Sharp Fronts for the Surface Quasi-Geostrophic Equation

By

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DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

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2020

To my family.

Contents

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Abstract

Piecewise-constant solutions of the surface quasi-geostrophic (SQG) equation support surface waves. We study two types of such solutions, called the front solutions and the two-front solutions. For fronts that are described as a graph, the formal contour dynamics equation does not converge. Using three different methods, we derive a well-formulated meaningful contour dynamics equation for the SQG fronts. This equation is nonlocal, quasi-linear, and has logarithmic dispersion relation. When the fronts have small slopes, we derive a cubically nonlinear approximate equation. We prove local well-posedness of the initial value problem for this approximate equation posed on the circle. Numerical solutions of the approximate equation provide evidence of the formation of finite-time singularities. We also prove that for sufficiently small and smooth initial data, the full SQG front equation posed on the real line has unique global solutions. For the SQG two-front solutions, the contour dynamics equations form a system with more complicated dispersion relations and quadratic nonlinearities. We use the contour dynamics equations to determine the linearized stability of the SQG shear flows that correspond to two flat fronts. We also prove local-in-time existence and uniqueness for small, smooth solutions.

Acknowledgments

My deepest gratitude should be sent to my advisor, Professor John K. Hunter, for his consistent support and advice. His profound mathematical insights, great attention to detail, and absolute enthusiasm in teaching have been a source of precious lessons for me. I cannot imagine finishing my doctoral studies without his encouragement and support, especially when I was experiencing hardships.

I am also extremely grateful to an efficient collaborator and great friend, Professor Qingtian Zhang, who gave me enormous invaluable advice both academically and non-academically. He showed me what kind of passion a young researcher should possess for mathematics.

I would like to extend my sincere thanks to my thesis committee, Professor Steve Shkoller and Professor Blake Temple, for reviewing my thesis. I also appreciate the opportunities of taking their graduate-level courses, which broadened my horizon on many important and intriguing problems and techniques. I thank Professor Joseph Biello, Professor Bruno Nachtergaele, and Professor Abigail Thompson for helpful advising, beneficial discussions, and friendships.

Thanks should also go to other people in the department. These include Kyle Chickering, Black Jiang, Ryan Christopher Moreno-Vasquez, Raaghav Ramani, Annalaura Stingo, Karry Wong, Bohan Zhou and many others. I enjoy discussions, hanging out after seminars, and being friends with them. I very much appreciate my officemates, Chao-Ping Lin, Xiaochen Liu, Yanwen Luo, and Jiawei Wang, for sharing space through the years, for being supportive in many aspects, and for playing card games on game days.

Additionally, I thank many great friends out of academia, who share with me different possibilities of life. These friends include Dong An, Sussan Finn, Xinyuan Liu, Yishan Xiang, and Da Zhang. I also thank the staff in the department, especially Tina Denena, Sarah Driver, and Victoria Whistler.

Last but not least, I am deeply indebted to my parents and grandparents for their eternal love.

CHAPTER 1

Introduction

By introducing you to yourself it enables you to discover for yourself the true meaning of life and thus enrich every moment of it.

– Venkatesananda Saraswati

1.1. Surface quasi-geostrophic equation

The surface quasi-geostrophic (SQG) equation arises from the quasi-geostrophic (QG) equation in atmosphere sciences. The QG equation describes stratified mid-to-high latitude synoptic scale dynamics in oceanic or atmosphere flows. One of the major hypotheses of the flows in this altitude range is that the long-scale dynamics of the fluids is governed by the near balance between the Coriolis force and horizontal pressure gradients [**Maj03**]. The SQG equation is an approximate equation for the QG equation confined near a surface [**Lap17**], and can also be derived from the 3D Navier-Stokes equations coupled with temperature via Boussinesq approximation under the smallness of Rossby and Ekman numbers as well as constant potential vorticity [**HPGS95**, **Ped79**, **Sal98**]. A simpler explanation is that the SQG equation describes quasi-geostrophic flows with a potential vorticity sheet [**Bie**, **Lap17**], as illustrate in Figure 1.1.

In mathematical literature, the (inviscid) SQG equation is usually written as

$$\theta_t + \mathbf{u} \cdot \nabla\theta = 0, \tag{1.1a}$$

$$\mathbf{u} = \nabla^\perp(-\Delta)^{-1/2}\theta. \tag{1.1b}$$

Here, $\theta(\mathbf{x}, t)$ with $\mathbf{x} = (x, y)$ is an unknown scalar field, $\nabla^\perp = (-\partial_y, \partial_x)$, and the velocity field $\mathbf{u}(\mathbf{x}, t)$ is determined nonlocally from θ by $\nabla^\perp(-\Delta)^{-1/2}$, which could be identified with a perpendicular Riesz transform

$$\mathbf{u} = -\mathbf{R}^\perp\theta.$$

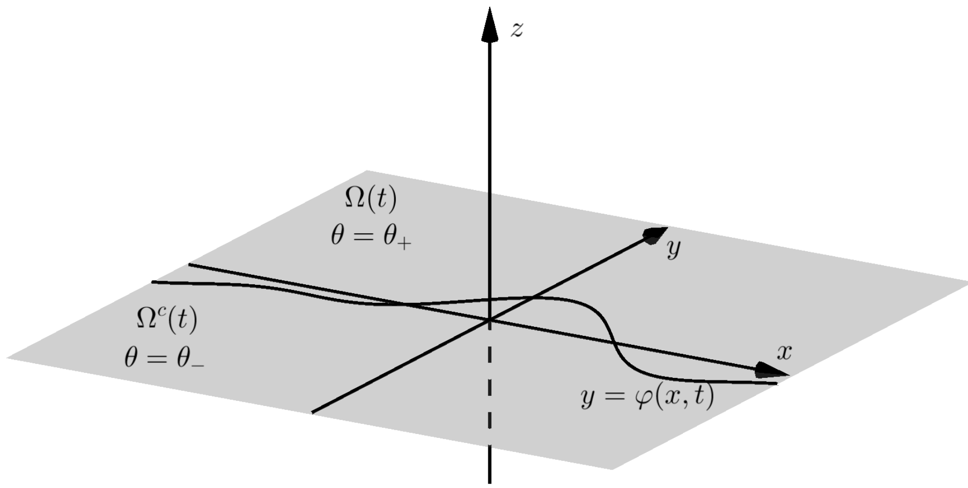


FIGURE 1.1. A quasi-geostrophic potential vorticity sheet with an SQG front on the sheet.

The Riesz transform also has a potential representation [**Ste70**, **Ste93**]

$$\mathbf{u}(\mathbf{x}) = \mathbf{R}^{\perp}\theta(\mathbf{x}) = \frac{1}{2\pi}\lim_{\epsilon\to 0+}\int_{\mathbb{R}^2\setminus B_\epsilon(\mathbf{x})} \frac{(\mathbf{x}-\mathbf{y})^{\perp}}{|\mathbf{x}-\mathbf{x}'|^3}\theta(\mathbf{x}')\,\mathrm{d}\mathbf{x}'.$$

More generally, the Riesz transform can also be defined in terms of a Neumann-Dirichlet map for the 3D Laplacian in the derivation of the 2D SQG equation from the 3D QG equation. See also Section 2.1.

The analytical study of (1.1) traces back to [**CMT94a**, **CMT94b**], where strong mathematical similarities between the SQG equation and the 3D Euler's equations are shown. This makes the SQG equation a useful 2D model for singularity formation in the 3D incompressible Euler equations. Indeed, if we take the skew gradient $\nabla^{\perp}$ of the equation (1.1a), we obtain

$$(\nabla^{\perp}\theta)_t + \mathbf{u}\cdot\nabla(\nabla^{\perp}\theta) = (\nabla^{\perp}\theta)\cdot\nabla\mathbf{u}.$$

In contrast, the vorticity-stream function formulation of the 3D Euler equation reads

$$\omega_t + \mathbf{u}\cdot\nabla\omega = \omega\cdot\nabla\mathbf{u},$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is a 3D velocity field, and $\omega = \operatorname{curl}\mathbf{u}$ is the vector vorticity of the fluid. This resemblance suggests that the 2D vector field $\nabla^{\perp}\theta$ in the SQG equation plays a similar role of ω in the 3D incompressible Euler equations. The SQG equation has global weak solutions in L^p-spaces

($p > 4/3$) [**Mar08**, **Res95**], and convex integration shows that low-regularity weak solutions need not be unique [**BSV19**]. A class of nontrivial global smooth solutions is constructed in [**CCGSar**], but — as for the 3D incompressible Euler equations — the question of whether general smooth solutions of the SQG equation remain smooth for all time or form singularities in finite time is open. See Table 1.1 for a summarization of some of the resemblance [**BKM84**, **BSV19**, **CMT94a**, **CMT94b**, **MB02**, **Shn97**].

3D Euler	2D SQG
$\omega = \nabla \times \mathbf{u}$	$\nabla^\perp \theta = (-\partial_y \theta, \partial_x \theta)$
$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \omega \cdot (\nabla \mathbf{u})$	$\partial_t (\nabla^\perp \theta) + \mathbf{u} \cdot \nabla (\nabla^\perp \theta) = (\nabla^\perp \theta) \cdot (\nabla \mathbf{u})$
$\nabla \cdot \omega = \nabla \cdot \mathbf{u} = 0$	$\nabla \cdot (\nabla^\perp \theta) = \nabla \cdot \mathbf{u} = 0$
$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathcal{K}_3(\mathbf{x}') \omega(\mathbf{x} - \mathbf{x}') \, \mathrm{d}\mathbf{x}'$	$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^2} \mathcal{K}_2(\mathbf{x}') \nabla^\perp \theta(\mathbf{x} - \mathbf{x}') \, \mathrm{d}\mathbf{x}'$
vortex lines move with the fluid	level sets of θ move with the fluid
$\|\mathbf{u}(t)\|_{L^2} = \|\mathbf{u}(0)\|_{L^2}$	$\|\mathbf{u}(t)\|_{L^2} = \|\mathbf{u}(0)\|_{L^2}$
local existence in H^s	local existence in H^s
BKM $\int_0^T \|\omega\|_{L^\infty} \, \mathrm{d}s \xrightarrow{T \to T_*} \infty$	$\int_0^T \|\nabla^\perp \theta\|_{L^\infty} \, \mathrm{d}s \xrightarrow{T \to T_*} \infty$
Nonuniqueness of weak solutions	Nonuniqueness of weak solutions

Table 1.1. Comparison between 3D Euler equations and 2D SQG equation. $\mathcal{K}_2$ and $\mathcal{K}_3$ are the Green's function of (generalized) Biot-Savart law in two-dimension and three-dimension, respectively.

It is also remarkable to compare the SQG equation with the 2D incompressible Euler equation in the vorticity-stream function formulation [**MT96**]. These two equations both fall into a family of *active scalar* equations

$$\begin{aligned} &\theta_t + \mathbf{u} \cdot \nabla \theta = 0, \\ &\mathbf{u} = \nabla^\perp (-\Delta)^{-\alpha/2} \theta, \end{aligned} \tag{1.2}$$

where $\alpha \in (0, 2]$ is a parameter. If $\alpha = 1$, (1.2) is the SQG equation, while if $\alpha = 2$, (1.2) corresponds to the Euler equation. When α takes other values, (1.2) is referred to as the generalized surface quasi-geostrophic (GSQG) equation, and they are a natural generalization of the other two cases. There are other values of α of interest [**KRara**, **KRarb**], but we do not mention them here. The operator $(-\Delta)^{-\alpha/2}$ is the Fourier multiplier operator with symbol $|\mathbf{k}|^{-\alpha}$:

$$(-\Delta)^{-\alpha/2}\int_{\mathbb{R}^2}\hat{f}(k,\ell)e^{ikx+i\ell y}\,\mathrm{d}k\,\mathrm{d}\ell=\int_{\mathbb{R}^2}(k^2+\ell^2)^{-\alpha/2}\hat{f}(k,\ell)e^{ikx+i\ell y}\,\mathrm{d}k\,\mathrm{d}\ell,$$

and the relation between $\mathbf{u}$ and θ is sometimes referred to as the *(generalized) Biot-Savart law*

$$\mathbf{u}(\mathbf{x})=g_\alpha\nabla^\perp G*\theta,$$

where G is the Green's function for $(-\Delta)^{\alpha/2}$ (Riesz potential if $0<\alpha<2$ [**MZ97**, **Ste93**]) or the two-dimensional Newtonian kernel if $\alpha=2$ [**MB02**]

$$G(x)=\begin{cases}-\frac{1}{2\pi}\log|x| & \text{if } \alpha=2,\\ |x|^{-(2-\alpha)} & \text{if } 0<\alpha<2,\end{cases}\qquad g_\alpha=\begin{cases}1 & \text{if } \alpha=2,\\ \frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{2^\alpha\pi\Gamma\left(\frac{\alpha}{2}\right)} & \text{if } 0<\alpha<2.\end{cases}\tag{1.3}$$

The endpoint case (Euler equation, $\alpha=2$) is mostly studied, and the scalar $\theta=\nabla^\perp\cdot\mathbf{u}$ is the scalar vorticity. It has long been established that the 2D Euler equation has global smooth solutions [**H3̈3**, **Wol33**]. Further results on the 2D Euler equation can be found in [**MB02**, **MP94**] and the references therein.

As for the other cases in the family, with $0<\alpha<1$ or $1<\alpha<2$, local existence of smooth solutions of these equations is proved in [**CCC$^+$12**], but the global existence of smooth solutions with general initial data is not known for any $0<\alpha<2$.

1.2. Piecewise constant solutions

We first review a notion of a weak solution of the GSQG equation [**Res95**].

Definition 1.2.1 (Weak solutions of the GSQG equation)**.** A bounded function θ is a *weak solution* of the GSQG equation if for any $\phi\in C_0^\infty(\mathbb{R}^2\times(0,T))$, we have

$$\int_{\mathbb{R}^2\times(0,T)}\left[\theta(\mathbf{x},t)\phi_t(\mathbf{x},t)+\theta(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\nabla\phi(\mathbf{x},t)\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t=0.$$

Equation (1.2) has a class of piecewise constant solutions of the form

$$\theta(\mathbf{x},t)=\sum_{k=1}^N\theta_k\mathbb{1}_{\Omega_k(t)}(\mathbf{x}),\tag{1.4}$$

where $N \geq 2$ is a positive integer, $\theta_1, \dots, \theta_N \in \mathbb{R}$ are constants, and $\Omega_1(t), \dots, \Omega_N(t) \subset \mathbb{R}^2$ are disjoint domains such that

$$\bigcup_{k=1}^{N} \overline{\Omega_k(t)} = \mathbb{R}^2,$$

and their boundaries $\partial\Omega_1(t), \partial\Omega_2(t), \dots, \partial\Omega_N(t)$ are smooth curves, whose components either coincide or are a positive distance apart. In (1.4), $\mathbb{1}_{\Omega_k(t)}$ denotes the indicator function of $\Omega_k(t)$. The transport equation (1.2) preserves the form of these weak solutions, at least locally in time, and to study their evolution, we only need to understand the dynamics of the boundaries $\partial\Omega_k(t)$ of the regions $\Omega_k(t)$.

Depending on the number of regions and the boundedness of each region, we distinguish the following three different types of solutions (see Figure 1.2). In this dissertation, we will be mainly concerned with the second and the third types, which we call *front* and *two-front* solutions, for the SQG equation (1.1), but in this chapter, we review also some of the results on GSQG equations (1.2).

1.2.1. Patches. Equation (1.4) is a patch solution if it satisfies the following assumptions:

(1) $N \geq 2$;

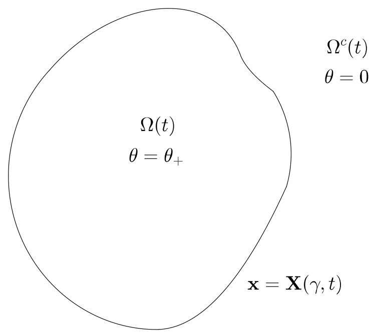
(2) $\theta_N = 0$, but $\theta_k \in \mathbb{R} \setminus \{0\}$ for each $1 \leq k \leq N-1$;

(3) for each $1 \leq k \leq N-1$, the region $\Omega_k(t)$ is bounded, and its boundary $\partial\Omega_k(t)$ is a smooth, simple, closed curve that is diffeomorphic to the circle $\mathbb{T}$;

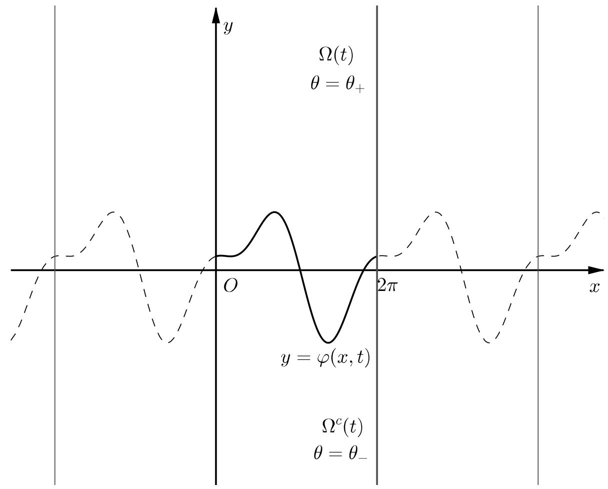
(4) the region $\Omega_N(t)$ is unbounded.

Under these assumptions, θ has compact support and contour dynamics equations for the motion of the patches are straightforward to derive, as was first done in [**ZHR79**] for the vortex-patch solutions of the Euler equation. The 2D Euler equation has global weak solutions with vorticity in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ [**MB02**, **Jud63**], and smooth vortex patch boundaries remain smooth and non-self-intersecting for all times [**BC93**, **Che93**, **Che98**, **Ser94**]. Some special types of nontrivial global-in-time smooth vortex patch solutions are constructed in [**Bur82**, **CCGS16b**, **dlHHMV16**, **HM16**, **HM17**, **HMV13**], mostly using a Crandall–Rabinowitz bifurcation theorem [**CR71**].

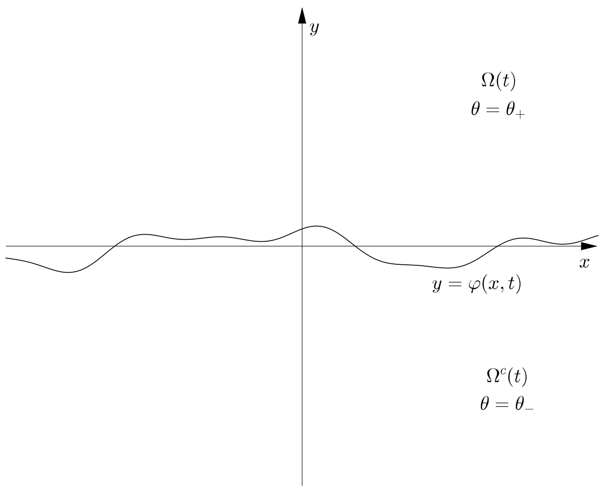
By use of a reparametrization-by-arclength technique, local well-posedness of the contour dynamics equations for SQG and GSQG patches is proved in [**CCC+12**, **CCG18**, **Gan08**, **GPar**]. The question of whether finite-time singularities can form in smooth boundaries of SQG or GSQG patches remains open, but it is proved in [**GS14**] that splash singularities cannot form. To be more



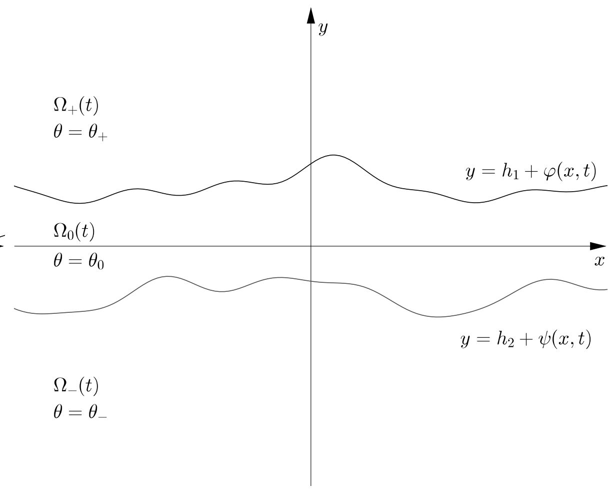
(a) Patch problem with $N = 2$.



(b) Spatially periodic front problem.



(c) Non-periodic front problem.



(d) Two-front problem.

FIGURE 1.2. Different types of problems.

specific, the boundary of an SQG patch cannot self-intersect in finite time as long as the boundary stays smooth. In [**CCGS16a**, **dlHHH16**, **GS19**, **HH15**, **HM17**], some particular classes of nontrivial global solutions for SQG and GSQG patches have been shown to exist. These solutions have some symmetries and are either rotating in time (time-periodic) or steady (the shape of the boundary do not change in time). See also [**GSPSYar**].

The local existence and the formation of finite-time singularities of smooth GSQG patches in the presence of a rigid boundary is shown in [**GPar**, **KYZ17**] for $\alpha \in (5/3, 2)$. By contrast, vortex patches in this setting (with $\alpha = 2$) have global regularity [**KRYZ16**].

Numerical solutions for vortex patches show that they form extraordinarily thin, high-curvature filaments [**Dri88**, **DM90**], although their boundaries remain smooth globally in time. On the other hand, numerical solutions for SQG patches suggest that complex, self-similar singularities can form in the boundary of a single patch [**SD14**, **SD19**] and provide evidence that two separated SQG patches can touch in finite time [**CFMR05**].

1.2.2. Fronts. Equation (1.4) is a front solution if it satisfies the following assumptions:

(1) $N = 2$;
(2) $\theta_1, \theta_2 \in \mathbb{R}$ are distinct constants;
(3) both $\Omega_1(t)$ and $\Omega_2(t)$ are unbounded and they share a boundary which is a simple, smooth curve diffeomorphic to $\mathbb{R}$.

When $1 \leq \alpha \leq 2$, the kernel of the (generalized) Biot-Savart law, which recovers the velocity field $\mathbf{u}$ from the scalar θ, decays too slowly at infinity for the formal front equations to converge. This differentiates the patch problems and the front problems, since there are no convergence issues at infinity in the case of patches with compactly supported θ. One goal of this dissertation is to present several methods deriving meaningful contour dynamics equations for the SQG fronts used in [**HS18**] and [**HSZara**].

The front problem for vorticity discontinuities in the Euler equation is studied in [**Ray96**], where it is shown that vorticity discontinuity is linearly stable, and that the surfaces waves propagate along discontinuity, but decay exponentially into the interior. It is also shown that the location of discontinuity can be described by the Burgers-Hilbert equation using asymptotic analysis [**BH10**, **HMVSZ**]. Local existence and uniqueness for spatially periodic SQG fronts is proved for C^∞ solutions in [**Rod05**] using a Nash–Moser iteration, and analytic solutions in [**FR11**] using a Cauchy–Kowalevski theorem. Spatially periodic almost SQG sharp fronts are studied in [**CFR04**, **FLR12**, **FR12**, **FR15**]. Smooth C^∞ solutions for spatially periodic GSQG fronts with $1 < \alpha < 2$ also exist locally in time [**CFMR05**].

In the non-periodic setting, smooth solutions to the GSQG front equations on $\mathbb{R}$ with $0 < \alpha < 1$ are shown to exist globally in time for small initial data in [**CGSI19**]. When $1 \leq \alpha \leq 2$, a regularization procedure is needed in the derivation of the front equations to account for the divergence of the naive contour dynamics equations at infinity [**HS18**], and smooth solutions also exist globally when $1 < \alpha < 2$ [**HSZarb**]. In this dissertation, we will also prove local and global well-posedness of an initial-value problem for the full SQG front equation for small, smooth initial data [**HSZarc**]. See also a review article [**Shu20**].

In this dissertation, without loss of generality and for simplicity, we write $\theta_+ = \theta_1$ and $\theta_- = \theta_2$, normalize the jump $\theta_+ - \theta_-$ to 2π, and consider only fronts that are a graph, located at

$$y = \varphi(x,t),$$

where $\varphi(\cdot,t)\colon \mathbb{R} \to \mathbb{R}$ is a smooth, bounded function. To be more specific, the transported scalar θ has following profile

$$\theta(x,y,t) = \begin{cases} 2\pi & \text{if } y > \varphi(x,t), \\ 0 & \text{if } y < \varphi(x,t). \end{cases} \tag{1.5}$$

1.2.3. Two-fronts. Equation (1.4) is a two-front solution if it satisfies the following assumptions:

(1) $N = 3$;

(2) $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ with $\theta_1 \neq \theta_2$ and $\theta_2 \neq \theta_3$;

(3) there is a diffeomorphism $\Psi_t\colon \mathbb{R}^2 \to \mathbb{R}^2$, satisfying $\Psi_t(\Omega_1(t)) = \mathbb{R} \times (1,\infty)$, $\Psi_t(\Omega_2(t)) = \mathbb{R} \times (-1,1)$, and $\Psi_t(\Omega_3(t)) = \mathbb{R} \times (-\infty,-1)$.

This case is mainly studied in [**HSZard**], and in this dissertation, we present the derivation of the SQG two-front system and prove well-posedness results for the resulting system. For notational simplification, we write $\Omega_+(t) = \Omega_1(t)$, $\Omega_0(t) = \Omega_2(t)$, $\Omega_-(t) = \Omega_3(t)$, with the same subscript changes applying to θ_+, θ_0, θ_-. We also define the jumps in θ across the fronts, scaled by a convenient factor g_α given in (1.3), by

$$\Theta_+ = g_\alpha(\theta_+ - \theta_0), \qquad \Theta_- = g_\alpha(\theta_0 - \theta_-). \tag{1.6}$$

Numerical solutions of the contour dynamics equations for spatially-periodic two-front solutions of the Euler equation and a study of the approximation of vortex sheets by a thin vortex layer are given in [**BS90**]. The two-front problem for the GSQG ($0 < \alpha \leq 2$) is studied in [**HSZard**].

In this dissertation, we only consider two-front solutions whose fronts are graphs located at

$$y = h_+ + \varphi(x,t), \qquad y = h_- + \psi(x,t),$$

where $\varphi, \psi \colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ denote the perturbations from the flat fronts $y = h_+$, $h = h_-$, and $h_+ > h_-$. To be more precise, the transported scalar θ admits following profile

$$\theta(x, y, t) = \begin{cases} \theta_+ & \text{if } y \in (h_+ + \varphi(x,t), \infty)\,, \\ \theta_0 & \text{if } y \in (h_- + \psi(x,t), h_+ + \varphi(x,t))\,, \\ \theta_- & \text{if } y \in (-\infty, h_- + \psi(x,t))\,. \end{cases} \tag{1.7}$$

1.3. Chapter summaries

The results in this dissertation are presented into five chapters, containing essentially selected results of papers [**HS18**, **HSZ18**, **HSZara**, **HSZarc**, **HSZard**]. This dissertation is organized as follows. In Chapter 2, we collect some fundamental facts and results that we will use throughout the whole dissertation. In Chapter 3, we present three methods to derive the contour dynamics equation for the SQG front equation — the regularization method proposed in [**HS18**], the decomposition method, and the modified Green's function method presented in [**HSZara**], and two methods to derive the contour dynamics system for the two-front SQG problem [**HSZard**]. In Chapter 4, we study an approximate equations for the SQG fronts. We present a *weak* local well-posedness result for a dispersionless version of the approximate equation posed on circle $\mathbb{T}$, in which solutions may lose Sobolev derivatives over time [**HS18**]. This analysis is of independent interest as it is an adaption of proofs for Gevrey-class solutions to the Sobolev solutions. We also prove a stronger short-time well-posedness result for the approximate SGQ equation posed on $\mathbb{T}$ using Weyl para-differential calculus [**HSZ18**], and present some numerical simulations for the approximate SQG front. In Chapter 5, we prove that the full SQG front equation admits global-in-time solutions for small and smooth initial data [**HSZarc**]. We finish the main part the of this dissertation by Chapter 6, where we first present the derivation of two-front SQG contour dynamics systems — the regularization method and the decomposition method, and then prove local well-posedness of the initial value problem for the system with small and smooth data [**HSZard**]. In appendices, we present an alternative formulation for the SQG front equation in Appendix A and prove some algebraic inequalities in Appendix B.

CHAPTER 2

Preliminaries

The beginner ... should not be discouraged if ... he finds that he does not have the prerequisites for reading the prerequisites.

– Paul Halmos

The goal of this chapter is devoted to summarize a number of theories and facts, as well as fix some notations that we will need to use throughout the dissertation. In Section 2.1, we state definitions and some properties of the Riesz transform, which relates L^∞ space and the BMO space. In Section 2.2, Fourier transform, Fourier multipliers, Sobolev spaces, and Weyl para-products are defined. We also prove some basic commutator estimates. In Section 2.3, we state some lemmas regarding a frequency cut-off Fourier multipliers and a multilinear estimate, which are widely used in Chapter 5. Finally, in Section 2.4, we review the definitions and some basic properties for the modified Bessel's functions.

2.1. Riesz transform, BMO space, and Dirichlet-Neumann maps

2.1.1. Riesz transform and BMO space. In this subsection, we recall some definitions and properties of the Riesz transform and the space of bounded mean oscillations (BMO). For more details, see [**Duo01**, **FS72**, **Ste70**, **Ste93**].

When $1 < p < \infty$, the Riesz transform $\mathbf{R}\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n;\mathbb{R}^n)$ is the bounded singular integral operator defined pointwise a.e. for $f \in L^p(\mathbb{R}^n)$ by [**Duo01**]

$$\begin{aligned} \mathbf{R}f(\mathbf{x}) &= C_n \text{p.v.} \int_{\mathbb{R}^n} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^{n+1}} f(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \\ &= C_n \lim_{\epsilon\to 0+} \int_{\mathbb{R}^n\setminus B_\epsilon(\mathbf{x})} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^{n+1}} f(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \\ C_n &= \frac{1}{\pi^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right), \end{aligned} \tag{2.1}$$

where $B_\epsilon(\mathbf{x})$ is the ball of radius ϵ centered at $\mathbf{x}$. One can also write $\mathbf{R} = -\nabla(-\Delta)^{-1/2}$.

For $f \in L^\infty(\mathbb{R}^n)$, the principal value integral on the right-hand side of (2.1) does not define $\mathbf{R}f$, unless it happens to converges absolutely at infinity. However, the Riesz transform can be extended to a bounded linear map $\mathbf{R}\colon L^\infty(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n;\mathbb{R}^n)$, where BMO denotes the Banach space of functions of bounded mean oscillation.

The BMO-norm of $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\|f\|_{\mathrm{BMO}} = \sup_{B\subset\mathbb{R}^n} \fint_B \left| f - \fint_B f \right|, \qquad \fint_B f = \frac{1}{|B|}\int_B f,$$

where B ranges over all balls and $\fint_B f$ denotes the average of f over B. The BMO-norm of a constant is equal to zero, and functions that differ by a constant are regarded as equivalent in BMO. The space BMO consists of equivalence classes of locally integrable functions with finite BMO-norms.

The Riesz transform of $f \in L^\infty(\mathbb{R}^n)$ can be defined by [**Duo01**]

$$\mathbf{R}f(\mathbf{x}) = \mathbf{R}[f\mathbb{1}_B](\mathbf{x}) + C_n \int_{\mathbb{R}^n\setminus B} \left[\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^{n+1}} - \frac{\mathbf{x}_0-\mathbf{x}'}{|\mathbf{x}_0-\mathbf{x}'|^{n+1}} \right] f(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{2.2}$$

where $\mathbf{x}_0 \in \mathbb{R}^n$ is a fixed point, B is a ball that contains $\mathbf{x}$ and $\mathbf{x}_0$, $\mathbb{1}_B$ is the characteristic function of B, and $\mathbf{R}[f\mathbb{1}_B]$ is defined as in (2.1). The integral on the right-hand side of (2.2) converges absolutely since the integrand is $O(|\mathbf{x}'|^{-(n+1)})$ as $|\mathbf{x}'| \to \infty$. Different choices of $\mathbf{x}_0$ and B lead to functions that differ by a constant, so they are equivalent in BMO, and it can be shown that $\mathbf{R}f \in \mathrm{BMO}$ for $f \in L^\infty$. In particular, $\mathbf{R}1 = 0$ in BMO. If the support E of $f \in L^\infty(\mathbb{R}^n)$ is a proper subset of $\mathbb{R}^n$ and $\mathbf{x}_0 \notin E$, then we can also define

$$\mathbf{R}f(\mathbf{x}) = C_n \mathrm{p.v.}\int_E \left[\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^{n+1}} - \frac{\mathbf{x}_0-\mathbf{x}'}{|\mathbf{x}_0-\mathbf{x}'|^{n+1}} \right] f(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{2.3}$$

since this expression agrees with (2.2) up to a constant.

2.2. Fourier transform, Sobolev space, and Weyl para-differential calculus

In this section, we first fix some notations for Fourier transforms, Fourier series, Sobolev spaces, and Weyl para-differential calculus, and then state several lemmas for the Fourier multiplier operators

$$L = \log|\partial_x|, \qquad D = -i\partial_x, \qquad |D|^s = |\partial_x|^s \tag{2.4}$$

with symbols $\lambda_L(\xi)$, ξ, and $|\xi|^s$, respectively, where

$$\lambda_L(\xi) = \begin{cases} \log|\xi| & \text{if } \xi \in \mathbb{Z}_*, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$ is the set of nonzero integers.

For $f\colon \mathbb{T} \to \mathbb{C}$, we denote the Fourier series of f by $\mathcal{F}f \equiv \hat{f}\colon \mathbb{T} \to \mathbb{C}$

$$f(x) = \sum_{\xi\in\mathbb{Z}} \hat{f}(\xi) e^{i\xi x}, \qquad \hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x) e^{-i\xi x}\,\mathrm{d}x.$$

We denote the Hilbert space of zero-mean, periodic functions with square-integrable weak derivatives of the order $s \in \mathbb{R}$ by

$$\begin{aligned} \dot{H}^s(\mathbb{T}) &= \left\{ f\colon \mathbb{T} \to \mathbb{R} | \hat{f}(0) = 0, \|f\|_{\dot{H}^s(\mathbb{R})} < \infty \right\}, \\ \|f\|_{\dot{H}^s(\mathbb{R})} &= \left(\sum_{\xi\in\mathbb{Z}_*} |\xi|^{2s} |\hat{f}(\xi)|^2 \right)^{1/2}. \end{aligned} \tag{2.5}$$

Respectively, for $f\colon \mathbb{R} \to \mathbb{C}$, by an abuse of notation concerning Fourier transforms and Fourier series, we denote the Fourier transform of f by $\hat{f}\colon \mathbb{R} \to \mathbb{C}$, where $\hat{f} = \mathcal{F}f$ is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x}\,\mathrm{d}\xi, \qquad \hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) e^{-i\xi x}\,\mathrm{d}x.$$

For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the space of Schwartz distributions f with $\|f\|_{H^s(\mathbb{R})} < \infty$, where

$$\|f\|_{H^s(\mathbb{R})} = \left[\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s |\hat{f}(\xi)|^2\,\mathrm{d}\xi \right]^{1/2}.$$

In this dissertation, we sometimes omit the space $\mathbb{T}$ or $\mathbb{R}$ when it is clear in the context. For $p \in [1, \infty]$, we denote by L^p the Lebesgue space of measurable functions satisfying $\|f\|_{L^p} < \infty$ where

$$\|f\|_{L^p} = \begin{cases} \left(\int |f(x)|^p\,\mathrm{d}x \right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \sup_x |f(x)| & \text{if } p = \infty. \end{cases}$$

In addition, if $\sigma \in \mathbb{N}$, we denote by $W^{\sigma,p}$ the Sobolev space function measurable function satisfying $\|f\|_{W^{\sigma,p}} < \infty$ where

$$\|f\|_{W^{\sigma,p}} = \sum_{k=0}^{\sigma} \|\partial_x^k f\|_{L^p}.$$

Throughout this dissertation, we use $A \lesssim B$ to mean there is a constant C such that $A \leq CB$, and $A \gtrsim B$ to mean there is a constant C such that $A \geq CB$. We use $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$. The notation $\mathcal{O}(f)$ denotes a term satisfying

$$\|\mathcal{O}(f)\|_{H^s} \lesssim \|f\|_{H^s}$$

whenever there exists $s \in \mathbb{R}$ such that $f \in H^s$. We also use $O(f)$ to denote a term satisfying $|O(f)| \lesssim |f|$ pointwise.

Let $\chi \colon \mathbb{R} \to \mathbb{R}$ be a smooth function supported in the interval $\{\xi \in \mathbb{R} \mid |\xi| \leq 1/10\}$ and equal to 1 on $\{\xi \in \mathbb{R} \mid |\xi| \leq 3/40\}$. If f is a Schwartz distribution on $\mathbb{R}$ and $a \colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a symbol, then we define the Weyl para-product $T_a f$ by

$$\mathcal{F}\left[T_a f\right](\xi) = \int_{\mathbb{R}} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \tilde{a}\Big(\xi - \eta, \frac{\xi + \eta}{2}\Big) \hat{f}(\eta) \,\mathrm{d}\eta, \tag{2.6}$$

where $\tilde{a}(\xi, \eta)$ denotes the partial Fourier transform of $a(x, \eta)$ with respect to x. For $r_1, r_2 \in \mathbb{N}_0$, we define a normed symbol space by

$$\mathcal{M}_{(r_1,r_2)} = \{a \colon \mathbb{R} \times \mathbb{R} \to \mathbb{C} : \|a\|_{\mathcal{M}_{(r_1,r_2)}} < \infty\},$$
$$\|a\|_{\mathcal{M}_{(r_1,r_2)}} = \sup_{(x,\eta)\in\mathbb{R}^2} \left\{\sum_{\alpha=0}^{r_1} \sum_{\beta=0}^{r_2} |\eta|^\beta \left|\partial_\eta^\beta \partial_x^\alpha a(x,\eta)\right|\right\}.$$

If $a \in \mathcal{M}_{(0,0)}$ and $f \in L^p$, with $1 \leq p \leq \infty$, then $T_a f \in L^p$ and

$$\|T_a f\|_{L^p} \lesssim \|a\|_{\mathcal{M}_{(0,0)}} \|f\|_{L^p}.$$

In particular, if $a \in \mathcal{M}_{(0,0)}$ is real-valued, then T_a is a self-adjoint, bounded linear operator on L^2. Weyl para-products admit the following decomposition

$$fg = T_f g + T_f g + R(f, g). \tag{2.7}$$

This decomposition is well-defined if, for example, $f \in W^{\sigma,\infty}$ and $g \in H^s$ with $s + \sigma > 0$, and then

$$\|R(f, g)\|_{H^{s+\sigma}} \lesssim \|f\|_{W^{\sigma,\infty}} \|g\|_{H^s}. \tag{2.8}$$

We can also define Weyl para-products on periodic functions, and above statements still hold. Further discussion of the Weyl calculus and para-products can be found in [**BCD11**, **Che98**, **Hö7**, **Tay00**].

Next, we prove some commutator estimates.

Lemma 2.2.1. *Suppose that* $f \in H^s(\mathbb{R})$, $a \in \mathcal{M}_{(1,0)}$, *and* $b, xb \in \mathcal{M}_{(0,0)}$. *Then*

$$\|[L, T_a]f\|_{H^s} \lesssim \|a\|_{\mathcal{M}_{(1,0)}} \|f\|_{H^{s-1}}, \tag{2.9}$$

$$\|[x, T_b]f\|_{H^s} \lesssim (\|b\|_{\mathcal{M}_{(0,0)}} + \|xb\|_{\mathcal{M}_{(0,0)}})\|f\|_{H^s}, \tag{2.10}$$

$$\|[x, L]f\|_{\dot{H}^s} \lesssim \|f\|_{\dot{H}^{s-1}} \qquad \text{if } s \geq 1. \tag{2.11}$$

Proof. **1.** We shall prove that

$$LT_a v = T_a L v + T_{Da} D^{-1} v + \mathcal{O}(T_{D^2 a} D^{-2} v).$$

Indeed, by the definition of Weyl para-product, we have for $\xi \neq 0$ that

$$\begin{aligned} \mathcal{F}\left[LT_a v\right](\xi) &= \log|\xi| \int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right) \tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) \hat{v}(\eta)\,\mathrm{d}\eta \\ &= \int_{\mathbb{R}} \log|\xi-\eta+\eta| \chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right) \tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right) \hat{v}(\eta)\,\mathrm{d}\eta. \end{aligned} \tag{2.12}$$

If (ξ, η) belongs to the support of $\chi(|\xi-\eta|/|\xi+\eta|)$, then

$$\left|\frac{\xi-\eta}{\eta}\right| \leq \frac{2}{9}. \tag{2.13}$$

To prove this claim, we use the fact that

$$|\xi-\eta| \leq \frac{1}{10}|\xi+\eta| \tag{2.14}$$

on the support of $\chi(|\xi-\eta|/|\xi+\eta|)$ and consider two cases.

- If $|\xi+\eta| \leq |\eta|$, then $|\xi-\eta| \leq |\eta|/10$, so
$$\left|\frac{\xi-\eta}{\eta}\right| \leq \frac{1}{10} < \frac{2}{9}.$$
- If $|\xi+\eta| > |\eta|$, then $\xi\eta > 0$, so $|\xi-\eta| = ||\xi|-|\eta||$, and can we rewrite (2.14) as
$$||\xi|-|\eta|| \leq \frac{1}{10}(|\xi|+|\eta|),$$

which implies that

$$\frac{9}{11}|\xi| \le |\eta| \le \frac{11}{9}|\xi|,$$

and (2.13) follows in this case also.

Using the Taylor expansion

$$\begin{aligned}\log|\xi-\eta+\eta| &= \log|\eta| + \log\left|1+\frac{\xi-\eta}{\eta}\right| \\ &= \log|\eta| + \frac{\xi-\eta}{\eta} + O\Big(\frac{|\xi-\eta|^2}{\eta^2}\Big)\end{aligned}$$

in (2.12), we get that

$$\begin{aligned}\mathcal{F}\left[LT_a v\right](\xi) &= \int_{\mathbb{R}}\left[\log|\eta| + \frac{\xi-\eta}{\eta} + O\Big(\frac{|\xi-\eta|^2}{\eta^2}\Big)\right]\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{a}\left(\xi-\eta,\frac{\xi+\eta}{2}\right)\hat{v}(\eta)\,\mathrm{d}\eta \\ &= \mathcal{F}\left[T_a L v + T_{Da}D^{-1}v + \mathcal{O}(T_{D^2 a}D^{-2}v)\right](\xi),\end{aligned}$$

and (2.9) follows directly from the assumption on a.

2. To prove (2.10), we compute that

$$\begin{aligned}&\mathcal{F}[x,T_b]f \\ =& -i\partial_\xi\int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta + \int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)i\partial_\eta\hat{f}(\eta)\,\mathrm{d}\eta \\ =& -i\int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\partial_\xi\Big[\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\Big]\hat{f}(\eta) + \partial_\xi\Big[\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\Big]\Big[\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\Big]\hat{f}(\eta)\,\mathrm{d}\eta \\ &+ \int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)i\partial_\eta\hat{f}(\eta)\,\mathrm{d}\eta.\end{aligned}$$

We rewrite the first integral above as

$$\begin{aligned}&\int_{\mathbb{R}}\partial_\xi\Big[\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\Big]\hat{f}(\eta)\,\mathrm{d}\eta \\ =&\int_{\mathbb{R}}(\partial_{\xi_1}+\partial_{\xi_2})\Big[\chi\left(\frac{|\xi_1-\eta|}{|\xi_2+\eta|}\right)\tilde{b}\Big(\xi_1-\eta,\frac{\xi_2+\eta}{2}\Big)\Big]\hat{f}(\eta)\,\mathrm{d}\eta\Big|_{\xi_1=\xi_2=\xi} \\ =&\int_{\mathbb{R}}(2\partial_{\xi_1}+\partial_\eta)\Big[\chi\left(\frac{|\xi_1-\eta|}{|\xi_2+\eta|}\right)\tilde{b}\Big(\xi_1-\eta,\frac{\xi_2+\eta}{2}\Big)\Big]\hat{f}(\eta)\,\mathrm{d}\eta\Big|_{\xi_1=\xi_2=\xi} \\ =&\int_{\mathbb{R}}2\partial_{\xi_1}\Big[\chi\left(\frac{|\xi_1-\eta|}{|\xi_2+\eta|}\right)\tilde{b}\Big(\xi_1-\eta,\frac{\xi_2+\eta}{2}\Big)\Big]\Big|_{\xi_1=\xi_2=\xi}\hat{f}(\eta) + \Big[\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\Big]\partial_\eta\hat{f}(\eta)\,\mathrm{d}\eta.\end{aligned}$$

Therefore

$$\mathcal{F}[x,T_b]f = -i\int_{\mathbb{R}} 2\partial_{\xi_1}\Big[\chi\left(\frac{|\xi_1-\eta|}{|\xi_2+\eta|}\right)\tilde{b}\Big(\xi_1-\eta,\frac{\xi_2+\eta}{2}\Big)\Big]\Big|_{\xi_1=\xi_2=\xi}\hat{f}(\eta)\,\mathrm{d}\eta$$

$$= -2i\int_{\mathbb{R}}\frac{\operatorname{sgn}(\xi-\eta)}{|\xi+\eta|}\chi'\left(\frac{|\xi_1-\eta|}{|\xi_2+\eta|}\right)\tilde{b}\Big(\xi_1-\eta,\frac{\xi_2+\eta}{2}\Big)\hat{f}(\eta)$$

$$+\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\partial_{\xi_1}\tilde{b}\Big(\xi_1-\eta,\frac{\xi_2+\eta}{2}\Big)\Big|_{\xi_1=\xi_2=\xi}\hat{f}(\eta)\,\mathrm{d}\eta,$$

and (2.10) follows.

3. Taking Fourier transforms, we get that

$$\mathcal{F}\Big([x,L]f\Big) = -i\partial_\xi[\log|\xi|\hat{f}(\xi)] - \log|\xi|(-i\partial_\xi\hat{f}(\xi)) = -\frac{i}{\xi}\hat{f}(\xi),$$

and (2.11) follows. □

Finally, we give an expansion of $|D|$ acting on para-products (cf. [**Li19**]).

Lemma 2.2.2. *If $a(x,\xi)\in\mathcal{M}_{(3,0)}$, $f\in H^s(\mathbb{R})$ and $s\in\mathbb{R}$, then*

$$|D|^sT_af = T_a|D|^sf + sT_{Da}|D|^{s-2}Df + \frac{s(s-1)}{2}T_{|D|^2a}|D|^{s-2}f + \mathcal{O}(T_{|D|^3a}|D|^{s-3}f),$$

where Da means that the differential operator D acts on the function $x\mapsto a(x,\xi)$ for fixed ξ, and similarly for $|D|^2a$ and $|D|^3a$.

Proof. By the definition of Weyl para-product

$$\mathcal{F}(|D|^sT_af)(\xi) = |\xi|^s\int_{\mathbb{R}}\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta$$
$$= \int_{\mathbb{R}}|\xi-\eta+\eta|^s\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{a}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta,$$

where $\tilde{a}$ denotes the partial Fourier transform of f in the first variable. On the support of $\chi(|\xi-\eta|/|\xi+\eta|)$ we have that

$$\left|\frac{\xi-\eta}{\eta}\right|\le\frac{2}{9},$$

and, using the Taylor expansion

$$|\xi-\eta+\eta|^s = |\eta|^s\left|1+\frac{\xi-\eta}{\eta}\right|^s = |\eta|^s\left(1+s\frac{\xi-\eta}{\eta}+\frac{s(s-1)}{2}\frac{(\xi-\eta)^2}{\eta^2}+O\Big(\frac{|\xi-\eta|^3}{\eta^3}\Big)\right)$$

in the expression for $\mathcal{F}\big[|D|^s T_a f\big]$, we get

$$\begin{aligned}
&\mathcal{F}\big[|D|^s T_a f\big](\xi)\\
&\quad= \int_{\mathbb{R}} |\eta|^s \left(1 + s\frac{\xi-\eta}{\eta} + \frac{s(s-1)}{2}\frac{(\xi-\eta)^2}{\eta^2} + O\Big(\frac{|\xi-\eta|^3}{\eta^3}\Big)\right)\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{a}\Big(\xi-\eta, \frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\,\mathrm{d}\eta\\
&\quad= \mathcal{F}\left[T_a|D|^s f + sT_{Da}|D|^{s-2}Df + \frac{s(s-1)}{2}T_{|D|^2 a}|D|^{s-2}f + \mathcal{O}(T_{|D|^3 a}|D|^{s-3}f)\right](\xi),
\end{aligned}$$

which proves the lemma. □

We remark that a similar result holds also for functions defined on $\mathbb{T}$ and have zero mean. Moreover, in the periodic case, we have following lemma, whose proofs we omit (see [**HSZ18**] for details).

Lemma 2.2.3. *If $u, v \in L^2$, then*

$$\begin{aligned}
L(uv) &= T_v Lu + T_{Dv}D^{-1}u - \frac{1}{2}T_{D^2 v}D^{-2}u + \frac{1}{3}T_{D^3 v}D^{-3}u + \mathcal{O}(T_{D^4 v}D^{-4}u)\\
&\quad + T_u Lv + T_{Du}D^{-1}v - \frac{1}{2}T_{D^2 u}D^{-2}v + \frac{1}{3}T_{D^3 u}D^{-3}v + \mathcal{O}(T_{D^4 u}D^{-4}v) + LR(u,v),
\end{aligned}$$

where the remainder terms satisfy

$$\|\mathcal{O}(T_{D^4 v}D^{-4}u)\|_{\dot{H}^s} \le C\|T_{D^4 v}D^{-4}u\|_{\dot{H}^s}, \qquad \|\mathcal{O}(T_{D^4 u}D^{-4}v)\|_{\dot{H}^s} \le C\|T_{D^4 u}D^{-4}v\|_{\dot{H}^s}.$$

Moreover, if $u, Lu \in W^{\sigma,\infty}$ for an integer $\sigma \ge 0$, and $v \in \dot{H}^s$ with $s+\sigma > 0$, then

$$\|LR(u,v)\|_{\dot{H}^{s+\sigma}} \le C(\|u\|_{W^{\sigma,\infty}} + \|Lu\|_{W^{\sigma,\infty}})\|v\|_{\dot{H}^s},$$

for some constant $C > 0$.

Setting $u = v$ in Lemma 2.2.3, we have the following corollary for Lu^2, which is of independent interest.

Corollary 2.2.4. *If $u \in L^\infty \cap \dot{H}^s$ with $Lu \in L^\infty$ and $s \ge 0$, then there exists a constant $C > 0$ such that*

$$\|Lu^2 - 2uLu\|_{\dot{H}^s} \le C(\|u\|_{L^\infty} + \|Lu\|_{L^\infty})\|u\|_{\dot{H}^s}.$$

Proof. By Lemma 2.2.3, we have that

$$L(u^2) = 2T_u Lu + 2T_{Du}D^{-1}u + \mathcal{O}(T_{D^2 u}D^{-2}u)$$

and

$$2uLu = 2T_u Lu + 2T_{Lu}u + 2R(Lu, u).$$

Taking the difference of above two equations yields

$$\begin{aligned}\|Lu^2 - 2uLu\|_{\dot{H}^s} &= \|2T_{Du}D^{-1}u + \mathcal{O}(T_{D^2u}D^{-2}u) - 2T_{Lu}u - 2R(Lu,u)\|_{\dot{H}^s} \\ &\le C(\|u\|_{L^\infty} + \|Lu\|_{L^\infty})\|u\|_{\dot{H}^s}.\end{aligned}$$

□

The following lemma gives an expansion of $L(uvw)$ and an estimates of the remainder terms.

Lemma 2.2.5. *If $u, v, w \in W^{3,\infty} \cap \dot{H}^s$, with $s \ge 0$, then*

$$L(uvw) = \sum_{u,v,w} T_v T_w Lu + (T_{Dv}T_w + T_v T_{Dw})D^{-1}u - \frac{1}{2}[T_{D^2v}T_w + T_{D^2w}T_v + 2T_{Dv}T_{Dw}]D^{-2}u + \textit{remainder},$$

where the summation is cyclic over u, v, w, and the remainder terms satisfy

$$\|\textit{remainder}\|_{\dot{H}^{s+2}} \le C\left(\|u\|_{W^{3,\infty}} + \|v\|_{W^{3,\infty}} + \|w\|_{W^{3,\infty}}\right)^2 \left(\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s} + \|w\|_{\dot{H}^s}\right),$$

for some constant $C > 0$.

Proof. By Lemma 2.2.3, we have

$$\begin{aligned}L[u(vw)] =& T_{vw}Lu + T_{D(vw)}D^{-1}u - \frac{1}{2}T_{D^2(vw)}D^{-2}u + \mathcal{O}(T_{D^3(vw)}D^{-3}u) \\ &+ T_u L(vw) + T_{Du}D^{-1}(vw) - \frac{1}{2}T_{D^2u}D^{-2}(vw) + \mathcal{O}(T_{D^3u}D^{-3}(vw)) \\ &+ LR(u, vw),\end{aligned} \tag{2.15}$$

with

$$\|LR(u, vw)\|_{\dot{H}^{s+2}} \le C(\|u\|_{W^{2,\infty}} + \|Lu\|_{W^{2,\infty}})\|vw\|_{\dot{H}^s},$$

where $\|Lu\|_{W^{2,\infty}} \le C\|u\|_{W^{3,\infty}}$ and $\|vw\|_{\dot{H}^s} \le C\left(\|v\|_{L^\infty}\|w\|_{\dot{H}^s} + \|w\|_{L^\infty}\|v\|_{\dot{H}^s}\right)$.

Using the fact that

$$\|T_{vw} - T_v T_w\|_{\dot{H}^s \to \dot{H}^{s+\sigma}} \le C(\|v\|_{W^{\sigma,\infty}}\|w\|_{L^\infty} + \|v\|_{L^\infty}\|w\|_{W^{\sigma,\infty}}),$$

and denoting the remainder terms by $\mathcal{R}_i$, we can expand each term in the above equation to get

$$\begin{aligned}
T_{vw}Lu &= T_v T_w Lu + \mathcal{R}_1,\\
T_{D(vw)}D^{-1}u &= (T_{Dv}T_w + T_v T_{Dw})D^{-1}u + \mathcal{R}_2,\\
T_{D^2(vw)}D^{-2}u &= [T_{D^2v}T_w + T_{D^2w}T_v + 2T_{Dv}T_{Dw}]D^{-2}u + \mathcal{R}_3,\\
D^{-1}(vw) &= D^{-1}(T_v w + T_w v + R(v,w))\\
&= T_v D^{-1}w - T_{Dv}D^{-2}w + \mathcal{O}(T_{D^2v}D^{-3}w)\\
&\quad + T_w D^{-1}v - T_{Dw}D^{-2}v + \mathcal{O}(T_{D^2w}D^{-3}v) + \mathcal{R}_4,\\
D^{-2}(vw) &= D^{-2}(T_v w + T_w v + R(v,w))\\
&= T_v D^{-2}w - 2T_{Dv}D^{-3}w + \mathcal{O}(T_{D^2v}D^{-4}w)\\
&\quad + T_w D^{-2}v - 2T_{Dw}D^{-3}v + \mathcal{O}(T_{D^2w}D^{-4}v) + \mathcal{R}_5,
\end{aligned} \tag{2.16}$$

with

$$\begin{aligned}
&\|\mathcal{R}_i\|_{\dot{H}^{s+2}} \le C\|v\|_{W^{3,\infty}}\|w\|_{W^{3,\infty}}\|u\|_{\dot{H}^s}, \quad \text{for } i = 1,2,3,\\
&\|\mathcal{R}_4\|_{\dot{H}^{s+2}} \le C\|v\|_{W^{1,\infty}}\|w\|_{\dot{H}^s} + \|w\|_{W^{1,\infty}}\|v\|_{\dot{H}^s},\\
&\|\mathcal{R}_5\|_{\dot{H}^{s+2}} \le \|v\|_{L^\infty}\|w\|_{\dot{H}^s} + \|w\|_{L^\infty}\|v\|_{\dot{H}^s}.
\end{aligned}$$

Then the lemma is proved by substituting (2.16) into (2.15). □

2.3. Fourier multipliers

Let $\varsigma\colon \mathbb{R} \to [0,1]$ be a smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define

$$\begin{aligned}
&\varsigma_k(\xi) = \varsigma(\xi/2^k) - \varsigma(\xi/2^{k-1}), \qquad \varsigma_{\le k}(\xi) = \varsigma(\xi/2^k), \qquad \varsigma_{\ge k}(\xi) = 1 - \varsigma(\xi/2^{k-1}),\\
&\tilde{\varsigma}_k(\xi) = \varsigma_{k-1}(\xi) + \varsigma_k(\xi) + \varsigma_{k+1}(\xi),
\end{aligned} \tag{2.17}$$

and denote by P_k, $P_{\le k}$, $P_{\ge k}$, and $\tilde{P}_k$ the Fourier multiplier operators with symbols $\varsigma_k, \varsigma_{\le k}, \varsigma_{\ge k}$, and $\tilde{\varsigma}_k$, respectively. Notice that $\varsigma_k(\xi) = \varsigma_0(\xi/2^k)$, $\tilde{\varsigma}_k(\xi) = \tilde{\varsigma}_0(\xi/2^k)$.

It is easy to check that

$$\|\varsigma_k\|_{L^2} \approx 2^{k/2}, \qquad \|\varsigma_k'\|_{L^2} \approx 2^{-k/2}. \tag{2.18}$$

We will need the following interpolation lemma, whose proof can be found in [**IP16**].

Lemma 2.3.1. *For any $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, we have*

$$\|\widehat{P_k f}\|_{L^\infty}^2 \lesssim \|P_k f\|_{L^1}^2 \lesssim 2^{-k}\|\hat{f}\|_{L^2_\xi}\left[2^k\|\partial_\xi \hat{f}\|_{L^2_\xi} + \|\hat{f}\|_{L^2_\xi}\right].$$

We will also use an estimate for multilinear Fourier multipliers proved in [**IP15**]. Before stating the estimate, we introduce some notation.

Define the class of symbols

$$S^\infty := \{\kappa\colon \mathbb{R}^d \to \mathbb{C}, \quad \kappa \text{ continuous and } \|\kappa\|_{S^\infty} := \|\mathcal{F}^{-1}(\kappa)\|_{L^1} < \infty\}. \tag{2.19}$$

Given $\kappa \in S^\infty$, we define a multilinear operator M_κ acting on Schwartz functions $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R})$ by

$$M_\kappa(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^m} e^{ix(\xi_1+\cdots+\xi_m)}\kappa(\xi_1, \dots, \xi_m)\hat{f}_1(\xi_1)\cdots\hat{f}_m(\xi_m)\,\mathrm{d}\xi_1\cdots\,\mathrm{d}\xi_m.$$

Lemma 2.3.2. *(i) If $\kappa_1, \kappa_2 \in S^\infty$, then $\kappa_1\kappa_2 \in S^\infty$.*

(ii) Suppose that $1 \le p_1, \dots, p_m \le \infty$, $1 \le p \le \infty$, satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

If $\kappa \in S^\infty$, then

$$\|M_\kappa\|_{L^{p_1}\times\cdots\times L^{p_m}\to L^p} \lesssim \|\kappa\|_{S^\infty}.$$

(iii) Assume $p, q, r \in [1, \infty]$ satisfy $1/p + 1/q + 1/r = 1$, and $m \in S^\infty_{\eta_1,\eta_2}L^\infty_\xi$. Then, for any $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, and $h \in L^r(\mathbb{R})$,

$$\left\|\int_{\mathbb{R}^2} m(\eta_1, \eta_2, \xi)\hat{f}(\eta_1)\hat{g}(\eta_2)\hat{h}(\xi - \eta_1 - \eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\|_{L^\infty_\xi} \lesssim \|m\|_{S^\infty_{\eta_1,\eta_2}L^\infty_\xi}\|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

Remark 2.3.3. For a symbol $m(\eta_1, \eta_2)$ in C_c^∞, by interpolation, we can estimate its S^∞ norm as

$$\begin{aligned} \|m\|_{S^\infty} &\lesssim \|m\|_{L^1}^{1/4}\|\partial_{\eta_2}^2 m\|_{L^1}^{1/2}\|\partial_{\eta_1}^2\partial_{\eta_2}^2 m\|_{L^1}^{1/4}, \\ \|m\|_{S^\infty} &\lesssim \|m\|_{L^1}^{1/4}\|\partial_{\eta_1}^2 m\|_{L^1}^{1/2}\|\partial_{\eta_1}^2\partial_{\eta_2}^2 m\|_{L^1}^{1/4}. \end{aligned} \tag{2.20}$$

2.4. Modified Bessel function of the second kind

In this section, we summarize some definitions and properties of modified Bessel functions, which can be found in [**OLBC10**, **Wat95**]. The modified Bessel function I_ν of the first kind is defined for $\nu \in \mathbb{R}$ by

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}.$$

The modified Bessel function K_ν of the second kind is defined for $\nu \notin \mathbb{Z}$ by

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},$$

and $K_n(x) = \lim_{\nu \to n} K_\nu(x)$ for $n \in \mathbb{Z}$. When $\nu > -1/2$ and $x > 0$, we can also write K_ν as

$$K_\nu(x) = \frac{\Gamma(\nu + \frac{1}{2})(2x)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos y}{(y^2 + x^2)^{\nu+1/2}} \, \mathrm{d}y. \tag{2.21}$$

In (2.21), and throughout this dissertation, $\Gamma(z)$ denotes the Gamma function.

