2.1 and recalled $\mathcal{D}_{f_1} \subset \mathcal{D}_{f_2}$, from *1.* of Lemma 2.1, and (iv) of Assumption 1.2. Hence we obtain (2.23).

From this and (2.21), for $A \in \mathcal{D}_f$, recalling $\tau^u_{\Phi(s)}(A) \in \mathcal{D}_{f_1}$ by *2.* of Lemma 2.1, we have

$$(\dot{\varphi_s})(A) = \int dt\omega_\gamma(t) \int_0^t du\varphi_s \circ \delta_{\dot{\Phi}(s)}\left(\tau^u_{\Phi(s)}(A)\right). \tag{2.27}$$

For any $X \in \mathcal{A}_{\rm loc}$, recall that $\alpha_s^{-1}(X) \in \alpha_s^{-1}(\mathcal{A}_{\rm loc}) \subset \mathcal{D}_f \subset \mathcal{D}_\zeta$ by *1.* of Lemma 2.1. From (2.17), (2.27) and *6.* of Lemma 2.1, we have

$$\begin{aligned}
&\frac{d}{ds}\left(\varphi_s \circ \alpha_s^{-1}(X)\right) = \dot{\varphi}_s \circ \alpha_s^{-1}(X) + \varphi_s \circ \frac{d}{ds}\alpha_s^{-1}(X) \\
&= \int dt\omega_\gamma(t) \int_0^t du\varphi_s \circ \delta_{\dot{\Phi}(s)}\left(\tau^u_{\Phi(s)} \circ \alpha_s^{-1}(X)\right) \\
&+ \int dt\omega_\gamma(t) \int_0^t du\varphi_s\left(\tau^u_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)}\left(\tau^{-u}_{\Phi(s)}\left(\alpha_s^{-1}(X)\right)\right)\right) = 0
\end{aligned} \tag{2.28}$$

Here we used the fact that ω_γ is an even function, and that φ_s is $\tau_{\Phi(s)}$-invariant because it is the $\tau_{\Phi(s)}$-ground state.

Hence we have proven the Theorem 1.3.

3. Proof of Proposition 2.2

Throughout this Section, we keep Assumption 1.2. We also continue to use the same $0 < \beta = \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and set $f, f_0, f_1, f_2, g, \zeta$, as given in Lemma 2.1.

Let $(\mathcal{H}_s, \pi_s, \Omega_s)$ be the GNS triple of φ_s. Let $H_s := H_{\varphi_s,\Phi(s)}$ be the associated bulk Hamiltonian. The key property of I_s we use is the following.

Lemma 3.1. *For any $A \in \mathcal{A}$, we have*

$$\pi_s\left(I_s\left(A\right)\right)\Omega_s = \varphi_s(A)\Omega_s. \tag{3.1}$$

Proof. As the Fourier transform $\hat{\omega}_\gamma$ of ω_γ has support in $[-\gamma, \gamma]$, (v) and (vi) of Assumption 1.2 and (1.16) implies:

$$\hat{\omega}_\gamma (H_s) = \frac{1}{\sqrt{2\pi}} |\Omega_s\rangle \langle \Omega_s| . \tag{3.2}$$

From the definition of I_s, substituting (3.2), we have

$$\begin{aligned} \pi_s (I_s (A)) \Omega_s &= \int dt\, \omega_\gamma(t) \pi_s \left(\tau^t_{\Phi(s)}(A) \right) \Omega_s \\ &= \int dt\, \omega_\gamma(t) e^{itH_s} \pi_s(A)\Omega_s = \sqrt{2\pi}\hat{\omega}_\gamma(H_s)\pi_s(A)\Omega_s = \varphi_s(A)\Omega_s. \quad \square \end{aligned} \tag{3.3}$$

From this, we immediately obtain the following decoupling.

Lemma 3.2. *For any $A, B \in \mathcal{A}$ and $s \in [0,1]$, we have*

$$\varphi_s (B^* I_s(A)) = \varphi_s(B^*)\varphi_s(A). \tag{3.4}$$

Lemma 3.3. *For each $s \in [0,1]$ and $A \in \mathcal{D}_f$, the integrand of*

$$V_s(A) := \int dt\, \omega_\gamma(t) \int_0^t du\, \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\dot{\Phi}(s)} \right) \circ \tau^u_{\Phi(s)}(A), \tag{3.5}$$

is continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$. For any $A \in D_f$, $[0,1] \ni s \to I_s(A) \in \mathcal{A}$ is differentiable with respect to $\|\cdot\|$ and

$$\frac{d}{ds} I_s(A) = V_s(A). \tag{3.6}$$

Proof. Let $A \in \mathcal{D}_f$. That the integrand of (3.5) is continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$, follow from *5.* and *7.*, of Lemma 2.1, respectively.

Next, recall the Duhamel formula

$$\tau^t_{\Phi(s)}(A) - \tau^t_{\Phi(s_0)}(A) = \int_0^t du\, \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\Phi(s)} - \delta_{\Phi(s_0)} \right) \circ \tau^u_{\Phi(s_0)}(A), \quad A \in \mathcal{D}_f. \tag{3.7}$$

Here we used the fact that $\tau^u_{\Phi(s_0)} (\mathcal{D}_f) \subset \mathcal{D}_\zeta \subset D \left(\delta_{\Phi(s)} \right)$, which follows from *2., 1., 3.,* of Lemma 2.1. By *5.* of Lemma 2.1, the integrand on the right hand side is continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$.

We multiply (3.7) by $\omega_\gamma(t)$ and integrate over $t \in \mathbb{R}$. Then we obtain

$$\begin{aligned} I_s(A) - I_{s_0}(A) &= \int dt\, \omega_\gamma(t)\tau^t_{\Phi(s)}(A) - \int dt\, \omega_\gamma(t)\tau^t_{\Phi(s_0)}(A) \\ &= \int dt\, \omega_\gamma(t) \int_0^t du\, \tau^{t-u}_{\Phi(s)} \circ \left(\delta_{\Phi(s)} - \delta_{\Phi(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A), \quad A \in \mathcal{D}_f. \end{aligned} \tag{3.8}$$

By *5.* of Lemma 2.1, all the integrands are continuous and the integral can be understood as a Bochner integral in Banach space $(\mathcal{A}, \|\cdot\|)$. For any $A \in \mathcal{D}_f$,

$$\begin{aligned} &\left\| \frac{I_s(A) - I_{s_0}(A)}{s - s_0} - V_{s_0}(A) \right\| \\ &\le \int dt\, \omega_\gamma(t) \int_{[0,t]} du \left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} \right) \circ \tau^u_{\Phi(s_0)}(A) - \tau^{t-u}_{\Phi(s_0)} \circ \left(\delta_{\dot{\Phi}(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \\ &\le \int dt\, \omega_\gamma(t) \int_{[0,t]} du \left(\begin{array}{c} \left\| \left(\tau^{t-u}_{\Phi(s)} - \tau^{t-u}_{\Phi(s_0)}\right) \circ \left(\delta_{\dot{\Phi}(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \\ + \left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} - \left(\delta_{\dot{\Phi}(s_0)}\right) \right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \end{array} \right). \end{aligned} \tag{3.9}$$

Here and after, $\int_{[0,t]} du$ always indicates Lebesgue integral (i.e. without sign) over the measurable set $[0, t]$. From *9.* of Lemma 2.1, for each t, u, we have

$$\lim_{s \to s_0} \left\| \left(\tau^{t-u}_{\Phi(s)} - \tau^{t-u}_{\Phi(s_0)}\right) \circ \left(\delta_{\dot{\Phi}(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| = 0, \quad A \in \mathcal{D}_f. \tag{3.10}$$

By *4.* of Lemma 2.1, for each t, u, we have

$$\begin{aligned} &\lim_{s \to s_0} \left\| \tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} - \left(\delta_{\dot{\Phi}(s_0)}\right) \right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \\ &\le \limsup_{s \to s_0} b(|s - s_0|) C^{(1)}_{f_2,\zeta} \left\| \tau^u_{\Phi(s_0)}(A) \right\|_{f_2} = 0, \quad A \in \mathcal{D}_f. \end{aligned} \tag{3.11}$$

Here we used $\tau^u_{\Phi(s_0)}(A) \in \mathcal{D}_{f_1} \subset \mathcal{D}_{f_2}$ which follows from Lemma 2.1, *1.*, *2.* Furthermore, from *2.*, *4.* of Lemma 2.1, for $A \in \mathcal{D}_f$,

$$\begin{aligned} &\left\| \left(\tau^{t-u}_{\Phi(s)} - \tau^{t-u}_{\Phi(s_0)}\right) \circ \left(\delta_{\dot{\Phi}(s_0)}\right) \circ \tau^u_{\Phi(s_0)}(A) \right\| \le 2C^{(1)}_{f_2,\zeta} \left\| \tau^u_{\Phi(s_0)}(A) \right\|_{f_2} \\ &\le 2C^{(1)}_{f_2,\zeta} \left(1 + \sup_N \frac{f_1(N)}{f_2(N)} \right) \left\| \tau^u_{\Phi(s_0)}(A) \right\|_{f_1} \le 2C^{(1)}_{f_2,\zeta} b_{f,f_1}(|u|) \left(1 + \sup_N \frac{f_1(N)}{f_2(N)} \right) \|A\|_f . \end{aligned} \tag{3.12}$$

Note that from $0 < \beta_3 < \beta_2 < 1$, we have $\sup_N \frac{f_1(N)}{f_2(N)} < \infty$. Similarly, from *2., 4.* of Lemma 2.1,

$$\begin{aligned}&\left\|\tau^{t-u}_{\Phi(s)} \circ \left(\frac{\delta_{\Phi(s)} - \delta_{\Phi(s_0)}}{s - s_0} - \left(\delta_{\dot{\Phi}(s_0)}\right)\right) \circ \tau^u_{\Phi(s_0)}(A)\right\| \\ &\leq b(1) C^{(1)}_{f_2,\zeta} b_{f,f_1}(|u|) \left(1 + \sup_N \frac{f_1(N)}{f_2(N)}\right) \|A\|_f \,.\end{aligned} \tag{3.13}$$

Combining this (2.2) in *2.* of Lemma 2.1, from Lebesgue's convergence theorem, we obtain

$$\lim_{s\to s_0} \left\| \frac{I_s(A) - I_{s_0}(A)}{s - s_0} - V_{s_0}(A)\right\| = 0, \quad A \in \mathcal{D}_f. \quad \square \tag{3.14}$$

Lemma 3.4. *For any $A, B \in \mathcal{D}_f$ and $s \in [0,1]$, $A, B^*, B^* I_s(A)$ belong to $\mathcal{D}_\zeta$ and we have*

$$\begin{aligned}&\dot{\varphi}_s\left(B^* I_s(A)\right) + \int dt\, \omega_\gamma(t) \int_0^t du \varphi_s \left(B^* \tau^{t-u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s)} \circ \tau^u_{\Phi(s)}(A)\right) \\ &= \dot{\varphi}_s(B^*)\varphi_s(A) + \varphi_s(B^*)\dot{\varphi}_s(A).\end{aligned} \tag{3.15}$$

Proof. For any $A, B \in \mathcal{D}_f \subset \mathcal{D}_\zeta$ and $s_0 \in [0,1]$, $B^* I_{s_0}(A)$ belongs to $\mathcal{D}_{f_1} \subset \mathcal{D}_\zeta$ (the inclusion *1.* of Lemma 2.1) because of *8.,* of Lemma 2.1 and Lemma B.1. Therefore, by (vii) of Assumption 1.2, $[0,1] \ni s \mapsto \varphi_s\left(B^* I_{s_0}(A)\right) \in \mathbb{C}$ is differentiable. For any $s, s_0 \in [0,1]$ with $s \neq s_0$, we have

$$\begin{aligned}&\frac{1}{s - s_0}\left(\varphi_s\left(B^* I_s(A)\right) - \varphi_{s_0}\left(B^* I_{s_0}(A)\right)\right) - \varphi_{s_0}\left(B^* V_{s_0}(A)\right) - \dot{\varphi}_{s_0}\left(B^* I_{s_0}(A)\right) \\ &= \varphi_s\left(B^*\left(\frac{I_s(A) - I_{s_0}(A)}{s - s_0} - V_{s_0}(A)\right)\right) - \dot{\varphi_{s_0}}\left(B^* I_{s_0}(A)\right) \\ &+ \frac{1}{s - s_0}\left(\varphi_s - \varphi_{s_0}\right)\left(B^* I_{s_0}(A)\right) + \left(\varphi_s - \varphi_{s_0}\right)\left(B^* V_{s_0}(A)\right).\end{aligned} \tag{3.16}$$

The right hand side goes to 0 as $s \to s_0$, because of Lemma 3.3 and the differentiability of $[0,1] \ni s \mapsto \varphi_s\left(B^* I_{s_0}(A)\right) \in \mathbb{C}$. On the other hand, the first part of the left hand side of (3.16) is

$$\frac{1}{s - s_0}\left(\varphi_s\left(B^* I_s(A)\right) - \varphi_{s_0}\left(B^* I_{s_0}(A)\right)\right) = \frac{1}{s - s_0}\left(\varphi_s\left(B^*\right)\varphi_s\left(A\right) - \varphi_{s_0}\left(B^*\right)\varphi_{s_0}\left(A\right)\right), \tag{3.17}$$

because of Lemma 3.2 and converges to

$$\dot{\varphi_{s_0}}(B^*)\varphi_{s_0}(A) + \varphi_{s_0}(B^*)\dot{\varphi_{s_0}}(A), \tag{3.18}$$

as $s \to s_0$. Hence we obtain (3.15). $\square$

For each $s \in [0,1]$, we introduce the left ideal $\mathcal{L}_s$ of $\mathcal{A}$ by

$$\mathcal{L}_s := \{A \in \mathcal{A} \mid \varphi_s(A^*A) = 0\}. \tag{3.19}$$

Lemma 3.5. *For any $A \in \mathcal{D}_f$ and $s \in [0,1]$, $I_s(A) - \varphi_s(A)\mathbb{I}$ belongs to $\mathcal{L}_s \cap \mathcal{L}_s^* \cap \mathcal{D}_{f_1}$.*

Proof. Let $A \in \mathcal{D}_f$. Let $(\mathcal{H}_s, \pi_s, \Omega_s)$ be the GNS triple of φ_s. That $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{D}_{f_1}$ is Lemma 2.1 *8*. To show $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{L}_s \cap \mathcal{L}_s^*$, recall Lemma 3.1. From the latter Lemma, we obtain

$$\pi_s\left(I_s(A) - \varphi_s(A)\right)\Omega_s = \pi_s\left(I_s(A^*) - \varphi_s(A^*)\right)\Omega_s = 0, \tag{3.20}$$

which means $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{L}_s \cap \mathcal{L}_s^*$, because $I_s(A)^* = I_s(A^*)$. $\square$

Lemma 3.6. *For any $A \in \mathcal{L}_s \cap \mathcal{D}_{f_1}$, there is a positive sequence $u_{N,A} \in \mathcal{A}_{\Lambda_N}$, $N \in \mathbb{N}$ with $\|u_{N,A}\| \le 1$ such that*

$$\|A(1 - u_{N,A})\|_g \to 0, \tag{3.21}$$

and

$$\lim_{N\to\infty} \varphi_s(u_{N,A}) = 0, \tag{3.22}$$

and

$$\operatorname{dist}\left(u_{N,A}, \mathcal{L}_s\right) := \inf_{x \in \mathcal{L}_s} \|x - u_{N,A}\| \to 0, \quad N \to \infty. \tag{3.23}$$

Proof. Choose $\beta_4 < \beta' < \beta_2$ and set $h(t) := e^{t^{\beta'}}$. Then we have

$$\lim_{N\to\infty} \frac{1}{g(N)\sqrt{h(N)}} = 0, \quad \lim_{N\to\infty} h(N) f_1(N) = 0. \tag{3.24}$$

