Wall-Crossing Structures in Cluster Algebras

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To my grandparents, 唐大炯 (Tang Dajiong) and 胡钦光 (Hu Qinguang)

Contents

Abstract

In this dissertation, we study the phenomenon of wall-crossing structures in cluster algebras of Fomin and Zelevinsky, with examples including cluster scattering diagrams of Gross, Hacking, Keel, and Kontsevich (GHKK) and stability scattering diagrams of Bridgeland. We show that in general, every consistent scattering diagram admits a canonical underlying cone complex structure. We describe mutations of the stability scattering diagram of a quiver with non-degenerate potential. Then we use this description to prove that the stability scattering diagram admits the so-called cluster complex structure. As a consequence, we verify if a quiver admits a reddening sequence, a conjecture of Kontsevich and Soibelman that the associated cluster scattering diagram is equivalent to the stability scattering diagram of the same quiver with a non-degenerate potential. We also give another proof of the Caldero–Chapoton formula of cluster monomials using scattering diagrams.

Skew-symmetrizable cluster algebras need extra care. We define a Langlands dual version of the cluster scattering diagram of GHKK and show that it admits a cluster complex structure that is Langlands dual to GHKK's version. We use it to describe the cluster monomials of skew-symmetrizable cluster algebras in terms of theta functions. Then we study the Hall algebra scattering diagram associated to the Geiss–Leclerc–Schröer algebra of an acyclic skew-symmetrizable matrix with a skew-symmetrizer. We show that it admits the same cluster complex structure as the aforementioned Langlands dual cluster scattering diagram. In the end, we extend the theory of scattering diagrams to Chekhov and Shapiro's generalized cluster algebras.

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CHAPTER 1

Introduction

The notion of *wall-crossing structures* has emerged from the work of Kontsevich and Soibelman [**KS06**], and Gross and Siebert [**GS11**] (under the name *consistent scattering diagrams*), on the Strominger-Yau-Zaslow approach to mirror symmetry, and in parallel, from the work of Kontsevich and Soibelman [**KS08**] on Donaldson-Thomas invariants of 3-Calabi-Yau categories. Since their first appearances, there are various applications of wall-crossing structures. Among them, a remarkable one is the construction of *cluster scattering diagrams* by Gross, Hacking, Keel, and Kontsevich (GHKK for short) [**GHKK18**], used to settle several long-standing conjectures about cluster algebras. On the other hand, another wall-crossing structure, the *motivic Hall algebra scattering diagram* is constructed by Bridgeland [**Bri17**] for any quiver with relation, encoding the wall-crossing-formula type information of the category of quiver representations. The theme of this dissertation centers around the relationship between these two types of wall-crossing structures.

Our work is motivated by the theory of *cluster algebras*, and in fact, part of our goal is to study the properties of cluster algebras using the technique of scattering diagrams. Cluster algebras, introduced by Fomin and Zelevinsky in [**FZ02**], are a class of commutative algebras generated in some Laurent polynomial ring by a distinguished set of Laurent polynomials (the *cluster variables*) grouped in overlapping subsets (the *clusters*) recursively obtained by operations called *mutations*. Gross, Hacking, Keel, and Kontsevich have proved important properties of cluster algebras regarding their bases in [**GHKK18**] where a class of wall-crossing structures named *cluster scattering diagram* plays a crucial role. In general, a scattering diagram is a (possibly infinite) cone complex in a vector space where its codimension one cones (*walls*) are decorated with certain transformations (elements in some group) referred as *wall-crossings*. The cluster scattering diagram associated with a cluster algebra has maximal cells (the *cluster chambers*) corresponding to clusters. The walls of the cluster chambers are decorated with wall-crossings in the automorphism group of some formal poisson torus to describe mutations. A nicely behaved basis of a cluster algebra (the *canonical*

basis) is constructed in [**GHKK18**] by counting in the associated cluster scattering diagram certain piecewise linear curves (the *broken lines*) which bend when crossing walls. The canonical basis contains monomials of cluster variables that belongs to the same cluster.

A large class of cluster algebras of interest is those associated to quivers. To cluster algebras of this type, there is a seemingly different approach that utilizes a categorification modeled on quiver representations. For more details in this approach (called *the additive categorification*), we refer the reader to the nice survey [**Kel08**]. In view of the additive categorification, clusters correspond to t-structures of the relevant triangulated category, and mutations are essentially tiltings of t-structures. In this framework, cluster monomials, part of the canonical basis, can be recovered by applying Caldero-Chapoton type formulas to certain quiver representations; see [**CC06**] for finite type quivers, generalizations in [**CK08**, **CK06**, **Pal08**], [**DWZ10**, **Pla11**] for arbitrary 2-acyclic quivers and [**Nag13**] for a point of view closest to this paper.

It is interesting to ask for the meanings of cluster scattering diagrams, wall-crossings and broken lines in the additive categorification. An important first step towards an answer is taken by Bridgeland, who constructs in [**Bri17**] a *Hall algebra scattering diagram* $\mathfrak{D}_{Q,I}^{\mathrm{Hall}}$ (Definition 6.2.1) for each quiver with relations (Q, I) by considering stability conditions on the abelian category of representations of (Q, I). If the ideal of relations arises from a potential W, we can apply an integration map to $\mathfrak{D}_{Q,I}^{\mathrm{Hall}}$ to get a *stability scattering diagram* $\mathfrak{D}_{Q,W}^{\mathrm{Stab}}$ valued in a much simpler group. He shows that for an acyclic quiver (thus with only zero potential), the stability scattering diagram is identical to the corresponding cluster scattering diagram $\mathfrak{D}_Q^{\mathrm{Cl}}$. In this case, the Caldero-Chapoton formulas for cluster monomials thus have interpretations in both the cluster and the stability scattering diagram. However, even for acyclic quivers, the representation-theoretic meaning of the canonical basis (apart from cluster monomials) is still unclear.

One goal of this dissertation is to further investigate the relationship between these two related, but a priori not necessarily equivalent, scattering diagrams $\mathfrak{D}_Q^{\mathrm{Cl}}$ and $\mathfrak{D}_{Q,W}^{\mathrm{Stab}}$, setting foundations towards a better understanding of the categorical meanings of the combinatorial objects extracted from cluster scattering diagrams, seeking a categorification of broken lines.

We also work to extend the *cluster-versus-stability* comparison to the *skew-symmetrizable* case. To a skew-symmetrizable integral matrix B (with a skew-symmetrizer D), there is an associated cluster algebra $\mathcal{A}(B)$. It is worth mentioning that GHKK's construction of cluster scattering diagrams do cover the skew-symmetrizable case. In this dissertation, we propose a scattering

diagram $\mathfrak{D}^{\mathrm{Cl}}_{B,D}$ that is Langlands dual to GHKK's cluster scattering diagram. We prove that our scattering diagrams have nice properties so that it can be used to describe the cluster monomials of $\mathcal{A}(B)$.

On the other hand, it seems harder to find an additive categorification for skew-symmetrizable cluster algebras than the skew-symmetric case. Besides other approaches such as [**Dem11**, **Rup15**] and many others, Geiss, Leclerc and Schröer [**GLS17**] have introduced a finite-dimensional algebra $H(B,D)$ associated to the pair (B,D) of an acyclic skew-symmetrizable matrix B with its left skew-symmetrizer D, and use certain $H(B,D)$-modules to describe the cluster monomials of $\mathcal{A}(B)$ when B is of Dynkin type [**GLS18**]. In a similar vein to the skew-symmetric case, we study the relationship between $\mathfrak{D}^{\mathrm{Hall}}_{H(B,D)}$ and $\mathfrak{D}^{\mathrm{Cl}}_{B,D}$. The main result we have along these lines is that these two scattering diagrams share the same cluster complex structure.

In the end, we extend GHKK's theory of cluster scattering diagrams to generalized cluster algebras of Chekhov and Shapiro [**CS14**]. The generalization is accomplished by modifying the wall-crossings on initial walls to allow more complicated reciprocal monic polynomials. We show that these *generalized cluster scattering diagrams* also admit a cluster complex structure. We hope that this framework is useful for solving the positivity conjecture of generalized cluster algebras and for understanding other properties. As an application, in a future work joint with Labardini Fragoso [**LFM**], we study the generalized cluster algebras associated to a class of orbifolds using this scattering diagram.

The rest of the introduction contains a more detailed account of our results and summarizes the content in each chapter. Some parts of Chapter 2, Chapter 4 and Chapter 6 overlap with the preprint [**Mou19**].

1.1. Wall-crossing structures and scattering diagrams

Despite its vast appearances in the literature in the study of mirror symmetry and Donaldson-Thomas invariants, e.g., [**KS06**, **GPS10**, **GS11**], the notion of wall-crossing structures was formalized by Kontsevich and Soibelman in [**KS14**]. This dissertation only concerns wall-crossing structures in vector spaces, which bear another name, *consistent scattering diagrams.*

We fix a lattice N of finite rank r with a chosen basis $\mathbf{e} = (e_1, e_2, \ldots, e_r)$. Set

$$M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}), \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}.$$

Define $N_{\mathbf{e}}^+ = N^+ \subset N$ to be the sub semigroup non-negatively generated by $\mathbf{e}$ without 0. A scattering diagram is valued in an N^+-graded Lie algebra

$$\mathfrak{g} = \bigoplus_{n \in N^+} \mathfrak{g}_n.$$

More precisely, we consider the pro-nilpotent Lie algebra

$$\hat{\mathfrak{g}} = \prod_{n \in N^+} \mathfrak{g}_n$$

as the completion of $\mathfrak{g}$ with respect to the grading. Taking formal exponentials, we obtain a pro-unipotent algebraic group:

$$\exp \colon \hat{\mathfrak{g}} \to \widehat{G}.$$

The multiplication in the group $\widehat{G}$ is defined formally through the Baker-Campbell-Hausdorff formula.

Definition 1.1.1. A *scattering diagram* valued in $\mathfrak{g}$ or a $\mathfrak{g}$-SD for short is a function

$$\Phi \colon M_{\mathbb{R}} \to \widehat{G}$$

such that for any $m \in M_{\mathbb{R}}$, we require that $\Phi(m)$ lies in the subgroup

$$\exp \left(\prod_{m(n)=0} \mathfrak{g}_n \right) \subset \widehat{G}.$$

In general, there are too many scattering diagrams with the above definition. The examples from mirror symmetry and Donaldson-Thomas theory and the application in cluster algebras all fall into a much restrictive class: *consistent scattering diagrams.*

To define a consistent scattering diagram, we first consider a Lie algebra $\mathfrak{g}$ with finite support, i.e., the set

$$S = S(\mathfrak{g}) := \{n \in N^+ \mid \mathfrak{g}_n \neq 0\}$$

is finite. Each normal vector $n \in S$ defines an orthogonal hyperplane $n^\perp \subset M_{\mathbb{R}}$, thus together cutting $M_{\mathbb{R}}$ into a cone complex $\mathfrak{S}_S$. If $\Phi \colon M_{\mathbb{R}} \to \widehat{G}$ is a $\mathfrak{g}$-SD, we see by definition that it is constructible with respect to the stratification of $M_{\mathbb{R}}$ induced by $\mathfrak{S}$, i.e., the function Φ is constant in the relative interior of any cone in $\mathfrak{S}_S$. In particular, if $\mathfrak{d}$ is a wall (a codimensional one cone) in

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$\mathfrak{S}_S$, we have a well-defined

$$\Phi(\mathfrak{d}) \in \widehat{G}$$

by evaluating Φ at any interior point in $\mathfrak{d}$.

Definition 1.1.2. For a Lie algebra $\mathfrak{g}$ with finite support, a $\mathfrak{g}$-scattering diagram $\Phi \colon M_\mathbb{R} \to \widehat{G}$ is *consistent* if for any sufficiently general path $\gamma \colon [0,1] \to M_\mathbb{R}$ with respect to $\mathfrak{S}_S$, the *path-ordered product*

$$\mathfrak{p}_\gamma := \prod_{i=1}^{k} \Phi(\mathfrak{d}_i)^{\pm} \in \widehat{G}$$

of the sequence of walls $(\mathfrak{d}_i)_{i=1}^{k}$ crossed in order by γ only depends on $\gamma(0)$ and $\gamma(1)$.

In order to extend the above definition to a Lie algebra $\mathfrak{g}$ with infinite support, we consider for each $d \in N$, the quotient Lie algebra

$$\mathfrak{g}^{<d} := \mathfrak{g}/\bigoplus_{|n|\geq d} \mathfrak{g}_n$$

and the induced group projection

$$\pi_d \colon \widehat{G} \to G^{<d} = \exp(\mathfrak{g}^{<d}).$$

Definition 1.1.3. For an N^+-graded Lie algebra $\mathfrak{g}$, we say a $\mathfrak{g}$-SD $\Phi \colon M_\mathbb{R} \to \widehat{G}$ consistent if for any $d \in \mathbb{N}$, the function

$$\Phi^{<d} = \pi_d \circ \Phi \colon M_\mathbb{R} \to G^{<d}$$

is a consistent $\mathfrak{g}^{<d}$-SD.

The subset of consistent $\mathfrak{g}$-SDs is much smaller than the set of all $\mathfrak{g}$-SDs. In fact, due to Theorem 2.2.5 of Kontsevich-Soibelman [**KS14**], consistent $\mathfrak{g}$-SDs are in bijection with the elements in the group $\widehat{G}$.

1.2. The canonical cone complex of a consistent scattering diagram

Our first observation is that the cone complex $\mathfrak{S}_S$ of hyperplane arrangements is too refined to well-capture the underlying cone complex structure of a consistent scattering diagram. We show in Section 2.3 that for a Lie algebra $\mathfrak{g}$ with finite support, any consistent $\mathfrak{g}$-SD admits a canonical underlying cone complex, coarsening $\mathfrak{S}_S$, where the cones are exactly the connected components of the level sets of the function $\Phi \colon M_\mathbb{R} \to \widehat{G}$.

THEOREM 1.2.1 (Theorem 2.3.1). *Let* Φ *be a consistent* $\mathfrak{g}$-*SD. Then we have*

(1) *for any* $h \in G$, *the preimage* $\Phi^{-1}(h)$ *is relatively open in a subspace in* $M_{\mathbb{R}}$,

(2) *each connected component of* $\Phi^{-1}(h)$ *is the relative interior of a rational polyhedral cone, and*

(3) *all such cones together form a complete finite cone complex of* $M_{\mathbb{R}}$, *which we refer to as the canonical cone complex of* Φ.

In many applications, the canonical cone complex is of interest. Thus we would like to record this information when we speak of a consistent scattering diagram. For example, in many cases, we use the name *consistent scattering diagram* to represent a pair

$$\mathfrak{D} = (\mathfrak{S}, \Phi \colon M_{\mathbb{R}} \to \widehat{G})$$

where Φ is the defining function as before, and $\mathfrak{S}$ is the canonical cone complex determined by Φ.

For a consistent SD valued in $\mathfrak{g}$ with infinite support, one does not always expect a finite cone complex. However, by considering the truncations $\Phi^{<d}$ for $d \in \mathbb{N}$, we get a filtration of finite cone complexes

$$\cdots \subset \mathfrak{S}^{<d} \subset \mathfrak{S}^{d+1} \subset \cdots$$

whose projective limit is a *profinite cone complex* in $M_{\mathbb{R}}$. For further details, see Section 2.3.2.

1.3. Cluster scattering diagrams versus stability scattering diagrams

Now we consider a class of consistent SDs with important applications in cluster algebras. Equip the lattice $N \cong \mathbb{Z}^r$ with a $\mathbb{Z}$-valued skew-symmetric bilinear form

$$\omega \colon N \times N \to \mathbb{Z}.$$

Let $I = \{1, \ldots, r\}$ be an indexing set. A *seed* $\mathbf{s}$ is a basis of N indexed by I. We define an N^+-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbf{s}}$ by setting

$$\mathfrak{g}_n = \mathbb{Q} \cdot x^n, \quad \{x^{n_1}, x^{n_2}\} = \omega(n_1, n_2) x^{n_1 + n_2}.$$

In [**GHKK18**], Gross, Hacking, Keel, and Kontsevich defined a consistent $\mathfrak{g}$-SD

$$\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}} = \left(\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}, \Phi_{\mathbf{s}}^{\mathrm{Cl}} \right)$$

by specifying the element $\Phi_{\mathbf{s}}^{\mathrm{Cl}}(0)$ (see Section 4.2). This scattering diagram admits many nice properties and is thus used to settle important conjectures about cluster algebras in [**GHKK18**].

The seed $\mathbf{s}$ determines a quiver $Q = Q(\mathbf{s})$ such that the vertices are identified with the indexing set I, and the negative of the adjacency matrix of Q is the pairing matrix of $\mathbf{s}$. A *seed with potential* $(\mathbf{s}, W)$ is a seed $\mathbf{s}$ with a potential $W \in \widehat{\mathbb{C}Q}$. Let

$$\mathcal{P}(Q, W) := \widehat{\mathbb{C}Q}/J(W)$$

be the (complete) Jacobian algebra of (Q, W). Associated to $\operatorname{mod} \mathcal{P}(Q, W)$, the abelian category of finite-dimensional $\mathcal{P}(Q, W)$-modules, there is a *motivic Hall algebra*

$$H(Q, W) = \bigoplus_{n \in N^{\oplus}} H(Q, W)_n$$

which is $N^{\oplus} = N^{+} \cup \{0\}$-graded and associative over $\mathbb{Q}$. In [**Bri17**], Bridgeland defined a consistent scattering diagram

$$\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Hall}} = \left(\mathfrak{S}_{\mathbf{s},W}^{\mathrm{Hall}}, \Phi_{\mathbf{s},W}^{\mathrm{Hall}} \colon M_{\mathbb{R}} \to H(Q, W) \right)$$

such that for any $m \in M_{\mathbb{R}}$, the value $\Phi(m)$ is the characteristic stack function of the subcategory of m-semistable $\mathcal{P}(Q, W)$-modules. With a properly defined integration map, the scattering diagram $\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Hall}}$ descends to the so-called *stability scattering diagram*

$$\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}} = \left(\mathfrak{S}_{\mathbf{s},W}^{\mathrm{Stab}}, \Phi_{\mathbf{s},W}^{\mathrm{Stab}} \right)$$

valued in the much simpler Lie algebra $\mathfrak{g}_{\mathbf{s}}$, the same Lie algebra as used by the cluster scattering diagram.

It is natural to ask the relationship between $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ and $\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}}$ as by definitions, they have the same wall-crossings at a generic point on any coordinate hyperplane $s_i^{\perp} \subset M_{\mathbb{R}}$. It is also not hard to see that both profinite cone complexes $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$ and $\mathfrak{S}_{\mathbf{s},W}^{\mathrm{Stab}}$ have a maximal simplicial cone, i.e., the cluster chamber

$$\mathcal{C}_{\mathbf{s}}^{+} := \{ m \in M_{\mathbb{R}} \mid m(s_i) \geq 0, \ \forall i \in I \},$$

whose facets have same the evaluations for $\Phi_{\mathbf{s}}^{\mathrm{Cl}}$ and $\Phi_{\mathbf{s},W}^{\mathrm{Stab}}$. In [**Bri17**], it is shown that if the quiver $Q(\mathbf{s})$ is acyclic, thus with only possible zero potential, then the two scattering diagrams are identical.

To further investigate their relationship, we first show that these two scattering diagrams share the same mutation behavior. Let $k \in I$ be an index. There are two operations $\mu_k^{\pm}$ that mutate $\mathbf{s}$ into another seed $\mu_k^{\pm}(\mathbf{s})$, giving the usual mutation of a quiver

$$Q\left(\mu_k^{\pm}(\mathbf{s})\right) = \mu_k(Q).$$

When the potential W is non-degenerate, Derksen, Weyman, and Zelevinsky [**DWZ08**] have lifted the mutations $\mu_k^{\pm}$ to seeds with potentials, thus obtaining $\mu_k^{\pm}(\mathbf{s}, W)$.

THEOREM 1.3.1 (Theorem 6.4.2). *The mutation* $\mathfrak{D}_{\mu_k^+(\mathbf{s},W)}^{\mathrm{Stab}}$ *is governed by the piecewise linear transformation*

$$T_k^+ \colon M_{\mathbb{R}} \to M_{\mathbb{R}}, \quad T_k^+(m) = \begin{cases} m, & \text{if } m(s_k) \leq 0 \\ m + m(s_k)p^*(s_k), & \text{if } m(s_k) > 0. \end{cases}$$

In particular, the profinite cone complexes are related by

$$\mathfrak{S}_{\mu_k^+(\mathbf{s},W)}^{\mathrm{Stab}} = T_k^+\left(\mathfrak{S}_{\mathbf{s},W}^{\mathrm{Stab}}\right).$$

The above theorem is the version in stability scattering diagrams of [**GHKK18**, theorem 1.24]. The following corollaries are immediate consequences of Theorem 1.3.1; see Section 6.4 for further details.

Corollary 1.3.2. *Let* $(\mathbf{s}, W)$ *be a non-degenerate seed with potential.*

(1) *The two cone decompositions* $\mathfrak{S}_{\mathbf{s},W}^{\mathrm{Stab}}$ *and* $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$ *contain a simplicial cone complex* $\Delta_{\mathbf{s}}^+$ *(the cluster complex) as a common sub-poset whose dual graph is isomorphic to the cluster exchange graph of* $Q(\mathbf{s})$ *where the positive chamber* $\mathcal{C}_{\mathbf{s}}^+$ *corresponds to the initial cluster.*

(2) *Moreover, the functions* $\Phi_{\mathbf{s},W}^{\mathrm{Stab}}$ *and* $\Phi_{\mathbf{s}}^{\mathrm{Cl}}$ *have the same values on the walls in* $\Delta_{\mathbf{s}}^+$.

(3) *If the quiver* $Q(\mathbf{s})$ *possesses a reddening sequence, then we have the equality of scattering diagrams*

$$\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}} = \mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}.$$

The condition of a quiver Q having a *reddening sequence* (see [**Mul16**] for a definition) is purely combinatorial, and many classes of quivers have been proven to have reddening sequences, including all acyclic quivers. Thus the part (3) of Corollary 1.3.2 is a generalization of the result

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of Bridgeland on acyclic quivers. We note that Qin also has a proof of (3) of Corollary 1.3.2 using *opposite* scattering diagrams [**Qin19**].

1.4. Scattering diagrams for skew-symmetrizable cluster algebras

We extend our cluster-versus-stability agenda to the skew-symmetrizable case. More generally, one can define a cluster algebra $\mathcal{A}(B)$ associated to a skew-symmetrizable integral matrix B which means there exists a diagonal positive integral matrix D such that

$$DB + B^T D^T = 0.$$

In our setting, we fix the lattice N with a chosen basis $\mathbf{s}$ and define a rational skew-symmetric form

$$\omega \colon N \times N \to \mathbb{Q}, \quad \omega(s_i, s_j) = d_j^{-1} b_{ij}.$$

Set the scaled seed $\tilde{\mathbf{s}} = (\tilde{s}_i)_{i \in I} = (d_i s_i)_{i \in I}$. Thus we have

$$\omega(s_i, \tilde{s}_j) = b_{ij}.$$

In [**GHKK18**], there is a *skew-symmetrizable* cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ defined for the data $(N, \omega, \mathbf{s}, D)$ generalizing the cluster scattering diagram mentioned in the last section where ω is integral and D is the identity matrix. In this dissertation, we propose a Langlands dual version of GHKK's cluster scattering diagram in Section 4.2, Chapter 4. We point out that this only makes a difference for non-trivial D, thus mainly in the skew-symmetrizable case.

Since in what follows, we will not use GHKK's version of the skew-symmetrizable cluster SD. Nevertheless, we will use the same notation $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$. It is worth mentioning the difference. For our scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$, at a generic point m on a coordinate hyperplane $s_i^{\perp}$, the wall-crossing is given by

$$\Phi(m) = \exp(-\operatorname{Li}_2(-x^{d_i s_i})/d_i),$$

whereas the GHKK's version has wall-crossing $\exp(-d_i \operatorname{Li}_2(-x^{s_i}))$. For further details and examples, see Section 4.2.2. The following theorem is Langlands dual to the GHKK's version in [**GHKK18**]. The proof also closely follow the scheme in loc. cit.

THEOREM 1.4.1 (Section 4.4). *The profinite cone complex $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$ contains a simplicial cone subcomplex $\Delta_{\mathbf{s}}^{+}$ dual to the cluster exchange graph of the matrix B. Each maximal cone of $\Delta_{\mathbf{s}}^{+}$ is generated by the g-vectors of the corresponding cluster of $\mathcal{A}(B)$.*

We note that the GHKK's version of $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$ has the same property corresponding to $B^{\vee} = -B^{T}$, the Langlands dual matrix of B.

The skew-symmetrizable version of the stability scattering diagram is yet to explore. The main reason is that as the matrix B does not naively determine a quiver, no quiver representations are available. The same difficulty exists for the additive categorification of skew-symmetrizable cluster algebras so that people have turned to new categories and objects, such as representations of species.

Suppose that the matrix B is acyclic and let D be its left symmetrizer. To the pair (B, D), Geiss, Leclerc and Schröer [**GLS17**] have defined a finite-dimensional algebra $H(B, D)$ whose representations are potential candidates that decategorify into the cluster monomials of $\mathcal{A}(B)$. Similar to the skew-symmetric case, there is an associated Hall algebra SD $\mathfrak{D}_{B,D}^{\mathrm{Hall}}$. We study this scattering diagram in Chapter 7. The main result regarding $\mathfrak{D}_{B,D}^{\mathrm{Hall}}$ is the following theorem. The proof relies on a description of c-vectors in terms of dimension vectors of modules of $H(B, D)$ by Geiss–Leclerc–Schröer [**GLS19**] and a stability–τ-tilting correspondence by Brüstle–Smith–Treffinger [**BST19**].

THEOREM 1.4.2 (Theorem 7.5.8). *The profinite cone complex $\mathfrak{S}_{B,D}^{\mathrm{Hall}}$ contains the same simplicial cone subcomplex $\Delta_{\mathbf{s}}^{+}$, as in the case of $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$.*

1.5. Scattering diagrams for Chekhov–Shapiro algebras

In Chapter 8, we extend the construction of cluster scattering diagrams to the case of Chekhov–Shapiro algebras [**CS14**]. These algebras generalize the ordinary cluster algebras of Fomin and Zelevinsky, thus also called *generalized cluster algebras*. Leaving the definitions and further details to later chapters (see Chapter 3 and Chapter 8), we explain below the main change from the cluster scattering diagram in order to obtain the generalized version.

We start again with the pair (B, D) and the corresponding data $(N, \omega, \mathbf{s}, D)$. Recall that for the cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$, the wall-crossing at (a generic point of) the hyperplane $s_i^{\perp}$ is given by

$$\Phi_{\mathbf{s}}^{\mathrm{Cl}}(s_i^{\perp}) = \exp(-\operatorname{Li}_2(-x^{d_i s_i})/d_i) \in \widehat{G}.$$

The group $\widehat{G}$ can be embedded into the automorphism group of the algebra

$$\mathbb{Q}[M] \otimes \mathbb{Q}[[N^{\oplus}]]$$

such that $\Phi_{\mathbf{s}}^{\mathrm{Cl}}(s_i^{\perp})$ acts by

$$\Phi_{\mathbf{s}}^{\mathrm{Cl}}(s_i^{\perp})(z^m) = z^m(1 + x^{d_i s_i})^{m(s_i)}.$$

We define generalized wall-crossings by setting

$$\Phi_{\mathbf{s}}^{\mathrm{CS}}(s_i^{\perp})(z^m) = z^m \rho_i(x^{s_i})^{m(s_i)}$$

where ρ_i is a degree d_i monic polynomial with the reciprocity $\rho_i(x) = x^{d_i}\rho_i(1/x)$. Here the polynomial ρ_i is of our choice, which is also part of the data defining a CS algebra Section 3.3.

With the polynomials $(\rho_i)_{i \in I}$ chosen, the CS scattering diagram

$$\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}} = (\mathfrak{S}_{\mathbf{s}}^{\mathrm{CS}}, \Phi_{\mathbf{s}}^{\mathrm{CS}})$$

is uniquely determined by requiring the hyperplanes $(s_i^{\perp})_{i \in I}$ to be the only incoming walls. It turns out that like the cluster scattering diagram, the CS scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$ behaves in the same way under mutations. As a consequence, we have

THEOREM 1.5.1 (Theorem 8.2.2). *The profinite cone complex $\mathfrak{S}_{\mathbf{s}}^{\mathrm{CS}}$ contains the cluster complex $\Delta_{\mathbf{s}}^{+}$ (also $\Delta_{\mathbf{s}}^{-}$) determined by the matrix B, the same one contained in $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$. Moreover, the generalized cluster variables can be obtained from path-ordered products of $\mathfrak{D}_{\mathbf{s}}^{CS}$ in the same way as the ordinary cluster variables are computed.*

Wall-Crossing structures and scattering diagrams

This chapter is an introduction to *wall-crossing structures* of Kontsevich and Soibelman [**KS14**], with an aim towards the applications in cluster algebras. In particular, we will define *consistent scattering diagrams* (in Section 2.2), the main object of study in this dissertation, as an example of wall-crossing structures.

The first two sections are devoted to the basic definitions and facts about wall-crossing structures and scattering diagrams. The last Section 2.3 contains our main results Theorem 2.3.1 and Theorem 2.3.10: every consistent scattering diagram admits a canonical underlying (profinite) cone complex structure.

2.1. Wall-crossing structures

In this section, we review the basics of Kontsevich-Soibelman's wall-crossing structures [**KS14**]. In the literature, the term *scattering diagram* often refers to some special class of wall-crossing structures, for example the *cluster scattering diagrams* in [**GHKK18**] and the *stability scattering diagrams* in [**Bri17**].

For experts, here we only touch on wall-crossing structures defined on a vector space.

2.1.1. Graded Lie algebras. Let $N \cong \mathbb{Z}^r$ be a lattice of rank r, i.e. a free abelian group of rank $r \in \mathbb{N}$. Fix a basis $\mathbf{s} = \{s_1, \ldots, s_r\}$ of N. We define $N^+ = N_{\mathbf{s}}^+$ to be the semi-subgroup (without 0) of N non-negatively generated by $\mathbf{s}$. The monoid $N_{\mathbf{s}}^{\oplus}$ is $N_{\mathbf{s}}^+ \cup \{0\}$. We will denote an N^+-*graded Lie algebra* by $\mathfrak{g}$. That is,

$$\mathfrak{g} = \bigoplus_{d \in N^+} \mathfrak{g}_d$$

as a free module over a commutative algebra over $\mathbb{Q}$ (usually a vector space over a field k of characteristic zero) with a Lie bracket such that $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] \subset \mathfrak{g}_{n_1+n_2}$ for any $n_1, n_2 \in N^+$. For a subset $S \subset N^+$, we will denote by $\mathfrak{g}_S$ the direct sum of the homogeneous spaces supported on S,

i.e.

$$\mathfrak{g}_S := \bigoplus_{d \in S} \mathfrak{g}_d.$$

Every ideal I of the semigroup N^+ gives an ideal $\mathfrak{g}_I$ of the Lie algebra $\mathfrak{g}$ and a quotient Lie algebra

$$\mathfrak{g}^{<I} := \mathfrak{g}/\mathfrak{g}_I.$$

Note that the Lie algebra $\mathfrak{g}^{<I}$ is still N^+-graded and is supported on the set $N^+ \setminus I$. If we have an inclusion of ideals $I \subset J$ of N^+, then there is an induced N^+-graded Lie algebra homomorphism $\rho_{I,J} \colon \mathfrak{g}^{<I} \to \mathfrak{g}^{<J}$.

When the Lie algebra $\mathfrak{g}$ is nilpotent, there is a corresponding unipotent algebraic group G that is in bijection with $\mathfrak{g}$ as sets. The product in the group G is given by the Baker-Campbell-Hausdorff formula. The bijection is denoted by $\exp \colon \mathfrak{g} \to G$. If the Lie algebra $\mathfrak{g}$ has finite support, i.e. when

$$\mathrm{Supp}(\mathfrak{g}) := \{d \in N^+ \mid \mathfrak{g}_d \neq 0\}$$

is a finite set, then it is nilpotent. We say an ideal I of N^+ is *cofinite* if $N^+ \setminus I$ is a finite set and denote the set of all cofinite ideals by $\mathrm{Cofin}(N^+)$. In this case, the quotient Lie algebra $\mathfrak{g}^{<I}$ has finite support and thus is nilpotent, giving the corresponding unipotent group $G^{<I}$. The inclusion of cofinite ideals $I \subset J$ induces a quotient map between groups, which we also denote by $\rho_{I,J} \colon G^{<I} \to G^{<J}$. In fact, we can define an order $J \leq I$ for $I \subset J$. Then the set of cofinite ideals $\mathrm{Cofin}(N^+)$ becomes a directed set and the associations $I \mapsto \mathfrak{g}^{<I}$ and $I \mapsto G^{<I}$ become inverse systems indexed by $\mathrm{Cofin}(N^+)$. Taking the projective limits, we obtain a pro-nilpotent Lie algebra and a corresponding pro-unipotent algebraic group:

(2.1.1)
$$\hat{\mathfrak{g}} := \varprojlim_I \mathfrak{g}^{<I} \cong \prod_{n \in N^+} \mathfrak{g}_n \quad \text{and} \quad \hat{G} := \varprojlim_I G^{<I}.$$

The group $\hat{G}$ is again in bijection with the Lie algebra $\hat{\mathfrak{g}}$ as sets.