The following lemma collects the properties of modified Bessel functions of the second kind that we need. Properties (i)–(iv) can be found in [**OLBC10**].

Lemma 2.4.1. *The modified Bessel functions of the second kind have following properties:*

(i) For each $\nu \geq 0$, $K_\nu(x)$ is a real-valued, analytic, strictly decreasing function on $(0, \infty)$.

(ii) For each fixed $x, \nu > 0$, $K_\nu(x) = K_{-\nu}(x)$.

(iii) If $\nu > 0$, then

$$K_\nu(x) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu}, \quad K_0(x) \sim -\log(x) \qquad \text{as } x \to 0^+.$$

(iv) If $\nu \geq 0$, then

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \qquad \text{as } x \to \infty.$$

(v) Let $m \geq 0$ be an integer, and define $f_m \colon \mathbb{R} \times (\frac{1}{2}, \infty) \to \mathbb{R}$ by

$$f_m(x, \nu) = |x|^{\nu+m} K_\nu(|x|).$$

Then $f_m(\cdot, \nu)$ attains its maximum, and if the maximum is attained at some $x_0 \in \mathbb{R}$, then

$$|x_0| \leq \sqrt{m^2 + (2\nu - 1)m}, \qquad 0 \leq f_m(x_0, \nu) \leq \frac{\left(m^2 + (2\nu-1)m\right)^{m/2} \Gamma(\nu)}{2^{\nu+1}}. \tag{2.22}$$

Proof of (v). It follows from (ii) that we only need to consider $x \geq 0$. We use the identities (see [**OLBC10**])

$$K'_\nu(x) = -\frac{K_{\nu-1}(x) + K_{\nu+1}(x)}{2},$$
$$xK_{\nu+1}(x) - xK_{\nu-1}(x) = 2\nu K_\nu(x),$$

to obtain

$$\begin{aligned}\frac{\partial}{\partial x} f_m(x,\nu) &= (\nu+m)x^{\nu+m-1}K_\nu(x) - \frac{1}{2}x^{\nu+m}\left(K_{\nu-1}(x) + K_{\nu+1}(x)\right)\\ &= x^{\nu+m-1}\left(mK_\nu(x) - xK_{\nu-1}(x)\right).\end{aligned}$$

When $m = 0$ and $\nu > 1/2$, we have $\partial_x f_0(x,\nu) \leq 0$. Thus f_0 is decreasing in x, and its maximum is attained at $x_0 = 0$ with

$$f_0(0,\nu) = \frac{\Gamma(\nu)}{2^{\nu+1}}.$$

When $m > 0$, it is clear that f_m is smooth in $x \in (0,\infty)$ with

$$f_m(0,\nu) = \lim_{x\to\infty} f_m(x,\nu) = 0,$$

so the maximum is attained at its critical points. Therefore, x_0 must satisfy

$$\frac{m}{x_0} = \frac{K_{\nu-1}(x_0)}{K_\nu(x_0)}.$$

For $\nu > \frac{1}{2}$ and $x > 0$, we have that [**Seg11**]

$$\frac{K_{\nu-1}(x)}{K_\nu(x)} > \frac{x}{\sqrt{x^2 + (\nu-1/2)^2} + \nu - 1/2},$$

which leads to the estimate of $|x_0|$ in (2.22). Then, using

$$f_m(x_0,\nu) = |x_0|^{\nu+m}K_\nu(x_0) = |x_0|^m f_0(x_0,\nu) \leq |x_0|^m f_0(0,\nu) = |x_0|^m \frac{\Gamma(\nu)}{2^{\nu+1}},$$

we obtain the upper bound for f_m. □

CHAPTER 3

Contour dynamics equations

The world is complex, dynamic, multidimensional; the paper is static, flat. How are we to represent the rich visual world of experience and measurement on mere flatland?

– *Edward Tufte*

In this Chapter, we derive contour dynamics equations for the front solutions (1.5) and two-front solutions (1.7) for the SQG equations (1.1).

One main result of this chapter is that, with the choice of boundary condition

$$\mathbf{u}(\mathbf{x}, t) = (2 \log |y|, 0) + o(1) \qquad \text{as } |y| \to \infty, \tag{3.1}$$

the evolutionary equation describing the location of an SQG front is

$$\begin{aligned} &\varphi_t(x,t) - 2\log|\partial_x|\varphi_x(x,t) \\ &\quad + \int_{\mathbb{R}} [\varphi_x(x,t) - \varphi_x(x+\zeta,t)] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + [\varphi(x,t) - \varphi(x+\zeta,t)]^2}} \right\} \mathrm{d}\zeta = 0, \end{aligned} \tag{3.2}$$

where $\log|\partial_x|$ is the Fourier multiplier operator defined in (2.4). This equation can be derived using three methods, a regularization method proposed in [**HS18**], a decomposition method, and a modified Green's function method [**HSZara**]. As is noted in [**HSZara**], the last two methods are essentially equivalent. This equation also possesses conservative form (3.12) and Hamiltonian structure (3.13)–(3.14). For spatially periodic SQG fronts, the contour dynamics equation is (3.43).

The second main result is regarding SQG two-front solutions (1.7). We show that, if $\Theta_+ = \Theta_-$, then the far-field velocity of in the two-front problem are bounded. However, in general, the far-field velocity is unbounded (as in the one-front case). In this case, we need to use the similar ideas as in the one-front case to derive well-formulated contour dynamics equations. We show that the contour dynamics equations for φ and ψ are

$$
\begin{aligned}
&\varphi_t(x,t) - (\Theta_+ - \Theta_-)(\gamma + \log h)\varphi_x(x,t) - 2\Theta_+ \log|\partial_x|\varphi_x(x,t) + 2\Theta_- K_0(2h|\partial_x|)\psi_x(x,t)\\
&+ \Theta_+ \int_{\mathbb{R}} \left[\varphi_x(x+\zeta,t) - \varphi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (\varphi(x+\zeta,t)-\varphi(x,t))^2}} - \frac{1}{|\zeta|}\right\}\mathrm{d}\zeta\\
&+ \Theta_- \int_{\mathbb{R}} \left[\psi_x(x+\zeta,t) - \varphi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (-2h+\psi(x+\zeta,t)-\varphi(x,t))^2}} - \frac{1}{\sqrt{\zeta^2+(2h)^2}}\right\}\mathrm{d}\zeta = 0,\\
&\psi_t(x,t) + (\Theta_+ - \Theta_-)(\gamma + \log h)\psi_x(x,t) - 2\Theta_- \log|\partial_x|\psi_x(x,t) + 2\Theta_+ K_0(2h|\partial_x|)\varphi_x(x,t)\\
&+ \Theta_- \int_{\mathbb{R}} \left[\psi_x(x+\zeta,t) - \psi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (\psi(x+\zeta,t)-\psi(x,t))^2}} - \frac{1}{|\zeta|}\right\}\mathrm{d}\zeta\\
&+ \Theta_+ \int_{\mathbb{R}} \left[\varphi_x(x+\zeta,t) - \psi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (2h+\varphi(x+\zeta,t)-\psi(x,t))^2}} - \frac{1}{\sqrt{\zeta^2+(2h)^2}}\right\}\mathrm{d}\zeta = 0,
\end{aligned}
\tag{3.3}
$$

where $\Theta_\pm$ is defined in (1.6) with $g_\alpha = g_1 = 1/(2\pi)$, and

$$h = \frac{h_+ - h_-}{2}.$$

This system also possesses Hamiltonian structure (3.53)–(3.54). Moreover, we also derive symmetric (with $\Theta_+ = \Theta_-$) and anti-symmetric (with $\Theta_+ = \Theta_-$) scalar reductions of these equations (see (3.56) and (3.57)).

The structure of this chapter is as follows. In Section 3.1 we use the three methods we mentioned above to derive (3.2), and in Section 3.2, we give two derivations using the regularization method [**HSZard**] and a sketch of the decomposition method.

3.1. Contour dynamics for SQG front solutions

3.1.1. Method of regularization. We begin by recalling the derivation of the contour dynamics equations for bounded patches (see e.g., [**CCG18, Gan08**]).

3.1.1.1. *Contour dynamics for patches.* Suppose that $\partial\Omega(t)$ is a smooth, simple, closed curve with bounded interior $\Omega(t) \subset \mathbb{R}^2$ and

$$\theta(\mathbf{x},t) = \begin{cases} 2\pi & \mathbf{x} \in \Omega(t), \\ 0 & \mathbf{x} \in \Omega^c(t). \end{cases} \tag{3.4}$$

The Green's function for the operator $(-\Delta)^{1/2}$ on $\mathbb{R}^2$ is given by the Reisz potential [**Ste70**]

$$\frac{1}{2\pi|\mathbf{x}|}.$$

Then, using (1.1b) and Green's theorem, one finds that the velocity field corresponding to (3.4) is

$$\mathbf{u}(\mathbf{x},t) = \int_{\partial\Omega(t)} \frac{\mathbf{n}^{\perp}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}s(\mathbf{x}'), \tag{3.5}$$

where $\mathbf{n} = (m,n)$ is the inward unit normal to $\Omega(t)$, $\mathbf{n}^{\perp} = (-n,m)$, and $s(\mathbf{x}')$ is arc-length on $\partial\Omega(t)$.

We suppose that $\partial\Omega(t)$ is given by the parametric equation $\mathbf{x} = \mathbf{X}(\zeta,t)$, where $\mathbf{X}(\cdot,t)\colon \mathbb{T} \to \mathbb{R}^2$ (see Figure 1.2(a)). Since θ satisfies the transport equation (1.1a), the curve $\partial\Omega(t)$ moves with normal velocity $\mathbf{X}_t \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}$. Notice that the tangential component of (3.5) is unbounded on $\partial\Omega(t)$ (in fact, $\mathbf{u} \in \mathrm{BMO}$), but the normal component is well-defined, and the motion of the curve is determined solely by its normal velocity. The equation for $\mathbf{X}$ is therefore

$$\mathbf{X}_t(\zeta,t) = c(\zeta,t)\mathbf{X}_\zeta(\zeta,t) + \int_{\mathbb{T}} \frac{1}{|\mathbf{X}(\zeta,t)-\mathbf{X}(\zeta',t)|}\left[\mathbf{X}_\zeta(\zeta,t) - \mathbf{X}_{\zeta'}(\zeta',t)\right]\,\mathrm{d}\zeta', \tag{3.6}$$

where $c(\cdot,t)\colon \mathbb{T} \to \mathbb{R}$ is an arbitrary smooth function that corresponds to a time-dependent reparametrization of the curve. The inclusion of the term proportional to the tangent vector $\mathbf{X}_\zeta$ in the integral ensures that the integral converges.

We note that there is a difficulty in extending the contour dynamics equation for a patch to an infinite front $y = \varphi(x,t)$ where $\varphi(\cdot,t)\colon \mathbb{R} \to \mathbb{R}$ (see Figure 1.2(c)). In that case, $\mathbf{X}(x,t) = (x,\varphi(x,t))$, and we get formally from (3.6) that $c = 0$ and

$$\varphi_t(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{(x-x')^2 + (\varphi(x,t)-\varphi(x',t))^2}}\left[\varphi_x(x,t) - \varphi_{x'}(x',t)\right]\mathrm{d}x'. \tag{3.7}$$

This equation does not make sense, since the integrand is not integrable at infinity and $\varphi_x(x,t)$ does not decay as $x' \to \infty$.

Roughly speaking, we have to regularize a short-distance "ultraviolet" singularity, caused by the infinite tangential velocity on the front, and simultaneously, a long-distance "infrared" singularity, caused by the slow decay of the Green's function.

To regularize the long-distance singularity, we introduce a long-range cutoff parameter λ, make a Galilean transformation into a reference frame moving with a suitable velocity $v(\lambda)$, where $v(\lambda) \to \infty$ as $\lambda \to \infty$, and take the limit $\lambda \to \infty$.

3.1.1.2. *Cutoff Regularization.* After a change of variables $x' = x + \zeta$ in (3.7), we introduce a large cutoff parameter $\lambda > 0$ to get the truncated equation

$$\varphi_t(x,t) = \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{\zeta^2 + (\varphi(x,t) - \varphi(x+\zeta,t))^2}} \left[\varphi_x(x,t) - \varphi_x(x+\zeta,t)\right] \mathrm{d}\zeta. \tag{3.8}$$

With the assumption that $\varphi(\cdot,t)\colon \mathbb{R} \to \mathbb{R}$ is a smooth bounded function with bounded first derivative, it is easy to see that the integral in (3.8) converges.

It is convenient to write (3.8) in the conservative form

$$\begin{aligned} \varphi_t(x,t) &= \partial_x \int_{-\lambda}^{\lambda} \int_0^{\varphi(x,t)-\varphi(x+\zeta,t)} \frac{1}{\sqrt{\zeta^2+s^2}} \,\mathrm{d}s\,\mathrm{d}\zeta \\ &= \partial_x \int_{-\lambda}^{\lambda} \sinh^{-1}\left(\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{\sqrt{\zeta^2+s^2}}\right) \mathrm{d}\zeta. \end{aligned}$$

To take the limit $\lambda \to \infty$, we write this equation as

$$\begin{aligned} \varphi_t(x,t) + \partial_x \int_{-\lambda}^{\lambda} \frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|} - \sinh^{-1}\left(\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right) \mathrm{d}\zeta \\ - \partial_x \int_{-\lambda}^{\lambda} \frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|} \mathrm{d}\zeta = 0. \end{aligned} \tag{3.9}$$

First, we consider the nonlinear term in (3.9). We observe that

$$\sinh^{-1}\left(\frac{y}{|x|}\right) \sim \frac{y}{|x|} + \mathcal{O}\left(\frac{1}{|x|^3}\right) \qquad \text{as } |x| \to \infty \text{ with } y \text{ fixed}, \tag{3.10}$$

so when $\varphi(\cdot,t)$ is bounded, we have

$$\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|} - \sinh^{-1}\left(\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right) = \mathcal{O}\left(\frac{1}{|\zeta|^3}\right) \qquad \text{as } |\zeta| \to \infty. \tag{3.11}$$

It follows that

$$\begin{aligned} &\lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|} - \sinh^{-1}\left(\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right) \mathrm{d}\zeta \\ &= \int_{\mathbb{R}} \frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|} - \sinh^{-1}\left(\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right) \mathrm{d}\zeta, \end{aligned}$$

since the integral converges on $\mathbb{R}$.

Next, we consider the linear term

$$\mathbf{L}_\lambda \varphi(x,t) = -\int_{-\lambda}^{\lambda} \frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|}\, \mathrm{d}\zeta.$$

Notice that the Green's function $1/|x|$ is nonintegrable at both 0 and ∞. We write

$$\begin{aligned}\mathbf{L}_\lambda \varphi(x,t) = v(\lambda)\varphi_x(x,t) &+ \int_{1<|\zeta|<\lambda} \frac{\varphi(x+\zeta,t)}{|\zeta|}\, \mathrm{d}\zeta - \int_{|\zeta|<1} \frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|}\, \mathrm{d}\zeta \\ &- \varphi_x(x,t)\left(\int_{|\zeta|>1} \frac{\cos\zeta}{|\zeta|}\, \mathrm{d}\zeta - \int_{|\zeta|<1} \frac{1-\cos\zeta}{|\zeta|}\, \mathrm{d}\zeta\right),\end{aligned}$$

where

$$v(\lambda) = -2\int_1^\lambda \frac{1}{|\zeta|}\, \mathrm{d}\zeta + 2\left(\int_1^\infty \frac{\cos\zeta}{\zeta}\, \mathrm{d}\zeta - \int_0^1 \frac{1-\cos\zeta}{\zeta}\, \mathrm{d}\zeta\right) = -2\log\lambda - 2\gamma,$$

where γ is the Euler-Mascheroni constant.

Making a Galilean transformation $x \mapsto x - v(\lambda)t$, and taking the limit of the resulting equation as $\lambda \to \infty$, we obtain

$$\begin{aligned}\mathbf{L}\varphi(x,t) = &\int_{|\zeta|>1} \frac{\varphi(x+\zeta,t)}{|\zeta|}\, \mathrm{d}\zeta \\ &- \int_{|\zeta|<1} \frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|}\, \mathrm{d}\zeta - \varphi(x,t)\left(\int_{|\zeta|>1} \frac{\cos\zeta}{\zeta}\, \mathrm{d}\zeta - \int_{|\zeta|<1} \frac{1-\cos\zeta}{\zeta}\, \mathrm{d}\zeta\right).\end{aligned}$$

Using Fourier transform, we have that the symbol for $\mathbf{L}$ is

$$\begin{aligned}&\int_{|\zeta|>1} \frac{e^{ik\zeta}}{|\zeta|}\, \mathrm{d}\zeta - \int_{|\zeta|<1} \frac{1-e^{ik\zeta}}{|\zeta|}\, \mathrm{d}\zeta - \left(\int_{|\zeta|>1} \frac{\cos\zeta}{\zeta}\, \mathrm{d}\zeta - \int_{|\zeta|<1} \frac{1-\cos\zeta}{\zeta}\, \mathrm{d}\zeta\right) \\ &= 2\left(\int_1^\infty \frac{\cos(k\zeta)}{\zeta}\, \mathrm{d}\zeta - \int_0^1 \frac{1-\cos(k\zeta)}{\zeta}\, \mathrm{d}\zeta\right) - 2\left(\int_1^\infty \frac{\cos\zeta}{\zeta}\, \mathrm{d}\zeta - \int_0^1 \frac{1-\cos\zeta}{\zeta}\, \mathrm{d}\zeta\right) \\ &= -2\log|k|,\end{aligned}$$

which concludes that $\mathbf{L} = -2\log|\partial_x|$.

Thus, the regularized equation for SQG fronts is

$$\varphi_t(x,t) + \partial_x \int_{\mathbb{R}} \left\{\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|} - \sinh^{-1}\left[\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right]\right\} \mathrm{d}\zeta = 2\log|\partial_x|\varphi_x(x,t), \tag{3.12}$$

and the non-conservative form of the equation is (3.2).

We remark that this equation also have Hamiltonian form

$$\varphi_t + \partial_x \left[\frac{\delta H}{\delta \varphi} \right] = 0, \tag{3.13}$$

where ∂_x is the Hamiltonian operator and the Hamiltonian is

$$H[\varphi] = \frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}} \int_0^{\varphi(x,t)-\varphi(x',t)} \frac{s}{|x-x'|} - \sinh^{-1}\left(\frac{s}{|x-x'|} \right) \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}x' - \int_{\mathbb{R}} \varphi(x,t) \log |\partial_x| \varphi(x,t) \, \mathrm{d}x. \tag{3.14}$$

The corresponding conserved momentum which generates spatial translations, is

$$\frac{1}{2} \int_{\mathbb{R}} \varphi^2 \, \mathrm{d}x.$$

3.1.2. Method of decomposition. We first not that for an SQG shear flow with $\theta = \theta(y)$, the Riesz transform $\mathbf{R}$ with respect to y reduces to the Hilbert transform $\mathbf{H}$, which is a Fourier multiplier operator with symbol $-i \operatorname{sgn} \xi$, and the corresponding velocity field is $\mathbf{u} = (u(y), 0)$ where $u = \mathbf{H}[\theta]$. In particular, if $\theta(y)$ is a step function with a jump of 2π

$$\theta(\mathbf{x}) = \theta(y) = \begin{cases} 2\pi & \text{if } y > 0, \\ 0 & \text{if } y < 0, \end{cases} \tag{3.15}$$

then we have the Hilbert-transform pair [**Duo01**]

$$\mathbf{H}[\theta](y) = 2 \log |y|,$$

which gives the planar front solution

$$\theta = \begin{cases} 2\pi & \text{if } y > 0, \\ 0 & \text{if } y < 0, \end{cases} \qquad \mathbf{u} = (2 \log |y|, 0). \tag{3.16}$$

3.1.2.1. *The quasi-geostrophic equation and Dirichlet-Neumann maps.* An equivalent way to describe the reconstruction of the SQG velocity field from the buoyancy is to return to the original derivation of the 2D SQG equation from the 3D QG equation.

In this subsection, to distinguish between the 2D and 3D variables, we use the notation

$$\mathbf{x} = (x, y, z), \quad \mathbf{x}_H = (x, y), \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

$$\Delta_H = \partial_x^2 + \partial_y^2, \quad \nabla_H^\perp = (-\partial_y, \partial_x), \quad \mathbf{R}_H^\perp = (-\mathbf{R}_y, \mathbf{R}_x).$$

The horizontal Riesz transform $-\mathbf{R}_H^\perp = \nabla_H^\perp(-\Delta_H)^{-1/2}$ in the SQG equation (1.1b) then arises as the orthogonal gradient of a Neumann-Dirichlet map $(-\Delta_H)^{-1/2}$ for the 3D Laplacian in the QG equation. The particular choice for the Riesz transform is determined by the far-field boundary conditions for the QG equation.

As has already been mentioned in Chapter 1, the QG equation provides an approximate description of nearly horizontal geostrophic flows in a vertically stratified fluid [**Maj03**, **Ped79**]. In suitably non-dimensionalized variables, the streamfunction $\Psi(\mathbf{x}, t)$ of the flow satisfies $\Delta\Psi = \mathrm{PV}$, where PV is the potential vorticity in the fluid. The horizontal velocity of the fluid is $\mathbf{U}_H = \nabla_H^\perp \Psi$. The streamfunction is proportional to the fluid pressure, and Ψ_z has the interpretation of a temperature perturbation or buoyancy, rather than a vertical velocity component.

The SQG equation describes quasi-geostrophic flows in a half-space $\mathbb{R}^2 \times \mathbb{R}^+$ with zero potential vorticity in $z > 0$ and a temperature jump, or surface buoyancy, $\theta(\mathbf{x}_H, t)$ at $z = 0$, which is transported by the velocity field $\mathbf{u}_H = \mathbf{U}_H|_{z=0}$ on the boundary [**Lap17**, **Ped79**]:

$$\begin{aligned} &\Delta\Psi = 0 \quad \text{in } z > 0, && -\partial_z \Psi|_{z=0} = \theta, \\ &\theta_t + \mathbf{u}_H \cdot \nabla_H \theta = 0, && \mathbf{u}_H = \nabla_H^\perp \, \Psi|_{z=0}. \end{aligned}$$

We omit an explicit indication of the time-variable. Then $\mathbf{u}_H = \nabla_H \, \Psi|_{z=0}$ and $\Psi|_{z=0}$ is related to θ by a solution of the Neumann problem

$$\Delta\Psi = 0 \quad \text{in } z > 0, \qquad -\partial_z \Psi|_{z=0} = \theta, \tag{3.17}$$

meaning that $\theta \mapsto \Psi|_{z=0}$ is a Neumann-Dirichlet map for the 3D Laplacian in the upper half space. From the point of view of potential theory, this problem is the same as finding the electrostatic potential Ψ of a semi-infinite charged plate located at $\{(x, y, 0) \in \mathbb{R}^3 : y > \varphi(x)\}$ with a constant surface charge density of 4π.

The solution of (3.17) is unique up to a harmonic function $\Psi'(\mathbf{x})$ in $z > 0$ with zero normal derivative on $z = 0$, which can be fixed by imposing suitable boundary conditions at infinity. For example, the addition of a linear harmonic function $\Psi' = Ax + By$ to Ψ does not change θ and adds a uniform velocity field $\mathbf{u}_H = (-B, A)$ to $\mathbf{u}_H$. On the other hand, if $\theta = C$ is constant, then the solution $\Psi' = Cz + D$ (corresponding to a uniform temperature in the QG equation) gives

$\Psi|_{z=0} = D$, so the addition of a constant to θ has no effect (only adds the vector field $\nabla_H \psi = (0,0)$) on the corresponding velocity field $\mathbf{u}_H$.

In particular, let us consider the QG solution that corresponds to the planar front solution in (3.16) with (3.15). Differentiating (3.17) with respect to z, we see that $\Phi = \Psi_z$ satisfies the Dirichlet problem

$$\Delta\Phi = 0 \quad \text{in } z > 0, \qquad -\Phi\Big|_{z=0} = \theta. \tag{3.18}$$

We look for solutions of (3.15)–(3.18) that are independent of x. Then $\Phi_{yy} + \Phi_{zz} = 0$, whose general solution for the Fourier transform of $\Phi(y,z)$ with respect to y,

$$\hat{\Phi}(\xi, z) = \frac{1}{2\pi}\int_{\mathbb{R}} \Phi(y,z) e^{-i\xi y}\,\mathrm{d}y,$$

is given by $\hat{\Phi}(\xi, z) = A(\xi)e^{-|\xi|z} + B(\xi)e^{|\xi|z}$.

We further require that $\hat{\Phi}(\xi, z) \to 0$ as $z \to \infty$ for $\xi \neq 0$, in which case $B = 0$ and $\hat{\Phi}(\xi, z) = \hat{\theta}(\xi)e^{-|\xi|z}$. Inverting this Fourier transform, we get that

$$\Phi(y,z) = \pi + 2\arctan\left(\frac{y}{z}\right),$$

and taking an antiderivative of Φ with respect to z, we get the streamfunction

$$\Psi(y,z) = -2y + y\log\left(y^2 + z^2\right) + 2z\arctan\left(\frac{y}{z}\right) + \pi z.$$

This function provides the appropriate far-field behavior as $y^2 + z^2 \to \infty$ of QG-front solutions in defining the Neumann-Dirichlet map from (3.17).

The boundary value of Ψ on $z = 0$ is

$$\Psi|_{z=0}(y) = \lim_{z\to 0^+} \Psi(y,z) = -2y + 2y\log|y|,$$

with the velocity field $\mathbf{u}_H = (2\log|y|, 0)$, as in the planar front solution (3.16).

3.1.2.2. *Derivation of the contour dynamics equation.* We now derive contour dynamics equations for the front solutions (1.5) by decomposing the solution into a planar shear flow and a perturbation whose velocity field approaches zero as $|y| \to \infty$. We temporarily restore dimensional variables, and consider fronts of the form

$$\theta(\mathbf{x},t) = \begin{cases} 2\pi\Theta & \text{if } \ y > \varphi(x,t), \\ 0 & \text{if } y < \varphi(x,t), \end{cases} \qquad \mathbf{u}(\mathbf{x},t) = \left(2\Theta \log\left(|y|/a\right), 0\right) + o(1) \qquad \text{as } |y| \to \infty. \tag{3.19}$$

Here $\Theta \neq 0$, with the dimensions of velocity, is proportional to the jump in θ across the front, and changes in the parameter $a > 0$, with the dimensions of length, correspond to the addition of a constant x-velocity to the flow, which leads to equivalent front dynamics by means of an appropriate Galilean transformation.

We denote the front $y = \varphi(x,t)$ by $\Gamma(t) = \partial\Omega(t)$, and consider its motion on a time interval $0 \leq t \leq T$ for some $T > 0$. We assume that:

$$\begin{aligned} &\text{(i) } \varphi(\cdot,t) \in C^{1,\alpha}(\mathbb{R}) \text{ for some } \alpha > 0 \text{ and } \varphi(x,t) \text{ is bounded on } \mathbb{R}\times[0,T]; \\ &\text{(ii) } \varphi_x(x,t) = O(|x|^{-\beta}) \text{ as } |x| \to \infty \text{ for some } \beta > 0. \end{aligned} \tag{3.20}$$

In that case, all of the integrals in the following converge.

We choose $h > 0$ such that $-h < \inf\{\varphi(x,t) : (x,t) \in \mathbb{R}\times[0,T]\}$, and let

$$\tilde{\theta}(\mathbf{x}) = \begin{cases} 2\pi\Theta & \text{if } \ y > -h, \\ 0 & \text{if } y < -h, \end{cases}, \qquad \tilde{\mathbf{u}}(\mathbf{x}) = \left(2\Theta\log\left(\frac{|y+h|}{a}\right), 0\right), \tag{3.21}$$

be the dimensionalized planar front solution (3.16) translated to $y = -h$.

We decompose the front solution (1.5) as

$$\theta(\mathbf{x},t) = \tilde{\theta}(\mathbf{x}) + \theta^*(\mathbf{x},t),$$

where $\tilde{\theta}$ is defined in (3.21), and

$$\theta^*(\mathbf{x},t) = \begin{cases} -2\pi\Theta & \text{if } -h < y < \varphi(x,t), \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

We denote the support of $\theta^*(\cdot,t)$ by $\Omega^*(t)$. The corresponding decomposition of the velocity field is

$$\mathbf{u}(\mathbf{x},t) = \tilde{\mathbf{u}}(\mathbf{x}) + \mathbf{u}^*(\mathbf{x},t), \tag{3.23}$$

where $\tilde{\mathbf{u}}$ is defined in (3.21) and $\mathbf{u}^* = -\mathbf{R}^\perp \theta^*$ is given by

$$\mathbf{u}^*(\mathbf{x}, t) = \mathrm{p.\,v.}\, \Theta \int_{\Omega^*(t)} \frac{(\mathbf{x} - \mathbf{x}')^\perp}{|\mathbf{x} - \mathbf{x}'|^3} \, \mathrm{d}\mathbf{x}', \qquad (x, y)^\perp = (-y, x). \tag{3.24}$$

Writing $\mathbf{x}' = (x', y')$, we see that the integrand is $O\left(|x'|^{-2}\right)$ as $|x'| \to \infty$ and compactly supported in y', so this principal value integral converges absolutely at infinity. It follows that

$$\mathbf{u}^*(\mathbf{x}) = \lim_{\lambda \to \infty} \mathbf{u}^*_\lambda(\mathbf{x}), \qquad \mathbf{u}^*_\lambda(\mathbf{x}) = \mathrm{p.\,v.}\, \Theta \int_{\Omega^*_\lambda(x,t)} \frac{(\mathbf{x} - \mathbf{x}')^\perp}{|\mathbf{x} - \mathbf{x}'|^3} \, \mathrm{d}\mathbf{x}', \tag{3.25}$$

where (see Figure 3.1)

$$\Omega^*_\lambda(x, t) = \left\{ \mathbf{x}' \in \mathbb{R}^2 : |x - x'| < \lambda, \, -h < y' < \varphi(x', t) \right\}. \tag{3.26}$$

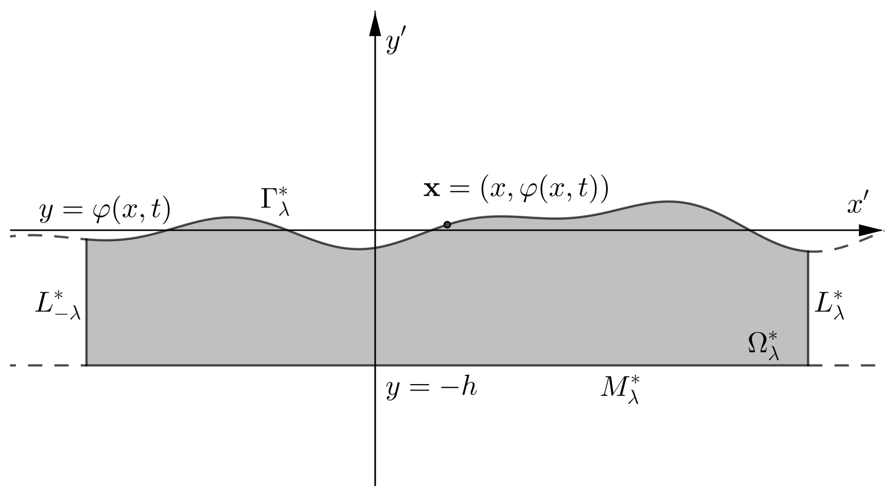


FIGURE 3.1. An illustration of the cut-off region Ω^*_λ in (3.26) with a point $\mathbf{x}$ on the front. The boundary $\partial\Omega^*_\lambda$ consists of the lines $L^*_{\pm\lambda} : x' = x \pm \lambda$ with $-h \leq y' \leq \varphi(x \pm \lambda)$, $M^*_\lambda : y' = -h$ with $|x - x'| \leq \lambda$, and the cut-off front $\Gamma^*_\lambda : y' = \varphi(x')$ with $|x - x'| \leq \lambda$. The function θ^* in (3.22) is equal to $-2\pi\Theta$ in the strip $-h < y < \varphi(x, t)$ and equal to 0 in $y < -h$ or $y > \varphi(x, t)$.

Let $\mathbf{x} = (x, \varphi(x))$ be a point on the front and denote by

$$\mathbf{n}(\mathbf{x}, t) = \frac{1}{\sqrt{1 + \varphi_x^2(x, t)}} (-\varphi_x(x, t), 1) \tag{3.27}$$

the unit upward normal to $\Gamma(t)$ at $\mathbf{x}$. The motion of the front is determined by the normal velocity $\mathbf{u} \cdot \mathbf{n}$, which is continuous and well-defined on the front. The tangential component of $\mathbf{u}$ diverges to infinity, but this does not affect the motion of the front.

We take the inner product of $\mathbf{u}^*_\lambda$ in (3.25) with $\mathbf{n}$, write

$$\frac{(\mathbf{x}-\mathbf{x}')^\perp}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla^\perp_{\mathbf{x}'} \frac{1}{|\mathbf{x}-\mathbf{x}'|},$$

and apply Green's theorem, to get that

$$\mathbf{u}^*_\lambda(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) = -\Theta \int_{\partial\Omega^*_\lambda(x,t)} \frac{\mathbf{t}(\mathbf{x}',t)\cdot\mathbf{n}(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}s(\mathbf{x}'), \tag{3.28}$$

where $\mathbf{t}$ is the negatively oriented unit tangent vector on $\partial\Omega^*_\lambda$ and $\mathrm{d}s(\mathbf{x}')$ is an element of arclength. Since $\mathbf{t}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)=0$, the assumed Hölder continuity of φ_x ensures that this integral converges at $\mathbf{x}'=\mathbf{x}$, so there is no contribution from the principal value at $\mathbf{x}'=(x,\varphi(x,t))$.

As illustrated in Figure 3.1, we decompose the boundary as $\partial\Omega^*_\lambda = \Gamma^*_\lambda \cup M^*_\lambda \cup L^*_\lambda \cup L^*_{-\lambda}$. On $L^*_{-\lambda}$, we have $\mathbf{t}(\mathbf{x}',t)=(0,1)$, $x'=-\lambda$, and $\mathrm{d}s(\mathbf{x}')=\mathrm{d}y'$, so

$$\int_{L^*_{-\lambda}} \frac{\mathbf{t}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}s(\mathbf{x}') = (0,1)I^*_\lambda(x,t),$$

where

$$\begin{aligned} I^*_\lambda(x,t) &= \int_{-h}^{\varphi(-\lambda)} \frac{1}{\sqrt{(x+\lambda)^2+(\varphi(x,t)-y')^2}}\,\mathrm{d}y' \\ &= -\log\left(\varphi(x,t)-\varphi(-\lambda,t)+\sqrt{(x+\lambda)^2+(\varphi(x,t)-\varphi(-\lambda,t))^2}\right) \\ &\qquad + \log\left(\varphi(x,t)+h+\sqrt{(x+\lambda)^2+(\varphi(x,t)+h)^2}\right) \\ &\to 0 \qquad \text{as } \lambda\to\infty, \end{aligned}$$

since φ is bounded. Similarly, the limit of the integral over L^*_λ as $\lambda\to\infty$ also vanishes, so the only contributions to $\mathbf{u}^*$ comes from Γ^*_λ and M^*_λ.

The tangent vector on Γ^*_λ is

$$\mathbf{t}(\mathbf{x}',t) = \frac{1}{\sqrt{1+\varphi^2_{x'}(x',t)}}\left(1,\varphi_{x'}(x',t)\right), \tag{3.29}$$

and the tangent vector on M^*_λ is $(-1,0)$. Using (3.27) and (3.29) in (3.28), and taking the limit $\lambda\to\infty$, we get that

$$\mathbf{u}^*(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) = -\lim_{\lambda\to\infty}\Theta\int_{\Gamma^*_\lambda\cup M^*_\lambda} \frac{\mathbf{t}(\mathbf{x}',t)\cdot\mathbf{n}(\mathbf{x},t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}s(\mathbf{x}') = \frac{\Theta}{\sqrt{1+\varphi_x^2(x,t)}}I^*(x,t),$$

$$I^*(x,t) = \int_{\mathbb{R}} \left\{ \frac{\varphi_x(x,t) - \varphi_{x'}(x',t)}{\sqrt{(x-x')^2 + (\varphi(x,t) - \varphi(x',t))^2}} - \frac{\varphi_x(x,t)}{\sqrt{(x-x')^2 + (\varphi(x,t)+h)^2}} \right\} \mathrm{d}x'.$$

Including the contribution from the background flow $\tilde{\mathbf{u}}$, and using the condition that the front $y = \varphi(x,t)$ moves with the upward normal velocity $\mathbf{u} \cdot \mathbf{n} = (\tilde{\mathbf{u}} + \mathbf{u}^*) \cdot \mathbf{n}$, we obtain that

$$\varphi_t(x,t) = \Theta I(x,t), \qquad I(x,t) = \frac{1}{\Theta} \left(\sqrt{1+\varphi_x^2} \right) \tilde{\mathbf{u}}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x},t) + I^*(x,t).$$

From (3.21) and (3.27), we have

$$\frac{1}{\Theta} \left(\sqrt{1+\varphi_x^2} \right) \tilde{\mathbf{u}}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x},t) = -2 \left(\log \frac{|\varphi(x,t)+h|}{a} \right) \varphi_x(x,t).$$

We then decompose I as

$$\begin{aligned}
I(x,t) &= I_1(x,t) + I_2(x,t) + I_3(x,t), \\
I_1(x,t) &= \int_{\mathbb{R}} \left\{ \frac{\varphi_x(x,t) - \varphi_x(x',t)}{\sqrt{(x-x')^2 + (\varphi(x,t) - \varphi(x',t))^2}} - \frac{\varphi_x(x,t) - \varphi_x(x',t)}{|x-x'|} \right\} \mathrm{d}x', \\
I_2(x,t) &= \int_{\mathbb{R}} \left\{ \frac{\varphi_x(x,t) - \varphi_x(x',t)}{|x-x'|} - \frac{\varphi_x(x,t)}{\sqrt{(x')^2 + a^2}} \right\} \mathrm{d}x', \\
I_3(x,t) &= \varphi_x(x,t) \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{(x')^2 + a^2}} - \frac{1}{\sqrt{(x-x')^2 + (\varphi(x,t)+h)^2}} \mathrm{d}x' - 2\log \frac{|\varphi(x,t)+h|}{a} \right\}.
\end{aligned} \tag{3.30}$$

All of these integrals converge in view of the assumed Hölder continuity and decay of φ_x.

Direct evaluation of the integral for I_3 yields

$$\begin{aligned}
I_3(x,t) = \varphi_x(x,t) &\Bigg\{ \left[\log \left(\frac{x' + \sqrt{(x')^2 + a^2}}{a} \right) - \log \left(\frac{x' - x + \sqrt{(x'-x)^2 + (\varphi(x,t)+h)^2}}{|\varphi(x,t)+h|} \right) \right]_{x'=-\infty}^{\infty} \\
&\quad -2\log \frac{|\varphi(x,t)+h|}{a} \Bigg\} \\
= 0.&
\end{aligned}$$

Thus, the equation for the front is

$$\varphi_t(x,t) = \Theta I_1(x,t) + \Theta I_2(x,t), \tag{3.31}$$

where I_1 is the nonlinear term and I_2 is the linear term in the equation.

To express I_2 as a Fourier multiplier operator, we note that $I_2 = 0$ if $\varphi = 1$, and if $\varphi(x) = e^{i\xi x}$ with $\xi \neq 0$, then

$$
\begin{aligned}
I_2(x) &= i\xi e^{i\xi x} \int_{\mathbb{R}} \left[\frac{1 - e^{i\xi(x'-x)}}{|x - x'|} - \frac{1}{\sqrt{(x')^2 + a^2}} \right] \mathrm{d}x' \\
&= i\xi e^{i\xi x} \int_{\mathbb{R}} \left[\frac{1 - e^{i\xi(x'-x)}}{|x - x'|} - \frac{1}{\sqrt{(x - x')^2 + a^2}} \right] \mathrm{d}x' \\
&= 2i\xi e^{i\xi x} \int_0^\infty \left[\frac{1 - \cos a\xi s}{s} - \frac{1}{\sqrt{s^2 + 1}} \right] \mathrm{d}s.
\end{aligned}
$$

Using the identity

$$
\int_0^\infty \left[\frac{1}{\sqrt{s^2 + 1}} - \frac{1}{\sqrt{s^2 + c^2}} \right] \mathrm{d}s = \log |c|, \tag{3.32}
$$

with $c = 1/a|\xi|$, the change of variable $s' = a|\xi|s$, and a cosine integral, we get that

$$
\begin{aligned}
I_2(x) &= 2i\xi e^{i\xi x} \left(\log a|\xi| + \int_0^\infty \left[\frac{1 - \cos s'}{s'} - \frac{1}{\sqrt{(s')^2 + 1}} \right] \mathrm{d}s' \right) \\
&= 2i\xi e^{i\xi x} \left(\log a|\xi| + \gamma - \log 2 \right).
\end{aligned}
$$

It follows that

$$
I_2 = 2 \log \left(a|\partial_x| \right) \varphi_x + 2(\gamma - \log 2)\varphi_x,
$$

where $\log(a|\partial_x|)$ is the Fourier multiplier operator with symbol $\log(a|\xi|)$. Thus, using the expressions for I_1, I_2 in (3.31), we get the front equation

$$
\begin{aligned}
&\varphi_t(x,t) + 2\Theta \left(\log 2 - \gamma \right) \varphi_x(x,t) - 2\Theta \log \left(a|\partial_x| \right) \varphi_x(x,t) \\
&\quad + \Theta \int_{\mathbb{R}} [\varphi_x(x,t) - \varphi_{x'}(x',t)] \left\{ \frac{1}{|x - x'|} - \frac{1}{\sqrt{(x - x')^2 + [\varphi(x,t) - \varphi(x',t)]^2}} \right\} \mathrm{d}x' = 0,
\end{aligned}
$$

After nondimensionalizing length and time scales so that $\Theta = 1$, $a = 1$, we get the front equation

$$
\begin{aligned}
&\varphi_t(x,t) + 2 \left(\log 2 - \gamma \right) \varphi_x(x,t) \\
&+ \int_{\mathbb{R}} [\varphi_x(x,t) - \varphi_{x'}(x',t)] \left\{ \frac{1}{|x - x'|} - \frac{1}{\sqrt{(x - x')^2 + [\varphi(x,t) - \varphi(x',t)]^2}} \right\} \mathrm{d}x' = 2 \log |\partial_x| \varphi_x(x,t),
\end{aligned}
$$

which agrees with (3.2) after a Galilean transformation $x \mapsto x - 2(\log 2 - \gamma)t$.

One can verify that the velocity perturbation (3.24) satisfies $\mathbf{u}^*(\mathbf{x}, t) = o(1)$ as $|y| \to \infty$. In fact,

$$2\log\frac{|y+h|}{a} = 2\log\frac{|y|}{a} + o(1) \qquad \text{as } |y| \to \infty,$$

then the velocity field $\mathbf{u}(\mathbf{x}, t)$ given by (3.23) and (3.21) has the asymptotic behavior stated in (3.19).

The appearance of a logarithm in the far-field boundary condition (3.1) breaks the scale-invariance of the SQG equation under $\mathbf{x} \mapsto \lambda\mathbf{x}$, $t \mapsto \lambda t$ for $\lambda > 0$. Instead, one sees from (3.19) that the appropriate scale-invariance is given by $\mathbf{x} \mapsto \lambda\mathbf{x}$, $t \mapsto \lambda t$, $a \mapsto \lambda a$, which leads to the invariance of (3.2) under a combined scaling-Galilean transformation [**HS18**]

$$x \mapsto \lambda[x + (2\log|\lambda|)t], \qquad y \mapsto \lambda y, \qquad t \mapsto \lambda t.$$

Similar issues are well-known in potential theory for unbounded charge distributions. For example, there is no length scale in the problem for the electrostatic potential of an infinite charged wire, which is given by the logarithmic Newtonian potential. The potential diverges at infinity, so one cannot normalize a zero point for the potential by requiring that the potential approaches zero at infinity (as one usually does for compact charge distributions). Instead, one picks an arbitrary radial distance $a > 0$ from the wire and requires that the potential vanish at a distance $r = a$, or $r = 1$ in spatial variables non-dimensionalized by a (see e.g., Sec. III.5 in [**Kel67**]). The problem is then invariant under spatial rescaling and an appropriate shift in the zero-point of the potential.

3.1.3. Method of modified Green's function. In this subsection, we give the third derivation of (3.2) based on the definition of the BMO-valued Riesz transform on L^∞ in (2.3). As before, we assume that φ satisfies (3.20).

From (1.1) and (2.3), with $n = 2$ and $C_2 = 1/(2\pi)$, a representative velocity field of the front solution (1.5) is given by

$$\mathbf{u}(\mathbf{x}, t) = -\,\mathrm{p.\,v.}\int_{\Omega(t)} \left[\frac{(\mathbf{x}-\mathbf{x}')^\perp}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{(\mathbf{x}_0-\mathbf{x}')^\perp}{|\mathbf{x}_0-\mathbf{x}'|^3}\right] \mathrm{d}\mathbf{x}' - \bar{\mathbf{u}}(t), \tag{3.33}$$

where $\Omega(t) = \left\{(x, y) \in \mathbb{R}^2 \mid y > \varphi(x, t)\right\}$ and $\mathbf{x}_0 \notin \bar{\Omega}(t)$. For definiteness, we choose $\mathbf{x}_0 = (0, -h)$ where $h > 0$ and $-h < \inf\{\varphi(x, t) \mid (x, t) \in \mathbb{R} \times [0, T]\}$. The spatially uniform velocity $\bar{\mathbf{u}}(t) = (\bar{u}(t), \bar{v}(t))$ in (3.33) will be chosen so that $\mathbf{u}(\mathbf{x}, t)$ satisfies the far-field condition (3.1). However, any such representative leads to equivalent dynamics for the fronts, since any uniform velocity $(\bar{u}(t), \bar{v}(t))$ can be removed by a translation $(x, y) \mapsto (x - a(t), y - b(t))$ where $(a, b)_t = (\bar{u}, \bar{v})$.

Since the integral in (3.33) converges absolutely at infinity, we have

$$\mathbf{u}(\mathbf{x}, t) = \lim_{\lambda \to \infty} \mathbf{u}_\lambda(\mathbf{x}, t) - \bar{\mathbf{u}}(t), \qquad \mathbf{u}_\lambda(\mathbf{x}, t) = -\,\mathrm{p.\,v.} \int_{\Omega_\lambda(x,t)} \left[\frac{(\mathbf{x} - \mathbf{x}')^\perp}{|\mathbf{x} - \mathbf{x}'|^3} - \frac{(\mathbf{x}_0 - \mathbf{x}')^\perp}{|\mathbf{x}_0 - \mathbf{x}'|^3} \right] \mathrm{d}\mathbf{x}',$$

where (See Figure 3.2)

$$\Omega_\lambda(x, t) = \left\{ \mathbf{x}' \in \mathbb{R}^2 : |x' - x| < \lambda,\, \varphi(x', t) < y' < \lambda \right\}. \tag{3.34}$$

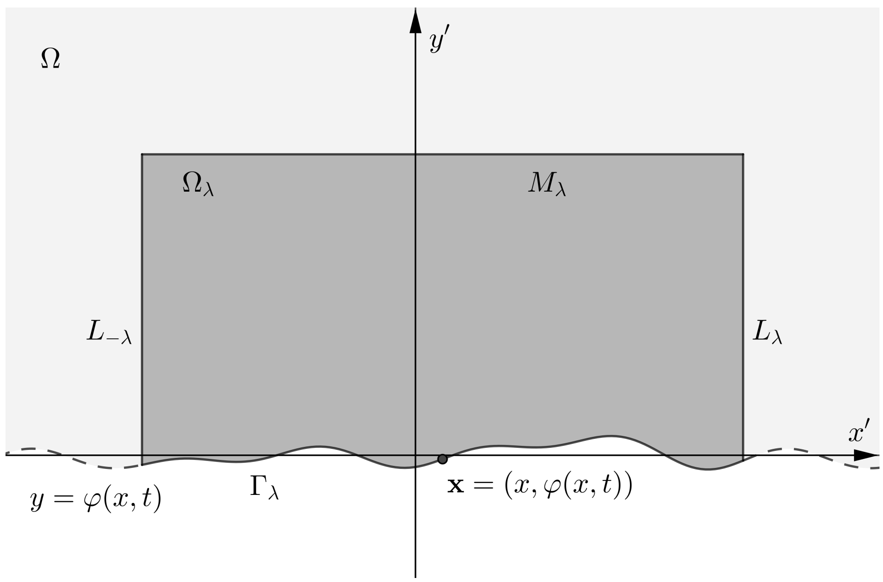


FIGURE 3.2. An illustration of the cut-off region Ω_λ in (3.34) with a point $\mathbf{x}$ on the front. The boundary $\partial\Omega_\lambda$ consists of the lines $L_{\pm\lambda} : x' = x \pm \lambda$ with $\varphi(x \pm \lambda) \le y' \le \lambda$, $M_\lambda : y' = \lambda$ with $|x - x'| \le \lambda$, and the cut-off front $\Gamma_\lambda : y' = \varphi(x')$ with $|x - x'| \le \lambda$.

First, we consider the case when $\mathbf{x} \notin \Gamma(t)$. We write

$$\frac{(\mathbf{x} - \mathbf{x}')^\perp}{|\mathbf{x} - \mathbf{x}'|^3} - \frac{(\mathbf{x}_0 - \mathbf{x}')^\perp}{|\mathbf{x}_0 - \mathbf{x}'|^3} = \nabla^\perp_{\mathbf{x}'} \left[\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x}_0 - \mathbf{x}'|} \right],$$

and apply Green's theorem to get that

$$\mathbf{u}_\lambda(\mathbf{x}, t) = -\int_{\partial\Omega_\lambda(x,t)} \left[\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x}_0 - \mathbf{x}'|} \right] \mathbf{t}(\mathbf{x}', t)\, \mathrm{d}s(\mathbf{x}'), \tag{3.35}$$

where $\mathbf{t}(\mathbf{x}', t)$ is the positively oriented unit tangent vector on $\partial\Omega_\lambda(x, t)$ and $\mathrm{d}s(\mathbf{x}')$ is an element of arclength. There is no contribution from the principal value, since the corresponding integral of $\mathbf{t}(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'|$ over $\partial B_\epsilon(\mathbf{x})$ is zero.

As illustrated in Figure 3.2, we decompose the boundary as $\partial\Omega_\lambda(x,t) = \Gamma_\lambda(x,t) \cup C_\lambda(x,t)$, where $C_\lambda(x,t)$ consist of the lines $L_{\pm\lambda}(x,t)$ and $M_\lambda(x,t)$, and $\Gamma_\lambda(x,t)$ is the cut-off front with tangent vector (3.29). The integrand in (3.35) is $O(\lambda^{-2})$ on C_λ, so taking the limit of (3.35) as $\lambda \to \infty$, we get for $\mathbf{x} \notin \Gamma$ that

$$\mathbf{u}(\mathbf{x},t) = -\int_{\Gamma(t)} \left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}_0-\mathbf{x}'|}\right] \mathbf{t}(\mathbf{x}',t)\,\mathrm{d}s(\mathbf{x}') - \bar{\mathbf{u}}(t). \tag{3.36}$$

Writing out (3.36) in components, we find that

$$u(x,y,t) = -\int_{\mathbb{R}} \left\{\frac{1}{\sqrt{(x-x')^2+(y-\varphi(x',t))^2}} - \frac{1}{\sqrt{(x')^2+(h+\varphi(x',t))^2}}\right\}\mathrm{d}x' - \bar{u}(t), \tag{3.37}$$

$$\begin{aligned} v(x,y,t) = &-\int_{\mathbb{R}} \left\{\frac{1}{\sqrt{(x-x')^2+(y-\varphi(x',t))^2}} - \frac{1}{\sqrt{(x')^2+(h+\varphi(x',t))^2}}\right\}\varphi_{x'}(x',t)\,\mathrm{d}x' \\ &- \bar{v}(t). \end{aligned} \tag{3.38}$$

Since $\varphi(\cdot,t)$ is bounded, we see by making a change of variables $x' \mapsto |y|s'$ that

$$\int_{\mathbb{R}} \left\{\frac{1}{\sqrt{(x-x')^2+(y-\varphi(x',t))^2}} - \frac{1}{\sqrt{(x')^2+y^2}}\right\}\mathrm{d}x' = o(1) \qquad \text{as } |y| \to \infty,$$

so (3.37) implies that

$$u(x,y,t) = -\int_{\mathbb{R}} \left\{\frac{1}{\sqrt{(x')^2+y^2}} - \frac{1}{\sqrt{(x')^2+(h+\varphi(x',t))^2}}\right\}\mathrm{d}x' - \bar{u}(t) + o(1) \qquad \text{as } |y| \to \infty.$$

Using the identity (3.32) with $c = |y|$, we get that

$$\begin{aligned} &-\int_{\mathbb{R}} \left\{\frac{1}{\sqrt{(x')^2+y^2}} - \frac{1}{\sqrt{(x')^2+(h+\varphi(x'))^2}}\right\}\mathrm{d}x' \\ &\qquad = 2\log|y| - \int_{\mathbb{R}} \left\{\frac{1}{\sqrt{(x')^2+1}} - \frac{1}{\sqrt{(x')^2+(h+\varphi(x'))^2}}\right\}\mathrm{d}x'. \end{aligned}$$

It follows that $u(x,y,t) = 2\log|y| + o(1)$ as $|y| \to \infty$ if

$$\bar{u}(t) = -\int_{\mathbb{R}} \left[\frac{1}{\sqrt{(x')^2+1}} - \frac{1}{\sqrt{(x')^2+(h+\varphi(x',t))^2}}\right]\mathrm{d}x'. \tag{3.39}$$

In view of the decay assumption (3.20) on φ_x, we see directly from (3.38) that

$$v(x,y,t) = \int_{\mathbb{R}} \frac{\varphi_{x'}(x',t)}{\sqrt{(x')^2+(h+\varphi(x',t))^2}}\,\mathrm{d}x' - \bar{v}(t) + o(1) \qquad \text{as } |y| \to \infty,$$

so $v(x,y,t) = o(1)$ as $|y| \to \infty$ if

$$\bar{v}(t) = \int_{\mathbb{R}} \frac{\varphi_{x'}(x',t)}{\sqrt{(x')^2 + (h+\varphi(x',t))^2}}\, \mathrm{d}x'. \tag{3.40}$$

The velocity field (3.36) therefore satisfies (3.1) when $\bar{\mathbf{u}} = (\bar{u}, \bar{v})$ is given by (3.39)–(3.40).

Next, let $\mathbf{x} = (x, \varphi(x))$ be a point on the front $\Gamma(t)$, with upward normal $\mathbf{n}$ in (3.27). We take the inner product of $\mathbf{u}_\lambda$ in (3.35) with $\mathbf{n}$ and take the limit $\lambda \to \infty$ as before, to get that

$$\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) = -\int_{\Gamma(t)} \left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}_0 - \mathbf{x}'|}\right] \mathbf{t}(\mathbf{x}',t)\cdot\mathbf{n}(\mathbf{x},t)\, \mathrm{d}s(\mathbf{x}') - \bar{\mathbf{u}}(t)\cdot\mathbf{n}(\mathbf{x}).$$

Writing out this integral explicitly and using the expressions for $\mathbf{t}$, $\mathbf{n}$, $\mathbf{x}$, and $\mathbf{x}_0$, we find that the normal velocity on the front is

$$\begin{aligned} \mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t) &= \frac{1}{\sqrt{1+\varphi_x^2(x,t)}}\left[J(x,t) + \varphi_x(x,t)\bar{u}(t) - \bar{v}(t)\right], \\ J(x,t) &= \int_{\mathbb{R}} \left[\frac{\varphi_x(x,t) - \varphi_{x'}(x',t)}{\sqrt{(x-x')^2 + (\varphi(x,t)-\varphi(x',t))^2}} - \frac{\varphi_x(x,t) - \varphi_{x'}(x',t)}{\sqrt{(x')^2 + (h+\varphi(x',t))^2}}\right] \mathrm{d}x'. \end{aligned} \tag{3.41}$$

The condition that the front $y = \varphi(x,t)$ moves with the normal velocity $\mathbf{u}\cdot\mathbf{n}$ implies that

$$\varphi_t(x,t) = J(x,t) + \varphi_x(x,t)\bar{u}(t) - \bar{v}(t). \tag{3.42}$$

We decompose the integral for J in (3.41) as

$$J(x,t) = I_1(x,t) + I_2(x,t) + J_3(x,t) + J_4(t),$$

where I_1, I_2 are given in (3.30) and

$$\begin{aligned} J_3(x,t) &= \varphi_x(x,t) \int_{\mathbb{R}} \left[\frac{1}{\sqrt{(x')^2+1}} - \frac{1}{\sqrt{(x')^2 + (h+\varphi(x',t))^2}}\right] \mathrm{d}x', \\ J_4(t) &= \int_{\mathbb{R}} \frac{\varphi_{x'}(x',t)}{\sqrt{(x')^2 + (h+\varphi(x',t))^2}}\, \mathrm{d}x'. \end{aligned}$$

From (3.39)–(3.40), we see that $J_3 = -\varphi_x \bar{u}$ and $J_4 = \bar{v}$, so (3.42) becomes (3.2).

3.1.4. Spatially periodic solutions. Equation (3.2) do not require that $\varphi(\cdot,t)$ is rapidly decreasing; in particular, they apply to smooth periodic solutions $\varphi(\cdot,t)\colon \mathbb{T} \to \mathbb{R}$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (see Figure 1.2(b)). The symbol of the linear operator $\mathbf{L}$ remains the same. Moreover, we can write the nonlinear term in (3.12) as

$$\int_{\mathbb{T}} K_p(\zeta, \varphi(x,t) - \varphi(x+\zeta,t))\,\mathrm{d}\zeta,$$

where

$$K_p(x,y) = \sum_{n\in\mathbb{Z}} \frac{y}{|x+2n\pi|} - \sinh^{-1}\left(\frac{y}{|x+2n\pi|}\right).$$

The sum defining K_p converges because of (3.11). The conservative form of the periodic front equation is then

$$\varphi_t(x,t) + \partial_x \int_{\mathbb{T}} K_p(\zeta, \varphi(x,t) - \varphi(x+\zeta,t))\,\mathrm{d}\zeta + \mathbf{L}\varphi_x(x,t) = 0.$$

The non-conservative form can be written as

$$\begin{aligned}\varphi_t(x,t) + \int_{\mathbb{T}} [\varphi_x(x,t) - \varphi_x(x+\zeta,t)] \Big\{ G_p(\zeta,0) - G_p\left(\zeta, \varphi(x,t) - \varphi(x+\zeta,t)\right)\Big\}\,\mathrm{d}\zeta \\ + \mathbf{L}\varphi_x(x,t) = 0,\end{aligned} \tag{3.43}$$

where

$$G_p(x,y) = \frac{1}{\sqrt{x^2+y^2}} + \sum_{n\in\mathbb{Z}_*}\left[\frac{1}{\sqrt{(x+2\pi n)^2+y^2}} - \frac{1}{2\pi|n|}\right].$$

One can verify that (3.43) is equivalent, up to a Galilean transformation, to the straightforward contour dynamics equation on a cylinder,

$$\varphi_t(x,t) - \int_{\mathbb{T}} [\varphi_x(x,t) - \varphi_x(x+\zeta,t)]\, G_p\left(\zeta, \varphi(x,t) - \varphi(x+\zeta,t)\right)\,\mathrm{d}\zeta = 0.$$

However, (3.43) explicitly separates the linear dispersive term from the cubic-order nonlinearity.

3.2. Contour dynamics for SQG two-fronts solutions

3.2.1. Method of regularization. Using the SQG equation (1.1) and Green's theorem, we find that the velocity field of the two front solution illustrated in Figure 1.2(d) is given formally by

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \nabla^\perp G * \theta(\mathbf{x},t) \\ &= \Theta_+ \int_{\partial\Omega_+(t)} \frac{\mathbf{n}^\perp(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}s_+(\mathbf{x}') + \Theta_- \int_{\partial\Omega_-(t)} \frac{\mathbf{n}^\perp(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}s_-(\mathbf{x}'),\end{aligned} \tag{3.44}$$

where the jumps $\Theta_\pm$ are defined in (1.6), $\mathbf{n} = (m, n)$ is the upward unit normal to $\partial\Omega_\pm(t)$, $\mathbf{n}^\perp = (-n, m)$, and $s_\pm(\mathbf{x}')$ is arc-length on $\partial\Omega_\pm(t)$.