Let $A \in \mathcal{L}_s \cap \mathcal{D}_{f_1}$. Set

$$u_{N,A} := \left(1 + h(N)\mathbb{E}_N(A^*A)\right)^{-1} h(N)\mathbb{E}_N(A^*A). \tag{3.25}$$

Clearly, $\|u_{N,A}\| \le 1$, and $0 \le u_{N,A} \le 1$. Then we have

$$\begin{aligned}
&\left\| u_{N,A} - \left(1 + h(N)(A^*A)\right)^{-1} h(N)(A^*A) \right\| \\
&= \left\| \left(1 + h(N)\mathbb{E}_N(A^*A)\right)^{-1} h(N)\mathbb{E}_N(A^*A) - \left(1 + h(N)(A^*A)\right)^{-1} h(N)(A^*A) \right\|
\end{aligned}$$

$$= \left\| (1 + h(N)\mathbb{E}_N(A^*A))^{-1} - (1 + h(N)(A^*A))^{-1} \right\| \tag{3.26}$$
$$= \left\| (1 + h(N)\mathbb{E}_N(A^*A))^{-1} \left(h(N)\left(A^*A - \mathbb{E}_N(A^*A)\right)\right)(1 + h(N)(A^*A))^{-1} \right\|$$
$$\le h(N) f_1(N) \left\| A^*A \right\|_{f_1} \to 0, \quad N \to \infty,$$

from (3.24). As $(1 + h(N)(A^*A))^{-1} h(N)(A^*A) \in \mathcal{L}_s$, we obtain (3.22), (3.23). We also have

$$\begin{aligned}
&\|A(1 - u_{N,A})\|^2 \le \|(1 - u_{N,A})(A^*A - \mathbb{E}_N(A^*A))(1 - u_{N,A})\| \\
&+ \|(1 - u_{N,A})(\mathbb{E}_N(A^*A))(1 - u_{N,A})\| \\
&\le \|A^*A\|_{f_1} f_1(N) + \left\|(1 + h(N)\mathbb{E}_N(A^*A))^{-1}\mathbb{E}_N(A^*A)(1 + h(N)\mathbb{E}_N(A^*A))^{-1}\right\| \\
&= \|A^*A\|_{f_1} f_1(N) \\
&+ \frac{1}{h(N)} \left\|(1 + h(N)\mathbb{E}_N(A^*A))^{-1}h(N)\mathbb{E}_N(A^*A)(1 + h(N)\mathbb{E}_N(A^*A))^{-1}\right\| \\
&\le \|A^*A\|_{f_1} f_1(N) + \frac{1}{h(N)} =: \varepsilon_N^2.
\end{aligned} \tag{3.27}$$

For $M > N$, we have

$$\begin{aligned}
&\frac{\|A(1 - u_{N,A}) - \mathbb{E}_M(A(1 - u_{N,A}))\|}{g(M)} = \frac{\|(A - \mathbb{E}_M(A))(1 - u_{N,A})\|}{g(M)} \\
&\le \|A\|_{f_1} \sup_{M>N} \left(\frac{f_1(M)}{g(M)}\right) =: \varepsilon'_N \to 0, \quad N \to \infty.
\end{aligned} \tag{3.28}$$

For $M \le N$, we have

$$\begin{aligned}
\frac{\|A(1 - u_{N,A}) - \mathbb{E}_M(A(1 - u_{N,A}))\|}{g(M)} &\le \frac{2\|A(1 - u_{N,A})\|}{g(N)} \frac{g(N)}{g(M)} \le \frac{2\|A(1 - u_{N,A})\|}{g(N)} \\
&\le \frac{2\varepsilon_N}{g(N)} \to 0, \quad N \to \infty,
\end{aligned} \tag{3.29}$$

from (3.24) and $0 < \beta_4 < \beta_2 < 1$. Hence we obtain,

$$\|A(1 - u_{N,A})\|_g \to 0, \tag{3.30}$$

proving the Lemma. □

Now we can prove Proposition 2.2.

Proof of Proposition 2.2. Fix $A \in \mathcal{D}_f$, and $s \in [0, 1]$. By Lemma 3.5, $I_s(A) - \varphi_s(A)\mathbb{I} \in \mathcal{L}_s \cap \mathcal{L}_s^* \cap \mathcal{D}_{f_1}$. Applying Lemma 3.6 to $(I_s(A) - \varphi_s(A)\mathbb{I})^* \in \mathcal{L}_s \cap \mathcal{L}_s^* \cap \mathcal{D}_{f_1}$ we obtain a sequence $u_N \in \mathcal{A}_{\Lambda_N}$, $N \in \mathbb{N}$ such that $\|u_N\| \le 1$

$$\left\|(1-u_N)^* \left(I_s(A)-\varphi_s(A)\mathbb{I}\right)\right\|_g = \left\|\left(I_s(A)-\varphi_s(A)\mathbb{I}\right)^* (1-u_N)\right\|_g \to 0, \tag{3.31}$$

$$\operatorname{dist}(u_N, \mathcal{L}_s) \to 0, \tag{3.32}$$

as $N \to 0$. Applying Lemma 3.4 to $u_N \in \mathcal{D}_f$ and $A \in \mathcal{D}_f$, we have

$$\begin{aligned}
&\dot{\varphi}_s \left(u_N^* \left(I_s(A)-\varphi_s(A)\mathbb{I}\right)\right) \\
&= -\int dt\, \omega_\gamma(t) \int_0^t du \varphi_s \left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right) + \varphi_s(u_N^*)\dot{\varphi}_s(A).
\end{aligned} \tag{3.33}$$

By (3.32), we have $\lim_{N\to\infty} \varphi_s \left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right) = 0$. On the other hand, from *2.*, and *4.*, of Lemma 2.1, since $\|u_N\| \le 1$, we have, as in (3.12), the bound

$$\left|\varphi_s \left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right)\right| \le C_{f_2,\zeta}^{(1)} b_{f,f_1}(|u|) \left(1+\sup_N \frac{f_1(N)}{f_2(N)}\right) \|A\|_f < \infty. \tag{3.34}$$

From *2.* of Lemma 2.1,

$$\int dt\, \omega_\gamma(t) \int_{[0,t]} du b_{f,f_1}(|u|) < \infty. \tag{3.35}$$

Therefore, by Lebesgue's convergence theorem, we have

$$\lim_{N\to\infty} \int dt\, \omega_\gamma(t) \int_0^t du \varphi_s \left(u_N^* \tau_{\Phi(s)}^{t-u} \circ \delta_{\dot{\Phi}(s)} \circ \tau_{\Phi(s)}^{u}(A)\right) = 0. \tag{3.36}$$

We also have $\lim_{N\to\infty} \varphi_s(u_N^*)\dot{\varphi}_s(A) = 0$, from (3.32). Therefore, the right hand side of (3.33) goes to 0 as $N \to \infty$. The left hand side of (3.33) goes to $\dot{\varphi}_s \left(\left(I_s(A)-\varphi_s(A)\mathbb{I}\right)\right)$ as $N \to \infty$, because of the continuity (1.10) of $\dot{\varphi}_s$ and (3.31). Clearly, $\dot{\varphi}_s(\mathbb{I}) = 0$. Therefore, we obtain $\dot{\varphi}_s \left(I_s(A)\right) = 0$. □

4. Technical lemmas

In this Section, we prove various lemmas used in this paper. We assume (i), (ii), (iii) of Assumption 1.2 throughout this section. For $t \in \mathbb{R}$, $[t]$ indicates the largest integer less than or equal to t.

4.1. Properties of $\tau_{\Phi(s)}$

First we recall several facts from [1] and [11]. Define positive functions $F(r)$ and $F_1(r)$ on $\mathbb{R}_{\ge 0}$ by $F(r) := (1+r)^{-(\nu+1)}$, $F_1(r) := (1+r)^{-(\nu+1)}e^{-r}$. For a path of interactions satisfying Assumption 1.2, there exist positive constants C_1', v satisfying the following Lieb-Robinson bound: For any $X, Y \in \mathfrak{S}_{\mathbb{Z}^\nu}$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$, $s \in [0,1]$ and $t \in \mathbb{R}$, we have

$$\left\| \left[\tau^t_{\Phi(s)}(A), B \right] \right\|, \quad \left\| \left[\tau^t_{\Phi(s),\Lambda}(A), B \right] \right\| \le C_1' e^{v|t|} \sum_{x \in X, y \in Y} F_1(d(x,y)) \|A\| \|B\| . \tag{4.1}$$

We fix the constant v and call it the Lieb-Robinson velocity. From this and Corollary 4.4. of [11] (Proposition A.1) we obtain the following.

Lemma 4.1. *There is a positive constant $C_1 > 0$ such that*

$$\begin{aligned} &\left\| \tau^t_{\Phi(s),\Lambda}(A) - \mathbb{E}_N \left(\tau^t_{\Phi(s),\Lambda}(A) \right) \right\|, \ \left\| \tau^t_{\Phi(s)}(A) - \mathbb{E}_N \left(\tau^t_{\Phi(s)}(A) \right) \right\| \\ &\le C_1 \left| \Lambda_M \right| e^{v|t| - (N-M)} \|A\| , \end{aligned} \tag{4.2}$$

for any $M, N \in \mathbb{N}$ with $M \le N$, $A \in \mathcal{A}_{\Lambda_M}$ and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$.

We also have the following (see Corollary 3.6 (3.80) of [11]).

Lemma 4.2. *There is a constant $C_4 > 0$ such that*

$$\begin{aligned} &\sup_{s \in [0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(B) - \tau^{-u}_{\Phi(s)}(B) \right\| \le C_4 \left| \Lambda_M \right| |u| e^{|u|v - (n-M)} \|B\| , \\ &\quad n \ge M, \quad u \in \mathbb{R}, \quad B \in \mathcal{A}_{\Lambda_M}. \end{aligned} \tag{4.3}$$

It is standard to derive the following from Lemma 4.2 (cf. [4]).

Lemma 4.3. *For any $A \in \mathcal{A}$,*

$$\sup_{s \in [0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right\| \to 0, \tag{4.4}$$

uniformly in compact $u \in \mathbb{R}$. In particular, for each $A \in \mathcal{A}$, $\mathbb{R} \times [0,1] \ni (u,s) \to \tau^{-u}_{\Phi(s)}(A) \in \mathcal{A}$ is continuous with respect to the norm $\|\cdot\|$.

Lemma 4.4. *Suppose $f_1, f_2 : (0,\infty) \to (0,\infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_i(t) = 0$, for $i = 1,2$. Suppose that we have*

$$\lim_{N \to \infty} \left(\frac{|\Lambda_{[\frac{N}{2}]}| e^{-(N - [\frac{N}{2}])}}{f_2(N)} \right) = 0, \tag{4.5}$$

and

$$\lim_{N\to\infty} \frac{f_1\left(\left[\frac{N}{2}\right]\right)}{f_2(N)} = 0. \tag{4.6}$$

Then

$$\sup_{s\in[0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right\|_{f_2} \to 0, \quad A \in \mathcal{D}_{f_1}, \tag{4.7}$$

uniformly in compact $u \in \mathbb{R}$. In particular, for each $A \in \mathcal{D}_{f_1}$, $\mathbb{R} \times [0,1] \ni (u,s) \to \tau^{-u}_{\Phi(s)}(A) \in \mathcal{D}_{f_2}$ is continuous with respect to the norm $\|\cdot\|_{f_2}$.

Proof. Let $A \in \mathcal{D}_{f_1}$. From Lemma 4.3, we have

$$\sup_{s\in[0,1]} \left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right\| \to 0. \tag{4.8}$$

Applying Lemma 4.2, for $N \le [\frac{n}{2}]$, we have

$$\begin{aligned}
&\left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right) \right\| \\
&\le \left\| \tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{n}{2}]}(A) \right) - \tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{n}{2}]}(A) \right) \right. \\
&\left. -\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{n}{2}]}(A) \right) - \tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{n}{2}]}(A) \right) \right) \right\| \\
&+ 4\left\| \mathbb{E}_{[\frac{n}{2}]}(A) - A \right\| \\
&\le 2C_4 \left| \Lambda_{[\frac{n}{2}]} \right| |u| e^{|u|v-(n-[\frac{n}{2}])} \|A\| + 4f_1\left(\left[\frac{n}{2}\right] \right) \|A\|_{f_1}
\end{aligned} \tag{4.9}$$

On the other hand, from Lemma 4.1 $N \ge \left[\frac{n}{2}\right]$,

$$\begin{aligned}
&\left\| \tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) - \mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) \right) \right\| \le C_1\|A\||\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])}, \\
&\left\| \tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) - \mathbb{E}_N\left(\tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) \right) \right\| \le C_1\|A\||\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])}.
\end{aligned} \tag{4.10}$$

Therefore, for $N \ge \left[\frac{n}{2}\right]$, we have

$$\begin{aligned}
&\left\| \tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}(A) - \tau^{-u}_{\Phi(s)}(A) \right) \right\| \\
&\le \left\| \tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) - \tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) \right. \\
&\left. -\mathbb{E}_N\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) - \tau^{-u}_{\Phi(s)}\left(\mathbb{E}_{[\frac{N}{2}]}(A) \right) \right) \right\| \\
&+ 4\left\| \mathbb{E}_{[\frac{N}{2}]}(A) - A \right\| \\
&\le 2C_1\|A\||\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])} + 4f_1\left(\left[\frac{N}{2}\right] \right) \|A\|_{f_1}
\end{aligned} \tag{4.11}$$

Hence we obtain

$$
\begin{aligned}
&\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|_{f_2}\\
&\leq \max\left\{\begin{array}{c}
2C_4|u|e^{|u|v}\left\|A\right\|\dfrac{\left|\Lambda_{[\frac{n}{2}]}\right|e^{-(n-[\frac{n}{2}])}}{f_2\left(\left[\frac{n}{2}\right]\right)}+4\dfrac{f_1\left(\left[\frac{n}{2}\right]\right)}{f_2\left(\left[\frac{n}{2}\right]\right)}\left\|A\right\|_{f_1},\\
2C_1\|A\|\sup_{N\geq[\frac{n}{2}]}\left(\dfrac{|\Lambda_{[\frac{N}{2}]}|e^{v|u|-(N-[\frac{N}{2}])}}{f_2(N)}\right)+\sup_{N\geq[\frac{n}{2}]}\left(\dfrac{4f_1\left(\left[\frac{N}{2}\right]\right)\left\|A\right\|_{f_1}}{f_2(N)}\right)
\end{array}\right\}\\
&\quad+\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|,
\end{aligned}
\tag{4.12}
$$

and $\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|_{f_2}$ converges to 0 as $n\to\infty$, uniformly in compact u. $\square$

Lemma 4.5. *Let $f, f_1 : (0,\infty)\to(0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t)=0$. Suppose that*

$$
\begin{aligned}
&\int\limits_{4v|t|\geq 1} dt\omega_\gamma(t)\frac{2|t|}{f_1(4v|t|)}<\infty,\\
&\sup_{N\in\mathbb{N}}\left(\frac{f(N-\left[\frac{N}{2}\right])}{f_1(N)}\right)<\infty,\\
&\sup_{N\in\mathbb{N}}\left(\frac{|\Lambda_N|e^{-\frac{\left[\frac{N}{2}\right]}{2}}}{f_1(N)}\right)<\infty.
\end{aligned}
\tag{4.13}
$$