We put

$$M := \mathrm{Hom}(N, \mathbb{Z}) \cong Z^n \quad \text{and} \quad M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \mathbb{R}^n.$$

For any $m \in M_{\mathbb{R}}$, there is a partition of N^+:

$$N^+ = P_{m,+} \sqcup P_{m,0} \sqcup P_{m,+}$$

where

$$(2.1.2) \qquad P_{m,\pm} := \{n \in N^+ \,|\, m(n) \gtrless 0\} \ \text{ and } P_{m,0} \colon = \{n \in N^+ \,|\, m(n) = 0\}.$$

This partition of N^+ induces a decomposition of $\mathfrak{g}$:

$$(2.1.3) \qquad \qquad \qquad \mathfrak{g} = \mathfrak{g}_{m,+} \oplus \mathfrak{g}_{m,0} \oplus \mathfrak{g}_{m,-}$$

where $\mathfrak{g}_{m,\bullet} := \mathfrak{g}_{P_{m,\bullet}}$ is a graded Lie subalgebra for $\bullet \in \{0, +, -\}$. We denote the corresponding pro-unipotent subgroups by $\hat{G}_{m,\bullet}$. In the following lemma, the element $m \in M_{\mathbb{R}}$ is fixed, thus being omitted in the subscript.

Lemma 2.1.1. *Fix some $m \in M_{\mathbb{R}}$. Then the decomposition (2.1.3) induces a unique factorization of any element $g \in \hat{G}$ into $g = g_+ \cdot g_0 \cdot g_-$ where $g_\bullet \in \hat{G}_\bullet$ for $\bullet \in \{0, +, -\}$. In other words, the map $\Phi \colon \hat{G}_+ \times \hat{G}_0 \times \hat{G}_- \to \hat{G}$ defined by*

$$\Phi(g_+, g_0, g_-) = g_+ \cdot g_0 \cdot g_-$$

is a set bijection.

PROOF. We first prove this for any $I \in \mathrm{Cofin}(N^+)$. Take a filtration of cofinite ideals

$$I = I_k \subset I_{k-1} \subset \cdots \subset I_0 = N^+$$

such that $I_i \setminus I_{i+1} = \{n_i\}$ contains only one element. We have the following commutative diagram

$$
\begin{array}{ccc}
G_+^{<I_{i+1}} \times G_0^{<I_{i+1}} \times G_-^{<I_{i+1}} & \xrightarrow{\Phi_{i+1}} & G^{<I_{i+1}} \\
\Big\downarrow{\scriptstyle f_i} & & \Big\downarrow{\scriptstyle h_i} \\
G_+^{<I_i} \times G_0^{<I_i} \times G_-^{<I_i} & \xrightarrow{\ \ \Phi_i\ \ } & G^{<I_i}
\end{array}
$$

where f_i and h_i are given by natural quotient maps induced by the inclusion $I_{i+1} \subset I_i$. Note that the maps f_i and g_i are both fibrations of $\exp(\mathfrak{g}_{n_i})$-torsors and the map Φ_{i+1} is $\exp(\mathfrak{g}_{n_i})$-equivariant. Here $\mathfrak{g}_{n_i}$ denotes the quotient Lie algebra $\mathfrak{g}^{<I_{i+1}}/\mathfrak{g}^{<I_i}$. Therefore, the map Φ_i being bijective would imply that Φ_{i+1} is bijective. Since Φ_0 is bijective, the map

$$\Phi_I := \Phi_k \colon G_+^{<I} \times G_0^{<I} \times G_-^{<I} \to G^{<I}$$

is bijective by induction.

Now we have similar commutative diagrams for any inclusion $I \subset J$

$$
\begin{array}{ccc}
G_+^{\leq I} \times G_0^{\leq I} \times G_-^{\leq I} & \xrightarrow{\Phi_I} & G^{<I} \\
\downarrow{f_{I,J}} & & \downarrow{\rho_{I,J}} \\
G_+^{\leq J} \times G_0^{\leq J} \times G_-^{\leq J} & \xrightarrow{\Phi_J} & G^{<J}.
\end{array}
$$

Then the bijections extend to a bijection between projective limits. $\qquad\square$

The factorization in the above lemma defines projection maps (of sets)

(2.1.4) $$\pi_{m,\bullet}\colon \hat{G} \to \hat{G}_{m,\bullet}$$

by sending g to $g_{m,\bullet}$. We will simply write π_m for $\pi_{m,0}$.

2.1.2. Cone complex. By a *cone* in the vector space $M_{\mathbb{R}}$, we mean a subset closed under scaling by $\mathbb{R}_{>0}$. A cone is *convex* if it is convex as a subset of $M_{\mathbb{R}}$. A *polyhedral cone* is a closed convex subset of $M_{\mathbb{R}}$ of the form

$$
\sigma = \left\{ \sum_{i=1}^k \lambda_i v_i \mid \lambda_i \in \mathbb{R}_{\geq 0}, \ v_i \in M_{\mathbb{R}} \right\}.
$$

It is called a *rational polyhedral cone* if v_i is in M for each i. A face of a cone σ is a subset of the form

$$
\sigma \cap n^{\perp} = \{ m \in \sigma \mid m(n) = 0 \}
$$

where $n \in N_{\mathbb{R}}$ satisfies $m(n) \geq 0$ for all $m \in \sigma$. A face of a cone is again a cone.

Definition 2.1.2. A *cone complex* $\mathfrak{S}$ in $M_{\mathbb{R}}$ is a collection of rational polyhedral cones in $M_{\mathbb{R}}$ such that

(1) for any $\sigma \in \mathfrak{S}$, if $\tau \subset \sigma$ is a face of σ, then $\tau \in \mathfrak{S}$;

(2) for any $\sigma_1, \sigma_2 \in \mathfrak{S}$, $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2.

Note that we do not require the cones in a cone complex to be *strictly convex*. For example, a closed half-space is allowed. We also do not require the collection to be finite, for which we call *a finite cone complex*. If the union of all cones $|\mathfrak{S}|$ equals $M_{\mathbb{R}}$, we say that the cone complex $\mathfrak{S}$ is *complete*. The complement of the union of all proper faces of σ in σ is called the *relative interior* of σ and is denoted by σ°. It is relatively open, i.e., open in the subspace in $M_{\mathbb{R}}$ spanned σ. The set of cones σ° for all $\sigma \in \mathfrak{S}$ is denoted by $\mathfrak{S}^{\circ}$.

A cone complex $\mathfrak{S}$ is a poset with the partial order $\sigma_1 \prec \sigma_2$ if and only if σ_1 is a face of σ_2. It can also be viewed as a category where the only morphisms are of the form $\sigma_1 \prec \sigma_2$. We define a set of cones associated to σ

$$(2.1.5) \qquad\qquad \text{Star}(\sigma) := \{\tau^\circ \in \mathfrak{S}^\circ \mid \sigma \prec \tau\}$$

and

$$|\text{Star}(\sigma)| := \bigcup_{\tau^\circ \in \text{Star}(\sigma)} \tau^\circ.$$

Example 2.1.3 (Hyperplane arrangements)**.** Let S be a finite subset of N. Consider a partition P of S into three disjoint subsets

$$S = P_+ \sqcup P_0 \sqcup P_-.$$

We define a closed cone associated to P

$$\sigma_P := \overline{\{m \in M_\mathbb{R} \mid m(P_+) > 0, \ m(P_0) = 0 \text{ and } m(P_-) < 0\}}.$$

One easily checks the (non-empty) cones σ_P for all such partitions of S form a complete cone complex $\mathfrak{S}_S$ in $M_\mathbb{R}$. We have $\sigma_{P_1} \prec \sigma_{P_2}$ if and only if

$$P_1 \prec P_2, \ \text{i.e. } P_{1,+} \subset P_{2,+}, \ P_{2,0} \subset P_{1,0}, \ \text{and } P_{1,-} \subset P_{2,-}.$$

2.1.3. A key lemma. Assume that $S = \text{Supp}(\mathfrak{g})$ is finite. Let $\sigma \in \mathfrak{S}_S$ and set

$$\sigma^\perp = \{d \in N \mid m(d) = 0 \quad \forall m \in \sigma\}.$$

We put $\mathfrak{g}_\sigma := \mathfrak{g}_{\sigma^\perp \cap S}$. Let σ_1 and σ_2 be two cones in $\mathfrak{S}_S$ such that $\sigma_1 \prec \sigma_2$, we define a map $\pi_{\sigma_1,\sigma_2} \colon \exp(\mathfrak{g}_{\sigma_1}) \to \exp(\mathfrak{g}_{\sigma_2})$ as follows. Suppose σ_1 and σ_2 are given by two partitions $P_1 \prec P_2$. Let m be in σ_2° and it gives (independent of the choice of m) a partition

$$P_{1,0} = (P_{1,0})_{m,+} \sqcup P_{2,0} \sqcup (P_{1,0})_{m,-}$$

as in (2.1.2). Note that $\mathfrak{g}_{\sigma_i} = \mathfrak{g}_{P_{i,0}}$. Then we define the map $\pi_{\sigma_1,\sigma_2} \colon \exp(\mathfrak{g}_{\sigma_1}) \to \exp(\mathfrak{g}_{\sigma_2})$ by the projection

$$\pi \colon \mathfrak{g}_{P_{1,0}} \to \mathfrak{g}_{P_{2,0}}$$

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exactly as in Lemma 2.1.1. For example, if σ_1 be the smallest cone in $\mathfrak{S}_S$ with respect to the order $\prec$, then $\pi_{\sigma_1,\sigma_2} = \pi_m$ for any $m \in \sigma_2^\circ$.

The following fundamental lemma will be useful later.

Lemma 2.1.4. *Let $\mathfrak{g}$ be an N^+-graded Lie algebra with finite support S. The assignment $\sigma \mapsto \exp(\mathfrak{g}_\sigma)$, $(\sigma_1 \prec \sigma_2) \mapsto \pi_{\sigma_1,\sigma_2}$ defines a functor from $\mathfrak{S}_S$ to Grp the category of groups.*

PROOF. The only thing we need to check is if $\sigma_0 \prec \sigma_1 \prec \sigma_2$, then

$$\pi_{\sigma_1,\sigma_2} \circ \pi_{\sigma_0,\sigma_1} = \pi_{\sigma_0,\sigma_2} : \mathfrak{g}_{\sigma_0} \to \mathfrak{g}_{\sigma_2}$$

Without loss of generality, we can assume that σ_0 is the origin. In fact, one can always restrict $\mathfrak{g}$ to $N \cap \sigma_0^\perp$ and correspondingly, take the quotient of M by the span of σ_0. That is, we have

$$(N \cap \sigma_0^\perp)^\vee \cong M/(\langle \sigma_0 \rangle \cap M).$$

Now $\mathfrak{g}_{\sigma_0} = \mathfrak{g}$ and for $m_i \in \sigma_i^\circ$, $\pi_{\sigma_0,\sigma_i} = \pi_{m_i}$ for $i = 1, 2$ (see (2.1.4)). Suppose σ_i comes from a partition P_i as before. We know $\pi_{\sigma_0,\sigma_1}(g)$ for $g \in G = \exp(\mathfrak{g})$ is the middle term in the factorization (Lemma 2.1.1)

$$g = \pi_{m_1,+}(g) \cdot \pi_{m_1}(g) \cdot \pi_{m_1,-}(g).$$

By factorizing $g_1 = \pi_{m_1}(g)$ further with respect to m_2, we get $\pi_{\sigma_1,\sigma_2} \circ \pi_{\sigma_0,\sigma_1}(g) = \pi_{m_2}(g_1)$:

$$g_1 = \pi_{m_2,+}(g_1) \cdot \pi_{m_2}(g_1) \cdot \pi_{m_2,-}(g_1).$$

Using previous notations, we have

$$P_{1,0} = (P_{1,0})_{m_2,+} \sqcup P_{2,0} \sqcup (P_{1,0})_{m_2,-}$$

and

$$P_{2,+} = P_{1,+} \sqcup (P_{1,0})_{m_2,+}, \quad P_{2,-} = P_{1,-} \sqcup (P_{1,0})_{m_2,-}.$$

This implies

$$\pi_{m_1,+}(g) \cdot \pi_{m_2,+}(g_1) \in \exp(\mathfrak{g}_{P_{2,+}}) = G_{m_2,+}$$

and

$$\pi_{m_2,-}(g_1) \cdot \pi_{m_1,-}(g) \in \exp(\mathfrak{g}_{P_{2,-}}) = G_{m_2,-}.$$

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Therefore we obtain a factorization

$$g = (\pi_{m_1,+}(g) \cdot \pi_{m_2,+}(g_1)) \cdot \pi_{\sigma_1,\sigma_2} \circ \pi_{\sigma_0,\sigma_1}(g) \cdot (\pi_{m_2,-}(g_1) \cdot \pi_{m_1,-}(g)).$$

Such a factorization is unique by Lemma 2.1.1. We conclude that the middle term $\pi_{\sigma_1,\sigma_2} \circ \pi_{\sigma_0,\sigma_1}(g)$ equals $\pi_{m_2}(g) = \pi_{\sigma_0,\sigma_2}(g)$. $\qquad\square$

2.1.4. Definition of wall-crossing structures. In this section, we define *wall-crossing structures* following [**KS14**]. Our scheme is to treat the finite support case first and then the infinite support case.

We first assume that $S = \mathrm{Supp}(\mathfrak{g})$ is finite. Define the following set (the étalé space)

$$\mathcal{G}^{\text{ét}} := \{(m, g') \mid m \in M_{\mathbb{R}},\, g' \in G_{m,0}\}$$

with a collection of subsets

$$W_{U,g} := \{(m, g') \mid m \in U,\, g' = \pi_m(g)\}$$

where g runs through G, and U runs through all open sets of $M_{\mathbb{R}}$.

Lemma 2.1.5. *The subsets $W_{U,g}$ give a base of topology on $\mathcal{G}^{\text{ét}}$. The projection $\mathcal{G}^{\text{ét}} \to M_{\mathbb{R}}$ defined by $(m, g') \mapsto m$ is a local homeomorphism.*

PROOF. First of all, the subsets $W_{U,g}$ cover $\mathcal{G}^{\text{ét}}$. Consider the cone complex $\mathfrak{S}_S$ defined in Example 2.1.3. Let $W_{U,g}$ and $W_{V,h}$ be two subsets of $\mathcal{G}^{\text{ét}}$. By Lemma 2.1.4, if $\pi_m(g) = \pi_m(h)$ for some $m \in U \cap V$, then $\pi_{m'}(g) = \pi_{m'}(h)$ for any m' in the interior of $|\mathrm{Star}(\sigma)|$ where σ is the unique cone in $\mathfrak{S}_S$ such that $m \in \sigma^\circ$. It follows that the set

$$R = \{m \in M_{\mathbb{R}} \mid \pi_m(g) = \pi_m(h)\}$$

is open in $M_{\mathbb{R}}$. So the intersection $W_{U,g} \cap W_{V,h} = W_{R,g}$ is a set of the same form. Thus the collection $W_{U,g}$ form a base of topology.

To prove the second statement, let $(m, g') \in \mathcal{G}^{\text{ét}}$. Then we have $\pi_m(g') = g'$. Take an arbitrary open neighborhood $U \subset M_{\mathbb{R}}$ of m. It is clear that the map $W_{U,g'} \to U$, $(u, \pi_u(g')) \mapsto u$ is a homeomorphism. $\qquad\square$

Definition 2.1.6. Let $\mathfrak{g}$ be a N^+-graded Lie algebra with finite support. The *sheaf of wall-crossing structures* $\mathcal{WCS}_\mathfrak{g}$ is defined to be the sheaf of sections of the local homeomorphism $\mathcal{G}^{\text{ét}} \to M_\mathbb{R}$ in Lemma 2.1.5, i.e $\mathcal{G}^{\text{ét}} \to M_\mathbb{R}$ is the étalé space of this sheaf.

For any $g \in G$, there is a section

$$s_g \colon M_\mathbb{R} \to \mathcal{G}^{\text{ét}}, \quad m \mapsto (m, \pi_m(g)).$$

The image of s_g is $W_{g,M_\mathbb{R}}$ and s_g is clearly a global section of the sheaf $\mathcal{WCS}_\mathfrak{g}$. By the construction of the étalé space $\mathcal{G}^{\text{ét}}$, the stalk $(\mathcal{WCS}_\mathfrak{g})_m$ is canonically identified with $G_{m,0}$. The germ of the section s_g at m is just given by $\pi_m(g) \in G_{m,0}$.

The above construction defines a map $s \colon G \to \Gamma(\mathcal{WCS}_\mathfrak{g}, M_\mathbb{R})$ sending g to the global section s_g. The following lemma follows from corollary 2.1.3 and lemma 2.1.7 in [**KS14**].

Lemma 2.1.7. *The map* $s \colon G \to \Gamma(\mathcal{WCS}_\mathfrak{g}, M_\mathbb{R})$, $g \mapsto s_g$ *is a bijection. Thus the set of global sections of* $\mathcal{WCS}_\mathfrak{g}$ *is in bijection with the group* G.

Now we remove the finiteness restriction on the support $\text{Supp}(\mathfrak{g})$. There is a directed inverse system of sheaves $\mathcal{WCS}_{\mathfrak{g}^{<I}}$ indexed by the directed set $\text{Cofin}(N^+)$

$$(2.1.6) \qquad\qquad \text{Cofin}(N^+) \to Sh(M_\mathbb{R}), \quad I \mapsto \mathcal{WCS}_{\mathfrak{g}^{<I}}$$

induced by natural quotient maps of Lie algebras.

Definition 2.1.8. In general, the *sheaf of wall-crossing structures* of the Lie algebra $\mathfrak{g}$ is defined to be the projective limit of the above mentioned inverse system $(2.1.6)$ in the category of sheaves of sets on $M_\mathbb{R}$:

$$\mathcal{WCS}_\mathfrak{g} := \varprojlim_I \mathcal{WCS}_{\mathfrak{g}^{<I}} \in Sh(M_\mathbb{R}).$$

It is standard that the space of sections of the projective limit of sheaves is in bijection with the projective limit of spaces of sections. So we have the identification of the space of global sections with the pro-unipotent group $\hat{G}$:

$$(2.1.7) \qquad\qquad \Gamma(\mathcal{WCS}_\mathfrak{g}, M_\mathbb{R}) = \varprojlim_I \Gamma(\mathcal{WCS}_{\mathfrak{g}^{<I}}, M_\mathbb{R}) = \varprojlim_I G^{<I} = \hat{G}.$$

The second equality follows from Lemma 2.1.7.

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Given a global section $g \in \hat{G}$, we define the following map:

$$(2.1.8) \qquad \Phi_g \colon M_{\mathbb{R}} \to \hat{G}, m \mapsto \pi_m(g) \in \hat{G}_{m,0} \subset \hat{G}$$

which records the germs of the global section g at every stalk.

Definition 2.1.9 ($\mathfrak{g}$-WCS cf. [**KS14**])**.** A global section of the sheaf $\mathcal{WCS}_{\mathfrak{g}}$ is called a $\mathfrak{g}$-*wall-crossing structure* ($\mathfrak{g}$-WCS in short). Thus the equality (2.1.7) shows that $\Gamma(\mathcal{WCS}_{\mathfrak{g}}, M_{\mathbb{R}})$, the set of all $\mathfrak{g}$-WCS', is in bijection with the pro-unipotent group $\hat{G}$.

2.2. Consistent scattering diagrams

The sheaf of wall-crossings $\mathcal{WCS}_{\mathfrak{g}}$ can be defined on more general spaces than vector spaces [**KS14**]. A *consistent scattering diagram* that we are about to define in this section is an equivalent notion of a wall-crossing structure on a vector space defined in the last section. We hope our use of terminology will not cause any confusion to the reader: the central subject to study in this dissertation is *consistent scattering diagrams*, which are equivalent to *wall-crossing structures* defined in Section 2.1.

2.2.1. Consistent scattering diagrams for $\mathfrak{g}$ with finite support. In this section we give the definition of a *consistent $\mathfrak{g}$-scattering diagram* (*consistent $\mathfrak{g}$-SD* in short). The relation between a consistent $\mathfrak{g}$-SD and a $\mathfrak{g}$-WCS will be explained in Proposition 2.2.6.

We first assume the set $S = \mathrm{Supp}(\mathfrak{g})$ to be finite.

Definition 2.2.1 ($\mathfrak{g}$-SD)**.** A $\mathfrak{g}$-*scattering diagram* ($\mathfrak{g}$-SD in short) $\mathfrak{D}$ with $S = \mathrm{Supp}(\mathfrak{g})$ is a pair $(\mathfrak{S}_S, \Phi_{\mathfrak{D}})$ consisting of the cone complex $\mathfrak{S}_S$ (see Example 2.1.3) and a function $\Phi_{\mathfrak{D}} \colon \mathfrak{S}_S \to G$ such that for any $\sigma \in \mathfrak{S}_S$, $\Phi_{\mathfrak{D}}(\sigma)$ is in the subgroup $\exp(\mathfrak{g}_\sigma) \subset G$. If codim $\sigma = 1$ (resp. > 1), we call $\Phi_{\mathfrak{D}}(\sigma)$ the *wall-crossing (resp. face-crossing)* of $\mathfrak{D}$ at σ.

For each cone $\sigma = \sigma_P$ in $\mathfrak{S}_S$ of codimension at least one, there is a maximal cell σ^+ relative to σ. It is the relative interior of the cone $\sigma_{P'}$ given by the partition P' such that

$$P'_+ = P_+ \sqcup P_0, \ P'_0 = \emptyset, \ \text{and} \ P'_- = P_-.$$

Similarly there is the negative maximal cell σ^-. For example, if σ is a wall (i.e. codim $\sigma = 1$), then σ^+ and σ^- are the two maximal cells on the two sides of σ.

Definition 2.2.2. An $\mathfrak{S}_S$-*path* is a smooth curve $\gamma\colon [0,1] \to M_\mathbb{R}$ such that

(1) the end points $\gamma(0)$ and $\gamma(1)$ are in maximal cells, i.e. not in some cones of codimension at least one.

(2) if for some $t \in [0,1]$, $\gamma(t) \in \sigma^\circ$ for some cone σ of codimension at least one in $\mathfrak{S}_S$, then there exists a neighborhood $(t-\varepsilon, t+\varepsilon)$ of t such that $\gamma((t-\varepsilon,t)) \subset \sigma^-$ and $\gamma((t,t+\varepsilon)) \subset \sigma^+$. We call t a *negative crossing* and denote it by t^- if $\varepsilon > 0$ and a *positive crossing* t^+ if $\varepsilon < 0$ respectively.

Let $\mathfrak{D}$ be a $\mathfrak{g}$-SD and γ be an $\mathfrak{S}_S$-path γ with finitely many crossings

$$0 < t_1^{\varepsilon_1} < t_2^{\varepsilon_2} < \cdots < t_k^{\varepsilon_k} < 1$$

with $\varepsilon_i \in \{-,+\}$. Record the cones at these crossings by σ_i and the corresponding wall-crossings or face-crossings by

$$g_i = \begin{cases} \Phi_{\mathfrak{D}}(\sigma_i) & \text{if } \varepsilon_i = + \\ \Phi_{\mathfrak{D}}(\sigma_i)^{-1} & \text{if } \varepsilon_i = - \end{cases}.$$

Definition 2.2.3. The *path-ordered product* for an $\mathfrak{S}_S$-path γ in $\mathfrak{D}$ is defined as

$$\mathfrak{p}_\gamma(\mathfrak{D}) := g_k \cdot \cdots \cdot g_2 \cdot g_1 \in G.$$

Now we are ready to define consistent $\mathfrak{g}$-scattering diagrams for $\mathfrak{g}$ with finite support. The consistency here means the path-ordered product is path-independent.

Definition 2.2.4 (Consistent $\mathfrak{g}$-SD)**.** A $\mathfrak{g}$-SD $\mathfrak{D}$ is said to be a *consistent* $\mathfrak{g}$-SD if the path-ordered product $\mathfrak{p}_\gamma(\mathfrak{D})$ for any $\mathfrak{S}_S$-path γ depends only on the end points $\gamma(0)$ and $\gamma(1)$. That is, if two $\mathfrak{S}_S$-paths γ_1 and γ_2 have the same end points, then $\mathfrak{p}_{\gamma_1}(\mathfrak{D}) = \mathfrak{p}_{\gamma_2}(\mathfrak{D})$.

There is a maximal cell $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) in $\mathfrak{S}_S$ that is the intersection of all positive (resp. negative) open half-spaces in the hyperplane arrangement in Example 2.1.3. Given $\mathfrak{D}$ a consistent $\mathfrak{g}$-SD, we define

$$\mathfrak{p}_{+,-}(\mathfrak{D}) := \mathfrak{p}_\gamma(\mathfrak{D})$$

for any $\mathfrak{S}_S$-path γ with $\gamma(0) \in \mathcal{C}^+$ and $\gamma(1) \in \mathcal{C}^-$. It does not depend on γ since $\mathfrak{D}$ is consistent. This defines a map $\mathfrak{p}_{+,-}$ from $\mathfrak{g}$-$\mathcal{SD}$ (the set of all consistent $\mathfrak{g}$-SD's) to $G = \exp(\mathfrak{g})$ by sending $\mathfrak{D}$ to $\mathfrak{p}_{+,-}(\mathfrak{D})$. The following theorem asserts that $\mathfrak{p}_{+,-}$ is a bijection. The original form of the theorem

is due to Kontsevich and Soibelman. Since our setting is slightly different from theirs (as we allow higher codimensional *face-crossings*), we give a complete proof for the reader's convenience.

THEOREM 2.2.5 (Kontsevich–Soibelman [**KS14**]). *The map* $\mathfrak{p}_{+,-} \colon \mathfrak{g}\text{-}\mathcal{SD} \to G$, $\mathfrak{D} \mapsto \mathfrak{p}_{+,-}(\mathfrak{D})$ *is a bijection of sets.*

PROOF. Let $g \in G$. We construct a $\mathfrak{g}$-SD $\mathfrak{D}_g$ as follows. Recall the function defined in (2.1.8)

$$\Phi_g \colon M_{\mathbb{R}} \to G, \ m \mapsto \pi_m(g).$$

Consider a $\mathfrak{g}$-SD $\mathfrak{D}_g = (\mathfrak{S}_S, \Phi_{\mathfrak{D}_g})$ with $\Phi_{\mathfrak{D}_g}(\sigma) = \Phi_g(m)$ for any $\sigma \in \mathfrak{S}_S$ and $m \in \sigma^\circ$. We show that it is consistent. In fact, a wall-crossing, or more generally face-crossing has the following description. Let $m^+ \in \sigma^+$ and $m^- \in \sigma^-$. By the definitions of $\sigma^\pm$ and the uniqueness in Lemma 2.1.1, we have

$$\pi_{m^+,+}(g) = \pi_{m,+}(g) \cdot \pi_m(g), \ \pi_{m^+}(g) = 1, \ \text{and} \ \pi_{m^+,-}(g) = \pi_{m,-}(g);$$

$$\pi_{m^-,+}(g) = \pi_{m,+}(g), \ \pi_{m^-}(g) = 1, \ \text{and} \ \pi_{m^-,-}(g) = \pi_m(g) \cdot \pi_{m,-}(g).$$

This gives

$$\Phi_g(m) = \pi_{m^-,+}^{-1}(g) \cdot \pi_{m^+,+}(g) \quad \text{and} \quad \Phi_g(m)^{-1} = \pi_{m^+,+}^{-1}(g) \cdot \pi_{m^-,+}(g).$$

By induction on the number of crossings, we have

(2.2.1) $$\mathfrak{p}_\gamma = \pi_{\gamma(1),+}^{-1}(g) \cdot \pi_{\gamma(0),+}(g)$$

for any $\mathfrak{S}_S$-path γ, which only depends on the end points, proving the consistency. Note that by construction $\mathfrak{p}_{+,-}(\mathfrak{D}_g) = \Phi_g(0) = g$. This shows the map $\mathfrak{p}_{+,-}$ is surjective.

We show next the map $\mathfrak{p}_{+,-}$ is also injective. Let $\mathfrak{D} \in \mathfrak{g}\text{-}\mathcal{SD}$, i.e. a consistent $\mathfrak{g}$-SD. Let $\sigma \in \mathfrak{S}_S$ and we choose $m_\sigma \in \sigma^\circ$ and $\lambda \in \mathcal{C}^+$. Consider the path $\gamma \colon (-\infty, +\infty) \to M_{\mathbb{R}}$ given by $\gamma(t) = m_\sigma - \lambda t$. Whenever γ meets some cone $\tau^\circ \in \mathfrak{S}_S^\circ$, it always goes from τ^+ to τ^-. Thus after rescaling, we get an $\mathfrak{S}_S$-path γ going from $\mathcal{C}^+$ to $\mathcal{C}^-$ with positive crossings at $\sigma_1, \cdots, \sigma_k$ in order where $\sigma = \sigma_l$ for some l. The path-ordered product is then

$$\mathfrak{p}_{+,-}(\mathfrak{D}) = \Phi_{\mathfrak{D}}(\sigma_k) \cdots \Phi_{\mathfrak{D}}(\sigma_l) \cdots \Phi_{\mathfrak{D}}(\sigma_1)$$

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If $i < l$, then $m_i := m_\sigma + t_i \cdot m^+ \in \sigma_i^\circ$ for some $t_i > 0$. If $n \in P_{m_i,0}$, i.e. $\langle m_i, n \rangle = 0$, then $\langle m_\sigma + t_i \cdot m^+, n \rangle = 0$ which implies $\langle m_\sigma, n \rangle < 0$, i.e. $n \in P_{m_\sigma,-}$. Thus we have $P_{m_i,0} \subset P_{m_\sigma,-}$ and $\Phi_{\mathfrak{D}}(\sigma_i) \in G_{m_\sigma,-}$. Similarly if $j > l$, we have $P_{m_j,0} \subset P_{m_\sigma,+}$ and $\Phi_{\mathfrak{D}}(\sigma_j) \in G_{m_\sigma,+}$. Thus we have,

$$\mathfrak{p}_{+,-}(\mathfrak{D}) = (\Phi_{\mathfrak{D}}(\sigma_k) \cdots \Phi_{\mathfrak{D}}(\sigma_{l+1})) \cdot \Phi_{\mathfrak{D}}(\sigma) \cdot (\Phi_{\mathfrak{D}}(\sigma_{l-1}) \cdots \Phi_{\mathfrak{D}}(\sigma_1))$$

as a factorization with respect to m in Lemma 2.1.1. This in particular shows

$$\Phi_{\mathfrak{D}}(\sigma) = \pi_{m_\sigma}(\mathfrak{p}_{+,-}(\mathfrak{D}))$$

and therefore, the consistent $\mathfrak{g}$-SD is entirely determined by $\mathfrak{p}_{+,-}(\mathfrak{D}) \in G$, i.e., the map $\mathfrak{p}_{+,-}$ is injective. $\qquad\square$

We clarify in the next proposition the relation between the notion of a $\mathfrak{g}$-WCS and that of a consistent $\mathfrak{g}$-SD when $\mathrm{Supp}(\mathfrak{g})$ is finite. It is an immediate consequence of the proof Theorem 2.2.5.

Proposition 2.2.6. *Let $g \in G$ and $\mathfrak{D}_g = (\mathfrak{S}_S, \Phi_{\mathfrak{D}_g})$ be $\mathfrak{p}_{+,-}^{-1}(g) \in \mathfrak{g}$-$\mathcal{SD}$, i.e. a consistent $\mathfrak{g}$-SD such that $\mathfrak{p}_{+,-}(\mathfrak{D}) = g$. By Lemma 2.1.7, there is also a $\mathfrak{g}$-WCS s_g and it is determined by the function $\Phi_g \colon M_{\mathbb{R}} \to G$ in (2.1.8). Then for any $\sigma \in \mathfrak{S}_S$, we have*

$$\Phi_{\mathfrak{D}_g}(\sigma) = \Phi_g(m)$$

for any $m \in \sigma^\circ$.

According to the above proposition, we will simply denote the $\mathfrak{g}$-SD $\mathfrak{p}_{+,-}^{-1}(g)$ by

$$\mathfrak{D}_g = (\mathfrak{S}_S, \Phi_g)$$

for $g \in G$. The function Φ_g then has domain $M_{\mathbb{R}}$ rather than $\mathfrak{S}_S$. However by $\Phi_g(\sigma)$, we always mean $\Phi_{\mathfrak{D}_g}(\sigma)$ without any ambiguity or equivalently one can interpret $\Phi_g(\sigma)$ as $\Phi_g(m)$ for some $m \in \sigma^\circ$.

2.2.2. Face-crossings are determined by wall-crossings. Now we explain that a consistent $\mathfrak{g}$-SD $\mathfrak{D}$ is completely determined by its wall-crossings (among more general face-crossings). In fact, most of the existing approaches to scattering diagrams, for example [**Bri17**, **GHKK18**], only encode wall-crossings into the definition of consistency.