We note that the integrals in (3.44) diverge. To obtain the front equations, we first cut-off the integration region to a λ-interval about some point $x \in \mathbb{R}$ and consider the limit $\lambda \to \infty$. If $\Theta_+ + \Theta_- \neq 0$, we also make a Galilean transformation $x \mapsto x - v'(\lambda)t$, where $v'(\lambda)$ is chosen to give well-defined limiting front equations and $|v'(\lambda)| \to \infty$ as $\lambda \to \infty$. We assume that the top and bottom fronts are smooth, approach $y = h_+$ and $y = h_-$ sufficiently rapidly as $|s_+(\mathbf{x}')| \to \infty$ and $|s_-(\mathbf{x}')| \to \infty$, respectively, and do not self-intersect or intersect each other.

Let the top and bottom fronts have parametric equations $\mathbf{x} = \mathbf{X}_1(\zeta, t)$ and $\mathbf{x} = \mathbf{X}_2(\zeta, t)$, where

$$\mathbf{X}_1(\cdot, t), \mathbf{X}_2(\cdot, t) \colon \mathbb{R} \to \mathbb{R}^2.$$

Since θ is transported by the velocity field, the fronts move with normal velocity

$$\partial_t \mathbf{X}_1 \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}, \qquad \partial_t \mathbf{X}_2 \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n},$$

so the cut-off equations for $\mathbf{X}_1$ and $\mathbf{X}_2$ are

$$\begin{aligned}
\partial_t \mathbf{X}_1(\zeta, t) = c_1(\zeta, t)\partial_\zeta \mathbf{X}_1(\zeta, t) - \Theta_+ \int_{\zeta-\lambda}^{\zeta+\lambda} \frac{\partial_{\zeta'} \mathbf{X}_1(\zeta', t)}{|\mathbf{X}_1(\zeta', t) - \mathbf{X}_1(\zeta, t)|} \, \mathrm{d}\zeta' \\
- \Theta_- \int_{\zeta-\lambda}^{\zeta+\lambda} \frac{\partial_{\zeta'} \mathbf{X}_2(\zeta', t)}{|\mathbf{X}_2(\zeta', t) - \mathbf{X}_1(\zeta, t)|} \, \mathrm{d}\zeta', \\
\partial_t \mathbf{X}_2(\zeta, t) = c_2(\zeta, t)\partial_\zeta \mathbf{X}_2(\zeta, t) - \Theta_+ \int_{\zeta-\lambda}^{\zeta+\lambda} \frac{\partial_{\zeta'} \mathbf{X}_1(\zeta', t)}{|\mathbf{X}_1(\zeta', t) - \mathbf{X}_2(\zeta, t)|} \, \mathrm{d}\zeta' \\
- \Theta_- \int_{\zeta-\lambda}^{\zeta+\lambda} \frac{\partial_{\zeta'} \mathbf{X}_2(\zeta', t)}{|\mathbf{X}_2(\zeta', t) - \mathbf{X}_2(\zeta, t)|} \, \mathrm{d}\zeta',
\end{aligned}$$

where $c_1(\zeta, t)$ and $c_2(\zeta, t)$ are arbitrary functions corresponding to time-dependent reparametrizations of the fronts.

If the fronts are given by graphs that are perturbations of $y = h_+$ and $y = h_-$, then the top front is located at $y = h_+ + \varphi(x, t)$ and the bottom front at $y = h_- + \psi(x, t)$, and we can solve for c_1 and c_2 to get

$$c_1(x, t) = \Theta_+ \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{\zeta^2 + \big(\varphi(x + \zeta, t) - \varphi(x, t)\big)^2}} \, \mathrm{d}\zeta$$

$$+ \Theta_- \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{\zeta^2 + \big(-2h + \psi(x+\zeta,t) - \varphi(x,t)\big)^2}} \, \mathrm{d}\zeta,$$

$$c_2(x,t) = \Theta_+ \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{\zeta^2 + \big(2h + \varphi(x+\zeta,t) - \psi(x,t)\big)^2}} \, \mathrm{d}\zeta$$

$$+ \Theta_- \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{\zeta^2 + \big(\psi(x+\zeta,t) - \psi(x,t)\big)^2}} \, \mathrm{d}\zeta.$$

We then obtain a coupled system for φ and ψ

$$\varphi_t(x,t) + \Theta_+ \int_{-\lambda}^{\lambda} \frac{\varphi_x(x+\zeta,t) - \varphi_x(x,t)}{\sqrt{\zeta^2 + \big(\varphi(x+\zeta,t) - \varphi(x,t)\big)^2}} \, \mathrm{d}\zeta$$

$$+ \Theta_- \int_{-\lambda}^{\lambda} \frac{\psi_x(x+\zeta,t) - \varphi_x(x,t)}{\sqrt{\zeta^2 + \big(-2h + \psi(x+\zeta,t) - \varphi(x,t)\big)^2}} \, \mathrm{d}\zeta = 0,$$

$$\psi_t(x,t) + \Theta_+ \int_{-\lambda}^{\lambda} \frac{\varphi_x(x+\zeta,t) - \psi_x(x,t)}{\sqrt{\zeta^2 + \big(2h + \varphi(x+\zeta,t) - \psi(x,t)\big)^2}} \, \mathrm{d}\zeta$$

$$+ \Theta_- \int_{-\lambda}^{\lambda} \frac{\psi_x(x+\zeta,t) - \psi_x(x,t)}{\sqrt{\zeta^2 + \big(\psi(x+\zeta,t) - \psi(x,t)\big)^2}} \, \mathrm{d}\zeta = 0.$$

Writing

$$G(x) = \frac{1}{|x|},$$

above system is equivalent to

$$\begin{aligned}
&\varphi_t(x,t) + \Theta_+ \partial_x \int_{-\lambda}^{\lambda} H_1\big(\zeta, \varphi(x+\zeta,t) - \varphi(x,t)\big) \, \mathrm{d}\zeta + \Theta_+ \partial_x \int_{-\lambda}^{\lambda} G(\zeta) \big[\varphi(x+\zeta,t) - \varphi(x,t)\big] \, \mathrm{d}\zeta \\
&\quad + \Theta_- \partial_x \int_{-\lambda}^{\lambda} H_2\big(\zeta, -2h + \psi(x+\zeta,t) - \varphi(x,t)\big) \, \mathrm{d}\zeta \\
&\qquad + \Theta_- \partial_x \int_{-\lambda}^{\lambda} G\left(\sqrt{\zeta^2 + (2h)^2}\right) \big[\psi(x+\zeta,t) - \varphi(x,t)\big] \, \mathrm{d}\zeta = 0, \\
&\psi_t(x,t) + \Theta_- \partial_x \int_{-\lambda}^{\lambda} H_1\big(\zeta, \psi(x+\zeta,t) - \psi(x,t)\big) \, \mathrm{d}\zeta + \Theta_- \partial_x \int_{-\lambda}^{\lambda} G(\zeta) \big[\psi(x+\zeta,t) - \psi(x,t)\big] \, \mathrm{d}\zeta \\
&\quad + \Theta_+ \partial_x \int_{-\lambda}^{\lambda} H_2\big(\zeta, 2h + \varphi(x+\zeta,t) - \psi(x,t)\big) \, \mathrm{d}\zeta \\
&\qquad + \Theta_+ \partial_x \int_{-\lambda}^{\lambda} G\left(\sqrt{\zeta^2 + (2h)^2}\right) \big[\varphi(x+\zeta,t) - \psi(x,t)\big] \, \mathrm{d}\zeta = 0
\end{aligned} \tag{3.45}$$

where

$$H_1(x,y) = -G(x)y + \int_0^y G\Big(\sqrt{x^2+s^2}\Big)\,\mathrm{d}s,$$
$$H_2(x,y) = -G\left(\sqrt{x^2+(2h)^2}\right)y + \int_0^y G\Big(\sqrt{x^2+s^2}\Big)\,\mathrm{d}s. \tag{3.46}$$

Therefore, we have for $j = 1, 2$ and fixed y that

$$H_j(x,y) = O\Big(\frac{1}{|x|^3}\Big) \quad \text{as } |x| \to \infty.$$

It follows that the nonlinear terms in (3.45) converge as $\lambda \to \infty$, so it suffices to consider linear terms in (3.45).

We only write out the computation for the first equation; the computation for the second equation is similar. The linear term

$$\mathbf{L}_{1,\lambda}\varphi(x,t) = \int_{-\lambda}^{\lambda} \frac{\varphi(x+\zeta,t)-\varphi(x,t)}{|\zeta|}\,\mathrm{d}\zeta$$

can be written as [**HS18**]

$$\mathbf{L}_{1,\lambda}\varphi(x,t) = v_1(\lambda)\varphi(x,t) + \mathbf{L}^*_{1,\lambda}\varphi(x,t),$$

where

$$v_1(\lambda) = -2\int_1^{\lambda} \frac{1}{|\zeta|}\,\mathrm{d}\zeta \tag{3.47}$$

and $\mathbf{L}^*_{1,\lambda}\varphi \to \mathbf{L}_1\varphi$ as $\lambda \to \infty$, where $\mathbf{L}_1$ is the Fourier multiplier with symbol

$$b_1(\xi) = \int_{|\zeta|>1} \frac{e^{i\xi\zeta}}{|\zeta|}\,\mathrm{d}\zeta - \int_{|\zeta|<1} \frac{1-e^{i\xi\zeta}}{|\zeta|}\,\mathrm{d}\zeta = -2\gamma - 2\log|\xi|. \tag{3.48}$$

As for the second linear term, we have

$$\begin{aligned}\mathbf{L}_{2,\lambda}\left[\varphi,\psi\right](x,t) &= \int_{-\lambda}^{\lambda} \frac{\psi(x+\zeta,t)-\varphi(x,t)}{\sqrt{\zeta^2+(2h)^2}}\,\mathrm{d}\zeta \\ &= v_2(\lambda)\varphi(x,t) + v_3(\lambda)\varphi(x,t) + \mathbf{L}^*_{2,\lambda}\psi(x,t),\end{aligned}$$

where $v_2(\lambda)$ is a divergent part and $v_3(\lambda)$ is a convergent part

$$v_2(\lambda) = -2\int_1^{\lambda} \frac{1}{\sqrt{\zeta^2+(2h)^2}}\,\mathrm{d}\zeta, \tag{3.49}$$

$$v_3(\lambda) = -2\int_0^1 \frac{1}{\sqrt{\zeta^2+(2h)^2}}\,\mathrm{d}\zeta, \tag{3.50}$$

and $\mathbf{L}_{2,\lambda}^*\psi \to \mathbf{L}_2\psi$ as $\lambda \to \infty$, where $\mathbf{L}_2$ is the Fourier multiplier with symbol

$$b_2(\xi) = \int_{\mathbb{R}} G\left(\sqrt{\zeta^2+(2h)^2}\right) e^{i\xi\zeta}\,\mathrm{d}\zeta = 2K_0\left(2h|\xi|\right), \tag{3.51}$$

where K_0 is the modified Bessel function of second kind defined in (2.21).

We denote by v_4 the limit

$$v_4 = \lim_{\lambda\to\infty} v_3(\lambda) = 2\log(2h) - 2\log\left(1+\sqrt{1+(2h)^2}\right),$$

where $v_3(\lambda)$ is given in (3.50).

The cut-off system (3.45) can then be written as

$$\begin{aligned}
&\varphi_t(x,t) + \left[\Theta_+ v_1(\lambda) + \Theta_- v_2(\lambda) + \Theta_- v_3(\lambda)\right]\varphi_x(x,t) + \Theta_+\mathbf{L}_{1,\lambda}^*\varphi_x(x,t) + \Theta_-\mathbf{L}_{2,\lambda}^*\psi_x(x,t)\\
&\quad + \Theta_+\partial_x \int_{-\lambda}^{\lambda} H_1\big(\zeta, \varphi(x+\zeta,t)-\varphi(x,t)\big)\,\mathrm{d}\zeta + \Theta_-\partial_x \int_{-\lambda}^{\lambda} H_2\big(\zeta, -2h+\psi(x+\zeta,t)-\varphi(x,t)\big)\,\mathrm{d}\zeta = 0,\\
&\psi_t(x,t) + \left[\Theta_- v_1(\lambda) + \Theta_+ v_2(\lambda) + \Theta_+ v_3(\lambda)\right]\psi_x(x,t) + \Theta_-\mathbf{L}_{1,\lambda}^*\psi_x(x,t) + \Theta_+\mathbf{L}_{2,\lambda}^*\varphi_x(x,t)\\
&\quad + \Theta_-\partial_x \int_{-\lambda}^{\lambda} H_1\big(\zeta, \psi(x+\zeta,t)-\psi(x,t)\big)\,\mathrm{d}\zeta + \Theta_+\partial_x \int_{-\lambda}^{\lambda} H_2\big(\zeta, 2h+\varphi(x+\zeta,t)-\psi(x,t)\big)\,\mathrm{d}\zeta = 0.
\end{aligned}$$

In the limit $\lambda \to \infty$, the possibly problematic terms in these equations are $\left[\Theta_+ v_1(\lambda) + \Theta_- v_2(\lambda)\right]\varphi_x(x,t)$ and $\left[\Theta_- v_1(\lambda) + \Theta_+ v_2(\lambda)\right]\psi_x(x,t)$. The only case when these two terms converge to finite limits are when $\Theta_+ = -\Theta_-$. Otherwise, we regularize the equations by choosing a suitable Galilean transformation. Indeed, if we choose

$$v'(\lambda) = \frac{\Theta_+ + \Theta_-}{2}\left(v_1(\lambda) + v_2(\lambda) + v_3(\lambda)\right),$$

and make a Galilean transformation $x \mapsto x - v'(\lambda)t$, then the system becomes

$$\begin{aligned}
&\varphi_t(x,t) + \frac{\Theta_+ - \Theta_-}{2}\left(v_1(\lambda) - v_2(\lambda) - v_3(\lambda)\right)\varphi_x(x,t) + \Theta_+\mathbf{L}_{1,\lambda}^*\varphi_x(x,t) + \Theta_-\mathbf{L}_{2,\lambda}^*\psi_x(x,t)\\
&\quad + \Theta_+\partial_x \int_{-\lambda}^{\lambda} H_1\big(\zeta, \varphi(x+\zeta,t)-\varphi(x,t)\big)\,\mathrm{d}\zeta + \Theta_-\partial_x \int_{-\lambda}^{\lambda} H_2\big(\zeta, -2h+\psi(x+\zeta,t)-\varphi(x,t)\big)\,\mathrm{d}\zeta = 0,\\
&\psi_t(x,t) - \frac{\Theta_+ - \Theta_-}{2}\left(v_1(\lambda) - v_2(\lambda) - v_3(\lambda)\right)\psi_x(x,t) + \Theta_-\mathbf{L}_{1,\lambda}^*\psi_x(x,t) + \Theta_+\mathbf{L}_{2,\lambda}^*\varphi_x(x,t)\\
&\quad + \Theta_-\partial_x \int_{-\lambda}^{\lambda} H_1\big(\zeta, \psi(x+\zeta,t)-\psi(x,t)\big)\,\mathrm{d}\zeta + \Theta_+\partial_x \int_{-\lambda}^{\lambda} H_2\big(\zeta, 2h+\varphi(x+\zeta,t)-\psi(x,t)\big)\,\mathrm{d}\zeta = 0.
\end{aligned}$$

The asymptotic behavior of $G(\zeta)$ and $G\left(\sqrt{\zeta^2+(2h)^2}\right)$ as $\zeta \to \infty$ is given by

$$G(\zeta) \sim \frac{1}{\zeta}$$
$$G\left(\sqrt{\zeta^2+(2h)^2}\right) \sim \frac{1}{\zeta} + O\left(\frac{h}{\zeta^3}\right).$$

Therefore, from (3.47) and (3.49), we see that $v_1(\lambda) - v_2(\lambda)$ converges as $\lambda \to \infty$, and we define

$$v_5 = \lim_{\lambda\to\infty} [v_1(\lambda) - v_2(\lambda)] = 2\log 2 - 2\log\left(1 + \sqrt{1+(2h)^2}\right).$$

Putting everything together and letting $\lambda \to \infty$, we get the regularized system in conservative form

$$\begin{aligned}
&\varphi_t(x,t) + v'\varphi_x(x,t) + \Theta_+\mathbf{L}_1\varphi_x(x,t) + \Theta_-\mathbf{L}_2\psi_x(x,t) \\
&+ \Theta_+\partial_x \int_{\mathbb{R}} H_1\big(\zeta, \varphi(x+\zeta,t) - \varphi(x,t)\big)\,\mathrm{d}\zeta + \Theta_-\partial_x \int_{\mathbb{R}} H_2\big(\zeta, -2h + \psi(x+\zeta,t) - \varphi(x,t)\big)\,\mathrm{d}\zeta = 0, \\
&\psi_t(x,t) - v'\psi_x(x,t) + \Theta_-\mathbf{L}_1\psi_x(x,t) + \Theta_+\mathbf{L}_2\varphi_x(x,t) \\
&+ \Theta_-\partial_x \int_{\mathbb{R}} H_1\big(\zeta, \psi(x+\zeta,t) - \psi(x,t)\big)\,\mathrm{d}\zeta + \Theta_+\partial_x \int_{\mathbb{R}} H_2\big(\zeta, 2h + \varphi(x+\zeta,t) - \psi(x,t)\big)\,\mathrm{d}\zeta = 0,
\end{aligned}$$

where H_1, H_2 are given in (3.46), the symbols of $\mathbf{L}_1$, $\mathbf{L}_2$ are given in (3.48)–(3.51), and

$$v' = -(\Theta_+ - \Theta_-)\log h. \tag{3.52}$$

One can also take the derivatives inside the integrals and apply an additional Galilean transformation $x \mapsto x + (\Theta_+ + \Theta_-)\gamma$ to obtain (3.3).

The system (3.3) has the Hamiltonian form

$$\varphi_t + J_+\frac{\delta\mathcal{H}}{\delta\varphi} = 0, \quad \psi_t + J_-\frac{\delta\mathcal{H}}{\delta\psi} = 0, \qquad J_+ = \frac{1}{\Theta_+}\partial_x, \quad J_- = \frac{1}{\Theta_-}\partial_x, \tag{3.53}$$

with the Hamiltonian

$$\begin{aligned}
H(\varphi,\psi) = \frac{1}{2}\int_{\mathbb{R}} &\left\{ v'\Theta_+\varphi^2 - v'\Theta_-\psi^2 - 2\Theta_+^2 \log|\partial_x|\varphi + 4\Theta_+\Theta_-\varphi K_0(2h|\partial_x|)\psi - 2\Theta_-^2\psi\log|\partial_x|\psi \right\}\mathrm{d}x \\
+ \frac{1}{2}\int_{\mathbb{R}^2} &\left\{ \Theta_+^2 F_1(x-x',\varphi-\varphi') + 2\Theta_+\Theta_- F_2(x-x', 2h+\varphi-\psi') + \Theta_-^2 F_1(x-x',\psi-\psi') \right\}\mathrm{d}x\,\mathrm{d}x',
\end{aligned} \tag{3.54}$$

where $\varphi = \varphi(x,t)$, $\varphi' = \varphi(x',t)$, $\psi = \psi(x,t)$, $\psi' = \psi(x',t)$, and the functions F_1, F_2 satisfy

$$\partial_y F_1(x,y) = H_1(x,y), \qquad \partial_y F_2(x,y) = H_2(x,y).$$

3.2.2. Method of decomposition. In this subsection, we sketch the derivation of (3.3) by the decomposition method as in the one-front case. We decompose the two-front solution (1.7) as

$$\theta(\mathbf{x},t) = \tilde{\theta}(\mathbf{x}) + \theta^*(\mathbf{x},t),$$

where the background field $\tilde{\theta}$ is defined as

$$\tilde{\theta}(y) = \begin{cases} \theta_+ & \text{if } y > h', \\ \theta_0 & \text{if } -h' < y < h', \\ \theta_- & \text{if } y < -h'. \end{cases}$$

Here, $h' > 0$ is fixed such that the locations of the fronts between θ_+, θ_0, θ_- satisfy

$$-h' < h_- + \psi(x,t) < h_+ + \varphi(x,t) < h'.$$

The perturbed field θ^* is then

$$\theta^*(\mathbf{x},t) = \begin{cases} 0 & y > h', \\ 2\pi\Theta_+ & h_+ + \varphi(x,t) < y < h', \\ 0 & h_- + \psi(x,t) < y < h_+ + \varphi(x,t), \\ -2\pi\Theta_- & -h' < y < h_- + \psi(x,t), \\ 0 & y < -h'. \end{cases}$$

The corresponding decomposition of the velocity field is

$$\mathbf{u}(\mathbf{x},t) = \tilde{\mathbf{u}}(\mathbf{x}) + \mathbf{u}^*(\mathbf{x},t),$$

and it is easy to use Hilbert transform $\mathbf{H}$ to calculate the background velocity field

$$\tilde{\mathbf{u}}(\mathbf{x}) = \left(2\Theta_+ \log|y-h'| + 2\Theta_- \log|y+h'|,\, 0\right), \tag{3.55}$$

where $\Theta_\pm$ is defined in (1.6).

We denote the support of $\theta^*(\cdot,t)$ by $\Omega^*(t)=\Omega^*_+(t)\cup\Omega^*_-(t)$ where

$$\Omega^*_+(t)=\left\{\mathbf{x}\in\mathbb{R}^2 : h_+ + \varphi(x,t) < y < h'\right\},$$
$$\Omega^*_-(t)=\left\{\mathbf{x}\in\mathbb{R}^2 : -h' < y < h_- + \psi(x,t)\right\}.$$

Use $\mathbf{u}^* = -\mathbf{R}^\perp\theta^*$, we have that

$$\mathbf{u}^*(\mathbf{x},t)=\frac{1}{2\pi}\,\mathrm{p.v.}\int_{\Omega^*(t)}\frac{(\mathbf{x}-\mathbf{x}')^\perp}{|\mathbf{x}-\mathbf{x}'|^3}\theta^*(\mathbf{x}',t)\,\mathrm{d}\mathbf{x}', \qquad (x,y)^\perp=(-y,x).$$

This integral converges absolutely, so that we can apply a far-field cutoff and use Green's theorem followed by taking the limits of cutoff parameter approaches infinity (as in the one-front case) to obtain $\mathbf{u}^*$. In order to study the evolution of the fronts, we need to calculate the normal component of $\mathbf{u}^*$ on the fronts.

For a point $\mathbf{x}=(x,h_+ + \varphi(x,t))$ on the top front (we denote by $\mathbf{n}_+$ the unit upward normal to the top front), we have that

$$\begin{aligned}
&\mathbf{u}^*(\mathbf{x},t)\cdot\mathbf{n}_+(\mathbf{x},t)\\
=&\frac{\Theta_+}{\sqrt{1+\varphi_x^2(x,t)}}\int_{\mathbb{R}}\frac{\varphi_x(x,t)-\varphi_{x'}(x',t)}{\sqrt{(x-x')^2+[\varphi(x,t)-\varphi(x',t)]^2}}-\frac{\varphi_x(x,t)}{\sqrt{(x-x')^2+[\varphi(x,t)+h_+-h']^2}}\,\mathrm{d}x'\\
&+\frac{\Theta_-}{\sqrt{1+\varphi_x^2(x,t)}}\int_{\mathbb{R}}\frac{\varphi_x(x,t)-\psi_{x'}(x',t)}{\sqrt{(x-x')^2+[\varphi(x,t)-\psi(x',t)+h_+-h_-]^2}}\\
&\qquad\qquad-\frac{\varphi_x(x,t)}{\sqrt{(x-x')^2+[\varphi(x,t)+h_++h']^2}}\,\mathrm{d}x'.
\end{aligned}$$

Similarly, for a point $\mathbf{x}=(x,h_- + \psi(x,t))$ on the bottom front (we denote by $\mathbf{n}_-$ the unit upward normal to the bottom front), we have

$$\begin{aligned}
&\mathbf{u}^*(\mathbf{x},t)\cdot\mathbf{n}_-(\mathbf{x},t)\\
=&\frac{\Theta_+}{\sqrt{1+\psi_x^2(x,t)}}\int_{\mathbb{R}}\frac{\psi_x(x,t)-\varphi_{x'}(x',t)}{\sqrt{(x-x')^2+[\psi(x,t)-\varphi(x',t)+h_--h_+]^2}}\\
&\qquad\qquad-\frac{\psi_x(x,t)}{\sqrt{(x-x')^2+[\psi(x,t)+h_--h']^2}}\,\mathrm{d}x'\\
&+\frac{\Theta_-}{\sqrt{1+\psi_x^2(x,t)}}\int_{\mathbb{R}}\frac{\psi_x(x,t)-\psi_{x'}(x',t)}{\sqrt{(x-x')^2+[\psi(x,t)-\psi(x',t)]^2}}\\
&\qquad\qquad-\frac{\psi_x(x,t)}{\sqrt{(x-x')^2+[\psi(x,t)+h_-+h']^2}}\,\mathrm{d}x'.
\end{aligned}$$

Including the contribution from the background flow $\tilde{\mathbf{u}}$, and using the condition that the fronts moves with the upward normal velocity, we obtain that

$$\varphi_t(x,t) = I_+(x,t), \qquad I_+(x,t) = \left(\sqrt{1+\varphi_x^2}\right)\left[\tilde{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{n}_+(\mathbf{x},t) + \mathbf{u}^*(\mathbf{x},t)\cdot\mathbf{n}_+(\mathbf{x},t)\right],$$
$$\psi_t(x,t) = I_-(x,t), \qquad I_-(x,t) = \left(\sqrt{1+\psi_x^2}\right)\left[\tilde{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{n}_-(\mathbf{x},t) + \mathbf{u}^*(\mathbf{x},t)\cdot\mathbf{n}_-(\mathbf{x},t)\right].$$

From (3.55) and definition of $\mathbf{n}_\pm$, we have

$$\begin{aligned}
\left(\sqrt{1+\varphi_x^2}\right)\tilde{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{n}_+(\mathbf{x},t) &= -2\Theta_+\log|\varphi(x,t)+h_+-h'|\varphi_x(x,t)\\
&\quad - 2\Theta_-\log|\varphi(x,t)+h_++h'|\varphi_x(x,t),\\
\left(\sqrt{1+\psi_x^2}\right)\tilde{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{n}_-(\mathbf{x},t) &= -2\Theta_+\log|\psi(x,t)+h_--h'|\psi_x(x,t)\\
&\quad - 2\Theta_-\log|\psi(x,t)+h_-+h'|\psi_x(x,t).
\end{aligned}$$

We then decompose I_+, I_- as

$$I_+(x,t) = I_{+,1}(x,t) + I_{+,2}(x,t) + I_{+,3}(x,t) + I_{+,4}(x,t) + I_{+,5}(x,t),$$
$$I_-(x,t) = I_{-,1}(x,t) + I_{-,2}(x,t) + I_{-,3}(x,t) + I_{-,5}(x,t) + I_{-,5}(x,t),$$

where

$$\begin{aligned}
I_{+,1}(x,t) &= \Theta_+\int_{\mathbb{R}}\left\{\frac{\varphi_x(x,t)-\varphi_{x'}(x',t)}{\sqrt{(x-x')^2+[\varphi(x,t)-\varphi(x',t)]^2}} - \frac{\varphi_x(x,t)-\varphi_{x'}(x',t)}{|x-x'|}\right\}\mathrm{d}x',\\
I_{+,2}(x,t) &= \Theta_+\int_{\mathbb{R}}\left\{\frac{\varphi_x(x,t)-\varphi_{x'}(x',t)}{|x-x'|} - \frac{\varphi_x(x,t)}{\sqrt{(x')^2+1}}\right\}\mathrm{d}x',\\
I_{+,3}(x,t) &= \Theta_-\int_{\mathbb{R}}\left\{\frac{\varphi_x(x,t)-\psi_{x'}(x',t)}{\sqrt{(x-x')^2+[\varphi(x,t)-\psi(x',t)+h_+-h_-]^2}}\right.\\
&\qquad\qquad\left. - \frac{\varphi_x(x,t)-\psi_{x'}(x',t)}{\sqrt{(x-x')^2+(h_+-h_-)^2}}\right\}\mathrm{d}x',\\
I_{+,4}(x,t) &= -\Theta_-\int_{\mathbb{R}}\frac{\psi_{x'}(x',t)}{\sqrt{(x-x')^2+(h_+-h_-)^2}}\mathrm{d}x',\\
I_{+,5}(x,t) &= \Theta_+\varphi_x(x,t)\left\{\int_{\mathbb{R}}\frac{1}{\sqrt{(x')^2+1}} - \frac{1}{\sqrt{(x-x')^2+[\varphi(x,t)+h_+-h']^2}}\mathrm{d}x'\right.\\
&\qquad\qquad\left. - 2\log|\varphi(x,t)+h_+-h'|\right\}
\end{aligned}$$

$$+\,\Theta_-\varphi_x(x,t)\bigg\{\int_{\mathbb{R}}\frac{1}{\sqrt{(x-x')^2+(h_+-h_-)^2}}-\frac{1}{\sqrt{(x-x')^2+[\varphi(x,t)+h_++h']^2}}\,\mathrm{d}x'$$
$$-2\log|\varphi(x,t)+h_++h'|\bigg\},$$

and similar expressions for $I_{-,j}$, $j=1,2,\dots,5$, which we omit.

Similar to the one-front case, using Fourier transform, and using $h_+-h_-=2h$, we get that

$$\begin{aligned}
&I_{+,2}(x,t)=2\Theta_+\log|\partial_x|\varphi_x(x,t)+2\Theta_+(\gamma-\log 2)\varphi_x(x,t),\\
&I_{+,4}(x,t)=-2\Theta_-K_0(2h|\partial_x|)\psi_x(x,t),\qquad I_{+,5}(x,t)=-2\Theta_-\log(2h)\varphi_x(x,t),\\
&I_{-,2}(x,t)=2\Theta_-\log|\partial_x|\psi_x(x,t)+2\Theta_-(\gamma-\log 2)\psi_x(x,t),\\
&I_{-,4}(x,t)=-2\Theta_+K_0(2h|\partial_x|)\psi_x(x,t),\qquad I_{-,5}(x,t)=-2\Theta_+\log(2h)\psi_x(x,t).
\end{aligned}$$

Thus, we obtain the contour dynamics equations for the two-front SQG equation (3.3) up to a Galilean transformation.

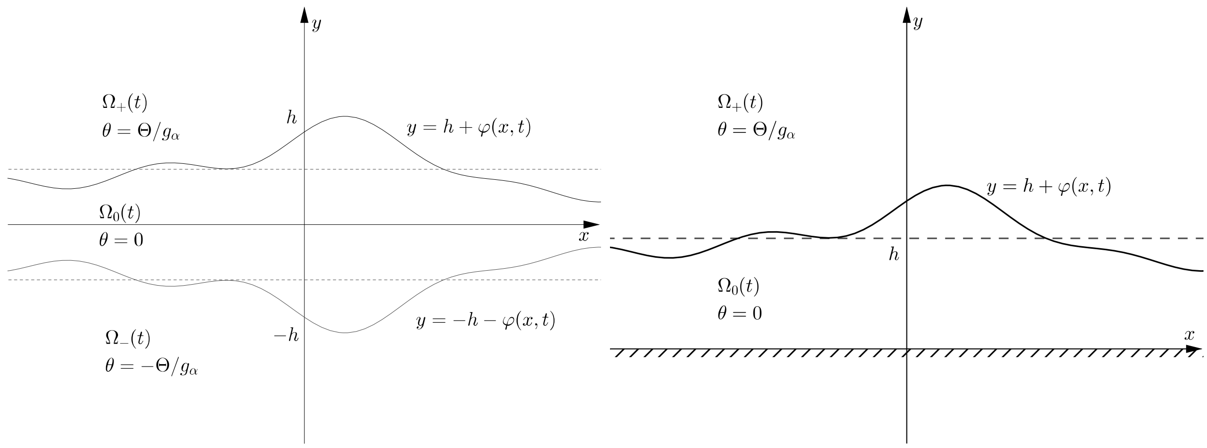
3.2.3. Scalar reductions. In this subsection, we write out two scalar equations that arise as reductions of the system (3.3) when the jumps are symmetric or anti-symmetric.

3.2.3.1. *Symmetric reduction.* If $\Theta_+=\Theta_-$, then $v'=0$ from (3.52), and the system (3.3) is compatible with solutions of the form $\psi(x,t)=-\varphi(x,t)$, when it reduces to a scalar equation for φ. Writing $\Theta=\Theta_+=\Theta_-$, we find that the equation becomes

$$\begin{aligned}
&\varphi_t(x,t)-2\Theta\left(\log|\partial_x|+K_0(2h|\partial_x|)\right)\varphi_x(x,t)\\
&+\Theta\int_{\mathbb{R}}[\varphi_x(x+\zeta,t)-\varphi_x(x,t)]\left\{\frac{1}{\sqrt{\zeta^2+[\varphi(x+\zeta,t)-\varphi(x,t)]^2}}-\frac{1}{|\zeta|}\right\}\mathrm{d}\zeta\\
&+\Theta\int_{\mathbb{R}}[\varphi_x(x+\zeta,t)+\varphi_x(x,t)]\left\{\frac{1}{\sqrt{\zeta^2+[2h+\varphi(x+\zeta,t)+\varphi(x,t)]^2}}-\frac{1}{\sqrt{\zeta^2+(2h)^2}}\right\}\mathrm{d}\zeta=0.
\end{aligned}\tag{3.56}$$

For the SQG equations (1.1) in the spatial upper half-plane $\mathbb{R}\times\mathbb{R}_+$ with no-flow boundary conditions on a rigid boundary $y=0$ (see Figure 3.3 and [**GPar**, **KRYZ16**, **KYZ17**]), we find by the method of images that

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}_+}\left\{\nabla^\perp_{\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\nabla^\perp_{\mathbf{x}}\frac{1}{|\mathbf{x}-\bar{\mathbf{x}}'|}\right\}\theta(\mathbf{x},t)\,\mathrm{d}\mathbf{x}',$$



(a) Symmetric GSQG fronts. (b) GSQG front with a rigid flat bottom.

FIGURE 3.3. Symmetric reduction of SQG system.

where $\bar{\mathbf{x}}' = (x', -y')$ if $\mathbf{x}' = (x', y')$. In this setting, if a front is located at $y = h + \varphi(x,t) > 0$, and

$$\theta(x,y,t) = \begin{cases} 2\pi\Theta & \text{if } y > h + \varphi(x,t), \\ 0 & \text{if } 0 < y < h + \varphi(x,t), \end{cases}$$

then the regularized contour dynamics equation for a front in the half-plane coincides with (3.56).

3.2.3.2. *Anti-symmetric reduction.* If $\Theta_+ = -\Theta_-$, then (3.3) is compatible with solutions of the form

$$\varphi(x,t) = \varphi(x,t), \qquad \psi(x,t) = -\varphi(-x,t),$$

and it reduces to a scalar equation for φ (see Figure 3.4). Writing $\Theta = \Theta_+ = -\Theta_-$ and making a Galilean transformation $x \mapsto x - 2\Theta(\gamma + \log h)t$, we find that the equation becomes

$$\begin{aligned} &\varphi_t(x,t) - 2\Theta \log|\partial_x|\varphi_x(x,t) - 2\Theta K_0(2h|\partial_x|)\varphi_x(-x,t) \\ &+ \Theta \int_{\mathbb{R}} [\varphi_x(x+\zeta,t) - \varphi_x(x,t)] \left\{ \frac{1}{\sqrt{\zeta^2 + [\varphi(x+\zeta,t) - \varphi(x,t)]^2}} - \frac{1}{|\zeta|} \right\} \mathrm{d}\zeta \\ &- \Theta \int_{\mathbb{R}} [\varphi_x(-x-\zeta,t) - \varphi_x(x,t)] \left\{ \frac{1}{\sqrt{\zeta^2 + [2h + \varphi(-x-\zeta,t) + \varphi(x,t)]^2}} - \frac{1}{\sqrt{\zeta^2 + (2h)^2}} \right\} \mathrm{d}\zeta = 0. \end{aligned} \tag{3.57}$$

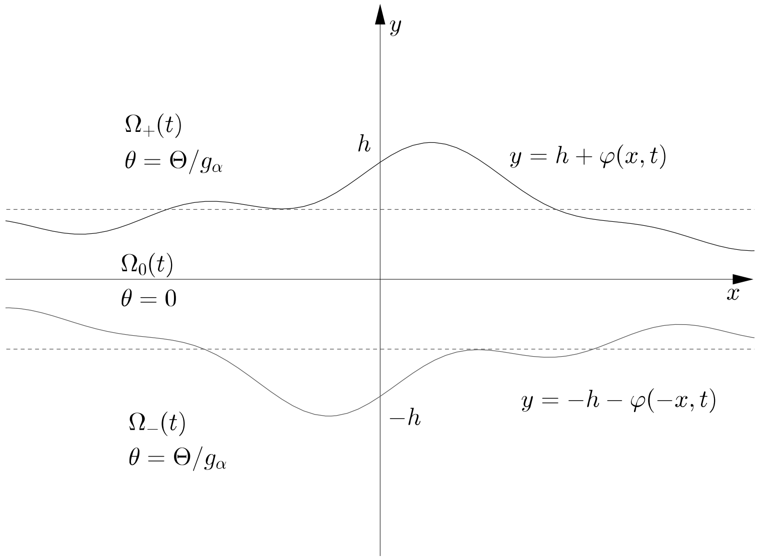


FIGURE 3.4. Anti-symmetric reduction of SQG system.

CHAPTER 4

Approximate SQG fronts

Between the approximation of the idea and the precision of reality there was a small gap of the unimaginable, and it was this hiatus that gave him no rest.

– *Milan Kundera*

This chapter is devoted to study a simplified version of (3.2) — a cubically nonlinear nonlocal equation — posed on the circle $\mathbb{T}$

$$\varphi_t + \frac{1}{2}\partial_x\left\{\varphi^2 \log|\partial_x|\varphi_{xx} - \varphi\log|\partial_x|(\varphi^2)_{xx} + \frac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\right\} = 2\log|\partial_x|\varphi_x. \tag{4.1}$$

We treat this equation as an approximate model equation for the full equation (3.2), under the assumption $\varphi_x \ll 1$. We also write down other equivalent forms of (4.1), including a conservative form (4.14) and a Hamiltonian form (4.15)–(4.16)).

We prove two theorems in this chapter. The first theorem concerns with the initial value problem for a dispersionless version of (4.1)

$$\begin{aligned} &\varphi_t + \frac{1}{2}\partial_x\left\{\varphi^2 \log|\partial_x|\varphi_{xx} - \varphi\log|\partial_x|(\varphi^2)_{xx} + \frac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\right\} + \mathbf{L}\varphi_x = 0, \\ &\varphi(x,0) = \varphi_0(x), \end{aligned} \tag{4.2}$$

where $\mathbf{L}$ is an arbitrary self-adjoint operator with symbol $b(k)$. The case $\mathbf{L} = -2\log|\partial_x|$ corresponds to the approximate SQG front equation (4.1).

To state the first main theorem, we fix some notations. The usual Sobolev space for mean-zero functions defined on circle $\mathbb{T}$ is defined in (2.5). In addition, we use a logarithmically-modified Hilbert space

$$\begin{aligned} \dot{H}^s_{\log}(\mathbb{T}) &= \left\{f\colon \mathbb{T}\to\mathbb{R} \mid \hat{f}(0) = 0,\ \|f\|_{\dot{H}^s_{\log}} < \infty\right\}, \\ \|f\|_{\dot{H}^s_{\log}} &= \left[\sum_{k\in\mathbb{Z}_*} \log(1+|k|)\cdot|k|^{2s}|\hat{f}(k)|^2\right]^{1/2}. \end{aligned} \tag{4.3}$$

If $\tau\colon [0,T_*) \to [0,\infty)$ is a decreasing function, then we denote by $L^\infty(0,T_*;\dot H^\tau(\mathbb{T}))$ the space of functions

$$\varphi\colon [0,T_*) \to \bigcup_{t\in[0,T_*)} \dot H^{\tau(t)}(\mathbb{T})$$

such that $\varphi(t) \in \dot H^{\tau(t)}(\mathbb{T})$, and for every $0 < T < T_*$

$$\varphi \in L^\infty(0,T;\dot H^{\tau(T)}(\mathbb{T})),$$

with analogous notation for other time-dependent Sobolev spaces.

Theorem 4.0.1. *Let the operator* $\mathbf{L}$ *have real-valued symbol* $b : \mathbb{Z} \to \mathbb{R}$ *and suppose that* $\tau_0 > 5/2$. *For every* $\varphi_0 \in \dot H^{\tau_0}(\mathbb{T})$, *there exists* $T_* > 0$ *and a differentiable, decreasing function* $\tau\colon [0,T_*) \to (5/2,\tau_0]$ *with* $\tau(0) = \tau_0$, *depending on* τ_0 *and* $\|\varphi_0\|_{\dot H^{\tau_0}(\mathbb{T})}$, *such that the initial value problem* (4.2) *has a solution with*

$$\varphi \in L^\infty(0,T_*;\dot H^\tau(\mathbb{T})) \cap L^2(0,T_*;\dot H^\tau_{\log}(\mathbb{T})).$$

Moreover, there exists a numerical constant $C > 0$ *such that*

$$\sup_{t\in[0,T]} \|\varphi(t)\|^2_{\dot H^{\tau(t)}} + C\|\varphi_0\|^2_{\dot H^{\tau_0}} \int_0^T \|\varphi(t)\|^2_{\dot H^{\tau(t)}_{\log}}\, \mathrm{d}t \le \|\varphi_0\|^2_{\dot H^{\tau_0}} \tag{4.4}$$

for every $0 < T < T_*$, *where the norms are defined in* (2.5), (4.3). *The solution is unique while* $\tau(t) > 9/2$.

The proof is purely hyperbolic in nature (c.f. [**Aus11**, **Hun06**, **Ifr12**]), but there is a loss of derivatives at logarithmic rate. Therefore, we adapt proofs for Gevrey-class solutions of nonlinear evolution equations (see e.g. [**FV11**, **KTVZ11**]), in which one uses time-dependent norms to compensate for the loss of regularity. The difference is that, since there is only a logarithmic derivative loss, we obtain solutions for initial data with finitely many derivatives, rather than C^∞ Gevrey-class initial data. The existence time in the theorem depends on the number of Sobolev derivatives possessed by the initial data, as well as in its Sobolev norm.

The second main result of this chapter is a local well-posedness theorem for the initial value problem for the approximate equation for SQG fronts

$$\begin{aligned} &\varphi_t + \frac{1}{2}\partial_x\left\{\varphi^2 \log|\partial_x|\varphi_{xx} - \varphi\log|\partial_x|(\varphi^2)_{xx} + \frac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\right\} = 2\log|\partial_x|\varphi_x, \\ &\varphi(x,0) = \varphi_0(x), \end{aligned} \tag{4.5}$$

and the theorem is as follows (Para-products are defined in (2.6)).

Theorem 4.0.2. *Let $s > 7/2$. If $\varphi_0 \in \dot{H}^s(\mathbb{T})$ satisfies $\|T^2_{\varphi_{0x}}\|_{L^2\to L^2} \leq C$ for some $0 < C < 2$, then there exists $T > 0$ depending only on $\|\varphi_0\|_{\dot{H}^s}$ and C such that the initial value problem (4.5) has a unique solution with $\varphi \in C([0,T];\dot{H}^s(\mathbb{T}))$. The solution map $U(t)\colon \dot{H}^s(\mathbb{T}) \to C([0,T];\dot{H}^s(\mathbb{T}))$ defined as $\varphi_0(x) \mapsto \varphi(x,t)$, is continuous on $\dot{H}^s(\mathbb{T})$ and Lipschitz continuous on $\dot{H}^r(\mathbb{T})$ for $0 \leq r < s-1$, meaning that if $\varphi, \tilde{\varphi} \in C([0,T];\dot{H}^s(\mathbb{T}))$ are solutions, then there exists a constant $M > 0$ depending on $\|\varphi\|_{C([0,T];\dot{H}^s)}$, $\|\tilde{\varphi}\|_{C([0,T];\dot{H}^s)}$ such that*

$$\|\varphi(\cdot,t) - \tilde{\varphi}(\cdot,t)\|_{H^r} \leq M\|\varphi(\cdot,0) - \tilde{\varphi}(\cdot,0)\|_{H^r} \qquad \text{for all } t \in [0,T]. \tag{4.6}$$

The idea of the proof is to define a weighted $\dot{H}^s$-energy (4.40), which uses the linear term to control the loss of derivatives from the nonlinearity.

An outline of this chapter is as follows. The derivation of equation (4.1) is in Section 4.1. In Section 4.2, we prove Theorem 4.0.1 and in Section 4.3, we prove Theorem 4.0.2. Finally, in Section 4.4, we show some numerical results for the approximate SQG equation, which suggest that smooth solutions break down in finite time, with the formation of oscillatory singularities in which the solutions remain continuous but their derivatives blows up.

4.1. Approximate equation

4.1.1. Conservative form. To derive equation (4.1), we first restrict our attention to the equation (3.2) posed on $\mathbb{R}$ (for definiteness) and assume that $\varphi_x \ll 1$. In fact, for the spatially periodic fronts, (4.1) can be obtained from (3.43) in a similar fashion, but we omit the details here.

It follows from (3.10) that

$$\sinh^{-1}\left(\frac{y}{|x|}\right) = \frac{y}{|x|} - \frac{1}{6}\frac{y^3}{|x|^3} + |x|\mathcal{O}\left(\frac{y^5}{|x|^5}\right) \qquad \text{as} \quad \frac{y}{|x|} \to 0.$$

Retaining the lowest order terms in y, we find the approximation

$$\frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|} - \sinh^{-1}\left(\frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|}\right) \sim \frac{1}{6}\frac{[\varphi(x,t) - \varphi(x+\zeta,t)]^3}{|\zeta|^3}.$$

Thus, the cubic approximation of the conservative equation (3.2) is

$$\varphi_t(x,t) + \frac{1}{6}\partial_x \int_{\mathbb{R}} \left[\frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|}\right]^3 \mathrm{d}\zeta = 2\log|\partial_x|\varphi_x(x,t). \tag{4.7}$$

Equation (4.7) is equivalent to (4.1), as we show by writing it in spectral form.

4.1.2. Spectral equation. We first write the nonlinear term in (4.7) as

$$\partial_x \int_{\mathbb{R}} \left[\frac{\varphi(x,t) - \varphi(x+\zeta,t)}{|\zeta|} \right]^3 \mathrm{d}\zeta = \int_{\mathbb{R}^3} T(k_2,k_3,k_4)\hat{\varphi}(k_2)\hat{\varphi}(k_3)\hat{\varphi}(k_4) e^{i(k_2+k_3+k_4)x} \,\mathrm{d}k_2 \,\mathrm{d}k_3 \,\mathrm{d}k_4, \tag{4.8}$$

where

$$\begin{aligned} T(k_2,k_3,k_4) &= \Re \int_{\mathbb{R}} \frac{(1-e^{ik_2\zeta})(1-e^{ik_3\zeta})(1-e^{ik_4\zeta})}{|\zeta|^3} \,\mathrm{d}\zeta \\ &= \Re \int_{\mathbb{R}} \frac{1}{|\zeta|^3} \Big\{ \left(1-e^{ik_2\zeta}\right) + \left(1-e^{ik_3\zeta}\right) + \left(1-e^{ik_4\zeta}\right) + \left(1-e^{i(k_2+k_3+k_4)\zeta}\right) \\ &\qquad - \left(1-e^{i(k_2+k_3)\zeta}\right) - \left(1-e^{i(k_2+k_4)\zeta}\right) - \left(1-e^{i(k_3+k_4)\zeta}\right) \Big\} \,\mathrm{d}\zeta. \end{aligned} \tag{4.9}$$

We now write T in (4.9) as

$$T(k_2,k_3,k_4) = \tilde{a}(k_2) + \tilde{a}(k_3) + \tilde{a}(k_4) + \tilde{a}(k_2+k_3+k_4) - \tilde{a}(k_2+k_3) - \tilde{a}(k_2+k_4) - \tilde{a}(k_3+k_4),$$

For $|\zeta| < 1$, we use the cancellation

$$k_2^2 + k_3^2 + k_4^2 + (k_2+k_3+k_4)^2 - (k_2+k_3)^2 - (k_2+k_4)^2 - (k_3+k_4)^2 = 0$$

and write $\tilde{a}$ as

$$\begin{aligned} \tilde{a}(k) &= 2\int_0^1 \frac{1 - \frac{1}{2}k^2\zeta^2 - \cos(k\zeta)}{\zeta^3} \,\mathrm{d}\zeta + 2\int_1^\infty \frac{1-\cos(k\zeta)}{\zeta^3} \,\mathrm{d}\zeta \\ &= 2k^2 \left(\int_0^{|k|} \frac{1-\frac{1}{2}\zeta^2 - \cos\zeta}{\zeta^3} \,\mathrm{d}\zeta + \int_{|k|}^\infty \frac{1-\cos\zeta}{\zeta^3} \,\mathrm{d}\zeta \right). \end{aligned}$$

Writing

$$\int_0^{|k|} \frac{1-\frac{1}{2}\zeta^2-\cos\zeta}{\zeta^3} \,\mathrm{d}\zeta + \int_{|k|}^\infty \frac{1-\cos\zeta}{\zeta^3} \,\mathrm{d}\zeta = \frac{1}{2}C - \frac{1}{2}\int_1^{|k|} \frac{\mathrm{d}\zeta}{\zeta}$$

where

$$C = 2\int_0^1 \frac{1-\frac{1}{2}\zeta^2 - \cos\zeta}{\zeta^3} \,\mathrm{d}\zeta + 2\int_1^\infty \frac{1-\cos\zeta}{\zeta^3} \,\mathrm{d}\zeta,$$

is a constant, we get that $\tilde{a}(k) = Ck^2 - k^2 \log|k|$.

In order to give a symmetric expression for T, it is convenient to introduce another variable k_1 and define $S\colon \mathbb{R}^4 \to \mathbb{R}$ by

$$S(k_1,k_2,k_3,k_4) = \tilde{a}(k_1) + \tilde{a}(k_2) + \tilde{a}(k_3) + \tilde{a}(k_4) - \frac{1}{2}\Big\{\tilde{a}(k_1+k_2) + \tilde{a}(k_1+k_3) + \tilde{a}(k_1+k_4) + \tilde{a}(k_2+k_3) + \tilde{a}(k_2+k_4) + \tilde{a}(k_3+k_4)\Big\}.$$

Then

$$T(k_2,k_3,k_4) = S(k_1,k_2,k_3,k_4) \qquad \text{on } k_1+k_2+k_3+k_4 = 0. \tag{4.10}$$

Notice that the term Ck^2 cancels out of the expression in (4.11) for $S(k_1,k_2,k_3,k_4)$ on $k_1 + k_2 + k_3 + k_4 = 0$, so that we can rewrite S as

$$S(k_1,k_2,k_3,k_4) = a(k_1) + a(k_2) + a(k_3) + a(k_4) - \frac{1}{2}\Big\{a(k_1+k_2) + a(k_1+k_3) + a(k_1+k_4) + a(k_2+k_3) + a(k_2+k_4) + a(k_3+k_4)\Big\}, \tag{4.11}$$

where

$$a = -k^2 \log|k|. \tag{4.12}$$

Substituting in (4.12) into (4.11), we obtain

$$\begin{aligned} &S(k_1,k_2,k_3,k_4) \\ =\ & -k_1^2\log|k_1| - k_2^2\log|k_2| - k_3^2\log|k_3| - k_4^2\log|k_4| \\ &+\frac{1}{2}\Big\{(k_1+k_2)^2\log|k_1+k_2| + (k_1+k_3)^2\log|k_1+k_3| + (k_1+k_4)^2\log|k_1+k_4| \\ &+ (k_2+k_3)^2\log|k_2+k_3| + (k_2+k_4)^2\log|k_2+k_4| + (k_3+k_4)^2\log|k_3+k_4|\Big\}. \end{aligned} \tag{4.13}$$

Using (4.8) and (4.10) in (4.7), we see that the spectral form of (4.7) is

$$\hat{\varphi}_t(k_1,t) + \frac{1}{6}ik_1\int_{\mathbb{R}^3}\delta(k_1+k_2+k_3+k_4)S(k_1,k_2,k_3,k_4)\hat{\varphi}^*(k_2,t)\hat{\varphi}^*(k_3,t)\hat{\varphi}^*(k_4,t)\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 - 2ik_1\log|k_1|\hat{\varphi}(k_1,t) = 0, \tag{4.14}$$

where δ denotes the delta-distribution, $\hat{\varphi}^*(k) = \hat{\varphi}(-k)$ denotes the complex conjugate of $\hat{\varphi}(k)$. Thus, using the convolution theorem and (4.13), we find that the approximate equation (4.7) has the real form (4.1).

4.1.3. Hamiltonian structure. We remark that the approximate equation (4.1) has the Hamiltonian form

$$\varphi_t + \partial_x \left[\frac{\delta H}{\delta \varphi}\right] = 0,$$

where, suppressing the time variable, we can write the Hamiltonian in equivalent forms as

$$H(\varphi) = \int_{\mathbb{R}} \left[\varphi \log|\partial_x|\varphi + \frac{1}{8}\varphi^2 \partial_x^2 \log|\partial_x|\varphi^2 - \frac{1}{6}\varphi \partial_x^2 \log|\partial_x|\varphi^3\right] \mathrm{d}x. \tag{4.15}$$

The spectral form of the Hamiltonian is

$$\begin{aligned} H(\hat{\varphi}) = &-\frac{1}{6\cdot 8}\int_{\mathbb{R}^4} \delta(k_1+k_2+k_3+k_4) S(k_1,k_2,k_3,k_4)\hat{\varphi}(k_1)\hat{\varphi}(k_2)\hat{\varphi}(k_3)\hat{\varphi}(k_4)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 \\ &- \int_{\mathbb{R}} \log|k|\hat{\varphi}^*(k)\hat{\varphi}(k)\,\mathrm{d}k. \end{aligned} \tag{4.16}$$

This Hamiltonian structure explains the symmetry of S in (4.11).

4.2. Weak local well-posedness

In this section, we prove Theorem 4.0.1. We will use the following consequence of Young's inequality

$$\sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{Z}_*\\ k_1+k_2+k_3+k_4=0}} \left|\hat{f}_1(k_1)\hat{f}_2(k_2)\hat{f}_3(k_3)\hat{f}_4(k_4)\right| \le \|\hat{f}_1\|_{\ell^2}\|\hat{f}_2\|_{\ell^1}\|\hat{f}_3\|_{\ell^1}\|\hat{f}_4\|_{\ell^2}, \tag{4.17}$$

and the Sobolev inequality

$$\|\hat{f}\|_{\ell^1} \le Z(s)\|f\|_{\dot{H}^s} \qquad \text{for } s > \frac{1}{2}, \tag{4.18}$$

where Z is given in terms of the Riemann-zeta function by

$$Z(s) = \left(\sum_{k\in\mathbb{Z}_*} \frac{1}{|k|^{2s}}\right)^{1/2} = \sqrt{2\zeta(2s)}.$$

Let $\rho\colon \mathbb{Z}_*^4 \to \mathbb{Z}_*^4$ be a map that permutes its entries and orders their absolute values. We denote the values of ρ by $(m_1, m_2, m_3, m_4) = \rho(k_1, k_2, k_3, k_4)$, where

$$(m_1, m_2, m_3, m_4) = (k_{\sigma 1}, k_{\sigma 2}, k_{\sigma 3}, k_{\sigma 4}) \quad \text{for some } \sigma \in S_4, \tag{4.19}$$

$$|m_1| \geq |m_2| \geq |m_3| \geq |m_4|. \tag{4.20}$$

Here, S_4 denotes the symmetric group on $\{1, 2, 3, 4\}$.

The proof depends crucially on the symmetry of the interaction coefficients that follows from the Hamiltonian structure of the equation. Consider the spectral form of the initial value problem (4.2) for a spatially-periodic function $\varphi(x,t)$, with Fourier coefficients $\hat{\varphi}(k,t)$, given by

$$\hat{\varphi}_t(k_1,t) + \frac{1}{6} i k_1 \sum_{\substack{k_2,k_3,k_4 \in \mathbb{Z}_* \\ k_2+k_3+k_4=-k_1}} S(k_1,k_2,k_3,k_4) \hat{\varphi}^*(k_2,t) \hat{\varphi}^*(k_3,t) \hat{\varphi}^*(k_4,t) - 2ik_1 b(k_1) \hat{\varphi}(k_1,t) = 0, \tag{4.21}$$

$$\hat{\varphi}(k_1, 0) = \hat{\varphi}_0(k_1).$$

When convenient, we omit the time variable and write $\hat{\varphi}(k) = \hat{\varphi}(k,t) = \hat{\varphi}^*(-k)$, $\varphi_j = \varphi(k_j)$. In (4.21), we assume that $S\colon \mathbb{Z}_*^4 \to \mathbb{R}$ satisfies

$$S(k_1, k_2, k_3, k_4) = S(-k_1, -k_2, -k_3, -k_4), \tag{4.22}$$

$$S(k_1, k_2, k_3, k_4) = S(k_{\sigma 1}, k_{\sigma 2}, k_{\sigma 3}, k_{\sigma 4}) \quad \text{for every } \sigma \in S_4. \tag{4.23}$$

It is easy to verity that S satisfies (4.22)–(4.23).

Moreover, we show in [**HS18**] that, if there exist $\mu, \nu \geq 0$ and a constant $C_S > 0$ such that

$$|S(k_1, k_2, k_3, k_4)| \leq C_S |m_3|^{\mu} |m_4|^{\nu} \qquad \text{for all } k_1, k_2, k_3, k_4 \in \mathbb{Z}_*, \tag{4.24}$$

where the $k_{\sigma j}$ and m_j are defined as in (4.19)–(4.20), which states that the growth of S is bounded by the smaller wavenumbers on which it depends, then the initial value problem is locally well-posed in Sobolev space. However, in the case of (4.11), the assumption (4.24) does not hold. Instead, S satisfies (see Corollary B.0.4)

$$|S(k_1, k_2, k_3, k_4)| \leq C_2 |m_3||m_4| \left[\log(1 + |m_1|) \log(1 + |m_2|)\right]^{1/2},$$

which explains the logarithmic loss of derivative we explained at the beginning of the chapter.

Proof of Theorem 4.0.1. First, we derive the *a priori* estimate (4.4). Let $\tau\colon [0,T] \to (5/2,\infty)$ be a differentiable function, and let φ be a smooth solution of (4.5). We define energies $E, F\colon [0,T] \to [0,\infty)$ by

$$E(t) = \|\varphi(t)\|^2_{\dot{H}^{\tau(t)}} = \sum_{k\in\mathbb{Z}_*} |k|^{2\tau(t)}|\hat{\varphi}(k,t)|^2,$$
$$F(t) = \|\varphi(t)\|^2_{\dot{H}^{\tau(t)}_{\log}} = \sum_{k\in\mathbb{Z}_*} \log(1+|k|)\cdot|k|^{2\tau(t)}|\hat{\varphi}(k,t)|^2.$$

We write the equation (4.2) in the spectral form (4.21) with kernel (4.13). Using the equation (4.21) with (4.23), Lemma B.0.1, and Corollary B.0.4 to estimate the time-derivative of E, we get that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= 2\dot{\tau}\sum_{k\in\mathbb{Z}_*}\log|k|\cdot|k|^{2\tau}|\hat{\varphi}(k)|^2 + \sum_{k\in\mathbb{Z}_*}|k|^{2\tau}\frac{\mathrm{d}}{\mathrm{d}t}|\hat{\varphi}(k)|^2\\
&\le 2\dot{\tau}F + \frac{1}{12}\sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{Z}_*\\ k_1+k_2+k_3+k_4=0}}\left|\left(k_1|k_1|^{2\tau}+k_2|k_2|^{2\tau}+k_3|k_3|^{2\tau}+k_4|k_4|^{2\tau}\right)S(k_1,k_2,k_3,k_4)\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4\right|\\
&\le 2\dot{\tau}F + \frac{4!}{12}C_0(\tau)C_2\sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{Z}_*\\ k_1+k_2+k_3+k_4=0}}\left[\log(1+|k_1|)\log(1+|k_2|)\right]^{1/2}|k_1|^{\tau}|k_2|^{\tau}|k_3|^2|k_4|\cdot|\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4|\\
&\le 2\dot{\tau}F + 2C_0(\tau)C_2F\cdot\left(\sum_{k_3\in\mathbb{Z}_*}|k_3|^2|\hat{\varphi}(k_3)|\right)\cdot\sum_{k_4\in\mathbb{Z}_*}|k_4||\hat{\varphi}(k_4)|,\end{aligned}$$

where a dot denotes a time derivative. Then, as long as $\tau > 5/2$, the Sobolev inequality (4.18) implies that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le 2\left[\dot{\tau} + C_3(\tau)E\right]F, \qquad C_3(s) = C_0(s)C_2Z(s-1)Z(s-2). \tag{4.25}$$

The function $C_3\colon (5/2,\infty)\to(0,\infty)$ is a smooth function such that $C_3(s)\to\infty$ as $s\to 5/2$ and $s\to\infty$. Thus, there is a numerical constant $C_4 > 0$ such that

$$C_3(s) \ge C_4 \qquad \text{for } 5/2 < s < \infty.$$

For example, if C_0, C_2 are given by (B.4), (B.8), then we find numerically that one can take $C_4 = 1000$.