Then $\tau^t_{\Phi(s)}(\mathcal{D}_f)\subset\mathcal{D}_{f_1}$ and there is a non-negative non-decreasing function on $\mathbb{R}_{\geq 0}$, $b_{f,f_1}(t)$ such that

$$
\int dt\ \omega_\gamma(t)\ |t|\cdot b_{f,f_1}(|t|)<\infty. \tag{4.14}
$$

$$
\sup_{n\in\mathbb{N}}\sup_{s\in[0,1]}\left\|\tau^t_{\Phi_n(s)}(A)\right\|_{f_1},\ \sup_{s\in[0,1]}\left\|\tau^t_{\Phi(s)}(A)\right\|_{f_1}\leq b_{f,f_1}(|t|)\left\|A\right\|_f,\quad A\in\mathcal{D}_f. \tag{4.15}
$$

Proof. Let $A\in\mathcal{D}_f$. We have to estimate

$$
\frac{\left\|\tau^t_{\Phi(s)}(A)-\mathbb{E}_N\left(\tau^t_{\Phi(s)}(A)\right)\right\|}{f_1(N)},\quad N\in\mathbb{N}. \tag{4.16}
$$

From Lemma 4.1 for $A\in\mathcal{D}_f$, $N,k\in\mathbb{N}$ with $k<N$, we obtain

$$
\begin{aligned}
&\left\|\tau^t_{\Phi(s)}(A)-\mathbb{E}_N\left(\tau^t_{\Phi(s)}(A)\right)\right\|\\
&\leq\left\|\tau^t_{\Phi(s)}\left(\mathbb{E}_k(A)\right)-\mathbb{E}_N\left(\tau^t_{\Phi(s)}\left(\mathbb{E}_k(A)\right)\right)\right\|+2\left\|A-\left(\mathbb{E}_k(A)\right)\right\|
\end{aligned}
\tag{4.17}
$$

$$\leq 2\left\|A\right\|_f f(k) + C_1\|A\||\Lambda_k|e^{v|t|-(N-k)}.$$

For $N \in \mathbb{N}$ with $4v\,|t| \leq N$, we use this bound with $k := N - \left[\frac{N}{2}\right]$ to estimate (4.16). Then we have

$$\begin{aligned}\left\|\tau^t_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(A)\right)\right\| &\leq 2\left\|A\right\|_f\left(f(N - \left[\frac{N}{2}\right])\right) + C_1\|A\||\Lambda_N|e^{v|t|-\left[\frac{N}{2}\right]} \\ &\leq 2\left\|A\right\|_f\left(f(N - \left[\frac{N}{2}\right])\right) + C_1\|A\||\Lambda_N|e^{-\frac{\left[\frac{N}{2}\right]}{2}+\frac{1}{2}}.\end{aligned} \tag{4.18}$$

On the other hand, for $N \in \mathbb{N}$ with $4v\,|t| > N$, we simply have

$$\left\|\tau^t_{\Phi(s)}(A) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(A)\right)\right\| \leq 2\left\|A\right\|. \tag{4.19}$$

Hence we obtain

$$\begin{aligned}&\left\|\tau^t_{\Phi(s)}(A)\right\|_{f_1} \\ &\leq \left(1 + \max\left\{\begin{matrix}2\sup_{N\in\mathbb{N}}\left(\frac{\left(f(N-\left[\frac{N}{2}\right])\right)}{f_1(N)}\right) + C_1\sup_{N\in\mathbb{N}}\left(\frac{|\Lambda_N|e^{-\frac{\left[\frac{N}{2}\right]}{2}+\frac{1}{2}}}{f_1(N)}\right), \\ \frac{2}{f_1\left((4v\,|t|)\right)}\mathbb{I}_{4v|t|\geq 1}\end{matrix}\right\}\right)\left\|A\right\|_f \\ &=: b_{f,f_1}(t)\left\|A\right\|_f,\end{aligned} \tag{4.20}$$

for $A \in \mathcal{D}_f$ and $t \in \mathbb{R}$, $s \in [0,1]$. Here $\mathbb{I}_{4v|t|\geq 1}$ is the characteristic function for $\{t \in \mathbb{R} \mid 4v\,|t| \geq 1\}$. From the assumptions and (1.16), $b_{f,f_1}(t)$ satisfies the required condition. The inequality for $\tau^t_{\Phi_n(s)}(A)$ can be proven in the same way. $\square$

Lemma 4.6. *Let $f, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_1(t) = 0$. Suppose that*

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\frac{f(N-\left[\frac{N}{2}\right])}{f_1(N)} < \infty, \\ &\sup_{N\in\mathbb{N}}\frac{|\Lambda_N|\,e^{-\frac{\left[\frac{N}{2}\right]}{2}}}{f_1(N)} < \infty, \\ &\sup_{N\in\mathbb{N}}\frac{W_\gamma\left(\frac{\left[\frac{N}{2}\right]}{2v}\right)}{f_1(N)} < \infty.\end{aligned} \tag{4.21}$$

(Recall (1.17).) For $s \in [0,1]$ and $A \in \mathcal{A}$, we set

$$I_s(A) := \int dt\;\omega_\gamma(t)\tau^t_{\Phi(s)}(A). \tag{4.22}$$

The integral can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$. *Then for any* $A \in \mathcal{D}_f$ *and* $s \in [0,1]$, *we have* $I_s(A) \in \mathcal{D}_{f_1}$.

Proof. That the integral can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$ is from the continuity of $\mathbb{R} \ni t \to \tau^t_{\Phi(s)}(A) \in \mathcal{A}$, Lemma 4.3 and $\omega_\gamma \in L^1(\mathbb{R})$.

From (4.2), we obtain

$$\|\tau^t_{\Phi(s)}(\mathbb{E}_k(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_k(A))\right)\| \le C_1 |\Lambda_k| e^{v|t|-(N-k)} \|A\|, \tag{4.23}$$

for any $A \in \mathcal{D}_f$, $s \in [0,1]$, $t \in \mathbb{R}$, $N, k \in \mathbb{N}$, with $k \le N$.

For any $A \in \mathcal{D}_f$, $s \in [0,1]$, $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\|I_s(A) - \mathbb{E}_N(I_s(A))\| \\
&\le \left\|I_s\left(\mathbb{E}_{N-[\frac{N}{2}]}(A)\right) - \mathbb{E}_N\left(I_s\left(\mathbb{E}_{N-[\frac{N}{2}]}(A)\right)\right)\right\| + 2\left\|A - \mathbb{E}_{N-[\frac{N}{2}]}(A)\right\| \\
&\le \int_{|t|\le\frac{[\frac{N}{2}]}{2v}} dt\omega_\gamma(t)\|\tau^t_{\Phi(s)}(\mathbb{E}_{N-[\frac{N}{2}]}(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_{N-[\frac{N}{2}]}(A))\right)\| \\
&+ \int_{|t|\ge\frac{[\frac{N}{2}]}{2v}} dt\omega_\gamma(t)\|\tau^t_{\Phi(s)}(\mathbb{E}_{N-[\frac{N}{2}]}(A)) - \mathbb{E}_N\left(\tau^t_{\Phi(s)}(\mathbb{E}_{N-[\frac{N}{2}]}(A))\right)\| + 2\|A\|_f f(N - \left[\frac{N}{2}\right]) \\
&\le \int_{|t|\le\frac{[\frac{N}{2}]}{2v}} dt\omega_\gamma(t) C_1 |\Lambda_N| e^{v|t|-[\frac{N}{2}]}\|A\| + \int_{|t|\ge\frac{[\frac{N}{2}]}{2v}} dt\omega_\gamma(t) 2\|A\| + 2\|A\|_f f(N - \left[\frac{N}{2}\right]) \\
&\le C_1 |\Lambda_N| e^{-\frac{[\frac{N}{2}]}{2}}\|A\| + 4\|A\| W_\gamma\left(\frac{\left[\frac{N}{2}\right]}{2v}\right) + 2\|A\|_f f(N - \left[\frac{N}{2}\right]).
\end{aligned} \tag{4.24}$$

For the first and the fourth inequality, we used (1.16). We used (4.23), with $k = N - [\frac{N}{2}]$, for the third inequality.

Hence we obtain

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \frac{\|I_s(A) - \mathbb{E}_N(I_s(A))\|}{f_1(N)} \\
&\le C_1\|A\| \sup_{N\in\mathbb{N}} \frac{|\Lambda_N| e^{-\frac{[\frac{N}{2}]}{2}}}{f_1(N)} + 4\|A\| \sup_{N\in\mathbb{N}} \frac{W_\gamma\left(\frac{[\frac{N}{2}]}{2v}\right)}{f_1(N)} + 2\|A\|_f \sup_{N\in\mathbb{N}} \frac{f(N - [\frac{N}{2}])}{f_1(N)} < \infty,
\end{aligned} \tag{4.25}$$

for any $A \in \mathcal{D}_f$ and $s \in [0,1]$. Hence we obtain $I_s(\mathcal{D}_f) \subset \mathcal{D}_{f_1}$, for any $s \in [0,1]$. □

4.2. Estimates on α_s

In the following, we prove estimates on quasi-locality of the automorphisms α_s and $\alpha_{s,\Lambda}$. To do this, we first recall a theorem from [1] on Lieb-Robinson bounds.

Define $\tilde{h}(x) = \frac{x}{\ln^2(x)}$ for $x > 1$. Define the weight function as:

$$h(x) = \begin{cases} \tilde{h}(e^2) & \text{if } 0 \le x \le e^2 \\ \tilde{h}(x) & \text{otherwise} \end{cases}.$$

The Lieb-Robinson bound for the automorphisms α_s is given as follows: there exists a constant $C_2 > 0$, $\eta_1 > 0$, $\tilde{a} > 0$ satisfying the following: setting $\hat{h}(x) := \eta_1 h(\tilde{a}x)$, we have

$$\|[\alpha_s(B), A]\|, \|[\alpha_{s,\Lambda_n}(B), A]\| \le \frac{C_2}{2}\|A\|\|B\||X|e^{-\hat{h}(d(X,Y))} \tag{4.26}$$

for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ with $X, Y \in \mathfrak{S}_{\mathbb{Z}^\nu}$, and $s \in [0,1]$. See Theorem 4.5 of [1] and Corollary 6.14 of [11]. (Note that in [1], Assumption 4.3 about a spectral gap is assumed but for the proof of (4.26), this assumption is not used.) From Corollary 3.6 (3.80) of [11], there is a constant $C_3 > 0$ such that

$$\begin{aligned} &\sup_{s\in[0,1]} \left\|\alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A)\right\| \le C_3 \left|\Lambda_M\right| e^{-\hat{h}(n-M)} \left\|A\right\|, \\ &\quad n \ge M, \quad M \in \mathbb{N}, \quad \text{and } A \in \mathcal{A}_{\Lambda_M}. \end{aligned} \tag{4.27}$$

From (4.26), we obtain the following.

Lemma 4.7. *For any $M, N \in \mathbb{N}$ with $M < N$, we have*

$$\|\alpha_s^{-1}(A) - \mathbb{E}_N(\alpha_s^{-1}(A))\| \le C_2 \left|\Lambda_M\right| \left\|A\right\| e^{-\hat{h}(N-M)}, \quad A \in \mathcal{A}_{\Lambda_M}. \tag{4.28}$$

Proof. If $A \in \mathcal{A}_{\Lambda_M}$ and $B \in \mathcal{A}_{\Lambda_N^c}$, then $B = \lim_{n\to\infty} B_n$ in norm for a sequence of local observables $B_n \in \mathcal{A}_{\Lambda_N^c} \cap \mathcal{A}_{\text{loc}}$ and:

$$\begin{aligned} \|[B, \alpha_s^{-1}(A)]\| = \|[\alpha_s(B), A]\| &\le \limsup_n \left(2\|A\|\|B - B_n\| + \frac{C_2}{2}\|A\||\Lambda_M|\|B_n\|e^{-\hat{h}(N-M)}\right) \\ &= \frac{C_2}{2}|\Lambda_M|\|A\|\|B\|e^{-\hat{h}(N-M)}. \end{aligned} \tag{4.29}$$

And so by Corollary 4.4. of [11] (Proposition A.1) we conclude (4.28). □

From this Lemma we immediately obtain the following:

Lemma 4.8. *Suppose $f : (0,\infty) \to (0,\infty)$ is a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. Suppose that for all $M \in \mathbb{N}$, we have*

$$\sup_n \frac{e^{-\hat{h}(n)}}{f(M+n)} < \infty \tag{4.30}$$

then $\alpha_s^{-1}(\mathcal{A}_{loc}) \subset D_f$.

Proof. Let $M \in \mathbb{N}$ and $A \in \mathcal{A}_{\Lambda_M}$. From (4.28), we have

$$\sup_{R\in\mathbb{N}} \left(\frac{\|\alpha_s^{-1}(A) - \mathbb{E}_{M+R}(\alpha_s^{-1}(A))\|}{f(M+R)} \right) \le \sup_{R\in\mathbb{N}} \left(C_2 \left|\Lambda_M\right| \frac{e^{-\hat{h}(R)}}{f(M+R)} \right) \|A\| < \infty. \tag{4.31}$$

Hence we obtain $\alpha_s^{-1}(A) \in \mathcal{D}_f$. □

Lemma 4.9. *Let $f_1, f_2 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f_i(t) = 0$, $i = 1, 2$. Suppose that*

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left(\frac{f_1\left(N - \left[\frac{N}{2}\right]\right)}{f_2(N)} \right) < \infty, \\ &\sup_{N\in\mathbb{N}} \left(\frac{e^{-\hat{h}(\left[\frac{N}{2}\right])} \left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|}{f_2(N)} \right) < \infty. \end{aligned} \tag{4.32}$$

Then we have $\alpha_s^{-1}(\mathcal{D}_{f_1}) \subset \mathcal{D}_{f_2}$, $\alpha_{s,\Lambda}^{-1}(\mathcal{D}_{f_1}) \subset \mathcal{D}_{f_2}$ for any $s \in [0,1]$, and $\Lambda \in \mathfrak{S}_{\mathbb{Z}^\nu}$. Furthermore we have the following inequalities:

$$\begin{aligned} &\sup_{s\in[0,1]} \left\|\alpha_s^{-1}(A)\right\|_{f_2}, \sup_{s\in[0,1]} \left\|\alpha_{s,\Lambda}^{-1}(A)\right\|_{f_2} \\ &\le \|A\|_{f_1} \left(1 + \sup_{N\in\mathbb{N}} \left(\frac{2f_1\left(N - \left[\frac{N}{2}\right]\right) + C_2 e^{-\hat{h}(\left[\frac{N}{2}\right])} \left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|}{f_2(N)} \right) \right), \end{aligned} \tag{4.33}$$

for any $A \in \mathcal{D}_{f_1}$.

Proof. This follows from the following inequality: for each $N \in \mathbb{N}$ and $A \in \mathcal{D}_{f_1}$,

$$\begin{aligned} &\left\|\alpha_s^{-1}(A) - \mathbb{E}_N\left(\alpha_s^{-1}(A)\right)\right\| \\ &\le \left\|\alpha_s^{-1}\left(A - \mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right) - \mathbb{E}_N\left(\alpha_s^{-1}\left(A - \mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right)\right)\right\| \\ &+ \left\|\alpha_s^{-1}\left(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right) - \mathbb{E}_N\left(\alpha_s^{-1}\left(\mathbb{E}_{N-\left[\frac{N}{2}\right]}(A)\right)\right)\right\| \\ &\le \|A\|_{f_1} \left(2f_1\left(N - \left[\frac{N}{2}\right]\right) + C_2 e^{-\hat{h}(\left[\frac{N}{2}\right])} \left|\Lambda_{N-\left[\frac{N}{2}\right]}\right| \right). \quad \square \end{aligned} \tag{4.34}$$

Lemma 4.10. *Suppose $f : (0,\infty) \to (0,\infty)$ is a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. Suppose that for all $M \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \sup_{N\geq n} \left(\frac{e^{-\hat{h}(N-M)}}{f(N)} \right) = 0. \tag{4.35}$$

Then we have

$$\sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_f \to 0, \quad A \in \mathcal{A}_{\mathrm{loc}}. \tag{4.36}$$

In particular, for each $A \in \mathcal{A}_{\mathrm{loc}}$, $\mathbb{R} \ni s \to \alpha_s^{-1}(A) \in \mathcal{D}_f$ is continuous with respect to the norm $\|\cdot\|_f$.