Let $\mathfrak{D}$ be a consistent $\mathfrak{g}$-SD. Let σ be some cone in $\mathfrak{S}_S$ with $\operatorname{codim}(\sigma) > 1$. There is an $\mathfrak{S}_S$-path γ going directly from σ^+ to σ^- by crossing σ°. Thus $\mathfrak{p}_\gamma(\mathfrak{D}) = \Phi_\mathfrak{D}(\sigma)$. However, we can always find an $\mathfrak{S}_S$-path γ' from σ^+ to σ^- with only wall-crossings. Thus the face-crossing $\Phi_\mathfrak{D}(\sigma)$ is equal to a product of wall-crossings because of consistency.

Deonte the subset of all cones of codimension one in $\mathfrak{S}_S$ by $\mathfrak{W}_S$. The following is yet another equivalent definition to Definition 2.2.4, which is more commonly adopted in the literature.

Definition 2.2.7. Let $\mathfrak{g}$ be an N^+-graded Lie algebra with finite support S. A consistent $\mathfrak{g}$-SD $\mathfrak{D}$ is the datum of a function $\Phi_\mathfrak{D} : \mathfrak{W}_S \to G$ such that

(1) $\Phi(\sigma) \in \exp(\mathfrak{g}_\sigma) \subset G$ for any $\sigma \in \mathfrak{W}_S$, and

(2) any path-ordered product for an $\mathfrak{S}_S$-path with only wall-crossings only depends on end points.

2.2.3. Consistent scattering diagrams for $\mathfrak{g}$ with infinite support. Now we remove the finiteness restriction of $S = \operatorname{Supp}(\mathfrak{g})$ to define consistent $\mathfrak{g}$-SD's in general. For $I \in \operatorname{Cofin}(N^+)$, the quotient Lie algebra $\mathfrak{g}^{<I}$ is supported on $S^{<I} = S \setminus I$. Recall that we have a group homomorphism $\rho^{<I} : \hat{G} \to G^{<I}$.

Definition 2.2.8. A $\mathfrak{g}$-SD $\mathfrak{D}$ is a function $\Phi_\mathfrak{D} : M_\mathbb{R} \to \hat{G}$ such that for each $I \in \operatorname{Cofin}(N^+)$, the induced function

$$\Phi_\mathfrak{D}^{<I} = \rho^{<I} \circ \Phi_\mathfrak{D} : M_\mathbb{R} \to G^{<I}$$

comes from a $\mathfrak{g}^{<I}$-SD $\mathfrak{D}^{<I} = (\mathfrak{S}_{S^{<I}}, \Phi_{\mathfrak{D}^{<I}})$, i.e.

$$\Phi_\mathfrak{D}^{<I}(m) = \Phi_{\mathfrak{D}^{<I}}(\sigma)$$

for any $\sigma \in \mathfrak{S}_{S^{<I}}$ and any $m \in \sigma^\circ$. It is said to be consistent if every $\mathfrak{D}^{<I}$ is consistent.

The following proposition is what we expect for the infinite support case extending Theorem 2.2.5.

Proposition 2.2.9. *The set $\mathfrak{g}$-$\mathcal{SD}$ of all consistent $\mathfrak{g}$-SDs is in bijection with $\hat{G}$ by sending $\mathfrak{D}$ to $\Phi_\mathfrak{D}(0) \in \hat{G}$.*

PROOF. Suppose $\mathfrak{D}$ is a consistent $\mathfrak{g}$-SD and let $g = \Phi_{\mathfrak{D}}(0)$. Then by definition, we have $\Phi_{\mathfrak{D}}^{<I}(0) = \rho^{<I}(g)$ for any $I \in \mathrm{Cofin}(N^+)$. Since $\mathfrak{D}^{<I}$ is consistent, for any $m \in M_{\mathbb{R}}$,

$$\rho^{<I}(\Phi_{\mathfrak{D}}(m)) = \Phi_{\mathfrak{D}^{<I}}(m) = \pi_m(\rho^{<I}(g)) = \rho^{<I}(\pi_m(g))$$

for any I. This shows $\Phi_{\mathfrak{D}}(m) = \pi_m(g)$ for any $m \in M_{\mathbb{R}}$. Thus $\mathfrak{D}$ is determined by $\Phi_{\mathfrak{D}}(0)$.

Conversely, for any $g \in \hat{G}$, we define a function $\Phi_g \colon M_{\mathbb{R}} \to \hat{G}$, $m \mapsto \pi_m(g)$. Note that $\Phi_0(g) = g$ by definition. For each $I \in \mathrm{Cofin}(N^+)$, the map $\rho^{<I} \circ \Phi_g = \Phi_{g^{<I}}$ defines a consistent $\mathfrak{g}^{<I}$-SD according to the proof of Theorem 2.2.5, thus defining a consistent $\mathfrak{g}$-SD in the projective limit. The bijection then follows. $\qquad\square$

2.3. The canonical cone complex

Let $\mathfrak{g}$ be an N^+-graded Lie algebra with finite support S. It often happens (for example, in the application in cluster algebras as in Section 7.4) that the cone complex $\mathfrak{S}_S$ induced by the arrangement of hyperplanes is too refined to capture particular features of a scattering diagram $\mathfrak{D}_g = (\mathfrak{S}_S, \Phi_g)$ in the sense that, for example, a path-connected component of $\Phi_g^{-1}(h)$ for some $h \in G$ may be a union of multiple cones in $\mathfrak{S}_S^\circ$. The following theorem is our main result in this chapter, which gives a canonical description of the underlying cone complex.

Fix $g \in G$. Consider the corresponding consistent $\mathfrak{g}$-SD $\mathfrak{D} = \mathfrak{D}_g$ with the map

$$\Phi = \Phi_g \colon M_{\mathbb{R}}, m \mapsto \pi_m(g).$$

THEOREM 2.3.1. *The level sets of the map $\Phi \colon M_{\mathbb{R}} \to G$ satisfy the following properties.*

(1) *For any $h \in G$, the level set $\Phi^{-1}(h)$ is contained in a rational subspace of $M_{\mathbb{R}}$ and relatively open.*

(2) *Each connected component of $\Phi^{-1}(h)$ is the relative interior of a rational polyhedral cone.*

(3) *These cones form a finite complete cone complex $\mathfrak{S}_g$ of $M_{\mathbb{R}}$.*

Remark 2.3.2. In the literature, for example in [**KS14**, **Bri17**, **GHKK18**], the codimension one skeleton of $\mathfrak{S}_g$ is corresponding to the *minimal or essential support* and codimension one cones are usually referred to as *walls*.

2.3.1. The proof of Theorem 2.3.1. We need some preparations before proving Theorem 2.3.1. Note that we assume $S = \mathrm{Supp}(\mathfrak{g})$ to be finite.

Definition 2.3.3. Let $g \in G$. We define $\mathrm{Supp}(g)$ to be the minimal subset of $\mathrm{Supp}(\mathfrak{g})$ such that $\mathfrak{g}_{\mathrm{Supp}(g)}$ is a Lie subalgebra of $\mathfrak{g}$ that contains $\log(g)$. We say $n \in \mathrm{Supp}(g)$ is *extreme* if it is not a positive linear combination of other elements in $\mathrm{Supp}(g)$. Denote the subset of all extreme elements by $E(g)$.

Recall that we know from Proposition 2.2.6 that the function Φ is constant on any $\sigma^{\circ} \in \mathfrak{S}_S^{\circ}$.

Lemma 2.3.4. *For any $h \in G$, the preimage $\Phi^{-1}(h)$, as a union of cones in $\mathfrak{S}_S^{\circ}$, is relatively open in the subspace $\mathrm{Supp}(h)^{\perp} \subset M_{\mathbb{R}}$.*

PROOF. First of all the set $\Phi^{-1}(h)$ is a union of cones in $\mathfrak{S}_S^{\circ}$ as Φ is constant on any σ°. We have $\Phi^{-1}(h) \subset \mathrm{Supp}(h)^{\perp}$ since $\log(h)$ is supported on $\mathrm{Supp}(h)$. We just need to show that for any σ° contained in $\Phi^{-1}(h)$,

$$|\mathrm{Star}(\sigma)| \cap \mathrm{Supp}(h)^{\perp} \subset \Phi^{-1}(h).$$

Let $m \in \sigma^{\circ}$ and $m' \in |\mathrm{Star}(\sigma)| \cap \mathrm{Supp}(h)^{\perp}$. By Lemma 2.1.4, we have

$$\Phi(m') = \pi_{m'}(g) = \pi_{m,m'}(\pi_m(g)) = \pi_{m,m'}(h).$$

The map $\pi_{m,m'}$ depends on the partition

$$S_{m,0} = (S_{m,0})_{m',+} \sqcup S_{m',0} \sqcup (S_{m,0})_{m',-}.$$

Note that by assumption we have $\mathrm{Supp}(h) \subset S_{m',0} \subset S_{m,0}$. Therefore $\Phi(m') = \pi_{m,m'}(h) = h$ which implies any such m' is contained in $\Phi^{-1}(h)$. This finishes the proof. $\square$

Lemma 2.3.5. *Let $\sigma \in \mathfrak{S}_S$. Suppose that $e \in E(\Phi(\sigma))$ is extreme. Then for any cone ρ in $\mathfrak{S}_S$ that is contained in $e^{\perp}$ and contains σ as a face, we have the component $\log(\Phi(\rho))_e \neq 0$, i.e. $e \in \mathrm{Supp}(\Phi(\rho))$.*

PROOF. Let $h = \Phi(\sigma)$. By Lemma 2.1.4, we have $\Phi(\rho) = \pi_{\sigma,\rho}(h)$ and

$$h = \pi_{m,+}(h) \cdot \Phi(\rho) \cdot \pi_{m,-}(h)$$

where $m \in \rho^{\circ}$. Note that $e \in S_{\rho,0} \subset S_{\sigma,0}$ and $\pi_{m,\pm}(h)$ are supported outside of $S_{\rho,0}$. Since e is extreme, it is an immediate consequence of the Baker–Campbell–Hausdorff formula that $\log(\Phi(\rho))_e \neq 0$. $\square$

PROOF OF THEOREM 2.3.1. Part (1) follows from Lemma 2.3.4. Let V be a connected component of $\Phi^{-1}(h)$. We know that the closure $\overline{V}$ is a union of cones in $\mathfrak{S}_S$. The convexity of $\overline{V}$ can be proved locally. In fact, it suffices to prove that for any $\sigma \in \mathfrak{S}_S$ in the boundary $\overline{V} \setminus V$, there exists some $e \in N$ such that any connected component τ of $|\mathrm{Star}(\sigma)| \cap V$ is contained in one of the two open halves of $\mathrm{Supp}(h)^\perp$ separated by the hyperplane $e^\perp$ which passes through σ. We prove this by finding some e such that for any $\rho \subset \mathrm{Supp}(h)^\perp \cap e^\perp$ that contains σ, $\Phi(\rho) \neq h$, which implies τ cannot cross the hyperplane $e^\perp$. In fact, since $\sigma \cap V = \emptyset$, $\Phi(\sigma) \neq \Phi(\tau) = h$. We just take some extreme element e in $(S_{\sigma,0} \setminus S_{\tau,0}) \cap E(\Phi(\sigma))$. It is clear that $e^\perp \cap \mathrm{Supp}(h)^\perp$ is a hyperplane in $\mathrm{Supp}(h)^\perp$ and we have $e \in \mathrm{Supp}(\Phi(\rho))$ by Lemma 2.3.5. Therefore we have $\Phi(\rho) \neq h$ since $e \notin \mathrm{Supp}(h)$. This proves part (2).

Let δ be a face of $\overline{V}$. It is a union of cones in $\mathfrak{S}_S$. The map Φ is constant on the relative interior of δ; otherwise, $\overline{V}$ would be split into two components. Suppose that Φ remains constant on some face σ' of σ. By Lemma 2.3.4, it remains constant on $|\mathrm{Star}(\sigma')| \cap \langle \sigma \rangle_{\mathbb{R}}$. This means Φ also extends constantly from V to some face of $\overline{V}$ that contains σ'. However, it contradicts the assumption that V is a connected component of $\Phi^{-1}(h)$. Thus, we conclude that the interior of any face of $\overline{V}$ is also a connected component of some level set of Φ. This proves part (3) that the set of all cones of the form $\overline{V}$ form a complete finite cone complex in $M_{\mathbb{R}}$. $\qquad\square$

2.3.2. Support-infinite case. Now we discuss the general case when the Lie algebra $\mathfrak{g}$ may have infinite support. Fix $g \in \hat{G}$ and consider the corresponding consistent $\mathfrak{g}$-SD $\mathfrak{D} = \mathfrak{D}_g$ with the function

$$\Phi = \Phi_g \colon M_{\mathbb{R}} \to \hat{G}, m \mapsto \pi_m(g).$$

For each $I \in \mathrm{Cofin}(N^+)$, the function

$$\Phi_I = \Phi_{g^{<I}} \colon M_{\mathbb{R}} \to G^{<I}, m \mapsto \pi_m(g^{<I})$$

admits a canonical complete finite cone complex $\mathfrak{S}_I \coloneqq \mathfrak{S}_{g^{<I}}$ by Theorem 2.3.1.

Lemma 2.3.6. *For two cofinite ideals $I \subset J \subset N^+$, the cone complex $\mathfrak{S}_I$ is a refinement of $\mathfrak{S}_J$, i.e. any cone $\sigma \in \mathfrak{S}_J^\circ$ is a disjoint union of cones in $\mathfrak{S}_I^\circ$.*

PROOF. Recall that we have

$$\Phi_J = \rho_{I,J} \circ \Phi_I \colon M_{\mathbb{R}} \to G^{<J}$$

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where $\rho_{I,J}$ denotes the projection from $G^{<I}$ to $G^{<J}$. Suppose σ is a cone in $\mathfrak{S}_J$, we know that Φ_J is constant on σ° and that

$$\Phi_J(\sigma^\circ) \neq \Phi_J(m)$$

for any $m \in \partial\sigma$ since σ° is characterized as a connected component of a level set. Since the map $\rho_{I,J}$ is surjective, we have that for any $m' \in \sigma^\circ$ and any $m \in \partial\sigma$,

$$\Phi_I(m') \neq \Phi_I(m).$$

Thus for any $\tau^\circ \in \mathfrak{S}_I^\circ$, either we have $\tau^\circ \subset \sigma^\circ$ or $\tau^\circ \cap \sigma^\circ = \varnothing$. It then follows that the cone σ° is a disjoint union of cones in $\mathfrak{S}_I^\circ$ since $\mathfrak{S}_I^\circ$ is a decomposition of $M_\mathbb{R}$. $\qquad\square$

In the situation of Lemma 2.3.6, we define a map

$$R_{I,J} \colon \mathfrak{S}_I \to \mathfrak{S}_J$$

as follows. In fact, to any $\sigma \in \mathfrak{S}_I$, there is a unique association of a cone in $\mathfrak{S}_J$ whose relative interior contains σ°. Thus this association defines a map $R_{I,J}$ from $\mathfrak{S}_I$ to $\mathfrak{S}_J$. We state in the following proposition that this map respects the poset structures on the two cone complexes.

Proposition 2.3.7. *The map $R_{I,J}$ respects the structures of cone complexes on $\mathfrak{S}_I$ and $\mathfrak{S}_J$. In other words, we have*

(1) *the map $R_{I,J}$ is a morphism between posets and*
(2) *for two cones σ and τ in $\mathfrak{S}_I$, we have*

$$R_{I,J}(\sigma \cap \tau) = R_{I,J}(\sigma) \cap R_{I,J}(\tau) \in \mathfrak{S}_J.$$

Moreover, we have for any $I \subset J \subset K$,

$$R_{I,K} = R_{J,K} \circ R_{I,J}$$

Now we take the projective limit poset (also called the inverse limit)

$$\varprojlim_{I \in \mathrm{Cofin}(N^+)} \mathfrak{S}_I = \left\{ (\sigma_I)_I \in \prod_I \mathfrak{S}_I \;\middle|\; \sigma_J = R_{I,J}(\sigma_I) \text{ for all } I \subset J \text{ in } \mathrm{Cofin}(N^+) \right\}$$

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of the inverse system $(\mathfrak{S}_I)_I$ indexed by $\mathrm{Cofin}(N^+)$. The partial order is described as $(\sigma_I)_I \prec (\tau_I)_I$ if and only if we have $\sigma_I \prec \tau_I$ in $\mathfrak{S}_I$ for every $I \in \mathrm{Cofin}(N^+)$. In this case, we say that $(\sigma_I)_I$ is a *face* of $(\tau_I)_I$.

To every tuple $(\sigma_I)_I$ in the projective limit, one can associate a unique non-empty closed convex cone

$$\bigcap_I \sigma_I \subset M_\mathbb{R}$$

by taking the intersection of the cones.

Definition 2.3.8. We define two collections of cones

$$\mathfrak{S}_g := \left\{ \bigcap_I \sigma_I \,\middle|\, (\sigma)_I \in \varprojlim_I \mathfrak{S}_I \right\} \text{ and } \mathfrak{S}_g^\circ := \left\{ \bigcap_I \sigma_I^\circ \,\middle|\, (\sigma)_I \in \varprojlim_I \mathfrak{S}_I \right\},$$

both of which inherit the poset structure from $\varprojlim_I \mathfrak{S}_I$.

Lemma 2.3.9. *For any $\sigma \in \mathfrak{S}_g$ of the form $\bigcap_I \sigma_I$, denote by σ° the cone $\bigcap_I \sigma_I^\circ$. Then we have*

(1) *$\sigma^\circ = \sigma \setminus |\{\tau \in \mathfrak{S}_g \mid \tau \prec \sigma \text{ and } \tau \neq \sigma\}|$ where $|\cdot|$ takes the union of all cones in a set, and*

(2) *σ is the closure of σ° (in the Euclidean topology) in $M_\mathbb{R}$, i.e. $\sigma = \overline{\sigma^\circ}$.*

PROOF. First, we note that this is, of course, true for polyhedral cones: the relative interior of a cone is the complement of the union of all proper faces. However, here the cone σ may no longer be polyhedral.

Let $m \in \sigma$ but not belong to σ°. For every I, there is a unique cone $\tau_I \in \mathfrak{S}_I$ such that $\tau_I \prec \sigma_I$ and $m \in \tau_I^\circ$. Then we have $m \in \tau^\circ := \cap_I \tau_I^\circ \in \mathfrak{S}_g$. Since m is not in σ°, the cone $\tau := \cap_I \tau_I$ is a proper face of σ. As the complement $\sigma \setminus \sigma^\circ$ is covered by the union of all proper faces, the result follows.

The part (2) follows from the fact that for every I, the rational polyhedral cone σ_I is the closure of σ_I°. $\qquad\square$

We will call the poset of cones $\mathfrak{S}_g$ a *profinite complete cone complex*. The following theorem justifies this terminology.

THEOREM 2.3.10. *Let $g \in \hat{G}$. Consider the posets of cones $\mathfrak{S}_g$ and $\mathfrak{S}_g^\circ$.*

(1) *There is a natural map of posets $R_I : \mathfrak{S}_g \to \mathfrak{S}_I$ such that for $\sigma = \cap_I \sigma_I$, we have $R_I(\sigma) = \sigma_I$. The poset $\mathfrak{S}_g$ with the maps $(R_I)_I$ is the projective limit of the inverse system $((\mathfrak{S}_I)_I, R_{I,J})$.*

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(2) *Any face of a cone $\sigma \in \mathfrak{S}_g$ is again in $\mathfrak{S}_g$.*

(3) *The intersection $\sigma \cap \tau$ of two cones in $\mathfrak{S}_g$ is a common face of them.*

(4) *The cones in $\mathfrak{S}_g^\circ$ are pairwise disjoint. Together they form a decomposition of $M_\mathbb{R}$.*

PROOF. Part (1) summarizes our previous discussion. We point out that the poset $\mathfrak{S}_g$ is isomorphic to $\varprojlim_I \mathfrak{S}_I$ by construction.

Part (2) is tautological by our definition of a face of a cone in $\mathfrak{S}_g$. Note that here the notion of a face can be very different from the finite case. It can happen, for example, in two dimensions, that σ is a strictly convex closed cone of dimension two and $\sigma^\circ = \sigma \setminus \{0\}$. Thus by Lemma 2.3.9, the only face of σ is the origin, and the two boundary rays do not count as faces.

For part (3), note that we have

$$\sigma \cap \tau = \bigcap_I (\sigma_I \cap \tau_I).$$

In $\mathfrak{S}_I$, the intersection $\sigma_I \cap \tau_I$ is common face and again in $\mathfrak{S}_I$. The tuples $(\sigma_I \cap \tau_I)_I$ constitute an element in the projective limit since we have by Proposition 2.3.7,

$$R_{I,J}(\sigma_I \cap \tau_I) = \sigma_J \cap \tau_J$$

for $I \subset J$. Thus by definition $\sigma \cap \tau$ is a common face of both σ and τ.

Suppose we have two cones $\bigcap_I \sigma_I^\circ$ and $\bigcap_I \tau_I^\circ$ in $\mathfrak{S}_g^\circ$. Then we have either for some I, $\sigma_I^\circ \cap \tau_I^\circ = \varnothing$ or for every I, $\sigma_I^\circ = \tau_I^\circ$. Thus the two cones are either disjoint or the same. Let m be any point in $M_\mathbb{R}$. For every I, there is a unique cone $\sigma_{m,I} \in \mathfrak{S}_I$ such that $m \in \sigma_{m,I}^\circ$ since $\mathfrak{S}_I$ is complete. The tuple $(\sigma_{m,I})_I$ is an element in $\mathfrak{S}_g$ and m is of course contained in $\bigcap_I \sigma_{m,I}^\circ$. This proves part (4).

$\square$

Each cone in the cone decomposition $\mathfrak{S}_g^\circ$ has the following characterization using the function $\Phi_g \colon M_\mathbb{R} \to \hat{G}$.

Proposition 2.3.11. *For any $h \in \hat{G}$, each path-connected component of the level set $\Phi_g^{-1}(h)$ is a convex cone in $M_\mathbb{R}$. These cones together form the cone decomposition $\mathfrak{S}_g^\circ$ of $M_\mathbb{R}$.*

PROOF. Let C be a path-connected component of $\Phi_g^{-1}(h)$ and $m \in C$. We define

(2.3.1)
$$\sigma_m^\circ := \bigcap_{I \in \mathrm{Cofin}(N^+)} \sigma_{m,I}^\circ.$$

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It is a convex cone and in particular path-connected. The map $\Phi_g \colon M_{\mathbb{R}} \to \hat{G}$ is obviously constant on σ_m° so we have $\sigma_m^\circ \subset C$. On the other hand, for every cofinite ideal I, the function Φ_I is constant on C so the path-connected set C is contained in the path-connected component $\sigma_{m,I}^\circ$. Thus we have $C = \sigma_m^\circ$. Therefore all such path-connected components form the collection $\mathfrak{S}_g^\circ$. $\qquad\square$

Now we update our definition of a consistent $\mathfrak{g}$-scattering diagram.

Definition 2.3.12 (Consistent $\mathfrak{g}$-SD updated)**.** Let $\mathfrak{g}$ be an N^+-graded Lie algebra and g be an element in the corresponding pro-unipotent group $\hat{G}$. The *consistent* $\mathfrak{g}$-*SD* corresponding to the group element $g \in \hat{G}$ now refers to the data

$$\mathfrak{D}_g = (\mathfrak{S}_g, \Phi_g)$$

consisting of the canonical profinite cone complex $\mathfrak{S}_g$ and the function $\Phi_g \colon M_{\mathbb{R}} \to \hat{G}$ (2.1.8) which is constant along each cone in the cone decomposition $\mathfrak{S}_g^\circ$.

Remark 2.3.13. We emphasize that the data of the profinite cone complex $\mathfrak{S}_g$ (as well as $\mathfrak{S}_g^\circ$) can be completely determined by the function Φ_g by combining Proposition 2.3.11, Theorem 2.3.10 and Lemma 2.3.9.

The following proposition will be useful later.

Proposition 2.3.14. *Let* $g \in \hat{G}$. *Suppose that* $\sigma \in \mathfrak{S}_g$ *is a rational polyhedral cone and it appears in the cone complex* $\mathfrak{S}_I$ *for some* I. *Then all the faces of* σ *are elements of* $\mathfrak{S}_g$.

PROOF. By Theorem 2.3.1, we have that $\mathfrak{S}_I$ is a cone complex, so the faces of σ are also cones in $\mathfrak{S}_I$. Since the cone σ is in $\mathfrak{S}_g$, it belongs to $\mathfrak{S}_J$ for any $J \subset I$ and so do its faces. Therefore all its faces are also elements in the projective limit $\mathfrak{S}_g$. $\qquad\square$

2.3.3. Induced scattering diagram. We set up some conventions that will be useful later. Let $f \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of N^+-graded Lie algebras. It induces a group homomorphism $F \colon \hat{G}_1 \to \hat{G}_2$ and consequently a map

$$F \colon \mathfrak{g}_1\text{-}\mathcal{SD} \to \mathfrak{g}_2\text{-}\mathcal{SD}, \quad \mathfrak{D}_g \mapsto \mathfrak{D}_{F(g)}$$

for $g \in \hat{G}_1$. Clearly, we have for the wall-crossing functions that

$$\Phi_{F(g)} = F \circ \Phi_g.$$

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There is also an induced map from the profinite cone complex $\mathfrak{S}_g$ to $\mathfrak{S}_{F(g)}$, which can be considered as a refinement.

2.3.4. Consistency revisited. In this section, we explain how to extend the notion of consistency in terms of path-ordered products to the canonical (profinite) cone complex $\mathfrak{S}_g$. As the cone complex $\mathfrak{S}_g$ may be infinite and a path can thus cross infinitely many walls, it needs extra care to define a path-ordered product.

First we assume $S = \mathrm{Supp}(\mathfrak{g})$ to be finite. Fix $g \in G$. Then Theorem 2.3.1 applies. In particular, the cone complex $\mathfrak{S}_g$ is a coarsening of $\mathfrak{S}_S$ (i.e., $\mathfrak{S}_S$ is a refinement of $\mathfrak{S}_g$). For each τ in $\mathfrak{S}_g$ of codimension at least one, there is a relatively positive maximal cell τ^+ in $\mathfrak{S}_g$ incident to τ as described in Section 2.2.1. Similarly, there is a relatively negative maximal cell τ^-. We then define $\mathfrak{S}_g$-paths and path-ordered products for $\mathfrak{S}_g$ in the exact same way as for $\mathfrak{S}_S$ in Section 2.2.1. Then the following proposition follows directly from the consistency of the scattering diagram $\mathfrak{D}_g$ (in terms of $\mathfrak{S}_S$ in Definition 2.2.4); see the proof of Theorem 2.2.5.

Proposition 2.3.15. *Let γ be an $\mathfrak{S}_g$-path. Then we have*

$$\mathfrak{p}_\gamma(\mathfrak{D}_g) = \pi_{\gamma(1),+}(g)^{-1} \cdot \pi_{\gamma(0),+}(g).$$

In particular, it only depends on the endpoints $\gamma(0)$ and $\gamma(1)$.

Now suppose that the support $\mathrm{Supp}(\mathfrak{g})$ may be infinite. Fix $g \in \hat{G}$. We extend the notion of a path-ordered product for the profinite cone complex $\mathfrak{S}_g$. We continue to use the notation $\mathfrak{D}_I = \mathfrak{D}_{g<I}$ and $\mathfrak{S}_I = \mathfrak{S}_{g<I}$.

Definition 2.3.16. A smooth curve $\gamma \colon [0,1] \to M_\mathbb{R}$ is said to be an $\mathfrak{S}_g$-*path* if it is an $\mathfrak{S}_I$-path (Definition 2.2.2) for any cofinite ideal $I \subset N^+$.

Lemma 2.3.17. *Let $\mathfrak{D}_g = (\mathfrak{S}_g, \Phi_g)$ be the consistent $\mathfrak{g}$-SD corresponding to $g \in \hat{G}$ and γ be an $\mathfrak{S}_g$-path. Then for any $I \subset J$, we have*

$$\rho_{I,J}(\mathfrak{p}_\gamma(\mathfrak{D}_I)) = \mathfrak{p}_\gamma(\mathfrak{D}_J).$$

PROOF. By definition, the $\mathfrak{S}_g$-path γ is also a $\mathfrak{D}_{g<I}$-path for any $I \in \mathrm{Cofin}(N^+)$. By Proposition 2.3.15, it amounts to show that

$$\rho_{I,J}\left(\pi_{\gamma(1),+}\left(g^{<I}\right)^{-1} \cdot \pi_{\gamma(0),+}\left(g^{<I}\right)\right) = \pi_{\gamma(1),+}\left(g^{<J}\right)^{-1} \cdot \pi_{\gamma(0),+}\left(g^{<J}\right),$$

which follows from the fact that the projection $\pi_{m,+}$ commutes with $\rho_{I,J}$. $\qquad\square$

The above lemma allows us to propose the following definition of the path-ordered product for an $\mathfrak{S}_g$-path γ in general.

Definition 2.3.18. The *path-ordered product* $\mathfrak{p}_\gamma(\mathfrak{D}_g)$ of γ is defined to be the projective limit in $\hat{G}$ of the path-ordered products $\mathfrak{p}_\gamma(\mathfrak{D}_{g<I})$ for $I \in \mathrm{Cofin}(N^+)$, i.e.

$$\mathfrak{p}_\gamma(\mathfrak{D}_g) := \varprojlim_I \mathfrak{p}_\gamma(\mathfrak{D}_{g<I}).$$

The consistency of $\mathfrak{p}_\gamma(\mathfrak{D}_g)$ follows directly from the definition. The following is the support-infinite version of Proposition 2.3.15.

Proposition 2.3.19. *For any $\mathfrak{D}_g$-path γ, we have*

$$\mathfrak{p}_\gamma(\mathfrak{D}_g) = \pi_{\gamma(1),+}(g)^{-1} \cdot \pi_{\gamma(0),+}(g).$$

In particular, it only depends on the endpoints $\gamma(0)$ and $\gamma(1)$.

Notice that in the case where an $\mathfrak{S}_g$-path γ crosses finitely many walls (or higher codimensional faces) in $\mathfrak{S}_g$, the path-ordered product $\mathfrak{p}_\gamma(\mathfrak{D}_g)$ can still be computed by a finite product of wall-crossings in the order of crossings. For two points m_1 and m_2 in $M_{\mathbb{R}}$, there might not be a $\mathfrak{S}_g$-path. Nevertheless, the formula in Proposition 2.3.19 still makes sense.

Cluster algebras

In this chapter, we introduce *cluster algebras* of Fomin and Zelevinsky [**FZ02**]. We choose an approach (in the framework of [**GHK13**]) to define cluster algebras that suits best the machinery of scattering diagrams developed in [**GHKK18**].

There is one slight change in our approach worth explanation. The cluster algebra we will define is with respect to some fixed data Γ (Section 3.1.3) with an initial seed $\mathbf{s}$. There is a construction of fixed data with an initial seed $(\Gamma, \mathbf{s})^\vee$ that is Langlands dual to the former one (see [**GHKK18**, Appendix A]). The cluster algebra we define for $(\Gamma, \mathbf{s})$ is isomorphic to GHKK's construction for $(\Gamma, \mathbf{s})^\vee$. Thus without going to the Langlands dual data, we can define two cluster algebras: ours is the cluster algebra $\mathcal{A}(B)$ while GHKK's is $\mathcal{A}(B^\vee)$. The precise meaning of these notions will be clear in the following sections.

3.1. Preliminaries

3.1.1. Oriented Cartan data. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a generalized symmetrizable Cartan matrix with a left symmetrizer D. That is, the matrix C satisfies

(1) $c_{ii} = 2$ for all $i \in I = \{1, 2, \ldots, n\}$;

(2) $c_{ij} \leq 0$ for all $i \neq j$;

(3) there exists a diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n) \in M_n(\mathbb{Z}_{>0})$ such that DC is symmetric.

An *orientation* of C is a set $\Omega \subset I \times I$ such that $|\Omega \cap \{(i, j), (j, i)\}| \leq 1$ and

$$|\Omega \cap \{(i, j), (j, i)\}| = 1 \iff c_{ij} < 0.$$

An orientation Ω is said to be *acyclic* if for each sequence $i_1, i_2, \ldots, i_k$ with $(i_j, i_{j+1}) \in \Omega$ for $j = 1, 2, \ldots, k$ we have $i_1 \neq i_{k+1}$.

We call the triple (C, D, Ω) *oriented Cartan data*, which is easily seen equivalent to a pair (B, D) where $B = (b_{ij}) \in M_n(\mathbb{Z})$ satisfies

(1) $b_{ij} = c_{ij} < 0$ and $b_{ji} = -c_{ij} > 0$ if $(i,j) \in \Omega$.

(2) $b_{ij} = 0$ if $|\Omega \cap \{(i,j), (j,i)\}| = 0$,

and D is a left skew-symmetrizer of B, i.e. $DB + (DB)^T = 0$. Such a matrix B is called *skew-symmetrizable*.