Fix a constant $M > 1$ and let τ be the solution of the initial value problem

$$\dot{\tau} + ME_0 C_3(\tau) = 0, \qquad \tau(0) = \tau_0 \tag{4.26}$$

on a maximal time-interval $[0, T_*)$ such that $\tau(t) > 5/2$, where $E_0 = E(0)$. Then it follows from (4.25)–(4.26) that E is decreasing on $[0, T_*)$ and

$$\frac{\mathrm{d}E}{\mathrm{d}t} + (M-1)C_3 E_0 F \leq 0.$$

Grönwall's inequality gives

$$E(t) + E_0(M-1)\int_0^t C_3(\tau(s))F(s)\,\mathrm{d}s \leq E_0,$$

so (4.4) follows for $0 \leq T < T_*$ with $C = (M-1)C_4$.

To estimate φ_t, we write (4.21) in spatial form as

$$\varphi_t + \partial_x \mathbf{Q}(\varphi, \varphi, \varphi) + \mathbf{L}\varphi_x = 0, \tag{4.27}$$

where the trilinear form $\mathbf{Q}$ is defined in terms of Fourier coefficients by

$$\hat{\mathbf{Q}}(\hat{\varphi}, \hat{\psi}, \hat{\xi})(k_1) = \frac{1}{6} \sum_{\substack{k_1,k_2,k_3,k_4 \in \mathbb{Z}_* \\ k_1+k_2+k_3+k_4=0}} S(k_1,k_2,k_3,k_4)\hat{\varphi}^*(k_2)\hat{\psi}^*(k_3)\hat{\xi}^*(k_4), \tag{4.28}$$

and S is given by (4.13).

The symmetry of S (4.23) implies that

$$q(\eta, \varphi, \psi, \chi) = \int_{\mathbb{T}} \eta \mathbf{Q}(\varphi, \psi, \chi)\,\mathrm{d}x$$

is a symmetric form. Moreover, using Corollary B.0.4, we get that for arbitrarily small $0 < \epsilon \ll 1$

$$\begin{aligned} &|q(\eta, \varphi, \psi, \chi)| \\ &\leq C \sum_{\substack{k_1,k_2,k_3,k_4 \in \mathbb{Z}_* \\ k_1+k_2+k_3+k_4=0}} |S(k_1,k_2,k_3,k_4)\hat{\eta}(k_1)\hat{\varphi}(k_2)\hat{\psi}(k_3)\hat{\chi}(k_4)| \\ &\leq C \sum_{\substack{k_1,k_2,k_3,k_4 \in \mathbb{Z}_* \\ k_1+k_2+k_3+k_4=0}} \left(|k_1|^{-s+2\epsilon}|\log(1+|m_1|)|\right)|k_1|^{s-2\epsilon}|m_3||m_4||\hat{\eta}(k_1)\hat{\varphi}(k_2)\hat{\psi}(k_3)\hat{\chi}(k_4)| \end{aligned} \tag{4.29}$$

On $k_1 + k_2 + k_3 + k_4 = 0$, we have

$$|k_1|^{s-2\epsilon} \leq C \left(|k_2|^{s-2\epsilon} + |k_3|^{s-2\epsilon} + |k_4|^{s-2\epsilon} \right).$$

From (4.19), we get that

$$\begin{aligned} &\left(|k_1|^{-s+2\epsilon} | \log(1 + |m_1|) | \right) |k_1|^{s-2\epsilon} |m_3||m_4| \\ &\leq C |k_1|^{-s+2\epsilon} \left[\log(1 + |k_2|) + \log(1 + |k_3|) + \log(1 + |k_4|) \right] \\ &\qquad \cdot \left[|k_2|^{s-2\epsilon} |k_3||k_4| + |k_3|^{s-2\epsilon} |k_2||k_4| + |k_4|^{s-2\epsilon} |k_2||k_3| \right] \\ &\quad + C |k_1|^{-s+2\epsilon} \log(1 + |k_1|) \left[|k_2|^{s-2\epsilon} |k_3||k_4| + |k_3|^{s-2\epsilon} |k_2||k_4| + |k_4|^{s-2\epsilon} |k_2||k_3| \right] \\ &\leq C |k_1|^{-s+\epsilon} \left[|k_2|^s |k_3|^{1+2\epsilon} |k_4|^{1+2\epsilon} + |k_3|^s |k_2|^{1+2\epsilon} |k_4|^{1+2\epsilon} + |k_4|^s |k_2|^{1+2\epsilon} |k_3|^{1+2\epsilon} \right]. \end{aligned}$$

Using this inequality in (4.29), followed by (4.17)–(4.18) with the assumption that

$$s > \frac{1}{2}, \tag{4.30}$$

and that the arbitrary choice of $0 < \epsilon \ll 1$ (independent of s), we get

$$|q(\eta, \varphi, \psi, \chi)| \leq C \|\eta\|_{\dot{H}^{-s+\epsilon}} \|\varphi\|_{\dot{H}^s} \|\psi\|_{\dot{H}^s} \|\chi\|_{\dot{H}^s}.$$

It follows by duality that (4.28) defines a bounded trilinear map

$$\mathbf{Q} \colon \dot{H}^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T}) \to \dot{H}^{s-\epsilon}(\mathbb{T}) \tag{4.31}$$

when s satisfies (4.30). Hence, (4.4) and (4.27) implies that if $0 < T < T_*$, then

$$\sup_{0 \leq t \leq T} \|\varphi_t(t)\|_{\dot{H}^{\tau(T)-1-\epsilon}} \leq C, \tag{4.32}$$

for some constant C depending on τ_0, T, and E_0.

The construction of the solution by the use of Galerkin approximations follows by standard arguments. We construct Galerkin approximations $\left\{\varphi^N : N \in \mathbb{N}\right\}$ by projecting the equations onto Fourier modes with $|k| \leq N$. These approximations satisfy the same estimates as the *a priori* estimates derived above, so from (4.4) and (4.32) we can extract a subsequence that converges weakly to a limit φ in $L^\infty(0, T; \dot{H}^{\tau(T)}(\mathbb{T})) \cap W^{1,\infty}(0, T; \dot{H}^{\tau(T)-1}(\mathbb{T}))$. By the Aubin-Lions lemma (see e.g., [**Ama00**]), a further subsequence converges strongly in $C([0, T]; \dot{H}^{\tau(T)-\epsilon}(\mathbb{T}))$ for sufficiently small $\epsilon > 0$, and by the continuity of the nonlinear term in (4.31), the limit is a solution of the equation. Then the *a priori* bounds (4.4) restores the regularity of the solutions.

Finally, if φ, $\tilde{\varphi}$ are solutions to (4.5) with initial data $\varphi(0) = \varphi_0$, $\tilde{\varphi}(0) = \tilde{\varphi}_0$, respectively, then write $u = \varphi - \tilde{\varphi}$ and let τ be the solution of (4.26) with $E_0 = \max\{\|\varphi_0\|^2_{H^{\tau_0}}, \|\tilde{\varphi}_0\|^2_{H^{\tau_0}}\}$, and we define

$$U(t) = \|\varphi(t) - \tilde{\varphi}(t)\|_{H^{\tau(t)-2}}, \qquad V(t) = \|\varphi(t) - \tilde{\varphi}(t)\|_{H^{\tau(t)-2}_{\log}},$$

where we assume that $\tau(t) - 2 > 5/2$.

We first write down the equation for u

$$u_t + \frac{1}{6}\partial_x \left[\mathbf{Q}(\varphi, \varphi, \varphi) - \mathbf{Q}(\tilde{\varphi}, \tilde{\varphi}, \tilde{\varphi})\right] + \mathbf{L}[u] = 0.$$

By bounds (4.4) for both φ and $\tilde{\varphi}$, we have that

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}t} - 2\dot{\tau}V &= -\frac{1}{6}\left[q(|\partial_x|^2\tau u_x, \varphi, \varphi, \varphi) - q(|\partial_x|^{2\tau}u_x, \tilde{\varphi}, \tilde{\varphi}, \tilde{\varphi})\right] \\ &= -\frac{1}{6}\left[q(|\partial_x|^{2\tau}u_x, u, \varphi, \varphi) + q(|\partial_x|^{2\tau}u_x, u, \varphi, \tilde{\varphi}) + q(|\partial_x|^{2\tau}u_x, u, \tilde{\varphi}, \tilde{\varphi})\right]\end{aligned}$$

Then a similar argument to the derivation of the energy estimate (4.25) whose details we omit, gives that

$$\frac{\mathrm{d}U}{\mathrm{d}t} \leq \left[2\dot{\tau} + E_0 C(\tau)\right] V + C(\tau)\left(\|\varphi\|_{H^\tau_{\log}} + \|\tilde{\varphi}\|_{H^\tau_{\log}}\right) U,$$

where $C(\tau) > 0$ is a continuous function of τ. If $0 < T < T_*$, then (4.26) implies that $\tau(t)$ is bounded independently of M on a time-interval $0 \leq t \leq T/M$. We choose M large enough that $MC_3(\tau) \geq C(\tau)$ on this interval. Then

$$\frac{\mathrm{d}U}{\mathrm{d}t} \leq C(\tau)\left(\|\varphi\|_{H^\tau_{\log}} + \|\tilde{\varphi}\|_{H^\tau_{\log}}\right) U$$

for $0 \leq t \leq T/M$, and Grönwall's inequality implies that the solution is unique. □

4.3. Local well-posedness

4.3.1. Para-linearization of the equation. In this subsection, we para-linearize the approximate SQG front equation (4.1) and for simplicity write $\log|\partial_x| = L$. We put (4.1) into a form that allows us to make weighted energy estimates. This form makes explicit the cancellation of second-order derivatives in the flux and extracts a nonlinear term $L(T^2_{\varphi_x}\varphi)$ from the flux that is responsible for the logarithmic loss of derivatives in the dispersionless equation.

In the following, we use $\mathcal{P}(\cdot)$ to denote a nondecreasing polynomial, which might change from line to line.

Lemma 4.3.1. *Suppose that $\varphi(\cdot, t) \in \dot{H}^s(\mathbb{T})$ with $s > 7/2$. Then (4.1) can be written as*

$$\varphi_t + \partial_x \left\{ \frac{1}{2} T_{B(\varphi)}\varphi + [T_{\varphi_x}, T_\varphi]\varphi_x \right\} + \mathcal{R} = L[(2 - T^2_{\varphi_x})\varphi]_x, \tag{4.33}$$

where

$$B(\varphi) = \varphi_x^2 - 3\varphi\varphi_{xx} - 2\varphi_{xx}L\varphi - 4\varphi_x L\varphi_x, \tag{4.34}$$

and the remainder term $\mathcal{R}$ satisfies the estimate

$$\|\mathcal{R}\|_{\dot{H}^s} \leq \mathcal{P}\left(\|\varphi\|_{\dot{H}^s}\right) \tag{4.35}$$

for a nondecreasing polynomial $\mathcal{P}$.

Proof. The nonlinear flux term in (4.1) is given by

$$\varphi^2 L\varphi_{xx} - \varphi L(\varphi^2)_{xx} + \frac{1}{3}L(\varphi^3)_{xx} = \varphi^2 L\varphi_{xx} - 2\varphi L(\varphi\varphi_{xx} + \varphi_x^2) + L(\varphi^2\varphi_{xx} + 2\varphi\varphi_x^2).$$

We will use the lemmas in Chapter 2 to expand this term.

1. Term $L(\varphi\varphi_{xx} + \varphi_x^2)$.

By Lemma 2.2.3, we have that

$$L(\varphi\varphi_{xx}) = T_\varphi L\varphi_{xx} + T_{D\varphi}D^{-1}\varphi_{xx} - \frac{1}{2}T_{D^2\varphi}D^{-2}\varphi_{xx} + T_{\varphi_{xx}}L\varphi + \mathcal{R}_1,$$
$$L(\varphi_x^2) = 2T_{\varphi_x}L\varphi_x + 2T_{D\varphi_x}D^{-1}\varphi_x + \mathcal{R}_2,$$

with

$$\|\mathcal{R}_1\|_{\dot{H}^{s+1}} \leq C\|\varphi\|_{W^{3,\infty}}\|\varphi\|_{\dot{H}^s}, \qquad \|\mathcal{R}_2\|_{\dot{H}^{s+1}} \leq C\|\varphi\|_{W^{3,\infty}}\|\varphi\|_{\dot{H}^s}.$$

2. Term $L(\varphi^2\varphi_{xx})$.

By Lemma 2.2.5, we have that

$$L(\varphi^2\varphi_{xx}) = T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{\varphi_{xx}} L\varphi + 2T_{D\varphi}T_\varphi D^{-1}\varphi_{xx} - \frac{1}{2}[2T_{D^2\varphi}T_\varphi + 2T_{D\varphi}T_{D\varphi}]D^{-2}\varphi_{xx} + \mathcal{R}_3,$$
$$= T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{\varphi_{xx}} L\varphi + 2T_{D\varphi}T_\varphi D^{-1}\varphi_{xx} - [T_{D^2\varphi}T_\varphi + T_{D\varphi}T_{D\varphi}]D^{-2}\varphi_{xx} + \mathcal{R}_3,$$

with

$$\|\mathcal{R}_3\|_{\dot{H}^{s+1}} \le C\|\varphi\|^2_{W^{3,\infty}}\|\varphi\|_{\dot{H}^s}.$$

3. Term $L(\varphi\varphi_x^2)$.

By Lemma 2.2.5, we have that

$$L(\varphi\varphi_x^2) = 2T_\varphi T_{\varphi_x} L\varphi_x + T_{\varphi_x}T_{\varphi_x}L\varphi + 2(T_{D\varphi}T_{\varphi_x} + T_\varphi T_{D\varphi_x})D^{-1}\varphi_x + \mathcal{R}_4,$$

with

$$\|\mathcal{R}_4\|_{\dot{H}^{s+1}} \le C\|\varphi\|^2_{W^{3,\infty}}\|\varphi\|_{\dot{H}^s}.$$

4. Term $\varphi^2 L\varphi_{xx}$.

By the decomposition (2.7), we can express $\varphi^2 L\varphi_{xx}$ as

$$\varphi^2 L\varphi_{xx} = T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{L\varphi_{xx}}\varphi + \mathcal{R}_5,$$

with

$$\|\mathcal{R}_5\|_{\dot{H}^{s+1}} \le C\|L\varphi\|^2_{W^{3,\infty}}\|\varphi\|_{\dot{H}^s}.$$

Collecting all the above expressions, we obtain that

$$\begin{aligned}
&\varphi^2 L\varphi_{xx} - 2\varphi L(\varphi\varphi_{xx} + \varphi_x^2) + L(\varphi^2\varphi_{xx} + 2\varphi\varphi_x^2) \\
&\quad = T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{L\varphi_{xx}}\varphi - 2\varphi\Big[T_\varphi L\varphi_{xx} + T_{D\varphi}D^{-1}\varphi_{xx} - \frac{1}{2}T_{D^2\varphi}D^{-2}\varphi_{xx} \\
&\qquad + T_{\varphi_{xx}}L\varphi + 2T_{\varphi_x}L\varphi_x + 2T_{D\varphi_x}D^{-1}\varphi_x\Big] + T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{\varphi_{xx}}L\varphi \\
&\qquad + 2T_{D\varphi}T_\varphi D^{-1}\varphi_{xx} - \big[T_{D^2\varphi}T_\varphi + T_{D\varphi}T_{D\varphi}\big]D^{-2}\varphi_{xx} + 4T_\varphi T_{\varphi_x}L\varphi_x \\
&\qquad + 2T_{\varphi_x}T_{\varphi_x}L\varphi + 4(T_{D\varphi}T_{\varphi_x} + T_\varphi T_{D\varphi_x})D^{-1}\varphi_x + \mathfrak{R}
\end{aligned}$$

$$
\begin{aligned}
&= T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{L\varphi_{xx}}\varphi - 2T_\varphi\Big[T_\varphi L\varphi_{xx} + T_{D\varphi}D^{-1}\varphi_{xx} - \frac{1}{2}T_{D^2\varphi}D^{-2}\varphi_{xx} \\
&\quad + T_{\varphi_{xx}}L\varphi + 2T_{\varphi_x}L\varphi_x + 2T_{D\varphi_x}D^{-1}\varphi_x\Big] - 2T_A\varphi + T_\varphi T_\varphi L\varphi_{xx} + 2T_\varphi T_{\varphi_{xx}}L\varphi \\
&\quad + 2T_{D\varphi}T_\varphi D^{-1}\varphi_{xx} - [T_{D^2\varphi}T_\varphi + T_{D\varphi}T_{D\varphi}]D^{-2}\varphi_{xx} + 4T_\varphi T_{\varphi_x}L\varphi_x \\
&\quad + 2T_{\varphi_x}T_{\varphi_x}L\varphi + 4(T_{D\varphi}T_{\varphi_x} + T_\varphi T_{D\varphi_x})D^{-1}\varphi_x + \mathfrak{R} - 2R(A,\varphi),
\end{aligned}
$$

where $R(\cdot,\cdot)$ is a term as in (2.8) and

$$
\begin{aligned}
&\mathfrak{R} = -2\varphi(\mathcal{R}_1 + \mathcal{R}_2) + \mathcal{R}_3 + 2\mathcal{R}_4 + \mathcal{R}_5, \\
&A = T_\varphi L\varphi_{xx} + T_{D\varphi}D^{-1}\varphi_{xx} - \frac{1}{2}T_{D^2\varphi}D^{-2}\varphi_{xx} + T_{\varphi_{xx}}L\varphi + 2T_{\varphi_x}L\varphi_x + 2T_{D\varphi_x}D^{-1}\varphi_x.
\end{aligned}
$$

Simplifying the above equation, we find that the higher order terms involving $L\varphi_{xx}$ and $L\varphi_x$ vanish, and

$$
\begin{aligned}
&\varphi^2 L\varphi_{xx} - \varphi L(\varphi^2)_{xx} + \frac{1}{3}L(\varphi^3)_{xx} \\
&\quad = 2T_\varphi T_{L\varphi_{xx}}\varphi - 2T_\varphi\left[\frac{1}{2}T_{D^2\varphi}\varphi + T_{\varphi_{xx}}L\varphi + 2T_{\varphi_{xx}}\varphi\right] - 2T_A\varphi + 2T_\varphi T_{\varphi_{xx}}L\varphi \\
&\qquad - [T_{\varphi_{xx}}T_\varphi + T_{\varphi_x}T_{\varphi_x}]\,\varphi + 2T_{\varphi_x}T_{\varphi_x}L\varphi + 4(T_{\varphi_x}T_{\varphi_x} + T_\varphi T_{\varphi_{xx}})\varphi \\
&\qquad + \mathfrak{R} - 2R(A,\varphi) + 2[T_{D\varphi}, T_\varphi]D^{-1}\varphi_{xx} \\
&\quad = 2T_{\varphi_x}T_{\varphi_x}L\varphi + 2T_\varphi T_{L\varphi_{xx}}\varphi + T_\varphi T_{\varphi_{xx}}\varphi - 2T_A\varphi - T_{\varphi_{xx}}T_\varphi\varphi + 3T_{\varphi_x}T_{\varphi_x}\varphi \\
&\qquad + \mathfrak{R} - 2R(A,\varphi) + 2[T_{D\varphi}, T_\varphi]D^{-1}\varphi_{xx} \\
&\quad = 2T_{\varphi_x}^2 L\varphi + T_B\varphi + 2[T_{D\varphi}, T_\varphi]D^{-1}\varphi_{xx} + \tilde{\mathcal{R}},
\end{aligned}
$$

where B is given by (4.34), and

$$
\begin{aligned}
&\tilde{\mathcal{R}} = \mathfrak{R} - 2R(A,\varphi) + (\tilde{B} - T_B)\varphi, \\
&\tilde{B} = 2T_\varphi T_{L\varphi_{xx}} + T_\varphi T_{\varphi_{xx}} - 2T_A - T_{\varphi_{xx}}T_\varphi + 3T_{\varphi_x}T_{\varphi_x}.
\end{aligned}
$$

By a Kato-Ponce type commutator estimate (see e.g., [**Li19**]), we have

$$
\|[T_{D\varphi}, T_\varphi]D^{-1}\varphi_{xx}\|_{\dot{H}^{s+1}} \le C\|\varphi\|_{W^{2,\infty}}^2\|\varphi\|_{\dot{H}^{s+1}}.
$$

In addition, using the estimates of the remainders $\mathcal{R}_i$, Sobolev embedding, and the estimate

$$\|(\tilde{B} - T_B)\varphi\|_{\dot{H}^{s+1}} \leq C(\|\varphi\|^2_{W^{3,\infty}})\|\varphi\|_{\dot{H}^s},$$

we get that

$$\|\tilde{\mathcal{R}}\|_{\dot{H}^{s+1}} \leq C\|\varphi\|^3_{\dot{H}^s}.$$

It follows that (4.1) can be written as

$$\varphi_t + \frac{1}{2}\partial_x \left\{2T^2_{\varphi_x} L\varphi + T_B\varphi + 2[T_{\varphi_x}, T_\varphi]\varphi_x + \tilde{\mathcal{R}}\right\} = 2L\varphi_x.$$

or

$$\varphi_t + \partial_x \left\{\frac{1}{2}T_B\varphi + [T_{\varphi_x}, T_\varphi]\varphi_x\right\} + \mathcal{R}_6 = [(2 - T^2_{\varphi_x})L\varphi]_x, \tag{4.36}$$

where $\|\mathcal{R}_6\|_{\dot{H}^s} \leq \mathcal{P}(\|\varphi\|_{\dot{H}^s})$ for $s > 7/2$.

Using Lemma 2.2.3 to expand the term $L(2 - T^2_{\varphi_x})\varphi$, we have the commutator estimate

$$\|[(2 - T^2_{\varphi_x}), L]\varphi\|_{H^{s+1}} \leq \mathcal{P}(\|\varphi\|_{\dot{H}^s}).$$

Hence, we can rewrite (4.36) as (4.33), with $L[(2 - T^2_{\varphi_x})\varphi]_x$ as the highest order term,

$$\partial_x \left\{\frac{1}{2}T_B\varphi + [T_{\varphi_x}, T_\varphi]\varphi_x\right\}$$

as the first order term, and $\mathcal{R}$ as the zeroth order term, which satisfies (4.35) and does not lose derivatives. $\square$

4.3.2. Energy estimate. In this subsection, we prove an *a priori* estimate for the initial value problem (4.5), which is stated in Proposition 4.3.3 below.

We first recall the following definition for fractional powers of operators. If $T : \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator on a Hilbert space $\mathcal{H}$ and $f \in C_c^\infty(\mathbb{R})$ is a function, then $f(T)$ may be defined by the Helffer-Sjöstrand formula [**Dav95**, **Hel13**] as

$$\begin{aligned} f(T) &= -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|\Im z| > \epsilon} \partial_{\bar{z}} \tilde{f}(z)(z - T)^{-1}\, \mathrm{d}\alpha\, \mathrm{d}\beta, \\ \tilde{f}(z) &= \left(f(\alpha) + i\beta f'(\alpha) + \frac{1}{2}(i\beta)^2 f''(\alpha)\right)\chi_0(\beta), \end{aligned} \tag{4.37}$$

where $z = \alpha + i\beta$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_\alpha + i\partial_\beta)$, and the cutoff-function $\chi_0 \in C_c^\infty(\mathbb{R})$ is equal to 1 in a neighborhood of 0. The function $\tilde{f}$ is an "almost analytic" extension of f since

$$\partial_{\bar{z}}\tilde{f}(z) = O(|\Im z|^2) \qquad \text{as } \Im z \to 0 \text{ with } \Re z \text{ fixed.} \tag{4.38}$$

Furthermore, if $U \subset \mathbb{R}$ is an open set that contains the spectrum $\sigma(T) \subset \mathbb{R}$ of T and $g \in C^\infty(U)$, then, by the resolution of identity form of the spectral theorem [**RS72**], we see that $g(T) = f(T)$, where $f = g\chi_1$ and $\chi_1 \in C_c^\infty(U)$ with $\chi_1 = 1$ on $\sigma(T)$.

In particular, if $\|T_{\varphi_x}^2\|_{L^2 \to L^2} < 2$, then $(2 - T_{\varphi_x}^2)$ is a positive, self-adjoint operator on L^2, and $(2 - T_{\varphi_x}^2)^s$ is well-defined for $s \in \mathbb{R}$ by (4.37) as $f(2 - T_{\varphi_x}^2)$, where

$$f(\alpha) = |\alpha|^s \chi_1(\alpha) \tag{4.39}$$

for $\chi_1 \in C_c^\infty(0, 2)$ such that $\chi_1 = 1$ on $\sigma(2 - T_{\varphi_x}^2)$. We can therefore define a weighted s-order energy by

$$E^{(s)}(t) = \int_{\mathbb{T}} |D|^s\varphi(x,t) \cdot \left(2 - T_{\varphi_x(x,t)}^2\right)^{2s+1} |D|^s\varphi(x,t)\,\mathrm{d}x. \tag{4.40}$$

In order to prove Proposition 4.3.3, we need the following lemma.

Lemma 4.3.2. *Suppose that $s > 7/2$. If φ is a smooth solution of* (4.33) *and $\tilde{\varphi} \in L^2$, then*

$$\partial_t(2 - T_{\varphi_x}^2)^s\tilde{\varphi} = (2 - T_{\varphi_x}^2)^s\tilde{\varphi}_t - s(2 - T_{\varphi_x}^2)^{s-1}(T_{\varphi_x}T_{\varphi_{xt}} + T_{\varphi_{xt}}T_{\varphi_x})\tilde{\varphi} + \mathcal{R}(\tilde{\varphi}),$$

where the remainder term satisfies

$$\|\mathcal{R}(\tilde{\varphi})\|_{\dot{H}^1} \leq \mathcal{P}\left(\|\varphi\|_{\dot{H}^s}\right)\|\tilde{\varphi}\|_{L^2} \tag{4.41}$$

for a nondecreasing polynomial $\mathcal{P}$.

Proof. For $z = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\begin{aligned}\left[\partial_t(z - 2 + T_{\varphi_x}^2)^{-1}\right](z - 2 + T_{\varphi_x}^2) + (z - 2 + T_{\varphi_x}^2)^{-1}(T_{\varphi_x}T_{\varphi_{xt}} + T_{\varphi_{xt}} + T_{\varphi_x})\\ = \partial_t\left[(z - 2 + T_{\varphi_x}^2)^{-1}(z - 2 + T_{\varphi_x}^2)\right] = \partial_t \,\mathrm{Id} = 0.\end{aligned}$$

It follows that

$$\begin{aligned}\partial_t(z-2+T_{\varphi_x}^2)^{-1} &= -(z-2+T_{\varphi_x}^2)^{-1}(T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}T_{\varphi_x})(z-2+T_{\varphi_x}^2)^{-1} \\ &= -(z-2+T_{\varphi_x}^2)^{-2}(T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}+T_{\varphi_x}) \\ &\qquad + (z-2+T_{\varphi_x}^2)^{-1}\left[(z-2+T_{\varphi_x}^2)^{-1}, T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}+T_{\varphi_x}\right] \\ &= \partial_z(z-2+T_{\varphi_x}^2)^{-1}(T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}+T_{\varphi_x}) \\ &\qquad + (z-2+T_{\varphi_x}^2)^{-1}\left[(z-2+T_{\varphi_x}^2)^{-1}, T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}+T_{\varphi_x}\right].\end{aligned}$$

Using (4.37), where f is defined by (4.39), and the previous equation, we get that

$$\begin{aligned}\partial_t(2-T_{\varphi_x}^2)^s\tilde{\varphi} &= \partial_t f(2-T_{\varphi_x}^2)\tilde{\varphi} \\ &= (2-T_{\varphi_x}^2)^s\tilde{\varphi}_t - \frac{1}{\pi}\left[\lim_{\epsilon\to 0^+}\int_{|\Im z|>\epsilon}\partial_{\bar{z}}\tilde{f}(z)\partial_t(z-2+T_{\varphi_x}^2)^{-1}\,\mathrm{d}\alpha\,\mathrm{d}\beta\right]\tilde{\varphi} \\ &= (2-T_{\varphi_x}^2)^s\tilde{\varphi}_t + T_1\tilde{\varphi} + \mathcal{R},\end{aligned}$$

where

$$\begin{aligned}T_1\tilde{\varphi} &= -\frac{1}{\pi}\left[\lim_{\epsilon\to 0^+}\int_{|\Im z|>\epsilon}\partial_{\bar{z}}\tilde{f}(z)\partial_z(z-2+T_{\varphi_x}^2)^{-1}\,\mathrm{d}\alpha\,\mathrm{d}\beta\right](T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}T_{\varphi_x})\tilde{\varphi}, \\ \mathcal{R} &= -\frac{1}{\pi}\left[\lim_{\epsilon\to 0^+}\int_{|\Im z|>\epsilon}\partial_{\bar{z}}\tilde{f}(z)(z-2+T_{\varphi_x}^2)^{-1}\left[(z-2+T_{\varphi_x}^2)^{-1}, T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}+T_{\varphi_x}\right]\mathrm{d}\alpha\,\mathrm{d}\beta\right]\tilde{\varphi}.\end{aligned}$$

Since $2-T_{\varphi_x}^2$ is self-adjoint, we have $\partial_{\bar{z}}(z-2+T_{\varphi_x}^2)^{-1}=0$ for $z\in\mathbb{C}\setminus\mathbb{R}$, so

$$\partial_z(z-2+T_{\varphi_x}^2)^{-1} = \partial_\alpha(z-2+T_{\varphi_x}^2)^{-1}.$$

We can then integrate by parts with respect to α in $T_1\tilde{\varphi}$ to get

$$\begin{aligned}T_1\tilde{\varphi} &= \frac{1}{\pi}\left[\lim_{\epsilon\to 0^+}\int_{|\Im z|>\epsilon}\partial_{\bar{z}}\widetilde{f'}(z)(z-2+T_{\varphi_x}^2)^{-1}\,\mathrm{d}\alpha\,\mathrm{d}\beta\right](T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}T_{\varphi_x})\tilde{\varphi} \\ &= -s(2-T_{\varphi_x}^2)^{s-1}(T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}T_{\varphi_x})\tilde{\varphi}.\end{aligned}$$

Finally, using a Kato-Ponce type estimate for commutators and (4.33) to estimate φ_{xt}, we have

$$\left\|(z-2+T_{\varphi_x}^2)^{-1}\left[(z-2+T_{\varphi_x}^2)^{-1}, T_{\varphi_x}T_{\varphi_{xt}}+T_{\varphi_{xt}}T_{\varphi_x}\right]\right\|_{L^2\to\dot{H}^1} \leq \mathcal{P}\left(\|\varphi\|_{\dot{H}^s}\right)|\Im z|^{-2}.$$

It follows that

$$\|\mathcal{R}\|_{\dot{H}^1} \leq \mathcal{P}\left(\|\varphi\|_{\dot{H}^s}\right)\|\tilde{\varphi}\|_{L^2}\left[\lim_{\epsilon\to 0^+}\int_{|\Im z|>\epsilon} |\partial_{\bar{z}}\tilde{f}(z)||\Im z|^{-2}\,\mathrm{d}\alpha\,\mathrm{d}\beta\right],$$

where the integral converges by (4.38). □

We now prove the following *a priori* estimate.

Proposition 4.3.3. *Suppose that $s > 7/2$ and φ is a smooth solution of (4.5) with $\varphi_0 \in \dot{H}^s$. If $\|T^2_{\varphi_{0x}}\|_{L^2\to L^2} \leq C$ for some constant $0 < C < 2$, then there exists a time $T > 0$ and a constant $M > 0$, depending on φ_0, such that*

$$\sup_{t\in[0,T]} E^{(s)}(t) \leq M,$$

where $E^{(s)}(t)$ is defined in (4.40).

Proof. We apply the operator $|D|^s$ to equation (4.33) to get

$$|D|^s\varphi_t + |D|^s\partial_x\left(\frac{1}{2}T_B\varphi + [T_{\varphi_x}, T_\varphi]\varphi_x\right) + |D|^s\mathcal{R}_7 = \partial_x L|D|^s\left[(2 - T^2_{\varphi_x})\varphi\right], \tag{4.42}$$

where $\mathcal{R}_7$ satisfies (4.35). Using Lemma 2.2.2 twice, we find that

$$\begin{aligned}|D|^s\left[(2 - T^2_{\varphi_x})\varphi\right] &= 2|D|^s\varphi - |D|^s(T^2_{\varphi_x}\varphi)\\ &= 2|D|^s\varphi - T^2_{\varphi_x}|D|^s\varphi + sT_{\varphi_x}T_{\varphi_{xx}}|D|^{s-2}\varphi_x + sT_{\varphi_{xx}}T_{\varphi_x}|D|^{s-2}\varphi_x + \mathcal{R}_8,\end{aligned}$$

where $\|\partial_x\mathcal{R}_8\|_{L^2} \leq C\|\varphi\|^2_{W^{3,\infty}}\|\varphi\|_{\dot{H}^s}$.

Thus, we write can the right-hand side of (4.42) as

$$\begin{aligned}&\partial_x L|D|^s\left[(2 - T^2_{\varphi_x})\varphi\right]\\ &\qquad= \partial_x L\left[(2 - T^2_{\varphi_x})|D|^s\varphi + sT_{\varphi_x}T_{\varphi_{xx}}|D|^{s-2}\varphi_x + sT_{\varphi_{xx}}T_{\varphi_x}|D|^{s-2}\varphi_x\right] + \mathcal{R}_0\\ &\qquad= L\big\{(2 - T^2_{\varphi_x})|D|^s\varphi_x - T_{\varphi_x}T_{\varphi_{xx}}|D|^s\varphi - T_{\varphi_{xx}}T_{\varphi_x}|D|^s\varphi\\ &\qquad\qquad - sT_{\varphi_x}T_{\varphi_{xx}}|D|^s\varphi - sT_{\varphi_{xx}}T_{\varphi_x}|D|^s\varphi\big\} + \mathcal{R}_{10}\\ &\qquad= L\left\{(2 - T^2_{\varphi_x})|D|^s\varphi_x - (s+1)(T_{\varphi_x}T_{\varphi_{xx}} + T_{\varphi_{xx}}T_{\varphi_x})|D|^s\varphi\right\} + \mathcal{R}_{10}.\end{aligned}$$

Applying $(2-T_{\varphi_x}^2)^s$ to (4.42), and commuting $(2-T_{\varphi_x}^2)^s$ with L up to a remainder term, as in the proof of Lemma 2.2.3, we obtain that

$$\begin{aligned}
&(2-T_{\varphi_x}^2)^s|D|^s\varphi_t + (2-T_{\varphi_x}^2)^s|D|^s\partial_x\left(\frac{1}{2}T_B\varphi + [T_{\varphi_x}, T_\varphi]\varphi_x\right) \\
&\qquad = L\left\{(2-T_{\varphi_x}^2)^{s+1}|D|^s\varphi_x - (s+1)(2-T_{\varphi_x})^s(T_{\varphi_x}T_{\varphi_{xx}} + T_{\varphi_{xx}}T_{\varphi_x})|D|^s\varphi\right\} + \mathcal{R}_{11} \\
&\qquad = \partial_x L\left\{(2-T_{\varphi_x}^2)^{s+1}|D|^s\varphi\right\} + \mathcal{R}_{12},
\end{aligned} \tag{4.43}$$

where $\|\mathcal{R}_{12}\|_{L^2} \le \mathcal{P}(\|\varphi\|_{\dot{H}^s})$.

By Lemma 4.3.2, with $\tilde{\varphi} = |D|^s\varphi$, the time derivative of $E^{(s)}(t)$ in (4.40) is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E^{(s)}(t) = &-\int_{\mathbb{T}}(2s+1)|D|^s\varphi\cdot(2-T_{\varphi_x}^2)^{2s}(T_{\varphi_x}T_{\varphi_{xt}} + T_{\varphi_{xt}}T_{\varphi_x})|D|^s\varphi\,\mathrm{d}x \\
&+ 2\int_{\mathbb{T}}|D|^s\varphi\cdot(2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi_t\,\mathrm{d}x + \int_{\mathbb{T}}\mathcal{R}_{13}\left(|D|^s\varphi\right)|D|^s\varphi\,\mathrm{d}x,
\end{aligned} \tag{4.44}$$

where $\mathcal{R}_{13}$ satisfies (4.41).

(1) Equation (4.33) implies that $\|\varphi_{xt}\|_{L^\infty} \le \mathcal{P}(\|\varphi\|_{\dot{H}^s})$, so the first term on the right-hand side of (4.44) can be estimated by

$$\begin{aligned}
&\left|\int_{\mathbb{T}}(2s+1)|D|^s\varphi\cdot(2-T_{\varphi_x}^2)^{2s}(T_{\varphi_x}T_{\varphi_{xt}} + T_{\varphi_{xt}}T_{\varphi_x})|D|^s\varphi\,\mathrm{d}x\right| \\
&\le C\|\varphi\|_{W^{1,\infty}}^3\|\varphi_{xt}\|_{L^\infty}\|\varphi\|_{\dot{H}^s}^2 \le \mathcal{P}(\|\varphi\|_{\dot{H}^s}).
\end{aligned}$$

In addition, from Lemma 4.3.2, the third term on the right-hand side of (4.44) can be estimated by

$$\int_{\mathbb{T}}\mathcal{R}_{13}\left(|D|^s\varphi\right)|D|^s\varphi\,\mathrm{d}x \le \mathcal{P}\left(\|\varphi\|_{\dot{H}^s}\right).$$

(2) To estimate the second term on the right-hand side (4.44), we multiply (4.43) by $(2-T_{\varphi_x}^2)^{s+1}|D|^s\varphi$, integrate the result with respect to x, and use the self-adjointness of $(2-T_{\varphi_x}^2)^{s+1}$, which gives

$$\int_{\mathbb{T}}|D|^s\varphi\cdot(2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi_t\,\mathrm{d}x = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I} &= -\int_{\mathbb{T}}|D|^s\varphi\cdot(2-T_{\varphi_x}^2)^{2s+1}|D|^s\partial_x\left(\frac{1}{2}T_B\varphi + [T_{\varphi_x}, T_\varphi]\varphi_x\right)\,\mathrm{d}x, \\
\mathrm{II} &= \int_{\mathbb{T}}(2-T_{\varphi_x}^2)^{s+1}|D|^s\varphi\cdot\partial_x L(2-T_{\varphi_x}^2)^{s+1}|D|^s\varphi\,\mathrm{d}x,
\end{aligned}$$

$$\mathrm{III} = \int_{\mathbb{T}} (2 - T_{\varphi_x}^2)^{s+1} |D|^s \varphi \cdot \mathcal{R}_{12} \, \mathrm{d}x.$$

We have $\mathrm{II} = 0$, since $\partial_x L$ is skew-symmetric, and

$$\mathrm{III} \leq \mathcal{P}(\|\varphi\|_{\dot{H}^s}),$$

since $\|\mathcal{R}_{12}\|_{L^2} \leq \mathcal{P}(\|\varphi\|_{\dot{H}^s})$ and $(2 - T_{\varphi_x}^2)^{s+1}$ is bounded on L^2.

Term I estimate. We write $\mathrm{I} = -\mathrm{I}_a + \mathrm{I}_b$, where

$$\begin{aligned}
\mathrm{I}_a &= \int_{\mathbb{T}} |D|^s \varphi \cdot (2 - T_{\varphi_x}^2)^{2s+1} \partial_x \left(\frac{1}{2} T_B |D|^s \varphi + [T_{\varphi_x}, T_\varphi] |D|^s \varphi_x \right) \mathrm{d}x, \\
\mathrm{I}_b &= \int_{\mathbb{T}} |D|^s \varphi \cdot (2 - T_{\varphi_x}^2)^{2s+1} \partial_x \left(\frac{1}{2} [T_B, |D|^s] \varphi + [[T_{\varphi_x}, T_\varphi], |D|^s] \, \varphi_x \right) \mathrm{d}x.
\end{aligned}$$

By a commutator estimate, the second integral satisfies $|\mathrm{I}_b| \leq \mathcal{P}(\|\varphi\|_{\dot{H}^s})$.

To estimate the first integral, we write it as

$$\mathrm{I}_a = \mathrm{I}_{a_1} - \frac{1}{2} \mathrm{I}_{a_2} - \mathrm{I}_{a_3},$$

where

$$\begin{aligned}
\mathrm{I}_{a_1} &= \int_{\mathbb{T}} |D|^s \varphi \cdot [(2 - T_{\varphi_x}^2)^{2s+1}, \partial_x] \left(\frac{1}{2} T_B |D|^s \varphi + [T_{\varphi_x}, T_\varphi] |D|^s \varphi_x \right) \mathrm{d}x, \\
\mathrm{I}_{a_2} &= \int_{\mathbb{T}} |D|^s \varphi_x \cdot (2 - T_{\varphi_x}^2)^{2s+1} \left(T_B |D|^s \varphi \right) \mathrm{d}x, \\
\mathrm{I}_{a_3} &= \int_{\mathbb{T}} |D|^s \varphi_x \cdot (2 - T_{\varphi_x}^2)^{2s+1} \left([T_{\varphi_x}, T_\varphi] |D|^s \varphi_x \right) \mathrm{d}x.
\end{aligned}$$

Term I_{a_1} estimate. A Kato-Ponce commutator estimate gives

$$|\mathrm{I}_{a_1}| \leq \mathcal{P} \left(\|\varphi\|_{\dot{H}^s} \right).$$

Term I_{a_2} estimate. We have

$$\begin{aligned}
\mathrm{I}_{a_2} &= \int_{\mathbb{T}} (T_B|D|^s\varphi)\cdot(2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi_x\,\mathrm{d}x \\
&= \int_{\mathbb{T}} (T_B|D|^s\varphi)\cdot\left\{\partial_x\left((2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi\right) - \left[\partial_x,(2-T_{\varphi_x}^2)^{2s+1}\right]|D|^s\varphi\right\}\,\mathrm{d}x \\
&= -\int_{\mathbb{T}} \partial_x\left(T_B|D|^s\varphi\right)\cdot(2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi\,\mathrm{d}x - \int_{\mathbb{T}} (T_B|D|^s\varphi)\cdot\left[\partial_x,(2-T_{\varphi_x}^2)^{2s+1}\right]|D|^s\varphi\,\mathrm{d}x \\
&= -\int_{\mathbb{T}} \left(T_B|D|^s\varphi_x + \left[\partial_x,T_B\right]|D|^s\varphi\right)\cdot(2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi\,\mathrm{d}x \\
&\quad\; -\int_{\mathbb{T}} T_B\left(|D|^s\varphi\right)\cdot\left[\partial_x,(2-T_{\varphi_x}^2)^{2s+1}\right]|D|^s\varphi\,\mathrm{d}x.
\end{aligned} \tag{4.45}$$

Using the commutator estimates

$$\left\|\left[\partial_x,(2-T_{\varphi_x}^2)^{2s+1}\right]|D|^s\varphi\right\|_{L^2} \le \mathcal{P}(\|\varphi\|_{\dot{H}^s}), \qquad \left\|\left[\partial_x,T_B\right]|D|^s\varphi\right\|_{L^2} \le \mathcal{P}(\|\varphi\|_{\dot{H}^s}),$$

and the fact that T_B is self-adjoint, we can rewrite (4.45) as

$$\mathrm{I}_{a_2} = -\mathrm{I}_{a_2} - \int_{\mathbb{T}} |D|^s\varphi\cdot\partial_x\left[(2-T_{\varphi_x}^2)^{2s+1},T_B\right]|D|^s\varphi\,\mathrm{d}x + \mathcal{R}_{14},$$

with $|\mathcal{R}_{14}| \le \mathcal{P}(\|\varphi\|_{H^s})$. Using the commutator estimate

$$\left\|\partial_x\left[(2-T_{\varphi_x}^2)^{2s+1},T_B\right]|D|^s\varphi\,\mathrm{d}x\right\|_{L^2} \le \mathcal{P}(\|\varphi\|_{\dot{H}^s}),$$

we conclude that $|\mathrm{I}_{a_2}| \le \mathcal{P}(\|\varphi\|_{\dot{H}^s})$.

Term I_{a_3} estimate. Using the self-adjointness of T_{φ_x} and T_φ, we obtain that

$$\begin{aligned}
\mathrm{I}_{a_3} &= \int_{\mathbb{T}} (2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi_x\cdot[T_{\varphi_x},T_\varphi]|D|^s\varphi_x\,\mathrm{d}x \\
&= -\int_{\mathbb{T}} [T_{\varphi_x},T_\varphi](2-T_{\varphi_x}^2)^{2s+1}|D|^s\varphi_x\cdot|D|^s\varphi_x\,\mathrm{d}x.
\end{aligned}$$

Since

$$\left\|\left[[T_{\varphi_x},T_\varphi],(2-T_{\varphi_x}^2)^{2s+1}\right]|D|^s\varphi_x\right\|_{L^2} \le \mathcal{P}(\|\varphi\|_{\dot{H}^s}),$$

we have that $|\mathrm{I}_{a_3}| \le \mathcal{P}(\|\varphi\|_{\dot{H}^s})$.

Collecting the above estimates, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{(s)} \le \mathcal{P}\left(\|\varphi\|_{\dot{H}^s}\right).$$

Finally, since $\|2 - T^2_{\varphi_{0x}}\|_{L^2 \to L^2} \geq 2 - C$ and $\|\varphi_x(t)\|_{L^\infty}$ is continuous in time, there exists $T > 0$ and $m > 0$, depending only on the initial data, such that

$$\|2 - T^2_{\varphi_x}\|_{L^2 \to L^2} \geq m \qquad \text{for } t \leq T.$$

We therefore obtain that

$$m^{2s+1}\||D|^s\varphi\|^2_{L^2} \leq E^{(s)} \leq 2^{2s+1}\||D|^s\varphi\|^2_{L^2},$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{(s)} \leq \mathcal{P}(E^{(s)}).$$

The result then follows by Grönwall's inequality. □

4.3.3. Local well-posedness. In this subsection, we finish the proof of Theorem 4.0.2. We first construct solutions of (4.5) by a Galerkin method. For $N \in \mathbb{N}$, let

$$J_N \colon L^2(\mathbb{T}) \to L^2(\mathbb{T}), \qquad J_N f(x) = \sum_{|\xi| \leq N} \hat{f}(\xi) e^{i\xi x} \tag{4.46}$$

denote the projection onto the first N Fourier modes. We define an approximate solution $\varphi^N(x,t)$ as the solution of the ODEs obtained by projection of (4.3.1),

$$\varphi^N_t + \partial_x J_N \left\{ \frac{1}{2} T_{B(\varphi^N)}\varphi^N + [T_{\varphi^N_x}, T_{\varphi^N}]\varphi^N_x \right\} + J_N \mathcal{R}(\varphi^N) = J_N L[(2 - T^2_{\varphi^N_x})\varphi^N]_x, \tag{4.47}$$

with initial data $\varphi^N(x,0) = J_N \varphi_0(x)$.

Repeating the previous estimates, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{(s)}(\varphi^N) \leq \mathcal{P}\left(E^{(s)}(\varphi^N)\right).$$

Thus, since $E^{(s)}(J_N\varphi_0) \lesssim \|\varphi_0\|^2_{\dot{H}^s}$, there exists $T > 0$ independent of N such that the solution of (4.47) exists for $t \in [0, T]$ and

$$\|\varphi^N(t)\|_{\dot{H}^s} \leq \mathcal{P}(\|\varphi_0\|_{\dot{H}^s}),$$

where $\mathcal{P}$ is an nondecreasing polynomial independent of N. The sequence of approximate solutions $\{\varphi^N\}$ is therefore bounded in $L^\infty(0, T; \dot{H}^s)$, so a subsequence converges weak-$*$ to a limit $\varphi \in L^\infty(0, T; \dot{H}^s)$.

Moreover, from (4.47), we see that $\{\varphi_t^N\}$ is bounded in $L^\infty(0,T;\dot{H}^{s-1-\delta})$ for $\delta > 0$. The Aubin-Lions lemma implies that a further subsequence converges strongly to φ in $C([0,T];\dot{H}^r)$ for any $r < s$. Taking the limit of (4.47) as $N \to \infty$, we find that φ is a solution of (4.3.1).

Since $\varphi \in L^\infty(0,T;\dot{H}^s) \cap C([0,T];\dot{H}^r)$, we see that $\varphi \in C_w([0,T];\dot{H}^s)$ is weakly continuous in $\dot{H}^s$. In addition, the Arzelà-Ascoli theorem implies that $E^{(s)}(\varphi)$ is continuous in time, since $E^{(s)}(\varphi^N)$ is continuous for each $N \in \mathbb{N}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{(s)}(\varphi^N)$$

is bounded uniformly in N. It follows that $\|\varphi\|_{\dot{H}^s}$ is continuous, so, by weak continuity and norm continuity, $\varphi \in C([0,T];\dot{H}^s)$ is strongly continuous in $\dot{H}^s$.

To prove the Lipschitz continuity (4.6) and uniqueness, we suppose that $\varphi, \tilde{\varphi} \in C([0,T];\dot{H}^s)$ are solutions of (4.5) with $s > 7/2$. Subtracting the evolution equations for φ and $\tilde{\varphi}$, we find that $u = \varphi - \tilde{\varphi}$ satisfies

$$\begin{aligned} \partial_t u + \partial_x \left\{ \frac{1}{2} T_{B(\varphi)} u + [T_{\varphi_x}, T_\varphi] u_x \right\} + \partial_x \left\{ \frac{1}{2} [T_{B(\varphi)} - T_{B(\tilde{\varphi})}] \tilde{\varphi} + \Big[[T_{\varphi_x}, T_\varphi] - [T_{\tilde{\varphi}_x}, T_{\tilde{\varphi}}] \Big] \tilde{\varphi}_x \right\} \\ = L[(2 - T_{\varphi_x}^2) u]_x - (L T_{\varphi_x}^2 - L T_{\tilde{\varphi}_x}^2) \tilde{\varphi}_x + \mathcal{R}(\varphi) - \mathcal{R}(\tilde{\varphi}). \end{aligned} \tag{4.48}$$

For $r \geq 0$, we define a weighted $\dot{H}^r$-norm by

$$E_\varphi^{(r)}(u(t)) = \int_{\mathbb{T}} |D|^r u(x,t) \cdot \left(2 - T_{\varphi_x(x,t)}^2 \right)^{2r+1} |D|^r u(x,t) \,\mathrm{d}x.$$

Applying ∂_x^r to (4.48), with $0 \leq r < s-1$, and carrying out energy estimates as before, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\varphi^{(r)}(u) \leq \mathcal{P}(\|\varphi\|_{H^s}, \|\tilde{\varphi}\|_{H^s}) \big[E_\varphi^{(r)}(u) + \|L\partial_x^{r+1}\tilde{\varphi}\|_{L^\infty} \|u\|_{\dot{H}^r}^2 \big],$$

where we have used the estimates

$$\begin{aligned} \|\partial_x^r (L T_{\varphi_x}^2 - L T_{\tilde{\varphi}_x}^2) \tilde{\varphi}_x\|_{L^2} &\lesssim \begin{cases} \|u_x\|_{L^\infty} (\|\varphi_x\|_{L^\infty} + \|\tilde{\varphi}_x\|_{L^\infty}) \|L\partial_x^{r+1}\tilde{\varphi}\|_{L^2}, & \text{when } \frac{3}{2} < r < s-1, \\ \|u_x\|_{L^2} (\|\varphi_x\|_{L^\infty} + \|\tilde{\varphi}_x\|_{L^\infty}) \|L\partial_x^{r+1}\tilde{\varphi}\|_{L^\infty}, & \text{when } 1 \leq r \leq \frac{3}{2}, \\ \|u\|_{\dot{H}^r} (\|\varphi_x\|_{L^\infty} + \|\tilde{\varphi}_x\|_{L^\infty}) \|L\partial_x^2\tilde{\varphi}\|_{L^\infty}, & \text{when } 0 \leq r < 1, \end{cases} \\ &\lesssim \|u\|_{H^r} \left(\|\varphi_x\|_{L^\infty} + \|\tilde{\varphi}_x\|_{L^\infty} \right) \|\tilde{\varphi}\|_{H^s}, \\ \|\partial_x^r [\mathcal{R}(\varphi) - \mathcal{R}(\tilde{\varphi})]\|_{L^2} &\lesssim \mathcal{P}(\|\varphi\|_{H^s}, \|\tilde{\varphi}\|_{H^s}) \|u\|_{\dot{H}^r}. \end{aligned}$$

It follows that

$$E^{(0)}_{\varphi}(u(t)) + E^{(r)}_{\varphi}(u(t)) \lesssim \left[E^{(0)}_{\varphi}(u(0)) + E^{(r)}_{\varphi}(u(0))\right] \int_0^t \mathcal{P}(\|\varphi\|_{H^s}, \|\tilde{\varphi}\|_{H^s})\,\mathrm{d}t,$$

and, since $E^{(0)}_{\varphi}(u) + E^{(r)}_{\varphi}(u)$ is equivalent to $\|u\|^2_{\dot{H}^r}$, the solution map is Lipschitz continuous on $\dot{H}^r$. In particular, the solution is unique.

Finally, we prove that the solution map is continuous on $\dot{H}^s$ by a Bona-Smith argument [**BS75**]. First, suppose that $\varphi \in C([0,T];\dot{H}^s)$, $\tilde{\varphi} \in C([0,T];\dot{H}^{s+1+\delta})$ are solutions, where $0 < \delta \ll 1$, and let $u = \varphi - \tilde{\varphi}$. In a similar way to before, we find that $E^{(s)}_{\varphi}(u)$ satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E^{(s)}_{\varphi}(u) &\leq \mathcal{P}(\|\varphi\|_{\dot{H}^s}, \|\tilde{\varphi}\|_{\dot{H}^s}) E^{(s)}_{\varphi}(u) \\ &\qquad + \|2 - T^2_{\varphi_x}\|^{2s+1}_{L^\infty} \left[\|\partial_x^s(LT^2_{\varphi_x} - LT^2_{\tilde{\varphi}_x})\tilde{\varphi}_x\|_{L^2} + \|\partial_x^s[\mathcal{R}_7(\varphi) - \mathcal{R}_7(\tilde{\varphi})]\|_{L^2}\right] \|u\|_{\dot{H}^s}. \end{aligned}$$

Using the estimates

$$\begin{aligned} \|\partial_x^s(LT^2_{\varphi_x} - LT^2_{\tilde{\varphi}_x})\tilde{\varphi}_x\|_{L^2} &\lesssim \|L\tilde{\varphi}\|_{\dot{H}^{s+1}} \|u\|_{\dot{H}^2} (\|\varphi\|_{\dot{H}^s} + \|\tilde{\varphi}\|_{\dot{H}^s}), \\ \|\partial_x^s[\mathcal{R}(\varphi) - \mathcal{R}(\tilde{\varphi})]\|_{L^2} &\lesssim \mathcal{P}(\|\varphi\|_{\dot{H}^s}, \|\tilde{\varphi}\|_{\dot{H}^s}) \|u\|_{\dot{H}^s}, \end{aligned}$$

we get that

$$E^{(s)}_{\varphi}(u(t)) \lesssim \mathcal{P}(\|\varphi\|_{L^\infty_t \dot{H}^s}, \|\tilde{\varphi}\|_{L^\infty_t \dot{H}^s}) \left[E^{(s)}_{\varphi}(u(0)) + \|u\|_{L^\infty_t \dot{H}^s} \|u\|_{L^\infty_t \dot{H}^2} \|L\tilde{\varphi}\|_{L^\infty_t \dot{H}^{s+1}}\right]. \tag{4.49}$$

The higher-order derivative term $\|L\tilde{\varphi}\|_{L^\infty_t \dot{H}^{s+1}}$, which obstructs Lipschitz continuity on $\dot{H}^s$, is compensated by the lower-order derivative factor $\|u\|_{L^\infty_t \dot{H}^2}$, and we treat it by approximating $\dot{H}^s$-solutions by smooth solutions.

Given $f \in L^2$ and $N \in \mathbb{N}$, let $f_N = J_N f$ where the projection J_N is defined in (4.46). If $f \in \dot{H}^s$, with $s \geq 2$, then $f_N \to f$ in $\dot{H}^s$ as $N \to \infty$, and

$$\|f_N - f\|_{\dot{H}^2} \lesssim \frac{1}{N^{s-2}} \|f\|_{\dot{H}^s}, \qquad \|f_N\|_{\dot{H}^{s+1+\delta}} \lesssim N^{1+\delta} \|f\|_{\dot{H}^s}. \tag{4.50}$$

Consider initial data $\varphi^n_0, \varphi_0 \in \dot{H}^s$ such that $\varphi^n_0 \to \varphi_0$ in $\dot{H}^s$ as $n \to \infty$, and let $\varphi^n, \varphi \in C([0,T];\dot{H}^s)$ denote the corresponding solutions. We approximate the initial data by $\varphi^n_{0,N}$, $\varphi_{0,N}$ and let φ^n_N, φ_N denote the corresponding solutions. Then

$$\|\varphi^n - \varphi\|_{\dot{H}^s} \leq \|\varphi^n - \varphi^n_N\|_{\dot{H}^s} + \|\varphi^n_N - \varphi_N\|_{\dot{H}^s} + \|\varphi_N - \varphi\|_{\dot{H}^s}. \tag{4.51}$$

Using (4.49) and the fact that $\|Lf\|_{L^2} \lesssim \|f\|_{\dot{H}^\delta}$, we get that

$$\|\varphi - \varphi_N\|^2_{\dot{H}^s} \lesssim \mathcal{P}(\|\varphi\|_{L^\infty_t \dot{H}^s}, \|\varphi_N\|_{L^\infty_t \dot{H}^s}) \cdot \left[\|\varphi_0 - \varphi_{0,N}\|^2_{\dot{H}^s} + \|\varphi - \varphi_N\|_{L^\infty_t \dot{H}^s}\|\varphi - \varphi_N\|_{L^\infty_t \dot{H}^2}\|\varphi_N\|_{L^\infty_t \dot{H}^{s+1+\delta}}\right],$$

with a similar estimate for $\|\varphi^n - \varphi^n_N\|^2_{\dot{H}^s}$. The Lipschitz continuity (4.6), with $r = 2$, and the approximation estimates (4.50) give

$$\|\varphi - \varphi_N\|_{L^\infty_t \dot{H}^2}\|\varphi_N\|_{L^\infty_t \dot{H}^{s+1+\delta}} \lesssim \|\varphi_0 - \varphi_{0,N}\|_{\dot{H}^2}\|\varphi_{0,N}\|_{\dot{H}^{s+1+\delta}} \lesssim \frac{1}{N^{s-3-\delta}}\|\varphi_0\|^2_{\dot{H}^s}.$$

Hence, since $s > 7/2$, we have for each $n \in \mathbb{N}$ that

$$\|\varphi - \varphi_N\|_{L^\infty_t \dot{H}^s} + \|\varphi^n - \varphi^n_N\|_{L^\infty_t \dot{H}^s} \to 0 \qquad \text{as } N \to \infty. \tag{4.52}$$

In addition, using (4.49), we get that

$$\|\varphi^n_N - \varphi_N\|^2_{\dot{H}^s} \lesssim \mathcal{P}(\|\varphi^n_N\|_{L^\infty_t \dot{H}^s}, \|\varphi_N\|_{L^\infty_t \dot{H}^s}) \cdot \left[\|\varphi^n_{0,N} - \varphi_{0,N}\|^2_{\dot{H}^s} + \|\varphi^n_N - \varphi_N\|_{L^\infty_t \dot{H}^s}\|\varphi^n_N - \varphi_N\|_{L^\infty_t \dot{H}^2}\|\varphi_N\|_{L^\infty_t \dot{H}^{s+1+\delta}}\right].$$

Since $\varphi^n_{0,N} \to \varphi_{0,N}$ as $n \to \infty$, equation (4.6) then implies that for each $N \in \mathbb{N}$, we have

$$\|\varphi^n_N - \varphi_N\|_{L^\infty_t \dot{H}^s} \to 0 \qquad \text{as } n \to \infty. \tag{4.53}$$

It follows from (4.51)–(4.53) that $\|\varphi^n - \varphi\|_{L^\infty_t \dot{H}^s} \to 0$ as $n \to \infty$, which proves that the solution map U is continuous on $\dot{H}^s$.

4.4. Numerical solutions

In this section, we show two numerical solutions of the initial value problem for the approximate SQG front equation in (4.5) that indicate the formation of singularities in finite time.

The first solution is for the initial data

$$\varphi_0(x) = \cos(x + \pi) + \frac{1}{2}\cos[2(x + \pi + 2\pi^2)]. \tag{4.54}$$

A surface plot of the solution, computed using a pseudo-spectral method with spectral viscosity, is shown in Figure 4.1. Numerical results suggest that an oscillatory singularity forms at $t \approx 0.06$ near $x \approx 2.15$, before there is an appreciable change in the global shape of the solution. The solution appears to be smooth before the singularity forms, and the numerical singularity formation time does not appear to change under further refinement. Moreover the structure of the solution remains similar as one increases the number of Fourier modes, although the number of oscillations and the x-location of their left endpoint increases.

One might conjecture that the formation of singularities in the approximate SQG front equation is associated with the breaking and filamentation of the front, rather than a loss of smoothness, but since we are using a graphical description of the front, we are unable to distinguish between the two. The numerical solutions suggest that it may be possible to continue smooth solutions of (4.5) by some type of weak solution after singularities form. These weak solutions appear to remain continuous, which could be associated with the extreme thinness of any filaments that form, as seems to occur in the case of the filamentation of vorticity fronts [**BH**, **BH10**].