Proof. Let $A \in \mathcal{A}_{\Lambda_M}$. From (4.27), for $n \geq N \geq M$, we have

$$\sup_{s\in[0,1]} \frac{\left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) - \mathbb{E}_N \left(\alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right) \right\|}{f(N)} \leq 2C_3 \left|\Lambda_M\right| \frac{e^{-\hat{h}(n-M)}}{f(n)} \|A\| . \tag{4.37}$$

On the other hand, for $M \leq n \leq N$, from (4.28)

$$\begin{aligned} &\sup_{s\in[0,1]} \frac{\left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) - \mathbb{E}_N \left(\alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right) \right\|}{f(N)} \\ &= \sup_{s\in[0,1]} \frac{\left\| \alpha_s^{-1}(A) - \mathbb{E}_N \left(\alpha_s^{-1}(A) \right) \right\|}{f(N)} \\ &\leq C_2 \left|\Lambda_M\right| \|A\| \frac{e^{-\hat{h}(N-M)}}{f(N)} \leq C_2 \left|\Lambda_M\right| \|A\| \sup_{N\geq n} \left(\frac{e^{-\hat{h}(N-M)}}{f(N)} \right). \end{aligned} \tag{4.38}$$

Furthermore, for $n \geq M > N$, we have

$$\begin{aligned} &\sup_{s\in[0,1]} \frac{\left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) - \mathbb{E}_N \left(\alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right) \right\|}{f(N)} \\ &\leq 2C_3 \left|\Lambda_M\right| \frac{e^{-\hat{h}(n-M)}}{f(M)} \|A\| . \end{aligned} \tag{4.39}$$

Hence we obtain

$$\begin{aligned} &\sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_f \\ &\leq \|A\| \left(1 + \max \left\{ 2C_3 \left|\Lambda_M\right| \frac{e^{-\hat{h}(n-M)}}{f(n)},\ C_2 \left|\Lambda_M\right| \sup_{N\geq n} \left(\frac{e^{-\hat{h}(N-M)}}{f(N)} \right), \right. \right. \end{aligned} \tag{4.40}$$

$$2C_3 \left|\Lambda_M\right| \frac{e^{-\hat{h}(n-M)}}{f(M)} \Bigg\}\Bigg) \to 0, \quad n \to \infty. \quad \square$$

Lemma 4.11. *Let $f, f_0, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_0(t) = \lim_{t\to\infty} f_1(t) = 0$. Suppose that for all $M \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \sup_{N\geq n} \left(\frac{e^{-\hat{h}(N-M)}}{f(N)} \right) = 0. \tag{4.41}$$

Suppose that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \frac{f_1(N - \left[\frac{N}{2}\right])}{f(N)} < \infty, \\ &\sup_{N\in\mathbb{N}} \frac{e^{-\hat{h}(\left[\frac{N}{2}\right])}}{f(N)} \left|\Lambda_{N-\left[\frac{N}{2}\right]}\right| < \infty. \end{aligned} \tag{4.42}$$

Suppose that

$$\lim_{N\to\infty} \frac{f_0(N)}{f_1(N)} = 0. \tag{4.43}$$

Then we have $\alpha_s^{-1}(\mathcal{D}_{f_0}) \subset \mathcal{D}_f$ and

$$\sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_f \to 0, \quad A \in \mathcal{D}_{f_0}. \tag{4.44}$$

In particular, for each $A \in \mathcal{D}_{f_0}$, $[0,1] \ni s \to \alpha_s^{-1}(A) \in \mathcal{D}_f$ is continuous with respect to the norm $\|\cdot\|_f$.

Proof. As

$$\sup_{N\in\mathbb{N}} \frac{f_0(N)}{f_1(N)} < \infty, \tag{4.45}$$

we have $\mathcal{D}_{f_0} \subset \mathcal{D}_{f_1}$. By Lemma 4.9 with (f_1, f_2) replaced by (f_1, f), we get $\alpha_s^{-1}(\mathcal{D}_{f_1}) \subset \mathcal{D}_f$. Hence we have $\alpha_s^{-1}(\mathcal{D}_{f_0}) \subset \mathcal{D}_f$. For any $A \in \mathcal{D}_{f_0}$,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}(A) - \alpha_s^{-1}(A) \right\|_f \\ &= \limsup_{n\to\infty} \sup_{s\in[0,1]} \Big\| \alpha_{s,\Lambda_n}^{-1}\left(A - \mathbb{E}_M(A)\right) - \alpha_s^{-1}\left(A - \mathbb{E}_M(A)\right) + \alpha_{s,\Lambda_n}^{-1}\left(\mathbb{E}_M(A)\right) \\ &-\alpha_s^{-1}\left(\mathbb{E}_M(A)\right) \Big\|_f \\ &\leq \limsup_{n\to\infty} \sup_{s\in[0,1]} \left\| \alpha_{s,\Lambda_n}^{-1}\left(\mathbb{E}_M(A)\right) - \alpha_s^{-1}\left(\mathbb{E}_M(A)\right) \right\|_f \end{aligned} \tag{4.46}$$

$$
\begin{aligned}
&+2\left\|A-\mathbb{E}_M(A)\right\|_{f_1}\left(\sup_{N\in\mathbb{N}}\frac{\left(2f_1\left(N-\left[\frac{N}{2}\right]\right)+C_2e^{-\hat{h}(\left[\frac{N}{2}\right])}\left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|\right)}{f(N)}+1\right)\\
&=2\left\|A-\mathbb{E}_M(A)\right\|_{f_1}\left(\sup_{N\in\mathbb{N}}\frac{\left(2f_1\left(N-\left[\frac{N}{2}\right]\right)+C_2e^{-\hat{h}(\left[\frac{N}{2}\right])}\left|\Lambda_{N-\left[\frac{N}{2}\right]}\right|\right)}{f(N)}+1\right)\to 0,\\
&\quad M\to\infty.
\end{aligned}
$$

For the inequality, we used Lemma 4.9. For the last line we used Lemma 4.10. As we have $\lim_{M\to\infty}\left\|A-\mathbb{E}_M(A)\right\|_{f_1}=0$ by Lemma B.3 with (f,f_1) replaced by (f_0,f_1), we have proven the claim. □

4.3. Properties of $\delta_{\Phi(s)}$, $\delta_{\dot{\Phi}(s)}$

Lemma 4.12. *Let $f_2:(0,\infty)\to(0,\infty)$ be a continuous decreasing function such that*

$$
\sum_{k=2}^{\infty}k^{\nu}f_2(k-1)<\infty. \tag{4.47}
$$

Let $f_3:(0,\infty)\to(0,\infty)$ be continuous decreasing function with $\lim_{t\to\infty}f_3(t)=0$ such that

$$
\lim_{N\to\infty}\frac{\sum_{k=N-R}^{\infty}k^{\nu}f_2(k-1)}{f_3(N)}=0. \tag{4.48}
$$

Then $\mathcal{D}_{f_2}\subset D(\delta_{\Phi(s)})\cap D(\delta_{\dot{\Phi}(s)})$, and there is a constant $C^{(1)}_{f_2,f_3}>0$ such that

$$
\sup_{s\in[0,1]}\left\|\delta_{\Phi(s)}(A)\right\|_{f_3},\ \sup_{N\in\mathbb{N}}\sup_{s\in[0,1]}\left\|\delta_{\Phi_N(s)}(A)\right\|_{f_3}\le C^{(1)}_{f_2,f_3}\left\|A\right\|_{f_2} \tag{4.49}
$$

$$
\sup_{s\in[0,1]}\left\|\delta_{\dot{\Phi}(s)}(A)\right\|_{f_3},\ \sup_{N\in\mathbb{N}}\sup_{s\in[0,1]}\left\|\delta_{\dot{\Phi}_N(s)}(A)\right\|_{f_3}\le C^{(1)}_{f_2,f_3}\left\|A\right\|_{f_2} \tag{4.50}
$$

for all $A\in\mathcal{D}_{f_2}$, and $\varepsilon>0$. If we assume Assumption 1.2 (iv) in addition, then we may also take $C^{(1)}_{f_2,f_3}>0$ so that

$$
\begin{aligned}
&\sup_{s,s_0\in[0,1],0<|s-s_0|\le\varepsilon}\left\|\delta_{\frac{\Phi(s)-\Phi(s_0)}{s-s_0}-\dot{\Phi}(s_0)}(A)\right\|_{f_3},\\
&\sup_{N\in\mathbb{N}}\sup_{s,s_0\in[0,1],0<|s-s_0|\le\varepsilon}\left\|\delta_{\frac{\Phi_N(s)-\Phi_N(s_0)}{s-s_0}-\dot{\Phi}_N(s_0)}(A)\right\|_{f_3}\\
&\le b(\varepsilon)C^{(1)}_{f_2,f_3}\left\|A\right\|_{f_2}.
\end{aligned}\tag{4.51}
$$

Proof. We prove (4.49). The proof of (4.50) and (4.51) are same. Note that there exists a constant $C_5 > 0$ such that

$$\left\|(H_{\Phi(s)})_{\Lambda_{N+R}}\right\| \le C_5 \left|\Lambda_{N+R}\right|, \quad s \in [0,1], \quad N \in \mathbb{N}. \tag{4.52}$$

Therefore, we have

$$\left\|\delta_{\Phi(s)}(A_N)\right\| = \left\|\left[(H_{\Phi(s)})_{\Lambda_{N+R}}, A_N\right]\right\| \le 2C_5 \left|\Lambda_{N+R}\right| \left\|A_N\right\|, \quad A_N \in \mathcal{A}_{\Lambda_N}, \ s \in [0,1]. \tag{4.53}$$

From this, for any $A \in \mathcal{D}_{f_2}$ and $N, M \in \mathbb{N}$ with $M > N$, we have

$$\begin{aligned}
&\left\|\delta_{\Phi(s)}\left(\mathbb{E}_N(A) - \mathbb{E}_M(A)\right)\right\| = \left\|\sum_{k=N+1}^{M} \delta_{\Phi(s)}\left(\mathbb{E}_k(A) - \mathbb{E}_{k-1}(A)\right)\right\| \\
&\le 2C_5 \sum_{k=N+1}^{M} \left|\Lambda_{k+R}\right| \left\|\mathbb{E}_k(A) - \mathbb{E}_{k-1}(A)\right\| \\
&\le 4C_5 \left\|A\right\|_{f_2} \sum_{k=N+1}^{M} \left|\Lambda_{k+R}\right| f_2(k-1).
\end{aligned} \tag{4.54}$$

Hence $\{\delta_{\Phi(s)}\left(\mathbb{E}_N(A)\right)\}_N$ with $A \in \mathcal{D}_{f_2}$ is a Cauchy sequence in $\mathcal{A}$, hence there exists a limit $\lim_{N\to\infty} \delta_{\Phi(s)}\left(\mathbb{E}_N(A)\right)$. On the other hand, $\mathbb{E}_N(A)$ converges to A in $\|\cdot\|$. By the closedness of $\delta_{\Phi(s)}$, $A \in \mathcal{D}_{f_2}$ belongs to the domain $D(\delta_{\Phi(s)})$ of $\delta_{\Phi(s)}$, and

$$\delta_{\Phi(s)}\left(A\right) = \lim_{N\to\infty} \delta_{\Phi(s)}\left(\mathbb{E}_N(A)\right). \tag{4.55}$$

Hence we get $\mathcal{D}_{f_2} \subset D(\delta_{\Phi(s)})$. From (4.54), we have

$$\begin{aligned}
&\left\|\delta_{\Phi(s)}\left(A\right)\right\| = \lim_{N\to\infty} \left\|\delta_{\Phi(s)}\left(\mathbb{E}_N(A)\right)\right\| = \lim_{N\to\infty} \left\|\delta_{\Phi(s)}\left(\mathbb{E}_N(A) - \mathbb{E}_1(A) + \mathbb{E}_1(A)\right)\right\| \\
&\le 4C_5 \left\|A\right\|_{f_2} \sum_{k=2}^{\infty} \left|\Lambda_{k+R}\right| f_2(k-1) + 2C_5 \left|\Lambda_{1+R}\right| \left\|A\right\| \\
&\le \left(4C_5 \sum_{k=2}^{\infty} \left|\Lambda_{k+R}\right| f_2(k-1) + 2C_5 \left|\Lambda_{1+R}\right|\right) \left\|A\right\|_{f_2},
\end{aligned} \tag{4.56}$$

for any $A \in \mathcal{D}_{f_2}$.

Next note that

$$\begin{aligned}
&\left\|\delta_{\Phi(s)}(A)-\mathbb{E}_N\left(\delta_{\Phi(s)}(A)\right)\right\|=\lim_{M\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_M(A)\right)-\mathbb{E}_N\left(\delta_{\Phi(s)}\left(\mathbb{E}_M(A)\right)\right)\right\|\\
&=\lim_{M\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)+\mathbb{E}_{N-R}(A)\right)-\mathbb{E}_N\left(\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)\right.\right.\right.\\
&\left.\left.\left.+\ \mathbb{E}_{N-R}(A)\right)\right)\right\|\\
&=\lim_{M\to\infty}\left\|\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)\right)-\mathbb{E}_N\left(\delta_{\Phi(s)}\left(\mathbb{E}_M(A)-\mathbb{E}_{N-R}(A)\right)\right)\right\|\\
&\le 8C_5\left\|A\right\|_{f_2}\sum_{k=N-R+1}^{\infty}\left|\Lambda_{k+R}\right|f_2(k-1),
\end{aligned}\tag{4.57}$$

for any $A\in\mathcal{D}_{f_2}$. Here, in the third line we used the fact that $\delta_{\Phi(s)}\left(\mathbb{E}_{N-R}(A)\right)\in\mathcal{A}_{\Lambda_N}$. In the fourth line, we used (4.54). Therefore, we obtain

$$\begin{aligned}
\left\|\delta_{\Phi(s)}(A)\right\|_{f_3}\le\Bigg(&8C_5\sup_{N\in\mathbb{N}}\frac{\sum_{k=N-R+1}^{\infty}\left|\Lambda_{k+R}\right|f_2(k-1)}{f_3(N)}\\
&+4C_5\sum_{k=2}^{\infty}\left|\Lambda_{k+R}\right|f_2(k-1)+2C_5\left|\Lambda_{1+R}\right|\Bigg)\left\|A\right\|_{f_2}.
\end{aligned}\tag{4.58}$$

The right hand side is finite from the assumptions. Hence we have shown (4.49). $\square$

Lemma 4.13. *Let $f,f_3:(0,\infty)\to(0,\infty)$ be continuous decreasing functions with $\lim_{t\to\infty}f(t)=\lim_{t\to\infty}f_3(t)=0$ such that*

$$\sum_{k=1}^{\infty}k^{\nu}\sqrt{f(k-1)}<\infty,\tag{4.59}$$

$$\lim_{N\to\infty}\frac{\sum_{k=N-R}^{\infty}k^{\nu}\sqrt{f(k-1)}}{\left(f_3(N)\right)^2}=0.\tag{4.60}$$

Then we have $\mathcal{D}_f\subset\mathcal{D}\left(\delta_{\dot{\Phi}(s)}\right)$, $\delta_{\dot{\Phi}(s)}\left(\mathcal{D}_f\right)\subset\mathcal{D}_{f_3}$ and

$$\lim_{N\to\infty}\sup_{s\in[0,1]}\left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A)\right\|_{f_3}=0,\quad A\in\mathcal{D}_f.\tag{4.61}$$

In particular, for each $A\in\mathcal{D}_f$, $[0,1]\ni s\to\delta_{\dot{\Phi}(s)}(A)\in\mathcal{D}_{f_3}$ is continuous with respect to the norm $\left\|\cdot\right\|_{f_3}$. The same statement, with $\delta_{\dot{\Phi}(s)}$ replaced by $\delta_{\Phi(s)}$ also holds.