3.1.2. Presymplectic lattice. Let N be a lattice of finite rank n, i.e. $N \cong \mathbb{Z}^n$. A pair (N, ω) of a lattice N and a $\mathbb{Q}$-valued skew-symmetric bilinear form

$$\omega \colon N \times N \to \mathbb{Q}$$

on N is called a $\mathbb{Q}$-*presymplectic* lattice (*symplectic* if the form is non-degenerate).

A $\mathbb{Q}$-presymplectic lattice can constructed from oriented Cartan data (C, D, Ω) as follows. Let (B, D) correspond to (C, D, Ω). Let $N = \mathbb{Z}^n$ where n is the rank of B with the standard basis $\{e_i \mid i = 1, \ldots, n\}$. Note that

$$\widetilde{B} = (\tilde{b}_{ij}) \colon = BD^{-1} \in M_n(\mathbb{Q})$$

is skew-symmetric. We define the form ω by putting

$$\omega(e_i, e_j) = \tilde{b}_{ij}.$$

3.1.3. Fixed data. We follow closely [**GHK13**, Section 2] and [**GHKK18**, Appendix A] on *fixed data*. Fixed data Γ consists of

    – a $\mathbb{Q}$-presymplectic lattice (N, ω);

    – an *unfrozen sublattice* $N_{\mathrm{uf}} \subset N$, a saturated sublattice of N;

    – an index set $I = \{1, 2, \ldots, r\}$ with $r = \operatorname{rank} N$ and a subset $I_{\mathrm{uf}} \subset I$ with $|I_{\mathrm{uf}}| = \operatorname{rank} N_{\mathrm{uf}}$;

    – positive integers d_i for $i \in I$;

    – a sublattice $N^\circ \subset N$ of finite index such that

$$\omega(N_{\mathrm{uf}}, N^\circ) \subset \mathbb{Z}, \text{ and } \omega(N, N_{\mathrm{uf}} \cap N^\circ) \subset \mathbb{Z};$$

    – $M = \operatorname{Hom}(N, \mathbb{Z})$ and $M^\circ = \operatorname{Hom}(N^\circ, \mathbb{Z})$;

A *seed* $\mathbf{s}$ for fixed data Γ (or a Γ-seed) is a subset of N indexed by I

$$\mathbf{s} = (s_i)_{i \in I}$$

such that $\{s_i \mid i \in I\}$ is a $\mathbb{Z}$-basis of N, $\{s_i \mid i \in I_{\mathrm{uf}}\}$ a basis of N_{uf}, and $\{d_i s_i \mid i \in I\}$ a basis of N°. We will call a pair $(\Gamma, \mathbf{s})$ of fixed data Γ and a Γ-seed $\mathbf{s}$ seed data. The seed $\mathbf{s}$ is often referred to as the initial seed.

For a seed $\mathbf{s}$, we put

$$\tilde{\mathbf{s}} = (\tilde{s}_i)_{i \in I}, \quad \tilde{s}_i = d_i s_i.$$

The *dual* of $\mathbf{s}$ is denoted by

$$\mathbf{s}^* = (s_i^*)_{i \in I}$$

where $\{s_i^* \mid i \in I\}$ is the dual basis of M. The dual of $\tilde{\mathbf{s}}$ is given by $\tilde{\mathbf{s}}^* = (\tilde{s}_i^*)_{i \in I}$ and we have

$$\tilde{s}_i^* = s_i^* / d_i.$$

These elements form a basis of the lattice M°.

We define a skew-symmetric matrix $\widetilde{B}(\mathbf{s}) = (\tilde{b}_{ij}) \in M_r(\mathbb{Q})$ by setting

$$\tilde{b}_{ij} = \omega(s_i, s_j)$$

and the matrix $B(\mathbf{s}) = (b_{ij}) \in M_r(\mathbb{Z})$ by setting

$$b_{ij} = \omega(s_i, \tilde{s}_j) = d_j \tilde{b}_{ij}.$$

The matrix $B(\mathbf{s})$ is skew-symmetrized by $D = \mathrm{diag}(d_i | i \in I)$, i.e. we have

$$DB + B^T D^T = 0.$$

We define a map for later use. We put

$$p^* \colon N \to M^\circ, \ n \mapsto \omega(-, n).$$

This is legitimate since for any $n \in N$, $\omega(-, n)$ is $\mathbb{Z}$-valued on N°. Note that one can restrict the map p^* to the sublattice N° and the image is then in M, as a sublattice of M°. For example, we have

(3.1.1) $$p^*(s_k) = -\sum_{i \in I} b_{ki} \tilde{s}_i^*, \quad p^*(\tilde{s}_k) = \sum_{i \in I} b_{ik} s_i^*.$$

3.1.4. Mutations of seeds. Suppose we are given fixed data Γ and a seed $\mathbf{e}$ of Γ. We define operations called *mutations* that iteratively generate seeds for the fixed data Γ.

Recall that we have an integer-valued matrix $B = B(\mathbf{e}) = (b_{ij})$. For an unfrozen index $k \in I_{\mathrm{uf}}$ and a sign $\varepsilon \in \{+, -\}$, we define the mutation $\mu_k^\varepsilon(\mathbf{e}) = (e_i'^{,\varepsilon})_{i \in I}$, a subset of N indexed by I, by setting

$$
e_i'^{,\varepsilon} := \begin{cases} e_i + [-\varepsilon b_{ik}]_+ e_k & \text{for } i \neq k \\ -e_i & \text{for } i = k \end{cases}.
$$

It is straightforward to check that the collection $\mu_k^\varepsilon(\mathbf{e})$ satisfies the requirements of a seed for Γ. This allows us to apply mutations to the new seed indefinitely. For a sequence $\mathbf{k} = (k_1, \ldots, k_l)$ of unfrozen indices and a sequence $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_l)$ of signs, we define an associated seed

$$
\mu_{\mathbf{k}}^{\underline{\varepsilon}}(\mathbf{e}) := \mu_{k_l}^{\varepsilon_l} \cdots \mu_{k_1}^{\varepsilon_1}(\mathbf{e}).
$$

In the following, we provide some concrete computations in terms of matrices of mutations. We denote the change-of-basis matrix of the mutation μ_k^ε by $E_k^\varepsilon(\mathbf{e})$, i.e. we have

$$
\mu_k^\varepsilon(\mathbf{e}) = E_k^\varepsilon(\mathbf{e}) \cdot \mathbf{e}
$$

where $\mathbf{e}$ is considered as a column vector with entries being e_i. Note that one can read off $E_k^\varepsilon(\mathbf{e})$ from the matrix B. Therefore we also denote it by $E_k^\varepsilon(B)$ in some cases.

As a matrix, $E_k^\varepsilon(B)$ is as follows:

$$
E_k^\varepsilon(B) = \begin{bmatrix}
1 & & & [-\varepsilon b_{1k}]_+ & & & \\
& \ddots & & \vdots & & & \\
& & 1 & [-\varepsilon b_{k-1,k}]_+ & & & \\
& & & -1 & & & \\
& & & [-\varepsilon b_{k+1,k}]_+ & 1 & & \\
& & & \vdots & & \ddots & \\
& & & [-\varepsilon b_{nk}]_+ & & & 1
\end{bmatrix}.
$$

Lemma 3.1.1. *We have for $\varepsilon \in \{+, -\}$ that*

$$
E_k^\varepsilon(\mathbf{e}) \cdot E_k^\varepsilon(\mathbf{e}) = I \text{ and } E_k^{-\varepsilon}(\mu_k^\varepsilon(\mathbf{e})) \cdot E_k^\varepsilon(\mathbf{e}) = I, \; i.e. \; \mu_k^{-\varepsilon} \circ \mu_k^\varepsilon(\mathbf{e}) = \mathbf{e}.
$$

PROOF. Direct computations through the above matrix presentation of E_k^ε. □

Now let $\mathbf{e}'^{,\varepsilon} = (e_i'^{,\varepsilon})_{i \in I}$ be $\mu_k^\varepsilon(\mathbf{e})$ where $\varepsilon = +$ or $-$. We omit the sign ε when there is no ambiguity. Scale $\mathbf{e}'$ by the diagonal matrix D and we get $\tilde{\mathbf{e}}' = D \cdot \mathbf{e}'$. Immediately we have

$$\tilde{\mathbf{e}}' = D \cdot \mathbf{e}' = DE_k^\varepsilon(\mathbf{e}) \cdot \mathbf{e} = DE_k^\varepsilon(\mathbf{e})D^{-1}\tilde{\mathbf{e}}.$$

We denote the transformation matrix by

$$F_k^\varepsilon(\mathbf{e})^T := DE_k^\varepsilon(\mathbf{e})D^{-1}.$$

As a matrix, $F_k^\varepsilon(B)$ is as follows:

$$F_k^\varepsilon(B) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ [\varepsilon b_{k1}]_+ & \cdots & -1 & \cdots & [\varepsilon b_{kn}]_+ \\ & & & \ddots & & \\ & & & & & 1 \end{bmatrix}.$$

A straightforward calculation shows the following lemma, known as the *tropical duality* [**NZ12**] (see also [**Kel08**]).

Lemma 3.1.2. *Let $B^\vee = -B^T$. Then we have*

$$E_k^\varepsilon(B) = F_k^\varepsilon(B^\vee)^T, \;\; F_k^\varepsilon(B) = E_k^\varepsilon(B^\vee)^T.$$

Lemma 3.1.3. *The set $\tilde{\mathbf{e}}'$ forms a basis of N°. This verifies in part that $\mathbf{e}'$ is indeed a seed for Γ.*

PROOF. This is because the transformation matrix from $\tilde{\mathbf{e}}$ to $\tilde{\mathbf{e}}'$ is given by $F_k^\varepsilon(B)^T$ and is equal to $E_k^\varepsilon(B^\vee)$ by the tropical duality. It is invertible in $M_n(\mathbb{Z})$. □

Lemma 3.1.4. *Let $\mu_k^\varepsilon(B) = (b_{ij}') \in M_n(\mathbb{Z})$ be the matrix such that $b_{ij}' = \omega\left(e_i'^{,\varepsilon}, \tilde{e}_j'^{,\varepsilon}\right)$. Then we have*

$$\mu_k^+(B) = \mu_k^-(B) = (b_{ij}') \in M_n(\mathbb{Z})$$

and

$$b_{ij}' = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

*Thus we denote them simply by $\mu_k(B)$. This is Fomin-Zelevinsky's mutation of a skew-symmetrizable matrix B (see [**FZ02**]).*

PROOF. The computation is straightforward. We only present the case where $i \neq k$ and $j \neq k$. By definition, we have

$$b'_{ij} = \omega(e_i + [-\varepsilon b_{ik}]_+ e_k, \tilde{e}_j + [\varepsilon b_{kj}]_+ \tilde{e}_k)$$

$$= b_{ij} + [\varepsilon b_{kj}]_+ b_{ik} + [-\varepsilon b_{ik}]_+ b_{kj}$$

$$= b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}.$$

$\square$

Remark 3.1.5. The main difference of our setting with [**GHK13**] and [**GHKK18**] is that we consider two types of mutations μ_k^+ and μ_k^- instead of choosing one of them. It turns out these two mutations both have interpretations in the context of additive categorification of cluster algebras.

3.1.5. Mutations of dual seeds. Denote by $\mathbf{e}^*$ the dual basis of $\mathbf{e}$ in M and by $\tilde{\mathbf{e}}^*$ the dual basis of $\tilde{\mathbf{e}}$ in M°. We denote by $\mu_k^\varepsilon(\mathbf{e}^*)$ the dual of $\mu_k^\varepsilon(\mathbf{e})$ and accordingly for $\tilde{\mathbf{e}}^*$. The mutation matrices are computed as follows.

Lemma 3.1.6. *we have that*

$$\mu_k^\varepsilon(\mathbf{e}^*) = \left(E_k^\varepsilon(\mathbf{e})^{-1} \right)^T \cdot \mathbf{e}^* = F_k^\varepsilon(B^\vee) \cdot \mathbf{e}^*$$

and

$$\mu_k^\varepsilon(\tilde{\mathbf{e}}^*) = F_k^\varepsilon(\mathbf{e})^{-1} \cdot \tilde{\mathbf{e}}^* = F_k^\varepsilon(B) \cdot \tilde{\mathbf{e}}^*.$$

As before, we could define mutations of dual seeds $\mu_{\mathbf{k}}^\varepsilon(\mathbf{e}^*)$ (and $\mu_{\mathbf{k}}^\varepsilon(\tilde{\mathbf{e}}^*)$) associated to sequences of unfrozen indices and signs.

3.2. Cluster algebras of Fomin and Zelevinsky

In this section, we define cluster algebras of Fomin and Zelevinsky [**FZ02**]. Our definition will not be as general as in the original paper [**FZ02**]. We fix the fixed data Γ. A cluster algebra $\mathcal{A} = \mathcal{A}([\mathbf{s}])$, as a subalgebra of $\mathbb{Z}[M]$ (caution: not $\mathbb{Z}[M^\circ]$) will be defined for the mutation class $[\mathbf{s}]$ of a seed $\mathbf{s}$ (the collection of seeds obtained by arbitrary iterative mutations to $\mathbf{s}$). We start with some preparations.

3.2.1. The infinite graph $\mathfrak{T}$. We consider an oriented graph $\mathfrak{T}$ defined as follows. First, let $\mathfrak{T}_0$ be a connected unoriented $2n$-regular tree. Replace each edge in $\mathfrak{T}_0$ with two edges with opposite orientations. For each vertex $v \in \mathfrak{T}_0$, label the $2n$ outgoing edges from v by (i, ε) where $i \in I$ and $\varepsilon \in \{+, -\}$ such that if v and w are two vertices connected by an edge in $\mathfrak{T}_0$, then the two oriented edges between them are labeled by (i, ε) and $(i, -\varepsilon)$. We call this infinite oriented graph $\mathfrak{T}$. A local picture when $n = 2$ is depicted in Figure 3.1.
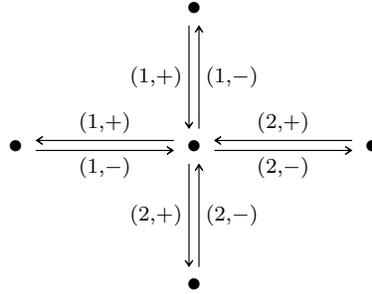
FIGURE 3.1. $n = 2$

Consider $(\Gamma, \mathbf{s})$ the fixed data Γ with an initial seed $\mathbf{s}$. Suppose $I_{\mathrm{uf}} = \{1, \ldots, n\} \subset I$. Pick a vertex v in $\mathfrak{T}$ and label it by the seed $\mathbf{s}$ to it. To each labeled edge, we associated corresponding mutations μ_i^ε. In this way, we can associate a seed to each vertex in $\mathfrak{T}$ such that any two adjacent seeds are related by mutations. A local picture of this association is depicted in Figure 3.2.
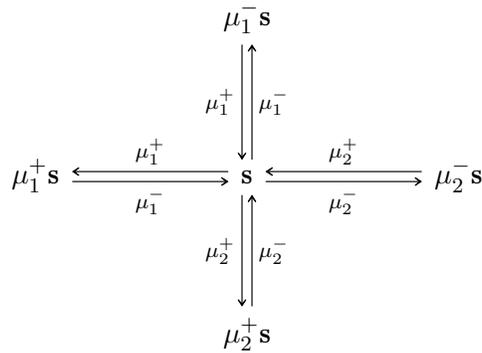
FIGURE 3.2. $n = 2$

We denote by $\mathfrak{T}_\mathbf{s}$ the graph $\mathfrak{T}$ with the labeling of seeds.

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3.2.2. Birational transformations.

Fix some field k of characteristic 0. We associate a torus

$$T_{\mathbf{e}} = T_N = \operatorname{Spec} k[M]$$

to each seed $\mathbf{e}$ and a I-tuple of coordinate functions

$$(A_i)_{i \in I} = \left(z^{f_i}\right)_{i \in I}$$

where $\mathbf{f} = (f_i)_{i \in I}$ is the dual seed of $\mathbf{e}$.

Suppose we have two adjacent vertices in the tree $\mathfrak{T}$ with labeling seeds $\mathbf{e}$ and $\mu_k^\varepsilon \mathbf{e}$

$$\mathbf{e} \xleftarrow[\mu_k^\varepsilon]{\mu_k^{-\varepsilon}} \mu_k^\varepsilon \mathbf{e} \ .$$

We define a birational map

$$\mu_k^\varepsilon \colon T_{\mathbf{e}} \dashrightarrow T_{\mu_k^\varepsilon \mathbf{e}}$$

via pull-back of functions

$$(\mu_k^\varepsilon)^* z^m = z^m \left(1 + z^{p^*(\varepsilon \tilde{e}_k)}\right)^{-m(e_k)} .$$

where $p^*(\varepsilon \tilde{e}_k)$ is explicitly computed as (see Equation (3.1.1))

$$p^*(\varepsilon \tilde{e}_k) = \{-, \varepsilon \tilde{e}_k\} = \sum_{i \in I} \{e_i, \varepsilon \tilde{e}_k\} f_i = \sum_{i \in I} \varepsilon b_{ik} f_i \in M.$$

Lemma 3.2.1. *Let $\mathbf{e}$ be a seed and $\mathbf{f}$ be the dual seed. The pull-backs of coordinates $(A_i'^{,\varepsilon})_{i \in I}$ of $\mathcal{A}_{\mu_k^\varepsilon \mathbf{e}}$ are computed as follows. In particular, they do not depend on the choice of ε.*

$$(\mu_k^\varepsilon)^* A_i'^{,\varepsilon} = \begin{cases} A_i & \text{if } i \neq k, \\ A_k^{-1} \left(\displaystyle\prod_{i:b_{ik}>0} A_i^{b_{ik}} + \prod_{j:b_{jk}<0} A_j^{-b_{jk}} \right) & \text{if } i = k. \end{cases}$$

PROOF. The coordinates of $\mathcal{A}_{\mu_k^\varepsilon \mathbf{e}}$ are given by

$$A_i'^{,\varepsilon} = z^{f_i'^{,\varepsilon}} = \begin{cases} z^{f_i} & \text{if } i \in I, \\ z^{-f_k + \sum_{i \in I} [-\varepsilon b_{ik}]_+ f_i} & \text{if } i = k. \end{cases}$$

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Then it is easy to see that $(\mu_k^\varepsilon)^* A_i'^{,\varepsilon} = A_i$ if $i \in I$. For the k-th coordinate, we have

$$
\begin{aligned}
A_k'^{,\varepsilon} = z^{f_k'^{,\varepsilon}} \left(1 + z^{p^*(\varepsilon \tilde{e}_k)}\right)^{-f_k'^{,\varepsilon}(e_k)} &= z^{-f_k + \sum_i [-\varepsilon b_{ik}] + f_i} \left(1 + z^{\sum_i \{e_i, \varepsilon \tilde{e}_k\} f_i}\right) \\
&= z^{-f_k} \left(z^{\sum_i [-\varepsilon b_{ik}] + f_i} + z^{\sum_i [-\varepsilon b_{ik}] + f_i + \sum_i \varepsilon b_{ik} f_i}\right) \\
&= z^{-f_k} \left(z^{\sum_{b_{ik} > 0} b_{ik} f_i} + z^{\sum_{b_{jk} < 0} (-b_{jk}) f_j}\right) \\
&= A_k^{-1} \left(\prod_{i:\, b_{ik} > 0} A_i^{b_{ik}} + \prod_{j:\, b_{jk} < 0} A_j^{-b_{jk}}\right).
\end{aligned}
$$

(3.2.1)

$\square$

The above lemma shows that the pull-backs of coordinates are regular functions in $k[M]$, i.e. Laurent polynomials. Note that for $i \in I \setminus I_{\mathrm{uf}}$, the variable A_i never gets changed under mutations. They are thus called frozen variables.

For any two vertices w, w' of the tree $\mathfrak{T}_{\mathbf{s}}$, there is essentially a unique path connecting them (without any 2 cycles, since μ_k^ε and $\mu_k^{-\varepsilon}$ are inverse to each other). We denote the seed associated to the vertex w by $\mathbf{s}_w$. Then we obtain a birational map

$$
\mu_{w, w'} : T_{\mathbf{s}_w} \dashrightarrow T_{\mathbf{s}_{w'}}
$$

between associated tori by composing the birational maps μ_k^ε in order along the path.

THEOREM 3.2.2 (Laurent phenomenon, [**FZ02**]). *For any two seeds related by a sequence of mutations, the pull-back of coordinates are Laurent polynomials with integer coefficients. More precisely, for vertices w, w' on $\mathfrak{T}$, denote the coordinates of $T_{\mathbf{s}_{w'}}$ by A_i' (A_i for $T_{\mathbf{s}_w}$) for $i \in I$ and we have*

$$
\mu_{w, w'}^*(A_i') \in \mathbb{Z}\left[A_j \mid j \in I \setminus I_{\mathrm{uf}}\right]\left[A_k^\pm \mid k \in I_{\mathrm{uf}}\right].
$$

3.2.3. Definitions. Now we are prepared to define cluster algebras and some other important related notions.

Definition 3.2.3 (Cluster algebras)**.** For fixed data Γ and an initial seed $\mathbf{s}$, we have constructed the infinite graph $\mathfrak{T}_{\mathbf{s}}$ with initial vertex v. We define the cluster algebra $\mathcal{A}(\mathbf{s})$ to be the subalgebra of $\mathbb{Z}[M]$ generated by the set

$$
\left\{\mu_{v, w}^*(A_i') \;\middle|\; A_i' = z^{s_{w, i}^*} \in \mathbb{Z}[M], \; i \in I, \; w \in \mathfrak{T}_{\mathbf{s}}\right\}.
$$

We also have some important notions.

(1) The Laurent polynomial $\mu^*_{v,w}(A'_i)$ is called a *cluster variable.*

(2) For each $w \in \mathfrak{T}_\mathbf{s}$, the tuple $\left(\mu^*_{v,w}(A'_i)\right)_{i \in I}$ is called a *cluster.*

(3) A monomial of cluster variables in one cluster is called a *cluster monomial.*

(4) For a skew-symmetrizable matrix B with a left skew-symmetrizer D (with a choice of unfrozen part), we can construct a corresponding fixed data with an initial seed as in Section 3.1.3. We call the cluster algebra of this data $\mathcal{A}(B)$, which in particular is independent of D.

(5) If we have $N_{\mathrm{uf}} = N$, then the corresponding cluster algebra is said to have no frozen variables or no coefficients.

If we take the seed $\mathbf{s}_w$ associated to a vertex w to be the initial seed, we can define the corresponding cluster algebra $\mathcal{A}(\mathbf{s}_w)$. The birational transformation between the tori $T_\mathbf{s}$ and $T_{\mathbf{s}_w}$ induces an isomorphism

$$\mu^*_{v,w} \colon \mathcal{A}(\mathbf{s}) \to \mathcal{A}(\mathbf{s}_w)$$

between two cluster algebras. Thus one can view that the cluster algebra is defined for the mutation class $[\mathbf{s}]$.

3.2.4. Cluster algebra with principal coefficients. In this section, we explain a particular case where the associated cluster algebra is said to have principal coefficients.

For fixed data Γ with $N_{\mathrm{uf}} = N$ with an initial seed $\mathbf{s}$, we define another fixed data Γ_{prin} as follows.

   – $\widetilde{N} := N \oplus M^\circ$ with the skew-symmetric form

$$\widetilde{\omega}((n_1, m_1), (n_2, m_2)) = \omega(n_1, n_2) + m_2(n_1) - m_1(n_2);$$

   – $\widetilde{N}_{\mathrm{uf}} := N \oplus 0;$

   – $\widetilde{N}^\circ := N^\circ \oplus M;$

   – The index set is $I := \{1, \ldots, r, r+1, \ldots, 2r\}$ with d_i unchanged and $d_{i+r} = d_i.$

   – The initial seed is

$$\bar{\mathbf{s}} := ((s_1, 0), \ldots, (s_r, 0), (0, \tilde{s}^*_1), \ldots, (0, \tilde{s}^*_r)).$$

The cluster algebra associated to $(\Gamma_{\mathrm{prin}}, \bar{\mathbf{s}})$ is denoted by $\mathcal{A}^{\mathrm{prin}}(\mathbf{s})$ and is said to have *principal coefficients*. Note that $\widetilde{M} = \mathrm{Hom}(\widetilde{N}, \mathbb{Z})$ is canonically identified with $M \oplus N^\circ$. Thus $\mathcal{A}^{\mathrm{prin}}(\mathbf{s})$ is a subalgebra of $\mathbb{Z}[M \oplus N^\circ]$.

3.2.5. GHKK's Langlands dual construction. In this section, we explain the difference in our definition of a cluster algebra with GHKK's construction [**GHK13, GHKK18**]. Suppose we have fixed data Γ and let $\mathbf{e}$ be a Γ-seed. Denote by $\mathbf{g} = (g_i)_{i \in I}$ the dual of $\tilde{\mathbf{e}}^*$. To $\mathbf{e}$, We associate a torus

$$T_{\mathbf{e}} := T_{N^\circ} = \mathrm{Spec}\, k[M^\circ]$$

with coordinates $A_i = z^{g_i}$. For a seed $\mathbf{e}$ and its mutation $\mu_k^\varepsilon \mathbf{e}$, we have a birational map between tori

$$\mu_k^\varepsilon \colon T_{\mathbf{e}} \dashrightarrow T_{\mu_k^\varepsilon \mathbf{e}}, \quad (\mu_k^\varepsilon)^*(z^m) = z^m \left(1 + z^{p^*(\varepsilon e_k)}\right)^{-m(\tilde{e}_k)}$$

for $m \in M^\circ$.

The cluster exchange relation is computed as follows. Let $\mathbf{e}$ be an initial seed and we start with the cluster

$$(A_1, A_2, \ldots, A_r) = (z^{g_1}, z^{g_2}, \ldots, z^{g_r})$$

associated to the seed $\mathbf{e}$. Let k be an unfrozen index. We put

$$\mu_k^\varepsilon(\mathbf{g}) = (g_i'^{,\varepsilon})_{i \in I}, \quad A_i'^{,\varepsilon} = z^{g_i'^{,\varepsilon}} \in \mathbb{Z}[M^\circ].$$

Then we have

$$
\begin{aligned}
(\mu_k^\varepsilon)^*(A_k'^{,\varepsilon}) &= z^{g_k'^{,\varepsilon}} \left(1 + z^{p^*(\varepsilon e_k)}\right) = z^{-g_k + \sum_{i \in I}[\varepsilon b_{ki}]_+ g_i} \left(1 + z^{\sum_i \{\tilde{e}_i, \varepsilon e_k\} g_i}\right) \\
&= z^{-g_k}\left(z^{\sum_{i \in I}[\varepsilon b_{ki}]_+ g_i} + z^{\sum_{i \in I}[\varepsilon b_{ki}]_+ g_i + \sum_i -\varepsilon b_{ki} g_i}\right) \\
&= z^{-g_k}\left(z^{\sum_{b_{ki}>0} b_{ki} g_i} + z^{\sum_{b_{kj}<0}(-b_{kj})g_j}\right) \\
&= A_k^{-1}\left(\prod_{i\,:\,b_{ki}>0} A_i^{b_{ki}} + \prod_{j\,:\,b_{kj}<0} A_j^{-b_{kj}}\right).
\end{aligned}
$$

The other cluster variables remain unchanged. One observes that the right-hand side does not depend on the sign ε.

One can perform mutations indefinitely and consequently obtain the set of cluster variables as in Section 3.2. For the fixed data Γ with an initial seed $\mathbf{s}$, GHKK's cluster algebra $\mathcal{A}^{\mathrm{GHKK}}(\mathbf{s})$ is

defined in a similar way as our $\mathcal{A}(\mathbf{s})$ to be the subalgebra of $\mathbb{Z}[M^\circ]$ (caution: instead of $\mathbb{Z}[M]$) generated by elements

$$\left\{\mu_{v,w}^*(z^{g_{w,i}}) \mid w \in \mathfrak{T}_v, i \in I\right\}$$

where $(g_{w,i})_{i \in I}$ is the dual of the scaled seed $\tilde{\mathbf{s}}_w$ associated to the vertex w.

The above exchange formula Section 3.2.5 should be compared to the previous one Equation (3.2.1). The difference is that the one in Equation (3.2.1) uses the k-th column of the matrix B instead of the k-th row of B used here. Thus the exchange relation now is the same as the previously defined one of the Langlands dual matrix $B^\vee = -B^T$. Therefore we have (with identified clusters and cluster variables)

$$\mathcal{A}^{\text{GHKK}}(B^\vee) \cong \mathcal{A}(B).$$

Of course, in skew-symmetric cases, this makes no difference.

3.3. Generalized cluster algebras of Chekhov and Shapiro

In this section, we explain how to define Chekhov-Shapiro's generalized cluster algebras [CS14] in the framework we have developed so far. These algebras will be referred to as *Chekhov-Shapiro cluster algebras, CS algebras* for short.

Again we are given fixed data Γ with an initial seed $\mathbf{s}$.

3.3.1. Integrality assumption. The first assumption we need is that the matrix

$$\widetilde{B} := BD^{-1}$$

is integer valued, i.e. $\widetilde{B} \in M_r(\mathbb{Z})$. Recall that in Section 3.1.3, $\widetilde{B}$ is the pairing matrix of the basis $\mathbf{e}$. So our assumption is that $\omega(e_i, e_j) = \tilde{b}_{ij} = b_{ij}/d_j$ is an integer for all i, j. This integrality is not required in general for ordinary cluster algebras.

3.3.2. Extra data on exchange polynomials. A CS cluster algebras depends on some additional data. Choose a polynomial $\theta_i(u, v)$ for each $i \in I$ such that

(1) $\theta_i(u, v) \in k[u, v]$ for some field k of characteristic zero;

(2) $\deg \theta_i = d_i$;

(3) θ_i is reciprocal, i.e. $\theta_i(u, v) = \theta_i(v, u)$, and the coefficient of the term u^{d_i} is 1.

3.3.3. Definition. Let $\mathbf{e}$ be a seed and $\mathbf{f}$ be its dual. Let

$$\rho_k(v) = \theta_k(1, v).$$

Similar to previous constructions of cluster algebras, we associate a torus $T_{\mathbf{e}} = \operatorname{Spec} k[M]$ to a seed $\mathbf{e}$ with coordinates $A_i = z^{f_i}$ for $i \in I$.

For $k \in I_{\mathrm{uf}}$ and $\varepsilon \in \{+, -\}$, we define a birational transformation between two tori

$$\mu_k^\varepsilon \colon T_{\mathbf{e}} \dashrightarrow T_{\mu_k^\varepsilon \mathbf{e}}, \quad (\mu_k^\varepsilon)^*(z^m) = z^m \cdot \rho_k\left(z^{p^*(\varepsilon e_k)}\right)^{-m(e_k)}.$$

Note that the monomial $z^{p^*(\varepsilon e_k)}$ is actually in $k[M]$ since

$$p^*(\varepsilon e_k) = \omega(-, \varepsilon e_k)$$

lies in M by our integrality assumption on the matrix $\widetilde{B}$.

We put

$$f_{k;>0}^\varepsilon := \sum_{i \in I} \left[\varepsilon \tilde{b}_{ik}\right]_+ f_i, \quad f_{k;<0}^\varepsilon := \sum_{i \in I} \left[-\varepsilon \tilde{b}_{ik}\right]_+ f_i.$$

A straightforward computation gives the following *CS cluster exchange relation* [**CS14**]:

$$
\begin{aligned}
A_k^{\prime,\varepsilon} := z^{f_k^{\prime,\varepsilon}} \rho_k\left(z^{p^*(\varepsilon e_k)}\right)^{-f_k^{\prime,\varepsilon}(e_k)} &= z^{-f_k + \sum_{i \in I}[-\varepsilon b_{ik}] + f_i} \rho_k\left(z^{\sum_i \{e_i, \varepsilon e_k\} f_i}\right) \\
&= z^{-f_k} \cdot z^{d_k f_{k;<0}^\varepsilon} \cdot \rho_k\left(z^{\sum_i \varepsilon \tilde{b}_{ik} f_i}\right) \\
&= z^{-f_k} \cdot z^{d_k f_{k;<0}^\varepsilon} \cdot \rho_k\left(z^{f_{k;>0}^\varepsilon - f_{k;<0}^\varepsilon}\right) \\
&= z^{-f_k} \cdot \theta_k\left(z^{f_{k;>0}^\varepsilon}, z^{f_{k;<0}^\varepsilon}\right),
\end{aligned}
$$

that is, we have

$$
(3.3.1) \qquad A_k^{\prime,\varepsilon} = A_k^{-1} \cdot \theta_k\left(\prod_{i \in I} A_i^{[\varepsilon \tilde{b}_{ik}]_+}, \ \prod_{i \in I} A_i^{[-\varepsilon \tilde{b}_{ik}]_+}\right).
$$

Since the polynomial $\theta_k(u, v)$ is symmetric on u and v, the polynomial $A_k^{\prime,\varepsilon}$ does not depend on the choice of sign ε.