In Figure 4.4–4.6, we show a solution of (4.5) with the initial data

$$\varphi_0(x) = \operatorname{sech}^2\left[\frac{5(x-\pi)}{2}\right]. \tag{4.55}$$

for $0 \le t \le 0.05$. The singularity formation time is $t \approx 0.02$. As in the previous case, a singularity forms before there is an appreciable change in the global shape of the solution, but in this case singularities form at two different locations, the first near the peak of the pulse and then, a little later, a second near the front of the pulse.

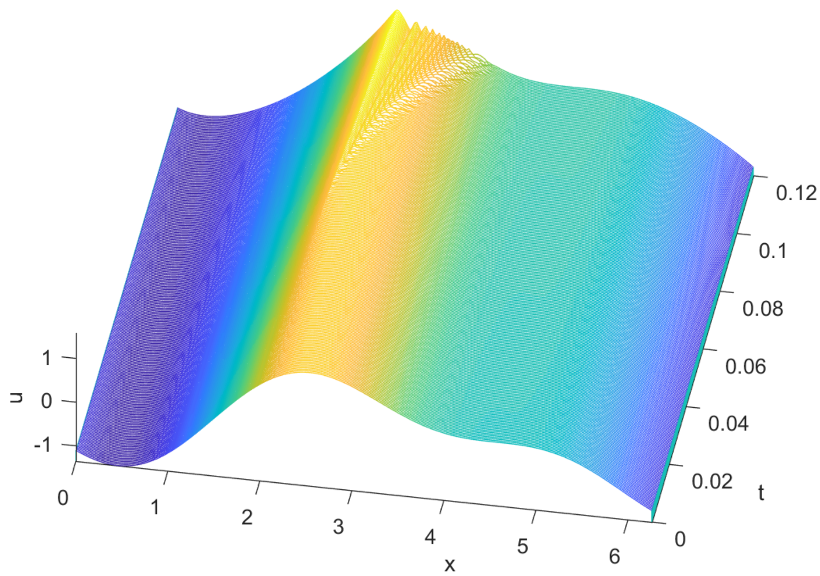


FIGURE 4.1. A surface plot of the solution of (4.5) with initial data (4.54) for $0 \leq t \leq 0.12$. The solution is computed by a pseudo-spectral method with 2^{14} Fourier modes. A small oscillatory singularity forms at $t \approx 0.06$ near $x \approx 2.15$.

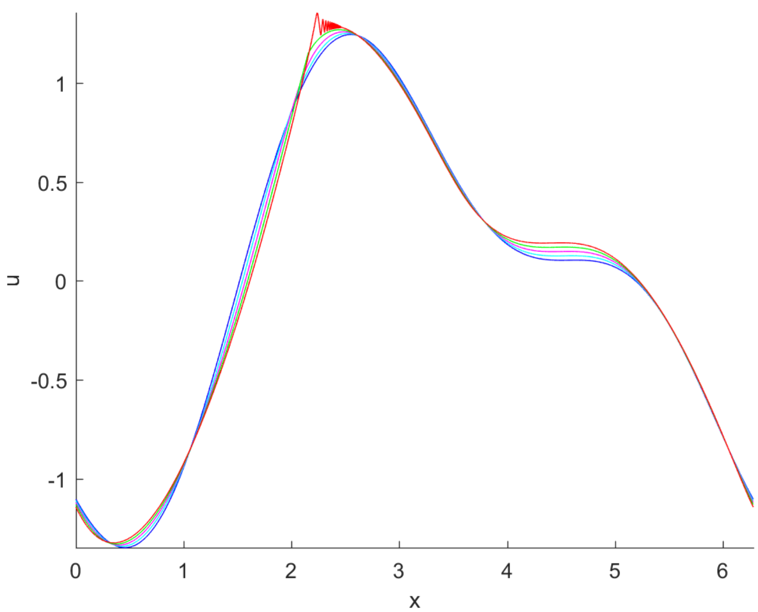


FIGURE 4.2. Graphs of the solution of (4.5) with initial data (4.54). The solution is shown at $t = 0$ (blue), $t = 0.01875$ (cyan), $t = 0.0375$ (magenta), $t = 0.05625$ (green), $t = 0.075$ (red). The solution is computed by a pseudo-spectral method with 2^{15} Fourier modes.

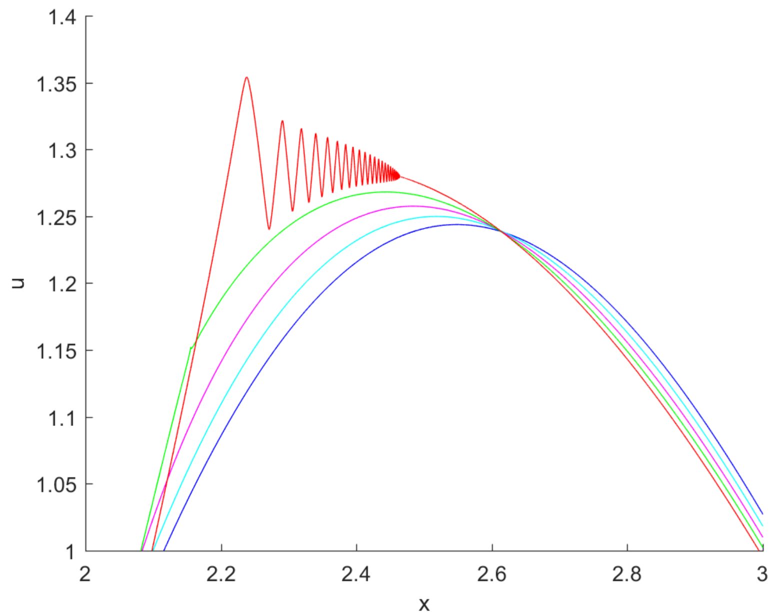


FIGURE 4.3. Detail of singularity formation in the solution of (4.5) with initial data (4.54) shown in Figure 4.2. The solution is shown at $t = 0$ (blue), $t = 0.01875$ (cyan), $t = 0.0375$ (magenta), $t = 0.05625$ (green), $t = 0.075$ (red).

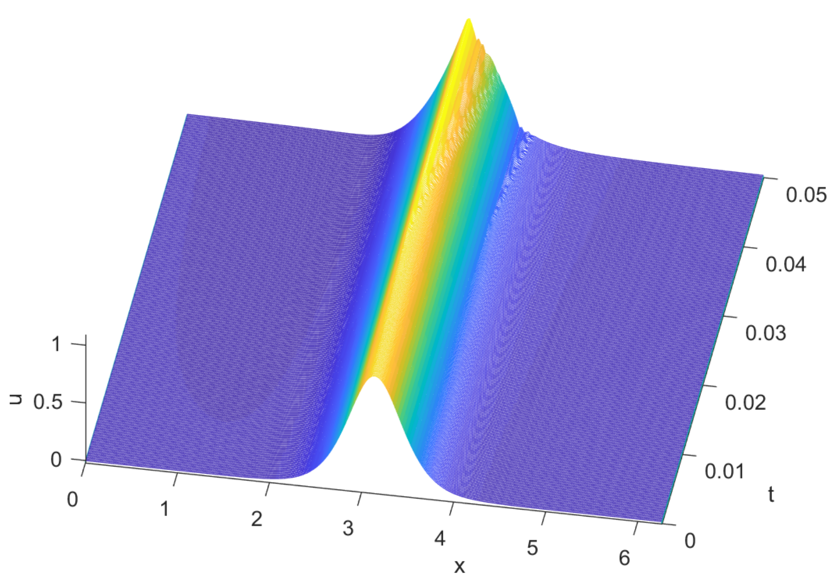


FIGURE 4.4. A surface plot of the solution of (4.5) with initial data (4.55) for $0 \leq t \leq 0.05$. The solution is computed by a pseudo-spectral method with 2^{15} Fourier modes.

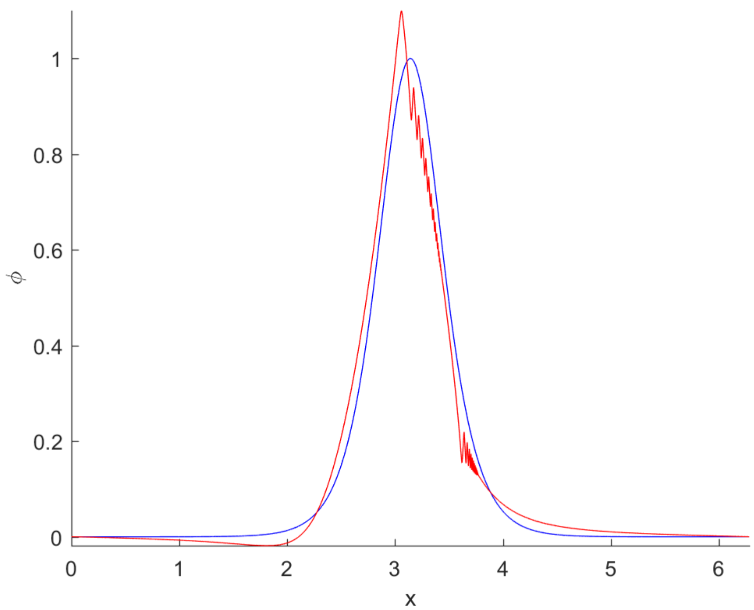


FIGURE 4.5. Graphs of the solution of (4.5) with initial data (4.55) for $t = 0$ (blue) and $t = 0.05$ (red). The solution is computed by a pseudo-spectral method with 2^{15} Fourier modes.

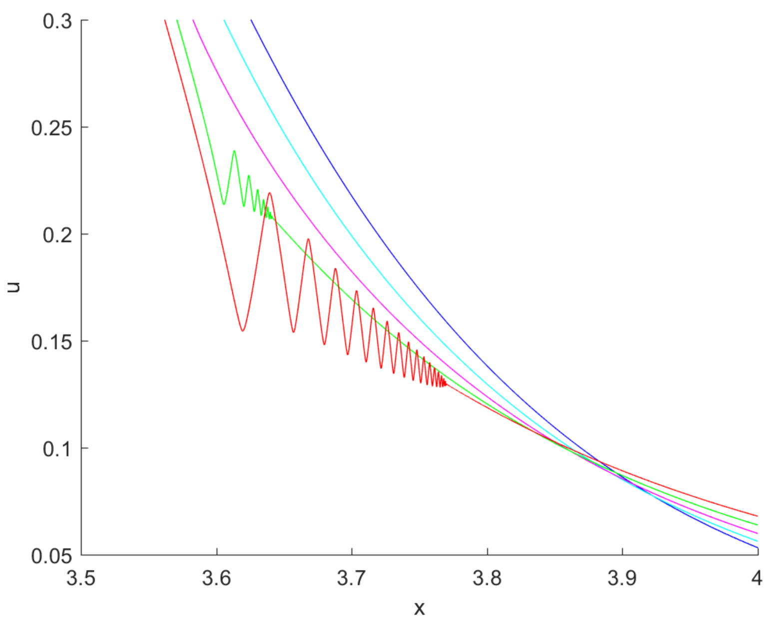


FIGURE 4.6. Detail of the singularity formation near the front of the pulse in the solution of (4.5) with initial data (4.55) shown in Figure 4.2. The solution is shown at $t = 0$ (blue), $t = 0.0125$ (cyan), $t = 0.025$ (magenta), $t = 0.0375$ (green), $t = 0.05$ (red).

CHAPTER 5

Global solutions to the SQG fronts

Nothing cannot exist forever.

– *Stephen Hawking*

This chapter concerns with the initial value problem for the full SQG front equation (3.2)

$$\begin{aligned} &\varphi_t(x,t) - 2\log|\partial_x|\varphi_x(x,t) \\ &\quad + \int_{\mathbb{R}} [\varphi_x(x,t) - \varphi_x(x+\zeta,t)] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + [\varphi(x,t) - \varphi(x+\zeta,t)]^2}} \right\} \mathrm{d}\zeta = 0, \\ &\varphi(x,0) = \varphi_0(x), \end{aligned} \tag{5.1}$$

where $\varphi\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is defined for $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, and $\log|\partial_x| = L$ is the Fourier multiplier operator with symbol $\log|\xi|$.

The main result of this chapter is the asymptotical stability of the planar steady SQG front $\varphi \equiv 0$, and is stated in the theorem below.

Theorem 5.0.1. *Let*

$$s = 1200, \qquad r = 7, \qquad p_0 = 10^{-4}. \tag{5.2}$$

There exists a constant $0 < \varepsilon \ll 1$, *such that if* $\varphi_0 \in H^s(\mathbb{R})$ *satisfies*

$$\|\varphi_0\|_{H^s} + \|x\partial_x\varphi_0\|_{H^r} \le \varepsilon_0$$

for some $0 < \varepsilon_0 \le \varepsilon$, *then there exists a unique global solution* $\varphi \in C([0,\infty); H^s(\mathbb{R}))$ *of* (5.1). *Moreover, this solution satisfies*

$$\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r} \lesssim \varepsilon_0 (t+1)^{p_0},$$

where $\mathcal{S}$ *is a vector field defined as*

$$\mathcal{S} = (x+2t)\partial_x + t\partial_t. \tag{5.3}$$

We remark that the operator $\mathcal{S}$ generates a scaling-Galilean transformation, and it follows from Lemma 5.5.1 that $\mathcal{S}$ commutes with the linearized equation.

The general strategy for proving the global existence of small solutions of dispersive equations is to prove an energy estimate together with a dispersive decay estimate. Energy estimates for (5.1) in the usual H^s-Sobolev spaces lead to a logarithmic loss of derivatives (see Lemma B.0.3). However, as shown in Section 4.3 for spatially periodic solutions of the cubic approximation, we can obtain good energy estimates in suitably weighted H^s-spaces by para-linearizing the equation and using the linear dispersive term to control the logarithmic loss of derivatives from the nonlinear term.

The proof of the dispersive estimates is more delicate. The linear part of the equation provides $t^{-1/2}$ decay for the L^∞-norm of the solution, but this is not sufficient to close the global energy estimates for the full equation, since the $O(t^{-1})$ contribution from the cubically nonlinear term is not integrable in time. We therefore need to analyze the nonlinear dispersive behavior in more detail. We do this by the method of space-time resonances introduced by Germain, Masmoudi and Shatah [**Ger10**, **GMS09**, **GMS12**], together with estimates for weighted L^∞_ξ-norms — the so-called Z-norms — developed by Ionescu and his collaborators [**CGSI19**, **DIP17**, **DIPP17**, **IP13**, **IP15**, **IP16**, **IP18**].

Our Z-norm estimates involve a detailed frequency-space analysis. The most difficult part is the estimate of the cubically nonlinear terms. In most regions of frequency space, these terms are nonresonant, and we can use integration-by-parts in either the spatial or temporal frequency variables to estimate the corresponding oscillatory integrals. In regions of space-time resonances, we use the method of modified scattering to account for the nonlinear, long-time asymptotics of the solutions [**IP14**, **Oza91**].

Our main reference is [**CGSI19**]. In that paper, the authors prove global well-posedness of the initial-value problem for the GSQG (1.2) front equation with $0 < \alpha < 1$. The linearized equation $\varphi_t = \partial_x|\partial_x|^{1-\alpha}\varphi$, with dispersion relation $\Lambda(\xi) = \xi|\xi|^{1-\alpha}$, has a scaling invariance and commutes with the vector field $x\partial_x+(2-\alpha)t\partial_t$, which provides a key ingredient in the dispersive estimates. The SQG equation considered here corresponds to the limiting case $\alpha = 1$, and its linearized dispersion relation is $\Lambda(\xi) = 2\xi \log|\xi|$. The linearized equation $\varphi_t = 2\log|\partial_x|\varphi_x$ is not scale-invariant, but it has a combined scaling-Galilean invariance and commutes with the scaling-Galilean vector field $\mathcal{S} = (x + 2t)\partial_x + t\partial_t$ defined in (5.3).

This chapter is organized as follows. In Section 5.1, we expand and para-linearize the nonlinear terms in the evolution equation. In Section 5.2, we derive the weighted energy estimates and state a local existence and uniqueness result in Theorem 5.2.3. In Section 5.3, we show that Theorem 5.0.1 is a consequence of a bootstrap argument (Proposition 5.3.1) . Finally, in Sections 5.4–5.6 we carry out the three key steps in the closing the bootstrap argument: linear dispersive estimates; scaling-Galilean estimates; and nonlinear dispersive estimates.

5.1. Reformulation of the equation

5.1.1. Expansion of the equation. In this subsection, we expand the nonlinearity in the SQG front equation (3.2) for fronts with small slopes $|\varphi_x| \ll 1$. As we will show, (3.2) can be rewritten as

$$\begin{aligned}\varphi_t(x,t) - \sum_{n=1}^{\infty} \frac{c_n}{2n+1}\partial_x \int_{\mathbb{R}^{2n+1}} \mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\cdots\hat{\varphi}(\eta_{2n+1},t)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n \\ = 2\log|\partial_x|\varphi_x(x,t),\end{aligned} \tag{5.4}$$

where $\boldsymbol{\eta}_n = (\eta_1, \eta_2, \ldots, \eta_{2n+1})$, and

$$\mathbf{T}_n(\boldsymbol{\eta}_n) = \int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta, \qquad c_n = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(n+1)}. \tag{5.5}$$

We remark that $c_n = O(n^{-1/2})$ as $n \to \infty$.

In fact, if we expand the nonlinearity in (3.2) around $\varphi_x(x,t) = 0$, we find that

$$\begin{aligned}&\int_{\mathbb{R}} \left[\frac{\varphi_x(x,t)-\varphi_x(x+\zeta,t)}{|\zeta|} - \frac{\varphi_x(x,t)-\varphi_x(x+\zeta,t)}{\sqrt{\zeta^2+(\varphi(x,t)-\varphi(x+\zeta,t))^2}}\right]\mathrm{d}\zeta \\ =& -\sum_{n=1}^{\infty} c_n \int_{\mathbb{R}} \frac{[\varphi_x(x,t)-\varphi_x(x+\zeta,t)]\cdot[\varphi(x,t)-\varphi(x+\zeta,t)]^{2n}}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta \\ =& -\sum_{n=1}^{\infty} \frac{c_n}{2n+1}\partial_x \int_{\mathbb{R}} \left[\frac{\varphi(x,t)-\varphi(x+\zeta,t)}{|\zeta|}\right]^{2n+1}\mathrm{d}\zeta.\end{aligned} \tag{5.6}$$

Writing

$$f_n(x) = \int_{\mathbb{R}} \left[\frac{\varphi(x)-\varphi(x+\zeta)}{|\zeta|}\right]^{2n+1}\mathrm{d}\zeta, \qquad \varphi(x) = \int_{-\infty}^{\infty} \hat{\varphi}(\eta)e^{i\eta x}\,\mathrm{d}\eta,$$

we have

$$f_n(x) = \int_{\mathbb{R}^{2n+1}} \mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n+1})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n, \tag{5.7}$$

which gives (5.4).

Isolating the lowest degree nonlinear term in (5.4), which is cubic, we can also write (3.2) as

$$\begin{aligned}\varphi_t(x,t) + \frac{1}{6}\partial_x \int_{\mathbb{R}^3} \mathbf{T}_1(\eta_1,\eta_2,\eta_3)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\hat{\varphi}(\eta_3,t)e^{i(\eta_1+\eta_2+\eta_3)x}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3 \\ + \mathcal{N}_{\geq 5}(\varphi)(x,t) = 2\log|\partial_x|\varphi_x(x,t),\end{aligned} \tag{5.8}$$

where $\mathcal{N}_{\geq 5}(\varphi)$ denotes the nonlinear terms of quintic degree or higher

$$\mathcal{N}_{\geq 5}(\varphi)(x,t) = -\sum_{n=2}^{\infty} \frac{c_n}{2n+1}\partial_x \int_{\mathbb{R}^{2n+1}} \mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\cdots\hat{\varphi}(\eta_{2n+1},t)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n. \tag{5.9}$$

Equation (5.8) will be used in Section 5.6 in order to carry out nonlinear dispersive estimates, where the main difficulty is controlling the slowest decay in time caused by the lowest degree, cubic nonlinearity.

In Appendix A, we evaluate the integrals in (5.5) and show that we can write (5.4) in the alternative form

$$\varphi_t + \partial_x \left\{ \sum_{n=1}^{\infty} \sum_{\ell=1}^{2n+1} (-1)^{\ell+1} d_{n,\ell} \varphi^{2n-\ell+1} \partial_x^{2n} \log|\partial_x| \varphi^{\ell} \right\} = 2\log|\partial_x|\varphi_x, \tag{5.10}$$

where the constants $d_{n,\ell}$ are given in (A.4). We will not use (5.10) in this dissertation since it makes sense classically only for C^∞-solutions and does not make explicit the fact that, owing to a cancelation of derivatives, the nonlinear flux in (5.10) involves at most logarithmic derivatives of φ.

5.1.2. Para-linearization of the equation. In this subsection, we para-linearize the SQG front equation (5.4) and put it in a form that allows us to make weighted energy estimates. This form extracts a nonlinear term $L(T_{B^{\log}[\varphi]}\varphi)$ from the flux that is responsible for the logarithmic loss of derivatives in the dispersionless equation.

We use Weyl para-differential calculus to decompose the nonlinearity in (3.2). In the following, we use $C(n,s)$ to denote a positive constant depending only on n and s, which may change from line to line.

Proposition 5.1.1. *Suppose that $\varphi(\cdot,t) \in H^s(\mathbb{R})$ with $s > 4$ and $\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}}$ is sufficiently small. Then (3.2) can be written as*

$$\varphi_t + \partial_x T_{B^0[\varphi]}\varphi + \mathcal{R} = L\big[(2 - T_{B^{\log}[\varphi]})\varphi\big]_x, \tag{5.11}$$

where the symbols $B^0[\varphi]$ and $B^{\log}[\varphi]$ are defined by

$$\begin{aligned}
B^{\log}[\varphi](\cdot,\xi) &= \sum_{n=1}^{\infty} B_n^{\log}[\varphi](\cdot,\xi), \quad B^0[\varphi](\cdot,\xi) = \sum_{n=1}^{\infty} B_n^0[\varphi](\cdot,\xi),\\
B_n^{\log}[\varphi](\cdot,\xi) &= -\mathcal{F}_\zeta^{-1}\left[2c_n \int_{\mathbb{R}^{2n}} \delta\Big(\zeta - \sum_{j=1}^{2n}\eta_j\Big)\prod_{j=1}^{2n}\Big(i\eta_j\hat{\varphi}(\eta_j)\chi\Big(\frac{(2n+1)\eta_j}{\xi}\Big)\Big)\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\right],\\
B_n^0[\varphi](\cdot,\xi) &= \mathcal{F}_\zeta^{-1}\left[2c_n \int_{\mathbb{R}^{2n}} \delta\Big(\zeta - \sum_{j=1}^{2n}\eta_j\Big)\prod_{j=1}^{2n}\Big(i\eta_j\hat{\varphi}(\eta_j)\chi\Big(\frac{(2n+1)\eta_j}{\xi}\Big)\Big)\right.\\
&\qquad\qquad \left.\cdot\int_{[0,1]^{2n}} \log\Big|\sum_{j=1}^{2n}\eta_j s_j\Big|\,\mathrm{d}\hat{\mathbf{s}}_n\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\right].
\end{aligned} \tag{5.12}$$

Here, c_n is given by (5.5), δ is the delta-distribution, χ is the cutoff function in (2.6), $\hat{\boldsymbol{\eta}}_n = (\eta_1, \eta_2, \ldots, \eta_{2n})$, and $\hat{\mathbf{s}}_n = (s_1, \ldots, s_{2n})$. The operators $T_{B^{\log}[\varphi]}$ and $T_{B^0[\varphi]}$ are self-adjoint and their symbols satisfy the estimates

$$\begin{aligned}
\|B^{\log}[\varphi]\|_{\mathcal{M}_{(1,1)}} &\lesssim \sum_{n=1}^{\infty} C(n,s)|c_n|\|\varphi_x\|_{W^{2,\infty}}^{2n},\\
\|B^0[\varphi]\|_{\mathcal{M}_{(1,1)}} &\lesssim \sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|L\varphi_x\|_{W^{2,\infty}}^{2n} + \|\varphi_x\|_{W^{2,\infty}}^{2n}\Big),
\end{aligned} \tag{5.13}$$

while the remainder term $\mathcal{R}$ satisfies

$$\|\mathcal{R}\|_{H^s} \lesssim \|\varphi\|_{H^s}\left\{\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right\}, \tag{5.14}$$

where the constants $C(n,s)$ have at most exponential growth in n.

Proof. We define

$$f_n(x) = \int_{\mathbb{R}^{2n+1}} \mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n+1})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n.$$

In view of (5.4) and the commutator estimate (2.9), we only need to prove that

$$-\sum_{n=1}^{\infty}\frac{c_n}{2n+1}\partial_x f_n(x)=\partial_x T_{B^0[\varphi]}\varphi+\partial_x[(T_{B^{\log}[\varphi]})L\varphi]+\mathcal{R},$$

where $\mathcal{R}$ satisfies (5.14), and to do this it suffices to prove for each n that

$$\frac{c_n}{2n+1}\partial_x f_n(x)=-\partial_x T_{B_n^0[\varphi]}\varphi-\partial_x[(T_{B_n^{\log}[\varphi]})L\varphi]+\mathcal{R}_n,$$
$$\|\mathcal{R}_n\|_{H^s}\lesssim C(n,s)|c_n|\left(\|\varphi_x\|_{W^{2,\infty}}^{2n}+\|L\varphi_x\|_{W^{2,\infty}}^{2n}\right)\|\varphi\|_{H^s}.$$

By symmetry, we can assume that $|\eta_{2n+1}|$ is the largest frequency in the expression of f_n. Then

$$\begin{aligned}\frac{c_n}{2n+1}\partial_x f_n(x)&=c_n\partial_x\int\limits_{\substack{|\eta_{2n+1}|\geq|\eta_j|\\ \text{for all } j=1,\dots,2n}}\mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n+1})e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n\\
&=c_n\partial_x\int_{\mathbb{R}}\int\limits_{\substack{|\eta_j|\leq|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}}\mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})e^{i\sum_{j=1}^{2n}\eta_j x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}\\
&=c_n\partial_x\int_{\mathbb{R}}\int\limits_{\substack{|\eta_j|\leq|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}}\mathbf{T}_n(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\left[\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)+1-\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\right]\hat{\varphi}(\eta_j)\\
&\qquad\cdot e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}.\end{aligned}\tag{5.15}$$

Next, we expand the product in the above integral, and consider two cases depending on whether a term in the expansion contains only factors of χ or contains at least one factor $1-\chi$. In the first case, the frequency η_{2n+1} is much larger than all of the other frequencies, and we can extract a logarithmic derivative acting on the highest frequency; in the second case at least one other frequency is comparable to η_{2n+1}, and we get a remainder term by distributing derivatives on comparable frequencies.

Case I. When we take only factors of χ in the expansion of the product, we get the integral

$$c_n\partial_x\int_{\mathbb{R}}\int\limits_{\substack{|\eta_j|\leq|\eta_{2n+1}|\\ \text{for all } j=1,\dots,2n}}\mathbf{T}_n(\boldsymbol{\eta}_n)\prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)\hat{\varphi}(\eta_j)e^{i\sum_{j=1}^{2n}\eta_j x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}.\tag{5.16}$$

From (5.5), we can write $\mathbf{T}_n$ as an integral with respect to $\mathbf{s}_n = (s_1, s_2, \dots, s_{2n+1})$,

$$\begin{aligned}
\mathbf{T}_n(\boldsymbol{\eta}_n) &= -\int_{\mathbb{R}} \operatorname{sgn}\zeta \int_{[0,1]^{2n+1}} \prod_{j=1}^{2n+1} i\eta_j e^{i\eta_j s_j \zeta} \,\mathrm{d}\mathbf{s}_n \,\mathrm{d}\zeta \\
&= 2(-1)^n \Big(\prod_{j=1}^{2n+1} \eta_j \Big) \int_{[0,1]^{2n+1}} \frac{1}{\sum_{j=1}^{2n+1} \eta_j s_j} \,\mathrm{d}\mathbf{s}_n \\
&= 2 \Big(\prod_{j=1}^{2n} (i\eta_j) \Big) \int_{[0,1]^{2n}} \log \Big| 1 + \sum_{j=1}^{2n} \frac{\eta_j}{\eta_{2n+1}} s_j \Big| - \log \Big| \sum_{j=1}^{2n} \frac{\eta_j}{\eta_{2n+1}} s_j \Big| \,\mathrm{d}\hat{\mathbf{s}}_n \\
&= 2 \log|\eta_{2n+1}| \cdot \prod_{j=1}^{2n} (i\eta_j) - 2 \Big(\prod_{j=1}^{2n} (i\eta_j) \Big) \int_{[0,1]^{2n}} \log \Big| \sum_{j=1}^{2n} \eta_j s_j \Big| \,\mathrm{d}\hat{\mathbf{s}}_n \\
&\quad + \Big(\prod_{j=1}^{2n} (i\eta_j) \Big) \int_{[0,1]^{2n}} \log \Big| 1 + \sum_{j=1}^{2n} \frac{\eta_j}{\eta_{2n+1}} s_j \Big| \,\mathrm{d}\hat{\mathbf{s}}_n.
\end{aligned}$$

Substitution of this expression into (5.16) gives the following three terms

$$c_n \partial_x \int_{\mathbb{R}} \int\limits_{\substack{|\eta_j| \le |\eta_{2n+1}| \\ \text{for all } j=1,\dots,2n}} \mathbf{T}_n^{\log}(\boldsymbol{\eta}_n) \prod_{j=1}^{2n} \chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i\sum_{j=1}^{2n} \eta_j x} \,\mathrm{d}\hat{\boldsymbol{\eta}}_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \,\mathrm{d}\eta_{2n+1}, \tag{5.17}$$

$$c_n \partial_x \int_{\mathbb{R}} \int\limits_{\substack{|\eta_j| \le |\eta_{2n+1}| \\ \text{for all } j=1,\dots,2n}} \mathbf{T}_n^{0}(\boldsymbol{\eta}_n) \prod_{j=1}^{2n} \chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i\sum_{j=1}^{2n} \eta_j x} \,\mathrm{d}\hat{\boldsymbol{\eta}}_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \,\mathrm{d}\eta_{2n+1}, \tag{5.18}$$

$$c_n \partial_x \int_{\mathbb{R}} \int\limits_{\substack{|\eta_j| \le |\eta_{2n+1}| \\ \text{for all } j=1,\dots,2n}} \mathbf{T}_n^{\le -1}(\boldsymbol{\eta}_n)(\boldsymbol{\eta}_n) \prod_{j=1}^{2n} \chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i\sum_{j=1}^{2n} \eta_j x} \,\mathrm{d}\hat{\boldsymbol{\eta}}_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \,\mathrm{d}\eta_{2n+1}, \tag{5.19}$$

where

$$\begin{aligned}
\mathbf{T}_n^{\log}(\boldsymbol{\eta}_n) &= 2\log|\eta_{2n+1}| \cdot \prod_{j=1}^{2n} (i\eta_j), \\
\mathbf{T}_n^{0}(\boldsymbol{\eta}_n) &= -2 \Big(\prod_{j=1}^{2n} (i\eta_j) \Big) \int_{[0,1]^{2n}} \log \Big| \sum_{j=1}^{2n} \eta_j s_j \Big| \,\mathrm{d}\hat{\mathbf{s}}_n, \\
\mathbf{T}_n^{\le -1}(\boldsymbol{\eta}_n) &= 2 \Big(\prod_{j=1}^{2n} (i\eta_j) \Big) \int_{[0,1]^{2n}} \log \Big| 1 + \sum_{j=1}^{2n} \frac{\eta_j}{\eta_{2n+1}} s_j \Big| \,\mathrm{d}\hat{\mathbf{s}}_n.
\end{aligned}$$

We claim that the terms (5.17) and (5.18) can be rewritten as

$$-\partial_x T_{B_n^{\log}[\varphi]} L\varphi + \mathcal{R}_1 \quad \text{and} \quad -\partial_x T_{B_n^0[\varphi]}\varphi + \mathcal{R}_2, \tag{5.20}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the estimate (5.14). Indeed,

$$\begin{aligned}\mathcal{F}[\partial_x T_{B_n^{\log}[\varphi]} L\varphi](\xi) = -2c_n i\xi \int_{\mathbb{R}} \chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\log|\eta| \int_{\mathbb{R}^{2n}} \delta\Big(\xi-\eta-\sum_{j=1}^{2n}\eta_j\Big) \\ \cdot \prod_{j=1}^{2n}\left(i\eta_j\hat{\varphi}(\eta_j)\chi\Big(\frac{2(2n+1)\eta_j}{\xi+\eta}\Big)\right) \mathrm{d}\hat{\boldsymbol{\eta}}_n \hat{\varphi}(\eta)\,\mathrm{d}\eta,\end{aligned}$$

while the Fourier transform of (5.17) is

$$\begin{aligned}2c_n i\xi \int_{\mathbb{R}} \int_{\substack{|\eta_j|\le|\eta_{2n+1}| \\ \text{for all } j=1,\dots,2n}} \delta\left(\xi - \sum_{j=1}^{2n+1}\eta_j\right)\log|\eta_{2n+1}| \\ \cdot \prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)(i\eta_j)\hat{\varphi}(\eta_j)\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})\,\mathrm{d}\eta_{2n+1}.\end{aligned}$$

The difference of the above two integrals is

$$\begin{aligned}2c_n i\xi \int_{\mathbb{R}^{2n+1}} \delta\left(\xi - \sum_{j=1}^{2n+1}\eta_j\right)\log|\eta_{2n+1}| \cdot \Bigg[\mathbb{I}_n \prod_{j=1}^{2n}\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)(i\eta_j)\hat{\varphi}(\eta_j) \\ - \chi\left(\frac{|\xi-\eta_{2n+1}|}{|\xi+\eta_{2n+1}|}\right)\prod_{j=1}^{2n}\left(i\eta_j\hat{\varphi}(\eta_j)\chi\Big(\frac{2(2n+1)\eta_j}{\xi+\eta_{2n+1}}\Big)\right)\Bigg]\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})\,\mathrm{d}\eta_{2n+1},\end{aligned} \tag{5.21}$$

where $\mathbb{I}_n$ is the function which is equal to 1 on $\{|\eta_j| \le |\eta_{2n+1}|,\ \text{for all } j = 1,\dots,2n\}$ and equal to zero otherwise.

When $\boldsymbol{\eta}_n$ satisfies

$$|\eta_j| \le \frac{1}{40}\frac{1}{2n+1}|\eta_{2n+1}| \qquad \text{for } j = 1,2,\dots,2n, \tag{5.22}$$

we have $\mathbb{I}_n = 1$ and $\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right) = 1$. In addition, since $\xi = \sum_{j=1}^{2n+1}\eta_j$, we have

$$\frac{|\xi-\eta_{2n+1}|}{|\xi+\eta_{2n+1}|} = \frac{\left|\sum_{j=1}^{2n}\eta_j\right|}{\left|\sum_{j=1}^{2n}\eta_j + 2\eta_{2n+1}\right|} \le \frac{\frac{1}{40}|\eta_{2n+1}|}{(2-\frac{1}{40})|\eta_{2n+1}|} = \frac{1}{79} < \frac{3}{40},$$

$$\frac{2(2n+1)|\eta_j|}{|\xi+\eta_{2n+1}|} \le \frac{\frac{1}{20}|\eta_{2n+1}|}{(2-\frac{1}{40})|\eta_{2n+1}|} = \frac{2}{79} < \frac{3}{40},$$

so

$$\chi\left(\frac{|\xi-\eta_{2n+1}|}{|\xi+\eta_{2n+1}|}\right) = 1, \qquad \chi\Big(\frac{2(2n+1)\eta_j}{\xi+\eta_{2n+1}}\Big) = 1.$$

Therefore the integrand of (5.21) is supported outside the set (5.22), and there exists $j_1 \in \{1,\ldots,2n\}$, such that $|\eta_{j_1}| > \frac{1}{40}\frac{1}{2n+1}|\eta_{2n+1}|$. Since $|\eta_{2n+1}|$ is the largest frequency, we see that $|\eta_{j_1}|$ and $|\eta_{2n+1}|$ are comparable in the error term. Therefore, the H^s-norm of (5.21) is bounded by

$$\|\varphi\|_{H^s} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big).$$

It follows that (5.17) can be written as in (5.20). A similar calculation applies to (5.18).

The symbols $B_n^{\log}[\varphi]$ and $B_n^0[\varphi]$ are real, so that $T_{B_n^{\log}[\varphi]}$ and $T_{B_n^0[\varphi]}$ are self-adjoint. Again, without loss of generality, we assume $|\eta_{2n}| = \max_{1\le j\le 2n}|\eta_j|$ and observe that

$$\begin{aligned}
&\int_{[0,1]^{2n}} \log\left|\sum_{j=1}^{2n}\eta_j s_j\right| \mathrm{d}\hat{\mathbf{s}}_n \\
&= \log|\eta_{2n}| + \int_{[0,1]^{2n-1}} \left\{\left(\sum_{j=1}^{2n-1}\frac{\eta_j}{\eta_{2n}}s_j\right)\log\left|1+\frac{1}{\sum_{j=1}^{2n-1}\frac{\eta_j}{\eta_{2n}}s_j}\right| + \log\left|1+\frac{1}{\sum_{j=1}^{2n-1}\frac{\eta_j}{\eta_{2n}}s_j}\right| - 1\right\}\mathrm{d}\mathbf{s}_{n-1} \\
&= \log|\eta_{2n}| + O(1).
\end{aligned}$$

Thus, using Young's inequality, we obtain from (5.12) the estimate (5.13), where the constants $C(n,s)$ have at most exponential growth in n.

To estimate the third term (5.19), we observe that on the support of the functions $\chi\left(\frac{(2n+1)\eta_j}{\eta_{2n+1}}\right)$, we have

$$\frac{|\eta_j|}{|\eta_{2n+1}|} \le \frac{1}{10(2n+1)}.$$

Since $s_j \in [0,1]$, a Taylor expansion gives

$$\left|\mathbf{T}_n^{\le -1}(\boldsymbol{\eta}_n)\right| \lesssim \frac{\left[\prod_{j=1}^{2n}|\eta_j|\right]\left[\sum_{j=1}^{2n}|\eta_j|\right]}{|\eta_{2n+1}|}.$$

Therefore the H^s-norm of (5.19) is bounded by $C(n,s)|c_n|\|\varphi\|_{H^s}\|\varphi_x\|_{W^{2,\infty}}^{2n}$, where $C(n,s)$ has at most exponential growth in n.

Case II. When there is at least one factor of the form $1-\chi$ in the expansion of the product in the integral (5.15), we get a term like

$$c_n\partial_x \int_{\mathbb{R}} \int_{\substack{|\eta_j|\le|\eta_{2n+1}| \\ \text{for all } j=1,\dots,2n}} \mathbf{T}_n(\boldsymbol{\eta}_n) \prod_{k=1}^{\ell}\left[1-\chi\left(\frac{(2n+1)\eta_{j_k}}{\eta_{2n+1}}\right)\right] \prod_{k=\ell+1}^{2n}\chi\left(\frac{(2n+1)\eta_{j_k}}{\eta_{2n+1}}\right) \\ \cdot\left(\prod_{j=1}^{2n}\hat{\varphi}(\eta_j)\right)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n})x}\,\mathrm{d}\hat{\boldsymbol{\eta}}_n\hat{\varphi}(\eta_{2n+1})e^{ix\eta_{2n+1}}\,\mathrm{d}\eta_{2n+1}, \tag{5.23}$$

where $1\le\ell\le 2n$ is an integer, and $\{j_k : k=1,\dots,2n\}$ is a permutation of $\{1,\dots,2n\}$.

Notice that $1-\chi\left(\frac{(2n+1)\eta_{j_1}}{\eta_{2n+1}}\right)$ is compactly supported on

$$\frac{|\eta_{j_1}|}{|\eta_{2n+1}|}\ge\frac{3}{40(2n+1)}.$$

By assumption, η_{2n+1} has the largest absolute value, so

$$\frac{3}{40(2n+1)}|\eta_{2n+1}|\le|\eta_{j_1}|\le|\eta_{2n+1}|,$$

meaning that the frequencies $|\eta_{j_1}|$ and $|\eta_{2n+1}|$ are comparable.

Without loss of generality, we assume that $|\eta_{j_1}|\le|\eta_{j_2}|\le\cdots\le|\eta_{j_{2n}}|\le|\eta_{2n+1}|$, define $\eta_{j_{2n+1}}=\eta_{2n+1}$, and split the integral of $\mathbf{T}_n$ (5.5) into three parts.

$$\mathbf{T}_n(\boldsymbol{\eta}_n)=\mathbf{T}_n^{low}(\boldsymbol{\eta}_n)+\sum_{k=1}^{2n}\mathbf{T}_n^{med,(k)}(\boldsymbol{\eta}_n)+\mathbf{T}_n^{high}(\boldsymbol{\eta}_n),$$

$$\mathbf{T}_n^{low}(\boldsymbol{\eta}_n)=\int_{|\eta_{2n+1}\zeta|<2}\frac{\prod_{j=1}^{2n+1}\left(1-e^{i\eta_j\zeta}\right)}{\zeta^{2n+1}}\operatorname{sgn}\zeta\,\mathrm{d}\zeta, \tag{5.24}$$

$$\mathbf{T}_n^{med,(k)}(\boldsymbol{\eta}_n)=\int_{\frac{2}{|\eta_{j_{k+1}}|}\le|\zeta|\le\frac{2}{|\eta_{j_k}|}}\frac{\prod_{j=1}^{2n+1}\left(1-e^{i\eta_j\zeta}\right)}{\zeta^{2n+1}}\operatorname{sgn}\zeta\,\mathrm{d}\zeta, \tag{5.25}$$

$$\mathbf{T}_n^{high}(\boldsymbol{\eta}_n)=\int_{|\eta_{j_1}\zeta|>2}\frac{\prod_{j=1}^{2n+1}\left(1-e^{i\eta_j\zeta}\right)}{\zeta^{2n+1}}\operatorname{sgn}\zeta\,\mathrm{d}\zeta. \tag{5.26}$$

To estimate (5.24), we notice that

$$|\mathbf{T}_n^{low}(\boldsymbol{\eta}_n)|\le\prod_{k=1}^{2n+1}|\eta_k|\cdot\int_{|\eta_{2n+1}\zeta|<2}\left(\prod_{k=1}^{2n+1}\frac{|1-e^{i\eta_{j_k}\zeta}|}{|\eta_{j_k}\zeta|}\right)\mathrm{d}\zeta\le C(n,s)\left(\prod_{k=1}^{2n}|\eta_{j_k}|\right).$$

For each $1 \le k \le 2n$, we consider two cases. If $k \neq 2n$, we estimate (5.25) as

$$\begin{aligned}
|\mathbf{T}_n^{med,(k)}(\boldsymbol{\eta}_n)| &\le \prod_{\ell=1}^{k} |\eta_{j_\ell}| \cdot \int_{\frac{2}{|\eta_{j_{k+1}}|} \le |\zeta| \le \frac{2}{|\eta_{j_k}|}} \left(\prod_{\ell=1}^{k} \frac{|1 - e^{i\eta_{j_\ell}\zeta}|}{|\eta_{j_\ell}\zeta|} \right) \cdot \frac{\prod_{\ell=k+1}^{2n+1} |1 - e^{i\eta_{j_\ell}\zeta}|}{|\zeta|^{2n+1-k}} \, \mathrm{d}\zeta \\
&\le 2^{2n+1-k} \prod_{\ell=1}^{k} |\eta_{j_\ell}| \cdot \int_{\frac{2}{|\eta_{j_{k+1}}|} \le |\zeta| \le \frac{2}{|\eta_{j_k}|}} |\zeta|^{-2n-1+k} \, \mathrm{d}\zeta \\
&\le \frac{2}{2n-k} \left(|\eta_{j_k}|^{2n-k} + |\eta_{j_{k+1}}|^{2n-k} \right) \prod_{\ell=1}^{k} |\eta_{j_\ell}| \\
&\le 2 \left(\prod_{k=1}^{2n} |\eta_{j_k}| \right).
\end{aligned}$$

If $k = 2n$, we have

$$\begin{aligned}
|\mathbf{T}_n^{med,(k)}(\boldsymbol{\eta}_n)| &\le 2 \prod_{\ell=1}^{2n} |\eta_{j_\ell}| \cdot \int_{\frac{2}{|\eta_{j_{2n+1}}|} \le |\zeta| \le \frac{2}{|\eta_{j_{2n}}|}} \frac{1}{|\zeta|} \, \mathrm{d}\zeta \\
&= 4 \prod_{\ell=1}^{2n} |\eta_{j_\ell}| \cdot \log \left| \frac{\eta_{j_{2n+1}}}{\eta_{j_{2n}}} \right| \le C(n,s) \prod_{\ell=1}^{2n} |\eta_{j_\ell}|,
\end{aligned}$$

where the last line follows from the fact that $|\eta_{j_{2n}}|$ and $|\eta_{j_{2n+1}}|$ are comparable.

As for (5.26), we have

$$\begin{aligned}
|\mathbf{T}_n^{high}(\boldsymbol{\eta}_n)| &\le |\eta_{j_1}| \int_{|\eta_{j_1}\zeta| > 2} \left(\prod_{k=2}^{2n+1} \frac{|1 - e^{i\eta_{j_k}\zeta}|}{|\zeta|} \right) \cdot \frac{|1 - e^{i\eta_{j_1}\zeta}|}{|\eta_{j_1}\zeta|} \, \mathrm{d}\zeta \\
&\le 2^{2n} |\eta_{j_1}| \int_{|\eta_{j_1}\zeta| > 2} \frac{\mathrm{d}\zeta}{|\zeta|^{2n}} \\
&\le \frac{4}{2n-1} \left(\prod_{k=1}^{2n} |\eta_{j_k}| \right).
\end{aligned}$$

Collecting these estimates yields

$$\mathbf{T}_n(\boldsymbol{\eta}_n) \le C(n,s) \left(\prod_{k=1}^{2n} |\eta_{j_k}| \right).$$

Using this inequality, we can bound the H^s-norm of (5.23) by

$$\|\varphi\|_{H^s} \left(\sum_{n=1}^{\infty} C(n,s) |c_n| \|\varphi_x\|_{W^{2,\infty}}^{2n} \right),$$

and the proposition follows. □

5.2. Energy estimates and local well-posedness

In this section, we prove an *a priori* estimate for the initial value problem (5.1), which is stated in Proposition 5.2.1. The content of this section is an analogy of the argument in Section 4.3 for the local well-posedness of the Cauchy problem for the approximate equation (4.1), but here we improve the estimates and include a blow-up criterion.

If $\|T_{B^{\log}[\varphi]}\|_{L^2\to L^2} < 2$, then $(2 - T_{B^{\log}[\varphi]})$ is a positive, self-adjoint operator on L^2. We can therefore define homogeneous and non-homogeneous weighted energies that are equivalent to the H^s-energies by

$$E^{(s)}(t) = \int_{\mathbb{R}} |D|^s\varphi(x,t)\cdot\left(2 - T_{B^{\log}[\varphi]}\right)^{2s+1}|D|^s\varphi(x,t)\,\mathrm{d}x, \qquad \tilde{E}^{(s)}(t) = \sum_{j=0}^{s} E^{(j)}(t). \tag{5.27}$$

For simplicity, we consider only integer norms with $s \in \mathbb{N}$.

In the following, we use F to denote an increasing, continuous, real-valued function, which might change from line to line.

Proposition 5.2.1. *Let $s \geq 5$ be an integer and φ a smooth solution of (5.1) with $\varphi_0 \in H^s(\mathbb{R})$. There exists a constant $\tilde{C} > 0$, depending only on s, such that if φ_0 satisfies*

$$\left\|T_{B^{\log}[\varphi_0]}\right\|_{L^2\to L^2} \leq C, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left(\|\partial_x\varphi_0\|_{W^{2,\infty}}^{2n} + \|L\partial_x\varphi_0\|_{W^{2,\infty}}^{2n}\right) < \infty$$

for some constant $0 < C < 2$, then there exists a time $T > 0$ such that

$$\left\|T_{B^{\log}[\varphi(t)]}\right\|_{L^2\to L^2} < 2, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left(\|\varphi_x(t)\|_{W^{2,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n}\right) < \infty$$

for all $t \in [0, T]$, and

$$\begin{aligned}\tilde{E}^{(s)}(t) \leq\ & \tilde{E}^{(s)}(0) \\ & + \int_0^t \left(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}\right)^2 F\left(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}\right)\tilde{E}^{(s)}(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{5.28}$$

where $\tilde{E}^{(s)}$ is defined in (5.27), and $F(\cdot)$ is an increasing, continuous, real-valued function such that

$$F\left(\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}}\right) \approx \sum_{n=0}^{\infty} \tilde{C}^n |c_n| \left(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\right). \tag{5.29}$$

Before proving this proposition, we first state a lemma.

Lemma 5.2.2. *Suppose that $s \geq 5$ is an integer. If φ is a smooth solution of* (5.11) *and $\psi \in C_t^1 L_x^2$, then*

$$\partial_t (2 - T_{B^{\log}[\varphi]})^s \psi = (2 - T_{B^{\log}[\varphi]})^s \psi_t - s(2 - T_{B^{\log}[\varphi]})^{s-1} T_{\partial_t B^{\log}[\varphi]} \psi + \mathcal{R}_2(\psi),$$

where the remainder term satisfies

$$\|\mathcal{R}_2(\psi)\|_{H^1} \lesssim \|\psi\|_{L^2} \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n} \Big) \right)$$

for constants $C(n,s)$ with at most exponential growth in n.

Proof. Since s is an integer, we can calculate the time derivative as

$$\begin{aligned} \partial_t (2 - T_{B^{\log}[\varphi]})^s \psi = {} & T_{\partial_t B^{\log}[\varphi]} (2 - T_{B^{\log}[\varphi]})^{s-1} \psi + (2 - T_{B^{\log}[\varphi]}) T_{\partial_t B^{\log}[\varphi]} (2 - T_{B^{\log}[\varphi]})^{s-2} \psi \\ & + \cdots + (2 - T_{B^{\log}[\varphi]})^{s-1} T_{\partial_t B^{\log}[\varphi]} \psi + (2 - T_{B^{\log}[\varphi]})^s \psi_t. \end{aligned}$$

Using (5.13), we have the commutator estimate

$$\Big\| [T_{\partial_t B^{\log}[\varphi]}, (2 - T_{B^{\log}[\varphi]})] f \Big\|_{H^1} \lesssim \|f\|_{L^2} \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n} \Big) \right).$$

By taking $f = (2 - T_{B^{\log}[\varphi]})^{s-2}\psi, (2 - T_{B^{\log}[\varphi]})^{s-3}\psi, \ldots, (2 - T_{B^{\log}[\varphi]})\psi$ and applying the commutator estimate repeatedly, we obtain the conclusion. □

Proof of Proposition 5.2.1. By continuity in time, there exists $T > 0$ such that

$$\sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\varphi_x(t)\|_{W^{2,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n} \Big) < \infty \qquad \text{for all } 0 \leq t \leq T.$$

We apply the operator $|D|^s$ to equation (5.11) to get

$$|D|^s \varphi_t + \partial_x |D|^s T_{B^0[\varphi]} \varphi + |D|^s \mathcal{R} = \partial_x L |D|^s \big[(2 - T_{B^{\log}[\varphi]}) \varphi \big]. \tag{5.30}$$

Using Lemma 2.2.2, we find that

$$\begin{aligned} |D|^s \Big[(2 - T_{B^{\log}[\varphi]}) \varphi \Big] &= 2|D|^s \varphi - |D|^s (T_{B^{\log}[\varphi]} \varphi) \\ &= 2|D|^s \varphi - T_{B^{\log}[\varphi]} |D|^s \varphi + s T_{\partial_x B^{\log}[\varphi]} |D|^{s-2} \varphi_x + \mathcal{R}_2, \end{aligned}$$

where

$$\|\partial_x \mathcal{R}_2\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \|\varphi_x\|_{W^{2,\infty}}^{2n} \right) \|\varphi\|_{H^{s-1}}.$$

Thus, we can write the right-hand side of (5.30) as

$$\begin{aligned}
&\partial_x L|D|^s \left[(2 - T_{B^{\log}[\varphi]})\varphi\right] \\
&\qquad = \partial_x L \left[(2 - T_{B^{\log}[\varphi]})|D|^s\varphi + sT_{\partial_x B^{\log}[\varphi]}|D|^{s-2}\varphi_x\right] + \mathcal{R}_3 \\
&\qquad = L\big\{(2 - T_{B^{\log}[\varphi]})|D|^s\varphi_x - T_{\partial_x B^{\log}[\varphi]}|D|^s\varphi - sT_{\partial_x B^{\log}[\varphi]}|D|^s\varphi\big\} + \mathcal{R}_3 \\
&\qquad = L\left\{(2 - T_{B^{\log}[\varphi]})|D|^s\varphi_x - (s+1)T_{\partial_x B^{\log}[\varphi]}|D|^s\varphi\right\} + \mathcal{R}_3,
\end{aligned}$$

where

$$\|\mathcal{R}_3\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}.$$

Applying $(2-T_{B^{\log}[\varphi]})^s$ to (5.30), and commuting $(2-T_{B^{\log}[\varphi]})^s$ with L up to remainder terms, we obtain that

$$\begin{aligned}
&(2 - T_{B^{\log}[\varphi]})^s|D|^s\varphi_t + (2 - T_{B^{\log}[\varphi]})^s\partial_x|D|^sT_{B^0[\varphi]}\varphi \\
&\quad = L\left\{(2 - T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi_x - (s+1)(2 - T_{B^{\log}[\varphi]})^sT_{\partial_x B^{\log}[\varphi]}|D|^s\varphi\right\} + \mathcal{R}_4 \\
&\quad = \partial_x L\left\{(2 - T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi\right\} + \mathcal{R}_5,
\end{aligned} \tag{5.31}$$

where $\|\mathcal{R}_5\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}$.

By Lemma 5.2.2, with $\psi = |D|^s\varphi$, the time derivative of $E^{(s)}(t)$ in (5.27) is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E^{(s)}(t) = &-\int_{\mathbb{R}} (2s+1)|D|^s\varphi \cdot (2 - T_{B^{\log}[\varphi]})^{2s}T_{\partial_t B^{\log}[\varphi]}|D|^s\varphi\,\mathrm{d}x \\
&+ 2\int_{\mathbb{R}} |D|^s\varphi \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_t\,\mathrm{d}x + \int_{\mathbb{R}} \mathcal{R}_2\left(|D|^s\varphi\right)|D|^s\varphi\,\mathrm{d}x.
\end{aligned} \tag{5.32}$$

We will estimate each of the terms on the right-hand side of (5.32), where the second term requires the most work.

Equation (5.11) implies that

$$\|\varphi_{xt}\|_{L^\infty} \lesssim \sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big),$$

so the first term on the right-hand side of (5.32) can be estimated by

$$\begin{aligned}
&\left|\int_{\mathbb{R}} (2s+1)|D|^s\varphi \cdot (2 - T_{B^{\log}[\varphi]})^{2s}T_{\partial_t B^{\log}[\varphi]}|D|^s\varphi\,\mathrm{d}x\right| \\
&\qquad \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}^2.
\end{aligned}$$

Using Lemma 5.2.2, we can estimate the third term on the right-hand side of (5.32) by

$$\int_{\mathbb{R}} \mathcal{R}_2\left(|D|^s\varphi\right)|D|^s\varphi\,\mathrm{d}x \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}\|\varphi\|_{H^{s-1}}.$$

To estimate the second term on the right-hand side (5.32), we multiply (5.31) by $(2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi$, integrate the result with respect to x, and use the self-adjointness of $(2-T_{B^{\log}[\varphi]})^{s+1}$, which gives

$$\int_{\mathbb{R}} |D|^s\varphi\cdot(2-T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_t\,\mathrm{d}x = \mathrm{I}+\mathrm{II}+\mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I} &= -\int_{\mathbb{R}} |D|^s\varphi\cdot(2-T_{B^{\log}[\varphi]})^{2s+1}|D|^s\partial_x T_{B^0[\varphi]}\varphi\,\mathrm{d}x,\\
\mathrm{II} &= \int_{\mathbb{R}} (2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi\cdot\partial_x L(2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi\,\mathrm{d}x,\\
\mathrm{III} &= \int_{\mathbb{R}} (2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi\cdot\mathcal{R}_5\,\mathrm{d}x.
\end{aligned}$$

We have $\mathrm{II}=0$, since $\partial_x L$ is skew-symmetric, and

$$\mathrm{III} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}^2,$$

since $\|\mathcal{R}_5\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n}+\|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}$ and $(2-T_{B^{\log}[\varphi]})^{s+1}$ is bounded on L^2.

Term I estimate. We write $\mathrm{I} = -\mathrm{I}_a + \mathrm{I}_b$, where

$$\begin{aligned}
\mathrm{I}_a &= \int_{\mathbb{R}} |D|^s\varphi\cdot(2-T_{B^{\log}[\varphi]})^{2s+1}\partial_x T_{B^0[\varphi]}|D|^s\varphi\,\mathrm{d}x,\\
\mathrm{I}_b &= \int_{\mathbb{R}} |D|^s\varphi\cdot(2-T_{B^{\log}[\varphi]})^{2s+1}\partial_x[T_{B^0[\varphi]},|D|^s]\varphi\,\mathrm{d}x.
\end{aligned}$$

By a commutator estimate and (5.13), the second integral satisfies

$$|\mathrm{I}_b| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}^2.$$

To estimate the first integral, we write it as

$$\mathrm{I}_a = \mathrm{I}_{a_1} - \mathrm{I}_{a_2},$$

where

$$\mathrm{I}_{a_1} = \int_{\mathbb{R}} |D|^s\varphi \cdot [(2 - T_{B^{\log}[\varphi]})^{2s+1}, \partial_x] \left(T_{B^0[\varphi]}|D|^s\varphi\right) \, \mathrm{d}x,$$
$$\mathrm{I}_{a_2} = \int_{\mathbb{R}} |D|^s\varphi_x \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \, \mathrm{d}x.$$

Term I_{a_1} estimate. A Kato-Ponce commutator estimate and (5.13) gives

$$|\mathrm{I}_{a_1}| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|^{2n}_{W^{2,\infty}} + \|L\varphi_x\|^{2n}_{W^{2,\infty}}\Big)\right)\|\varphi\|^2_{H^s}.$$

Term I_{a_2} estimate. We have

$$\begin{aligned}
\mathrm{I}_{a_2} &= \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_x \, \mathrm{d}x \\
&= \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot \Big\{\partial_x \Big((2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi\Big) - \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi\Big\} \, \mathrm{d}x \\
&= -\int_{\mathbb{R}} \partial_x \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi \, \mathrm{d}x \\
&= -\int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi_x + \left[\partial_x, T_{B^0[\varphi]}\right] |D|^s\varphi\right) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi \, \mathrm{d}x.
\end{aligned} \tag{5.33}$$

Using commutator estimates and (5.13), we get that

$$\left\|\left[\partial_x, T_{B^0[\varphi]}\right] |D|^s\varphi\right\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \left(\|\varphi_x\|^2_{W^{2,\infty}} + \|L\varphi_x\|^2_{W^{2,\infty}}\right)\right)\|\varphi\|_{H^s},$$
$$\left\|\Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi\right\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|^{2n}_{W^{2,\infty}} + \|L\varphi_x\|^{2n}_{W^{2,\infty}}\Big)\right)\|\varphi\|_{H^s},$$
$$\left\|\partial_x \Big[(2 - T_{B^{\log}[\varphi]})^{2s+1}, T_{B^0[\varphi]}\Big] |D|^s\varphi\right\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|^{2n}_{W^{2,\infty}} + \|L\varphi_x\|^{2n}_{W^{2,\infty}}\Big)\right)\|\varphi\|^2_{H^s}.$$

Since $T_{B^0[\varphi]}$ is self-adjoint, we can rewrite (5.33) as

$$\mathrm{I}_{a_2} = -\mathrm{I}_{a_2} + \mathcal{R}_6,$$

with

$$|\mathcal{R}_6| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi_x\|^{2n}_{W^{2,\infty}} + \|L\varphi_x\|^{2n}_{W^{2,\infty}}\Big)\right)\|\varphi\|^2_{H^s},$$

and we conclude that

$$|\mathrm{I}_{a_2}| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}^2.$$

This completes the estimate of the terms on the right hand side of (5.32). Collecting the above estimates and using the interpolation inequalities for $E^{(0)}$ and $E^{(s)}$, we obtain that

$$\begin{aligned} \tilde{E}^{(s)}(t) \leq \tilde{E}^{(s)}(0) \int_0^t \left(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}} \right)^2 \\ \cdot F\left(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}} \right) \|\varphi(\tau)\|_{H^s}^2 \, \mathrm{d}\tau, \end{aligned} \tag{5.34}$$

with

$$F\left(\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}} \right) \approx \sum_{n=0}^{\infty} C(n,s)|c_n| \Big(\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n} \Big).$$

We observe that there exists a constant $\tilde{C}(s) > 0$ such that $C(n,s) \lesssim \tilde{C}(s)^n$. The series in (5.29) then converges whenever $\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}}$ is sufficiently small, and we can choose F to be an increasing, continuous, real-valued function that satisfies (5.29).