Proof. We prove the claim for $\delta_{\dot{\Phi}(s)}$. The proof for $\delta_{\Phi(s)}$ is the same. Set $f_2(t):=\sqrt{f(t)}$ and $f_4(t):=(f_3(t))^2$. As we have $\sup_N\frac{f(N)}{f_2(N)}<\infty$, $\sup_N\frac{f_4(N)}{f_3(N)}<\infty$ we have $\mathcal{D}_f\subset\mathcal{D}_{f_2}$ and $\mathcal{D}_{f_4}\subset\mathcal{D}_{f_3}$. From Lemma 4.12 with (f_2,f_3) replaced by $(f_2=\sqrt{f},f_4=f_3^2)$, we have $\mathcal{D}_f\subset\mathcal{D}_{f_2}\subset D(\delta_{\dot{\Phi}(s)})$, and $\delta_{\dot{\Phi}(s)}(\mathcal{D}_f)\subset\delta_{\dot{\Phi}(s)}(\mathcal{D}_{f_2})\subset\mathcal{D}_{f_4}\subset\mathcal{D}_{f_3}$. From Lemma 4.12, with (f_2,f_3) replaced by (f_2,f_4) for $N>R$, we have

$$
\begin{aligned}
&\left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A)\right\| \leq \left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A-\mathbb{E}_{N-R}(A))\right\| \\
&+\left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)\mathbb{E}_{N-R}(A)\right\| \\
&=\left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A-\mathbb{E}_{N-R}(A))\right\| \leq 2C^{(1)}_{f_2f_4}\left\|A-\mathbb{E}_{N-R}(A)\right\|_{f_2} \\
&=2C^{(1)}_{f_2f_4}\left(\begin{array}{c}\left\|A-\mathbb{E}_{N-R}(A)\right\| \\ +\sup\limits_{M\in\mathbb{N}}\dfrac{\left\|A-\mathbb{E}_{N-R}(A)-\mathbb{E}_M\left(A-\mathbb{E}_{N-R}(A)\right)\right\|}{f_2(M)}\end{array}\right) \\
&\leq 2C^{(1)}_{f_2f_4}\left(\begin{array}{c}\left\|A-\mathbb{E}_{N-R}(A)\right\| \\ +\max\left\{\begin{array}{c}\sup\limits_{N-R\leq M\in\mathbb{N}}\dfrac{\left\|A-\mathbb{E}_M(A)\right\|}{f_2(M)}, \\ \sup\limits_{N-R>M\in\mathbb{N}}\dfrac{\left\|A-\mathbb{E}_{N-R}(A)\right\|}{f_2(N-R)}\end{array}\right\}\end{array}\right) \\
&\leq 2C^{(1)}_{f_2f_4}\left(\begin{array}{c}f(N-R)\left\|A\right\|_f \\ +\max\left\{\begin{array}{c}\sup\limits_{N-R\leq M\in\mathbb{N}}\left\|A\right\|_f\dfrac{f(M)}{f_2(M)}, \\ \sup\limits_{N-R>M\in\mathbb{N}}\left\|A\right\|_f\dfrac{f(N-R)}{f_2(N-R)}\end{array}\right\}\end{array}\right) \\
&=2C^{(1)}_{f_2f_4}\left(f(N-R)+\sup_{N-R\leq L}\left(\frac{f(L)}{f_2(L)}\right)\right)\left\|A\right\|_f \\
&=2C^{(1)}_{f_2f_4}\left(f(N-R)+\sqrt{f(N-R)}\right)\left\|A\right\|_f .
\end{aligned}
\tag{4.62}
$$

Here $C^{(1)}_{f_2f_4}$ is a constant independent of N, s. Therefore, we have

$$
\lim_{N\to\infty}\sup_{s\in[0,1]}\left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A)\right\|=0, \quad A\in\mathcal{D}_f. \tag{4.63}
$$

Furthermore, for $A\in\mathcal{D}_f$, we have

$$
\frac{\left\|\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A)-\mathbb{E}_M\left(\left(\delta_{\dot{\Phi}(s),N}-\delta_{\dot{\Phi}(s)}\right)(A)\right)\right\|}{f_3(M)}
$$

$$\leq \begin{cases} \frac{f_4(M)}{f_3(M)} \left(\left\| \delta_{\dot{\Phi}(s),N}(A) \right\|_{f_4} + \left\| \delta_{\dot{\Phi}(s)}(A) \right\|_{f_4} \right) \leq 2C^{(1)}_{f_2 f_4} \frac{f_4(M)}{f_3(M)} \|A\|_{f_2}, \\ \leq 2 f_3(N-R) C^{(1)}_{f_2 f_4} \|A\|_{f_2} \\ \qquad \text{for } M > N-R, \\ \frac{4C^{(1)}_{f_2 f_4} \left(f(N-R) + \sqrt{f(N-R)} \right)}{f_3(M)} \|A\|_f \\ \leq \frac{4C^{(1)}_{f_2 f_4} \left(f(N-R) + \sqrt{f(N-R)} \right)}{f_3(N-R)} \|A\|_f, \\ \qquad \text{for } M \leq N-R. \end{cases} \tag{4.64}$$

For $M > N-R$, we used Lemma 4.12, with (f_2, f_3) replaced by (f_2, f_4). For $M \leq N-R$, we used (4.62). As

$$\lim_{N\to\infty} 2f_3(N-R) C^{(1)}_{f_2 f_4} \|A\|_{f_2} = \lim_{N\to\infty} \frac{4C^{(1)}_{f_2 f_4} \left(f(N-R) + \sqrt{f(N-R)} \right)}{f_3(N-R)} \|A\|_f = 0, \tag{4.65}$$

we get

$$\lim_{N\to\infty} \sup_{M\in\mathbb{N}} \left(\frac{\left\| \left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A) - \mathbb{E}_M \left(\left(\delta_{\dot{\Phi}(s),N} - \delta_{\dot{\Phi}(s)} \right)(A) \right) \right\|}{f_3(M)} \right) = 0, \quad A \in \mathcal{D}_f. \tag{4.66}$$

From this and (4.62), we have shown the claim of the Lemma. □

4.4. Proof of Lemma 2.1

Below, we use the following facts repeatedly: for any $0 < \beta < \beta' \leq 1$, $0 < c, c'$, $0 < a, a'$, $s \in \mathbb{R}$, $l \in \mathbb{N}$, $r = 0, 1$, and $k \in \mathbb{Z}$, we have

$$\lim_{t\to\infty} \frac{t^k e^{-\hat{h}(t-s)}}{e^{-t^\beta}} = \lim_{t\to\infty} \frac{t^k e^{-\hat{h}([\frac{t}{2}])}}{e^{-t^\beta}} = \lim_{t\to\infty} \frac{t^k e^{-\hat{h}(t-[\frac{t}{2}])}}{e^{-t^\beta}} = 0, \tag{4.67}$$

$$\lim_{t\to\infty} \frac{e^{-t^\beta}}{e^{-(\frac{t}{2})^\beta}} = 0, \tag{4.68}$$

$$\lim_{t\to\infty} \frac{t^k e^{-t^{\beta'}}}{e^{-(t)^\beta}} = \lim_{t\to\infty} \frac{t^k e^{-c([\frac{t}{2}])^{\beta'}}}{e^{-t^\beta}} = \lim_{t\to\infty} \frac{t^k e^{-(t-[\frac{t}{2}])^{\beta'}}}{e^{-t^\beta}} = 0, \tag{4.69}$$

$$\sum_{m=1}^{\infty} m^k e^{-c(m-r)^\beta} < \infty, \tag{4.70}$$

$$\lim_{N\to\infty}\frac{\sum_{m=N-l}^{\infty} m^k e^{-c(m-r)^{\beta'}}}{e^{-c'N^{\beta}}} \le \sum_{m=1}^{\infty} m^k e^{-\frac{c}{2}(m-r)^{\beta'}} \lim_{N\to\infty}\frac{e^{-\frac{c}{2}(N-l-r)^{\beta'}}}{e^{-c'N^{\beta}}} = 0. \quad (4.71)$$

We also note that for $0<\beta<1$, $0<c,c'$, and $l\in\mathbb{N}$, $|t|^l e^{-\hat{h}(ct)}/e^{-(c't)^{\beta}}$ is integrable with respect to $t>0$. From this and (1.14), for any $0<\beta<1$, $0<c$, and $l\in\mathbb{N}$, we have

$$\int_{-\infty}^{\infty} dt\omega_{\gamma}(t)\,|t|^l e^{(c|t|)^{\beta}} < \infty. \quad (4.72)$$

We also have for any $0<\beta<1$ and $c>0$

$$\sup_{t\ge 1}\frac{W_{\gamma}(c\left[\frac{t}{2}\right])}{e^{-t^{\beta}}} < \infty, \quad (4.73)$$

from (1.15).

Lemma 4.14. *Fix $0<\beta_5<\beta_1<1$ and set $f(t) := \frac{\exp(-t^{\beta_1})}{t}$, and $\zeta(t) := \exp(-t^{\beta_5})$. Then for any $A\in\mathcal{D}_f$, and $(s',u',s'',s''')\in[0,1]\times\mathbb{R}\times[0,1]\times[0,1]$, we have $\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A)\in\mathcal{D}_{f_2}\subset\mathcal{D}_{\zeta}\subset D(\delta_{\Phi(s')})\cap D(\delta_{\dot{\Phi}(s')})$ and $\delta_{\Phi(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A), \delta_{\dot{\Phi}(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A)\in\mathcal{D}_{\zeta}$. For any $A\in\mathcal{D}_f$ and any compact intervals $[a,b]$, $[c,d]$ of $\mathbb{R}$, the maps*

$$\begin{aligned}&[a,b]\times[0,1]\times[0,1]\times[c,d]\times[0,1]\times[0,1]\ni(u,s,s',u',s'',s''')\\&\mapsto \tau_{\Phi(s)}^{u}\circ\delta_{\Phi(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A)\in\mathcal{A}\end{aligned} \quad (4.74)$$

and

$$\begin{aligned}&[a,b]\times[0,1]\times[0,1]\times[c,d]\times[0,1]\times[0,1]\ni(u,s,s',u',s'',s''')\\&\mapsto \tau_{\Phi(s)}^{u}\circ\delta_{\dot{\Phi}(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A)\in\mathcal{A}\end{aligned} \quad (4.75)$$

are uniformly continuous with respect to $\|\cdot\|$, and the maps

$$[0,1]\times[c,d]\times[0,1]\times[0,1]\ni(s',u',s'',s''')\mapsto\delta_{\Phi(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A)\in\mathcal{D}_{\zeta} \quad (4.76)$$

and

$$[0,1]\times[c,d]\times[0,1]\times[0,1]\ni(s',u',s'',s''')\mapsto\delta_{\dot{\Phi}(s')}\circ\tau_{\Phi(s'')}^{-u'}\circ\alpha_{s'''}^{-1}(A)\in\mathcal{D}_{\zeta} \quad (4.77)$$

are uniformly continuous with respect to $\|\cdot\|_{\zeta}$. For any $A\in\mathcal{D}_f$, the integral

$$\int dt\omega_{\gamma}(t)\int_0^t du\tau_{\Phi(s)}^{u}\circ\delta_{\dot{\Phi}(s)}\left(\tau_{\Phi(s)}^{-u}(A)\right), \quad (4.78)$$

and

$$\int dt\ \omega_\gamma(t) \int_0^t du\ \tau_{\Phi(s)}^{t-u} \circ \left(\delta_{\dot{\Phi}(s)}\right) \circ \tau_{\Phi(s)}^{u}(A), \tag{4.79}$$

are well-defined as Bochner integrals of $(\mathcal{A}, \|\cdot\|)$. *Furthermore, for any* $A \in \mathcal{D}_f$, $\alpha_s^{-1}(A)$ *and* $\alpha_s(A)$ *are differentiable with respect to* $\|\cdot\|$ *and*

$$\frac{d}{ds}\alpha_s^{-1}(A) = \int dt\omega_\gamma(t) \int_0^t du\tau_{\Phi(s)}^{u} \circ \delta_{\dot{\Phi}(s)} \left(\tau_{\Phi(s)}^{-u}\left(\alpha_s^{-1}(A)\right)\right). \tag{4.80}$$

The right hand side can be understood as a Bochner integral of $(\mathcal{A}, \|\cdot\|)$ *and there is a constant* $C_{9,f} > 0$ *such that*

$$\left\|\frac{d}{ds}\alpha_s^{-1}(A)\right\|, \left\|\frac{d}{ds}\alpha_s(A)\right\| \leq C_{9,f}\|A\|_f, \quad A \in \mathcal{D}_f. \tag{4.81}$$

Remark 4.15. As mentioned in the introduction, α_s is the same automorphism given in [1] and [11]. In particular, if a C^1-path of interactions satisfy *Condition B* in [13] except for the time reversal condition (iii) *6*, for each $s \in [0, 1]$, the unique ground state φ_s is given by $\varphi_s = \varphi_0 \circ \alpha_s$, with the α_s. Lemma 4.14 implies for any $A \in \mathcal{D}_f$, $\varphi_s(A) = \varphi_0 \circ \alpha_s(A)$ is differentiable and the derivative is bounded by $C_{9,f}\|A\|_f$, corresponding to Assumption 1.2 (vii). It is well known that the local gap implies the existence of the gap in the bulk in the sense of Assumption 1.2 (vi), [12].

Proof. We prove the continuity for (4.75) and (4.77). The proof for (4.74) and (4.76) are the same. We also prove only (4.78). The proof for (4.79) is the same. We prove (4.81) only for α_s^{-1}. The proof for α_s is analogous.

Choose real numbers $\beta_4, \beta_3, \beta_2$ so that $0 < \beta_5 < \beta_4 < \beta_3 < \beta_2 < \beta_1 < 1$ and fix. Define $f_0(t) := \exp(-t^{\beta_1})$, $f_1(t) := \exp(-t^{\beta_2})$, $f_2(t) := t^{-2(\nu+2)}\exp(-t^{\beta_3})$, $g(t) := \exp(-t^{\beta_4})$.

Note that $f_1, f, f_0 : (0, \infty) \to (0, \infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_0(t) = 0$. From (4.67), we have

$$\lim_{N\to\infty}\left(\frac{e^{-\hat{h}(N-M)}}{f_1(N)}\right) = 0, \text{ for all } M \in \mathbb{N}, \tag{4.82}$$

$$\sup_{N\in\mathbb{N}} \frac{e^{-\hat{h}([\frac{N}{2}])}}{f_1(N)}\left|\Lambda_{N-[\frac{N}{2}]}\right| < \infty. \tag{4.83}$$

Furthermore, from (4.69) and $0 < \beta_2 < \beta_1 < 1$, we have

$$\sup_{N\in\mathbb{N}} \frac{f_0(N - [\frac{N}{2}])}{f_1(N)} < \infty. \tag{4.84}$$

We also have

$$\lim_{M\to\infty}\frac{f(M)}{f_0(M)}=\lim_{M\to\infty}\frac{1}{M}=0. \tag{4.85}$$

Therefore, from Lemma 4.11 with (f, f_0, f_1) replaced by (f_1, f, f_0), we have $\alpha_s^{-1}(\mathcal{D}_f)\subset\mathcal{D}_{f_1}$ and

$$\sup_{s\in[0,1]}\left\|\alpha_{s,\Lambda_n}^{-1}(A)-\alpha_s^{-1}(A)\right\|_{f_1}\to 0,\quad A\in\mathcal{D}_f. \tag{4.86}$$

Therefore, for each $A\in\mathcal{D}_f$, $[0,1]\ni s\to\alpha_s^{-1}(A)\in\mathcal{D}_{f_1}$ is continuous with respect to the norm $\|\cdot\|_{f_1}$.