Example 3.3.1. Suppose that $d_k = 2$ and $\theta_k(u, v) = u^2 + uv + v^2$, thus we have

$$\rho_k(v) = 1 + v + v^2.$$

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The new generalized cluster variable under mutation at $k \in I$ is

$$z^{f'_k, \varepsilon} \left(1 + z^{p^*(\varepsilon e_k)} + z^{2p^*(\varepsilon e_k)} \right) = z^{-f_k} \left(z^{2f^\varepsilon_{k;>0}} + z^{f^\varepsilon_{k;>0} + f^\varepsilon_{k;<0}} + z^{2f^\varepsilon_{k;<0}} \right).$$

We construct the graph $\mathfrak{T}_\mathbf{s}$ with the labeling of seeds as before (see Section 3.2). Each vertex of $\mathfrak{T}_\mathbf{s}$ is associated with a torus. Between two tori associated to adjacent vertices, there is a birational transformation described above generalizing the one of ordinary cluster algebras. In this way, we obtain the set of cluster variables as before.

The Laurent phenomenon of ordinary cluster algebras has been extended to CS algebras in [**CS14**].

THEOREM 3.3.2 (Laurent phenomenon, [**CS14**, Theorem 2.5]). *For any two seeds related by a sequence of mutations, the pull-back of coordinates under CS generalized birational transformations are Laurent polynomials with integer coefficients. In other words, for vertices w, w' of the tree $\mathfrak{T}$, denote the coordinates of $T_{\mathbf{s}_{w'}}$ by A'_i (A_i for $T_{\mathbf{s}_w}$) for $i \in I$ and we have*

$$\mu^*_{w,w'}(A'_i) \in k[A^\pm_1, A^\pm_2, \dots, A^\pm_n].$$

We define Chekhov-Shapiro algebra $\mathcal{CS}(\mathbf{s})$ in the same way as ordinary cluster algebras, that is $\mathcal{CS}(\mathbf{s})$ is defined to be the subalgebra in $k[M]$ generated by the set of Laurent polynomials

$$\left\{ \mu^*_{v,w}(A_{w,i}) \mid w \in \mathfrak{T}, i \in I \right\}$$

where v is the initial vertex and $A_{w,i} \coloneqq z^{f_{w,i}}$ is the i-th coordinate function on the torus $T_{\mathbf{s}_w}$.

Example 3.3.3. We give an example of CS algebra of type B_2 here. Take

$$\widetilde{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let

$$\rho_1(v) = 1 + v, \quad \rho_2(v) = 1 + v + v^2.$$

The initial seed is given by (e_1, e_2). Denote the initial cluster variables by

$$A_1 = z^{e^*_1}, \quad A_2 = z^{e^*_2}.$$

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The mutation at $k = 2$ gives

$$\mu_2(A_1, A_2) = \left(A_1, A_2^{-1}(1 + A_1 + A_1^2)\right).$$

Same as the ordinary cluster algebra of type B_2, in total, there are 6 clusters and 6 generalized cluster variables.

Cluster scattering diagrams

In this chapter, we introduce the cluster scattering diagrams, an essential technical tool used in [**GHKK18**] to prove the positivity conjecture of cluster algebras.

A cluster scattering diagram will be defined for any fixed data Γ with an initial seed **s**. They are characterized by the so-called *initial data* or equivalently the set of incoming walls, explained in Section 4.1. The definition of a cluster scattering diagram is in Section 4.2.

4.1. Initial data and incoming walls

Let Γ be a fixed data with no frozen indices (i.e. $N_{\mathrm{uf}} = N$) and **s** be a Γ-seed. We pick up our notions from Chapter 2 for scattering diagrams.

Definition 4.1.1. An N^+-graded Lie algebra $\mathfrak{g}$ is *skew-symmetric* if we have

$$\omega(d_1, d_2) = 0 \implies [\mathfrak{g}_{d_1}, \mathfrak{g}_{d_2}] = 0$$

for any $d_1, d_2 \in N^+$.

This feature ensures that the so-called *initial data* is able to determine a consistent $\mathfrak{g}$-scattering diagram (see [**KS14**, section 3]). We briefly review this important point of view. Recall that the set of consistent $\mathfrak{g}$-SDs is in bijection with the group $\widehat{G}$. We introduce a way to parametrize elements in $\widehat{G}$ as described in [**KS14**, section 3] and also in [**GHKK18**, section 1.2].

Define a map

$$p^*: N \to M_{\mathbb{R}}, \quad n \mapsto \omega(-, n) \in M_{\mathbb{R}}.$$

Note that we do not require the map p^* to be injective. We define the set of *primitive elements* in N^+ as

$$\mathrm{Prim}(N^+) := \left\{ n \in N^+ \mid n/k \notin N^+ \text{ for any } k \in \mathbb{N}_{>1} \right\}.$$

Let $n \in \mathrm{Prim}(N^+)$. Consider the decomposition of $\mathfrak{g}$ with respect to $p^*(n)$ as in Lemma 2.1.1

$$(4.1.1) \qquad \mathfrak{g} = \mathfrak{g}_{p^*(n),+} \oplus \mathfrak{g}_{p^*(n),0} \oplus \mathfrak{g}_{p^*(n),-},$$

which induces a factorization of the corresponding pro-nilpotent group

$$\widehat{G} = \widehat{G}_{p^*(n),+} \cdot \widehat{G}_{p^*(n),0} \cdot \widehat{G}_{p^*(n),-}.$$

The Lie subalgebra $\mathfrak{g}_{p^*(n),0}$ further decomposes into

$$\mathfrak{g}_{p^*(n),0} = \mathfrak{g}_n^{\parallel} \oplus \mathfrak{g}_{p^*(n),0}^{\nparallel}$$

where

$$\mathfrak{g}_n^{\parallel} := \mathfrak{g}_{\mathbb{N} \cdot n} = \bigoplus_{k \in \mathbb{N}} \mathfrak{g}_{kn}, \quad \mathfrak{g}_{p^*(n),0}^{\nparallel} := \bigoplus_{p^*(n)(d)=0,\, d \notin \mathbb{N} \cdot n} \mathfrak{g}_d.$$

Note that the Lie subalgebra $\mathfrak{g}_n^{\parallel}$ is central in $\mathfrak{g}_{p^*(n),0}$ and $\mathfrak{g}_{p^*(n),0}^{\nparallel}$ is an ideal of $\mathfrak{g}_{p^*(n),0}$. The quotient map

$$\mathfrak{g}_{p^*(n),0} \to \mathfrak{g}_{p^*(n),0}/\mathfrak{g}_{p^*(n),0}^{\nparallel} \cong \mathfrak{g}_n^{\parallel}$$

induces a group homomorphism projection

$$r_n \colon \hat{G}_{p^*(n),0} \to \hat{G}_n^{\parallel} = \exp(\hat{\mathfrak{g}}_n^{\parallel}).$$

Given an element $g \in \widehat{G}$, for each primitive $n \in N^+$, we define

$$\psi_n(g) := r_n \circ \Pi_{p^*(n),0}(g) = r_n(g_{p^*(n),0}) \in \hat{G}_n^{\parallel}.$$

This defines a map (of sets)

(4.1.2) $$\psi \colon \widehat{G} \to \prod_{n \in \mathrm{Prim}(N^+)} \hat{G}_n^{\parallel}, \quad \psi(g) = (\psi_n(g))_{n \in \mathrm{Prim}(N^+)}.$$

Proposition 4.1.2 (Proposition 3.3.2 in [**KS14**])**.** *In the case that the Lie algebra* $\mathfrak{g}$ *is skew-symmetric in the sense of Definition 4.1.1, the map* ψ *is a bijection of sets.*

This proposition provides another way (other than the bijection $\widehat{G} \longleftrightarrow \prod_{d \in N^+} \mathfrak{g}_d$) to express an element g in $\widehat{G}$ by its components in each $\widehat{G}_n^{\parallel}$ under ψ. This expression of g is called the *initial data* of the corresponding consistent $\mathfrak{g}$-SD $\mathfrak{D}_g$.

Definition 4.1.3 (Initial data)**.** The initial data of the scattering diagram $\mathfrak{D}_g$ is the image under the map ψ of g, i.e., the tuple $(\psi_n(g))_{n \in \mathrm{Prim}(N^+)}$.

The fact that a consistent $\mathfrak{g}$-SD is determined by its initial data is sometimes known as "a consistent scattering diagram is determined by its incoming walls", e.g. see [**GHKK18**].

4.2. Cluster scattering diagrams

4.2.1. Cluster SDs. Continue on with the data $(\Gamma, \mathbf{s})$. Consider the N-graded Poisson algebra (often called the torus Lie algebra) defined as follows:

$$T := \mathbb{Q}[N] = \bigoplus_{d \in N} \mathbb{Q} \cdot x^d \quad \text{and} \quad \left[x^{d_1}, x^{d_2}\right] = \omega(d_1, d_2) x^{d_1 + d_2}, \ d_1, d_2 \in N.$$

We consider a Lie subalgebra

$$\mathfrak{g} = \mathfrak{g}_{\mathbf{s}} := T_{N_{\mathbf{s}}^+} \subset T.$$

It is $N_{\mathbf{s}}^+$-graded and skew-symmetric in the sense of Definition 4.1.1.

The dilogarithm series $\mathrm{Li}_2(x) \in \mathbb{Q}[[x]]$ *is defined by*

$$\mathrm{Li}_2(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^2}. \tag{4.2.1}$$

We define the following element $g_{\mathbf{s}}$ in $\widehat{G}$ by its initial data, i.e. the image under the bijective map ψ in (4.1.2). We define the initial data

$$\psi(g_{\mathbf{s}}) = (g_n)_{n \in \mathrm{Prim}(N^+)} \in \prod_{n \in \mathrm{Prim}(N^+)} \widehat{G}_n^{\parallel} \tag{4.2.2}$$

by putting

$$g_{s_i} = \exp\left(-\mathrm{Li}_2(-x^{d_i s_i})/d_i\right) = \exp\left(\frac{1}{d_i} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k\tilde{s}_i}}{k^2}\right) \in \widehat{G}_{s_i}^{\parallel}$$

and $g_n = \mathrm{id}$ for any other primitive n. We will use the notation

$$\mathbb{E}(n) := \exp(-\mathrm{Li}_2(-x^n)/d)$$

where d is the positive integer such that n/d is primitive. For example, we have $g_{s_i} = \mathbb{E}(\tilde{s}_i)$ for $i \in I$.

Definition 4.2.1 (Cluster scattering diagram)**.** The *cluster scattering diagram* $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ is defined to be the unique consistent $\mathfrak{g}$-SD corresponding to the group element $g_{\mathbf{s}} \in \widehat{G}$. The defining function is written as

$$\Phi_{\mathbf{s}}^{\mathrm{Cl}} : M_{\mathbb{R}} \to \widehat{G}.$$

The canonical cone complex is denoted by $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$.

Remark 4.2.2. For fixed data Γ, any seed for Γ will define a cluster scattering diagram. Note that the Lie algebra $\mathfrak{g}$ depends on the choice of the seed as the semigroup $N_{\mathbf{s}}^+$ depends on $\mathbf{s}$.

What we define in Definition 4.2.1 is different from (in fact, Langlands dual to) GHKK's version of a cluster scattering diagram in the skew-symmetrizable case. The difference will be explained in Section 4.2.2.

Example 4.2.3. Let $N = \mathbb{Z}^2$ with the standard basis $\mathbf{e} = (e_1, e_2)$. Let the form ω be given by the skew-symmetric matrix $\widetilde{B} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ in the standard basis. Let $\tilde{\mathbf{e}} = (be_1, ce_2)$ where b and c are two positive integers. We have $B = \left[\begin{smallmatrix} 0 & -c \\ b & 0 \end{smallmatrix}\right]$. For some special values of b and c, the corresponding cluster scattering diagrams are computed as follows in Figure 4.1 and Figure 4.2.
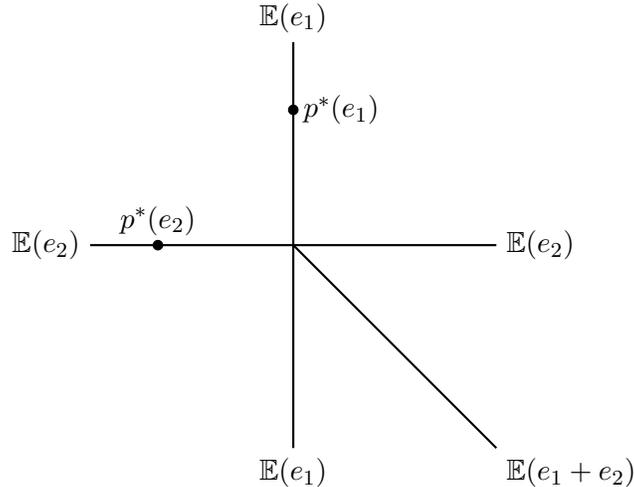
FIGURE 4.1. $b = 1, c = 1$

Remark 4.2.4. These diagrams live in $M_{\mathbb{R}}$ and are depicted in the basis (e_1^*, e_2^*).

4.2.2. GHKK's version. We describe GHKK's cluster scattering diagram for fixed data Γ with $\mathbf{s}$. Consider the initial data $(g_n)_{n \in \mathrm{prim}(N^+)}$ such that for any $s_i \in \mathbf{s}$, we put

$$g_{s_i} = \mathbb{E}(s_i)^{d_i}, \quad i \in I$$

and $g_n = \mathrm{id}$ otherwise.

The unique consistent scattering diagram corresponding to this initial data is what Gross, Hacking, Keel, and Kontsevich call the cluster scattering diagram in [**GHKK18**]. One may also

$$\mathbb{E}(e_1)$$

$$\mathbb{E}(2e_2) \qquad\qquad \mathbb{E}(2e_2)$$

$$\mathbb{E}(e_1 + 2e_2)$$

$$\mathbb{E}(e_1) \qquad\qquad \mathbb{E}(2e_1 + 2e_2)$$
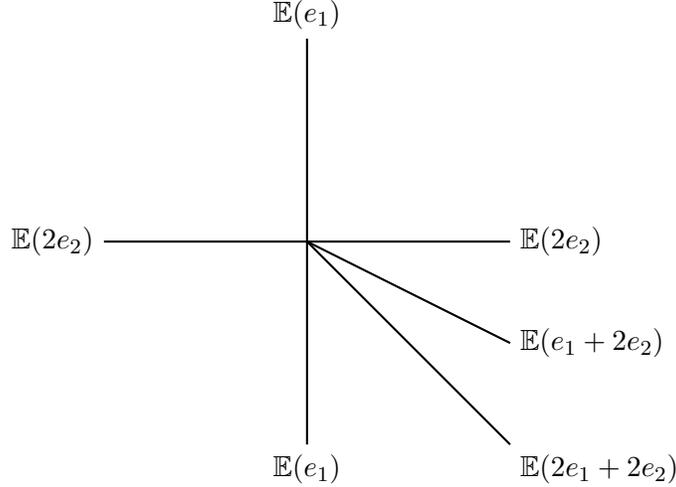
FIGURE 4.2. $b = 1$, $c = 2$

notice that when considering initial data, they used $p^*(-n)$ instead of $p^*(n)$. This difference, however, is superficial. The convention we stick to is the one from [**FZ07**]: the cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\text{Cl}}$ should corresponds to the cluster algebra $\mathcal{A}(B(\mathbf{s}))$. In particular, this requires that the cluster complex $\Delta_{\mathbf{s}}^+$ (Section 4.4) coincides with the cone complex of g-vectors defined in [**FZ07**].

Let $B = B(\mathbf{s})$. It turns out GHKK's version corresponds to the cluster algebra $\mathcal{A}(B^\vee)$ of the Langlands dual matrix $B^\vee = -B^T$ while ours (Definition 4.2.1) corresponds to $\mathcal{A}(B)$.

We present some examples of GHKK's cluster scattering diagrams in rank two; see Figure 4.3, 4.4 and 4.5. Some of them are also computed in [**GHKK18**, Section 1].

Example 4.2.5. Let ω be given by the skew-symmetric matrix $\tilde{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in the standard basis $\mathbf{e} = (e_1, e_2)$. Let $\tilde{\mathbf{e}} = (be_1, ce_2)$ where b and c are two positive integers. We have $B = \begin{bmatrix} 0 & -c \\ b & 0 \end{bmatrix}$. For some special values of b and c, the corresponding cluster scattering diagrams are computed as follows.

Remark 4.2.6. The three diagrams live in $M_\mathbb{R}$ and are depicted in the basis $(\tilde{e}_1^*, \tilde{e}_2^*)$ dual to $\tilde{\mathbf{e}}$. See also [**GHKK18**, Figure 1.2.] for the case $b = 1$ and $c = 3$ where the wall-crossing $\mathbb{E}(n)^k$ is represented by $(1 + z^{p^*(n)})^k$ and corresponds to an action that sends z^m to

$$z^m \left(1 + z^{p^*(n)}\right)^{km(n)}$$

for $n \in N$ and $m \in M^\circ$. Note that $m(n)$ is not always an integer for $m \in M^\circ$ and the coefficient k is crucial for making $z^m \left(1 + z^{p^*(n)}\right)^{km(n)}$ a polynomial. For example, the wall-crossing $\mathbb{E}(e_1 + 2e_2)$ is

$$\mathbb{E}(e_1)$$

$$\mathbb{E}(e_2) \quad\quad\quad \mathbb{E}(e_2)$$

$$\mathbb{E}(e_1) \quad\quad \mathbb{E}(e_1 + e_2)$$

FIGURE 4.3. $b = 1$, $c = 1$

$$\mathbb{E}(e_1)$$

$$\mathbb{E}(e_2)^2 \quad\quad\quad \mathbb{E}(e_2)^2$$

$$\mathbb{E}(e_1) \quad \mathbb{E}(e_1 + e_2)^2 \quad \mathbb{E}(e_1 + 2e_2)$$
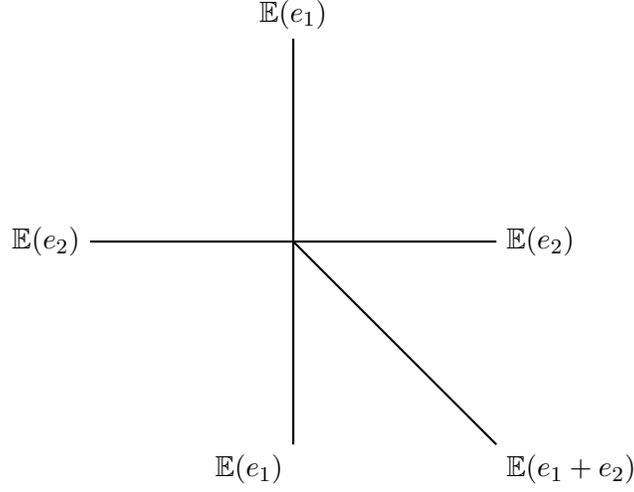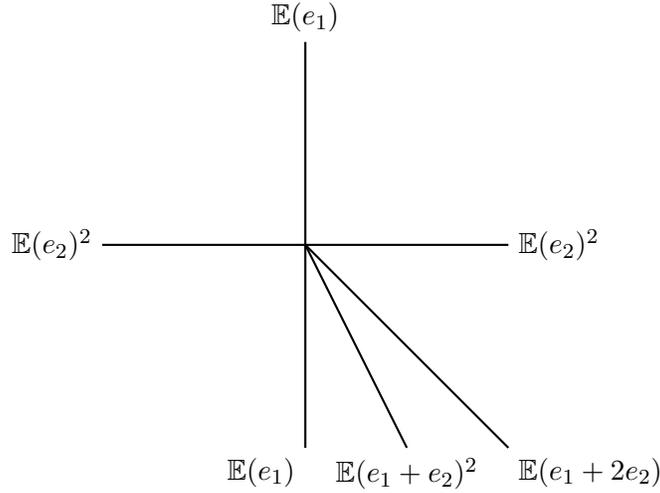
FIGURE 4.4. $b = 1$, $c = 2$

represented by the function $(1 + A_1^{-2} A_2^3)$ where $A_i = z^{\tilde{e}_i^*}$. We will come back to this interpretation in more details in Section 4.5.

4.3. Mutation invariance

The structure of the scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ (in particular, the underlying cone complex $\mathfrak{S}_{\mathbf{s}}^{\mathrm{C}}$) can be studied by comparing it with the one associated to the mutated seed. In this section, we describe the relation between $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ and $\mathfrak{D}_{\mu_{\tilde{k}}^{\varepsilon}\mathbf{s}}^{\mathrm{Cl}}$ in this section following [**GHKK18**].

The element $s_k \in \mathbf{s}$ defines a hyperplane

$$s_k^{\perp} := \{ m \in M_{\mathbb{R}} \mid m(s_k) = 0 \},$$

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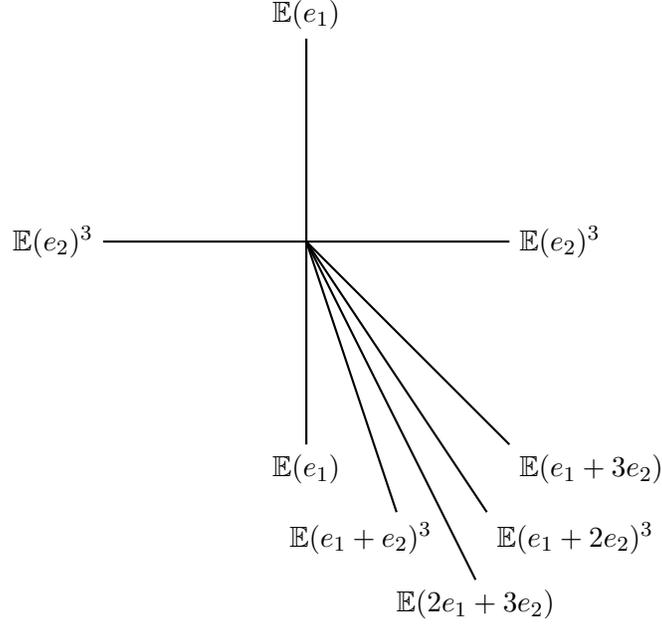
FIGURE 4.5. $b = 1$, $c = 3$

which separates the space $M_\mathbb{R}$ into two open half spaces

$$\mathcal{H}_{\mathbf{s}}^{k,+} := \{m \in M_\mathbb{R} \mid m(s_k) > 0\}, \quad \mathcal{H}_{\mathbf{s}}^{k,-} := \{m \in M_\mathbb{R} \mid m(s_k) < 0\}.$$

THEOREM 4.3.1 (GHKK [**GHKK18**, Theorem 1.24]). *Consider the cluster scattering diagram* $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ *and its mutation* $\mathfrak{D}_{\mu_k^\varepsilon \mathbf{s}}^{\mathrm{Cl}}$ *with* $k \in I$ *and* $\varepsilon \in \{+, -\}$.

(1) *At a generic* $m \in s_k^\perp$, *the wall-crossing is given by*

$$\Phi_{\mathbf{s}}^{\mathrm{Cl}}(m) = \mathbb{E}(\tilde{s}_k).$$

(2) *For any* $m \in \mathcal{H}_{\mathbf{s}}^{k,-\varepsilon}$, *we have*

$$\Phi_{\mu_k^\varepsilon \mathbf{s}}^{\mathrm{Cl}}(m) = \Phi_{\mathbf{s}}^{\mathrm{Cl}}(m).$$

Remark 4.3.2. In fact, the two functions $\Phi_{\mathbf{s}}^{\mathrm{Cl}}$ and $\Phi_{\mu_k^\varepsilon \mathbf{s}}^{\mathrm{Cl}}$ take values in different groups. We define

$$\mathfrak{g}_{\mathbf{s} \cap \mu_k^\varepsilon \mathbf{s}} := \mathfrak{g}_{\mathbf{s}} \cap \mathfrak{g}_{\mu_k^\varepsilon \mathbf{s}}.$$

This Lie algebra has a well-defined completion by gradings. The last equality in the above theorem in fact takes value in the group $\exp\left(\hat{\mathfrak{g}}_{\mathbf{s} \cap \mu_k^\varepsilon \mathbf{s}}\right)$.

PROOF. We closely follow the proof of Theorem 1.24 in [**GHKK18**], so we only explain a few necessary modifications as our version of cluster scattering diagram is slight different form GHKK's.

First of all, the *slab* in Definition 1.27 in [**GHKK18**] is now the pair

$$\mathfrak{d}_k = \left(s_k^\perp, \mathbb{E}(\tilde{s}_k) \right).$$

The group element $\mathbb{E}(\tilde{s}_k)$ has a different action on the monomial z^m described in Lemma 4.5.5. However, Theorem 1.28 in [**GHKK18**] still holds with the new slab $\mathfrak{d}_k$. Our theorem follows from a formal check of consistency as in Step II of the proof of Theorem 1.24. In our case, the consistency is due to the identity, for any $m \in M$ (instead of M°)

$$z^m \left(1 + z^{p^*(\tilde{s}_k)}\right)^{-m(s_k)} = z^{m-m(s_k)p^*(\tilde{s}_k)} \left(1 + z^{p^*(-\tilde{s}_k)}\right)^{-m(s_k)}.$$

$\square$

Theorem 4.3.1 relates the cluster scattering diagram associated to a seed $\mathbf{s}$ to the ones associated to the neighboring seeds $\mu_k^\varepsilon \mathbf{s}$ for any $k \in I$. We will see in the following that this observation leads to the so-called *cluster complex structure* of $\mathfrak{D}_\mathbf{s}^{\mathrm{Cl}}$.

4.4. Cluster complex structure

In this section we explain the *cluster complex structure* of $\mathfrak{D}_\mathbf{s}^{\mathrm{Cl}}$, obtained by using the mutation invariance Theorem 4.3.1 in the last section. In this section, the fixed data Γ will remain unchanged. Any scattering diagram in consideration will be a cluster scattering diagram, so we will omit the superscript, simply denoting one by $\mathfrak{D}_\mathbf{s}$.

4.4.1. Mutation of cluster scattering diagram. We put

$$\mathbf{u} = (u_i)_{i \in I} = \mu_k^+ \mathbf{s}, \quad \mathbf{v} = (v_i)_{i \in I} = \mu_k^- \mathbf{s}.$$

The dual seeds are denoted by

$$\mathbf{u}^* = (u_i^*)_{i \in I}, \quad \mathbf{v}^* = (v_i^*)_{i \in I}.$$

There is a linear map that transforms the basis $\mathbf{v}^*$ to $\mathbf{u}^*$

$$T_k \colon M \to M, \quad T_k(v_i^*) = u_i^*.$$

It is a shear transformation, explicitly given by

$$T_k(m) = m + m(s_k)p^*(\tilde{s}_k).$$

Recall the formulas of mutations of dual seeds in Section 3.1.5. Explicitly, we have

$$T_k(v_k^*) = v_k^* + v_k^*(s_k) \sum_{i \in I} (b_{ik}s_i^*) = u_k^*$$

and

$$T_k(v_i^*) = s_i^* = u_i^*, \quad i \neq k.$$

The map T_k induces a dual linear map on N

$$T_k^* : N \to N, \quad T_k^*(u_i) = v_i.$$

Since we have $\widetilde{B}(\mathbf{u}) = \widetilde{B}(\mathbf{v})$, the map T_k^* preserves the form ω. Therefore the map T_k^* actually induces an isomorphism between the seed data $(\Gamma, \mathbf{v})$ and $(\Gamma, \mathbf{u})$. Thus the associated cluster scattering diagrams should also be isomorphic, i.e.

$$\mathfrak{D}_\mathbf{u} \cong T_k(\mathfrak{D}_\mathbf{v}).$$

The precise meaning of this isomorphism is as follows. The linear map T_k^* induces an isomorphism between graded Lie algebras

$$T_k^* : \mathfrak{g}_\mathbf{u} \to \mathfrak{g}_\mathbf{v},$$

which can be extended to completions. We denote the corresponding isomorphism between pronilpotent groups also by T_k^*. Then we have for any $m \in M_\mathbb{R}$,

(4.4.1) $$\Phi_\mathbf{v}(m) = T_k^* \circ \Phi_\mathbf{u} \circ T_k(m) \in \widehat{G}_\mathbf{v}.$$

In view of the associated canonical cone complexes of $\mathfrak{D}_\mathbf{v}$ and $\mathfrak{D}_\mathbf{u}$, we have

$$\mathfrak{S}_\mathbf{u} = T_k(\mathfrak{S}_\mathbf{v}),$$

that is, the cones in $\mathfrak{S}_\mathbf{u}$ are obtained by applying the linear isomorphism T_k (extended to $M_\mathbb{R}$) to the cones in $\mathfrak{S}_\mathbf{v}$.

The above discussion gives a way to describe $\mathfrak{D}_\mathbf{v} = \mathfrak{D}_{\mu_k^+\mathbf{s}}$ in terms of $\mathfrak{D}_\mathbf{s}$. We define the following piecewise linear map on $M_\mathbb{R}$

$$T_k^+ : M_\mathbb{R} \to M_\mathbb{R},$$

$$T_k^+(m) := \begin{cases} T_k(m) = m + m(s_k)p^*(\tilde{s}_k) & \text{if } m \in \mathcal{H}_\mathbf{s}^{k,+}, \\ m & \text{if } m \in \mathcal{H}_\mathbf{s}^{k,-}. \end{cases}$$

THEOREM 4.4.1 (Mutation of cluster scattering diagram). *The cluster scattering diagram* $\mathfrak{D}_{\mu_k^+(\mathbf{s})}^{\mathrm{Cl}}$ *has the following description in terms of* $\mathfrak{D}_\mathbf{s}^{\mathrm{Cl}}$.

(1) *At a generic* $m \in s_k^\perp$, *the wall-crossing is given by*

$$\Phi_{\mu_k^+\mathbf{s}}(m) = \mathbb{E}(-\tilde{s}_k).$$

(2) *On* $\mathcal{H}_\mathbf{s}^{k,+} \cup \mathcal{H}_\mathbf{s}^{k,-} \subset M_\mathbb{R}$, *we have*

$$\Phi_\mathbf{s}(m) = \left(T_k^+\right)^* \circ \Phi_{\mu_k^+(\mathbf{s})} \circ T_k^+(m),$$

where $(T_k^+)^*$ *denotes the induced group homomorphism on its domain of linearity.*

(3) *The piecewise linear map* T_k^+ *induces an isomorphism from the canonical profinite cone complex* $\mathfrak{S}_\mathbf{s}$ *to* $\mathfrak{S}_{\mu_k^+\mathbf{s}}$.

PROOF. For part (1), we use Theorem 4.3.1. In fact, we have in the seed $\mathbf{u} = \mu_k^+\mathbf{s}$,

$$\tilde{u}_k = -\tilde{s}_k.$$

The second part (2) needs more explanation. Note that the map $T_k^+ : M_\mathbb{R} \to M_\mathbb{R}$ is piecewise linear and is linear on the two open halves $\mathcal{H}_\mathbf{s}^{k,\pm}$ respectively. On $\mathcal{H}_\mathbf{s}^{k,-}$, the map T_k^+ is identity. Thus on this domain, the equality in (2) follows from Theorem 4.3.1, i.e. we have for any $m \in \mathcal{H}_\mathbf{s}^{k,-}$,

$$\Phi_\mathbf{s}(m) = \Phi_{\mu_k^+\mathbf{s}}(m).$$

Restricted on $\mathcal{H}_\mathbf{s}^{k,+}$, we have $T_k^+ = T_k$. Then according to Theorem 4.3.1 and Equation (4.4.1), we have for $m \in \mathcal{H}_\mathbf{s}^{k,+}$,

$$\Phi_\mathbf{s}(m) = \Phi_{\mu_k^-\mathbf{s}}(m) = T_k^* \circ \Phi_{\mu_k^+\mathbf{s}} \circ T_k(m).$$

Combining the results on two domains of linearity proves part (2).

For (3), we consider the cone decompositions $\mathfrak{S}_{\mathbf{s}}^{\circ}$ and $\mathfrak{S}_{\mu_k^+\mathbf{s}}^{\circ}$. First of all, because of (1) of Theorem 4.3.1, a cone $\sigma^{\circ} \in \mathfrak{S}_{\mathbf{s}}^{\circ}$ is either contained in $s_k^{\perp}$ or in one of the halve spaces. The same is true for the other cone decomposition $\mathfrak{S}_{\mu_k^+\mathbf{s}}^{\circ}$. Thus according to the characterization of a cone as a path-connected component of a level set (Proposition 2.3.11), we conclude that the map T_k^+ maps the cones in $\mathfrak{S}_{\mathbf{s}}^{\circ}$ that avoid the hyperplane $s_k^{\perp}$ bijectively to the ones avoiding $s_k^{\perp}$ in $\mathfrak{S}_{\mu_k^+\mathbf{s}}^{\circ}$. Note that the cones contained in the hyperplane $s_k^{\perp}$ are determined by the cones in $\mathcal{H}_{\mathbf{s}}^{k,\pm}$. More precisely, at a finite level l, we have that $\mathfrak{S}_{\mathbf{s}}^{\leq l}$ is a finite cone complex so the cones in $s_k^{\perp}$ are just faces of cones in the open halves. Taking the projective limit, we get cones in $\mathfrak{S}_{\mathbf{s}}$. Thus we conclude that

$$\mathfrak{S}_{\mathbf{s}} \cap s_k^{\perp} = \mathfrak{S}_{\mu_k^+\mathbf{s}} \cap s_k^{\perp}.$$

Therefore we have

$$\mathfrak{S}_{\mu_k^+\mathbf{s}} = T_k^+(\mathfrak{S}_{\mathbf{s}}),$$

proving part (3). $\qquad\square$

Remark 4.4.2. There is a version of the above theorem of the other mutation μ_k^-. One can repeat the discussion with μ_k^- replacing μ_k^+ and the piecewise linear map T_k^+ replaced by another map T_k^- defined as

$$T_k^-(m) := \begin{cases} m, & \text{if } m \in \mathcal{H}_{\mathbf{s}}^{k,+} \\ m - m(s_k)p^*(\tilde{s}_k) & \text{if } m \in \mathcal{H}_{\mathbf{s}}^{k,-} \end{cases}$$

We leave the details to the reader as an exercise.