Finally, since $\|2 - T_{B^{\log}[\varphi_0]}\|_{L^2 \to L^2} \geq 2 - C$, and $\|B^{\log}[\varphi](\cdot,t)\|_{\mathcal{M}_{(0,0)}}$ and $F\left(\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}} \right)$ are continuous in time, there exist $T > 0$ and $m > 0$, depending only on the initial data, such that

$$\|2 - T_{B^{\log}[\varphi(t)]}\|_{L^2 \to L^2} \geq m \qquad \text{for } 0 \leq t \leq T.$$

We therefore obtain that

$$m^{2s+1} \|\varphi\|_{H^s}^2 \leq \tilde{E}^{(s)} \leq 2^{2s+1} \|\varphi\|_{H^s}^2,$$

so (5.34) implies (5.28). □

Proposition 5.2.1 leads to the following local existence, uniqueness and breakdown result.

Theorem 5.2.3. *Let $s \geq 5$ be an integer. Suppose that $\varphi_0 \in H^s(\mathbb{R})$ satisfies*

$$\|T_{B^{\log}[\varphi_0]}\|_{L^2 \to L^2} \leq C, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\partial_x \varphi_0\|_{W^{2,\infty}}^{2n} + \|L\partial_x \varphi_0\|_{W^{2,\infty}}^{2n} \Big) < \infty$$

for some constant $0 < C < 2$, where $\tilde{C}$ is the same constant as the one in Proposition 5.2.1 and the symbol $B^{\log}[\varphi_0]$ is defined in (5.12). Then there exists a maximal time of existence $0 < T_{\max} \leq \infty$ depending only on $\|\varphi_0\|_{H^s}$, C, and $\tilde{C}$ such that the initial value problem (5.1) has a unique solution *with* $\varphi \in C([0,T_{\max});H^s(\mathbb{R}))$*. If* $T_{\max}<\infty$*, then either*

$$\lim_{t\uparrow T_{\max}} \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\varphi_x(t)\|_{W^{2,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n} \Big) = \infty \quad \textit{or} \quad \lim_{t\uparrow T_{\max}} \big\| T_{B^{\log}[\varphi(\cdot,t)]} \big\|_{L^2\to L^2} = 2. \tag{5.35}$$

Proof. By Sobolev embedding, we have $\|\varphi\|_{W^{3,\infty}} \lesssim \|\varphi\|_{H^s}$ and $\|L\varphi\|_{W^{3,\infty}} \lesssim \|\varphi\|_{H^s}$, so we obtain from (5.28) that

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{(j)}(t) \lesssim F\left([\tilde{E}^{(s)}(t)]^{1/2} \right) [\tilde{E}^{(j)}(t)]^2.$$

Thus, there exists a time $T>0$ such that $E^{(s)}(t)$ is bounded when $t\in[0,T]$. Therefore, by classical C_0-semigroup theory for local existence [**Paz83**], there is a unique solution $\varphi \in C([0,T];H^s(\mathbb{R}))$.

By applying Grönwall's inequality to (5.28), we get that

$$\tilde{E}^{(s)}(t) \le \tilde{E}^{(s)}(0) \exp\left[\int_0^t (\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 F\left(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}} \right) \mathrm{d}\tau \right], \tag{5.36}$$

where F is given by (5.29). It follows from the local existence result that the solution can be continued so long as $\|2 - T_{B^{\log}[\varphi(t)]}\|_{L^2\to L^2}$ remains bounded away from zero, and

$$\sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left(\|\varphi_x(t)\|_{W^{2,\infty}}^{2n} + \|L\varphi_x(t)\|_{W^{2,\infty}}^{2n} \right)$$

remains bounded, so the breakdown criterion (5.35) follows. □

The front equation is invariant under $(x,t)\mapsto(-x,-t)$, so the same result holds backward in time. One could use a Bona-Smith argument, as in Section 4.3, to prove that the solution depends continuously on the initial data, but we will not carry out the details here.

5.3. Global solution for small initial data

Beginning with this section, we address the proof of Theorem 5.0.1. From now on, the parameters s, r, and p_0 are fixed as in (5.2). We also introduce the notation

$$h(x,t) = e^{-2t\partial_x \log|\partial_x|}\varphi(x,t), \qquad \hat{h}(\xi,t) = e^{-2it\xi\log|\xi|}\hat{\varphi}(\xi,t) \tag{5.37}$$

for the function h obtained by removing the action of the linearized evolution group on φ. When convenient, we write $h(\cdot,t)=h(t)$, $\varphi(\cdot,t)=\varphi(t)$.

Given local existence, we only need to prove the global *a priori* bound. In order to do this, we introduce the Z-norm of a function $f \in L^2(\mathbb{R})$, defined by

$$\|f\|_Z = \left\| (|\xi| + |\xi|^{r+3}) \hat{f}(\xi) \right\|_{L^\infty_\xi}, \tag{5.38}$$

and prove the global bound by use of the following bootstrap argument.

Proposition 5.3.1 (Bootstrap)**.** *Let $T > 1$ and suppose that $\varphi \in C([0,T]; H^s)$ is a solution of* (5.1)*, where the initial data satisfies*

$$\|\varphi_0\|_{H^s} + \|x\partial_x \varphi_0\|_{H^r} \leq \varepsilon_0$$

for some $0 < \varepsilon_0 \ll 1$. If there exists $\varepsilon_0 \ll \varepsilon_1 \lesssim \varepsilon_0^{1/3}$ such that the solution satisfies

$$(t+1)^{-p_0} \left(\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r}\right) + \|\varphi\|_Z \leq \varepsilon_1$$

for every $t \in [0,T]$, then the solution satisfies an improved bound

$$(t+1)^{-p_0} \left(\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r}\right) + \|\varphi\|_Z \lesssim \varepsilon_0.$$

Theorem 5.0.1 then follows from combining this bootstrap proposition with the local existence and blow-up result in Theorem 5.2.3. We call the assumptions in Proposition 5.3.1 the *bootstrap assumptions*. To prove Proposition 5.3.1, we need the following lemmas, some of whose proofs are deferred to the next sections.

Lemma 5.3.2 (Sharp pointwise decay)**.** *Under the bootstrap assumptions,*

$$\|\varphi_x(t)\|_{W^{r,\infty}} + \|L\varphi_x(t)\|_{W^{r,\infty}} \lesssim \varepsilon_1 (t+1)^{-1/2}.$$

Lemma 5.3.3 (Scaling-Galilean estimate)**.** *Under the bootstrap assumptions,*

$$(t+1)^{-p_0} \|\mathcal{S}\varphi(t)\|_{H^r} \lesssim \varepsilon_0.$$

Lemma 5.3.4. *Under the bootstrap assumptions,*

$$(t+1)^{-p_0} (\|\varphi(t)\|_{H^s} + \|x\partial_x \varphi(t)\|_{H^r}) \lesssim \varepsilon_0.$$

Proof. Recall the energy estimate (5.36)

$$\tilde{E}^{(s)}(t) \lesssim \tilde{E}^{(s)}(0) e^{\int_0^t F(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}})^2 \, d\tau}.$$

From Lemma 5.3.2, we have

$$F(\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}}) \lesssim 1,$$

$$\|\varphi_x(\tau)\|_{W^{2,\infty}} + \|L\varphi_x(\tau)\|_{W^{2,\infty}} \lesssim (t+1)^{-1/2}\varepsilon_1,$$

which implies that

$$\tilde{E}^{(s)}(t) \lesssim \varepsilon_0^2 (t+1)^{C\varepsilon_1^2}$$

for some constant C, so once $\varepsilon_1^2 \ll p_0$, we have

$$(t+1)^{-p_0} \|\varphi\|_{H^s} \lesssim \varepsilon_0.$$

Next, we observe that we can use $\|\mathcal{S}\varphi\|_{H^r}$ to control $\|x\partial_x h\|_{H^r}$. It follows from (5.37), the definition of $\mathcal{S}$, and (5.4) that

$$\begin{aligned}
\mathcal{F}_x[x\partial_x h](\xi) &= -\partial_\xi\big(\xi \hat{h}(\xi)\big) = -\hat{h}(\xi) - \xi\partial_\xi \hat{h}(\xi), \\
\xi\partial_\xi \hat{h}(\xi,t) &= \xi e^{-2it\xi \log|\xi|} \left(-2it(\log|\xi|+1)\hat{\varphi}(\xi,t) + \partial_\xi\hat{\varphi}(\xi,t)\right) \\
&= e^{-2it\xi\log|\xi|}\left[\xi\partial_\xi\hat{\varphi}(\xi,t) - (2it\xi - 1)\hat{\varphi}(\xi,t) - t\hat{\varphi}_t(\xi,t) - t\widehat{\mathcal{N}}(\xi,t) - \hat{\varphi}(\xi,t)\right] \\
&= e^{-2it\xi\log|\xi|}\left[-\widehat{\mathcal{S}\varphi}(\xi,t) - \hat{\varphi}(\xi,t) - t\widehat{\mathcal{N}}(\xi,t)\right],
\end{aligned} \tag{5.39}$$

where $\mathcal{N}$ denotes the nonlinear term in (5.4), which satisfies the estimate

$$\||\partial_x|^j \mathcal{N}\|_{L^2} \lesssim \sum_{n=1}^{\infty} (\|\varphi_x\|_{W^{2,\infty}}^{2n} + \|L\varphi_x\|_{W^{2,\infty}}^{2n}) \|\varphi\|_{H^{j+2}} \qquad \text{for all } j = 0, \dots, r. \tag{5.40}$$

By the bootstrap assumptions, Lemma 5.3.2, and Lemma 5.3.3 we then find that

$$(t+1)^{-p_0} \|x\partial_x h(t)\|_{H^r} \lesssim \varepsilon_0,$$

and the same estimate holds for φ in view of (5.37). □

Lemma 5.3.5 (Nonlinear dispersive estimate)**.** *Under the bootstrap assumptions, the solution of* (5.1) *satisfies*

$$\|\varphi(t)\|_Z \lesssim \varepsilon_0.$$

Proposition 5.3.1 then follows by combining Lemmas 5.3.2–5.3.5.

5.4. Linear dispersive estimate

In this section, we prove a dispersive estimate for the linearized evolution operator $e^{2t\partial_x \log|\partial_x|}$ defined in (5.37) and use it to prove Lemma 5.3.2. We recall that P_k and $\tilde{P}_k$ are the frequency-localization operators with symbols ς_k and $\tilde{\varsigma}_k$, respectively (see (2.17)).

Lemma 5.4.1. *For $t > 0$ and $f \in L^2$, we have the linear dispersive estimates*

$$\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} \lesssim (t+1)^{-1/2} 2^{k/2} \|\widehat{P_k f}\|_{L^\infty_\xi} + (t+1)^{-3/4} 2^{-k/4} \left[\|P_k(x\partial_x f)\|_{L^2} + \|\tilde{P}_k f\|_{L^2}\right]. \tag{5.41}$$

Proof. Using the inverse Fourier transform, we can write the solution as

$$e^{2t\partial_x \log|\partial_x|} P_k f = \int_{\mathbb{R}} e^{ix\xi + 2i(\xi \log|\xi|)t} \varsigma_k(\xi) \hat{f}(\xi)\, \mathrm{d}\xi.$$

Since

$$\partial_\xi e^{ix\xi + 2i(\xi\log|\xi|)t} = [ix + 2it(\log|\xi| + 1)] e^{ix\xi + 2i(\xi\log|\xi|)t}, \tag{5.42}$$

we can integrate by parts to get

$$\begin{aligned}
\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} &= \left\|\int_{\mathbb{R}} e^{ix\xi + 2i(\xi\log|\xi|)t} \hat{f}(\xi)\varsigma_k(\xi)\, \mathrm{d}\xi\right\|_{L^\infty} \\
&= \left\|\int_{\mathbb{R}} \frac{1}{[ix + 2it(\log|\xi| + 1)]} \partial_\xi e^{ix\xi + 2i(\xi\log|\xi|)t} \hat{f}(\xi)\varsigma_k(\xi)\, \mathrm{d}\xi\right\|_{L^\infty} \\
&= \left\|\int_{\mathbb{R}} e^{ix\xi + 2i(\xi\log|\xi|)t} \partial_\xi \left(\frac{1}{[ix + 2it(\log|\xi| + 1)]} \hat{f}(\xi)\varsigma_k(\xi)\right) \mathrm{d}\xi\right\|_{L^\infty} \\
&= \Bigg\| \int_{\mathbb{R}} e^{ix\xi + 2i(\xi\log|\xi|)t} \Bigg(\frac{-2it}{\xi[ix + 2it(\log|\xi| + 1)]^2} \hat{f}(\xi)\varsigma_k(\xi) \\
&\qquad + \frac{1}{[ix + 2it(\log|\xi| + 1)]} \varsigma_k(\xi)\partial_\xi \hat{f}(\xi) + \frac{1}{[ix + 2it(\log|\xi| + 1)]} \hat{f}(\xi)\varsigma_k'(\xi)\Bigg) \mathrm{d}\xi \Bigg\|_{L^\infty}.
\end{aligned}$$

1. If $|ix + 2it(\log|\xi| + 1)| \gtrsim (t+1)$, we use (2.18) and get

$$\begin{aligned}
\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} &\lesssim \frac{1}{(t+1)} \int_{\mathbb{R}} \left|\xi^{-1}\hat{f}(\xi)\varsigma_k(\xi) + \varsigma_k(\xi)\partial_\xi \hat{f}(\xi) + \hat{f}(\xi)\varsigma_k'(\xi)\right| \mathrm{d}\xi \\
&\lesssim \frac{1}{(t+1)} \left[2^{-k}\|\widehat{P_k f}\|_{L^2_\xi} + 2^{-k/2}\|P_k \mathcal{F}^{-1}(\xi\partial_\xi \hat{f})\|_{L^2} + 2^{-k/2}\|\tilde{P}_k f\|_{L^2}\right].
\end{aligned}$$

Then (5.41) follows when $(t+1)^{-1} \lesssim 2^k$. Otherwise, when $t+1 \lesssim 2^{-k}$, we have

$$\|e^{2t\partial_x \log|\partial_x|} P_k f\|_{L^\infty} \lesssim 2^k \|\widehat{P_k f}\|_{L^\infty_\xi} \lesssim (t+1)^{-1/2} 2^{k/2} \|\widehat{P_k f}\|_{L^\infty_\xi}.$$

2. Next we prove estimates for the case when $|ix + 2it(\log|\xi| + 1)| \ll (t+1)$. Let

$$\xi_0^\pm = \pm e^{-1-x/2t}$$

be the solutions of $x + 2t(\log|\xi| + 1) = 0$. Since ς_k is supported in an annulus with radius around 2^k, we only need to consider the case when $|\xi_0^\pm| \approx 2^k$ and ς_k is supported on the neighborhood of the stationary phase point $\xi_0^\pm$. We decompose the integral and estimate it as

$$\left| \int_{\mathbb{R}} e^{ix\xi + 2i(\xi \log|\xi|)t} \hat{f}(\xi) \varsigma_k(\xi) \, \mathrm{d}\xi \right| \lesssim \sum_{l \le k+N} \left[|J_l^+| + |J_l^-| \right],$$

with

$$J_l^\pm = \int_{\mathbb{R}} e^{ix\xi + 2i(\xi \log|\xi|)t} \hat{f}(\xi) \varsigma_k(\xi) \mathbf{1}_\pm(\xi) \varsigma_l(\xi - \xi_0^\pm) \, \mathrm{d}\xi,$$

where $\mathbf{1}_\pm$ is the indicator function supported on $\mathbb{R}_\pm$ and N is large enough that the support of ς_k is covered by the set $\bigcup_{l \le k+N} \{\xi \mid \varsigma_l(\xi - \xi_0^\pm) = 1\}$.

When $2^l \le 2^{k/2}(t+1)^{-1/2}$, we have

$$\sum_{2^l \le 2^{k/2}(t+1)^{-1/2}} |J_l^\pm| \lesssim \sum_{2^l \le 2^{k/2}(t+1)^{-1/2}} 2^l \|\widehat{P_k f}\|_{L^\infty} \le 2^{k/2}(t+1)^{-1/2} \|\widehat{P_k f}\|_{L^\infty}.$$

When $2^{k/2}(t+1)^{-1/2} \le 2^l \le 2^{k+N}$, since $|\xi - \xi_0| \approx 2^l$ and $|\xi_0| \approx 2^k$, we get the estimate

$$x + 2t(\log|\xi| + 1) = 2t \log\left|\frac{\xi}{\xi_0}\right| \approx 2t \log\left|1 \pm \frac{2^l}{2^k}\right|.$$

Using (5.42) and integration by parts, we have

$$\begin{aligned} |J_l^\pm| &\lesssim \frac{2^{k-l}}{(t+1)} \int_{\mathbb{R}} \left(|\partial_\xi \hat{f}(\xi)| + 2^{-l} |\hat{f}(\xi)| \right) \varsigma_l(\xi - \xi_0^\pm) \, \mathrm{d}\xi \\ &\lesssim \frac{2^{k-l}}{(t+1)} \|\hat{f}\|_{L^\infty} + \frac{2^{k-\frac{l}{2}}}{(t+1)} \|\partial_\xi \hat{f}\|_{L^2}. \end{aligned}$$

Then we take the sum of J_l over $2^l \ge 2^{k/2}(t+1)^{-1/2}$ to get the estimates (5.41). □

Proof of Lemma 5.3.2. Take the function f in Lemma 5.4.1 to be $\partial_x^j h$, $j = 1, 2, \dots, r+1$. Since $e^{2t\partial_x \log|\partial_x|}$ and P_k commute, and

$$x\partial_x^{j+1} h = \partial_x^j (x\partial_x h) - j\partial_x^j h,$$

we have that

$$\|P_k\partial_x^j\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}\|\mathcal{F}(P_k|\partial_x|^{\frac{1}{2}+j}\varphi)\|_{L^\infty_\xi} + (t+1)^{-3/4}[2^{\frac{3}{4}k}\||\partial_x|^{j-1}P_k(x\partial_x h)\|_{L^2} + \|\tilde{P}_k(|\partial_x|^{j-\frac{1}{4}}\varphi)\|_{L^2}].$$

It follows from (5.39) that

$$\||\partial_x|^{j-1}P_k(x\partial_x h)\|_{L^2} \lesssim \||\partial_x|^{j-1}P_k\varphi\|_{L^2} + \||\partial_x|^{j-1}P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k|\partial_x|^{j-1}\mathcal{N}\|_{L^2}.$$

We first observe that $k \in \mathbb{Z}_-$ automatically leads to $(t+1)^{-1/4+p_0}2^{\frac{3}{4}k} \lesssim 1$, and then we have

$$\begin{aligned}\|P_k\partial_x^j\varphi\|_{L^\infty} &\lesssim (t+1)^{-1/2}2^{k/2}\|\varsigma_k(\xi)|\xi|^j\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\ &\quad + (t+1)^{-1/2-p_0}[\||\partial_x|^{j-1}\tilde{P}_k\varphi\|_{L^2} + \||\partial_x|^{j-1}P_k\mathcal{S}\varphi\|_{L^2} + t\||\partial_x|^{j-1}P_k\mathcal{N}\|_{L^2}].\end{aligned} \tag{5.43}$$

For $k \in \mathbb{Z}_+$ and $(t+1)^{-1/4+p_0}2^{\frac{3}{4}k} \lesssim 1$, we have

$$\begin{aligned}\|P_k\partial_x^j\varphi\|_{L^\infty} &\lesssim (t+1)^{-1/2}2^{-3k/2}\|\varsigma_k(\xi)|\xi|^{j+2}\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\ &\quad + (t+1)^{-1/2-p_0}[\||\partial_x|^{j-1}\tilde{P}_k\varphi\|_{L^2} + \||\partial_x|^{j-1}P_k\mathcal{S}\varphi\|_{L^2} + t\||\partial_x|^{j-1}P_k\mathcal{N}\|_{L^2}].\end{aligned} \tag{5.44}$$

Finally, for $k \in \mathbb{Z}_+$ and $(t+1)^{-1/4+p_0}2^{\frac{3}{4}k} \gtrsim 1$, that is $2^{-k} \lesssim (t+1)^{-\frac{1}{3}+\frac{4}{3}p_0}$, we have

$$\begin{aligned}\|P_k\partial_x^j\varphi\|_{L^\infty} &\lesssim \||\xi|^j\varsigma_k(\xi)\hat{\varphi}(\xi)\|_{L^1_\xi} \lesssim \||\xi|^{j-s}\varsigma_k(\xi)\|_{L^2}\|\tilde{P}_k\varphi\|_{H^s} \\ &\lesssim 2^{(j-s-\frac{1}{2})k}\|\tilde{P}_k\varphi\|_{H^s} \lesssim (t+1)^{-1-p_0}\|\tilde{P}_k\varphi\|_{H^s}.\end{aligned} \tag{5.45}$$

Summing over $k \in \mathbb{Z}$, using (5.40), the bootstrap assumptions, and (5.43)–(5.45) in the corresponding ranges of k, and we obtain that

$$\|\varphi_x(t)\|_{W^{r+1,\infty}} \lesssim \varepsilon_1(t+1)^{-1/2}.$$

Similarly, to estimate $\|L\varphi_x\|_{W^{r+1,\infty}}$, we take the function f in Lemma 5.4.1 to be $L\partial_x^j h$ for $j = 1, 2, \dots, r+1$, and obtain

$$\|P_k\partial_x^j L\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}\|\mathcal{F}(P_k|\partial_x|^{\frac{1}{2}+j}L\varphi)\|_{L^\infty_\xi}$$

$$+ (t+1)^{-3/4}[2^{-k/4}\|P_k(x\partial_x^{j+1}Lh)\|_{L^2} + \|\tilde{P}_k(|\partial_x|^{j-\frac{1}{4}}L\varphi)\|_{L^2}].$$

Using

$$x\partial_x^{j+1}Lh = L(x\partial_x^{j+1}h) - [L,x]\partial_x^{j+1}h = L[\partial_x^j(x\partial_x h) - j\partial_x^j h] - [L,x]\partial_x^{j+1}h,$$

and (5.39), we get that

$$\begin{aligned}\|P_k\partial_x^j L\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}\|\varsigma_k(\xi)|\xi|^{\frac{1}{2}+j}\log|\xi|\hat{\varphi}(\xi)\|_{L^\infty_\xi} + (t+1)^{-3/4}2^{\frac{3}{4}k}\big[\||\partial_x|^{j-1}L\tilde{P}_k\varphi\|_{L^2} \\ + \||\partial_x|^{j-1}LP_k\mathcal{S}\varphi\|_{L^2} + \||\partial_x|^{j-1}P_k\varphi\|_{L^2} + t\||\partial_x|^{j-1}LP_k\mathcal{N}\|_{L^2}\big].\end{aligned}$$

If $j = 1, 2, \ldots, r+1$, then

$$|\xi|^{\frac{1}{2}+j}\log|\xi| \lesssim |\xi| + |\xi|^{r+3},$$

so for $k \in \mathbb{Z}_-$ and $(t+1)^{-1/4+p_0}2^{\frac{3}{4}k}|k| \lesssim 1$, we have

$$\begin{aligned}\|P_k\partial_x^j L\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}(2^{k/2}|k|)\|\varsigma_k(\xi)|\xi|^j\hat{\varphi}(\xi)\|_{L^\infty_\xi} + (t+1)^{-1/2-p_0}[\||\partial_x|^{j-1}\tilde{P}_k\varphi\|_{L^2} \\ + \||\partial_x|^{j-1}P_k\mathcal{S}\varphi\|_{L^2} + \||\partial_x|^{j-1}P_k\varphi\|_{L^2} + t\||\partial_x|^{j-1}P_k\mathcal{N}\|_{L^2}].\end{aligned} \tag{5.46}$$

For $k \in \mathbb{Z}_+$ and $(t+1)^{-1/4+p_0}2^{\frac{3}{4}k}|k| \lesssim 1$, we have

$$\begin{aligned}\|P_k\partial_x^j L\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}(2^{-3k/2}|k|)\|\varsigma_k(\xi)|\xi|^{j+2}\hat{\varphi}(\xi)\|_{L^\infty_\xi} + (t+1)^{-1/2-p_0}[\||\partial_x|^{j-1}\tilde{P}_k\varphi\|_{L^2} \\ + \||\partial_x|^{j-1}P_k\mathcal{S}\varphi\|_{L^2} + \||\partial_x|^{j-1}P_k\varphi\|_{L^2} + t\||\partial_x|^{j-1}P_k\mathcal{N}\|_{L^2}].\end{aligned} \tag{5.47}$$

Finally, for $k \in \mathbb{Z}_+$ and $(t+1)^{-1/4+p_0}2^{\frac{3}{4}k}|k| \gtrsim 1$, we have

$$\begin{aligned}\|P_k\partial_x^j L\varphi\|_{L^\infty} &\lesssim (|k|+1)\||\xi|^j\varsigma_k(\xi)\hat{\varphi}(\xi)\|_{L^1_\xi} \lesssim \||\xi|^{j-s}\varsigma_k(\xi)\|_{L^2}\|\tilde{P}_k\varphi\|_{H^s} \\ &\lesssim 2^{(j-s-\frac{1}{2})k}\|\tilde{P}_k\varphi\|_{H^s} \lesssim (t+1)^{-1}\|\tilde{P}_k\varphi\|_{H^s}.\end{aligned} \tag{5.48}$$

Summing over $k \in \mathbb{Z}$, using (5.40), the bootstrap assumptions, and (5.46)–(5.48) in the corresponding range of k, and we obtain that

$$\|L\varphi_x(t)\|_{W^{r+1,\infty}} \lesssim \varepsilon_1(t+1)^{-1/2},$$

which concludes the proof of the lemma. □

5.5. Scaling-Galilean estimate

In this section, we prove the scaling-Galilean estimate in Lemma 5.3.3.

First, we summarize some commutator identities for the scaling-Galilean operator $\mathcal{S}$ defined in (5.3) and $L = \log|\partial_x|$. The straightforward proofs follow by use of the Fourier transform and are omitted.

Lemma 5.5.1. *Let $\varphi(x,t)$ be a Schwartz distribution on $\mathbb{R}^2$ such that $L\varphi(x,t)$ is a Schwartz distribution. Then*

$$[\mathcal{S},\partial_x]\varphi = -\partial_x\varphi, \qquad [\mathcal{S},L]\varphi = -\varphi, \qquad [\mathcal{S},L\partial_x]\varphi = -\varphi_x - L\partial_x\varphi,$$
$$[\mathcal{S},\partial_t]\varphi = -2\partial_x\varphi - \partial_t\varphi, \qquad [\mathcal{S},\partial_t - 2L\partial_x]\varphi = -\partial_t\varphi + 2L\partial_x\varphi.$$

Next, we prove a weighted energy estimate for $\mathcal{S}\varphi$.

Proof of Lemma 5.3.3. Applying $\mathcal{S}$ to equation (5.11) and using Lemma 5.5.1, we get

$$(\mathcal{S}\varphi)_t - 2L\partial_x(\mathcal{S}\varphi) + \partial_x T_{B^0[\varphi]}\mathcal{S}\varphi + L[T_{B^{\log}[\varphi]}\mathcal{S}\varphi]_x + \mathcal{S}\mathcal{R} = \text{commutators},$$

where the commutators are

$$\partial_x[\mathcal{S},T_{B^0[\varphi]}]\varphi, \qquad [\mathcal{S},\partial_x]T_{B^0[\varphi]}\varphi, \qquad [\mathcal{S},L\partial_x]\big(T_{B^{\log}[\varphi]}\varphi\big), \qquad L\partial_x\Big([\mathcal{S},T_{B^{\log}[\varphi]}]\varphi\Big).$$

By the commutator estimate Lemma 2.2.1 and (5.13), we obtain for $k \le r$ that

$$\begin{aligned}
\|\partial_x[\mathcal{S},T_{B^0[\varphi]}]\varphi\|_{H^k} &\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})\|\varphi_x\|_{W^{r,\infty}}\|\mathcal{S}\varphi\|_{H^r}\\
&\lesssim F(\|L\varphi\|_{W^{2,\infty}} + \|\varphi\|_{W^{2,\infty}})(\|L\varphi\|_{W^{r,\infty}} + \|\varphi\|_{W^{r,\infty}})^2\|\mathcal{S}\varphi\|_{H^r},\\
\|[\mathcal{S},\partial_x]T_{B^0[\varphi]}\varphi\|_{H^k} &= \|T_{B^0[\varphi]}\varphi\|_{H^{k+1}}\\
&\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})^2\|\varphi\|_{H^{k+1}},\\
\big\|[\mathcal{S},L\partial_x]\big(T_{B^{\log}[\varphi]}\varphi\big)\big\|_{H^k} &= \big\|\big(T_{B^{\log}[\varphi]}\varphi\big)_x + L\partial_x\big(T_{B^{\log}[\varphi]}\varphi\big)\big\|_{H^k}\\
&\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})^2\\
&\qquad\qquad \cdot(\|\varphi\|_{H^{k+1}} + \|L\varphi\|_{H^{k+1}}),\\
\Big\|L\partial_x\Big([\mathcal{S},T_{B^{\log}[\varphi]}]\varphi\Big)\Big\|_{H^k} &\lesssim F(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}})^2\|\mathcal{S}\varphi\|_{H^r}.
\end{aligned}$$

Thus, the evolution equation for $\mathcal{S}\varphi$ can be written as

$$(\mathcal{S}\varphi)_t + \partial_x T_{B^0[\varphi]}\mathcal{S}\varphi + \mathcal{R}_{\mathcal{S}} = L\big[(2 - T_{B^{\log}[\varphi]})\mathcal{S}\varphi\big],$$

where the remainder $\mathcal{R}_{\mathcal{S}}$ satisfies

$$\|\mathcal{R}_{\mathcal{S}}\|_{H^r} \lesssim \left(\|\varphi\|_{W^{r+1,\infty}} + \|L\varphi\|_{W^{r+1,\infty}}\right)^2 \left(\|\mathcal{S}\varphi\|_{H^r} + \|\varphi\|_{H^s}\right).$$

As in (5.27), we define a weighted energy for $\mathcal{S}\varphi$ by

$$\begin{aligned} E_{\mathcal{S}}^{(j)}(t) &= \int_{\mathbb{R}} |D|^j \mathcal{S}\varphi(x,t) \cdot \left(2 - T_{B^{\log}[\varphi]}\right)^{2j+1} |D|^j \mathcal{S}\varphi(x,t)\, \mathrm{d}x, \qquad j = 0, 1, \cdots, r, \\ \tilde{E}_{\mathcal{S}}^{(r)}(t) &= \sum_{j=0}^{r} E_{\mathcal{S}}^{(j)}(t), \end{aligned}$$

and repeat similar estimates to the ones in the proof of Proposition 5.2.1 to get

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathcal{S}}^{(j)}(t) \lesssim (\|\varphi_x\|_{W^{r,\infty}} + \|L\varphi_x\|_{W^{r,\infty}})^2 F\left(\|\mathcal{S}\varphi\|_{H^r} + \|\varphi\|_{H^s}\right) \|\mathcal{S}\varphi\|_{H^r}.$$

Using Lemma 5.3.2 and the equivalence of $\tilde{E}_{\mathcal{S}}^{(r)}$ and $\|\mathcal{S}\varphi\|_{H^r}^2$ when $\|2 - T_{B^{\log}[\varphi]}\|_{L^2 \to L^2}$ is bounded away from zero, we find by integrating in t that

$$\tilde{E}_{\mathcal{S}}^{(r)}(t) \lesssim \varepsilon_0^2 (t+1)^{2p_0},$$

which proves the lemma. □

5.6. Nonlinear dispersive estimate

In this section, we prove the estimate in Lemma 5.3.5 for the Z-norm $\|\varphi\|_Z$ defined in (5.38).

When $|\xi| < (t+1)^{-p_0}$, Lemma 2.3.1 and the bootstrap assumptions give

$$\begin{aligned} |(|\xi| + |\xi|^{r+3})\hat{\varphi}(\xi,t)|^2 &\lesssim (|\xi| + |\xi|^{r+3})^2 |\xi|^{-1} \|\hat{\varphi}\|_{L^2_\xi} (|\xi| \|\partial_\xi \hat{\varphi}\|_{L^2_\xi} + \|\hat{\varphi}\|_{L^2_\xi}) \\ &\lesssim (|\xi| + |\xi|^{r+3}) \|\varphi\|_{L^2} (\|\mathcal{S}\varphi\|_{L^2} + \|\varphi\|_{L^2}) \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

Let $p_1 = 10^{-6}$. When $|\xi| \geq (t+1)^{p_1}$, Lemma 2.3.1 and the bootstrap assumptions, with the parameter values (5.2), give

$$\begin{aligned} |(|\xi| + |\xi|^{r+3})\hat{\varphi}(\xi,t)|^2 &\lesssim \frac{(|\xi| + |\xi|^{r+3})^2}{|\xi|^{s+1}} \|\varphi\|_{H^s} (\|\mathcal{S}\varphi\|_{L^2} + \|\varphi\|_{L^2}) \\ &\lesssim |\xi|^{2r+5-s} \varepsilon_0^2 (t+1)^{2p_0} \\ &\lesssim \varepsilon_0^2. \end{aligned}$$

Thus, we only need to consider the frequency range

$$(t+1)^{-p_0} \leq |\xi| \leq (t+1)^{p_1}. \tag{5.49}$$

In the following, we fix ξ in this range, and denote by $\mathfrak{d}(\xi,t)$ a smooth cut-off function compactly supported on a small neighborhood of $\{(\xi,t) : (t+1)^{-p_0} < |\xi| < (t+1)^{p_1}\}$.

Taking the Fourier transform of (5.8), we obtain that

$$\hat{\varphi}_t(\xi) + \frac{1}{6} i\xi \iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{\varphi}(\xi-\eta_1-\eta_2)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 + \widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi) = 2i\xi \log|\xi|\hat{\varphi}(\xi), \tag{5.50}$$

where $\mathcal{N}_{\geq 5}(\varphi)$ is given by (5.9). From (A.3),

$$\begin{aligned}\mathbf{T}_1(\eta_1,\eta_2,\eta_3) = &-\eta_1^2\log|\eta_1| - \eta_2^2\log|\eta_2| - \eta_3^2\log|\eta_3| - (\eta_1+\eta_2+\eta_3)^2\log|\eta_1+\eta_2+\eta_3| \\ &+ \left\{(\eta_1+\eta_2)^2\log|\eta_1+\eta_2| + (\eta_1+\eta_3)^2\log|\eta_1+\eta_3| + (\eta_2+\eta_3)^2\log|\eta_2+\eta_3|\right\}.\end{aligned}$$

with

$$\begin{aligned}\partial_{\eta_1}[\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)] = -2\Big\{&\eta_1\log|\eta_1| - (\eta_1+\eta_2)\log|\eta_1+\eta_2| \\ &+ (\xi-\eta_1)\log|\xi-\eta_1| - (\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2|\Big\}, \\ \partial_{\eta_2}[\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)] = -2\Big\{&\eta_2\log|\eta_2| - (\eta_1+\eta_2)\log|\eta_1+\eta_2| \\ &+ (\xi-\eta_2)\log|\xi-\eta_2| - (\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2|\Big\}.\end{aligned} \tag{5.51}$$

5.6.1. Modified scattering. Nonlinearity leads to a cumulative frequency shift in the long-time behavior of the Fourier components of the solution due to space-time resonances of the form $\xi+\xi-\xi=\xi$. To account for this effect, we use the method of modified scattering and introduce a phase correction

$$\Theta(\xi,t) = -2t\xi\log|\xi| + \xi\int_0^t [\beta_1(t)\mathbf{T}_1(\xi,\xi,-\xi) + \beta_2(t)\mathbf{T}_1(\xi,-\xi,\xi) + \beta_3(t)\mathbf{T}_1(-\xi,\xi,\xi)]|\hat{\varphi}(\xi,\tau)|^2\,\mathrm{d}\tau,$$

where $\beta_1(t), \beta_2(t)$, and $\beta_3(t)$ are real-valued functions of t to be determined later. We then let

$$\hat{v}(\xi,t) = e^{i\Theta(\xi,t)}\hat{\varphi}(\xi,t).$$

Using (5.50), we find that

$$
\begin{aligned}
\hat{v}_t(\xi,t) &= e^{i\Theta(\xi,t)}[\hat{\varphi}_t(\xi,t) + i\Theta_t(\xi,t)\hat{\varphi}(\xi,t)] \\
&= U_1(\xi,t) + U_2(\xi,t) - e^{i\Theta(\xi,t)}\widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi,t),
\end{aligned}
\tag{5.52}
$$

where

$$
\begin{aligned}
U_1(\xi,t) = e^{i\Theta(\xi,t)}\bigg\{ &-\frac{1}{6}i\xi \iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{\varphi}(\xi-\eta_1-\eta_2,t)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&+ i\xi\left[\beta_1(t)\mathbf{T}_1(\xi,\xi,-\xi) + \beta_2(t)\mathbf{T}_1(\xi,-\xi,\xi) + \beta_3(t)\mathbf{T}_1(-\xi,\xi,\xi)\right]|\hat{\varphi}(\xi,t)|^2\hat{\varphi}(\xi,t)\bigg\}, \\
U_2(\xi,t) = \hat{v}(\xi,t)\bigg\{ &i\xi\int_0^t \left[\beta_1'(t)\mathbf{T}_1(\xi,\xi,-\xi) + \beta_2'(t)\mathbf{T}_1(\xi,-\xi,\xi) + \beta_3'(t)\mathbf{T}_1(-\xi,\xi,\xi)\right]|\hat{\varphi}(\xi,\tau)|^2\,\mathrm{d}\tau\bigg\}.
\end{aligned}
$$

The coefficient of $\hat{v}$ in the term U_2 is purely imaginary, so it leads to a phase shift in $\hat{v}$ that does not affect its norm, and we get from (5.52) that

$$
\begin{aligned}
\|\varphi\|_Z &= \|(|\xi| + |\xi|^{r+3})\hat{\varphi}(\xi,t)\|_{L^\infty_\xi} = \|(|\xi| + |\xi|^{r+3})\hat{v}(\xi,t)\|_{L^\infty_\xi} \\
&\lesssim \int_0^t \|(|\xi| + |\xi|^{r+3})U_1(\xi,\tau)\|_{L^\infty_\xi} + \|(|\xi| + |\xi|^{r+3})\widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi,\tau)\|_{L^\infty_\xi}\,\mathrm{d}\tau.
\end{aligned}
$$

We will estimate the cubic terms involving U_1 in Sections 5.6.2–5.6.5 and the higher-degree terms involving $\widehat{\mathcal{N}_{\geq 5}(\varphi)}$ in Section 5.6.6. We do not need to consider the terms in U_1 that involve the β_j until we come to an analysis of the space-time resonances in Section 5.6.5.

To begin with, we recall that $h = e^{-2t\partial_x \log|\partial_x|}\varphi$ is defined in (5.37). From (5.50), we find that $\hat{h}$ satisfies

$$
\begin{aligned}
\hat{h}_t(\xi,t) + \frac{1}{6}i\xi \iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi-\eta_1-\eta_2,t)\hat{h}(\eta_1,t)\hat{h}(\eta_2,t)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
+ e^{-2it\xi\log|\xi|}\widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi,t) = 0,
\end{aligned}
\tag{5.53}
$$

where

$$
\Phi(\xi,\eta_1,\eta_2) = 2(\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2| + 2\eta_1\log|\eta_1| + 2\eta_2\log|\eta_2| - 2\xi\log|\xi|. \tag{5.54}
$$

Suppressing the dependence on the time variable t, we can write the integral in U_1 involving φ in terms of h as

$$\begin{aligned}\iint_{\mathbb{R}^2} &\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)\hat{\varphi}(\xi - \eta_1 - \eta_2)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\ &= \iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi - \eta_1 - \eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.\end{aligned}$$

Carrying out a dyadic decomposition, with $h_j = P_j h$ and $\varphi_j = P_j\varphi$ where P_j is the Fourier multiplier with symbol ς_j defined in (2.17), we rewrite this integral in each dyadic block as

$$\iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2. \tag{5.55}$$

In the following subsections, we estimate this integral in various regions of frequency-space.

In Section 5.6.2, we estimate the integral for high frequencies (large j_1, j_2, and j_3). In Section 5.6.3, we estimate the integral for nonresonant frequencies, using oscillatory integral estimates with respect to the frequency variables together with multilinear estimates to get sufficient time decay.

In Section 5.6.4, we consider frequencies that are close to the resonant frequencies. In that case, the bounds for the multilinear symbols are worse, so we cannot obtain sufficient time decay by the method used for the nonresonant frequencies. We resolve this issue by an additional dyadic decomposition centered at each resonant point and a refinement of the symbol estimates.

Finally, in Section 5.6.5, we consider frequencies that are at the space resonance or space-time resonances. For the space resonance, we estimate the integral in a region about the space resonance point that shrinks in time, using an oscillatory integral estimate with respect to time and the equation to eliminate the time-derivative of the solution. For the space-time resonances, we take advantage of the modified scattering phase correction and estimate the integral on shrinking regions about the space-time resonance points.

5.6.2. High frequencies. When $\max\{j_1, j_2, j_3\} \gtrsim 10^{-3}\log_2|t+1| > 0$, we can estimate the nonlinear terms (5.55) by using Lemma 2.3.2, with the L^∞-norm placed on the lowest derivative term. There are, in total, $r + 6 = 13$ derivatives shared by three factors of φ. Thus, we can ensure that the term with least derivatives has at most four derivatives, with or without a logarithmic derivative.

To be more specific, using Hölder's inequality, Sobolev embedding, and the bootstrap assumptions, we obtain the estimate

$$
\begin{aligned}
&\left\|\xi(|\xi|+|\xi|^{r+3})\mathfrak{d}(\xi,t)\iint_{\mathbb{R}^2}\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\|_{L^\infty_\xi}\\
&\lesssim (t+1)^{(r+7-s)10^{-3}}\|\varphi_{\min}\|_{L^2}(\|\varphi_{\text{med}}\|_{L^\infty}+\|L\partial_x\varphi_{\text{med}}\|_{W^{r,\infty}})\|\varphi_{\max}\|_{H^s}\\
&\lesssim (t+1)^{(r+7-s)10^{-3}}\|\varphi_{j_1}\|_{H^s}\|\varphi_{j_2}\|_{H^s}\|\varphi_{j_3}\|_{H^s},
\end{aligned}
$$

where max, med, min represent the maximum, median, and the minimum of j_1, j_2, j_3, and $\mathfrak{d}$ is the cut-off function for the frequency range (5.49). From (5.2), we have $(r+7-s)10^{-3} < -1.1$, so the right-hand-side is summable over j_1, j_2, j_3, and the sum is integrable for $t \in (0,\infty)$.

5.6.3. Nonresonant frequencies. We now only need to consider when $\max\{j_1, j_2, j_3\} < 10^{-3}\log_2(t+1)$. The regions $|j_1 - j_3| > 1$ or $|j_2 - j_3| > 1$ correspond to nonresonant frequencies. Without loss of generality, we assume $|j_1 - j_3| > 1$.

Notice that by (5.54), we have

$$
\partial_{\eta_1}\Phi = 2\log|\eta_1| - 2\log|\xi-\eta_1-\eta_2|. \tag{5.56}
$$

Since $|\eta_1|$ and $|\xi-\eta_1-\eta_2|$ are in different dyadic blocks, we have $\big||\eta_1|-|\xi-\eta_1-\eta_2|\big| \gtrsim \max\{|\eta_1|, |\xi-\eta_1-\eta_2|\}$. Therefore, $|\partial_{\eta_1}\Phi| \gtrsim 1$.

After integrating by part, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^2}\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\\
=&\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\\
=&-W_1-W_2-W_3,
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(\xi,t) &= \iint_{\mathbb{R}^2}\partial_{\eta_1}\left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,\\
W_2(\xi,t) &= \iint_{\mathbb{R}^2}\left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\partial_{\eta_1}\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,\\
W_3(\xi,t) &= \iint_{\mathbb{R}^2}\left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right]e^{it\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}
$$

Estimate of W_1. Since

$$\|W_1\|_{L^\infty_\xi} \lesssim \|\mathcal{F}^{-1}(W_1)\|_{L^1}, \tag{5.57}$$

it suffices to estimate the L^1_x norm of

$$\iiint_{\mathbb{R}^3} e^{i\xi x}\partial_{\eta_1}\left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)}\right] e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\xi.$$

Notice that by (5.56)

$$\partial_{\eta_1}\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)} = \kappa_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2) - \frac{\kappa_2(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{2},$$

where

$$\kappa_1(\eta_1,\eta_2,\eta_3) = \frac{\partial_{\eta_1}\mathbf{T}_1(\eta_1,\eta_2,\eta_3) - \partial_{\eta_3}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\log|\eta_1| - \log|\eta_3|},$$
$$\kappa_2(\eta_1,\eta_2,\eta_3) = \mathbf{T}_1(\eta_1,\eta_2,\eta_3)\frac{\frac{1}{\eta_1}+\frac{1}{\eta_3}}{(\log|\eta_1|-\log|\eta_3|)^2}.$$

Making a change of variable $\eta_3 = \xi - \eta_1 - \eta_2$, we need to estimate the trilinear form

$$\frac{1}{it}\iiint_{\mathbb{R}^3} e^{i\xi x}\left[\kappa_1(\eta_1,\eta_2,\eta_3)+\kappa_2(\eta_1,\eta_2,\eta_3)\right]\hat{\varphi}_{j_1}(\eta_1)\hat{\varphi}_{j_2}(\eta_2)\hat{\varphi}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3,$$

with symbol

$$\left[\kappa_1(\eta_1,\eta_2,\eta_3)+\kappa_2(\eta_1,\eta_2,\eta_3)\right]\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3).$$

According to Lemma 2.3.2, this trilinear operator is bounded on $L^2\times L^2\times L^\infty \to L^1$ by

$$\begin{aligned}
&\|\left[\kappa_1(\eta_1,\eta_2,\eta_3)+\kappa_2(\eta_1,\eta_2,\eta_3)\right]\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3)\|_{S^\infty}\\
\lesssim&\Big(\|\partial_{\eta_1}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\|_{S^\infty}\\
&\qquad+\|\partial_{\eta_3}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\|_{S^\infty}\Big)\cdot\left\|\frac{\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3)}{\log|\eta_1|-\log|\eta_3|}\right\|_{S^\infty}\\
&+\left(\left\|\frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\right\|_{S^\infty}\right.\\
&\qquad\left.+\left\|\frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_3}\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\right\|_{S^\infty}\right)\cdot\left\|\frac{\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3)}{(\log|\eta_1|-\log|\eta_3|)^2}\right\|_{S^\infty}.
\end{aligned}\tag{5.58}$$

Lemma 5.6.1. *Suppose that $|j_1 - j_3| > 1$. Then for any $m\in\mathbb{Z}_+$,*

$$\left\|\frac{1}{(\log|\eta_1|-\log|\eta_3|)^m}\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3)\right\|_{S^\infty}\lesssim 1.$$

Proof. By the definition of the S^∞-norm (2.19) and the definition of ς_k (2.17), we have that

$$\begin{aligned}
&\left\| \frac{\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3)}{(\log|\eta_1| - \log|\eta_3|)^m} \right\|_{S^\infty} \\
=& \left\| \iiint_{\mathbb{R}^3} \frac{\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\varsigma_{j_3}(\eta_3)}{(\log|\eta_1| - \log|\eta_3|)^m} e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, \mathrm{d}\eta_3 \right\|_{L^1} \\
=& \iiint_{\mathbb{R}^3} \left| \iiint_{\mathbb{R}^3} \frac{\varsigma_0(2^{-j_1}\eta_1)\varsigma_0(2^{-j_2}\eta_2)\varsigma_0(2^{-j_3}\eta_3)}{(\log|\eta_1| - \log|\eta_3|)^m} e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, \mathrm{d}\eta_3 \right| \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3 \\
\lesssim& \ 1,
\end{aligned}$$

where the last inequality comes from oscillatory integral estimates, using the fact that $|j_1 - j_3| > 1$ and the support of ς_0 is $(-\frac{8}{5}, -\frac{5}{8}) \cup (\frac{5}{8}, \frac{8}{5})$. ☐

For the estimates of other symbols in (5.58), we have the following lemma.

Lemma 5.6.2. *For any $j_1, j_2, j_3 \in \mathbb{Z}$, we have*

$$\|\partial_{\eta_1}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\|_{S^\infty} \lesssim 2^{\max\{j_2,j_3\}}, \tag{5.59}$$

$$\|\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\|_{S^\infty} \lesssim 2^{\max\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}}, \tag{5.60}$$

and

$$\left\| \frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_2,j_3\}}. \tag{5.61}$$

Furthermore, since $\mathbf{T}_1$ is symmetric, we also have

$$\|\partial_{\eta_3}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\|_{S^\infty} \lesssim 2^{\max\{j_1,j_2\}},$$

$$\left\| \frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_3}\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_1,j_2\}}.$$

Proof. **1.** We prove (5.59) first. Using inverse Fourier transform in (η_1,η_2,η_3), we obtain

$$\begin{aligned}
&\mathcal{F}^{-1}[\partial_{\eta_1}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)] \\
=& \iiint_{\mathbb{R}^3} e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\partial_{\eta_1}\left[\int_{\mathbb{R}} \frac{\prod_{j=1}^3(1-e^{i\eta_j\zeta})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3 \\
=& \iiint_{\mathbb{R}^3}\left[\int_{\mathbb{R}} \frac{-i\zeta e^{i\eta_1(\zeta+y_1)}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})(e^{iy_3\eta_3}-e^{i\eta_3(\zeta+y_3)})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3 \\
=& \int_{\mathbb{R}} \frac{-i\zeta}{|\zeta|^3}\cdot\left[\mathcal{F}^{-1}[\tilde{\varsigma}_{j_1}](y_1+\zeta)\right]\cdot\left[\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](y_2)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](\zeta+y_2)\right]\left[\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](y_3)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](\zeta+y_3)\right]\mathrm{d}\zeta.
\end{aligned}$$

Notice that

$$\left|\mathcal{F}^{-1}[\tilde{\varsigma}_{j_1}](y_1+\zeta)\right| = 2^{j_1}\left|\mathcal{F}^{-1}[\tilde{\varsigma}_0](2^{j_1}(y_1+\zeta))\right|,$$
$$\left|\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](y_2)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](\zeta+y_2)\right| = 2^{j_2}\left|\mathcal{F}^{-1}[\tilde{\varsigma}_0](2^{j_2}y_2)-\mathcal{F}^{-1}[\tilde{\varsigma}_0](2^{j_2}(\zeta+y_2))\right|,$$
$$\left|\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](y_3)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](\zeta+y_3)\right| = 2^{j_3}\left|\mathcal{F}^{-1}[\tilde{\varsigma}_0](2^{j_3}y_3)-\mathcal{F}^{-1}[\tilde{\varsigma}_0](2^{j_3}(\zeta+y_3))\right|,$$

and that

$$\int_{\mathbb{R}}\left|\mathcal{F}^{-1}[\tilde{\varsigma}_0](2^{j_1}(y_1+\zeta))\right|\,\mathrm{d}y_1 \lesssim 2^{-j_1},$$
$$\int_{\mathbb{R}}\left|\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](2^{j_2}y_2)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](2^{j_2}(\zeta+y_2))\right|\,\mathrm{d}y_2 \lesssim \min\{2^{-j_2},|\zeta|\},$$
$$\int_{\mathbb{R}}\left|\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](2^{j_3}y_3)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](2^{j_3}(\zeta+y_3))\right|\,\mathrm{d}y_3 \lesssim \min\{2^{-j_3},|\zeta|\}.$$

Therefore, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}[\partial_{\eta_1}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)]\right\|_{L^1}\\
\lesssim\ &\int_{\mathbb{R}}\frac{1}{|\zeta|^2}2^{j_2+j_3}\min\{2^{-j_2},|\zeta|\}\min\{2^{-j_3},|\zeta|\}\,\mathrm{d}\zeta\\
=\ &2^{j_2+j_3}\Bigg(\int_{|\zeta|>\max\{2^{-j_2},2^{-j_3}\}}\frac{1}{|\zeta|^2}2^{-j_2-j_3}\,\mathrm{d}\zeta+\int_{\min\{2^{-j_2},2^{-j_3}\}<|\zeta|<\max\{2^{-j_2},2^{-j_3}\}}\frac{1}{|\zeta|}\min\{2^{-j_2},2^{-j_3}\}\,\mathrm{d}\zeta\\
&\quad+\int_{|\zeta|<\min\{2^{-j_2},2^{-j_3}\}}1\,\mathrm{d}\zeta\Bigg)\\
\lesssim\ &2^{\max\{j_2,j_3\}}.
\end{aligned}$$

2. Next, we prove (5.60) and (5.61). The estimate of (5.60) is similarly to (5.59). We first use inverse Fourier transform and write

$$\begin{aligned}
&\mathcal{F}^{-1}\left[\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\right]\\
=\ &\iiint_{\mathbb{R}^3}e^{i(y_1\eta_1+y_2\eta_2+y_3\eta_3)}\left[\int_{\mathbb{R}}\frac{\prod_{j=1}^3(1-e^{i\eta_j\zeta})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\\
=\ &\iiint_{\mathbb{R}^3}\left[\int_{\mathbb{R}}\frac{(e^{iy_1\eta_1}-e^{i\eta_1(\zeta+y_1)})(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})(e^{iy_3\eta_3}-e^{i\eta_3(\zeta+y_3)})}{|\zeta|^3}\,\mathrm{d}\zeta\right]\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\\
=\ &\int_{\mathbb{R}}\frac{1}{|\zeta|^3}\left[\mathcal{F}^{-1}[\tilde{\varsigma}_{j_1}](y_1)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_1}](\zeta+y_1)\right]\\
&\quad\cdot\left[\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](y_2)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_2}](\zeta+y_2)\right]\cdot\left[\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](y_3)-\mathcal{F}^{-1}[\tilde{\varsigma}_{j_3}](\zeta+y_3)\right]\,\mathrm{d}\zeta.
\end{aligned}$$

Taking the L^1-norm, we obtain

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}[\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)]\right\|_{L^1} \\
\lesssim\ & \int_{\mathbb{R}} 2^{j_1+j_2+j_3}\frac{1}{|\zeta|^3}\min\{2^{-j_1},|\zeta|\}\min\{2^{-j_2},|\zeta|\}\min\{2^{-j_3},|\zeta|\}\,\mathrm{d}\zeta \\
\lesssim\ & \int_{|\zeta|>\max\{2^{-j_1},2^{-j_2},2^{-j_3}\}}\frac{1}{|\zeta|^3}\,\mathrm{d}\zeta + \int_{|\zeta|<\min\{2^{-j_1},2^{-j_2},2^{-j_3}\}} 2^{j_1+j_2+j_3}\,\mathrm{d}\zeta \\
&+ \int_{\min\{2^{-j_1},2^{-j_2},2^{-j_3}\}<|\zeta|<\mathrm{med}\{2^{-j_1},2^{-j_2},2^{-j_3}\}} 2^{\mathrm{med}\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}}\frac{1}{|\zeta|}\,\mathrm{d}\zeta \\
&+ \int_{\mathrm{med}\{2^{-j_1},2^{-j_2},2^{-j_3}\}<|\zeta|<\max\{2^{-j_1},2^{-j_2},2^{-j_3}\}} 2^{\min\{j_1,j_2,j_3\}}\frac{1}{|\zeta|^2}\,\mathrm{d}\zeta \\
\lesssim\ & 2^{2\min\{j_1,j_2,j_3\}} + 2^{\max\{j_1,j_2,j_3\}+\mathrm{med}\{j_1,j_2,j_3\}} + 2^{\max\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}} + 2^{\min\{j_1,j_2,j_3\}+\mathrm{med}\{j_1,j_2,j_3\}} \\
\lesssim\ & 2^{\max\{j_1,j_2,j_3\}+\min\{j_1,j_2,j_3\}},
\end{aligned}
$$

which proves (5.60).

As for (5.61), we define

$$\tilde{\tilde{\varsigma}}_k(\eta) := \sum_{j=k-3}^{k+3}\varsigma_j(\eta).$$

Then it follows from the support of ς_k and the fact that ς_k forms a partition of unity that

$$\frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3) = [\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)]\cdot\left[\frac{1}{\eta_1}\tilde{\tilde{\varsigma}}_{j_1}(\eta_1)\tilde{\tilde{\varsigma}}_{j_2}(\eta_2)\tilde{\tilde{\varsigma}}_{j_3}(\eta_3)\right].$$

By Lemma 2.3.2, we have

$$
\begin{aligned}
&\left\|\frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{\eta_1}\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\right\|_{S^\infty} \\
\lesssim\ & \|\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\tilde{\varsigma}_{j_1}(\eta_1)\tilde{\varsigma}_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3)\|_{S^\infty}\left\|\frac{\tilde{\tilde{\varsigma}}_{j_1}(\eta_1)\tilde{\tilde{\varsigma}}_{j_2}(\eta_2)\tilde{\tilde{\varsigma}}_{j_3}(\eta_3)}{\eta_1}\right\|_{S^\infty}.
\end{aligned}
\tag{5.62}
$$

In view of (5.60), we only need to estimate the second term. To this end, we have

$$\left\|\frac{1}{\eta_1}\tilde{\tilde{\varsigma}}_{j_1}(\eta_1)\tilde{\tilde{\varsigma}}_{j_2}(\eta_2)\tilde{\tilde{\varsigma}}_{j_3}(\eta_3)\right\|_{S^\infty} = \left\|\int_{\mathbb{R}}\eta_1^{-1}\tilde{\tilde{\varsigma}}_{j_1}(\eta_1)e^{i\eta_1 y_1}\,\mathrm{d}\eta_1\mathcal{F}^{-1}[\tilde{\tilde{\varsigma}}_{j_2}](y_2)\mathcal{F}^{-1}[\tilde{\tilde{\varsigma}}_{j_3}](y_3)\right\|_{L^1}\lesssim 2^{-j_1}.$$

Therefore, by (5.62) and considering all the possible relations between j_1, j_2, and j_3, we obtain (5.61). □

Applying the above lemmas to (5.58) and (5.57), we obtain

$$\|W_1\|_{L^\infty_\xi} \lesssim (t+1)^{-1}\big[\|\partial_x \varphi_{\max\{j_1,j_2\}}\|_{L^\infty}\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2} + \|\partial_x \varphi_{\max\{j_2,j_3\}}\|_{L^\infty}\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\big].$$

Since the two terms are symmetric in j_1 and j_3, it suffices to estimate one of them, as the other one is similar. We use Lemma 5.4.1 and get

$$\|\partial_x \varphi_{\max\{j_1,j_2\}}\|_{L^\infty} \lesssim (t+1)^{-1/2}\||\xi|^{1.5}\hat{h}_{\max\{j_1,j_2\}}\|_{L^\infty_\xi} + (t+1)^{-3/4}\left[\||\partial_x|^{0.75}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{0.75}h_{\max\{j_1,j_2\}}\|_{L^2}\right].$$

Therefore,

$$\begin{aligned}
&\|W_1\|_{L^\infty_\xi}\\
&\lesssim (t+1)^{-1.5}\Big(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{0.5\max\{j_1,j_2\}}\||\xi|\hat{h}_{\max\{j_1,j_2\}}\|_{L^\infty_\xi}\\
&\quad + \mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-1.5-r)\max\{j_1,j_2\}}\||\xi|^{r+3}\hat{h}_{\max\{j_1,j_2\}}\|_{L^\infty_\xi}\Big)\cdot\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\quad + (t+1)^{-1.75}\Big(\||\partial_x|^{0.75}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{0.75}h_{\max\{j_1,j_2\}}\|_{L^2}\Big)\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\quad + (t+1)^{-1.5}\Big(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{0.5\max\{j_2,j_3\}}\||\xi|\hat{h}_{\max\{j_2,j_3\}}\|_{L^\infty_\xi}\\
&\quad + \mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-1.5-r)\max\{j_2,j_3\}}\||\xi|^{r+3}\hat{h}_{\max\{j_2,j_3\}}\|_{L^\infty_\xi}\Big)\cdot\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\\
&\quad + (t+1)^{-1.75}\Big(\||\partial_x|^{0.75}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{0.75}h_{\max\{j_2,j_3\}}\|_{L^2}\Big)\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\\
&\lesssim (t+1)^{-1.5}\Big(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{0.5\max\{j_1,j_2\}} + \mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-1.5-r)\max\{j_1,j_2\}}\Big)\\
&\qquad\qquad\cdot\|h_{\max\{j_1,j_2\}}\|_Z\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\quad + (t+1)^{-1.75}\Big(\||\partial_x|^{0.75}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{0.75}h_{\max\{j_1,j_2\}}\|_{L^2}\Big)\cdot\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\quad + (t+1)^{-1.5}\Big(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{0.5\max\{j_2,j_3\}} + \mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-1.5-r)\max\{j_2,j_3\}}\Big)\\
&\qquad\qquad\cdot\|h_{\max\{j_2,j_3\}}\|_Z\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\\
&\quad + (t+1)^{-1.75}\Big(\||\partial_x|^{0.75}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2} + \||\partial_x|^{0.75}h_{\max\{j_2,j_3\}}\|_{L^2}\Big)\cdot\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}.
\end{aligned}$$

Estimate of W_2 and W_3. We rewrite W_2 as

$$\iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{it\partial_{\eta_1}\Phi(\xi,\eta_1,\eta_2)(\xi-\eta_1-\eta_2)}\right] e^{it\Phi(\xi,\eta_1,\eta_2)} \cdot \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\left[(\xi-\eta_1-\eta_2)\partial_{\eta_1}\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\right] \mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

In view of the multilinear estimate Lemma 2.3.2, we need to estimate the S^∞-norm of the symbol

$$\frac{\mathbf{T}_1(\eta_1,\eta_2,\eta_3)}{(\log|\eta_1|-\log|\eta_3|)\eta_3}\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\tilde{\varsigma}_{j_3}(\eta_3).$$

Similar to the estimates of W_1, using Lemma 5.6.1 and Lemma 5.6.2, we obtain

$$\|W_2\|_{L^\infty_\xi} \lesssim (t+1)^{-1}\|\partial_x\varphi_{\max\{j_1,j_2\}}\|_{L^\infty}\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}.$$

Using Lemma 5.4.1, we have

$$\begin{aligned}\|W_2\|_{L^\infty_\xi} \lesssim{}& (t+1)^{-1.5}\left(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{0.5\max\{j_1,j_2\}}+\mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-1.5-r)\max\{j_1,j_2\}}\right)\\ &\cdot\|h_{\max\{j_1,j_2\}}\|_Z\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\ &+(t+1)^{-1.75}\left(\||\partial_x|^{0.75}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{0.75}h_{\max\{j_1,j_2\}}\|_{L^2}\right)\\ &\cdot\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|W_3\|_{L^\infty_\xi} \lesssim{}& (t+1)^{-1.5}\left(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{0.5\max\{j_2,j_3\}}+\mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-1.5-r)\max\{j_2,j_3\}}\right)\\ &\cdot\|h_{\max\{j_2,j_3\}}\|_Z\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\\ &+(t+1)^{-1.75}\left(\||\partial_x|^{0.75}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{0.75}h_{\max\{j_2,j_3\}}\|_{L^2}\right)\\ &\cdot\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}.\end{aligned}$$

In conclusion, for nonresonant frequencies,

$$\begin{aligned}&\left\|\xi(|\xi|+|\xi|^{r+3})\mathfrak{d}(\xi,t)\iint_{\mathbb{R}^2}\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\|_{L^\infty_\xi}\\ &\lesssim (t+1)^{(r+4)p_1}\left(\|W_1\|_{L^\infty_\xi}+\|W_2\|_{L^\infty_\xi}+\|W_3\|_{L^\infty_\xi}\right)\end{aligned}$$

$$
\begin{aligned}
&\lesssim (t+1)^{-1.5+(r+4)p_1}\Big(\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}+\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\Big)\\
&\qquad\qquad\cdot\Big(\mathbf{1}_{\max\{j_2,j_3\}\le 0}2^{0.5\max\{j_2,j_3\}}+\mathbf{1}_{\max\{j_2,j_3\}>0}2^{(-1.5-r)\max\{j_2,j_3\}}\Big)\|h_{\max\{j_2,j_3\}}\|_Z\\
&\quad+(t+1)^{-1.5+(r+4)p_1}\Big(\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}+\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\Big)\\
&\qquad\qquad\cdot\Big(\mathbf{1}_{\max\{j_1,j_2\}\le 0}2^{0.5\max\{j_1,j_2\}}+\mathbf{1}_{\max\{j_1,j_2\}>0}2^{(-1.5-r)\max\{j_1,j_2\}}\Big)\|h_{\max\{j_1,j_2\}}\|_Z\\
&\quad+(t+1)^{-1.75+(r+4)p_1}\\
&\qquad\cdot\Big[\Big(\||\partial_x|^{0.75}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{0.75}h_{\max\{j_1,j_2\}}\|_{L^2}\Big)\|\varphi_{j_3}\|_{L^2}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\qquad\quad+\Big(\||\partial_x|^{0.75}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{0.75}h_{\max\{j_2,j_3\}}\|_{L^2}\Big)\|\varphi_{j_1}\|_{L^2}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\\
&\qquad\quad+\Big(\||\partial_x|^{0.75}P_{\max\{j_1,j_2\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{0.75}h_{\max\{j_1,j_2\}}\|_{L^2}\Big)\|\xi\partial_\xi\hat{h}_{j_3}\|_{L^2_\xi}\|\varphi_{\min\{j_1,j_2\}}\|_{L^2}\\
&\qquad\quad+\Big(\||\partial_x|^{0.75}P_{\max\{j_2,j_3\}}(x\partial_x h)\|_{L^2}+\||\partial_x|^{0.75}h_{\max\{j_2,j_3\}}\|_{L^2}\Big)\|\xi\partial_\xi\hat{h}_{j_1}\|_{L^2_\xi}\|\varphi_{\min\{j_2,j_3\}}\|_{L^2}\Big].
\end{aligned}
$$

By the bootstrap assumptions and Lemma 5.3.4, the right-hand-side is summable for j_1, j_2, j_3 and the sum is integrable for $t \in (0, \infty)$.