Note that $f, f_1 : (0,\infty)\to(0,\infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_1(t)=\lim_{t\to\infty} f(t)=0$. From (4.69), and $0<\beta_2<\beta_1<1$, we have

$$\sup_{N\in\mathbb{N}}\left(\frac{f\left(N-\left[\frac{N}{2}\right]\right)}{f_1(N)}\right)<\infty. \tag{4.87}$$

From this and (4.83), Lemma 4.9 with (f_1, f_2) replaced by (f, f_1) implies the existence of a constant $C_{8,f,f_1}>0$ such that

$$\sup_{s\in[0,1]}\left\|\alpha_s^{-1}(A)\right\|_{f_1}\le C_{8,f,f_1}\|A\|_f,\quad A\in\mathcal{D}_f. \tag{4.88}$$

The functions $f_1, f_2 : (0,\infty)\to(0,\infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f_i(t)=0$, $i=1,2$. From (4.69), we have

$$\lim_{N\to\infty}\left(\frac{|\Lambda_{\left[\frac{N}{2}\right]}|e^{-(N-\left[\frac{N}{2}\right])}}{f_2(N)}\right)=0. \tag{4.89}$$

From (4.69) and $0<\beta_3<\beta_2<1$, we have

$$\lim_{N\to\infty}\frac{f_1\left(\left[\frac{N}{2}\right]\right)}{f_2(N)}=0. \tag{4.90}$$

Therefore, from Lemma 4.4, we have

$$\sup_{s\in[0,1]}\left\|\tau^{-u}_{\Phi(s),\Lambda_n}(A)-\tau^{-u}_{\Phi(s)}(A)\right\|_{f_2}\to 0,\quad A\in\mathcal{D}_{f_1}, \tag{4.91}$$

uniformly in compact $u\in\mathbb{R}$. Therefore, for each $A\in\mathcal{D}_{f_1}$, $\mathbb{R}\times[0,1]\ni(u,s)\to\tau^{-u}_{\Phi(s)}(A)\in\mathcal{D}_{f_2}$ is continuous with respect to the norm $\|\cdot\|_{f_2}$.

Note that $f_2, \zeta : (0,\infty)\to(0,\infty)$ are continuous decreasing functions with $\lim_{t\to\infty} f(t)=\lim_{t\to\infty}\zeta(t)=0$. From (4.70) and (4.71), and $0<\beta_5<\beta_3<1$, we have

$$\sum_{k=1}^{\infty} k^{\nu} \sqrt{f_2(k)} < \infty, \tag{4.92}$$

$$\lim_{N\to\infty} \frac{\sum_{k=N-R}^{\infty} k^{\nu} \sqrt{f_2(k)}}{\zeta(N)^2} = 0. \tag{4.93}$$

Therefore applying Lemma 4.13 with (f, f_3) replaced by (f_2, ζ), we have $\delta_{\dot{\Phi}(s)}(\mathcal{D}_{f_2}) \subset \mathcal{D}_{\zeta}$ and

$$\lim_{N\to\infty} \sup_{s\in[0,1]} \left\| \left(\delta_{\dot{\Phi}_N(s)} - \delta_{\dot{\Phi}(s)} \right) (A) \right\|_{\zeta} = 0, \quad A \in \mathcal{D}_{f_2}. \tag{4.94}$$

Therefore, for each $A \in \mathcal{D}_{f_2}$, $[0,1] \ni s \to \delta_{\dot{\Phi}(s)}(A) \in \mathcal{D}_{\zeta}$ is continuous with respect to the norm $\|\cdot\|_{\zeta}$.

Note that $f_2 : (0,\infty) \to (0,\infty)$ is a continuous decreasing function with $\lim_{t\to\infty} f_2(t) = 0$. From (4.72), we have

$$\int_{(4v|t|)\geq 1} dt\, \omega_{\gamma}(t) \frac{|t|}{f_2((4v\,|t|))} < \infty. \tag{4.95}$$

We also have

$$\sup_{N\in\mathbb{N}} \frac{f_1(N - \left[\frac{N}{2}\right])}{f_2(N)} < \infty, \tag{4.96}$$

$$\sup_{N\in\mathbb{N}} \frac{|\Lambda_N|\, e^{-\frac{\left[\frac{N}{2}\right]}{2}}}{f_2(N)} < \infty, \tag{4.97}$$

from (4.69) with $0 < \beta_3 < \beta_2 < 1$ and (4.67). Therefore, from Lemma 4.5, with (f, f_1) replaced by (f_1, f_2) we have $\tau^t_{\Phi(s)}(\mathcal{D}_{f_1}) \subset \mathcal{D}_{f_2}$ and there is a non-negative non-decreasing function on $\mathbb{R}_+$, $b_{1,f_1,f_2}(t)$ such that

$$\int dt\, \omega_{\gamma}(t)\, |t| \cdot b_{1,f_1,f_2}(|t|) < \infty \tag{4.98}$$

and

$$\sup_{s\in[0,1]} \left\| \tau^t_{\Phi(s)}(A) \right\|_{f_2}, \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\| \tau^t_{\Phi_N(s)}(A) \right\|_{f_2} \leq b_{1,f_1,f_2}(|t|)\, \|A\|_{f_1}, \quad A \in \mathcal{D}_{f_1}. \tag{4.99}$$

Note that $f_2, \zeta : (0,\infty) \to (0,\infty)$ are continuous decreasing functions such that $\lim_{t\to\infty} f_2(t) = \lim_{t\to\infty} \zeta(t) = 0$. By (4.70) and (4.71) with $0 < \beta_5 < \beta_3 < 1$, we have

$$\sum_{k=2}^{\infty} k^{\nu} f_2(k-1) < \infty, \tag{4.100}$$

$$\limsup_N \frac{\sum_{k=N-R}^{\infty} k^{\nu} f_2(k)}{\zeta(N)} = 0. \tag{4.101}$$

Therefore, from Lemma 4.12 with (f_2, f_3) replaced by (f_2, ζ), we have $\mathcal{D}_{f_2} \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)}) \cap D(\delta_{\frac{\Phi(s)-\Phi(s_0)}{s-s_0} - \dot{\Phi}(s_0)})$, and there exists a constant $C^{(1)}_{7,f_2,\zeta} > 0$ such that

$$\sup_{s\in[0,1]} \left\|\delta_{\Phi(s)}(A)\right\|_{\zeta}, \quad \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\|\delta_{\Phi_N(s)}(A)\right\|_{\zeta} \le C^{(1)}_{7,f_2,\zeta} \|A\|_{f_2} \tag{4.102}$$

$$\sup_{s\in[0,1]} \left\|\delta_{\dot{\Phi}(s)}(A)\right\|_{\zeta}, \quad \sup_{N\in\mathbb{N}} \sup_{s\in[0,1]} \left\|\delta_{\dot{\Phi}_N(s)}(A)\right\|_{\zeta} \le C^{(1)}_{7,f_2,\zeta} \|A\|_{f_2} \tag{4.103}$$

for all $A \in \mathcal{D}_{f_2}$ and $\varepsilon > 0$.

We claim that for any compact intervals $[a,b]$, $[c,d]$ of $\mathbb{R}$ and $A \in \mathcal{D}_f$,

$$\begin{aligned}&[a,b]\times[0,1]\times[0,1]\times[c,d]\times[0,1]\times[0,1] \ni (u,s,s',u',s'',s''')\\ &\mapsto \tau^{u}_{\Phi(s)} \circ \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{A}\end{aligned} \tag{4.104}$$

is continuous with respect to $\|\cdot\|$. We also claim that

$$[0,1]\times[c,d]\times[0,1]\times[0,1] \ni (s',u',s'',s''') \mapsto \delta_{\dot{\Phi}(s')} \circ \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s'''}(A) \in \mathcal{D}_{\zeta} \tag{4.105}$$

is continuous with respect to $\|\cdot\|_{\zeta}$.

To see this, let $A \in \mathcal{D}_f$ and fix any $\varepsilon > 0$. Note that from the continuity of $[0,1] \ni s''' \mapsto \alpha^{-1}_{s'''}(A) \in \mathcal{D}_{f_1}$ in $\|\cdot\|_{f_1}$, there exists a finite sequence $s_0 = 0 < s_1 < \cdots < s_{N_\varepsilon} = 1$ such that

$$\left\|\alpha^{-1}_{s'''}(A) - \alpha^{-1}_{s_i}(A)\right\|_{f_1} < \varepsilon, \text{for all } s''' \in [s_{i-1}, s_{i+1}], \text{and } i = 1,\dots,N_\varepsilon - 1. \tag{4.106}$$

For $\alpha^{-1}_{s_i}(A) \in \mathcal{D}_{f_1}$, $i = 0,\dots,N_\varepsilon$, from the continuity of $(u', s'') \mapsto \tau^{-u'}_{\Phi(s'')} \circ \alpha^{-1}_{s_i}(A) \in \mathcal{D}_{f_2}$, in $\|\cdot\|_{f_2}$ we get $\tilde{s}_0 = 0 < \tilde{s}_1 < \cdots < \tilde{s}_{\tilde{N}_\varepsilon} = 1$ and $u_0 = c < u_1 < \cdots < u_{M_\varepsilon} = d$ such that

$$\begin{aligned}&\left\|\left(\tau^{-u'}_{\Phi(s'')} - \tau^{-u_k}_{\Phi(\tilde{s}_j)}\right) \circ \alpha^{-1}_{s_i}(A)\right\|_{f_2} < \varepsilon,\\ &\text{for all } s'' \in [\tilde{s}_{j-1}, \tilde{s}_{j+1}], \text{and } j = 1,\dots,\tilde{N}_\varepsilon - 1,\\ &\text{for all } u' \in [u_{k-1}, u_{k+1}], \text{and } k = 1,\dots,M_\varepsilon - 1,\\ &\text{and } i = 1,\dots,N_\varepsilon - 1.\end{aligned} \tag{4.108}$$

From the continuity of $[0,1] \ni s' \to \delta_{\dot{\Phi}(s')} \circ \tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \in \mathcal{D}_{\zeta}$ for $\tau^{-u_k}_{\Phi(\tilde{s}_j)} \circ \alpha^{-1}_{s_i}(A) \in \mathcal{D}_{f_2}$ in $\|\cdot\|_{\zeta}$, there exists a finite sequence $\hat{s}_0 = 0 < \hat{s}_1 < \cdots < \hat{s}_{\hat{N}_\varepsilon} = 1$ such that

$$\begin{aligned}
&\left\|\left(\delta_{\dot{\Phi}(s')}-\delta_{\dot{\Phi}(\hat{s}_l)}\right)\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)\right\|_\zeta<\varepsilon.\\
&\text{for all } s'\in[\hat{s}_{l-1},\hat{s}_{l+1}], \text{and } l=1,\dots,\hat{N}_\varepsilon-1,\\
&\text{and } j=1,\dots,\tilde{N}_\varepsilon-1, \text{and } k=1,\dots,M_\varepsilon-1,\\
&\text{and } i=1,\dots,N_\varepsilon-1.
\end{aligned}\tag{4.109}$$

Finally, from the continuity of $\mathbb{R}\times[0,1]\ni(u,s)\to\tau^u_{\Phi(s)}\left(\delta_{\dot{\Phi}(\hat{s}_l)}\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)\right)\in\mathcal{A}$ in the norm $\|\cdot\|$, (Lemma 4.3) we have finite sequences $\check{s}_0=0<\check{s}_1<\cdots<\check{s}_{\check{N}_\varepsilon}=1$ and $\hat{u}_0=a<\hat{u}_1<\cdots<\hat{u}_{\hat{M}_\varepsilon}=b$ such that

$$\begin{aligned}
&\left\|\left(\tau^u_{\Phi(s)}-\tau^{\hat{u}_x}_{\Phi(\check{s}_y)}\right)\circ\delta_{\dot{\Phi}(\hat{s}_l)}\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)\right\|<\varepsilon,\\
&\text{for all } s\in[\check{s}_{y-1},\check{s}_{y+1}], \text{and } y=1,\dots,\check{N}_\varepsilon-1,\\
&\text{and } u\in[\hat{u}_{x-1},\hat{u}_{x+1}], \text{and } x=1,\dots,\hat{M}_\varepsilon-1,\\
&\text{and } l=1,\dots,\hat{N}_\varepsilon-1,\\
&\text{and } j=1,\dots,\tilde{N}_\varepsilon-1, \text{and } k=1,\dots,M_\varepsilon-1,\\
&\text{and } i=1,\dots,N_\varepsilon-1.
\end{aligned}\tag{4.110}$$

Now for any $(u,s,s',u',s'',s''')\in[a,b]\times[0,1]\times[0,1]\times[c,d]\times[0,1]\times[0,1]$, there is (x,y,l,k,j,i) such that

$$\begin{aligned}
&u\in[\hat{u}_{x-1},\hat{u}_{x+1}], s\in[\check{s}_{y-1},\check{s}_{y+1}], s'\in[\hat{s}_{l-1},\hat{s}_{l+1}], u'\in[u_{k-1},u_{k+1}],\\
&s''\in[\tilde{s}_{j-1},\tilde{s}_{j+1}], s'''\in[s_{i-1},s_{i+1}].
\end{aligned}\tag{4.111}$$

For any such (x,y,l,k,j,i), we have

$$\begin{aligned}
&\left\|-\tau^{\hat{u}_x}_{\Phi(\check{s}_y)}\circ\delta_{\dot{\Phi}(\hat{s}_l)}\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)+\tau^u_{\Phi(s)}\circ\delta_{\dot{\Phi}(s')}\circ\tau^{-u'}_{\Phi(s'')}\circ\alpha^{-1}_{s'''}(A)\right\|\\
&\le\left\|\left(\tau^u_{\Phi(s)}-\tau^{\hat{u}_x}_{\Phi(\check{s}_y)}\right)\circ\delta_{\dot{\Phi}(\hat{s}_l)}\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)\right\|\\
&+\left\|\tau^u_{\Phi(s)}\circ\left(-\delta_{\dot{\Phi}(\hat{s}_l)}+\delta_{\dot{\Phi}(s')}\right)\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)\right\|\\
&+\left\|\tau^u_{\Phi(s)}\circ\delta_{\dot{\Phi}(s')}\circ\left(-\tau^{-u_k}_{\Phi(\tilde{s}_j)}+\tau^{-u'}_{\Phi(s'')}\right)\circ\alpha^{-1}_{s_i}(A)\right\|\\
&+\left\|\tau^u_{\Phi(s)}\circ\delta_{\dot{\Phi}(s')}\circ\tau^{-u'}_{\Phi(s'')}\circ\left(-\alpha^{-1}_{s_i}(A)+\alpha^{-1}_{s'''}(A)\right)\right\|\\
&\le 2\varepsilon+C^{(1)}_{7,f_2,\zeta}\varepsilon+C^{(1)}_{7,f_2,\zeta}\sup_{u\in[c,d]}b_{1,f_1,f_2}(|u|)\varepsilon.
\end{aligned}\tag{4.112}$$

We also have

$$\begin{aligned}
&\left\|-\delta_{\dot{\Phi}(\hat{s}_l)}\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)+\delta_{\dot{\Phi}(s')}\circ\tau^{-u'}_{\Phi(s'')}\circ\alpha^{-1}_{s'''}(A)\right\|_\zeta\\
&\le\left\|\left(-\delta_{\dot{\Phi}(\hat{s}_l)}+\delta_{\dot{\Phi}(s')}\right)\circ\tau^{-u_k}_{\Phi(\tilde{s}_j)}\circ\alpha^{-1}_{s_i}(A)\right\|_\zeta
\end{aligned}\tag{4.113}$$

$$+\left\|\delta_{\dot{\Phi}(s')}\circ\left(-\tau^{-u_k}_{\Phi(\tilde{s}_j)}+\tau^{-u'}_{\Phi(s'')}\right)\circ\alpha^{-1}_{s_i}(A)\right\|_\zeta+\left\|\delta_{\dot{\Phi}(s')}\circ\tau^{-u'}_{\Phi(s'')}\circ\left(-\alpha^{-1}_{s_i}(A)+\alpha^{-1}_{s'''}(A)\right)\right\|_\zeta$$
$$\le\varepsilon+C^{(1)}_{7,f_2,\zeta}\varepsilon+C^{(1)}_{7,f_2,\zeta}\sup_{u\in[c,d]}b_{1,f_1,f_2}(|u|)\varepsilon.$$

As b_{1,f_1,f_2} is an $\mathbb{R}$-valued nondecreasing function, $\sup_{u\in[c,d]}b_{1,f_1,f_2}(|u|)$ is finite. Hence we have proven the continuity of (4.75) and (4.77).