4.4.2. Cluster complex structure. In this section, we explain the so-called cluster complex structure of a cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}$.

Lemma 4.4.3. *The cones*

$$\mathcal{C}_{\mathbf{s}}^+ := \{m \in M_{\mathbb{R}} \mid m(s_i) \geq 0, \ i \in I\}$$

and

$$\mathcal{C}_{\mathbf{s}}^- := \{m \in M_{\mathbb{R}} \mid m(s_i) \leq 0, \ i \in I\}$$

and all their faces are elements in the canonical profinite cone complex $\mathfrak{S}_{\mathbf{s}}$.

PROOF. We observe that for $\mathfrak{g}_{\mathbf{s}}^{\leq 1}$, the canonical cone complex is induced by the hyperplane arrangement of all coordinate hyperplanes. Thus both $\mathcal{C}_{\mathbf{s}}^{\pm}$ and their faces are in $\mathfrak{S}_{\mathbf{s}}^{\leq 1}$. For $l >$

1, these cones cannot be further subdivided in $\mathfrak{S}_{\mathbf{s}}^{\leq l}$. Therefore these cones are elements in the projective limit. $\qquad\square$

Remark 4.4.4. In fact, not only we know that the cones described in Lemma 4.4.3 are in $\mathfrak{S}_{\mathbf{s}}$, it is also clear that the wall-crossing on a facet contained in $s_i^\perp$ is

$$\mathbb{E}(\tilde{s}_i) = \exp(-\operatorname{Li}_2(-x^{\tilde{s}_i})/d_i).$$

However, it is usually hard to determine the face-crossing at higher codimensional cones.

To a pair $(\mathbf{k}, \underline{\varepsilon})$ of a sequence of indices $\mathbf{k} = (k_1, k_2, \ldots, k_l)$ and a sequence of signs $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ of the same length, we can associate a seed

$$\mu_{\mathbf{k}}^{\underline{\varepsilon}}(\mathbf{s}) := \mu_{k_l}^{\varepsilon_l} \cdots \mu_{k_2}^{\varepsilon_2} \mu_{k_1}^{\varepsilon_1}(\mathbf{s})$$

obtained by applying iterative mutations to the initial seed $\mathbf{s}$. There is a cluster scattering diagram of the seed $\mu_{\mathbf{k}}^{\underline{\varepsilon}}(\mathbf{s})$ and we denote it by $\mathfrak{D}_{(\mathbf{k},\underline{\varepsilon})}$. It also has the two maximal cones described in Lemma 4.4.3, i.e the positive and negative chambers

$$\mathcal{C}_{\mathbf{k},\underline{\varepsilon}}^+, \ \mathcal{C}_{\mathbf{k},\underline{\varepsilon}}^- \subset M_\mathbb{R}.$$

We define the following piecewise linear map

$$T_{\mathbf{k}}^{\underline{\varepsilon}} := T_{k_l}^{\varepsilon_l} \circ \cdots \circ T_{k_2}^{\varepsilon_2} \circ T_{k_1}^{\varepsilon} : M_\mathbb{R} \to M_\mathbb{R}.$$

More precisely, we point out that the definition of the map $T_{k_i}^{\varepsilon_i}$ is with respect to the seed

$$\mu_{k_{i-1}}^{\varepsilon_{i-1}} \cdots \mu_{k_2}^{\varepsilon_2} \mu_{k_1}^{\varepsilon_1}(\mathbf{s}).$$

Proposition 4.4.5. *Let σ be a face of $\mathcal{C}_{\mathbf{k},\underline{\varepsilon}}^+$ or $\mathcal{C}_{\mathbf{k},\underline{\varepsilon}}^-$ (including themselves). Then the cone*

$$(T_{\mathbf{k}}^{\underline{\varepsilon}})^{-1}(\sigma)$$

belongs to the profinite cone complex $\mathfrak{S}_{\mathbf{s}}$. It does not depend on the sequence of signs ε.

PROOF. We prove this by induction on the length of the sequence $\mathbf{k}$. When $l = 0$, this follows from Lemma 4.4.3. Now let

$$\mathbf{k}' = (k_2, \ldots, k_l), \quad \varepsilon' = (\varepsilon_2, \ldots \varepsilon_l).$$

Then by induction, we have that the cone

$$(T_{\mathbf{k}'}^{\varepsilon})^{-1}(\sigma)$$

belongs to $\mathfrak{S}_{\mu_{k_1}^{\varepsilon_1}(\mathbf{s})}$. Then by part (3) of Theorem 4.4.1, we have that

$$(T_{\mathbf{k}}^{\varepsilon})^{-1}(\sigma) = (T_{k_1}^{\varepsilon_1})^{-1}(T_{\mathbf{k}'}^{\varepsilon})^{-1}(\sigma)$$

is in $\mathfrak{S}_{\mathbf{s}}$. The independency is also proved similarly by induction. $\qquad\qquad\square$

We put

$$G_{\mathbf{k}}^+ := (T_{\mathbf{k}}^{\varepsilon})^{-1}(\mathcal{C}_{\mathbf{k},\varepsilon}^+), \quad G_{\mathbf{k}}^- := (T_{\mathbf{k}}^{\varepsilon})^{-1}(\mathcal{C}_{\mathbf{k},\varepsilon}^-)$$

for a sequence of vertices $\mathbf{k}$. They are simplicial cones of dimension n in $M_{\mathbb{R}}$. If

$$\mathbf{k}' = (\mathbf{k}, k_{l+1}),$$

then it is easy to see that the cones $G_{\mathbf{k}}^{\varepsilon}$ and $G_{\mathbf{k}'}^{\varepsilon}$ share a common facet of codimension one. We call the maximal cone $G_{\mathbf{k}}^+$ the *cluster chamber* corresponding to $\mathbf{k}$.

Definition 4.4.6. We define the *positive cluster complex* $\Delta_{\mathbf{s}}^+$ to be the cone complex consisting of the cluster chambers $G_{\mathbf{k}}^+$ of all sequences $\mathbf{k}$ and all their faces. The *negative cluster complex* $\Delta_{\mathbf{s}}^-$ is defined similarly by considering the cones $G_{\mathbf{k}}^-$.

According to Proposition 4.4.5, both of the positive and negative cluster complexes are cone subcomplexes of $\mathfrak{S}_{\mathbf{s}}$.

The following proposition concerns the wall-crossings on walls of $\Delta_{\mathbf{s}}^+$ and $\Delta_{\mathbf{s}}^-$. Let $\mathbf{k}$ be a sequence of indices. Then the simplicial cone $G_{\mathbf{k}}^+$ is generated by vectors

$$\mathbf{g}_{\mathbf{k}} = (g_{\mathbf{k},i})_{i \in I} \in M^I.$$

The indexing by I is natural from iterative mutations. These vectors are called the g-vectors of the cluster associated to $\mathbf{k}$. These are the images under the map $(T_{\mathbf{k}}^{\varepsilon})^{-1}$ of the dual seed of $\mu_{\mathbf{k}}^{\varepsilon}(\mathbf{s})$. Let

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the D-scaled dual basis of $\mathbf{g_k}$ in N be

$$\mathbf{c_k} = (c_{\mathbf{k},i})_{i \in I} \in (N^\circ)^I,$$

i.e. $c_{\mathbf{k},i} = d_i g^*_{\mathbf{k},i}$ for $i \in I$. These vectors are called the c-vectors of the cluster associated to $\mathbf{k}$.

Proposition 4.4.7. *Let σ be the facet dual to the c-vector $c_{\mathbf{k},i}$ of the cone $G^+_{\mathbf{k}}$. Then we have*

$$\Phi_{\mathbf{s}}(\sigma) = \mathbb{E}(|c_i|).$$

PROOF. Note that the normal vector c_i must be in N^+ or $-N^+$ as the walls in a scattering diagram can only have normal vectors in N^+ or $-N^+$. The vector c_i corresponds to the i-th basis element e_i in the seed $\mu^\varepsilon_{\mathbf{k}}(\mathbf{s})$. In the scattering diagram $\mathfrak{D}_{\mathbf{k},\varepsilon}$, the wall-crossing function at $e^\perp_i$ is given by $\mathbb{E}(d_i e_i)$. Then by Theorem 4.4.1, under the map $(T^\varepsilon_{\mathbf{k}})^{-1}$, the wall-crossing function at σ can only be of the form $\mathbb{E}(|c_i|)$. □

Remark 4.4.8. Combining Proposition 4.4.5 and Proposition 4.4.7, we have the so-called *cluster complex structure* of $\mathfrak{D}_{\mathbf{s}}$. That is, we have a simplicial cone complex (also called a simplicial fan in toric geometry) $\Delta^+_{\mathbf{s}}$ (also the negative version $\Delta^-_{\mathbf{s}}$) defined in Definition 4.4.6 with wall-crossings on facets described in Proposition 4.4.7.

4.4.3. Combinatorics of g-vectors. In this section, we show an iterative way to compute the g-vectors of cluster chambers. This algorithm may be well-known to experts but it is not easy to find a reference.

For two sequences $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_m)$ of indices or signs, we define their concatenation as

$$\mathbf{p} \sqcup \mathbf{q} := (p_1, \ldots, p_n, q_1, \ldots, q_m).$$

Proposition 4.4.9. *Let $\mathbf{k}'$ be a sequence of indices of length $l-1$ and $\mathbf{k} = \mathbf{k}' \sqcup (k_l)$. Then we have that $\mathbf{c_k}$ is a D-scaled seed of fixed data Γ, and $\mathbf{g_k}$ is the dual seed of $\mathbf{c}_k/D$. They can be computed iteratively as (recall the mutations of seeds and dual seeds in Section 3.1.4)*

$$\mathbf{g_k} = \mu^{\varepsilon_l}_{k_l}(\mathbf{g_{k'}}), \quad \mathbf{c_k} = \mu^{\varepsilon_l}_{k_l}(\mathbf{c_{k'}})$$

for

$$\varepsilon_l = \begin{cases} +, & \text{if } c_{\mathbf{k}',k_l} \in N_{\mathbf{s}}^+ \\[2mm] -, & \text{if } c_{\mathbf{k}',k_l} \in -N_{\mathbf{s}}^+. \end{cases}$$

For a sequence $\varnothing$ of length 0, we use the convention that

$$\mathbf{g}_\varnothing = \mathbf{s}^*, \quad \mathbf{c}_\varnothing = \tilde{\mathbf{s}}.$$

PROOF. We prove this lemma by induction. First of all, for the initial seed $\mathbf{s}$ and its neighbors, by Theorem 4.3.1, we have for any $i \in I$,

$$\mathbf{g}_{(i)} = \mu_i^+(\mathbf{s}^*), \quad \mathbf{c}_{(i)} = \mu_i^+(\tilde{\mathbf{s}}).$$

Now suppose that the lemma is true for all sequences with length no greater than $l-1$. We choose a sequence of signs

$$\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$$

recursively by the rule indicated by the lemma. By definition and induction, we have

$$\mathbf{g}_{\mathbf{k}} = \left(T_{\mathbf{k}}^{\underline{\varepsilon}}\right)^{-1}\left(\mu_{k_l}^{\varepsilon_l}(\mathbf{g}_{\mathbf{k}'})\right), \quad \mathbf{g}_{\mathbf{k}'} = \left(T_{\mathbf{k}'}^{\underline{\varepsilon}'}\right)^{-1}(\mathbf{g}_{\mathbf{k}'})$$

where $\mathbf{k} = \mathbf{k}' \sqcup k_l$ and $\underline{\varepsilon} = \underline{\varepsilon}' \sqcup \varepsilon_l$.

We want to show that

$$\mathbf{g}_{\mathbf{k}} = \mu_{k_l}^{\varepsilon_l}(\mathbf{g}_{\mathbf{k}'}).$$

The dual statement for c-vectors follows.

Then we pull back both $\mathbf{g}_{\mathbf{k}'}$ and $\mu_{k_l}^{\varepsilon_l}(\mathbf{g}_{\mathbf{k}'})$ back to the scattering diagram after one step of mutation. More precisely, let us consider the seed and its dual after one step of mutation at k_1:

$$\mathbf{s}_{(k_1)} := \mu_{k_1}^+(\mathbf{s}), \quad \mathbf{g}_{(k_1)} = \mu_{k_1}^+(\mathbf{s}^*).$$

We pull back $\mathbf{g}_{\mathbf{k}'}$ and $\mu_{k_l}^{\varepsilon_l}(\mathbf{g}_{\mathbf{k}'})$ (and their duals) to $\mathfrak{D}_{\mathbf{s}_{(k_1)}}$, i.e. we define (collections of vectors)

$$\Sigma_1 := \left(T_{\mathbf{k}\backslash k_1}^{\underline{\varepsilon}\backslash \varepsilon_1}\right)^{-1}(\mu_{k_l}^{\varepsilon_l}(\mathbf{g}_{\mathbf{k}'}))$$

and

$$\Sigma_2 := \left(T_{\mathbf{k}'\backslash k_1}^{\underline{\varepsilon}'\backslash \varepsilon_1}\right)^{-1}(\mathbf{g}_{\mathbf{k}'})$$

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where $\mathbf{k} \setminus k_1$ is the sequence such that $\mathbf{k} = (k_1) \sqcup \mathbf{k} \setminus k_1$; $\mathbf{k}' \setminus k_1$ similarly defined.

By induction, we have

$$\Sigma_1 = \mu_{k_l}^{\delta}(\Sigma_2)$$

where the sign δ is the sign of $c_{\mathbf{k}' \setminus k_1, k_l}$ with respect to $\mathbf{c}_{(k_1)}$. There are two cases to deal with.

(1) The cone of Σ_1 and the cone of Σ_2 are separated by $s_{k_1}^{\perp}$. This means the vector $c_{\mathbf{k}' \setminus k_1, k_l}$ is in the direction of s_{k_1}. Suppose we have

$$c_{\mathbf{k}' \setminus k_1, k_l} / d_{k_l} = -s_{k_1}.$$

This means $\mathrm{Cone}(\Sigma_2)$ and $G_{(k_1)}^+$ are in the same half space where $(T_{k_1}^{\varepsilon_1})^{-1} = (T_{k_1}^+)^{-1}$ acts by identity. On the other hand Σ_1 and $G_{\varnothing}^+ = \mathcal{C}^+$ are in the other half. In this case $\delta = +$ (and thus $\varepsilon_l = -$) and we have

$$(T_{k_1}^{\varepsilon_1})^{-1}(\Sigma_1) = (T_{k_1}^+)^{-1}\left(\mu_{k_l}^+(\Sigma_2)\right) = \mu_{k_l}^-(\Sigma_2) = \mu_{k_l}^-\left((T_{k_1}^+)^{-1}\Sigma_2\right).$$

By definition, we have

$$\mathbf{g_k} = (T_{k_1}^{\varepsilon_1})^{-1}(\Sigma_1), \quad \mathbf{g_{k'}} = (T_{k_1}^{\varepsilon_1})^{-1}(\Sigma_2)$$

and hence the desired

$$\mathbf{g_k} = \mu_{k_l}^-(\mathbf{g_{k'}}).$$

The case where $c_{\mathbf{k}' \setminus k_1, k_l} / d_{k_l} = s_{k_1}$ ($\delta = -$) can be proved similarly.

(2) Both the cones generated by Σ_1 and by Σ_2 are on the same side of $s_{k_1}^{\perp}$. Then the piecewise linear map $(T_{k_1}^{\varepsilon_1})^{-1}$ transforms Σ_1 and Σ_2 by the same linear map (either an identity or a shear transformation) and will keep them on the same side of $s_{k_1}^{\perp}$. Recall that $(T_{k_1}^{\varepsilon_1})$ restricted on its linear domain is an automorphism of (N, ω). Thus we have

$$\mu_{k_l}^{\delta}((T_{k_1}^{\varepsilon_1})^{-1}(\Sigma_2)) = (T_{k_1}^{\varepsilon_1})^{-1}(\mu_{k_l}^{\delta}(\Sigma_2))$$

and hence

$$(T_{k_1}^{\varepsilon_1})^{-1}(\Sigma_1) = \mu_{k_l}^{\delta}((T_{k_1}^{\varepsilon_1})^{-1}(\Sigma_2))$$

i.e. the desired

$$\mathbf{g_k} = \mu_{k_l}^{\delta}(\mathbf{g_{k'}}).$$

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Note that in this case, the sign of $c_{\mathbf{k}'\backslash k_1, k_l}$ with respect to $\mathbf{c}_{(k_1)}$ coincide with the sign of $c_{\mathbf{k}', k_l}$ with respect to $\mathbf{s}$.

$\square$

4.5. Cluster variables via scattering diagram

4.5.1. Wall-crossings as poisson automorphisms. In this section, we explain how to obtain a cluster variable from the cluster scattering diagram.

We now consider the following commutative algebra

$$\widehat{T} = \widehat{T}_{\mathbf{s}} := \mathbb{Q}[M^\circ] \otimes \mathbb{Q}[[N_{\mathbf{s}}^\oplus]]$$

whose monomials are denoted by

$$z^m x^n = z^m \otimes x^n.$$

Equip this algebra with the poisson bracket determined by

$$\{x^{n_1}, x^{n_2}\} = \omega(n_1, n_2) x^{n_1 + n_2}, \quad \{z^{m_1}, z^{m_2}\} = 0, \quad \{x^n, z^m\} = m(n) z^m \otimes x^n.$$

Lemma 4.5.1. *The algebra $\widehat{T}$ with the bracket $\{,\}$ as defined above is a poisson algebra.*

PROOF. We check for monomials x^{n_1}, x^{n_2} and z^m the Jacobian identity. The details of other requirements are left to the reader. We have

$$\{x^{n_1}, \{x^{n_2}, z^m\}\} = m(n_2)(\omega(n_1, n_2) + m(n_1)) z^m x^{n_1 + n_2}$$

and then

$$\{x^{n_1}, \{x^{n_2}, z^m\}\} - \{x^{n_2}, \{x^{n_1}, z^m\}\} = m(n_1 + n_2)\omega(n_1, n_2) z^m x^{n_1 + m_2} = \{\{x^{n_2}, x^{n_1}\}, z^m\}.$$

$\square$

Lemma 4.5.2. *There is an injective Lie algebra homomorphism*

$$f \colon \hat{\mathfrak{g}} \to \mathrm{Der}(\widehat{T}), \quad x^n \mapsto \{x^n, -\}.$$

PROOF. This follows from the more general consideration that $\hat{\mathfrak{g}}$ is a Lie subalgebra of $\widehat{T}$. $\square$

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Proposition 4.5.3. *The Lie algebra homomorphism f induces an injective group homomorphism*

$$\mathrm{Ad}\colon \widehat{G} \to \mathrm{Aut}(\widehat{T})$$

by exponentiating.

PROOF. This proposition follows directly from the above Lemma 4.5.2 and completions. We briefly explain how to compute the automorphism by exponentiating. For example, for $\exp(a) \in \widehat{G}$, the action is given by for any $t \in \widehat{T}$,

$$\mathrm{Ad}\exp(a)(t) := \exp(\{a,-\})(t) = \sum_{k=0}^{\infty} \frac{1}{k!}(\{a,-\})^k(t).$$

The inverse of $\mathrm{Ad}\exp(a)$ is simply given by $\mathrm{Ad}\exp(-a)$. $\qquad\square$

Recall that the 2-form ω defines a group homomorphism

$$p^*\colon N \to M^\circ, \quad p^*(n) = \omega(-,n).$$

We consider the injective group homomorphism

$$\tilde{p}^*\colon N \to M^\circ \oplus N, \quad \tilde{p}^*(n) = (p^*(n), n).$$

Next we consider the (complete) algebra

$$\widehat{\mathbb{T}} = \widehat{\mathbb{T}}_{\mathbf{s}} := \bigoplus_{m \in M^\circ} \mathbb{Q}[[p^*(N_{\mathbf{s}}^\oplus)]] \cdot z^m \subset \prod_{(m,n) \in M^\circ \oplus N} \mathbb{Q} \cdot z^m x^n.$$

The completion is only viable because the map $\tilde{p}^*$ is injective. There is an algebra isomorphism, induced by $\tilde{p}^*$,

$$P\colon \widehat{T} \to \widehat{\mathbb{T}}, \quad P(z^m) = z^m, \quad P(x^n) = z^{p^*(n)}x^n.$$

Thus from the view of Proposition 4.5.3, the group $\widehat{G}$ also acts on $\widehat{\mathbb{T}}$ as it acts on $\widehat{T}$, which we denote by for $\exp(a) \in \widehat{G}$,

$$\mathrm{Ad}_{\widehat{\mathbb{T}}}\exp(a)\colon \widehat{\mathbb{T}} \longrightarrow \widehat{\mathbb{T}}.$$

For example, let $z^m \in \widehat{T}$ and $\exp(a) \in \widehat{G}$. Then we have

$$\mathrm{Ad}\exp(a)(z^m) = z^m h_{a,m}$$

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for some formal series $h_{a,m} \in \mathbb{Q}[[N^{\oplus}]]$ while the action on $\widehat{\mathbb{T}}$ is given by

$$\mathrm{Ad}_{\widehat{\mathbb{T}}}\exp(a)(z^m) = z^m P(h_{a,m}).$$

The following straightforward lemma is useful in later applications.

Lemma 4.5.4. *For any* $\exp(a) \in \widehat{G}$ *and any* $b \in \widehat{T}$*, we have*

$$P(\mathrm{Ad}\exp(a)(b)) = \mathrm{Ad}_{\widehat{\mathbb{T}}}\exp(a)(P(b)) \in \widehat{\mathbb{T}}.$$

Lemma 4.5.5. *For some* $n_0 \in N^+$*, let*

$$a = \sum_{k=1}^{\infty} a_k x^{kn_0} \in \widehat{\mathfrak{g}}.$$

Then we have

$$\mathrm{Ad}\exp(a)(z^m) = z^m h_{a,m} = z^m \cdot \exp\left(m(n_0)\sum_{k=1}^{\infty} ka_k x^{kn_0}\right) \in \widehat{T}.$$

PROOF. Since the actions $\{a_k x^{kn_0}, -\}$ for different k commute, we compute each of them sep-arately. We have

$$\mathrm{Ad}\exp(a_k x^{kn_0})(z^m) = \sum_{i\geq 0}\frac{1}{i!}\left\{a_k x^{kn_0}, -\right\}^i (z^m)$$

$$= \sum_{i\geq 0} z^m \cdot \frac{1}{i!}\left(m(n_0)ka_k x^{kn_0}\right)^i$$

$$= z^m \cdot \exp\left(m(n_0)ka_k x^{kn_0}\right).$$

The result then follows. $\qquad\square$

Example 4.5.6. If $h_{a,m} = (1 + x^{n_0})^{m(n_0)}$, then we have by Lemma 4.5.5,

$$\exp(a) = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}x^{kn_0}\right) = \exp(-\mathrm{Li}_2(-x^{n_0})) = \mathbb{E}(n_0).$$

Note that

$$P(h_{a,m}) = \left(1 + z^{\omega(-,n_0)}x^{n_0}\right)^{m(n_0)}.$$

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4.5.2. Cluster monomial by path-ordered product. Fix the data $(\Gamma, \mathbf{s})$. For a sequence $\mathbf{k}$ of indices, there is a corresponding cluster consisting of cluster variables

$$(A_{\mathbf{k},1}, A_{\mathbf{k},2}, \ldots, A_{\mathbf{k},n}).$$

There is also a maximal cone $G_{\mathbf{k}}^+$ in $\mathfrak{D}_{\mathbf{s}}$, with generators (what we call *g-vectors*)

$$\mathbf{g}_{\mathbf{k}} = (g_{\mathbf{k},1}, g_{\mathbf{k},2}, \ldots, g_{\mathbf{k},n}).$$

Pick a generic point θ in $G_{\mathbf{k}}^+$ and another generic point in the positive chamber $\mathcal{C}^+$. The two points define a path-ordered product (for a path γ going from θ to $\mathcal{C}+$) that we denoted by

$$\mathfrak{p}_{\gamma}^{\mathrm{Cl}} := \mathfrak{p}_{\gamma}(\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}) \in \widehat{G}.$$

Note that we can always choose an actual path going from $\mathcal{C}^+$ to $G_{\mathbf{k}}^+$ that only crosses finitely many walls of cluster chambers with the reverse order of the sequence $\mathbf{k}$. So the element $\mathfrak{p}_{\theta,+}$ can be expressed as a finite product of wall-crossings of the form

$$\mathbb{E}\left(|c_{\mathbf{k}(i),k_i}|\right)^{\pm} \in \widehat{G}, \quad |c_{\mathbf{k}(i),k_i}| \in N^+ \cap N^\circ$$

where $\mathbf{k}(i)$ is the subsequence

$$(k_1, k_2, \ldots, k_i).$$

We are ready to state the main result Theorem 4.5.7 of this section. This theorem is natural from the point of view of [**GHKK18**] in terms of theta functions and broken lines. The difference here is that the cluster scattering diagram we use is Langlands dual to GHKK's version and that we do not use the notion of broken lines (thus not the result in [**CPS10**]) in our proof.

Recall the notations in Section 4.5.1.

THEOREM 4.5.7. *For any $i \in I$, we have the following identities for cluster variables with principal coefficients*

$$A_{\mathbf{k},i}^{\mathrm{Prin}} = \mathrm{Ad}_{\widehat{\mathbb{T}}} \, \mathfrak{p}_{\gamma}^{\mathrm{Cl}}(z^{g_{\mathbf{k},i}}) \in \mathbb{Z}[M \oplus N^\circ] \subset \widehat{\mathbb{T}},$$

and for cluster variables without coefficients

$$A_{\mathbf{k},i} = \mathrm{Ad}_{\widehat{\mathbb{T}}} \, \mathfrak{p}_{\gamma}^{\mathrm{Cl}}(z^{g_{\mathbf{k},i}}) \,|_{x^n=1} \in \mathbb{Z}[M].$$

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PROOF. We prove this theorem by induction on the length of $\mathbf{k}$. Assume it is true for sequences of length no longer than $l - 1$. Suppose $\mathbf{k} = \mathbf{k}' \sqcup (k_l)$ and $k_l = i \in i$. Then the chambers $G_\mathbf{k}$ and $G_{\mathbf{k}'}$ share a common facet $\mathfrak{d}$ with the normal vector

$$c_{\mathbf{k}',i} = -c_{\mathbf{k},i}.$$

Choose a point $\theta' \in G_{\mathbf{k}'}$ and let γ' be a path from θ' to $\mathcal{C}^+$. We define for n in N^+ or $-N^+$,

$$\operatorname{sgn}(n) = \begin{cases} +, & \text{if } n \in N^+; \\ -, & \text{if } n \in -N^+. \end{cases}$$

Let $\delta = \operatorname{sgn}(c_{\mathbf{k}',i})$. Since the type of crossing (Definition 2.2.2) going from θ to θ' is the opposite of δ, then we have

$$\mathfrak{p}_\gamma^{\mathrm{Cl}} = \mathfrak{p}_{\gamma'}^{\mathrm{Cl}} \cdot \Phi(\mathfrak{d})^{-\delta} = \mathfrak{p}_{\gamma'}^{\mathrm{Cl}} \cdot \mathbb{E}\left(|c_{\mathbf{k}',i}|\right)^{-\delta}.$$

By Proposition 4.4.9, we have $\mathbf{g}_\mathbf{k} = \mu_i^\delta(\mathbf{g}_{\mathbf{k}'})$ and thus (from Section 3.1.4)

$$g_{\mathbf{k},i} = -g_{\mathbf{k}',i} + \sum_{j \in I} [-\delta b_{ji}]_+ g_{\mathbf{k}',j},$$

where $b_{ji} := \omega(c_{\mathbf{k}',j}/d_j, c_{\mathbf{k}',i})$ for $i, j \in I$. By Lemma 4.5.5, the action is computed as

$$\operatorname{Ad}_{\widehat{\mathbb{T}}} \mathbb{E}(|c_{\mathbf{k}',i}|)^{-\delta}(z^{g_{\mathbf{k},i}}) = z^{g_{\mathbf{k},i}} \left(1 + x^{\delta c_{\mathbf{k}',i}} z^{p^*\left(\delta c_{\mathbf{k}',i}\right)}\right)^{-\delta(g_{\mathbf{k},i}(\delta c_{\mathbf{k}',i}/d_i))}$$

$$= z^{-g_{\mathbf{k}',i}} \prod_{j \in I} z^{[-\delta b_{ji}]_+ g_{\mathbf{k}',j}} \left(1 + x^{\delta c_{\mathbf{k}',i}} \prod_{j \in I} z^{\delta b_{ji} g_{\mathbf{k}',j}}\right)$$

$$= z^{-g_{\mathbf{k}',i}} \left(\prod_{j \in I} z^{[-\delta b_{ji}]_+ g_{\mathbf{k}',i}} + x^{\delta c_{\mathbf{k}',i}} \prod_{i \in I} z^{[\delta b_{ji}]_+ g_{\mathbf{k}',i}}\right) \in \mathbb{Z}[M \oplus N^\circ] \subset \widehat{\mathbb{T}}.$$

The above equation is exactly the cluster exchange relation for cluster variables with principal coefficients. By induction, we have

$$\mathrm{Ad}_{\widehat{\mathbb{T}}}\,\mathfrak{p}^{\mathrm{Cl}}_{\gamma}\left(z^{g_{\mathbf{k},i}}\right) = \mathrm{Ad}_{\widehat{\mathbb{T}}}\,\mathfrak{p}^{\mathrm{Cl}}_{\gamma'}\left(\mathbb{E}(|c_{\mathbf{k}',i}|)^{\delta}(z^{g_{\mathbf{k},i}})\right)$$

$$= \mathrm{Ad}_{\widehat{\mathbb{T}}}\,\mathfrak{p}^{\mathrm{Cl}}_{\gamma'}\left(z^{-g_{\mathbf{k}',i}}\right)\cdot \mathrm{Ad}_{\widehat{\mathbb{T}}}\,\mathfrak{p}^{\mathrm{Cl}}_{\gamma'}\left(\prod_{j\in I}z^{[-\delta b_{ji}]+g_{\mathbf{k}',i}} + x^{\delta c_{\mathbf{k}',i}}\prod_{j\in I}z^{[\delta b_{ji}]+g_{\mathbf{k}',i}}\right)$$

$$= \left(A^{\mathrm{Prin}}_{\mathbf{k}',i}\right)^{-1}\cdot\left(\prod_{j\in I}\left(A^{\mathrm{Prin}}_{\mathbf{k}',j}\right)^{[-\delta b_{ji}]+} + x^{\delta c_{\mathbf{k}',i}}\prod_{j\in I}\left(A^{\mathrm{Prin}}_{\mathbf{k}',j}\right)^{[\delta b_{ji}]+}\right)$$

$$= A^{\mathrm{Prin}}_{\mathbf{k},i}\in\widehat{\mathbb{T}}.$$

Note that the vector $c_{\mathbf{k},i}$ for any sequence $\mathbf{k}$ and $i\in I$ is in N°. By the Laurent phenomenon Theorem 3.2.2, we have that $A^{\mathrm{Prin}}_{\mathbf{k},i}$ lies in $\mathbb{Z}[M\oplus N^{\circ}]$. The result on cluster variables without coefficients follows. $\qquad\square$

In [**GHKK18**, Section 3], GHKK defined the so-called *theta functions* for cluster scattering diagrams. These are elements of the form

$$\vartheta_{Q,m}\in\mathbb{Q}[[p^{*}(N^{\oplus})]]\cdot z^{m}\in\widehat{\mathbb{T}}$$

that depends on an endpoint Q (s.t. $\Phi_{\mathfrak{D}}(Q)=\mathrm{id}$) and $m\in M$. In fact, we have for $m\in\Delta^{+}_{\mathbf{s}}$ and $Q\in(\mathcal{C}^{+}_{\mathbf{s}})^{\circ}$,

$$\vartheta_{Q,m}=\mathrm{Ad}_{\widehat{\mathbb{T}}}\,\mathfrak{p}^{\mathrm{Cl}}_{\gamma}(z^{m})$$

where θ is in the interior of a cluster chamber that contains m.

Remark 4.5.8. We note that Proposition 4.4.9 provides a simple algorithm to compute the g-vectors. In our context, the g-vectors are the generators of the rays of cluster chambers. There is also a definition through the cluster variables themselves. Then Proposition 4.4.9 implies our notion of g-vectors coincide with the usual notion of g-vectors in cluster algebras (for example, see [**Kel08**]). In view of Theorem 4.5.7, the monomial $z^{g_{\mathbf{k},i}}$ should be viewed as a leading term of $A_{\mathbf{k},i}$.