5.6.4. Close to the resonance. When

$$
\max\{j_1, j_2, j_3\} < 10^{-3}\log_2(t+1), \qquad |j_3 - j_2| \le 1, \qquad |j_3 - j_1| \le 1, \tag{5.63}
$$

we need to consider the following two cases:

(i) Frequencies η_1, η_2 and $\xi - \eta_1 - \eta_2$ have the same sign.

By the definition of cut-off function ψ, we have

$$
\frac{5}{8}2^{j_1} \le |\eta_1| \le \frac{8}{5}2^{j_1}, \quad \frac{5}{8}2^{j_2} \le |\eta_2| \le \frac{8}{5}2^{j_2}, \quad \frac{5}{8}2^{j_3} \le |\xi - \eta_1 - \eta_2| \le \frac{8}{5}2^{j_3},
$$

and thus,

$$
\frac{5}{8}(2^{j_1} + 2^{j_2} + 2^{j_3}) \le |\xi| \le \frac{8}{5}(2^{j_1} + 2^{j_2} + 2^{j_3}).
$$

This corresponds to the region near the *space resonance* $\eta_1 = \eta_2 = \xi - \eta_1 - \eta_2 = \xi/3$.

(ii) Frequencies η_1, η_2 and $\xi - \eta_1 - \eta_2$ do not have the same sign.

This corresponds to the region near the *space-time resonances* $(\eta_1, \eta_2) = (\xi, \xi)$, $(\xi, -\xi)$, or $(-\xi, \xi)$ separately. Since the symbol $\mathbf{T}_1'(\eta_1, \eta_2, \eta_3)$ is symmetric in η_1, η_2, and η_3, it suffices to (5.55) in the region near (ξ, ξ).

To estimate (5.55) in the region (5.63), we decompose the region further. Denoting $(\xi_1, \xi_2, \xi_3) = (\xi, \xi, -\xi)$ or $(\frac{\xi}{3}, \frac{\xi}{3}, \frac{\xi}{3})$, we decompose (5.63) using the new cut-off functions ψ_{k_1} and ψ_{k_2}. Using the fact that

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2) = 1,$$

we write the integral (5.55) as

$$\iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \cdot \left[\sum_{k_1=-\infty}^{\max\{j_1,j_3\}+1} \psi_{k_1}(\eta_1 - \xi_1)\right] \cdot \left[\sum_{k_2=-\infty}^{\max\{j_2,j_3\}+1} \psi_{k_2}(\eta_2 - \xi_2)\right] \mathrm{d}\eta_1\, \mathrm{d}\eta_2,$$

where

$$\left[\sum_{k_1=-\infty}^{\max\{j_1,j_3\}+1} \psi_{k_1}(\eta_1 - \xi_1)\right] \cdot \left[\sum_{k_2=-\infty}^{\max\{j_2,j_3\}+1} \psi_{k_2}(\eta_2 - \xi_2)\right] = 1$$

on the support of $\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)$. Thus, we need to consider

$$\begin{aligned} &\iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \\ &\quad \cdot \psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2. \end{aligned} \tag{5.64}$$

In this subsection, we restrict our attention to

$$k_1 \geq \log_2[\varrho(t)] \qquad \text{or} \qquad k_2 \geq \log_2[\varrho(t)],$$

where

$$\varrho(t) = (t+1)^{-0.49}. \tag{5.65}$$

The case of $k_1 < \log_2[\varrho(t)]$ and $k_2 < \log_2[\varrho(t)]$, related to the resonant frequencies, will be discussed in Section 5.6.5.

Since these expressions are symmetric in η_1 and η_2, we assume without loss of generality that $j_1 \geq k_1 \geq k_2 \geq \log_2[\varrho(t)]$. The other case can be discussed in the similar way.

Using integrating by parts, we can write (5.64) as

$$\iint_{\mathbb{R}^2} \frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{2it(\log|\eta_1| - \log|\xi - \eta_1 - \eta_2|)} \partial_{\eta_1} e^{it\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \cdot \psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2$$

$$= \frac{i}{2t}(V_1 + V_2 + V_3 + V_4),$$

where

$$V_1(\xi,t) = \iint_{\mathbb{R}^2} \partial_{\eta_1}\left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1| - \log|\xi-\eta_1-\eta_2|}\right] e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot \psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$

$$V_2(\xi,t) = \iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1| - \log|\xi-\eta_1-\eta_2|}\right] e^{it\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot \psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$

$$V_3(\xi,t) = \iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1| - \log|\xi-\eta_1-\eta_2|}\right] e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\partial_{\eta_1}\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot \psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$

$$V_4(\xi,t) = \iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\log|\eta_1| - \log|\xi-\eta_1-\eta_2|}\right] e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) \cdot \partial_{\eta_1}\psi_{k_1}(\eta_1-\xi_1)\psi_{k_2}(\eta_2-\xi_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

Estimate of V_1. We first denote the symbol for V_1 as

$$\begin{aligned} &m(\eta_1,\eta_2,\xi) \\ =& \frac{-2}{\log|\eta_1| - \log|\xi-\eta_1-\eta_2|}\cdot\Big[\eta_1\log|\eta_1| - (\eta_1+\eta_2)\log|\eta_1+\eta_2| \\ &\qquad + (\xi-\eta_1)\log|\xi-\eta_1| - (\xi-\eta_1-\eta_2)\log|\xi-\eta_1-\eta_2|\Big] \\ &- \frac{\eta_1^{-1} + (\xi-\eta_1-\eta_2)^{-1}}{(\log|\eta_1| - \log|\xi-\eta_1-\eta_2|)^2}\cdot\Big[-\eta_1^2\log|\eta_1| - \eta_2^2\log|\eta_2| - \eta_3^2\log|\eta_3| \\ &\quad - (\eta_1+\eta_2+\eta_3)^2\log|\eta_1+\eta_2+\eta_3| + (\eta_1+\eta_2)^2\log|\eta_1+\eta_2| \\ &\quad + (\eta_1+\eta_3)^2\log|\eta_1+\eta_3| + (\eta_2+\eta_3)^2\log|\eta_2+\eta_3|\Big]. \end{aligned}$$

Denote $\upsilon_i = \eta_i - \xi_i$, $i = 1, 2$, and it suffices to estimate

$$\left\|\iint_{\mathbb{R}^2} m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)e^{it\Phi(\xi,\upsilon_1+\xi_1,\upsilon_2+\xi_2)}\hat{h}_{j_1}(\upsilon_1+\xi_1)\hat{h}_{j_2}(\upsilon_2+\xi_2)\hat{h}_{j_3}(\xi_3-\upsilon_1-\upsilon_2) \cdot \psi_{k_1}(\upsilon_1)\psi_{k_2}(\upsilon_2)\,\mathrm{d}\upsilon_1\,\mathrm{d}\upsilon_2\right\|_{L^\infty_\xi}.$$

Using Lemma 2.3.2, we have

$$\|V_1\|_{L^\infty_\xi} \lesssim \|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{S^\infty_{\upsilon_1,\upsilon_2}L^\infty_\xi}$$
$$\cdot \|\hat{\varphi}_{j_1}(\upsilon_1+\xi_1)\psi_{k_1}(\upsilon_1)\|_{L^2_{\upsilon_1}L^\infty_\xi}\|\hat{\varphi}_{j_2}(\upsilon_2+\xi_2)\psi_{k_2}(\upsilon_2)\|_{L^2_{\upsilon_2}L^\infty_\xi}\|\varphi_{j_3}\|_{L^\infty},$$

where

$$\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi) = \tilde{\psi}_{k_1}(\upsilon_1)\tilde{\psi}_{k_2}(\upsilon_2)\tilde{\psi}_{j_1}(\upsilon_1+\xi_1)\tilde{\psi}_{j_2}(\upsilon_2+\xi_2)\tilde{\psi}_{j_3}(\xi_3-\upsilon_1-\upsilon_2)\chi(\xi).$$

(i) If $(\xi_1,\xi_2,\xi_3)=(\xi/3,\xi/3,\xi/3)$, since S^∞-norm is rotational and scaling invariant, setting $w_1=\upsilon_1$, $w_2=-2\upsilon_1-\upsilon_2$, and using (2.20), we have

$$\begin{aligned}
&\|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{S^\infty_{\upsilon_1,\upsilon_2}L^\infty_\xi}\\
&= \|\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)\|_{S^\infty_{w_1,w_2}L^\infty_\xi}\\
&\lesssim \|\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)\|^{1/4}_{L^1_{w_1w_2}L^\infty_\xi}\\
&\quad\cdot \|\partial^2_{w_1}[\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)]\|^{1/2}_{L^1_{w_1w_2}L^\infty_\xi}\\
&\quad\cdot \|\partial^2_{w_1}\partial^2_{w_2}[\chi^{k_1,k_2}_{j_1,j_3}(w_1,-2w_1-w_2,\xi)m(w_1+\xi_1,-2w_1-w_2+\xi_2,\xi)]\|^{1/4}_{L^1_{w_1w_2}L^\infty_\xi}\\
&\lesssim (1+|j_1|)(2^{j_1+k_1}2^{j_1})^{1/4}(2^{-j_1+k_1}2^{j_1})^{1/2}(2^{-j_1-k_1}2^{j_1})^{1/4}\\
&= (1+|j_1|)\cdot 2^{(j_1+k_1)/2},
\end{aligned}$$

where we have used the estimate

$$\left|\frac{\chi^{k_1,k_2}_{j_1,j_3}}{\log|w_1+\frac{\xi}{3}|-\log|\frac{\xi}{3}+w_1+w_2|}\right| \lesssim 2^{j_1-k_1},$$
$$\left|\chi^{k_1,k_2}_{j_1,j_3}\partial^2_{w_1}\frac{1}{\log|w_1+\frac{\xi}{3}|-\log|\frac{\xi}{3}+w_1+w_2|}\right| \lesssim 2^{3(j_1-k_1)}2^{2(-2j_1+k_1)} = 2^{-j_1-k_1},$$
$$\left|\chi^{k_1,k_2}_{j_1,j_3}\partial^2_{w_1}\partial^2_{w_2}\frac{1}{\log|w_1+\frac{\xi}{3}|-\log|\frac{\xi}{3}+w_1+w_2|}\right| \lesssim 2^{5(j_1-k_1)}2^{-2j_1}2^{2(-2j_1+k_1)} = 2^{-j_1-3k_1}.$$

Therefore, using (2.18), (5.41), and (5.63), we obtain

$$\begin{aligned}\|V_1\|_{L^\infty_\xi} &\lesssim (1+|j_1|)2^{(j_1+k_1)/2-j_3}(t+1)^{-1}\|\hat\varphi_{j_1}(\upsilon_1+\xi_1)\psi_{k_1}(\upsilon_1)\|_{L^2_{\upsilon_1}L^\infty_\xi}\\ &\qquad\cdot\|\hat\varphi_{j_2}(\upsilon_2+\xi_2)\psi_{k_2}(\upsilon_2)\|_{L^2_{\upsilon_2}L^\infty_\xi}\|\partial_x\varphi_{j_3}\|_{L^\infty}\\ &\lesssim (1+|j_1|)2^{-0.5j_1+k_1+0.5k_2}\|\psi_{k_1}\hat\varphi_{j_1}\|_{L^\infty_\xi}\|\psi_{k_2}\hat\varphi_{j_2}\|_{L^\infty_\xi}\Big\{(t+1)^{-1.5}\||\xi|^{3/2}\hat h^{k_1,k_2}_{j_3}\|_{L^\infty_\xi}\\ &\qquad+(t+1)^{-1.75}\big[\||\partial_x|^{3/4}P^{k_1,k_2}_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h^{k_1,k_2}_{j_3}\|_{L^2}\big]\Big\}.\end{aligned}\tag{5.66}$$

(ii) If $(\xi_1,\xi_2,\xi_3)=(\xi,\xi,-\xi)$, we use (2.20) to obtain

$$\begin{aligned}&\|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|_{S^\infty_{\upsilon_1,\upsilon_2}L^\infty_\xi}\\ &\lesssim \|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)\|^{1/4}_{L^1_{\upsilon_1\upsilon_2}}\|\partial^2_{\upsilon_1}[\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)]\|^{1/2}_{L^1_{\upsilon_1\upsilon_2}}\\ &\quad\cdot\|\partial^2_{\upsilon_1}\partial^2_{\upsilon_2}[\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)m(\upsilon_1+\xi_1,\upsilon_2+\xi_2,\xi)]\|^{1/4}_{L^1_{\upsilon_1\upsilon_2}}\\ &\lesssim (1+|j_1|)(2^{j_1+k_1}2^{j_1})^{1/4}(2^{-j_1+k_1}2^{j_1})^{1/2}(2^{-j_1-2k_2+k_1}2^{j_1})^{1/4}\\ &=(1+|j_1|)\cdot 2^{\frac12 j_1+k_1-\frac12 k_2},\end{aligned}$$

where we have used the estimates

$$\begin{aligned}\left|\frac{\chi^{k_1,k_2}_{j_1,j_3}}{\log|\upsilon_1+\xi|-\log|-\xi-\upsilon_1-\upsilon_2|}\right| &\lesssim 2^{j_1-k_2},\\ \left|\chi^{k_1,k_2}_{j_1,j_3}\partial^2_{\upsilon_1}\frac{1}{\log|\upsilon_1+\xi|-\log|-\xi-\upsilon_1-\upsilon_2|}\right| &\lesssim 2^{3(j_1-k_2)}2^{2(-2j_1+k_2)}=2^{-j_1-k_2},\\ \left|\chi^{k_1,k_2}_{j_1,j_3}\partial^2_{\upsilon_1}\partial^2_{\upsilon_2}\frac{1}{\log|\upsilon_1+\xi|-\log|-\xi-\upsilon_1-\upsilon_2|}\right| &\lesssim 2^{5(j_1-k_2)}2^{-2j_1}2^{2(-2j_1+k_2)}=2^{-j_1-3k_2}.\end{aligned}$$

Therefore, using (2.18), (5.41), and (5.63)

$$\begin{aligned}\|V_1\|_{L^\infty_\xi} &\lesssim (1+|j_1|)2^{0.5j_1-j_3+k_1-0.5k_2}(t+1)^{-1}\|\hat\varphi_{j_1}(\upsilon_1+\xi_1)\psi_{k_1}(\upsilon_1)\|_{L^2_{\upsilon_1}L^\infty_\xi}\\ &\qquad\cdot\|\hat\varphi_{j_2}(\upsilon_2+\xi_2)\psi_{k_2}(\upsilon_2)\|_{L^2_{\upsilon_2}L^\infty_\xi}\|\partial_x\varphi_{j_3}\|_{L^\infty}\\ &\lesssim (1+|j_1|)2^{-0.5j_1+1.5k_1}\|\psi_{k_1}\hat\varphi_{j_1}\|_{L^\infty_\xi}\|\psi_{k_2}\hat\varphi_{j_2}\|_{L^\infty_\xi}\Big\{(t+1)^{-1.5}\||\xi|^{3/2}\hat h_{j_3}\|_{L^\infty_\xi}\\ &\qquad+(t+1)^{-1.75}\big[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}\big]\Big\}.\end{aligned}\tag{5.67}$$

Estimates of V_2–V_4. The estimates for V_2–V_4 are similar to V_1. We omit the details here. The resulting estimates are as follows.

(i) If $(\xi_1,\xi_2,\xi_3)=(\frac{\xi}{3},\frac{\xi}{3},\frac{\xi}{3})$, the symbol can be estimated as

$$\left\|\frac{\mathbf{T}_1'(\xi_1+\upsilon_1,\xi_2+\upsilon_2,\xi_3-\upsilon_1-\upsilon_2)}{\log|\xi_1+\upsilon_1|-\log|\xi_3-\upsilon_1-\upsilon_2|}\right\|_{S^\infty_{\upsilon_1\upsilon_2}L^\infty_\xi}\lesssim 2^{1.5j_1+0.5k_1}.$$

(ii) If $(\xi_1,\xi_2,\xi_3)=(\xi,\xi,-\xi)$, the symbol can be estimated as

$$\begin{aligned}&\left\|\chi^{k_1,k_2}_{j_1,j_3}(\upsilon_1,\upsilon_2,\xi)\frac{\mathbf{T}_1'(\xi_1+\upsilon_1,\xi_2+\upsilon_2,\xi_3-\upsilon_1-\upsilon_2)}{\log|\xi_1+\upsilon_1|-\log|\xi_3-\upsilon_1-\upsilon_2|}\right\|_{S^\infty_{\upsilon_1\upsilon_2}L^\infty_\xi}\\ &\lesssim(2^{j_1+k_1}2^{2j_1})^{1/4}(2^{-j_1+k_1}2^{2j_1})^{1/2}(2^{-j_1-2k_2+k_1}2^{2j_1})^{1/4}\\ &=(1+|j_1|)\cdot 2^{1.5j_1+k_1-0.5k_2}.\end{aligned}$$

In either case, we have the following estimates

$$\begin{aligned}\|V_2\|_{L^\infty_\xi}\lesssim{}&(1+|j_1|)2^{-0.5j_1+k_1}\|\eta_1\partial_{\eta_1}\hat{\varphi}_{j_1}(\eta_1)\|_{L^2_{\eta_1}}\|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}\\ &\cdot\Big\{(t+1)^{-1.5}\||\xi|^{3/2}\hat{h}_{j_3}\|_{L^\infty_\xi}+(t+1)^{-1.75}\big[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}\big]\Big\},\end{aligned}\tag{5.68}$$

$$\begin{aligned}\|V_3\|_{L^\infty_\xi}\lesssim{}&(1+|j_1|)2^{-0.5j_1+k_1}\|\eta_3\partial_{\eta_3}\hat{\varphi}_{j_3}(\eta_3)\|_{L^2_{\eta_3}}\|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}\\ &\cdot\Big\{(t+1)^{-1.5}\||\xi|^{3/2}\hat{h}_{j_1}\|_{L^\infty_\xi}+(t+1)^{-1.75}\big[\||\partial_x|^{3/4}P_{j_1}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_1}\|_{L^2}\big]\Big\},\end{aligned}\tag{5.69}$$

$$\begin{aligned}\|V_4\|_{L^\infty_\xi}\lesssim{}&(1+|j_1|)2^{0.5j_1+0.5k_1}\|\psi_{k_1}\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}\\ &\cdot\Big\{(t+1)^{-1.5}\||\xi|^{3/2}\hat{h}_{j_3}\|_{L^\infty_\xi}+(t+1)^{-1.75}\big[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}\big]\Big\}.\end{aligned}\tag{5.70}$$

Now we take the summation over $\log_2[\varrho(t)]\le k_1,k_2\le\max\{j_1,j_3\}+1$, and combine the estimates (5.66)–(5.70) to get

$$\begin{aligned}\Bigg\|\xi(|\xi|+|\xi|^{r+3})\mathfrak{d}(\xi,t)\iint_{\mathbb{R}^2}&\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\\ &\cdot\left[\sum_{k_1=\log_2[\varrho(t)]}^{\max\{j_1,j_3\}+1}\psi_{k_1}(\eta_1-\xi_1)\right]\cdot\left[\sum_{k_2=\log_2[\varrho(t)]}^{\max\{j_2,j_3\}+1}\psi_{k_2}(\eta_2-\xi_2)\right]\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Bigg\|_{L^\infty_\xi}\end{aligned}$$

$$\lesssim (1+|j_1|)\left[\max\{j_1,j_3\}-\log_2[\varrho(t)]\right]^2 (t+1)^{(r+3)p_1}\cdot\Big[\||\xi|\psi_{k_1}\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\||\xi|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}$$
$$+\|\eta_1\partial_{\eta_1}\hat{\varphi}_{j_1}(\eta_1)\|_{L^2_{\eta_1}}\||\xi|\psi_{k_2}\hat{\varphi}_{j_2}\|_{L^\infty_\xi}+\||\xi|\psi_{k_1}\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|\eta_2\partial_{\eta_2}\hat{\varphi}_{j_2}(\eta_2)\|_{L^2_{\eta_2}}\Big]$$
$$\cdot\Big\{(t+1)^{-1.5}\||\xi|^{3/2}\hat{h}_{j_3}\|_{L^\infty_\xi}+(t+1)^{-1.75}\big[\||\partial_x|^{3/4}P_{j_3}(x\partial_x h)\|_{L^2}+\||\partial_x|^{3/4}h_{j_3}\|_{L^2}\big]\Big\}.$$

The right-hand-side is summable with respect to j_1, j_2, j_3 under $|j_3-j_2|\le 1$ and $|j_3-j_1|\le 1$, since we can write

$$\||\xi|^{3/2}\hat{h}_j\|_{L^\infty_\xi}\lesssim(\mathbf{1}_{j\le 0}2^{j/2}+\mathbf{1}_{j>0}2^{(-r-3/2)j})\|h_j\|_Z,$$

and the resulting sum is integrable for $t\in(0,\infty)$.

5.6.5. Resonant frequencies. In this subsection, we estimate (5.64) in the region

$$|j_1-j_3|\le 1,\qquad |j_2-j_3|\le 1,\qquad k_1<\log_2[\varrho(t)],\qquad k_2<\log_2[\varrho(t)],$$

and then sum over $k_1, k_2<\log_2(\varrho(t))$.

After taking the sum, the cut-off function of the integrand is

$$\mathfrak{b}(\xi,\eta_1,\eta_2,t):=\varsigma\left(\frac{|\eta_1|-|\xi-\eta_1-\eta_2|}{\varrho(t)}\right)\cdot\varsigma\left(\frac{|\eta_2|-|\xi-\eta_1-\eta_2|}{\varrho(t)}\right).$$

The support of this cut-off function is

$$\left\{(\eta_1,\eta_2)\mid \big||\eta_1|-|\xi-\eta_1-\eta_2|\big|<\frac{8}{5}\varrho(t),\ \big||\eta_2|-|\xi-\eta_1-\eta_2|\big|<\frac{8}{5}\varrho(t)\right\},$$

which can be rewritten as the union of four disjoint sets $A_1\cup A_2\cup A_3\cup A_4$, where

$$A_1:=\left\{(\eta_1,\eta_2)\,\middle|\,\left|2\left(\eta_1-\frac{\xi}{3}\right)+\left(\eta_2-\frac{\xi}{3}\right)\right|<\frac{8}{5}\varrho(t),\quad\left|\left(\eta_1-\frac{\xi}{3}\right)+2\left(\eta_2-\frac{\xi}{3}\right)\right|<\frac{8}{5}\varrho(t)\right\},$$
$$A_2:=\left\{(\eta_1,\eta_2)\,\middle|\,\big|\eta_2-\xi\big|<\frac{8}{5}\varrho(t),\quad\big|\eta_1-\xi\big|<\frac{8}{5}\varrho(t)\right\},$$
$$A_3:=\left\{(\eta_1,\eta_2)\,\middle|\,\big|2(\eta_1-\xi)+(\eta_2-(-\xi))\big|<\frac{8}{5}\varrho(t),\quad\big|\eta_1-\xi\big|<\frac{8}{5}\varrho(t)\right\},$$
$$A_4:=\left\{(\eta_1,\eta_2)\,\middle|\,\big|\eta_2-\xi\big|<\frac{8}{5}\varrho(t),\quad\big|(\eta_1+\xi)+2(\eta_2-\xi)\big|<\frac{8}{5}\varrho(t)\right\}.$$

We notice that A_1, A_2, A_3, A_4 are parallelogram centered at $(\xi/3,\xi/3)$, (ξ,ξ), $(\xi,-\xi)$, and $(-\xi,\xi)$, respectively. A_1 corresponds to the space resonance, while A_2, A_3 and A_4 correspond to the space-time resonances.

5.6.5.1. *Space resonances.* When $(\eta_1, \eta_2) \in A_1$, we can expand $\mathbf{T}_1/\Phi$ around $(\xi, \xi/3, \xi/3)$ as

$$\frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} = \left(\frac{1}{2} - \frac{2\log 2}{3\log 3}\right)\xi + O\left(\left|\eta_1 - \frac{\xi}{3}\right|^2 + \left|\eta_2 - \frac{\xi}{3}\right|^2\right). \tag{5.71}$$

After writing

$$e^{i\tau\Phi(\xi,\eta_1,\eta_2)} = \frac{1}{i\Phi(\xi,\eta_1,\eta_2)}\left[\partial_\tau e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\right],$$

and integrating by parts with respect to τ, we get that

$$\begin{aligned}
&\int_1^t i\xi \iint_{\mathbb{R}^2} \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad \cdot \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)\mathfrak{b}(\xi, \eta_1, \eta_2, \tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau \\
=&\int_1^t \xi \iint_{\mathbb{R}^2} \frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi,\eta_1,\eta_2)}\partial_\tau\left[e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\right] \\
&\qquad \cdot \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)\mathfrak{b}(\xi, \eta_1, \eta_2, \tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau \\
=&J_1 - \int_1^t J_2(\tau) + J_3(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \iint_{\mathbb{R}^2} \xi\frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi,\eta_1,\eta_2)} e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad \cdot \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)\mathfrak{b}(\xi, \eta_1, \eta_2, \tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Big|_{\tau=1}^{\tau=t}, \\
J_2(\tau) &= \xi \iint_{\mathbb{R}^2} \frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi,\eta_1,\eta_2)} e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad \cdot \partial_\tau\left[\hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)\right]\mathfrak{b}(\xi, \eta_1, \eta_2, \tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
J_3(\tau) &= \xi \iint_{\mathbb{R}^2} \partial_\tau\mathfrak{b}(\xi, \eta_1, \eta_2, \tau)\frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi,\eta_1,\eta_2)} e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad \cdot \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}$$

For J_1, for any $\tau \geq 1$, we have from (5.71) that

$$\begin{aligned}
&\Bigg|(|\xi| + |\xi|^{r+3}) \iint_{\mathbb{R}^2} \mathfrak{b}(\xi, \eta_1, \eta_2, t)\xi\frac{\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad \cdot \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left|(|\xi|+|\xi|^{r+3})\iint_{\mathbb{R}^2}\mathfrak{b}(\xi,\eta_1,\eta_2,t)\xi^2\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right| \\
&\quad + (|\xi|+|\xi|^{r+3})\iint_{\mathbb{R}^2}\mathfrak{b}(\xi,\eta_1,\eta_2,t)[\varrho(\tau)]^2\left|\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right|\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&\lesssim (\tau+1)^{2p_0+(r+2)p_1}\||\xi|\hat{h}_{j_1}\|_{L^\infty_\xi}\||\xi|\hat{h}_{j_2}\|_{L^\infty_\xi}\||\xi|\hat{h}_{j_3}\|_{L^\infty_\xi}\left([\varrho(\tau)]^2+[\varrho(\tau)]^4\right).
\end{aligned}
$$

Notice that in A_1, the number of summations of j_1, j_2, and j_3 is or order $\log(t+1)$, and therefore, the right-hand-side is bounded for $\tau \geq 1$ after the summation in j_1, j_2, and j_3.

After taking the time derivative, the term J_2 can be written as a sum of three terms.

$$
\begin{aligned}
&\xi\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad\qquad\cdot\left[\partial_\tau\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
&\xi\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad\qquad\cdot\left[\hat{h}_{j_1}(\eta_1,\tau)\partial_\tau\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\tau,\xi-\eta_1-\eta_2)\right]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\xi\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad\qquad\cdot\left[\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\partial_\tau\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right]\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}
$$

Notice that by (5.53),and the bootstrap assumptions and Lemma 5.3.2, we have

$$
\begin{aligned}
&\|\partial_t\hat{h}\|_{L^\infty_\xi} \\
&\lesssim \left\|\xi\iint_{\mathbb{R}^2}\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\|_{L^\infty_\xi}+\left\|\widehat{\mathcal{N}_{\geq 5}(\varphi)}\right\|_{L^\infty_\xi} \\
&\lesssim \left\|\partial_x\left\{\varphi^2\log|\partial_x|\varphi_{xx}-\varphi\log|\partial_x|(\varphi^2)_{xx}+\frac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\right\}\right\|_{L^1}+\|\mathcal{N}_{\geq 5}(\varphi)\|_{L^1} \\
&\lesssim \|\varphi\|^2_{H^s}\cdot\sum_{j=0}^{\infty}\left(\|\varphi_x\|^{2j+1}_{W^{2,\infty}}+\|L\varphi_x\|^{2j+1}_{W^{2,\infty}}\right) \\
&\lesssim \varepsilon_1^3(t+1)^{2p_0-\frac{1}{2}}.
\end{aligned}
$$

Therefore, we obtain

$$
\left|(|\xi|+|\xi|^{r+3})J_2(\tau)\right| \lesssim \sum\|h_{\ell_1}\|_Z\|\partial_\tau\hat{h}_{\ell_2}\|_{L^\infty_\xi}\|h_{\ell_3}\|_Z[\varrho(\tau)]^2 \lesssim \varepsilon_1^3(\tau+1)^{p_0-\frac{1}{2}}[\varrho(t)]^2\sum\|h_{\ell_1}\|_Z\|h_{\ell_3}\|_Z
$$

where we sum over all permutations (ℓ_1, ℓ_2, ℓ_3) of (j_1, j_2, j_3) in the space resonance region A_1. Again, we notice that the number of summations is of order $\log(\tau+1)$, and the resulting sum is integrable for $\tau \in (1, \infty)$.

As for the term J_3, by the definition of the cut-off function, we have

$$\left|\partial_\tau \left[\varsigma_{\leq \log_2(\varrho(\tau))}(|\eta_1| - |\xi - \eta_1 - \eta_2|) \cdot \varsigma_{\leq \log_2(\varrho(\tau))}(|\eta_2| - |\xi - \eta_1 - \eta_2|)\right]\right| \lesssim \varrho_1'(t)(\varrho(t))^{-1} \lesssim \frac{1}{t+1}.$$

The area of its support is of the order of $[\varrho(t)]^2$. Then, using (5.71), we get that

$$\left|(|\xi| + |\xi|^{r+3}) J_3(\tau)\right| \lesssim (\tau+1)^{2p_0 + (r+2)p_1 - 1} [\varrho(\tau)]^2 \sum \|\xi \hat{h}_{\ell_1}\|_{L^\infty_\xi} \|\xi \hat{h}_{\ell_2}\|_{L^\infty_\xi} \|\xi \hat{h}_{\ell_3}\|_{L^\infty_\xi},$$

where the summation is taken over permutations (ℓ_1, ℓ_2, ℓ_3) of (j_1, j_2, j_3), so the sum converges and is integrable for $\tau \in (1, \infty)$.

5.6.5.2. *Space-time resonances.* We now use modified scattering to consider the term

$$\frac{1}{6} \iint_{A_2 \bigcup A_3 \bigcup A_4} i\xi \mathfrak{b}(\xi, \eta_1, \eta_2, t) \mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}_{j_1}(\eta_1) \hat{\varphi}_{j_2}(\eta_2) \hat{\varphi}_{j_3}(\xi - \eta_1 - \eta_2) \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2$$
$$- i\xi \left[\beta_1(t) \mathbf{T}_1(\xi, \xi, -\xi) + \beta_2(t) \mathbf{T}_1(\xi, -\xi, \xi) + \beta_3(t) \mathbf{T}_1(-\xi, \xi, \xi)\right] |\hat{\varphi}(\tau, \xi)|^2 \hat{\varphi}(\tau, \xi).$$

For A_2, we take

$$\beta_1(t) = \frac{1}{6} \iint_{A_2} \mathfrak{b}(\xi, \eta_1, \eta_2, t) \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2.$$

Therefore, using a Taylor expansion and (5.51), we obtain

$$\Bigg|(|\xi| + |\xi|^{r+3}) \frac{1}{6} i\xi \iint_{A_2} \mathfrak{b}(\xi, \eta_1, \eta_2, t)$$
$$\Big[\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}_{j_1}(\eta_1) \hat{\varphi}_{j_2}(\eta_2) \hat{\varphi}_{j_3}(\xi - \eta_1 - \eta_2) - \mathbf{T}_1(\xi, \xi, -\xi) |\hat{\varphi}(\tau, \xi)|^2 \hat{\varphi}(\tau, \xi)\Big] \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2\Bigg|$$
$$\lesssim (|\xi| + |\xi|^{r+3})(i\xi) \iint_{A_2} \left|\partial_{\eta_1} \left[\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}_{j_1}(\eta_1) \hat{\varphi}_{j_2}(\eta_2) \hat{\varphi}_{j_3}(\xi - \eta_1 - \eta_2)\right]\Big|_{\eta_1 = \eta_1'} (\xi - \eta_1)\right|$$
$$+ \left|\partial_{\eta_2} \left[\mathbf{T}_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}_{j_1}(\eta_1) \hat{\varphi}_{j_2}(\eta_2) \hat{\varphi}_{j_3}(\xi - \eta_1 - \eta_2)\right]\Big|_{\eta_2 = \eta_2'} (\xi - \eta_2)\right| \mathrm{d}\eta_1 \, \mathrm{d}\eta_2$$
$$\lesssim (t+1)^{(r+2)p_1} \|\xi \hat{\varphi}_{j_1}\|_{L^\infty_\xi} \|\xi \hat{\varphi}_{j_2}\|_{L^\infty_\xi} \|\xi \hat{\varphi}_{j_3}\|_{L^\infty_\xi} [\varrho(t)]^3 + \sum \|\xi \hat{\varphi}_{\ell_1}\|_{L^\infty_\xi} \|\xi \hat{\varphi}_{\ell_2}\|_{L^\infty_\xi} \|\mathcal{S} \varphi_{\ell_3}\|_{H^r} [\varrho(t)]^{5/2},$$

where η_1' (or η_2') in the first inequality is some number between ξ and η_1 (or η_2), and the summation in the second inequality is over permutations (ℓ_1, ℓ_2, ℓ_3) of (j_1, j_2, j_3). The estimates for A_3 and A_4 follow by a similar argument.

Taking a summation over j_1, j_2, j_3 and using the estimates in the above subsections together with the time-decay of $\varrho(t)$ in (5.65), we conclude that

$$\int_0^\infty \|(|\xi| + |\xi|^{r+3})U_1\|_{L^\infty_\xi} \, \mathrm{d}t \lesssim \varepsilon_0.$$

5.6.6. Higher-degree terms. In this subsection, we prove that

$$\left\|(|\xi| + |\xi|^{r+3})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\right\|_{L^\infty_\xi}$$

is integrable in time. We begin by proving an estimate for the symbol $\mathbf{T}_n$. We have

$$\begin{aligned}
&\mathcal{F}^{-1}\left[\mathbf{T}_n(\eta_1, \eta_2, \dots, \eta_{2n+1})\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\cdots\varsigma_{j_{2n+1}}(\eta_{2n+1})\right] \\
&= \iiint_{\mathbb{R}^{2n+1}} e^{i(y_1\eta_1 + y_2\eta_2 + \dots + y_{2n+1}\eta_{2n+1})}\left[\int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1}(1 - e^{i\eta_j\zeta})}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta\right]\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\cdots\varsigma_{j_{2n+1}}(\eta_{2n+1}) \, \mathrm{d}\boldsymbol{\eta}_n \\
&= \iiint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \frac{(e^{iy_1\eta_1} - e^{i\eta_1(\zeta+y_1)})\cdots(e^{iy_{2n+1}\eta_{2n+1}} - e^{i\eta_{2n+1}(\zeta+y_{2n+1})})}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta\right]\varsigma_{j_1}(\eta_1)\cdots\varsigma_{j_{2n+1}}(\eta_{2n+1}) \, \mathrm{d}\boldsymbol{\eta}_n \\
&= \int_{\mathbb{R}} \frac{1}{|\zeta|^{2n+1}}\left[\mathcal{F}^{-1}[\varsigma_{j_1}](y_1) - \mathcal{F}^{-1}[\varsigma_{j_1}](\zeta + y_1)\right]\cdots\left[\mathcal{F}^{-1}[\varsigma_{j_{2n+1}}](y_{2n+1}) - \mathcal{F}^{-1}[\varsigma_{j_{2n+1}}](\zeta + y_{2n+1})\right] \, \mathrm{d}\zeta,
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\mathbf{T}_n(\eta_1, \eta_2, \dots, \eta_{2n+1})\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\cdots\varsigma_{j_{2n+1}}(\eta_{2n+1})\right]\right\|_{L^1} \\
&\quad\lesssim \int_{\mathbb{R}} 2^{j_1 + \dots + j_{2n+1}} \frac{1}{|\zeta|^{2n+1}} \min\{2^{-j_1}, |\zeta|\}\min\{2^{-j_2}, |\zeta|\}\cdots\min\{2^{-j_{2n+1}}, |\zeta|\} \, \mathrm{d}\zeta.
\end{aligned}$$

Let $\ell_1, \ell_2, \dots, \ell_{2n+1}$ be a permutation of $j_1, j_2, \dots, j_{2n+1}$ satisfying $2^{-\ell_1} \leq 2^{-\ell_2} \leq \dots \leq 2^{-\ell_{2n+1}}$. Then

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\mathbf{T}_n(\eta_1, \eta_2, \dots, \eta_{2n+1})\varsigma_{j_1}(\eta_1)\varsigma_{j_2}(\eta_2)\cdots\varsigma_{j_{2n+1}}(\eta_{2n+1})\right]\right\|_{L^1} \\
&\quad\lesssim \int_{|\zeta| > 2^{-\ell_{2n+1}}} \frac{1}{|\zeta|^{2n+1}} \, \mathrm{d}\zeta + \int_{2^{-\ell_{2n}} < |\zeta| < 2^{-\ell_{2n+1}}} \frac{2^{\ell_1}}{|\zeta|^{2n}} \, \mathrm{d}\zeta \\
&\quad\quad + \dots + \int_{2^{-\ell_1} < |\zeta| < 2^{-\ell_2}} \frac{2^{\ell_1 + \dots + \ell_{2n}}}{|\zeta|} \, \mathrm{d}\zeta + \int_{|\zeta| < 2^{-\ell_1}} 2^{\ell_1 + \dots + \ell_{2n+1}} \, \mathrm{d}\zeta \\
&\quad\lesssim 2^{\ell_2 + \dots + \ell_{2n+1}}.
\end{aligned}$$

Therefore, by Lemma 2.3.2, we have

$$\left\| (|\xi| + |\xi|^{r+3}) \widehat{\mathcal{N}_{\geq 5}(\varphi)} \right\|_{L^\infty_\xi} \lesssim (t+1)^{(r+3)p_1} \|\mathcal{N}_{\geq 5}(\varphi)\|_{L^1} \lesssim \|\varphi\|_{H^1}^2 \sum_{n=2}^{\infty} \left(\|\varphi_x\|_{L^\infty}^{2n-1} + \|L\varphi_x\|_{L^\infty}^{2n-1} \right).$$

Using the dispersive estimate Lemma 5.3.2, we see that the right-hand-side is integrable in t, which leads to

$$\int_0^\infty \left\| (|\xi| + |\xi|^{r+3}) \widehat{\mathcal{N}_{\geq 5}(\varphi)} \right\|_{L^\infty_\xi} \mathrm{d}t \lesssim \varepsilon_0.$$

This completes the proof of Theorem 5.0.1.

CHAPTER 6

Two-front SQG solutions

Twice and thrice over, as they say, good is it to repeat and review what is good.

– Plato

In Chapter 3, we derive contour dynamics equations (3.3) describing the evolution of two SQG fronts located at

$$y = h_+ + \varphi(x,t) \qquad \text{and} \qquad y = h_- + \psi(x,t).$$

In this chapter, we mainly focus on the initial value problem for (3.3)

$$\begin{aligned}
&\varphi_t(x,t) - (\Theta_+ - \Theta_-)(\gamma + \log h)\varphi_x(x,t) - 2\Theta_+ \log|\partial_x|\varphi_x(x,t) + 2\Theta_- K_0(2h|\partial_x|)\psi_x(x,t)\\
&+ \Theta_+ \int_{\mathbb{R}} \left[\varphi_x(x+\zeta,t) - \varphi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (\varphi(x+\zeta,t) - \varphi(x,t))^2}} - \frac{1}{|\zeta|}\right\} \mathrm{d}\zeta\\
&+ \Theta_- \int_{\mathbb{R}} \left[\psi_x(x+\zeta,t) - \varphi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (-2h + \psi(x+\zeta,t) - \varphi(x,t))^2}} - \frac{1}{\sqrt{\zeta^2 + (2h)^2}}\right\} \mathrm{d}\zeta = 0,\\
&\psi_t(x,t) + (\Theta_+ - \Theta_-)(\gamma + \log h)\psi_x(x,t) - 2\Theta_- \log|\partial_x|\psi_x(x,t) + 2\Theta_+ K_0(2h|\partial_x|)\varphi_x(x,t) \qquad (6.1)\\
&+ \Theta_- \int_{\mathbb{R}} \left[\psi_x(x+\zeta,t) - \psi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (\psi(x+\zeta,t) - \psi(x,t))^2}} - \frac{1}{|\zeta|}\right\} \mathrm{d}\zeta\\
&+ \Theta_+ \int_{\mathbb{R}} \left[\varphi_x(x+\zeta,t) - \psi_x(x,t)\right]\left\{\frac{1}{\sqrt{\zeta^2 + (2h + \varphi(x+\zeta,t) - \psi(x,t))^2}} - \frac{1}{\sqrt{\zeta^2 + (2h)^2}}\right\} \mathrm{d}\zeta = 0,\\
&\varphi(x,0) = \varphi_0(x), \qquad \psi(x,t) = \psi(x).
\end{aligned}$$

A main theorem we prove is the local well-posedness of (6.1) with small and smooth initial data.

Theorem 6.0.1. *Let $s \geq 4$ be an integer, and suppose that $\varphi_0, \psi_0 \in H^s(\mathbb{R})$ satisfy: (i)*

$$\|T_{B^{\log}[\varphi_0]}\|_{L^2\to L^2} \leq C, \quad \|T_{B^{\log}[\psi_0]}\|_{L^2\to L^2} \leq C$$

for some constant $0 < C < 2$, where the symbol $B^{\log}[f]$ is defined in (5.12)*; (ii)*

$$\sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\varphi_0\|^{2n}_{W^{3,\infty}} + \|L\varphi_0\|^{2n}_{W^{3,\infty}}\Big) < \infty, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\psi_0\|^{2n}_{W^{3,\infty}} + \|L\psi_0\|^{2n}_{W^{3,\infty}}\Big) < \infty,$$

where $L = \log|\partial_x|$ is the Fourier multiplier with symbol $\log|\xi|$, c_n is given by (5.5), and $\tilde{C} > 1$ is the constant depending only on s and h in Proposition 6.2.2. Then there exists $T > 0$, depending only on $\|\varphi_0\|_{H^s}$, $\|\psi_0\|_{H^s}$, C, and $\tilde{C}$, such that the initial value problem for (6.1) with $\varphi(x,0) = \varphi(x)$, $\psi(x,0) = \psi_0(x)$ has a unique solution with $\varphi, \psi \in C([0,T); H^s(\mathbb{R}))$.

Remark 6.0.2. The smallness conditions in this theorem arise from the fact that the nonlinear terms in the front equations lose derivatives, and we use a multilinear expansion of the nonlinearity to extract the terms responsible for the loss of derivatives. This expansion can only be done when the solutions are sufficiently small and requires Condition (ii). We then use the linear terms to control these nonlinear terms in a weighted energy space, but our weight may degenerate if Condition (i) fails.

Condition (ii) also implies that the initial data satisfies the non-intersection condition

$$2h - \psi_0(x,t) + \varphi_0(x,t) > 0 \text{ for all } x \in \mathbb{R},$$

since it guarantees that $|\psi_0(x) - \varphi_0(x)| < 2h$ for all $x \in \mathbb{R}$.

The strategy to prove this theorem is similar to the proof of Theorem 5.2.3, where we need to para-linearize the system and use a weighted energy to prevent loss of derivatives, but with complication in dealing with the interactive terms between φ and ψ, as well as the Bessel K function.

We analyze the linear stability of unperturbed flat two front solutions to the system (3.3) in Section 6.1. And in Section 6.2, we prove Theorem 6.0.1.

6.1. Linear stability

We have mentioned in Section 3.2 that the SQG equation (1.1) admits shear flow solution

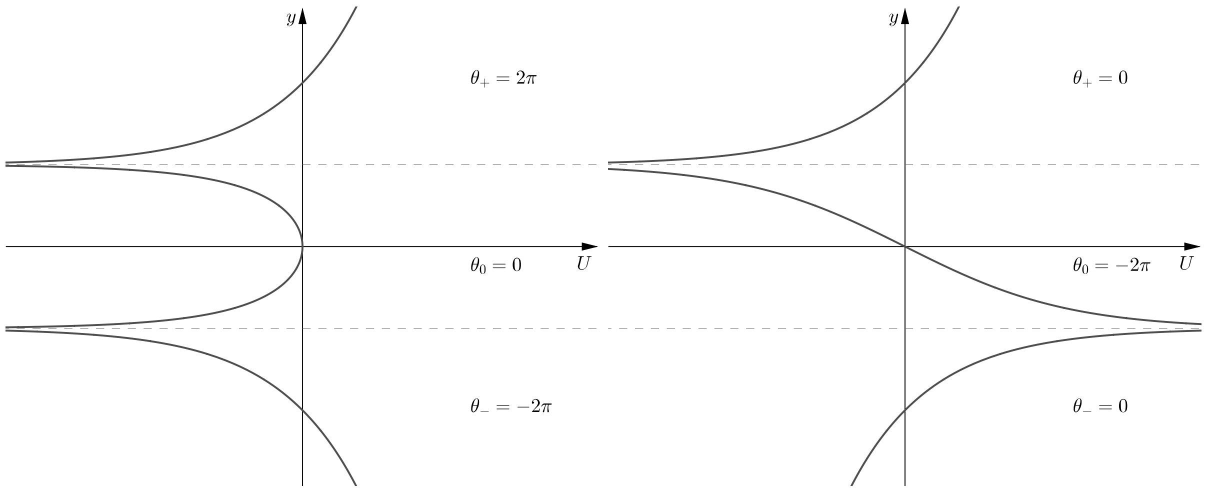
$$\bar{\theta}(y) = \begin{cases} \theta_+ & \text{if } y > h_+, \\ \theta_0 & \text{if } h_- < y < h_+, \\ \theta_- & y < h_-, \end{cases}$$

whose corresponding velocity field is

$$U(y) = 2\Theta_+ \log|y - h_+| + 2\Theta_- \log|y - h_-|. \tag{6.2}$$

Recall that $\Theta_\pm$ are normalized constants defined in (1.6).

This shear-flow solution is the SQG analog of the piecewise linear shear flow that is often considered for the Euler equation (see Figure 6.1). We observe that the tangential velocity of the shear flow on the fronts diverges to infinity and that $U(y) \to 0$ as $|y| \to \infty$ if $\Theta_+ + \Theta_- = 0$; otherwise $|U(y)| \to \infty$ as $|y| \to \infty$.



(a) Symmetric SQG shear flow. (b) Anti-symmetric SQG shear flow.

FIGURE 6.1. SQG shear flows. The symmetric flows have scaled jumps $\Theta_+ = \Theta_- = 1$, and the anti-symmetric flows have $\Theta_+ = -\Theta_- = 1$.

There do not appear to be many studies of the stability of SQG shear flows $\mathbf{u} = (U(y), 0)$. However, as noted in [**FS05**] for SQG shear flows, the classical necessary conditions for the linearized instability of Euler shear flows — the Rayleigh and Fjørtoft criteria — carry over directly to sufficiently smooth flows: If there are linear modes with exponential growth in time, then $|\partial_y|U = (-\partial_y^2)^{1/2}U$ must change sign, and for any constant U_*, the function $(U - U_*) \cdot |\partial_y|U$ must be strictly positive for some values of y. Conversely, Friedlander and Shvydkoy [**FS05**] prove that the SQG shear flow with $U(y) = \sin y$ is linearly unstable.

To study the stability of the two-front SQG shear flows (6.2) by contour dynamics, we linearize the system (3.3) about $\varphi = \psi = 0$ to get

$$\begin{aligned} &\varphi_t - (\Theta_+ - \Theta_-)(\gamma + \log h)\varphi_x - 2\Theta_+ \log|\partial_x|\varphi_x + 2\Theta_- K_0(2h|\partial_x|)\psi_x = 0, \\ &\psi_t + (\Theta_+ - \Theta_-)(\gamma + \log h)\psi_x - 2\Theta_- \log|\partial_x|\psi_x + 2\Theta_+ K_0(2h|\partial_x|)\varphi_x = 0, \end{aligned} \tag{6.3}$$

where γ is the Euler-Mascheroni constant and K_0 is the modified Bessel's function of second kind defined in (2.21).

Taking the Fourier transform of (6.3) with respect to x, we get the system

$$\begin{bmatrix} \hat{\varphi} \\ \hat{\psi} \end{bmatrix}_t = \begin{bmatrix} i\xi\left[(\Theta_+ - \Theta_-)(\gamma + \log h) + 2\Theta_+ \log|\xi|\right] & -2\Theta_- i\xi K_0(2h|\xi|) \\ -2\Theta_+ i\xi K_0(2h|\xi|) & i\xi\left[(\Theta_- - \Theta_+)(\gamma + \log h) + 2\Theta_- \log|\xi|\right] \end{bmatrix} \begin{bmatrix} \hat{\varphi} \\ \hat{\psi} \end{bmatrix}, \tag{6.4}$$

The characteristic polynomial (in μ) of the coefficient matrix in (6.4) is

$$\mu^2 - 2i\xi \log|\xi|(\Theta_+ + \Theta_-)\mu + 4\Theta_+\Theta_-\xi^2 K_0^2(2h|\xi|)$$
$$- \xi^2\left[(\Theta_+ - \Theta_-)(\gamma + \log h) + 2\Theta_+ \log|\xi|\right]\left[(\Theta_- - \Theta_+)(\gamma + \log h) + 2\Theta_- \log|\xi|\right],$$

with roots

$$\mu_\pm(\xi) = \frac{1}{2}\left\{2i\xi \log|\xi|\,(\Theta_+ + \Theta_-) \pm \sqrt{\triangle(\xi)}\right\}, \tag{6.5}$$

where the discriminant $\triangle$ is given by

$$\triangle(\xi) = -4(\Theta_+ - \Theta_-)^2\,(\gamma + \log h + \log|\xi|)^2\,\xi^2 - 4\Theta_+\Theta_-\xi^2 K_0^2(2h|\xi|).$$

If $\Theta_+\Theta_- > 0$, then $\triangle(\xi) \leq 0$ for all $\xi \in \mathbb{R}$, so the roots of the characteristic polynomial are imaginary and the SQG shear flow is linearly stable. In particular, the symmetric SQG shear flows shown in Figure 6.1(a) are linearly stable.

On the other hand, if $\triangle(\xi) > 0$ for some $\xi \in \mathbb{R}$, then there is a mode with positive growth rate, and the shear flow is linearly unstable. For the anti-symmetric SQG shear flow shown in Figure 6.1(b), we find that

$$\triangle(\xi) = 16\xi^2 K_0^2(2h|\xi|) - 16\xi^2\left[\log(h|\xi|) + \gamma\right]^2, \tag{6.6}$$

where γ is the Euler-Mascheroni constant. A numerical plot of the corresponding growth rates and wave speeds is shown in Figure 6.2 (c.f. [**Val17**] for the Euler equation). The instability results from an interaction between negative and positive energy waves on the fronts that leads to an exponential growth in time when the horizontal wavelengths of the waves are sufficiently large in comparison with the distance between the fronts.

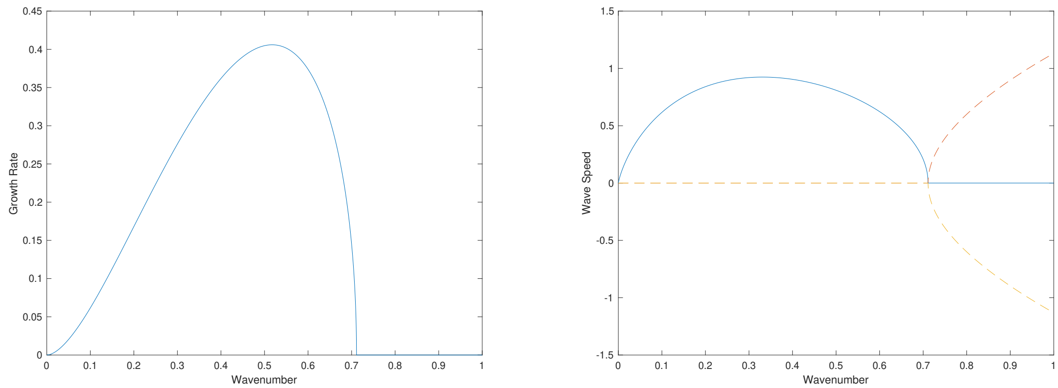


FIGURE 6.2. Left: Growth rate $\Im\mu$ for anti-symmetric SQG flow in Figure 6.1 with $\Theta_+ = 1$, $\Theta_- = -1$, and $h = 1$, calculated from (6.5) and (6.6). Right: Real (dashed) and imaginary (solid) wave speeds $c = \mu/\xi$. The flow is unstable for $0 < h|\xi| \lesssim 0.71129$, with the maximum growth rate occurring at $h|\xi| \approx 0.51756$.

6.2. Local well-posedness

In this section, we prove the local well-posedness of (6.1) posed on $\mathbb{R}$. Theorem 6.0.1 follows from *a priori* estimates and classical C_0-semigroup theory for local existence (see, e.g. [**Paz83**]). Therefore, we only derive the *a priori* estimates (Proposition 6.2.2).

6.2.1. Expansion and para-linearization of the system.

The final goal of the subsection is to para-linearize the equations in (3.3) and extract a term that accounts for the loss of derivatives. To start with, we carry out a multilinear expansion of the nonlinearities in (3.3) assuming small amplitude and small slope, i.e., $|\varphi|, |\psi| \ll h$ and $|\varphi_x|, |\psi_x| \ll 1$. We will use the expanded system in the local existence proof, and the smallness condition (ii) in Theorem 6.0.1 is sufficient to justify the expansion.

The first nonlinear terms in each equation of (3.3) are the same as the nonlinear term in (3.2), then (5.6) and (5.7) apply. As for the second nonlinear terms, we take Fourier transforms and use (2.21) to get

$$\int_{\mathbb{R}} \frac{\big(\psi(x+\zeta,t)\big)^m}{(\zeta^2+(2h)^2)^{n+\frac{1}{2}}}\,\mathrm{d}\zeta = \begin{cases} \Gamma\left(\dfrac{\sqrt{\pi}\,\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}\right)(2h)^{-2n} & \text{if } m=0,\\[2ex] \dfrac{2\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)(4h)^n}|\partial_x|^n K_n(2h|\partial_x|)\big(\psi(x,t)\big)^m & \text{if } m \geq 1. \end{cases}$$

Then, using the Taylor expansion

$$(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} c_n x^n,$$

where c_n is defined in (5.5),we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\psi_x(x+\zeta,t) - \varphi_x(x,t)\right] \left\{ \frac{1}{(\zeta^2 + (-2h + \psi(x+\zeta,t) - \varphi(x,t))^2)^{\frac{1}{2}}} - \frac{1}{(\zeta^2 + (2h)^2)^{\frac{1}{2}}} \right\} \mathrm{d}\zeta \\
&= \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}} \frac{\psi_x(x+\zeta,t) - \varphi_x(x,t)}{(\zeta^2 + (2h)^2)^{n+\frac{1}{2}}} \Big[-4h\big(\psi(x+\zeta,t) - \varphi(x,t)\big) + \big(\psi(x+\zeta,t) - \varphi(x,t)\big)^2 \Big]^n \mathrm{d}\zeta \\
&= \sum_{n=1}^{\infty} \sum_{\ell=0}^{n} d_{n,\ell} \partial_x \int_{\mathbb{R}} \frac{\big(\psi(x+\zeta,t) - \varphi(x,t)\big)^{2n-\ell+1}}{(\zeta^2 + (2h)^2)^{n+\frac{1}{2}}} \mathrm{d}\zeta \\
&= \sum_{n=1}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{2n-\ell+1} d_{n,\ell,m} \partial_x \Big\{ \big(\varphi(x,t)\big)^{2n-\ell+1-m} \int_{\mathbb{R}} \frac{\big(\psi(x+\zeta,t)\big)^m}{(\zeta^2 + (2h)^2)^{n+\frac{1}{2}}} \mathrm{d}\zeta \Big\} \\
&= \sum_{n=1}^{\infty} \sum_{\ell=0}^{n} d_{n,\ell,0,1} \partial_x \left\{ \big(\varphi(x,t)\big)^{2n-\ell+1} \right\} \\
&\quad + \sum_{n=1}^{\infty} \sum_{\ell=0}^{n} \sum_{m=1}^{2n-\ell+1} d_{n,\ell,m,1} \partial_x \Big\{ \big(\varphi(x,t)\big)^{2n-\ell+1-m} |\partial_x|^n K_n(2h|\partial_x|) \big(\psi(x,t)\big)^m \Big\},
\end{aligned}$$

where

$$\begin{aligned}
d_{n,\ell} &= \frac{\sqrt{\pi}(-4h)^\ell}{(2n-\ell+1)\Gamma(\ell+1)\Gamma(n+1-\ell)\Gamma\left(\frac{1}{2}-n\right)}, \\
d_{n,\ell,m} &= \frac{(-1)^{2n+1-m}\sqrt{\pi}\Gamma(2n+2-\ell)(4h)^\ell}{(2n-\ell+1)\Gamma(\ell+1)\Gamma(n+1-\ell)\Gamma\left(\frac{1}{2}-n\right)\Gamma(m+1)\Gamma(2n+2-m-\ell)}, \\
d_{n,\ell,m,1} &= \begin{cases} d_{n,\ell,0} \cdot \dfrac{2\sqrt{\pi}\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)(4h)^{n+\frac{1}{2}}} & \text{if } m = 0, \\[2ex] d_{n,\ell,m} \cdot \dfrac{2\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)(4h)^n} & \text{if } m \geq 1. \end{cases}
\end{aligned}$$

The computation for the second nonlinear term in the second equation of the systems (3.3) is similar. We only need to replace φ by ψ, multiply $d_{n,\ell}$ and $d_{n,\ell,m}$ by $(-1)^\ell$, and replace $d_{n,\ell,m,1}$ by $d_{n,\ell,m,2}$ where

$$d_{n,\ell,m,2} = (-1)^\ell d_{n,\ell,m,1}.$$

To para-linearize (3.3), Proposition 5.1.1 applies to the first nonlinear term in each equation. As for the second nonlinear terms, we need the following proposition.