Furthermore, for any $A\in\mathcal{D}_f$, we have

$$\sup_{s\in[0,1]}\int dt\ \omega_\gamma(t)\int_{[0,t]}du\left\|\tau^u_{\Phi(s)}\circ\delta_{\dot{\Phi}(s)}\left(\tau^{-u}_{\Phi(s)}\left(\alpha^{-1}_s(A)\right)\right)\right\|$$
$$\le\sup_{s\in[0,1]}\int dt\ \omega_\gamma(t)\int_{[0,t]}duC^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\left\|A\right\|_f \tag{4.114}$$
$$\le C^{(1)}_{7,f_2,\zeta}C_{8,f,f_1}\left\|A\right\|_f\int dt\omega_\gamma(t)b_{1,f_1,f_2}(|t|)|t|<\infty.$$

In the last line we used the fact that b_{1,f_1,f_2} is nondecreasing and (4.98). Therefore, the right hand side of (4.80) is a well-defined Bochner integral of $(\mathcal{A},\|\cdot\|)$ for any $A\in\mathcal{D}_f$. By the same argument, (4.78) is a well-defined Bochner integral of $(\mathcal{A},\|\cdot\|)$ for any $A\in\mathcal{D}_f$. By the definition of α_{s,Λ_n}, we have

$$\frac{d}{ds}\alpha^{-1}_{s,\Lambda_n}(A)=i\left[D_{\Lambda_n}(s),\alpha^{-1}_{s,\Lambda_n}(A)\right]=i\int dt\omega_\gamma(t)\int_0^t du\left[\tau^u_{\Phi(s),\Lambda_n}\left(H_{\dot{\Phi}(s),\Lambda_n}\right),\alpha^{-1}_{s,\Lambda_n}(A)\right]$$
$$=\int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(s),\Lambda_n}\circ\delta_{\dot{\Phi}_n(s)}\left(\tau^{-u}_{\Phi(s),\Lambda_n}\left(\alpha^{-1}_{s,\Lambda_n}(A)\right)\right),\quad A\in\mathcal{D}_f. \tag{4.115}$$

Hence we obtain

$$\alpha^{-1}_{s,\Lambda_n}(A)-\alpha^{-1}_{s_0,\Lambda_n}(A)=\int_{s_0}^s dv\int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot{\Phi}_n(v)}\left(\tau^{-u}_{\Phi(v),\Lambda_n}\left(\alpha^{-1}_{v,\Lambda_n}(A)\right)\right),$$
$$A\in\mathcal{D}_f. \tag{4.116}$$

For each (u,v), for any $A\in\mathcal{D}_f$, we have

$$\left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot{\Phi}_n(v)}\circ\tau^{-u}_{\Phi(v),\Lambda_n}\circ\alpha^{-1}_{v,\Lambda_n}(A)-\tau^u_{\Phi(v)}\circ\delta_{\dot{\Phi}(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha^{-1}_v(A)\right\|$$
$$\le\left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot{\Phi}_n(v)}\circ\tau^{-u}_{\Phi(v),\Lambda_n}\left(\alpha^{-1}_{v,\Lambda_n}(A)-\alpha^{-1}_v(A)\right)\right\|$$
$$+\left\|\tau^u_{\Phi(v),\Lambda_n}\circ\delta_{\dot{\Phi}_n(v)}\circ\left(\tau^{-u}_{\Phi(v),\Lambda_n}-\tau^{-u}_{\Phi(v)}\right)\alpha^{-1}_v(A)\right\|$$

$$
\begin{aligned}
&+\left\|\tau^{u}_{\Phi(v),\Lambda_n}\circ\left(\delta_{\dot{\Phi}_n(v)}-\delta_{\dot{\Phi}(v)}\right)\left(\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\| \qquad (4.117)\\
&+\left\|\left(\tau^{u}_{\Phi(v),\Lambda_n}-\tau^{u}_{\Phi(v)}\right)\circ\left(\delta_{\dot{\Phi}(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\|\\
&\le C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)\left\|\alpha^{-1}_{v,\Lambda_n}(A)-\alpha_v^{-1}(A)\right\|_{f_1}+C^{(1)}_{7,f_2,\zeta}\left\|\left(\tau^{-u}_{\Phi(v),\Lambda_n}-\tau^{-u}_{\Phi(v)}\right)\alpha_v^{-1}(A)\right\|_{f_2}\\
&+\left\|\left(\delta_{\dot{\Phi}_n(v)}-\delta_{\dot{\Phi}(v)}\right)\left(\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\|\\
&+\left\|\left(\tau^{u}_{\Phi(v),\Lambda_n}-\tau^{u}_{\Phi(v)}\right)\circ\left(\delta_{\dot{\Phi}(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right)\right\|.
\end{aligned}
$$

From (4.86), (4.91), (4.94) and Lemma 4.3, the last part converges to 0 as $n\to\infty$. Furthermore, we have

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\left\|\tau^{u}_{\Phi(v),\Lambda_n}\circ\delta_{\dot{\Phi}_n(v)}\circ\tau^{-u}_{\Phi(v),\Lambda_n}\circ\alpha^{-1}_{v,\Lambda_n}(A)-\tau^{u}_{\Phi(v)}\circ\delta_{\dot{\Phi}(v)}\circ\tau^{-u}_{\Phi(v)}\circ\alpha_v^{-1}(A)\right\|\\
&\le 2C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\left\|A\right\|_f, \qquad (4.118)
\end{aligned}
$$

with

$$\int_0^1 ds\int dt\ \omega_\gamma(t)\int_{[0,t]} du2C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\left\|A\right\|_f<\infty. \qquad (4.119)$$

Therefore, applying Lebesgue's convergence theorem for (4.116), we obtain

$$\alpha_s^{-1}(A)-\alpha_{s_0}^{-1}(A)=\int_{s_0}^{s}dv\int dt\omega_\gamma(t)\int_0^t du\tau^{u}_{\Phi(v)}\circ\delta_{\dot{\Phi}(v)}\left(\tau^{-u}_{\Phi(v)}\left(\alpha_v^{-1}(A)\right)\right),\quad A\in\mathcal{D}_f. \qquad (4.120)$$

From this, for $A\in\mathcal{D}_f$, we get

$$
\begin{aligned}
&\left\|\frac{\alpha_s^{-1}(A)-\alpha_{s_0}^{-1}(A)}{s-s_0}-\int dt\omega_\gamma(t)\int_0^t du\tau^{u}_{\Phi(s_0)}\circ\delta_{\dot{\Phi}(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right\|\\
&\le\int dt\omega_\gamma(t)\int_{[0,t]}du\left\|\frac{1}{s-s_0}\int_{s_0}^{s}dv\left(\tau^{u}_{\Phi(v)}\circ\delta_{\dot{\Phi}(v)}\left(\tau^{-u}_{\Phi(v)}\left(\alpha_v^{-1}(A)\right)\right)\right.\right. \qquad (4.121)\\
&\left.\left.-\tau^{u}_{\Phi(s_0)}\circ\delta_{\dot{\Phi}(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right)\right\|.
\end{aligned}
$$

By the continuity of $(s,u)\to\tau^{u}_{\Phi(s)}\circ\delta_{\dot{\Phi}(s)}\left(\tau^{-u}_{\Phi(s)}\left(\alpha_s^{-1}(A)\right)\right)\in\mathcal{A}$ with respect to $\|\cdot\|$ for $A\in\mathcal{D}_f$, we have

$$\lim_{s\to s_0} \frac{1}{s-s_0}\int_{s_0}^{s} dv \left(\tau^u_{\Phi(v)}\circ\delta_{\dot{\Phi}(v)}\left(\tau^{-u}_{\Phi(v)}\left(\alpha_v^{-1}(A)\right)\right)\right.$$
$$\left.-\tau^u_{\Phi(s_0)}\circ\delta_{\dot{\Phi}(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right)=0, \tag{4.122}$$

for each u. On the other hand, we have

$$\left\|\frac{1}{s-s_0}\int_{s_0}^{s} dv \left(\tau^u_{\Phi(v)}\circ\delta_{\dot{\Phi}(v)}\left(\tau^{-u}_{\Phi(v)}\left(\alpha_v^{-1}(A)\right)\right)-\tau^u_{\Phi(s_0)}\circ\delta_{\dot{\Phi}(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right)\right\|$$
$$\leq 2C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\left\|A\right\|_f, \tag{4.123}$$

with (4.119). From Lebesgue's convergence theorem, we obtain

$$\lim_{s\to s_0}\left\|\frac{\alpha_s^{-1}(A)-\alpha_{s_0}^{-1}(A)}{s-s_0}-\int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(s_0)}\circ\delta_{\dot{\Phi}(s_0)}\left(\tau^{-u}_{\Phi(s_0)}\left(\alpha_{s_0}^{-1}(A)\right)\right)\right\|=0,$$
$$\text{for } A\in\mathcal{D}_f. \tag{4.124}$$

Hence for $A\in\mathcal{D}_f$, $[0,1]\ni s\mapsto \alpha_s^{-1}(A)$ is differentiable with respect to $\|\cdot\|$, and we have

$$\frac{d}{ds}\alpha_s^{-1}(A)=\int dt\omega_\gamma(t)\int_0^t du\tau^u_{\Phi(s)}\circ\delta_{\dot{\Phi}(s)}\circ\tau^{-u}_{\Phi(s)}\left(\alpha_s^{-1}(A)\right). \tag{4.125}$$

From this formula, we obtain

$$\left\|\frac{d}{ds}\alpha_s^{-1}(A)\right\|\leq\left(\int dt\omega_\gamma(t)\int_{[0,t]} C^{(1)}_{7,f_2,\zeta}b_{1,f_1,f_2}(|u|)C_{8,f,f_1}\right)\left\|A\right\|_f=:C_{9,f}\left\|A\right\|_f. \qquad\square \tag{4.126}$$

Now we prove Lemma 2.1.

Proof of Lemma 2.1.

1. The inclusions $\mathcal{D}_f\subset\mathcal{D}_{f_0}\subset\mathcal{D}_{f_1}\subset\mathcal{D}_{f_2}\subset\mathcal{D}_g\subset\mathcal{D}_\zeta$ follow by the monotone choice of the β_i, $i=1,\dots,5$. From (4.67), we can see that f satisfies the condition required in Lemma 4.8. Therefore, from Lemma 4.8, we have $\alpha_s^{-1}(\mathcal{A}_{loc})\subset\mathcal{D}_f$ for all $s\in[0,1]$.

2. This is from Lemma 4.5. From (4.72), (4.69) (f,f_1) satisfies the conditions required in Lemma 4.5.

3. Fix $0 < \beta_6 < \beta_5$ and set $\zeta_0(t) := e^{-t^{\beta_6}}$ for $t > 0$. We apply Lemma 4.12, replacing (f_2, f_3) in it by (ζ, ζ_0). To see that (ζ, ζ_0) satisfy the required conditions in Lemma 4.12, we recall (4.70) and (4.71). Hence from Lemma 4.12, we obtain $\mathcal{D}_\zeta \subset D(\delta_{\Phi(s)}) \cap D(\delta_{\dot{\Phi}(s)})$.

4. This also follows by Lemma 4.12 with (f_2, f_3) replaced by (f_2, ζ). The required conditions in Lemma 4.12 can be checked by (4.70) and (4.71).

5., 6., and 7. are proven in Lemma 4.14.

8. This follows from Lemma 4.6 for (f, f_1). The conditions for (f, f_1) can be checked from (4.69) and (4.73).

9. This is Lemma 4.3.

10. For any $A \in \mathcal{D}_f$, from *5.* above, $(u, s) \mapsto \delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A) \in \mathcal{D}_\zeta$ is continuous with respect to $\|\cdot\|_\zeta$. Furthermore, from *4., 2.,* above, as in (3.12), we have

$$\begin{aligned}
&\left\| \delta_{\Phi(s)} \circ \tau^u_{\Phi(s)}(A) \right\|_\zeta \leq C^{(1)}_{f_2,\zeta} \left\| \tau^u_{\Phi(s)}(A) \right\|_{f_2} \leq C^{(1)}_{f_2,\zeta} \left(1 + \sup_{N \in \mathbb{N}} \frac{f_1(N)}{f_2(N)} \right) \left\| \tau^u_{\Phi(s)}(A) \right\|_{f_1} \\
&\leq C^{(1)}_{f_2,\zeta} b_{f,f_1}(|u|) \left(1 + \sup_N \frac{f_1(N)}{f_2(N)} \right) \|A\|_f .
\end{aligned} \tag{4.127}$$

From *2.* above, the inequality (2.2) holds and (2.14) is well-defined as the Bochner integral with respect to $(\mathcal{D}_\zeta, \|\cdot\|_\zeta)$. □

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Appendix A. Conditional expectation $\mathbb{E}_N$

We now briefly describe a family of conditional expectations $\{\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N} \mid N \in \mathbb{N}\}$ are used extensively in this paper. Let $N \in \mathbb{N}$ be fixed and let Λ denote any finite set containing Λ_N. Define:

$$\mathbb{E}^\Lambda_N = \mathrm{id}_{\Lambda_N} \otimes \rho_{\Lambda \setminus \Lambda_N} \tag{A.1}$$

where ρ_X is the product state whose factors are normalized trace:

$$\rho_X = \frac{1}{d^{|X|}} \bigotimes_{x \in X} \mathrm{tr}_x. \tag{A.2}$$

Each $\mathbb{E}_N^\Lambda$ is bounded and linear, and as $\Lambda \subset \Sigma$ implies $\mathbb{E}_N^\Sigma|_{\mathcal{A}_\Lambda} = \mathbb{E}_N^\Lambda$, there exists a unique bounded map and conditional expectation $\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N}$ such that for all Λ containing Λ_N:

$$\mathbb{E}_N|_{\mathcal{A}_\Lambda} = \mathbb{E}_N^\Lambda \tag{A.3}$$

Furthermore, by the definition (A.1) of the finite-volume maps, $\mathbb{E}_N(A^*) = \mathbb{E}_N(A)^*$ for all $A \in \mathcal{A}$ and if $M \in \mathbb{N}$ and $M \geq N$,

$$\mathbb{E}_M \mathbb{E}_N = \mathbb{E}_N \mathbb{E}_M = \mathbb{E}_N. \tag{A.4}$$

The family $\{\mathbb{E}_N\}$ provides local approximations of quasi-local observables. For completeness, we record this as the following proposition and refer to [11] for the proof.

Proposition A.1. *Let $\varepsilon \geq 0$. Suppose $A \in \mathcal{A}$ is such that for all $B \in \bigcup_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^\nu} \\ X \cap \Lambda_N = \emptyset}} \mathcal{A}_X$:*

$$\|[A, B]\| \leq \varepsilon \|B\|. \tag{A.5}$$

Then $\|A - \mathbb{E}_N(A)\| \leq 2\varepsilon$.