CHAPTER 5

Categorification of skew-symmetric cluster algebras

In this chapter, we give a brief review of the so-called additive categorification of cluster algebras. The review is far from being comprehensive. For example, we do not even mention the remarkable cluster categories initiated in [**BMR$^+$06**]. Our goal is to introduce necessary ingredients that are the most relevant to stability scattering diagrams. We focus on conveying the idea that certain quiver representations decategorify into cluster monomials; see Section 5.3. As the title suggests, we only deal with the skew-symmetric case, that is, the matrix B is skew-symmetric.

5.1. Quivers with potentials and mutations

5.1.1. Quivers with potentials. The key player is *quivers*.

Definition 5.1.1. A quiver Q is a quadruple (Q_0, Q_1, s, t) composed of

(1) A finite set Q_0 of *vertices*;

(2) A finite set Q_1 of *edges*;

(3) A map $s\colon Q_1 \to Q_0$ that specifies the *source* of an edge;

(4) A map $t\colon Q_1 \to Q_0$ that specifies the *target* of an edge.

Fix a field K. For a quiver Q, we have its *vertex span* $R = K^{Q_0}$ and *arrow span* $A = K^{Q_1}$ as K-valued functions on Q_0 and Q_1.

The vector space R is a commutative K-algebra under the pointwise multiplication of functions. The arrow span A is an R-bimodule where the action on the left is by acting on the target of an edge.

Definition 5.1.2. The *path algebra* of Q (also denoted by KQ) is defined as the (graded) tensor algebra

$$R\langle A \rangle = \bigoplus_{d=0}^{\infty} A^d$$

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where A^d denote the R-bimodule

$$A^d = \underbrace{A \otimes_R \cdots \otimes_R A}_{d}.$$

Definition 5.1.3. The *complete path algebra* of Q (also denoted by $\widehat{KQ}$) is defined as the complete graded tensor algebra

$$R\langle\langle A \rangle\rangle = \prod_{d=0}^{\infty} A^d.$$

A *potential* is an element in $R\langle\langle A \rangle\rangle$ such that each component is a linear combination of cyclic paths. Thus potentials live in a closed subspace $R\langle\langle A \rangle\rangle_{\mathrm{cyc}} \subset R\langle\langle A \rangle\rangle$.

For every potential W, its *(complete) Jacobian ideal* $J(W)$ is defined to be the closure of the two-sided ideal generated by cyclic derivatives; see [**DWZ08**] for the definition.

A pair (Q, W) of a quiver Q and a potential $W \in \widehat{KQ}$ is called a *quiver with potential*, QP in short.

Definition 5.1.4. The *(complete) Jacobian algebra* of a QP (Q, W) is defined as the quotient

$$\mathcal{P}(Q, W) := \widehat{KQ}/J(W).$$

5.1.2. Mutations of quivers. We first review the mutations of quivers. These are purely combinatorial constructions.

Let Q be 2-acyclic, that is, there is no oriented path of length 2 in Q. For a vertex $k \in Q_0$, we define the quiver $\tilde{\mu}_k(Q)$ obtained from Q by the following two steps:

Step I For each pair of incoming arrow $a \colon i \to k$ and outgoing arrow $b \colon k \to j$ at k, create one arrow $[ba] \colon i \to j$.

Step II Replace each incoming arrow $a \colon i \to k$ with a new outgoing arrow $\alpha^\star \colon k \to i$; replace each outgoing arrow $b \colon k \to j$ with a new incoming arrow $b^\star \colon j \to k$.

Now we perform the last step to get a 2-acyclic quiver $\mu_k(Q)$ from $\tilde{\mu}_k(Q)$.

Step III Delete a maximal collection of disjoint 2-cycles in $\tilde{\mu}_k(Q)$.

It is easy to see the mutation μ_k is an involution, i.e. $\mu_k^2(Q) \cong Q$.

5.1.3. Mutations of QPs of Derksen–Weyman–Zelevinsky. We explain in this section the mutations of QPs of Derksen–Weyman–Zelevinsky [**DWZ08**]. This is a fundamental notion in (various) additive categorifications (e.g. [**DWZ10**] and [**Ami09**]) of cluster algebras.

Definition 5.1.5 (Right-equivalence, c.f. [**DWZ08**, definition 4.2])**.** Let (Q, W) and (Q', W') be two QPs with the same vertex set Q_0. We say that they are *right-equivalent* if there is an isomorphism $\varphi \colon \widehat{KQ} \to \widehat{KQ'}$ preserving Q_0 and taking W to a potential $\varphi(W)$ cyclic equivalent to W'.

Note that the isomorphism φ also induces an isomorphism between complete Jacobian algebras $\mathcal{P}(Q, W) \cong \mathcal{P}(Q', W')$.

A potential is called *reduced* if it contains no cycle of length less than or equal to 2; it is called *trivial* if the corresponding Jacobian algebra is trivial. For two QPs (Q_1, W_1) and (Q_2, W_2) with the identified set of vertices, their *direct sum* is the QP

$$(Q_1, W_1) \oplus (Q_2, W_2) := (Q_1 \oplus Q_2, W_1 + W_2)$$

where $Q_1 \oplus Q_2$ is the quiver on the same set of vertices but taking disjoint union of arrows of Q_1 and Q_2.

THEOREM 5.1.6 ([**DWZ08**, theorem 4.6]). *Every QP (Q, W) is right-equivalent to the direct sum of a trivial QP and a reduced QP*

$$(Q_{\mathrm{triv}}, W_{\mathrm{triv}}) \oplus (Q_{\mathrm{red}}, W_{\mathrm{red}})$$

where each direct summand is determined up to right-equivalence by the right equivalent class of (Q, W). The Jacobian algebra of (Q, W) is then isomorphic to the Jacobian algebra of the reduced *part $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ via the embedding of $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ in $(Q_{\mathrm{triv}}, W_{\mathrm{triv}}) \oplus (Q_{\mathrm{red}}, W_{\mathrm{red}})$.*

Note that the quivers Q_{triv} and Q_{red} (up to isomorphisms fixing the set of vertices) are canonically constructed from (Q, W); see [**DWZ08**, 4.4] whereas it is less trivial to compute the potential W_{red}.

Now let Q be a 2-acyclic quiver and W be a potential such that no term in its expansion starts at k (if not, replace W with a cyclic equivalent one). We construct a potential for the quiver $\tilde{\mu}_k(Q)$ as follows. For each pair of incoming a and outgoing b at vertex k, replace any occurrences of ba in W with $[ba]$. We thus get a potential $\overline{W}$ in $\widehat{K\tilde{\mu}_k(Q)}$. Define

$$\tilde{\mu}_k(W) := \overline{W} + \sum_{a,b \in Q_1 \,:\, t(a)=s(b)=k} [ba] a^\star b^\star,$$

where the sum is taken over all pairs of incoming α and outgoing β at k. We thus have obtained a QP $(\tilde{\mu}_k(Q), \tilde{\mu}_k(W))$.

Definition 5.1.7 (Mutation of QP). The mutation $\mu_k(Q, W)$ of a 2-cyclic QP (Q, W) is defined to be the *reduced part* of $(\tilde{\mu}_k(Q), \tilde{\mu}_k(W))$ given by Theorem 5.1.6:

$$\mu_k(Q, W) := (\tilde{\mu}_k(Q)_{\text{red}}, \tilde{\mu}_k(W)_{\text{red}}).$$

Definition 5.1.8 (k-mutable QPs). We say that a 2-acyclic QP (Q, W) is *mutable* at vertex k (or *k-mutable*) if the quiver $\tilde{\mu}_k(Q)_{\text{red}}$ is equal to $\mu_k(Q)$. Therefore we have the justified notation $(\mu_k(Q), \mu_k(W))$ of $\mu_k(Q, W)$ for k-mutable QPs.

The following proposition is an immediate corollary of theorem 4.5 in [**DWZ08**].

Proposition 5.1.9. *The mutation μ_k is an involution on the set of right-equivalent classes of k-mutable QPs, i.e. $\mu_k(Q, W)$ is also k-mutable and $\mu_k^2(Q, W)$ is right-equivalent to (Q, W).*

5.1.4. Mutations of seeds with potentials. In this section, we lift mutations of QPs to SPs. Suppose we have fixed data Γ with an initial seed $\mathbf{s}$. We assume that $I_{\text{uf}} = I$ and $N^\circ = N$. As in Section 3.1.3, the *adjacency matrix* $B(\mathbf{s}) = (b_{ij})$ is given by $b_{ij} = \omega(s_i, s_j)$, which is skew-symmetric.

Definition 5.1.10 (The quiver of a seed). The matrix $B(\mathbf{s})$ determines a 2-acyclic quiver $Q(\mathbf{s})$ such that $Q_0 = I$ and there are $[b_{ij}]_+$ arrows from i to j for each $i, j \in I$.

Example 5.1.11. Let $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The quiver is then given by $1 \longleftarrow 2$.

Lemma 3.1.4 shows $B(\mu_k^+(\mathbf{s})) = B(\mu_k^-(\mathbf{s}))$, i.e. two types of mutations of a seed give the same quiver:

(5.1.1) $$\mu_k(Q(\mathbf{s})) \cong Q(\mu_k^+(\mathbf{s})) \cong Q(\mu_k^-(\mathbf{s})).$$

A pair $(\mathbf{s}, W)$ consisting of a seed $\mathbf{s}$ and a potential of $Q(\mathbf{s})$ is called a *seed with potential*, SP in short. Two SPs $(\mathbf{s}, W)$ and $(\mathbf{s}', W')$ are said to be *right-equivalent* if $\mathbf{s} = \mathbf{s}'$ and $(Q(\mathbf{s}), W)$ is right-equivalent to $(Q(\mathbf{s}'), W')$ respecting the identification on vertices. An SP is said to be *k-mutable* if the associated QP is k-mutable (Definition 5.1.8).

Definition 5.1.12 (Mutation of SP)**.** For a k-mutable seed with potential $(\mathbf{s}, W)$, we define the following two *mutations* at vertex k:

$$\mu_k^+(\mathbf{s}, W) \coloneqq (\mu_k^+(\mathbf{s}), \mu_k(W)), \quad \mu_k^-(\mathbf{s}, W) \coloneqq (\mu_k^-(\mathbf{s}), \mu_k(W)).$$

Proposition 5.1.13. *The mutations of SPs μ_k^+ and μ_k^- are inverse to each other on the set of right equivalent classes of k-mutable SPs.*

PROOF. The statement follows directly from Equation (5.1.1) and Proposition 5.1.9. $\qquad\square$

5.2. Ginzburg's differential graded algebras

For any QP (Q, W), there is an associated (complete) Ginzburg differential graded algebra $\Gamma = \Gamma(Q, W)$ (see [**Gin06**]) whose derived category $\mathcal{D}\Gamma$ (see [**Kel94**] for the definition of the derived category of an dg algebra) plays an essential role in the additive categorification of cluster algebras.

The (complete) Ginzburg dga $\Gamma = \Gamma(Q, W)$ is non-positively graded with a differential d of degree 1. We refer to [**Nag13**, Section 1.1] for the precise definition. We have $H^0(\Gamma) \cong \mathcal{P} = \mathcal{P}(Q, W)$. An right-equivalence between (Q, W) and (Q', S') induces an dg algebra isomorphism between the corresponding Ginzburg dg algebras [**KY11**].

Let $\mathcal{D}\Gamma$ be the derived category of Γ, whose objects are left dg Γ-modules. It is triangulated with the apparent shift functor. The Ginzburg dg algebra $_\Gamma\Gamma$ itself, as a left dg Γ-module, is an object in $\mathcal{D}\Gamma$. We have a decomposition of Γ

$$\Gamma = \bigoplus_{i \in I} \Gamma e_i$$

into dg Γ-modules $\Gamma_i \coloneqq \Gamma_{e_i}$. There is an embedding from $\operatorname{Mod}\mathcal{P}$ (the category of left $\mathcal{P}$-modules) into $\mathcal{D}\Gamma$

$$\iota\colon \operatorname{Mod}\mathcal{P} \to \mathcal{D}\Gamma,$$

whose image is the heart of the natural t-structure. Taking H^0, we get a functor

$$H^0\colon \mathcal{D}\Gamma \to \operatorname{Mod}\mathcal{P}$$

such that $H^0 \circ \iota = \operatorname{id}$.

The *perfect derived category* per Γ is the smallest full subcategory of $\mathcal{D}\Gamma$ containing the object Γ that is stable under shifts, forming cones, and taking direct summands.

The *finite-dimensional derived category* $\mathcal{D}_{\mathrm{fd}}\Gamma$ is the full subcategory of $\mathcal{D}\Gamma$ consisting of objects with finite total cohomology. We have two functors,

$$\iota \colon \operatorname{mod}\mathcal{P} \subset \mathcal{D}_{\mathrm{fd}}\Gamma, \quad H^0 \colon \mathcal{D}_{\mathrm{fd}}\Gamma \to \operatorname{mod}\mathcal{P}$$

where $\operatorname{mod}\mathcal{P}$ denotes the category of finite-dimensional modules of $\mathcal{P}$.

5.2.1. Keller-Yang's derived equivalences. Let (Q, W) be a k-mutable quiver with potential. We set $(\widetilde{Q}, \widetilde{W}) = \tilde{\mu}_k(Q, W)$, and $(Q', W') = \mu_k(Q, W) = (\widetilde{Q}, \widetilde{W})_{\mathrm{red}}$. We also set $\widetilde{\Gamma} = \Gamma(\widetilde{Q}, \widetilde{W})$, and $\Gamma' = \Gamma(Q', W')$.

We consider a dg Γ-module $T = \bigoplus_{i \in I} T_i$ where $T_i := \Gamma_i$ for $i \neq k$ and

$$T_k := \operatorname{cone}\left(\Gamma_k \to \bigoplus_{\alpha \in Q_1 \colon t(\alpha) = k} \Gamma_{s(\alpha)}\right).$$

Then we have the exact triangle

(5.2.1) $$\Gamma_k \to \bigoplus_{\alpha \in Q_1 \colon t(\alpha) = k} \Gamma_{s(\alpha)} \to T_k \to \Gamma_k[1].$$

It is shown in [**KY11**] that T has a left dg $\widetilde{\Gamma}^{\mathrm{op}}$-module structure realized by a dg algebra homomorphism from $\widetilde{\Gamma}^{\mathrm{op}}$ to $\mathcal{H}om_{\Gamma}(T, T)$ ([**KY11**, Proposition 3.5]). Thus we can regard T as a dg Γ-$\widetilde{\Gamma}$-bimodule. We consider two functors $\widetilde{\mathcal{F}}_k^+ \colon \mathcal{D}\Gamma \to \mathcal{D}\widetilde{\Gamma}$ defined by $\mathcal{R}\mathcal{H}om(T, -)$ and respectively $\widetilde{\mathcal{G}}_k^- \colon \mathcal{D}\widetilde{\Gamma} \to \mathcal{D}\Gamma$ defined by $T \overset{L}{\otimes}_{\widetilde{\Gamma}} -$.

THEOREM 5.2.1 (c.f. [**KY11**]).

(1) *The two functors $\widetilde{\mathcal{F}}_k^+$ and $\widetilde{\mathcal{G}}_k^-$ are quasi-inverse triangle equivalences.*

(2) *We have that*
$$\widetilde{\mathcal{G}}_k^+(\widetilde{\Gamma}_i) = \begin{cases} \Gamma_i & \text{if } i \neq k, \\ T_k & \text{if } i = k. \end{cases}$$

(3) *The functor $\widetilde{\mathcal{F}}_k^+$ restricts to triangle equivalences from* $\operatorname{per}\Gamma$ *to* $\operatorname{per}\widetilde{\Gamma}$ *and from* $\mathcal{D}_{\mathrm{fd}}\Gamma$ *to* $\mathcal{D}_{\mathrm{fd}}\widetilde{\Gamma}$.

(4) *The equivalences $\widetilde{\mathcal{F}}_k^+$ extend (by the isomorphism between Γ' and $\widetilde{\Gamma}$ induced by right equiv-*
   alence) further to triangle equivalences

$$\mathcal{F}_k^+ \colon \mathcal{D}\Gamma \to \mathcal{D}\Gamma', \quad \mathcal{F}_k^+ \colon \operatorname{per}\Gamma \to \operatorname{per}\Gamma', \quad \mathcal{F}_k^+ \colon \mathcal{D}_{\mathrm{fd}}\Gamma \to \mathcal{D}_{\mathrm{fd}}\Gamma'.$$

We denote the quasi-inverse functor by $\mathcal{F}_k^-$.

Remark 5.2.2. Since μ_k is an involution on QPs up to right equivalence, we have $\Gamma \cong \mu_k(\Gamma')$ and thus the functor $\mathcal{F}_k^-$ above is from $\mathcal{D}\Gamma'$ to $\mathcal{D}\mu_k(\Gamma')$. Therefore one can think that there are two triangle equivalences $\mathcal{F}_k^\pm$ from $\mathcal{D}\Gamma$ to $\mathcal{D}\Gamma'$.

Consider the Grothendieck groups $K_0(\mathcal{D}_{\mathrm{fd}}\Gamma)$ and $K_0(\operatorname{per}\Gamma)$. The group $K_0(\mathcal{D}_{\mathrm{fd}}\Gamma)$ has a basis given by the classes of simple modules $(S_i)_{i\in I}$ and the group $K_0(\operatorname{per}\Gamma)$ has a basis by the classes of $(\Gamma_i)_{i\in I}$. Here I is the indexing set identified with the set of vertices of Q. The two groups and their bases are naturally dual to each other.

Following our notations on fixed data Section 3.1.3, let $N = K_0(\mathcal{D}_{\mathrm{fd}}\Gamma) \cong \mathbb{Z}^r$ (suppose the rank of the quiver Q is r) and thus $M = K_0(\operatorname{per}\Gamma)$. The form ω on N is determined by the quiver Q in the basis of simples. The initial seed $\mathbf{s}$ is given by simples $([S_i])_{i\in I}$. The seed mutations in Section 3.1.4 are categorified as follows. Let $\mathbf{s}' = (s_i')_{i\in I} = \mu_k^\varepsilon(\mathbf{s})$ for $\varepsilon \in \{+, -\}$. Denote the simples in $\operatorname{mod}\mathcal{P}' = \operatorname{mod}\mathcal{P}(\mu_k(Q,W))$ by S_i' for $i \in I$. We have from Theorem 5.2.1

$$s_i' = \left[(\mathcal{F}_k^\varepsilon)^{-1}(S_i')\right] \in K_0\left(\mathcal{D}_{\mathrm{fd}}\Gamma\right) = N$$

and

$$\left(s_i'\right)^* = \left[(\mathcal{F}_k^\varepsilon)^{-1}(\Gamma_i')\right] \in K_0\left(\operatorname{per}\Gamma\right) = M.$$

Moreover, assuming that (Q, W) is non-degenerate (Definition 6.4.3), if there is a sequence of vertices $\mathbf{k}$ and signs $\underline{\varepsilon}$, we have the triangle equivalence

$$\mathcal{F}_{\mathbf{k}}^{\underline{\varepsilon}} \colon \mathcal{D}\Gamma \to \mathcal{D}(\mu_{\mathbf{k}}\Gamma)$$

by composing the equivalences $\mathcal{F}_{k_i}^{\varepsilon_i}$ in order. Then we have

$$\mu_{\mathbf{k}}^{\underline{\varepsilon}}(\mathbf{s}) = \left(\left[(\mathcal{F}_{\mathbf{k}}^{\underline{\varepsilon}})^{-1} S_{\mathbf{k},i}\right]\right)_{i\in I} \in N^I$$

and

$$\mu_{\mathbf{k}}^{\varepsilon}(\mathbf{s}^*) = \left(\left[(\mathcal{F}_{\mathbf{k}}^{\varepsilon})^{-1} \Gamma_{\mathbf{k},i} \right] \right)_{i \in I} \in M^I$$

where $S_{\mathbf{k},i}$ denotes the simples of $\operatorname{mod} \mathcal{P}(\mu_{\mathbf{k}}(Q, W))$ and $\Gamma_{\mathbf{k},i} := \Gamma(\mu_{\mathbf{k}}(Q, W))_i$.

5.3. Caldero–Chapoton formula

Here we explain Nagao's description of the Caldero–Chapoton formula for cluster monomials [**Nag13**]. Recall that for a sequence of indices $\mathbf{k}$, we have the cluster

$$(A_{\mathbf{k},1}, A_{\mathbf{k},2}, \ldots, A_{\mathbf{k},r})$$

of cluster variables in $\mathbb{Z}[M]$. The goal is to find modules over $\mathcal{P}$ to express these cluster variables.

For any $\mathbf{k}$, there is an associated sequence of signs $\underline{\varepsilon}(\mathbf{k})$ uniquely determined by $\mathbf{k}$ as in Proposition 4.4.9. Following [**Nag13**], we define the following $\mathcal{P}$-module

$$R_{\mathbf{k},i} := H^1\left(\left(\mathcal{F}_{\mathbf{k}}^{\varepsilon(\mathbf{k})} \right)^{-1} (\Gamma_{\mathbf{k},i}) \right), \quad i \in I.$$

Here the cohomology is taken with respect to the t-structure with the heart $\operatorname{mod} \mathcal{P}$. It is proved in [**Nag13**, Lemma 5.1] that for any sequence $\mathbf{k}$ and any $i \in I$, the module $R_{\mathbf{k},i}$ is finite-dimensional.

Moreover, we define the class

$$[\Gamma_{\mathbf{k},i}] := \left[\left(\mathcal{F}_{\mathbf{k}}^{\varepsilon(\mathbf{k})} \right)^{-1} (\Gamma_{\mathbf{k},i}) \right]$$

considered in $M = K_0(\operatorname{per} \Gamma)$. According to the iterative description in the last section and Proposition 4.4.9, we have that in M,

$$[\Gamma_{\mathbf{k},i}] = g_{\mathbf{k},i}.$$

THEOREM 5.3.1 (Caldero–Chapoton formula). *The i-th cluster variable of the $\mathbf{k}$-th cluster is computed by*

$$A_{\mathbf{k},i} = z^{[\Gamma_{\mathbf{k},i}]} \cdot \left(\sum_{n \in N^{\oplus}} \chi(\operatorname{Gr}(R_{\mathbf{k},i}, n)) z^{p^*(n)} \right)$$

where $\operatorname{Gr}(R_{\mathbf{k},i}, n)$ is the quiver Grassmannian parametrizing quotients of dimension vector n of the module $R_{\mathbf{k},i}$ and $\chi(\cdot)$ takes the Euler characteristic of analytic topology.

The Caldero–Chapoton formula has been proved for various generality by different people; see the introduction chapter for the relevant references. We also note that the characterizations of the

module $R_{\mathbf{k},i}$ are different in different approaches. In the next chapter, Section 6.8, we will give a proof of the above theorem by interpreting a cluster monomial as a θ-function (Theorem 4.5.7) and using stability scattering diagrams following Nagao's approach in [**Nag13**]. The advantage of our proof is that it does not explicitly rely on the so-called multiplication formula. An application of this strategy of proof is in [**LFM**] to prove a Caldero–Chapoton type formula for generalized cluster algebras.

CHAPTER 6

Stability scattering diagrams of quivers with potentials

This chapter is devoted to scattering diagrams of quivers with potentials. In [**Bri17**], Bridgeland defined the motivic Hall algebra scattering diagram associated to a quiver with (polynomial) potential. By using an integration map of Joyce [**Joy07**], this scattering diagram descends to the so-called stability scattering diagram, valued in the same Lie algebra as of the cluster scattering diagram. We study in detail the structures of stability scattering diagrams of quivers with potentials and their mutations.

6.1. Motivic Hall algebras

In this section, we introduce motivic Hall algebras of quivers with relations, following Bridgeland [**Bri17**].

6.1.1. Moduli space of modules. Let Q be a quiver, and $I \in \mathbb{C}Q$ be an ideal. Let $A = \mathbb{C}Q/I$. There is a $\mathbb{C}$-stack $\mathfrak{M} = \mathfrak{M}(Q, I)$ parametrizing finite-dimensional A-modules. Over a $\mathbb{C}$-scheme S, the groupoid $\mathfrak{M}(S)$ consists of objects

$$(\mathcal{E}, \rho \colon \mathbb{C}Q/I \to \mathrm{End}(\mathcal{E}))$$

where $\mathcal{E}$ is a finite rank locally free sheaf on S and ρ is a $\mathbb{C}$-algebra homomorphism. For a morphism $f \colon S' \to S$, we choose a pullback $f^{-1}\mathcal{E}$ for every $\mathcal{E}$ on S, and $f^{-1}\rho$ is defined such that $f^{-1}\rho(a) = \rho(f^{-1}a)$ for any $a \in \mathbb{C}Q/I$.

The stack $\mathfrak{M}$ has the following decomposition

$$\mathfrak{M} = \coprod_{d \in \mathbb{N}^{Q_0}} \mathfrak{M}_d.$$

where $\mathfrak{M}_d$ (parametrizing A-modules of dimension d) is algebraic, of finite type and with affine diagonal. In fact, we have that the stack $\mathfrak{M}_d$ is equivalent to the quotient stack $[\mathrm{Rep}(Q, I)_d / \mathrm{GL}_d]$. Here $\mathrm{Rep}(Q, I)_d$ is the closed subscheme of $\mathrm{Rep}(Q)_d$ cut out by the ideal induced by the relations I. In particular, when $d = 0$, the moduli stack $\mathfrak{M}_0$ is isomorphic to $\mathrm{Spec}(\mathbb{C})$.

6.1.2. The motivic Hall algebra. We refer to [**Bri17**, section 5] and [**Nag13**, section 7] for details of the following definitions. Let $K(\mathrm{St}/\mathfrak{M})$ be the *relative Grothendieck group of stacks* (algebraic of finite type over $\mathbb{C}$ with affine stabilizers) over $\mathfrak{M} = \mathfrak{M}(Q, I)$. It is naturally a module over $K(\mathrm{St}/\mathbb{C})$, the *Grothendieck ring of stacks* over $\mathbb{C}$. Furthermore, one can define a convolution type product $*$ ([**Joy07**, theorem 4.1]) on $K(\mathrm{St}/\mathfrak{M})$ so that it becomes an associative $K(\mathrm{St}/\mathbb{C})$-algebra graded by $\mathbb{N}^{|Q_0|}$; see also [**Bri17**, 5.4]. We describe this convolution product briefly.

There is an algebraic stack $\mathfrak{M}^{(2)}$ of isomorphism classes of short exact sequences in $\mathrm{mod}\, A$, the category of finite-dimensional left A-modules. There are natural forgetful maps from $\mathfrak{M}^{(2)}$, sending a short exact sequence to its constituents, summarized in the following diagram. All the maps are of finite type and note that π_2 (sending a short exact sequence to its middle term) is presentable and proper.

$$
\begin{array}{ccc}
\mathfrak{M}^{(2)} & \xrightarrow{\ \pi_2\ } & \mathfrak{M} \\
{\scriptstyle (\pi_3, \pi_1)}\downarrow & & \\
\mathfrak{M} \times \mathfrak{M} & &
\end{array}
$$

Then the convolution product is then defined as

$$
[X \xrightarrow{\ f\ } \mathfrak{M}] * [Y \xrightarrow{\ g\ } \mathfrak{M}] = (\pi_2)_*(\pi_3, \pi_1)^*[X \times Y \xrightarrow{(f,g)} \mathfrak{M} \times \mathfrak{M}]
$$

where the pushforward and pullback are well-defined in the current situation. Note that π_1 sends a short exact sequence to the kernel.

The algebra $(K(\mathrm{St}/\mathfrak{M}), *)$ is the *motivic Hall algebra* of (Q, I) and will be denoted by $H(Q, I)$.

6.1.3. A regular subalgebra. In this section, we consider a subalgebra of $H(Q, I)$.

Consider the subalgebra

$$
K_{\mathrm{reg}}(\mathrm{St}/\mathbb{C}) \coloneqq K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1},\ [\mathbb{P}^n]^{-1} \colon n \in \mathbb{N}] \subset K(\mathrm{St}/\mathbb{C})
$$

where $\mathbb{L} = [\mathbb{A}^1] \in K(\mathrm{Var}/\mathbb{C})$. We define $H_{\mathrm{reg}}(Q, I)$ to be the submodule of $H(Q, I)$ generated over $K_{\mathrm{reg}}(\mathrm{St}/\mathbb{C})$ by classes of maps $[X \xrightarrow{\ f\ } \mathfrak{M}]$ with X a variety. By the following theorem, the submodule $H_{\mathrm{reg}}(Q, I)$ is a subalgebra of $H(Q, I)$.

THEOREM 6.1.1 ([**Bri12**, Theorem 5.1]). *The submodule $H_{\mathrm{reg}}(Q, I)$ is closed under the convolution product:*

$$
H_{\mathrm{reg}}(Q, I) * H_{\mathrm{reg}}(Q, I) \subset H_{\mathrm{reg}}(Q, I).
$$

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The quotient

$$H_{sc}(Q, I) = H_{reg}(Q, I)/(\mathbb{L} - 1)H_{reg}(Q, I)$$

is a commutative $K(\mathrm{Var}/\mathbb{C})$-algebra.

It follows that one can define a Poisson bracket on $H_{reg}(Q, I)$ by the formula

$$\{f, g\} = (\mathbb{L} - 1)^{-1}(f * g - g * f).$$

6.2. Hall algebra scattering diagrams

We define the Hall algebra scattering diagrams associated to quivers with relations following [**Bri17**].

6.2.1. Definition. Let (Q, I) be a quiver with relations. Let $\mathfrak{g} = H(Q, I)_{>0}$ with the commutator bracket. Let $N = \mathbb{Z}^n$ where $n = |Q_0|$. The set of dimension vectors is identified with $\mathbb{N}^n \subset \mathbb{Z}^n$ and $N^+ = \mathbb{N}^n \setminus \{0\}$. We have that $\mathfrak{g}$ is N^+-graded. Consider the complete motivic Hall algebra

$$\widehat{H}(Q, I) = \prod_{d \in \mathbb{N}^n} K(\mathrm{St}/\mathfrak{M}_d).$$

The formal group $\hat{G}$ is identified with a multiplicative subgroup of $\widehat{H}$ by taking exponentials of elements in $\hat{\mathfrak{g}}$,

$$\exp \colon \hat{\mathfrak{g}} \to 1 + \widehat{H}(Q, I)_{>0} \subset \widehat{H}(Q, I), \quad x \mapsto \sum_{i=0}^{\infty} \frac{x^i}{n!}.$$

The characteristic function of the whole moduli stack $1_{\mathfrak{M}} = [\mathfrak{M} \xrightarrow{\mathrm{id}} \mathfrak{M}]$ belongs to $1 + \widehat{H}(Q, I)_{>0}$.

Definition 6.2.1. The *Hall algebra scattering diagram* $\mathfrak{D}_{Q,I}^{\mathrm{Hall}}$ of (Q, I) is defined to be the consistent $\mathfrak{g}$-SD corresponding to the group element $1_{\mathfrak{M}}$.

6.2.2. Wall-crossings. We denote the wall-crossing function by

$$\Phi = \Phi_{Q,I}^{\mathrm{Hall}} \colon M_{\mathbb{R}} \to \exp(\hat{\mathfrak{g}}).$$

We would like to describe for any $m \in M_{\mathbb{R}}$, the wall-crossing $\Phi(m)$. It turns out this is related to stability conditions on $\mathrm{mod}\,\mathbb{C}Q/I$.

Let θ be a King's stability condition on $\mathrm{mod}\,\mathbb{C}Q/I$. Precisely, the stability condition θ is an element in $M_{\mathbb{R}}$, viewed as a linear function on dimension vectors.

Definition 6.2.2 (King [**Kin94**]). A $\mathbb{C}Q/I$ module M is said to be θ-semistable (resp. θ-stable) if

(1) $\theta(\dim M) = 0$;

(2) for any proper non-zero submodule $M' \subset M$, $\theta(\dim M) \le 0$ (resp. $\theta(\dim M) < 0$).

All θ-semistable $\mathbb{C}Q/I$-modules form a full subcategory of $\mathrm{mod}\,\mathbb{C}Q/I$, parametrized by an open substack

$$\mathfrak{M}^{\theta\text{-ss}} = \coprod_{d \in N^{\oplus}} \mathfrak{M}_d^{\theta\text{-ss}}$$

of $\mathfrak{M}$ (see [**Kin94**]). The inclusion defines in the motivic Hall algebra the characteristic stack function of θ-semistable modules

$$1_{\mathfrak{M}^{\theta\text{-ss}}} := \left[\coprod_{d \in N^{\oplus}} \mathfrak{M}_d^{\theta\text{-ss}} \hookrightarrow \mathfrak{M} \right] \in 1 + H(Q, I)_{>0}.$$

THEOREM 6.2.3 (Bridgeland [**Bri17**]). *The wall-crossing function* $\Phi\colon M_{\mathbb{R}} \to 1 + \widehat{H}(Q, I)_{>0}$ *of the Hall algebra scattering diagram* $\mathfrak{D}_{Q,I}^{\mathrm{Hall}}$ *satisfies*

$$\Phi(\theta) = 1_{\mathfrak{M}^{\theta\text{-ss}}}$$

for any $\theta \in M_{\mathbb{R}}$.