Proposition 6.2.1. *Suppose that $\varphi(\cdot,t), \psi(\cdot,t) \in H^s(\mathbb{R})$ with $s \geq 4$ and $\|\varphi\|_{W^{3,\infty}} + \|\psi\|_{W^{3,\infty}}$ is sufficiently small. Then we can write*

$$\sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\sum_{m=1}^{2n-\ell+1} d_{n,\ell,m,1}\partial_x \left\{ \big(\varphi(x,t)\big)^{2n-\ell+1-m} |\partial_x|^n K_n(2h|\partial_x|)\big(\psi(x,t)\big)^m \right\} = T_{\mathfrak{B}_1^{(1)}[\varphi,\psi]}\varphi_x + \mathcal{R}_2,$$

where

$$\begin{aligned} \mathfrak{B}_1^{(1)}[\varphi,\psi] &= \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\sum_{m=1}^{2n-\ell} d_{n,\ell,m,1}(2n-\ell+1-m)\mathfrak{B}_{1,n,\ell,m}^{(1)}[\varphi,\psi], \\ \mathfrak{B}_{1,n,\ell,m}^{(1)}[\varphi,\psi] &= \varphi^{2n-\ell-m}|\partial_x|^n K_n(2h|\partial_x|)\psi^m. \end{aligned} \tag{6.7}$$

The symbol $\mathfrak{B}_1^{(1)}[\varphi,\psi]$ and remainder $\mathcal{R}_2$ satisfy symbol estimates

$$\|\mathfrak{B}_1^{(1)}[\varphi,\psi]\|_{\mathcal{M}_{(1,1)}} \lesssim \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\sum_{m=1}^{2n-\ell} C(n,s)h^{\ell-2n}\|\varphi\|_{W^{1,\infty}}^{2n-\ell-m}\|\psi\|_{W^{1,\infty}}^{m},$$

$$\begin{aligned} \|\mathcal{R}_2\|_{H^s} \lesssim\ & \|\varphi\|_{H^s}\sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\sum_{m=1}^{2n-\ell} C(n,s)h^{\ell-2n}\|\psi\|_{W^{1,\infty}}^{m}\|\varphi\|_{W^{1,\infty}}^{2n-\ell-m} \\ &+ \|\psi\|_{H^s}\sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\sum_{m=1}^{2n-\ell} C(n,s)h^{\ell-2n-1}\|\psi\|_{W^{1,\infty}}^{m-1}\|\varphi\|_{W^{1,\infty}}^{2n-\ell+1-m} \\ &+ \|\psi\|_{H^s}\sum_{n=1}^{\infty}\sum_{\ell=0}^{n} C(n,s)h^{\ell-2n-\frac{3}{2}}\|\psi\|_{W^{1,\infty}}^{2n-\ell}. \end{aligned} \tag{6.8}$$

A similar result holds with φ and ψ exchanged.

Proof. We suppress the dependence of variables of φ and ψ for simplicity. By the product rule and the decomposition (2.7), we see that for $m < 2n-\ell+1$,

$$\begin{aligned} &\partial_x\left\{\varphi^{2n-\ell+1-m}|\partial_x|^n K_n(2h|\partial_x|)\psi^m\right\} \\ =\ &\Big[(2n-\ell+1-m)\varphi^{2n-\ell-m}|\partial_x|^n K_n(2h|\partial_x|)\psi^m\Big]\varphi_x(x,t) + \varphi^{2n-\ell+1-m}\partial_x|\partial_x|^n K_n(2h|\partial_x|)\psi^m \\ =\ &(2n-\ell+1-m)T_{\mathfrak{B}_{1,n,\ell,m}^{(1)}[\varphi,\psi]}\varphi_x + \mathcal{R}_{2,n,\ell,m}, \end{aligned}$$

where $\mathfrak{B}_{1,n,\ell,m}^{(1)}[\varphi,\psi]$ is defined as in (6.7), and, by Lemma 2.4.1,

$$\begin{aligned} \|\mathcal{R}_{2,n,\ell,m}\|_{H^s} \leq\ & C(n,s)\Gamma(n)h^{-n}\|\varphi\|_{H^s}\|\psi\|_{W^{1,\infty}}^{m}\|\varphi\|_{W^{1,\infty}}^{2n-\ell-m} \\ &+ C(n,s)\Gamma(n)h^{-n-1}\|\psi\|_{H^s}\|\psi\|_{W^{1,\infty}}^{m-1}\|\varphi\|_{W^{1,\infty}}^{2n-\ell+1-m}. \end{aligned}$$

When $m = 2n - \ell + 1$, again, by Lemma 2.4.1, we have

$$\mathcal{R}_{2,n,\ell,2n-\ell+1} = \partial_x |\partial_x|^n K_n(2h|\partial_x|)\psi^{2n-\ell+1},$$

$$\|\mathcal{R}_{2,n,\ell,2n-\ell+1}\|_{H^s} \leq C(n,s)\Gamma(n)h^{-n-1}\|\psi\|_{H^s}\|\psi\|_{W^{1,\infty}}^{2n-\ell}.$$

The estimates (6.8) for $\mathfrak{B}_1^{(1)}[\varphi,\psi]$ and

$$\mathcal{R}_2 = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\sum_{m=1}^{2n-\ell+1} \mathcal{R}_{2,n,\ell,m},$$

then follow from the above estimates and Stirling's formula applied to the Γ-function coefficients.

□

By Proposition 5.1.1 and Proposition 6.2.1, we can write (3.3) in the following para-linearized form

$$\begin{aligned}
&\varphi_t - (\Theta_+ - \Theta_-)(\gamma + \log h)\varphi_x + T_{\mathfrak{B}_\varphi^{(1)}}\varphi_x + \mathcal{R}_1 + 2\Theta_- K_0(2h|\partial_x|)\psi_x = \Theta_+ L\big[(2 - T_{B^{\log}[\varphi]})\varphi\big]_x,\\
&\psi_t + (\Theta_+ - \Theta_-)(\gamma + \log h)\psi_x + T_{\mathfrak{B}_\psi^{(1)}}\psi_x + \mathcal{R}_2 + 2\Theta_+ K_0(2h|\partial_x|)\varphi_x = \Theta_- L\big[(2 - T_{B^{\log}[\psi]})\psi\big]_x,
\end{aligned} \tag{6.9}$$

where

$$\begin{aligned}
\mathfrak{B}_\varphi^{(1)} &= \Theta_- \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}(2n-\ell+1)d_{n,\ell,0,1}\varphi^{2n-\ell} + \Theta_-\mathfrak{B}_1^{(1)}[\varphi,\psi] + \Theta_+ B^0[\varphi],\\
\mathfrak{B}_\psi^{(1)} &= \Theta_+ \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}(2n-\ell+1)d_{n,\ell,0,1}\psi^{2n-\ell} + \Theta_+\mathfrak{B}_1^{(1)}[\psi,\varphi] + \Theta_- B^0[\psi],
\end{aligned}$$

and $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded by

$$\|\mathcal{R}_i\|_{H^s} \lesssim (\|\varphi\|_{H^s} + \|\psi\|_{H^s})F(\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}} + \|\psi\|_{W^{3,\infty}} + \|L\psi\|_{W^{3,\infty}}), \quad i = 1,2, \tag{6.10}$$

where F is a positive polynomial.

6.2.2. Energy estimates. We define homogeneous and non-homogeneous weighted energies that are equivalent to the H^s-energies by

$$\begin{aligned}
E^{(j)}(t) = \int_{\mathbb{R}} &|\Theta_+||D|^j\varphi(x,t)\cdot\Big(2 - T_{B^{\log}[\varphi]}\Big)^{2j+1}|D|^j\varphi(x,t)\\
&+ |\Theta_-||D|^j\psi(x,t)\cdot\Big(2 - T_{B^{\log}[\psi]}\Big)^{2j+1}|D|^j\psi(x,t)\,\mathrm{d}x,
\end{aligned}$$

$$\tilde{E}^{(s)}(t) = \|\varphi\|^2_{L^2(\mathbb{R})} + \|\psi\|^2_{L^2(\mathbb{R})} + \sum_{j=1}^{s} E^{(j)}(t).$$

For simplicity, we consider only integer norms with $s \in \mathbb{N}$.

We now are ready to prove the following *a priori* estimates.

Proposition 6.2.2. *Let $s > 4$ be an integer and φ, ψ a smooth solution of* (6.9) *with $\varphi_0, \psi_0 \in H^s(\mathbb{R})$. There exists a constant $\tilde{C} > 0$, depending only on s, such that if φ_0, ψ_0 satisfies*

$$\big\|T_{B^{\log}[\varphi_0]}\|_{L^2\to L^2} \leq C, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\varphi_0\|^{2n}_{W^{3,\infty}} + \|L\varphi_0\|^{2n}_{W^{3,\infty}} \Big) < \infty,$$

$$\big\|T_{B^{\log}[\psi_0]}\|_{L^2\to L^2} \leq C, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\psi_0\|^{2n}_{W^{3,\infty}} + \|L\psi_0\|^{2n}_{W^{3,\infty}} \Big) < \infty,$$

for some constant $0 < C < 2$, then there exists a time $T > 0$ such that

$$\big\|T_{B^{\log}[\varphi(t)]}\|_{L^2\to L^2} < 2, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\varphi(t)\|^{2n}_{W^{3,\infty}} + \|L\varphi(t)\|^{2n}_{W^{3,\infty}} \Big) < \infty,$$

$$\big\|T_{B^{\log}[\psi(t)]}\|_{L^2\to L^2} < 2, \qquad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\psi(t)\|^{2n}_{W^{3,\infty}} + \|L\psi(t)\|^{2n}_{W^{3,\infty}} \Big) < \infty,$$

for all $t \in [0, T]$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{E}^{(s)}(t) \leq F\left(\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}} + \|\psi\|_{W^{3,\infty}} + \|L\psi\|_{W^{3,\infty}} \right) \tilde{E}^{(s)}(t), \tag{6.11}$$

where $F(\cdot)$ is an increasing, continuous, real-valued function.

Proof. Observe that $\|\varphi\|^2_{L^2(\mathbb{R})} + \|\psi\|^2_{L^2(\mathbb{R})}$ is conserved by the system. So we only need to estimate the higher-order energy. By direct calculation, for $f = \varphi$ or ψ,

$$\partial_t (2 - T_{B^{\log}[f]})^s f = (2 - T_{B^{\log}[f]})^s f_t - s(2 - T_{B^{\log}[f]})^{s-1} T_{\partial_t B^{\log}[f]} \psi + \mathcal{R}(f), \tag{6.12}$$

where the remainder term $\mathcal{R}$ is bounded by (6.10).

By continuity in time, there exists $T > 0$ such that

$$\sum_{n=1}^{\infty} \tilde{C}^n |c_n| \Big(\|\varphi(t)\|^{2n}_{W^{3,\infty}} + \|L\varphi(t)\|^{2n}_{W^{3,\infty}} + \|\psi(t)\|^{2n}_{W^{3,\infty}} + \|L\psi(t)\|^{2n}_{W^{3,\infty}} \Big) < \infty \qquad \text{for all } 0 \leq t \leq T.$$

We apply the operator $|D|^s$ to the first equation of (6.9) to get

$$\begin{aligned}|D|^s\varphi_t - (\Theta_+ - \Theta_-)(\gamma + \log h)|D|^s\varphi_x + |D|^s T_{\mathfrak{B}^{(1)}_\varphi}\varphi_x \\ + |D|^s\mathcal{R}_1 + 2\Theta_-|D|^s K_0(2h|\partial_x|)\psi_x(x,t) = |D|^s\partial_x L\big[(2 - T_{B^{\log}[\varphi]})\varphi\big].\end{aligned} \tag{6.13}$$

Using Lemma 2.2.2, we find that

$$\begin{aligned}|D|^s\left[(2 - T_{B^{\log}[\varphi]})\varphi\right] &= 2|D|^s\varphi - |D|^s(T_{B^{\log}[\varphi]}\varphi) \\ &= 2|D|^s\varphi - T_{B^{\log}[\varphi]}|D|^s\varphi + sT_{\partial_x B^{\log}[\varphi]}|D|^{s-2}\varphi_x + \mathcal{R}_3,\end{aligned}$$

where

$$\|\partial_x\mathcal{R}_3\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\|\varphi\|^{2n}_{W^{3,\infty}}\right)\|\varphi\|_{H^{s-1}}.$$

Thus, we can write the right-hand side of (6.13) as

$$\begin{aligned}&\partial_x L|D|^s\left[(2 - T_{B^{\log}[\varphi]})\varphi\right] \\ &= \partial_x L\left[(2 - T_{B^{\log}[\varphi]})|D|^s\varphi + sT_{\partial_x B^{\log}[\varphi]}|D|^{s-2}\varphi_x\right] + \mathcal{R}_4 \\ &= L\big\{(2 - T_{B^{\log}[\varphi]})|D|^s\varphi_x - T_{\partial_x B^{\log}[\varphi]}|D|^s\varphi - sT_{\partial_x B^{\log}[\varphi]}|D|^s\varphi\big\} + \mathcal{R}_4 \\ &= L\left\{(2 - T_{B^{\log}[\varphi]})|D|^s\varphi_x - (s+1)T_{\partial_x B^{\log}[\varphi]}|D|^s\varphi\right\} + \mathcal{R}_4,\end{aligned}$$

where

$$\|\mathcal{R}_4\|_{L^2} \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\left(\|\varphi\|^{2n}_{W^{3,\infty}} + \|L\varphi\|^{2n}_{W^{3,\infty}}\right)\right)\|\varphi\|_{H^s}.$$

Applying $(2 - T_{B^{\log}[\varphi]})^s$ to (6.13), and commuting $(2 - T_{B^{\log}[\varphi]})^s$ with L up to remainder terms, we obtain that

$$\begin{aligned}&(2 - T_{B^{\log}[\varphi]})^s|D|^s\varphi_t - (\Theta_+ - \Theta_-)(\gamma + \log h)(2 - T_{B^{\log}[\varphi]})^s|D|^s\varphi_x \\ &\quad + (2 - T_{B^{\log}[\varphi]})^s\partial_x|D|^s T_{\mathfrak{B}^{(1)}_\varphi}\varphi + 2\Theta_-(2 - T_{B^{\log}[\varphi]})^s|D|^s K_0(2h|\partial_x|)\psi_x(x,t) \\ &\qquad = L\left\{(2 - T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi_x - (s+1)(2 - T_{B^{\log}[\varphi]})^s T_{\partial_x B^{\log}[\varphi]}|D|^s\varphi\right\} + \mathcal{R}_5 \\ &\qquad = \partial_x L\left\{(2 - T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi\right\} + \mathcal{R}_5,\end{aligned} \tag{6.14}$$

where $\|\mathcal{R}_5\|_{L^2}$ is bounded by the right-hand-side of (6.10).

By (6.12), the time derivative of $E^{(s)}(t)$ is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E^{(s)}(t) = &-\int_{\mathbb{R}} (2s+1)|D|^s\varphi \cdot (2-T_{B^{\log}[\varphi]})^{2s} T_{\partial_t B^{\log}[\varphi]} |D|^s\varphi \,\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} |D|^s\varphi \cdot (2-T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_t \,\mathrm{d}x + \int_{\mathbb{R}} \mathcal{R}\left(|D|^s\varphi\right)|D|^s\varphi \,\mathrm{d}x. \end{aligned} \tag{6.15}$$

We will estimate each of the terms on the right-hand side of (6.15).

Equation (6.9) implies that

$$\|\varphi_{xt}\|_{L^\infty} \lesssim \sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n}\Big),$$

so the first term on the right-hand side of (6.15) can be estimated by

$$\begin{aligned} &\left|\int_{\mathbb{R}} (2s+1)|D|^s\varphi \cdot (2-T_{B^{\log}[\varphi]})^{2s} T_{\partial_t B^{\log}[\varphi]} |D|^s\varphi \,\mathrm{d}x\right| \\ &\qquad \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}^2. \end{aligned}$$

We can estimate the third term on the right-hand side of (6.15) by

$$\int_{\mathbb{R}} \mathcal{R}\left(|D|^s\varphi\right)|D|^s\varphi \,\mathrm{d}x \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n|\Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n}\Big)\right)\|\varphi\|_{H^s}\|\varphi\|_{H^{s-1}}.$$

To estimate the second term on the right-hand side (6.15), we multiply (6.14) by $(2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi$, integrate the result with respect to x, and use the self-adjointness of $(2-T_{B^{\log}[\varphi]})^{s+1}$, which gives

$$\int_{\mathbb{R}} |D|^s\varphi \cdot (2-T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_t \,\mathrm{d}x = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$\begin{aligned} \mathrm{I} &= -\int_{\mathbb{R}} |D|^s\varphi \cdot (2-T_{B^{\log}[\varphi]})^{2s+1}|D|^s\partial_x T_{\mathfrak{B}^{(1)}_\varphi}\varphi \,\mathrm{d}x, \\ \mathrm{II} &= \int_{\mathbb{R}} (2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi \cdot \partial_x L(2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi \,\mathrm{d}x, \\ \mathrm{III} &= \int_{\mathbb{R}} (2-T_{B^{\log}[\varphi]})^{s+1}|D|^s\varphi \cdot \left(\mathcal{R}_1 + 2\Theta_- K_0(2h|\partial_x|)\psi_x\right)\,\mathrm{d}x, \\ \mathrm{IV} &= -\int_{\mathbb{R}} |D|^s\varphi \cdot (\Theta_+ - \Theta_-)(\gamma + \log h)(2-T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_x. \end{aligned}$$

We have $\mathrm{II} = 0$, since $\partial_x L$ is skew-symmetric, and

$$|\mathrm{III}| \lesssim \left(\sum_{n=0}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n} \Big) \right) (\|\varphi\|_{H^s}^2 + \|\psi\|_{H^s}^2).$$

Because $(2 - T_{B^{\log}[\varphi]})$ is self-adjoint,

$$\begin{aligned}
\mathrm{IV} &= -(\Theta_+ - \Theta_-)(\gamma + \log h) \int_{\mathbb{R}} (2 - T_{B^{\log}[\varphi]})^{2s+1} |D|^s \varphi \cdot |D|^s \varphi_x \, \mathrm{d}x \\
&= (\Theta_+ - \Theta_-)(\gamma + \log h) \int_{\mathbb{R}} \partial_x (2 - T_{B^{\log}[\varphi]})^{2s+1} |D|^s \varphi \cdot |D|^s \varphi \, \mathrm{d}x \\
&= -\mathrm{IV} + (\Theta_+ - \Theta_-)(\gamma + \log h) \int_{\mathbb{R}} [\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}] |D|^s \varphi \cdot |D|^s \varphi \, \mathrm{d}x.
\end{aligned}$$

By a commutator estimate,

$$\left| \int_{\mathbb{R}} [\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}] |D|^s \varphi \cdot |D|^s \varphi \, \mathrm{d}x \right| \lesssim \|\varphi\|_{H^s}^2 F(\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}}).$$

Therefore

$$|\mathrm{IV}| \lesssim \|\varphi\|_{H^s}^2 F(\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}}).$$

Term I estimate. We write $\mathrm{I} = -\mathrm{I}_a + \mathrm{I}_b$, where

$$\begin{aligned}
\mathrm{I}_a &= \int_{\mathbb{R}} |D|^s \varphi \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1} \partial_x T_{\mathfrak{B}_\varphi^{(1)}} |D|^s \varphi \, \mathrm{d}x, \\
\mathrm{I}_b &= \int_{\mathbb{R}} |D|^s \varphi \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1} \partial_x [T_{\mathfrak{B}_\varphi^{(1)}}, |D|^s] \varphi \, \mathrm{d}x.
\end{aligned}$$

By a commutator estimate and (5.13), the second integral satisfies

$$|\mathrm{I}_b| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}^2.$$

To estimate the first integral, we write it as

$$\mathrm{I}_a = \mathrm{I}_{a_1} - \mathrm{I}_{a_2},$$

where

$$\begin{aligned}
\mathrm{I}_{a_1} &= \int_{\mathbb{R}} |D|^s \varphi \cdot [(2 - T_{B^{\log}[\varphi]})^{2s+1}, \partial_x] \left(T_{\mathfrak{B}_\varphi^{(1)}} |D|^s \varphi \right) \mathrm{d}x, \\
\mathrm{I}_{a_2} &= \int_{\mathbb{R}} |D|^s \varphi_x \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1} \left(T_{\mathfrak{B}_\varphi^{(1)}} |D|^s \varphi \right) \mathrm{d}x.
\end{aligned}$$

Term I_{a_1} estimate. A Kato-Ponce commutator estimate and (5.13) gives

$$|\mathrm{I}_{a_1}| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}^2.$$

Term I_{a_2} estimate. We have

$$\begin{aligned}
\mathrm{I}_{a_2} &= \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi_x \,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot \Big\{ \partial_x \Big((2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi \Big) - \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi \Big\} \,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \partial_x \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi \,\mathrm{d}x \\
&= -\int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi_x + \left[\partial_x, T_{B^0[\varphi]}\right] |D|^s\varphi\right) \cdot (2 - T_{B^{\log}[\varphi]})^{2s+1}|D|^s\varphi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \left(T_{B^0[\varphi]}|D|^s\varphi\right) \cdot \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi \,\mathrm{d}x.
\end{aligned} \tag{6.16}$$

Using commutator estimates and (5.13), we get that

$$\begin{aligned}
\left\| \left[\partial_x, T_{B^0[\varphi]}\right] |D|^s\varphi \right\|_{L^2} &\lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \left(\|\varphi\|_{W^{3,\infty}}^2 + \|L\varphi\|_{W^{3,\infty}}^2 \right) \right) \|\varphi\|_{H^s}, \\
\left\| \Big[\partial_x, (2 - T_{B^{\log}[\varphi]})^{2s+1}\Big] |D|^s\varphi \right\|_{L^2} &\lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}, \\
\left\| \partial_x \Big[(2 - T_{B^{\log}[\varphi]})^{2s+1}, T_{B^0[\varphi]}\Big] |D|^s\varphi \right\|_{L^2} &\lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}^2.
\end{aligned}$$

Since $T_{B^0[\varphi]}$ is self-adjoint, we can rewrite (6.16) as

$$\mathrm{I}_{a_2} = -\mathrm{I}_{a_2} + \mathcal{R}_6,$$

with

$$|\mathcal{R}_6| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}^2,$$

and we conclude that

$$|\mathrm{I}_{a_2}| \lesssim \left(\sum_{n=1}^{\infty} C(n,s)|c_n| \Big(\|\varphi\|_{W^{3,\infty}}^{2n} + \|L\varphi\|_{W^{3,\infty}}^{2n} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n} \Big) \right) \|\varphi\|_{H^s}^2.$$

By a similar procedure, we can obtain the estimate for ψ. This completes the estimate of the terms on the right hand side of (6.15). Collecting the above estimates and using the interpolation inequalities, we obtain that

$$\tilde{E}^{(s)}(t) \le \tilde{E}^{(s)}(0) + \int_0^t F\left(\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}} + \|\psi\|_{W^{3,\infty}} + \|L\psi\|_{W^{3,\infty}}\right)\|\varphi\|_{H^s}^2 \,\mathrm{d}t', \qquad (6.17)$$

where F is a positive, increasing, continuous, real-valued function.

We observe that there exists a constant $\tilde{C}(s) > 0$ such that $C(n,s) \lesssim \tilde{C}(s)^n$. The series in F then converges whenever $\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}} + \|\psi\|_{W^{3,\infty}}^{2n} + \|L\psi\|_{W^{3,\infty}}^{2n}$ is sufficiently small, and we can choose F to be an increasing, continuous, real-valued function.

Finally, since $\|2-T_{B^{\log}[\varphi_0]}\|_{L^2\to L^2} \ge 2-C$, and $\|B^{\log}[\varphi](\cdot,t)\|_{\mathcal{M}_{(0,0)}}$ and $F\left(\|\varphi\|_{W^{3,\infty}} + \|L\varphi\|_{W^{3,\infty}}\right)$ are continuous in time, there exist $T > 0$ and $m > 0$, depending only on the initial data, such that

$$\|2 - T_{B^{\log}[\varphi(t)]}\|_{L^2\to L^2} \ge m \qquad \text{for } 0 \le t \le T.$$

We therefore obtain that

$$m^{2s+1}(\|\varphi\|_{H^s}^2 + \|\varphi\|_{H^s}^2) \le \tilde{E}^{(s)} \le 2^{2s+1}(\|\varphi\|_{H^s}^2 + \|\varphi\|_{H^s}^2),$$

so (6.17) implies (6.11). $\square$

APPENDIX A

Alternative formulation of the SQG front equation

Face front, true believers!

– Stan Lee "Marvel Masterworks: Fantastic Four Vol. 4"

We first prove an algebraic identity that will be used in deriving (5.10).

Lemma A.0.1. *Let $N \geq 2$ be an integer. Then for any integer $1 \leq p \leq N-1$ and any $\eta_j \in \mathbb{R}$, $j = 1, 2, \ldots, N$*

$$\sum_{\ell=1}^{N} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq N} (-1)^\ell (\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell})^p = 0. \tag{A.1}$$

Proof. A general term in the expansion of left-hand-side of (A.1) is proportional to

$$\eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_N^{\alpha_N}, \tag{A.2}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are nonnegative integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_N = p$. It suffices to show that the coefficients of the monomials (A.2) are zero. Let $1 \leq M \leq N-1$ denote the number of nonzero terms in the list $(\alpha_1, \alpha_2, \ldots, \alpha_N)$. Using the multinomial theorem, we see that the coefficient of (A.2) is

$$\binom{p}{\alpha_1, \ldots, \alpha_N} \cdot \sum_{j=0}^{N-M} (-1)^{M+j} \binom{N-M}{j} = \binom{p}{\alpha_1, \ldots, \alpha_N} \cdot (-1)^M (1-1)^{N-M} = 0.$$

□

To compute $\mathbf{T}_n(\boldsymbol{\eta}_n)$ in (5.5), we first expand the product

$$\begin{aligned}\Re \prod_{j=1}^{2n+1} (1 - e^{i\eta_j \zeta}) &= 1 + \sum_{\ell=1}^{2n+1} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq 2n+1} (-1)^\ell \cos\left((\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell})\zeta\right) \\ &= \sum_{\ell=1}^{2n+1} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq 2n+1} (-1)^{\ell+1} \left[1 - \cos\left((\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell})\zeta\right)\right].\end{aligned}$$

We replace the integral over $\mathbb{R}$ in (5.5) by an integral over $\mathbb{R} \setminus (-\epsilon, \epsilon)$, where $\epsilon \ll 1$, and decompose the expression for $\mathbf{T}_n$ into a sum of terms of the form

$$\int_{\epsilon<|\zeta|<\infty} \frac{1-\cos(\eta\zeta)}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta = \int_{\epsilon<|\zeta|\le 1/|\eta|} \frac{1+\sum_{j=1}^{n}\frac{(-1)^j(\eta\zeta)^{2j}}{(2j)!}-\cos(\eta\zeta)}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta + \int_{|\zeta|>1/|\eta|}\frac{1-\cos(\eta\zeta)}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta$$
$$- \sum_{j=1}^{n}\frac{(-1)^j\eta^{2j}}{(2j)!}\int_{\epsilon<|\zeta|\le 1/|\eta|}\frac{1}{|\zeta|^{2n-2j+1}}\,\mathrm{d}\zeta$$
$$= C_{n,1}\eta^{2n} - \sum_{j=1}^{n}\frac{(-1)^j\eta^{2j}}{(2j)!}\int_{\epsilon<|\zeta|\le 1/|\eta|}\frac{1}{|\zeta|^{2n-2j+1}}\,\mathrm{d}\zeta + o(1),$$

where

$$C_{n,1} = \int_{|\theta|\le 1}\frac{1+\sum_{j=1}^{n}\frac{(-1)^j(\theta)^{2j}}{(2j)!}-\cos(\theta)}{|\theta|^{2n+1}}\,\mathrm{d}\theta + \int_{|\theta|>1}\frac{1-\cos(\theta)}{|\theta|^{2n+1}}\,\mathrm{d}\theta$$

is some constant that depends only on n.

We have

$$\sum_{j=1}^{n}\frac{(-1)^j\eta^{2j}}{(2j)!}\int_{\epsilon<|\zeta|\le 1/|\eta|}\frac{1}{|\zeta|^{2n-2j+1}}\,\mathrm{d}\zeta = C_{n,2}^{\epsilon}\eta^{2n} + \sum_{j=1}^{n-1}C_{n,3}^{j,\epsilon}\eta^{2j} + C_{n,4}\eta^{2n}\log|\eta|,$$

where

$$C_{n,2}^{\epsilon} = \sum_{j=1}^{n-1}\frac{(-1)^{j+1}}{(n-j)(2j)!} + 2\frac{(-1)^{n+1}\log\epsilon}{(2n)!}, \qquad C_{n,3}^{j,\epsilon} = \frac{(-1)^j\epsilon^{2j-2n}}{(n-j)(2j)!}, \qquad C_{n,4} = 2\frac{(-1)^{n+1}}{(2n)!}.$$

Thus, we conclude that

$$\int_{\epsilon<|\zeta|\le 1/|\eta|}\frac{1-\cos(\eta\zeta)}{|\zeta|^{2n+1}}\,\mathrm{d}\zeta = \left(C_{n,1}-C_{n,2}^{\epsilon}\right)\eta^{2n} - \sum_{j=1}^{n-1}C_{n,3}^{j,\epsilon}\eta^{2j} - C_{n,4}\eta^{2n}\log|\eta|.$$

We use these results in the expression for $\mathbf{T}_n$ and take the limit as $\epsilon \to 0^+$. The singularity at $\epsilon = 0$ does not enter into the final result because of the cancelation in Lemma A.0.1, and we find that

$$\mathbf{T}_n(\boldsymbol{\eta}_n) = 2\frac{(-1)^{n+1}}{(2n)!}\sum_{\ell=1}^{2n+1}\sum_{1\le m_1<m_2<\cdots<m_\ell\le 2n+1}(-1)^\ell(\eta_{m_1}+\cdots+\eta_{m_\ell})^{2n}\log|\eta_{m_1}+\eta_{m_2}+\cdots+\eta_{m_\ell}|. \tag{A.3}$$

It follows that

$$f_n = 2\frac{(-1)^n}{(2n)!}\sum_{\ell=1}^{2n+1}\binom{2n+1}{\ell}(-1)^\ell\varphi^{2n-\ell+1}\partial^{2n}\log|\partial|(\varphi^l).$$

Therefore, we conclude that

$$\int_{\mathbb{R}} \left[\frac{\varphi_x(x,t) - \varphi_x(x+\zeta,t)}{|\zeta|} - \frac{\varphi_x(x,t) - \varphi_x(x+\zeta,t)}{\sqrt{\zeta^2 + (\varphi(x,t) - \varphi(x+\zeta,t))^2}} \right] \mathrm{d}\zeta$$

$$= -\sum_{n=1}^{\infty} \frac{2c_n(-1)^n}{\Gamma(2n+2)} \partial_x \left\{ \sum_{\ell=1}^{2n+1} \begin{pmatrix} 2n+1 \\ \ell \end{pmatrix} (-1)^\ell \varphi^{2n-\ell+1}(x,t) \partial_x^{2n} \log|\partial_x| \big(\varphi^l(x,t) \big) \right\}$$

$$= \sum_{n=1}^{\infty} \sum_{\ell=1}^{2n+1} (-1)^{\ell+1} d_{n,\ell} \partial_x \Big\{ \varphi^{2n-\ell+1}(x,t) \partial_x^{2n} \log|\partial_x| \big(\varphi^l(x,t) \big) \Big\},$$

where

$$d_{n,\ell} = \frac{2\sqrt{\pi}}{\left|\Gamma\left(\frac{1}{2} - n\right)\right| \Gamma(\ell+1)\Gamma(2n+2-\ell)\Gamma(n+1)} > 0. \tag{A.4}$$

Using this expansion in (3.2), we get (5.10).

APPENDIX B

Some algebraic inequalities

They usually develop in unequal proportion, in one love, in the other ambition and avarice. But now we, you and I, at our present ages, can to some extent do something ourselves one way or another to keep the things inside us in order.

– *Vincent van Gogh*

In this appendix, we prove the inequalities used in the local well-posedness proofs. We use $\{k_1, k_2, k_3, k_4\}$ to denote a quadruple of real numbers such that

$$k_1 + k_2 + k_3 + k_4 = 0,$$

and, as in (4.19)–(4.20), we denote by (m_1, m_2, m_3, m_4) a permutation of (k_1, k_2, k_3, k_4) such that

$$|m_1| \geq |m_2| \geq |m_3| \geq |m_4|.$$

If, as we assume, the k_j are not identically zero, then $m_1, m_2 \neq 0$, and we define

$$x = -\frac{m_2}{m_1}, \qquad y = -\frac{m_3}{m_1}, \qquad 1 - x - y = -\frac{m_4}{m_1}. \tag{B.1}$$

Since $m_1 + m_2 + m_3 + m_4 = 0$, the ordering of the $|m_j|$ implies that $0 \leq y \leq x \leq 1$ and $|1 - x - y| \leq y$, so $(x, y) \in R$, where the feasible region

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1 \text{ and } 1 \leq x + 2y \leq 2\right\} \tag{B.2}$$

is shown in Figure B.1. We note that $m_3 = 0$ corresponds to the point $(x, y) = (1, 0)$, and $m_4 = 0$ corresponds to the line $x + y = 1$. The ratio m_4/m_1 changes sign across this line: if $x + y > 1$, then m_1, m_4 have the same sign and the opposite sign to m_2, m_3; while if $x + y < 1$, then m_2, m_3, m_4 have the same sign and the opposite sign to m_1.

We begin with the following inequality for a symmetric function of fractional powers.

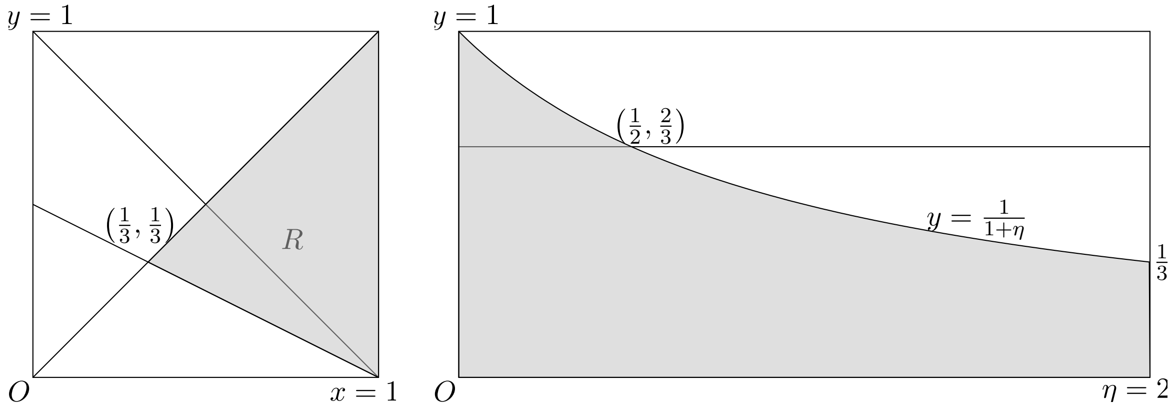


FIGURE B.1. Left: The feasible region R for (x, y)-variables in (B.2). Right: The feasible region for (η, y)-variables in (B.3).

Lemma B.0.1. *If $k_j, m_j \in \mathbb{R}$ with $j = 1, 2, 3, 4$ are defined as above, then for every $s > 0$ there exists a constant $C_0(s)$, depending only on s, such that*

$$\left| k_1|k_1|^{2s} + k_2|k_2|^{2s} + k_3|k_3|^{2s} + k_4|k_4|^{2s} \right| \leq C_0(s)|m_1|^s|m_2|^s|m_3|.$$

Proof. Both sides of the inequality are zero if $m_3 = 0$, when $m_1 = -m_2$ and $m_4 = 0$, so we may assume that $m_3 \neq 0$. Using (B.1), and the fact that the k_j are a permutation of the m_j, we get that

$$\frac{k_1|k_1|^{2s} + k_2|k_2|^{2s} + k_3|k_3|^{2s} + k_4|k_4|^{2s}}{|m_1|^s|m_2|^s m_3} = f(x, y),$$

where the continuous function $f \colon R \setminus \{(1, 0)\} \to \mathbb{R}$ is given by

$$f(x, y) = \frac{1 - x^{2s+1} - y^{2s+1} + (x + y - 1)|x + y - 1|^{2s}}{x^s y}.$$

The only place where f could fail to be bounded is near $(1, 0)$. Writing

$$x = 1 - \eta y, \qquad \text{with } 0 \leq \eta \leq 2, \tag{B.3}$$

and Taylor expanding f as $y \to 0^+$, we get that

$$f(1 - \eta y, y) = (2s + 1)\eta + \mathcal{O}(y + y^{2s})$$

uniformly in $0 \leq \eta \leq 2$. It follows that

$$\lim_{y \to 0^+} \sup_{0 \leq \eta \leq 2} |f(1 - \eta y, y)| = 2 \cdot (2s + 1),$$

which proves the lemma. ∎

Numerical computations show that the supremum of $|f|$ on $R \setminus \{(1,0)\}$ is attained at $(x,y) = (1/3, 1/3)$ if $s \geq s_0$, where $s_0 \approx 0.6365$ is the positive value of s at which $3^{s+1} - 3^{1-s} = 2(2s+1)$. In that case, we may take

$$C_0(s) = 3^{s+1} - 3^{1-s}. \tag{B.4}$$

Next, we estimate the SQG kernel defined in (4.13). From (4.11), these kernels have the form

$$S(k_1, k_2, k_3, k_4) = |m_1|^2 \left[g\left(-\frac{m_2}{m_1}, -\frac{m_3}{m_1}\right) + h\left(-\frac{m_2}{m_1}, -\frac{m_3}{m_1}\right)\right], \tag{B.5}$$

$$\begin{aligned} g(x,y) &= a(y) + a(x+y-1) - a(1-x), \\ h(x,y) &= a(1) + a(x) - a(1-y) - a(x+y). \end{aligned} \tag{B.6}$$

First, we estimate h.

Lemma B.0.2. *Let a be given by* (4.12)*, and let h be given by* (B.6)*. There exists $C > 0$ such that*

$$|h(x,y)| \leq C|x+y-1|y \qquad \text{for all } (x,y) \in R.$$

Proof. Using coordinates (B.3), we have

$$\begin{aligned} h(1-\eta y, y) &= a(1) + a(1-\eta y) - a(1-y) - a(1+(1-\eta)y) \\ &= -\int_{1-y}^{1} \int_0^{(1-\eta)y} a''(t+s)\, \mathrm{d}s\, \mathrm{d}t. \end{aligned} \tag{B.7}$$

If $0 \leq y \leq 2/3$ and $(x,y) \in R$, then $1/3 \leq t+s \leq 5/3$ in (B.7). Since $|a''(x)| \leq M$ is bounded on this interval, we get that

$$|h(1-\eta y, y)| \leq M|1-\eta|y^2.$$

If $2/3 \leq y \leq 1$ and $(x,y) \in R$, then $0 \leq \eta \leq 1/2$, and it follows that

$$\begin{aligned} |h(1-\eta y, y)| &\leq \int_0^1 \int_0^1 |a''(t+s)|\, \mathrm{d}s\, \mathrm{d}t \\ &\leq \frac{9}{2}\left(\int_0^1 \int_0^1 |a''(t+s)|\, \mathrm{d}s\, \mathrm{d}t\right) |1-\eta|y^2. \end{aligned}$$

Since the integral converges and $(1-\eta)y = x+y-1$, we obtain the result. ∎

We now estimate the SQG kernel.

Lemma B.0.3. *Let S be given by* (4.13). *If $k_j, m_j \in \mathbb{R} \setminus \{0\}$ with $j = 1, 2, 3, 4$ are defined as above, then there exists a numerical constant C_2 such that*

$$|S(k_1, k_2, k_3, k_4)| \le C_2 |m_3||m_4| \log\left(1 + \left|\frac{m_2}{m_3}\right|\right).$$

Proof. The kernel S is given by (B.5)–(B.6) with $a(x) = -x^2 \log |x|$. We have

$$\begin{aligned} |g(1-\eta y, y)| &= \left|y^2 \log y + (1-\eta)^2 y^2 \log |(1-\eta)y| - \eta^2 y^2 \log \eta y\right| \\ &= \left|2(1-\eta) y^2 \log y + \left[(1-\eta)^2 \log|1-\eta| - \eta^2 \log \eta\right] y^2\right| \\ &\le C|1-\eta| y^2 \left[1 + \log(1/y)\right]. \end{aligned}$$

Since $x \ge 1/3$ and $x/y \ge 1$, it follows that

$$|g(x,y)| \le C|x+y-1| y \log\left(1 + \frac{x}{y}\right).$$

Using this inequality and Lemma B.0.2, we get that

$$|g(x,y) + h(x,y)| \le C|x+y-1| y \log\left(1 + \frac{x}{y}\right),$$

and the use of this inequality in (B.5) proves the lemma. □

Numerical computations show that in Lemma B.0.3 we can take, for example,

$$C_2 = 5. \tag{B.8}$$

The worst case for the growth of S is when two wavenumbers are in the same "shell" with much larger and almost equal absolute values than the other two wavenumbers, which happens near the point $(x, y) = (1, 0)$ in R. For example, suppose that

$$k_1 = k + a, \qquad k_2 = -(k+b), \qquad k_3 = -a, \qquad k_4 = b,$$

and consider the limit $k \to \infty$ with $a, b > 0$ fixed. Then one finds that

$$S(k_1, k_2, k_3, k_4) = -2ab \log |k| + \mathcal{O}(1) = 2 m_3 m_4 \log \left|\frac{m_2}{m_3}\right| + \mathcal{O}(1).$$

Thus, the logarithmic factor in Lemma B.0.3 cannot be improved upon.

We end this appendix with a corollary of Lemma B.0.3 for the SQG kernel as a function of integer wavenumbers. This Lemma uses the fact that the $|k_j|$ are bounded away from zero, so it does not apply in the spatial case with $k_j \in \mathbb{R} \setminus \{0\}$.

Corollary B.0.4. *Let S be given by (4.13). If $k_j, m_j \in \mathbb{Z}_*$ with $j = 1, 2, 3, 4$ are defined as above, then there exists a constant C_2 such that*

$$|S(k_1, k_2, k_3, k_4)| \leq C_2 |m_3||m_4| \left[\log(1 + |m_1|) \log(1 + |m_2|)\right]^{1/2}.$$

Proof. The result follows immediately from Lemma B.0.3, since $|m_2| \leq |m_1|$ and $|m_3| \geq 1$. □

Bibliography

[Ama00] H. Amann, *Compact embeddings of vector-valued Sobolev and Besov spaces*, vol. 35(55), 2000, pp. 161–177, Dedicated to the memory of Branko Najman.

[Aus11] L. A. A. Austria, *Nonlinear hyperbolic surface waves in variational boundary-value problems*, ProQuest LLC, Ann Arbor, MI, 2011, Thesis (Ph.D.)–University of California, Davis.

[BC93] A. L. Bertozzi and P. Constantin, Global regularity for vortex patches, *Comm. Math. Phys.* **152** (1993), no. 1, 19–28.

[BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011.

[BH] J. A. Biello and J. K. Hunter, Unpublished contour dynamics simulations.

[BH10] J. A. Biello and J. K. Hunter, Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities, *Comm. Pure Appl. Math.* **63** (2010), no. 3, 303–336.

[Bie] J. A. Biello, private communication, 2017.

[BKM84] J. T. Beale, T. Kato, and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Comm. Math. Phys.* **94** (1984), no. 1, 61–66.

[BS75] J. L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equation, *Philos. Trans. Roy. Soc. London Ser. A* **278** (1975), no. 1287, 555–601.

[BS90] G. R. Baker and M. J. Shelley, On the connection between thin vortex layers and vortex sheets, *J. Fluid Mech.* **215** (1990), 161–194.

[BSV19] T. Buckmaster, S. Shkoller, and V. Vicol, Nonuniqueness of weak solutions to the SQG equation, *Comm. Pure Appl. Math.* **72** (2019), no. 9, 1809–1874.

[Bur82] J. Burbea, Motions of vortex patches, *Lett. Math. Phys.* **6** (1982), no. 1, 1–16.

[CCC$^+$12] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, and J. Wu, Generalized surface quasi-geostrophic equations with singular velocities, *Comm. Pure Appl. Math.* **65** (2012), no. 8, 1037–1066.

[CCG18] A. Córdoba, D. Córdoba, and F. Gancedo, Uniqueness for SQG patch solutions, *Trans. Amer. Math. Soc. Ser. B* **5** (2018), 1–31.

[CCGS16a] A. Castro, D. Córdoba, and J. Gómez-Serrano, Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations, *Duke Math. J.* **165** (2016), no. 5, 935–984.

[CCGS16b] ———, Uniformly rotating analytic global patch solutions for active scalars, *Ann. PDE* **2** (2016), no. 1, Art. 1, 34.

[CCGSar] ———, Global smooth solutions for the inviscid SQG equation, *Mem. Amer. Math. Soc.* (to appear).

[CFMR05] D. Córdoba, M. A. Fontelos, A. M. Mancho, and J. L. Rodrigo, Evidence of singularities for a family of contour dynamics equations, *Proc. Natl. Acad. Sci. USA* **102** (2005), no. 17, 5949–5952.

[CFR04] D. Córdoba, C. Fefferman, and J. L. Rodrigo, Almost sharp fronts for the surface quasi-geostrophic equation, *Proc. Natl. Acad. Sci. USA* **101** (2004), no. 9, 2687–2691.

[CGSI19] D. Córdoba, J. Gómez-Serrano, and A. D. Ionescu, Global solutions for the generalized SQG patch equation, *Arch. Ration. Mech. Anal.* **233** (2019), no. 3, 1211–1251.

[Che93] J.-Y. Chemin, Persistance de structures géométriques dans les fluides incompressibles bidimensionnels, *Ann. Sci. École Norm. Sup. (4)* **26** (1993), no. 4, 517–542.

[Che98] ———, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 14, The Clarendon Press, Oxford University Press, New York, 1998, Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[CMT94a] P. Constantin, A. J. Majda, and E. G. Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, *Nonlinearity* **7** (1994), no. 6, 1495–1533.

[CMT94b] ———, Singular front formation in a model for quasigeostrophic flow, *Phys. Fluids* **6** (1994), no. 1, 9–11.

[CR71] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Functional Analysis* **8** (1971), 321–340.

[Dav95] E. B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.

[DIP17] Y. Deng, A. D. Ionescu, and B. Pausader, The Euler-Maxwell system for electrons: global solutions in 2D, *Arch. Ration. Mech. Anal.* **225** (2017), no. 2, 771–871.

[DIPP17] Y. Deng, A. D. Ionescu, B. Pausader, and F. Pusateri, Global solutions of the gravity-capillary water-wave system in three dimensions, *Acta Math.* **219** (2017), no. 2, 213–402.

[dlHHH16] F. de la Hoz, Z. Hassainia, and T. Hmidi, Doubly connected V-states for the generalized surface quasi-geostrophic equations, *Arch. Ration. Mech. Anal.* **220** (2016), no. 3, 1209–1281.

[dlHHMV16] F. de la Hoz, T. Hmidi, J. Mateu, and J. Verdera, Doubly connected V-states for the planar Euler equations, *SIAM J. Math. Anal.* **48** (2016), no. 3, 1892–1928.

[DM90] D. G. Dritschel and M. E. McIntyre, Does contour dynamics go singular?, *Phys. Fluids A* **2** (1990), no. 5, 748–753.

[Dri88] D. G. Dritschel, Contour surgery: a topological reconnection scheme for extended integrations using contour dynamics, *J. Comput. Phys.* **77** (1988), no. 1, 240–266.

[Duo01] J. Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001, Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[FLR12] C. Fefferman, G. Luli, and J. Rodrigo, The spine of an SQG almost-sharp front, *Nonlinearity* **25** (2012), no. 2, 329–342.

[FR11] C. Fefferman and J. L. Rodrigo, Analytic sharp fronts for the surface quasi-geostrophic equation, *Comm. Math. Phys.* **303** (2011), no. 1, 261–288.

[FR12] ______, Almost sharp fronts for SQG: the limit equations, *Comm. Math. Phys.* **313** (2012), no. 1, 131–153.

[FR15] C. L. Fefferman and J. L. Rodrigo, Construction of almost-sharp fronts for the surface quasi-geostrophic equation, *Arch. Ration. Mech. Anal.* **218** (2015), no. 1, 123–162.

[FS72] C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), no. 3-4, 137–193.

[FS05] S. Friedlander and R. Shvydkoy, The unstable spectrum of the surface quasi-geostropic equation, *J. Math. Fluid Mech.* **7** (2005), no. suppl. 1, S81–S93.

[FV11] S. Friedlander and V. Vicol, On the ill/well-posedness and nonlinear instability of the magneto-geostrophic equations, *Nonlinearity* **24** (2011), no. 11, 3019–3042.

[Gan08] F. Gancedo, Existence for the α-patch model and the QG sharp front in Sobolev spaces, *Adv. Math.* **217** (2008), no. 6, 2569–2598.

[Ger10] P. Germain, Space-time resonances, *Journées équations aux dérivées partielles* (2010) (en).

[GMS09] P. Germain, N. Masmoudi, and J. Shatah, Global solutions for 3D quadratic Schrödinger equations, *Int. Math. Res. Not. IMRN* (2009), no. 3, 414–432.

[GMS12] P. Germain, N. Masmoudi, and J. Shatah, Global solutions for the gravity water waves equation in dimension 3, *Ann. of Math. (2)* **175** (2012), no. 2, 691–754.

[GPar] F. Gancedo and N. Patel, On the local existence and blow-up for generalized SQG patches, preprint, `arXiv:1811.00530`.

[GS14] F. Gancedo and R. M. Strain, Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem, *Proc. Natl. Acad. Sci. USA* **111** (2014), no. 2, 635–639.

[GS19] J. Gómez-Serrano, On the existence of stationary patches, *Adv. Math.* **343** (2019), 110–140.

[GSPSYar] J. Gómez-Serrano, J. Park, J. Shi, and Y. Yao, Symmetry in stationary and uniformly-rotating solutions of active scalar equations, preprint, `arXiv:1908.01722`.

[H3̈3] E. Hölder, Über die unbeschränkte Fortsetzbarkeit einer stetigen ebenen Bewegung in einer unbegrenzten inkompressiblen Flüssigkeit, *Math. Z.* **37** (1933), no. 1, 727–738.

[H0̈7] L. Hörmander, *The analysis of linear partial differential operators. III*, Classics in Mathematics, Springer, Berlin, 2007, Pseudo-differential operators, Reprint of the 1994 edition.

[Hel13] B. Helffer, *Spectral theory and its applications*, Cambridge Studies in Advanced Mathematics, vol. 139, Cambridge University Press, Cambridge, 2013.

[HH15] Z. Hassainia and T. Hmidi, On the V-states for the generalized quasi-geostrophic equations, *Comm. Math. Phys.* **337** (2015), no. 1, 321–377.

[HM16] T. Hmidi and J. Mateu, Degenerate bifurcation of the rotating patches, *Adv. Math.* **302** (2016), 799–850.

[HM17] ———, Existence of corotating and counter-rotating vortex pairs for active scalar equations, *Comm. Math. Phys.* **350** (2017), no. 2, 699–747.

[HMV13] T. Hmidi, J. Mateu, and J. Verdera, Boundary regularity of rotating vortex patches, *Arch. Ration. Mech. Anal.* **209** (2013), no. 1, 171–208.

[HMVSZ] J. K. Hunter, R. C. Moreno-Vasquez, J. Shu, and Q. Zhang, On the approximation of vorticity fronts by the Burgers-Hilbert equation, preprint.

[HPGS95] I. M. Held, R. T. Pierrehumbert, S. T. Garner, and K. L. Swanson, Surface quasi-geostrophic dynamics, *J. Fluid Mech.* **282** (1995), 1–20.

[HS18] J. K. Hunter and J. Shu, Regularized and approximate equations for sharp fronts in the surface quasi-geostrophic equation and its generalizations, *Nonlinearity* **31** (2018), no. 6, 2480–2517.

[HSZ18] J. K. Hunter, J. Shu, and Q. Zhang, Local well-posedness of an approximate equation for SQG fronts, *J. Math. Fluid Mech.* **20** (2018), no. 4, 1967–1984.

[HSZara] ———, Contour dynamics for surface quasi-geostrophic fronts, *Nonlinearity* (to appear).

[HSZarb] ———, Global solutions for a family of GSQG front equations, preprint, `arXiv:2005.09154`.

[HSZarc] ———, Global solutions of a surface quasi-geostrophic front equation, preprint, `arXiv:1808.07631`.

[HSZard] ———, Two-front solutions of the SQG equations and its generalizations, *Commun. Math. Sci.* (to appear).

[Hun06] J. K. Hunter, Short-time existence for scale-invariant Hamiltonian waves, *J. Hyperbolic Differ. Equ.* **3** (2006), no. 2, 247–267.

[Ifr12] M. Ifrim, *Normal form transformations for quasilinear wave equations*, 2012, p. 85, Thesis (Ph.D.)–University of California, Davis.

[IP13] A. D. Ionescu and B. Pausader, The Euler-Poisson system in 2D: global stability of the constant equilibrium solution, *Int. Math. Res. Not. IMRN* (2013), no. 4, 761–826.

[IP14] A. D. Ionescu and F. Pusateri, Nonlinear fractional Schrödinger equations in one dimension, *J. Funct. Anal.* **266** (2014), no. 1, 139–176.

[IP15] ———, Global solutions for the gravity water waves system in 2d, *Invent. Math.* **199** (2015), no. 3, 653–804.

[IP16] ———, Global analysis of a model for capillary water waves in two dimensions, *Comm. Pure Appl. Math.* **69** (2016), no. 11, 2015–2071.

[IP18] ———, Global regularity for 2D water waves with surface tension, *Mem. Amer. Math. Soc.* **256** (2018), no. 1227, v+124.

[Jud63] V. I. Judovič, Non-stationary flows of an ideal incompressible fluid, *Ž. Vyčisl. Mat. i Mat. Fiz.* **3** (1963), 1032–1066.

[Kel67] O. D. Kellogg, *Foundations of potential theory*, Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31, Springer-Verlag, Berlin-New York, 1967.

[KRara] C. Khor and J. L. Rodrigo, Local existence of analytic sharp fronts for singular SQG, preprint, `arXiv:2001.10412`.

[KRarb] ———, On sharp fronts and almost-sharp fronts for singular SQG, preprint, `arXiv:2001.10332`.

[KRYZ16] A. Kiselev, L. Ryzhik, Y. Yao, and A. Zlatoš, Finite time singularity for the modified SQG patch equation, *Ann. of Math. (2)* **184** (2016), no. 3, 909–948.

[KTVZ11] I. Kukavica, R. Temam, V. C. Vicol, and M. Ziane, Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain, *J. Differential Equations* **250** (2011), no. 3, 1719–1746.

[KYZ17] A. Kiselev, Y. Yao, and A. Zlatoš, Local regularity for the modified SQG patch equation, *Comm. Pure Appl. Math.* **70** (2017), no. 7, 1253–1315.

[Lap17] G. Lapeyre, Surface quasi-geostrophy, *Fluids* **2** (2017), no. 1.

[Li19] D. Li, On Kato-Ponce and fractional Leibniz, *Rev. Mat. Iberoam.* **35** (2019), no. 1, 23–100.

[Maj03] A. Majda, *Introduction to PDEs and waves for the atmosphere and ocean*, Courant Lecture Notes in Mathematics, vol. 9, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[Mar08] F. Marchand, Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces L^p or $\dot{H}^{-1/2}$, *Comm. Math. Phys.* **277** (2008), no. 1, 45–67.

[MB02] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002.

[MP94] C. Marchioro and M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, Applied Mathematical Sciences, vol. 96, Springer-Verlag, New York, 1994.

[MT96] A. J. Majda and E. G. Tabak, A two-dimensional model for quasigeostrophic flow: comparison with the two-dimensional Euler flow, *Phys. D* **98** (1996), no. 2-4, 515–522, Nonlinear phenomena in ocean dynamics (Los Alamos, NM, 1995).

[MZ97] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, vol. 51, American Mathematical Society, Providence, RI, 1997.

[OLBC10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST handbook of mathematical functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010, With 1 CD-ROM (Windows, Macintosh and UNIX).

[Oza91] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, *Comm. Math. Phys.* **139** (1991), no. 3, 479–493.

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[Ped79] J. Pedlosky, *Geophysical fluid dynamics*, Springer-Verlag, New York, 1979.

[Ray96] L. Rayleigh, On the propagation of waves upon the plane surface separating two portions of fluid of different vorticities, *Proc. Lond. Math. Soc.* **27** (1895/96), 13–18.

[Res95] S. G. Resnick, *Dynamical problems in non-linear advective partial differential equations*, Ph.D. thesis, 1995, Thesis (Ph.D.)–The University of Chicago, p. 76.

[Rod05] J. L. Rodrigo, On the evolution of sharp fronts for the quasi-geostrophic equation, *Comm. Pure Appl. Math.* **58** (2005), no. 6, 821–866.

[RS72] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York-London, 1972.

[Sal98] R. Salmon, *Lectures on geophysical fluid dynamics*, Oxford University Press, New York, 1998.

[SD14] R. Scott and D. Dritschel, Numerical simulation of a self-similar cascade of filament instabilities in the surface quasigeostrophic system, *Physical Review Letters* **112** (2014), no. 14 (English).

[SD19] R. K. Scott and D. G. Dritschel, Scale-invariant singularity of the surface quasigeostrophic patch, *J. Fluid Mech.* **863** (2019), R2, 12.

[Seg11] J. Segura, Bounds for ratios of modified Bessel functions and associated Turán-type inequalities, *J. Math. Anal. Appl.* **374** (2011), no. 2, 516–528.

[Ser94] P. Serfati, Une preuve directe d'existence globale des vortex patches 2D, *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), no. 6, 515–518.

[Shn97] A. Shnirelman, On the nonuniqueness of weak solution of the Euler equation, *Comm. Pure Appl. Math.* **50** (1997), no. 12, 1261–1286.

[Shu20] J. Shu, *Fronts for the SQG equation: a review*, Hyperbolic Problems: Theory, Numerics, Applications, AIMS Ser. Appl. Math., vol. 10, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2020, pp. 630–638.

[Ste70] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[Ste93] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[Tay00] M. E. Taylor, *Tools for PDE*, Mathematical Surveys and Monographs, vol. 81, American Mathematical Society, Providence, RI, 2000, Pseudodifferential operators, paradifferential operators, and layer potentials.

[Val17] G. K. Vallis, *Atmospheric and oceanic fluid dynamics: Fundamentals and large-scale circulation*, 2 ed., Cambridge University Press, 2017.

[Wat95] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.

[Wol33] W. Wolibner, Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long, *Math. Z.* **37** (1933), no. 1, 698–726.

[ZHR79] N. J. Zabusky, M. H. Hughes, and K. V. Roberts, Contour dynamics for the Euler equations in two dimensions, *J. Comput. Phys.* **30** (1979), no. 1, 96–106.