Proof. See Corollary 4.4 of [11]. □

Appendix B. Properties of $\mathcal{D}_f$

The map $\|\cdot\|_f : \mathcal{D}_f \to \mathbb{R}_{\geq 0}$ is a norm on $\mathcal{D}_f$. Note that $\|A^*\|_f = \|A\|_f$, and $\|\mathbb{E}_N(A)\|_f \leq \|A\|_f$. Furthermore, if $\sup_{N \in \mathbb{N}} \frac{f(N)}{g(N)} < \infty$, then $\mathcal{D}_f \subset \mathcal{D}_g$.

Lemma B.1. *Let $f : (0, \infty) \to (0, \infty)$ be a continuous decreasing function with $\lim_{t \to \infty} f(t) = 0$. The set $\mathcal{D}_f$ is a $*$-algebra which is a Banach space with respect to the norm $\|\cdot\|_f$.*

Proof. That $\mathcal{D}_f$ is $*$-closed is trivial from $\|A^*\|_f = \|A\|_f$. To see that $\mathcal{D}_f$ is closed under multiplication, let $A, B \in \mathcal{D}_f$. For each $N \in \mathbb{N}$, we have

$$\begin{aligned} &\|AB - \mathbb{E}_N(AB)\| \leq \|(A - \mathbb{E}_N(A)) \cdot B\| + \|-\mathbb{E}_N((A - \mathbb{E}_N(A)) \cdot B)\| \\ &+ \|\mathbb{E}_N(A) \cdot (B - \mathbb{E}_N(B))\| \\ &\leq \left(2\|A\|_f \|B\| + \|A\| \|B\|_f\right) f(N) \leq 3\|A\|_f \|B\|_f f(N). \end{aligned} \tag{B.1}$$

Hence we obtain $AB \in \mathcal{D}_f$, and $\mathcal{D}_f$ is closed under the multiplication.

To prove that $\mathcal{D}_f$ is complete with respect to $\|\cdot\|_f$, let $\{A_n\}_n$ be a Cauchy sequence in $\mathcal{D}_f$ with respect to $\|\cdot\|_f$. As $\{A_n\}_n$ is Cauchy with respect to $\|\cdot\|$ as well, there is an $A \in \mathcal{A}$ such that $\lim_{n\to\infty} \|A - A_n\| = 0$. This A belongs to $\mathcal{D}_f$ because

$$\sup_{N\in\mathbb{N}} \frac{\|A - \mathbb{E}_N(A)\|}{f(N)} = \sup_{N\in\mathbb{N}} \left(\lim_{M\to\infty} \frac{\|A_M - \mathbb{E}_N(A_M)\|}{f(N)} \right) \le \sup_M \|A_M\|_f < \infty. \tag{B.2}$$

Furthermore, we have

$$\begin{aligned} \sup_N \frac{\|A - A_m - \mathbb{E}_N(A - A_m)\|}{f(N)} &= \sup_N \lim_{n\to\infty} \left(\frac{\|A_n - A_m - \mathbb{E}_N(A_n - A_m)\|}{f(N)} \right) \\ &\le \limsup_{n\to\infty} \|A_n - A_m\|_f . \end{aligned} \tag{B.3}$$

Therefore, A_m converges to $A \in \mathcal{D}_f$ in $\|\cdot\|_f$-norm. □

Lemma B.2. *Let $f : (0,\infty) \to (0,\infty)$ be a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$ with $M \in \mathbb{N}$. For any $A \in \mathcal{D}_f$ and $B \in \mathcal{A}_{\Lambda_M}$ and $M \in \mathbb{N}$ we have*

$$\|BA\|_f \le \left(1 + \max\left\{\frac{2}{f(M)}, 1\right\}\right) \|B\| \, \|A\|_f . \tag{B.4}$$

Proof. This follows from the following inequality:

$$\begin{aligned} &\|BA - \mathbb{E}_N(BA)\| \\ &\le \begin{cases} 2\|B\|\,\|A\|, & N \le M, \\ \|B(A - \mathbb{E}_N(A))\|, & N > M. \end{cases} \\ &\le \begin{cases} 2\|B\|\,\|A\|, & N \le M, \\ \|B\|\,\|A\|_f\, f(N) & N > M. \end{cases} \end{aligned} \tag{B.5}$$

□

Lemma B.3. *Let $f, f_1 : (0,\infty) \to (0,\infty)$ be continuous decreasing functions. Suppose that and*

$$\lim_{N\to\infty} \frac{f(N)}{f_1(N)} = 0. \tag{B.6}$$

Then we have

$$\lim_{M\to\infty} \|A - \mathbb{E}_M(A)\|_{f_1} = 0, \quad A \in \mathcal{D}_f. \tag{B.7}$$

Proof. Let $A \in \mathcal{D}_f$. By the definition of $\mathcal{A}$, we have $\lim_{M\to\infty} \|A - \mathbb{E}_M(A)\| = 0$. We note that for $N \in \mathbb{N}$,

$$
\begin{aligned}
&\frac{\|A - \mathbb{E}_M(A) - \mathbb{E}_N\left(A - \mathbb{E}_M(A)\right)\|}{f_1(N)} = \begin{cases} \dfrac{\|A - \mathbb{E}_N\left(A\right)\|}{f_1(N)}, & M \le N, \\ \dfrac{\|A - \mathbb{E}_M(A)\|}{f_1(N)}, & M > N, \end{cases} \\
&= \begin{cases} \dfrac{\|A - \mathbb{E}_N\left(A\right)\|}{f(N)} \dfrac{f(N)}{f_1(N)}, & M \le N, \\ \dfrac{\|A - \mathbb{E}_M(A)\|}{f(M)} \dfrac{f(M)}{f_1(N)}, & M > N \end{cases} \\
&\le \|A\|_f \begin{cases} \dfrac{f(N)}{f_1(N)}, & M \le N, \\ \dfrac{f(M)}{f_1(M)}, & M > N \end{cases} \\
&\le \|A\|_f \sup_{M \le L \in \mathbb{N}} \left(\frac{f(L)}{f_1(L)} \right) \to 0, \ M \to \infty.
\end{aligned} \tag{B.8}
$$

Hence we obtain (B.7). □

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6 | Appendix

Assumption 1.2 of [20] specifies assumptions to prove a quasi-adiabatic theorem for unique gapped ground states. In this appendix, we will show that Condition (vii) of Assumption 1.2 is satisfied by generic paths of ground states of uniformly locally gapped, rapidly decaying QSS interactions. Let $\Psi_s : P_f(\mathbb{Z}^D) \to \mathfrak{A}_{\text{loc}}$ be the interaction which generates α_s (see Theorem 5.5 of [3]) and which satisfies $\|\Psi\|_{F_\Psi} < \infty$, for the superpolynomially decaying F_Ψ given in [3]. Let $\delta_s = \overline{\delta_s^{loc}}$, where δ_s^{loc} is the derivation with domain $\mathfrak{A}_{\text{loc}}$ defined by $\delta_s^{\text{loc}}(A) = \lim_{\Lambda \to \mathbb{Z}^\nu} i[H_\Lambda(\Psi_s), A]$. Note that δ_s^{loc} is closeable since:

$$\sup_{x \in \mathbb{Z}^D} \sum_{\substack{X \in P_f(\mathbb{Z}^D) \\ x \in X}} \|\Psi_s(X)\| \leq \|\Psi\|_{F_\Psi} F_\Psi(0) < \infty. \tag{6.1}$$

Lemma 6.0.1. *There exists $C > 0$ such that for all $A \in \mathcal{D}_\zeta$,*

$$\left| \frac{d}{ds} \alpha_s(A) \right| \leq C \|A\|_\zeta \tag{6.2}$$

Proof. This will follow from the computation below, which will be justified by an interchange of limits and the derivative formula of Theorem 3.9 of [24] for local observables:

$$\begin{aligned}\frac{d}{ds}\alpha_s(A) &= \frac{d}{ds}\lim_{N\to\infty}\alpha_s(\mathbb{E}_N(A))\\ &= \lim_{N\to\infty}\frac{d}{ds}\alpha_s(\mathbb{E}_N(A))\\ &= \lim_{N\to\infty}\alpha_s(\delta_s(\mathbb{E}_N(A)))\\ &= \alpha_s(\delta_s(A))\end{aligned} \tag{6.3}$$

in which case then, by Equation (6.17) below, there will exists $C > 0$ such that:

$$\|\alpha_s(\delta_s(A))\| \leq \|\delta_s(A)\| \leq C\|A\|_\zeta. \tag{6.4}$$

First we justify the interchange of limits in the second equality of (6.3). For fixed $s \in [0,1)$, define the continuous function:

$$\begin{aligned} &f_n : [-s, 1-s] \to \mathfrak{A}\\ &f_n(h) = \begin{cases} \frac{1}{h}(\alpha_{s+h}(\mathbb{E}_n(A)) - \alpha_s(\mathbb{E}_n(A))) & \text{if } h \neq 0\\ \alpha_s(\delta_s(\mathbb{E}_n(A))) & \text{else}\end{cases}.\end{aligned} \tag{6.5}$$

The case $s = 1$ is similar using a one-sided left limit. We show that (f_n) is uniformly Cauchy. For any $B \in \mathfrak{A}_{\text{loc}}$ and $h \neq 0$,

$$\|\frac{1}{h}(\alpha_{s+h}(B) - \alpha_s(B))\| = \frac{1}{|h|}\|\int_{\min\{s,s+h\}}^{\max\{s+h\}} dr\ \frac{d}{dr}\alpha_r(B)\| \leq \frac{1}{|h|}\int_{\min\{s,s+h\}}^{\max\{s+h\}} dr\ \|\alpha_r(\delta_r(B))\| \tag{6.6}$$

Denote $\Lambda_n = [-n,n]^D$, $\mathbb{E}_n = \mathbb{E}_{\Lambda_n}$ and $H_n = H_{\Lambda_n}(\Psi_r)$. Denote $B = \mathbb{E}_n(A) - \mathbb{E}_m(A)$ for $n > m$, and $B_j = \mathbb{E}_j(A) - \mathbb{E}_{j-1}(A)$. Then:

$$\|\delta_r(B)\| \leq \sum_{j=m+1}^{n}\|\delta_r(B_j)\| \tag{6.7}$$

For a single summand:

$$
\begin{aligned}
\|\delta_r(B_j)\| &= \lim_{\substack{N\to\infty \\ N>2j}} \|[H_N, B_j]\| \\
&\leq \|[H_{2j}, B_j]\| + \|[\sum_{k=1}^{\infty} H_{2j+k} - H_{2j+k-1}, B_j]\| \\
&\leq 4\|\Psi\|_{F_\Psi} F_\Psi(0)\|A\|_\zeta |\Lambda_{2j}|\zeta(j-1) + \\
&\qquad\qquad + \sum_{k=1}^{\infty} \|[\sum \left\{\Psi(X;s) : X \subset \Lambda_{2j+k}, X \cap (\Lambda_{2j+k} \setminus \Lambda_{2j+k-1}) \neq \emptyset, X \cap \Lambda_j \neq \emptyset\right\}, B_j]\|.
\end{aligned}
\tag{6.8}
$$

The sets in the kth summand of:

$$
\sum_{k=1}^{\infty} \|[\sum \left\{\Psi(X;s) : X \subset \Lambda_{2j+k}, X \cap (\Lambda_{2j+k} \setminus \Lambda_{2j+k-1}) \neq \emptyset, X \cap \Lambda_j \neq \emptyset\right\}, B_j]\| \tag{6.9}
$$

have diameter at least $j + k$, and so:

$$
\begin{aligned}
(0.8) &\leq \sum_{k=1}^{\infty} \sum_{x \in (\Lambda_{2j+k} \setminus \Lambda_{2j+k-1})} \sum_{y \in \Lambda_j} \|\Psi\|_{F_\Psi} F_\Psi(\|x-y\|)\|B_j\| \\
&\leq \sum_{k=1}^{\infty} \sum_{x \in (\Lambda_{2j+k} \setminus \Lambda_{2j+k-1})} \|\Psi\|_{F_\Psi} |\Lambda_j| F_\Psi(d(x, \Lambda_j))\|B_j\| \\
&\leq \sum_{k=1}^{\infty} \|\Psi\|_{F_\Psi} |\Lambda_j||\Lambda_{2j+k}| F_\Psi(j+k)\|B_j\| \\
&\leq 72^D \big(\sum_{k=1}^{\infty} k^D F_\Psi(k)\big) j^{2D} \|B_j\| \\
&\leq C_\Psi j^{2D} \zeta(j-1)\|A\|_\zeta
\end{aligned}
\tag{6.10}
$$

for a constant $C_\Psi > 0$ independent of n and m. Hence:

$$
\|\delta_r(B)\| \leq \left(\sum_{j=m+1}^{\infty} 4\|\Psi\|_{F_\Psi} F_\Psi(0)|\Lambda_{2j}|\zeta(j-1) + C_\Psi j^{2D}\zeta(j-1) \right) \|A\|_\zeta \tag{6.11}
$$

and by decay of ζ, the function $m \mapsto \sum_{j=m+1}^{\infty} j^{2D}\zeta(j-1)$ decays to 0 as $m \to \infty$, uniformly in r.

This also shows that at $h = 0$:

$$\|f_n(0) - f_m(0)\| \leq \sup_{s \in [0,1)} \|\delta_s(B)\| \to 0 \text{ as } m \to \infty. \tag{6.12}$$

Conclude that (f_n) is uniformly convergent on $[-s, 1-s]$, and that:

$$\lim_{n\to\infty} \lim_{h\to 0} f_n(s) = \lim_{h\to 0} \lim_{n\to\infty} f_n(s) \tag{6.13}$$

In particular, $\frac{d}{ds}\alpha_s(A)$ exists and is equal to $\alpha_s(\delta_s(A))$. Next, we show there exists $C > 0$ such that $\|\frac{d}{ds}\alpha_s(A)\| \leq C\|A\|_\zeta$. Setting $m = 1$, so that $B = \mathbb{E}_n(A) - \mathbb{E}_1(A)$, equation (6.11) yields:

$$\|\delta_r(\mathbb{E}_n(A))\| \leq \left(\sum_{j=2}^{\infty} 4\|\Psi\|_{F_\Psi} F_\Psi(0)|\Lambda_{2j}|\zeta(j-1) + C_\Psi j^{2D}\zeta(j-1)\right)\|A\|_\zeta + \|\delta_r(\mathbb{E}_1(A))\| \tag{6.14}$$

The norm $\|\delta_r(\mathbb{E}_1(A))\|$ is bounded independently of n by:

$$\begin{aligned}\|\delta_r(\mathbb{E}_1(A))\| &\leq \|[H_1, \mathbb{E}_1(A)]\| + \sum_{j=2}^{\infty} \|[H_j - H_{j-1}, \mathbb{E}_1(A)]\| \\ &\leq 2\|H_1\|\|A\| + \sum_{j=2}^{\infty} \|\Psi\|_{F_\Psi}|\Lambda_j|F_\Psi(j-1)\|A\| \\ &\leq \left(2\|H_1\| + \sum_{j=2}^{\infty} \|\Psi\|_{F_\Psi}|\Lambda_j|F_\Psi(j-1)\right)\|A\|_\zeta\end{aligned} \tag{6.15}$$

So let:

$$C = \left(\sum_{j=2}^{\infty} 4\|\Psi\|_{F_\Psi} F_\Psi(0)|\Lambda_{2j}|\zeta(j-1) + C_\Psi j^{2D}\zeta(j-1)\right) + 2\sup_{s\in[0,1]} \|H_{\Lambda_1}(\Psi_s)\| + \sum_{j=2}^{\infty} \|\Psi\|_{F_\Psi}|\Lambda_j|F_\Psi(j-1). \tag{6.16}$$

Since C is independent of n, passing to the limit yields:

$$\|\delta_r(A)\| \leq C\|A\|_\zeta. \tag{6.17}$$

$\square$

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