Remark 6.2.4. The statement in [**Bri17**] is slightly different from Theorem 6.2.3 as it only considers wall-crossings for generic points on walls, whereas we describe them for all points in $M_{\mathbb{R}}$. However, the proof in [**Bri17**] still works here without any change.

6.2.3. Path-ordered product. Let $\mathcal{C}^+$ be the cone in $M_{\mathbb{R}}$ consisting of points $\theta \in M_{\mathbb{R}}$ such that $\theta(n) > 0$ for any $n \in N^+$. It is clear that $\Phi(\theta) = 1$ for any $\theta \in \mathcal{C}^+$, there is no non-zero semistable θ-modules.

Now let θ^+ be any point in $\mathcal{C}^+$ and $m \in M_{\mathbb{R}}$. Consider the path-ordered product $\mathfrak{p}_\gamma\left(\mathfrak{D}_{Q,I}^{\mathrm{Hall}} \right)$ in the sense of Proposition 2.3.19 where γ is a path from θ^+ to m. It has a moduli-theoretic interpretation.

Proposition 6.2.5 (Bridgeland [**Bri17**]). *Define a torsion class*

$$\mathcal{T}(m) := \{E \in \mathrm{mod}\,\mathbb{C}Q/I\colon \text{any quotient object } E \twoheadrightarrow F \text{ satisfies } m(\dim F) > 0\}.$$

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Then we have

$$\mathfrak{p}_\gamma \left(\mathfrak{D}_{Q,I}^{\mathrm{Hall}} \right) = 1_{\mathcal{T}(\theta)} := [\mathfrak{M}(\mathcal{T}(m)) \hookrightarrow \mathfrak{M}]$$

where $\mathfrak{M}(\mathcal{T}(m))$ is the moduli substack that parametrizes objects in the torsion class $\mathcal{T}(m)$.

6.3. Cases for quivers with potentials and the integration map

In this section, we deal with the case where the ideal I of relations is given by the Jacobian ideal $J(W)$ of some potential $W \in \widehat{\mathbb{C}Q}$. This case is relevant to cluster algebras and Donaldson-Thomas theory, and thus has been intensively studied (e.g., in [**DWZ08**, **JS11**]). We point out that the potential W may be formal in which case some additional care is needed; see Section 6.3.1. At the end of this section, we define the stability scattering diagram of a quiver with potential.

6.3.1. Modules of complete Jacobian algebras. In the context of this paper, we are working with the topological algebra $\mathcal{P}(Q, W)$ (see [**DWZ08**]) whose representation theory is slightly different from path algebras with relations. We construct moduli stacks of $\mathcal{P}(Q, W)$-modules in this section based on the construction in the last section.

The following lemma shows any finite-dimensional $\mathcal{P}(Q, W)$-module is nilpotent.

Lemma 6.3.1. *[**DWZ08**] For any dimension vector d, there exists some positive integer N_d, such that every d-dimensional $\widehat{\mathbb{C}Q}$-module M is annihilated by $\mathfrak{m}^{N_d}$, i.e.*

$$\mathfrak{m}^{N_d} M = 0.$$

Let ∂W be the finite set of all cyclic derivatives of W. Take truncations $\partial W(N_d)$ by $\mathfrak{m}^{N_d}$. The set $\partial W(N_d) \subset \mathbb{C}Q$ and $\mathfrak{m}^{N_d}$ together generate a 2-sided ideal $I(N_d)$ of $\mathbb{C}Q$. The following lemma is straightfoward.

Lemma 6.3.2. *We have an equivalence between the category of d-dimensional representations of $(Q, I(N_d))$ and the category of d-dimensional $\mathcal{P}(Q, W)$-modules, i.e.*

$$\mathrm{mod}_d\, \mathbb{C}Q/I(N_d) \cong \mathrm{mod}_d\, \mathcal{P}(Q, W).$$

We define $\mathfrak{M}(Q, W)_d$ the moduli stack of $\mathrm{mod}_d\, \mathcal{P}(Q, W)$ to be $\mathfrak{M}(Q, I(N_d))_d$. Note that by this definition, the stack $\mathfrak{M}(Q, W)_d$ depends on the choice of N_d. However, the set of $\mathbb{C}$-points of $\mathfrak{M}(Q, W)_d$ is in bijection with the set of isomorphism classes of objects in $\mathrm{mod}_d\, \mathcal{P}(Q, W)$. Thus this does not make any difference in view of motivic Hall algebra.

We define the motivic Hall algebra as before (with the convolution product)

$$H(Q, W) := \bigoplus_{d \in \mathbb{N}^n} K(\mathrm{St}/\mathfrak{M}(Q, W)_d).$$

6.3.2. Hall algebra scattering diagrams for quivers with potentials. We consider fixed data Γ with $N^\circ = N$. Let $\mathbf{s}$ be a seed for Γ and W be a potential for $Q = Q(\mathbf{s})$.

The set of dimension vectors is identified with $N^\oplus = N_{\mathbf{s}}^\oplus \subset N$. The Hall algebra $H(\mathbf{s}, W) := H(Q, W)$ is thus graded by $N^\oplus$. We take

$$\mathfrak{g} = \mathfrak{g}_{\mathbf{s},W}^{\mathrm{Hall}} := H(\mathbf{s}, W)_{>0}$$

an N^+-graded Lie algebra with commutator bracket.

As in Section 6.2, we have the characteristic function

$$1_{\mathfrak{M}(\mathbf{s},W)} = 1_{\mathfrak{M}(Q,W)} := \left[\mathfrak{M}(Q, W) \xrightarrow{\ \mathrm{id}\ } \mathfrak{M}(Q, W) \right]$$

representing the whole module category $\mathrm{mod}\,\mathcal{P}(Q, W)$.

Definition 6.3.3. The *Hall algebra scattering diagram* $\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Hall}}$ is defined to be the unique consistent $\mathfrak{g}$-SD corresponding to the group element

$$1_{\mathfrak{M}(\mathbf{s},W)} \in \hat{G} = 1 + \widehat{H}(Q, W)_{>0}.$$

6.3.3. The integration map. We now review the integration map that we will apply to define stability scattering diagrams.

Recall that we have the subalgebra

$$H_{\mathrm{reg}}(\mathbf{s}, W) \subset H(\mathbf{s}, W).$$

We define

$$\mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}} := \frac{H_{\mathrm{reg}}(\mathbf{s}, W)_{>0}}{\mathbb{L} - 1}.$$

It follows from the properties of $H_{\mathrm{reg}}(\mathbf{s}, W)$ that $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}}$ is an N^+-graded Lie subalgebra of $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{Hall}}$. To summarize, we have the following theorem.

THEOREM 6.3.4 ([**JS11**], [**Bri17**], [**Nag13**, theorem 7.4]). *We have the following properties regarding $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}}$ and $H_{\mathrm{reg}}(\mathbf{s}, W)$.*

(1) *The submodule $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}}$ is an $N_{\mathbf{s}}^{+}$-graded Lie subalgebra of $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{Hall}}$.*

(2) *The submodule $H_{\mathrm{reg}}(\mathbf{s},W)$ is a subalgebra of $H(\mathbf{s},W)$. It is a Poisson algebra with the bracket*

$$\{a,b\} = \frac{a*b - b*a}{\mathbb{L}-1}.$$

(3) *There is an $N_{\mathbf{s}}^{\oplus}$-graded Poisson homomorphism (the integration map)*

$$I \colon H_{\mathrm{reg}}(\mathbf{s},W) \to \mathbb{Q}[N_{\mathbf{s}}^{\oplus}], \quad I\left([X \to \mathfrak{M}_d]\right) = e(X)x^d$$

where $e(X)$ is the Euler characteristic of X_{an}.

By the above theorem, we have that $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}}$ is isomorphic to $(H_{\mathrm{reg}}(\mathbf{s},W)_{>0}, \{\,,\,\})$ as Lie algebras by identifying $x/(\mathbb{L}-1)$ with $x \in H_{\mathrm{reg}}(\mathbf{s},W)_{>0}$. Therefore the integration map I induces a Lie algebra homomorphism

(6.3.1) $$\mathcal{I} \colon \mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}} \to \mathfrak{g}_{\mathbf{s}} = \mathbb{Q}[N_{\mathbf{s}}^{+}], \quad [X \to \mathfrak{M}_d]/(\mathbb{L}-1) \mapsto e(X)x^d.$$

6.3.4. Absence of poles. The following *absence of poles* theorem is due to Joyce; see also section 3.2 and definition 7.15 in [**JS11**].

THEOREM 6.3.5 ([**Joy07**, theorem 8.7]). *Let $(\mathbf{s},W)$ be a seed with potential. For any $m \in M_{\mathbb{R}}$, we write the characteristic function of θ-semistable modules $1_{\mathfrak{M}^{\theta\text{-ss}}} = 1 + \sigma$ for $\sigma \in \hat{\mathfrak{g}}_{\mathbf{s},W}^{\mathrm{Hall}}$. Then we have that*

$$\log\left(1_{\mathfrak{M}^{\theta\text{-ss}}}\right) := \sigma - \frac{1}{2}\sigma * \sigma + \cdots + \frac{(-1)^{n-1}}{n}\sigma * \cdots * \sigma + \cdots$$

belongs to the Lie subalgebra $\hat{\mathfrak{g}}_{\mathbf{s},W}^{\mathrm{reg}} \subset \hat{\mathfrak{g}}_{\mathbf{s},W}^{\mathrm{Hall}}$.

In the last section, $\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Hall}}$ is defined as a $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{Hall}}$-SD. The above theorem, in particular, shows that $\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Hall}}$ is also a $\mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}}$-SD.

6.3.5. Stability scattering diagrams. Recall Equation (6.3.1) that we have an $N_{\mathbf{s}}^{+}$-graded Lie algebra homomorphism

$$\mathcal{I} \colon \mathfrak{g}_{\mathbf{s},W}^{\mathrm{reg}} \to \mathfrak{g}_{\mathbf{s}}.$$

By abuse of notation, we will denote the maps of corresponding pro-unipotent groups still by

$$\mathcal{I} \colon \exp(\hat{\mathfrak{g}}_{\mathbf{s},W}^{\mathrm{reg}}) \to \exp(\hat{\mathfrak{g}}_{\mathbf{s}})$$

Recall the notation we set up in Section 2.3.3.

Definition 6.3.6 (Stability scattering diagrams)**.** Let $(\mathbf{s}, W)$ be a seed with potential. We define the *stability scattering diagram* of $(\mathbf{s}, W)$ to be the consistent $\mathfrak{g}_{\mathbf{s}}$-SD

$$\mathfrak{D}^{\mathrm{Stab}}_{\mathbf{s},W} := \mathcal{I}\left(\mathfrak{D}^{\mathrm{Hall}}_{\mathbf{s},W}\right),$$

i.e. the consistent $\mathfrak{g}_{\mathbf{s}}$-SD corresponding to $\mathcal{I}\left(1_{\mathfrak{M}(\mathbf{s},W)}\right) \in \exp(\hat{\mathfrak{g}}_{\mathbf{s}})$.

Remark 6.3.7. The stability scattering diagram $\mathfrak{D}^{\mathrm{Stab}}_{\mathbf{s},W}$ is defined by Bridgeland in [**Bri17**, section 11] for a polynomial potential W. However, the definition can be easily extended to any formal potential, as previously discussed in this section, especially in Section 6.3.1.

Example 6.3.8. Choosing a generic point m in $[S_i]^\perp \subset M_{\mathbb{R}}$ for some vertex $i \in \{1, \dots, n\}$ of the quiver, the subcategory of m-semistable $\mathcal{P}(Q, W)$-modules is generated by the simple module S_i. Then we have

$$(6.3.2) \qquad 1_{\mathfrak{M}^{m\text{-ss}}} = \left[\coprod_{k \geq 0} BGL_k \to \coprod_{k \geq 0} \mathfrak{M}_{ks_i}\right]$$

where the map is an isomorphism. Note that we have that

$$[BGL \to \mathfrak{M}_k] = [\mathrm{GL}_k]^{-1} \cdot [\mathrm{pt} \to \mathfrak{M}_k]$$

and that in $K(\mathrm{Var}/\mathbb{C})$,

$$[\mathrm{GL}_k] = \mathbb{L}^{k(k-1)/2}(\mathbb{L} - 1)^k \prod_{i=1}^{k-1}[\mathbb{P}^i].$$

We also have that in the motivic Hall algebra

$$[\mathrm{pt} \to \mathfrak{M}_{d_1}] * [\mathrm{pt} \to \mathfrak{M}_{d_2}] = \mathbb{L}^{-d_1 d_2}[\mathrm{pt} \to \mathfrak{M}_{d_1+d_2}]$$

and thus

$$\mathbb{L}^{-d_1^2/2}[\mathrm{pt} \to \mathfrak{M}_{d_1}] * \mathbb{L}^{-d_2^2/2}[\mathrm{pt} \to \mathfrak{M}_{d_2}] = \mathbb{L}^{-(d_1+d_2)^2/2}[\mathrm{pt} \to \mathfrak{M}_{d_1+d_2}].$$

Let $x = \mathbb{L}^{-1/2}[\mathrm{pt} \to \mathfrak{M}_1]$ and we can rewrite (6.3.2) as

$$1_{\mathfrak{M}^{m\text{-ss}}} = \sum_{k=0}^{\infty} \frac{\mathbb{L}^{k^2/2}x^k}{[\mathrm{GL}_k]} = \sum_{k=0}^{\infty} \frac{\mathbb{L}^{k/2}x^k}{(\mathbb{L}-1)^k \prod_{i=1}^{k-1}[\mathbb{P}^i]}.$$

By q-binomial theorem, we have the following identity

$$\sum_{k=0}^{\infty} \frac{\mathbb{L}^{k/2} x^k}{(\mathbb{L}-1)^k \prod_{i=1}^{k-1}[\mathbb{P}^i]} = \prod_{k=0}^{\infty} \frac{1}{(1+\mathbb{L}^{k+1/2}x)}.$$

Taking logarithm, we have

$$\log\left(1_{\mathfrak{M}^{m\text{-ss}}}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\mathbb{L}^{-k(k-1)/2}[\text{pt} \to \mathfrak{M}_k]}{k(\mathbb{L}^k-1)},$$

which belongs to $\hat{\mathfrak{g}}_{\mathbf{s},W}^{\text{reg}}$ guaranteed by Joyce's absence of poles Theorem 6.3.5. Applying the integration map, we conclude that

$$\mathcal{I}(\log(1_{\mathfrak{M}^{m\text{-ss}}})) = -\text{Li}_2(-x^{s_i}).$$

6.4. Mutations of stability scattering diagrams

In this section, we state the main result Theorem 6.4.1 of this chapter: the mutation invariance of stability scattering diagrams.

Recall that we have fixed data Γ such that $N^\circ = N$. Let $\mathbf{s}$ be a Γ-seed and W be a potential for $Q(\mathbf{s})$. Suppose that $(\mathbf{s}, W)$ is k-mutable for some $k \in I$. Consider the stability scattering diagrams $\mathfrak{D}_{\mathbf{s},W}^{\text{Stab}}$ and $\mathfrak{D}_{\mu_k^{\varepsilon}(\mathbf{s},W)}^{\text{Stab}}$ for $\varepsilon \in \{+,-\}$. Denote their wall-crossing functions by

$$\Phi_{\mathbf{s},W}^{\text{Stab}} \colon M_{\mathbb{R}} \to \exp(\hat{\mathfrak{g}}_{\mathbf{s}}), \quad \Phi_{\mu_k^{\varepsilon}(\mathbf{s},W)}^{\text{Stab}} \colon M_{\mathbb{R}} \to \exp\left(\hat{\mathfrak{g}}_{\mu_k(\mathbf{s})}\right)$$

respectively.

THEOREM 6.4.1 (Stability mutation invariance). *We have the following results regarding* $\mathfrak{D}_{\mathbf{s},W}^{\text{Stab}}$ *and* $\mathfrak{D}_{\mu_k^{\varepsilon}(\mathbf{s},W)}^{\text{Stab}}$.

(1) *At a generic point* $m \in s_k^{\perp}$, *the wall-crossing is given by*

$$\Phi_{\mathbf{s},W}^{\text{Stab}}(m) = \mathbb{E}(s_k).$$

(2) *For any* $m \in \mathcal{H}_{\mathbf{s}}^{k,-\varepsilon}$, *we have*

$$\Phi_{\mathbf{s},W}^{\text{Stab}}(m) = \Phi_{\mu_k^{\varepsilon}(\mathbf{s},W)}^{\text{Stab}}(m).$$

The part (1) follows directly from the computation in Example 6.3.8. The proof of part (2) is postponed to Section 6.7.

Recall the mutation invariance Theorem 4.3.1 of cluster scattering diagrams. One sees that the cluster SD $\mathfrak{D}_\mathbf{s}^\text{Cl}$ behaves in the same pattern as the stability one. Thus we have the following theorem. Recall the relevant notations in Section 4.4.1.

THEOREM 6.4.2 (Mutation of stability scattering diagram). *The cluster scattering diagram* $\mathfrak{D}_{\mu_k^-(\mathbf{s})}^\text{Stab}$ *has the following description in terms of* $\mathfrak{D}_{\mathbf{s},W}^\text{Stab}$.

(1) *At a generic* $m \in s_k^\perp$, *the wall-crossing is given by*

$$\Phi_{\mu_k^+(\mathbf{s},W)}^\text{Stab}(m) = \mathbb{E}(-s_k).$$

(2) *On* $\mathcal{H}_\mathbf{s}^{k,+} \cup \mathcal{H}_\mathbf{s}^{k,-} \subset M_\mathbb{R}$, *we have*

$$\Phi_{\mathbf{s},W}^\text{Stab}(m) = \left(T_k^+\right)^* \circ \Phi_{\mu_k^+(\mathbf{s},W)}^\text{Stab} \circ T_k^+(m),$$

where $(T_k^+)^*$ *denotes the induced group homomorphism on its domain of linearity.*

(3) *The piecewise linear map* T_k^+ *induces an isomorphism between the canonical profinite cone complexes* $\mathfrak{S}_{\mathbf{s},W}^\text{Stab}$ *and* $\mathfrak{S}_{\mu_k^+(\mathbf{s},W)}^\text{Stab}$.

PROOF. We refer to the proof of Theorem 4.4.1 for comparison. Part (1), the same as part (1) of Theorem 6.4.1, follows directly from Example 6.3.8. Note that here we are in the skew-symmetric case with $N^\circ = N$. Thus we have $s_k = \tilde{s}_k$.

Part (2) follows from part (2) of Theorem 6.4.1 and an isomorphism between the stability SDs $\mathfrak{D}_{\mu_k^+(\mathbf{s},W)}^\text{Stab}$ and $\mathfrak{D}_{\mu_k^+(\mathbf{s},W)}^\text{Stab}$. See a discussion of this type of isomorphism in the cluster case in Section 4.4.1.

The proof of part (3) is the same as the proof of part (3) of Theorem 4.4.1. □

Notice that in the above theorems, the potential W is required to be k-mutable. In order to mutate an SP $(\mathbf{s}, W)$ indefinitely, we need to require it to be *non-degenerate*.

Definition 6.4.3. Let $\mathbf{k} = \{k_1, \ldots, k_l\}$ be a sequence of indices. We say a QP (Q, W) to be $\mathbf{k}$-*mutable* if for any $1 \leq l' \leq l$, the QP $\mu_{k_{l'-1}} \ldots \mu_{k_1}(Q, W)$ is $k_{l'}$-mutable. We say (Q, W) to be *non-degenerate* if it is $\mathbf{k}$-mutable for any sequence $\mathbf{k}$. An SP is said to be $\mathbf{k}$-mutable or non-degenerate if the associated QP is.

For a non-degenerate SP $(\mathbf{s}, W)$, it turns out the stability SD $\mathfrak{D}_{\mathbf{s},W}^\text{Stab}$ possesses the same cluster complex structure as of the cluster SD $\mathfrak{D}_\mathbf{s}^\text{Cl}$.

THEOREM 6.4.4. *Let* $(\mathbf{s}, W)$ *be a non-degenerate SP.*

(1) *The positive and negative cluster complexes* $\Delta_{\mathbf{s}}^{\pm}$ *(see Definition 4.4.6) are both cone subcomplexes of the profinite cone complex* $\mathfrak{S}_{\mathbf{s},W}^{\mathrm{Stab}}$.

(2) *Let* $\mathbf{k}$ *be a sequence of indices and* $i \in I$. *The wall-crossing at the facet dual to* $c_{\mathbf{k},i}$ *of the maximal cone* $G_{\mathbf{k}}^{+}$ *is given by* $\mathbb{E}(|c_{\mathbf{k},i}|)$.

Definition 6.4.5. We say a seed $\mathbf{s}$ has a *reddening sequence* if the negative chamber $\mathcal{C}_{\mathbf{s}}^{-}$ belongs to the positive cluster complex $\Delta_{\mathbf{s}}^{+}$. In fact, this means there is a sequence $\mathbf{k}$ of indices such that $G_{\mathbf{k}}^{+} = \mathcal{C}_{\mathbf{s}}^{-}$. The sequence $\mathbf{k}$ is the reddening sequence.

Corollary 6.4.6. *If a seed* $\mathbf{s}$ *has a reddening sequence, then for any non-degenerate potential* W *of* $Q(\mathbf{s})$, *we have*

$$\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}} = \mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}}.$$

PROOF. Let γ be a path going from the positive chamber to the negative chamber by crossing the chamber walls in the order of $\mathbf{k}$. By (2) of Theorem 6.4.4, we have

$$\mathfrak{p}_{\gamma}\left(\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}\right) = \mathfrak{p}_{\gamma}\left(\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}}\right).$$

However, we know from Theorem 2.2.5 that this element uniquely determines a consistent scattering diagram. The result follows. $\qquad\square$

6.5. Generalized reflection functors

In this section, we introduce the generalized reflection functors Definition 6.5.3 between module categories

$$F_k^{+}, \ F_k^{-}: \ \mathrm{mod}\,\mathcal{P}(Q,W) \to \mathrm{mod}\,\mathcal{P}(\mu_k(Q,W)).$$

They are necessary ingredients in the proof of Theorem 6.4.1. These functors are inspired by operations called *mutations of decorated representations* in [**DWZ08**] and generalize the original reflection functors of [**BGP73**].

Recall that there is an intermediate QP $\tilde{\mu}_k(Q,W)$ (Definition 5.1.7) in the construction of $\mu_k(Q,W)$. There is an equivalence between module categories of the algebras $\mathcal{P}(\tilde{\mu}_k(Q,W))$ and $\mathcal{P}(\mu_k(Q,W))$ induced by the isomorphism of algebras in Theorem 5.1.6. The functors $F_k^{\pm}$ will be

defined to be the following functors

$$\widetilde{F}_k^{\pm} \colon \operatorname{mod} \mathcal{P}(Q, W) \to \operatorname{mod} \mathcal{P}(\tilde{\mu}_k(Q, W))$$

post-composed by this equivalence. The functors $\widetilde{F}_k^{\pm}$ are constructed as follows.

6.5.1. The construction of $\widetilde{F}_K^{+}$. Let M be a $\mathcal{P}(Q, W)$-module regarded as a finite-dimensional nilpotent representation of Q annihilated by cyclic derivatives

$$\partial W = \{\partial_a W \mid a \in Q_1\}.$$

See [**DWZ08**, Definition 10.1] for the relevant definitions.

We denote by M_i the vector space $e_i M$ for a vertex $i \in Q_0$ (where $e_i \in J$ is the idempotent corresponding to the vertex i), and by a_M the action of an arrow $a \in Q_1$ on M (or the restriction linear map on $M_{s(a)}$ with the target $M_{t(a)}$).

There is a diagram of vector spaces:

where

$$M_{\text{in}} \coloneqq \bigoplus_{a \in Q_1 \colon t(a)=k} M_{s(a)}, \quad M_{\text{out}} \coloneqq \bigoplus_{\beta \in Q_1, \ s(\beta)=k} M_{t(\beta)}$$

and (the components of) the linear maps of the upper triangle are given by

$$\alpha \coloneqq (a_M)_{a \colon t(a)=k}, \quad \beta \coloneqq (b_M)_{b \colon s(b)=k}, \quad \gamma \coloneqq (\partial_{ba} W)_{(a,b) \colon t(a)=s(b)=k}.$$

Here $\partial_{ba} W$ is obtained by taking the cyclic derivative with respect to the composition ba. One can check that

$$\alpha \circ \gamma = 0, \quad \gamma \circ \beta = 0;$$

see [**DWZ08**, Lemma 10.6]. This implies the map γ factors through q, i.e. we have

$$\gamma = \phi \circ q$$

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for a unique map

$$\phi\colon \operatorname{coker}\beta \to M_{\text{in}}.$$

Note that there are natural embeddings $\iota_b\colon M_{t(b)} \to M_{\text{out}}$ and projections $\pi_a\colon M_{\text{in}} \to M_{s(a)}$.

We first define $M' = \widetilde{F}_k^+(M)$ as a representation of $Q' = \tilde{\mu}_k(Q)$ as follows.

(1) We put $M_k' := \operatorname{coker}\beta$ and $M_i' := M_i$ for $i \neq k$.

(2) For each $b^\star\colon j \to k$, let $b_{M'}^\star$ be the composition (mind the sign $-q$)

$$-q \circ \iota_b\colon M_j \xrightarrow{\iota_b} M_{\text{out}} \xrightarrow{-q} \operatorname{coker}\beta;$$

For each $a^\star\colon k \to i$, let $a_{M'}^\star$ be the composition

$$\pi_a \circ \phi\colon \operatorname{coker}\beta \xrightarrow{\phi} M_{\text{in}} \xrightarrow{\pi_a} M_i.$$

(3) For each $[ba]\colon i \to j$, we put

$$[ba]_{M'} := b_M \circ a_M.$$

(4) For any arrow c not incident to k, let

$$c_{M'} := c_M.$$

Proposition 6.5.1. *The above construction does define a representation M' of $\tilde{\mu}_k(Q, W)$, i.e. M' is a $\mathcal{P}(\tilde{\mu}_k(Q, W))$-module. Moreover, this naturally induces an additive functor*

$$\widetilde{F}_k^+\colon \operatorname{mod}\mathcal{P}(Q, W) \to \operatorname{mod}\mathcal{P}(\tilde{\mu}_k(Q, W))$$

such that $\widetilde{F}_k^+(M) = M'$.

PROOF. By construction, M' is a finite-dimensional representation of $Q' = \tilde{\mu}_k Q$. Furthermore, it is nilpotent (which follows from the fact that M is nilpotent over $\mathbb{C}Q$), thus a module over the complete path algebra $\widehat{\mathbb{C}Q'}$.

Now we need to show that M' is annihilated by $\partial_\gamma W'$ for any arrow γ of Q'. This is essentially checked in [**DWZ08**, proposition 10.7] although the *mutation of decorated representations* define there is different from ours (in particular on the vector space M_k'). We prove this fact in the following.

If $\gamma \in Q_1'$ comes from $\gamma \in Q_1$ that is not incident to k, then we have that

$$(\partial_\gamma W')_{M'} \colon M'_{s(\gamma)} \to M'_{t(\gamma)}$$

is equal to $(\partial_\gamma W)_{M'} = 0$.

If γ is of the form $[ba]$ for $a \colon i \to k$ and $b \colon k \to j$, then we have

$$\partial_{[ba]} W' = \partial_{ba} W + a^\star b^\star.$$

Note that $a^\star_{M'} b^\star_{M'}$ is precisely given by

$$(-\partial_{ba} W)_M = (-\partial_{ba} W)_{M'} \colon M'_j \to M'_i.$$

Thus we have $(\partial_{[ba]} W')_{M'} = 0$.

If $\gamma = a^\star$ for some $a \colon i \to k$ in Q_1, then we have

$$(\partial_{a^\star} W')_{M'} = \sum_{b \colon s(b)=k} b^\star_{M'} [ba]_{M'} = \left(\sum_{b \colon s(b)=k} b^\star_{M'} b_M \right) a_M \colon M'_i \to M'_k$$

in which the map

$$\sum_{b \colon s(b)=k} b^\star_{M'} b_M \colon M_k \to M'_k = \operatorname{coker} \beta$$

goes through the exact sequence

$$M_k \xrightarrow{\beta} \bigoplus_{b \colon s(b)=k} M_{s(b)} \xrightarrow{q} \operatorname{coker} \beta \to 0,$$

that is

$$\sum_{b \colon s(b)=k} b^\star_{M'} b_M = (-q) \circ \beta = 0$$

Therefore we have $(\partial_{a^\star} W')_{M'} = 0$.

If $\gamma = b^\star$ for some $b \colon k \to j$ in Q_1, then we have

$$(\partial_{b^\star} W')_{M'} = \sum_{a \colon t(a)=k} [ba]_{M'} a^\star_{M'} = b_M \left(\sum_{a \colon t(a)=k} a_M a^\star_{M'} \right) \colon M'_k \to M'_j.$$

Note that we have

$$\sum_{a \colon t(a)=k} a_M a^\star_{M'} = \sum_{a \colon t(a)=k} a_M \left(\sum_{c \colon s(c)=k} \partial_{ca} W \right)_M = \left(\sum_{c \colon s(c)=k} \partial_c W \right)_M = 0.$$

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Thus $(\partial_{b^\star} W')_{M'} = 0$.

To define the action of $\widetilde{F}_k^+$ on the space of morphisms, let $f\colon U \to V$ be a morphism between $\mathcal{P}(Q,W)$-modules, and we construct a morphism $\widetilde{F}_k^+ f\colon \widetilde{F}_k^+ U \to \widetilde{F}_k^+ V$ by giving maps between vector spaces associated to the vertices of Q'. We keep the maps $f_i\colon M_i \to V_i$ unchanged if $i \neq k$ and let

$$(\widetilde{F}_k^+ f)_k \colon (\widetilde{F}_k^+ U)_k \to (\widetilde{F}_k^+ V)_k$$

be the map naturally induced between cokernels. Then by construction, these maps between vector spaces intertwine with the actions of arrows in Q' and thus form a morphism $\widetilde{F}_k^+(f)$ of representations.

Other requirements of an additive functor are easy to check. $\qquad\square$

6.5.2. The construction of $\widetilde{F}_k^-$. To define $M^\circ = \widetilde{F}_k^-(M) \in \operatorname{mod} \mathcal{P}(Q',W')$, we use the following diagram of vector spaces.

$$
\begin{array}{ccc}
 & M_k & \\
{\scriptstyle \alpha}\nearrow & & \searrow{\scriptstyle \beta} \\
M_{\text{in}} \xleftarrow{\quad\gamma\quad} & & M_{\text{out}} \\
{\scriptstyle r}\nwarrow & & \swarrow{\scriptstyle \psi} \\
 & \ker \alpha &
\end{array}
$$

As mentioned earlier, we have

$$\alpha \circ \gamma = 0.$$

Thus the map γ factors through $\ker \alpha$, i.e. we have

$$\gamma = r \circ \psi$$

for a unique map

$$\psi\colon M_{\text{out}} \to \ker \alpha.$$

Define a representation M° of Q' as follows.

(1) We put $M_k^\circ := \ker \alpha$ and $M_i^\circ := M_i$ for $i \neq k$.

(2) For each $a^\star \colon k \to i$, let $a_{M^\circ}^\star$ be the composition

$$\pi_a \circ r\colon \ker \alpha \xrightarrow{r} M_{\text{in}} \xrightarrow{\pi_a} M_i;$$

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For each $b^\star\colon j \to k$, let $b^\star_{M^\circ}$ be the composition

$$(-\psi) \circ \iota_b\colon M_j \xrightarrow{\iota_b} M_{\text{out}} \xrightarrow{-\psi} \ker\alpha.$$

(3) For each $[ba]\colon i \to j$, we put

$$[ba]_{M^\circ} := b_M \circ a_M.$$

(4) For any arrow c not incident to k, let

$$c_{M^\circ} := c_M.$$

Proposition 6.5.2. *The above construction gives a representation M° of $\tilde{\mu}_k(Q, W)$, i.e. M° is a $\mathcal{P}(\tilde{\mu}_k(Q, W))$-module. Moreover, this naturally induces an additive functor*

$$\widetilde{F}_k^-\colon \operatorname{mod}\mathcal{P}(Q, W) \to \operatorname{mod}\mathcal{P}(\tilde{\mu}_k(Q, W))$$

such that $\widetilde{F}_k^-(M) = M^\circ$.

PROOF. The proof is analogous to the proof of Proposition 6.5.1. We leave the details to the reader. Similarly, this defines a functor since the map between kernels is naturally induced. $\square$

6.5.3. Generalized reflection functors and their properties. Now we have two functors

$$\widetilde{F}_k^\pm\colon \operatorname{mod}\mathcal{P}(Q, W) \to \operatorname{mod}\mathcal{P}(\tilde{\mu}_k(Q, W)).$$

To extend the target of these functors to $\operatorname{mod}\mathcal{P}(\mu_k(Q, W))$, one uses the following equivalence

$$R\colon \operatorname{mod}\mathcal{P}(\tilde{\mu}_k(Q, W)) \to \operatorname{mod}\mathcal{P}(\mu_k(Q, W))$$

induced by algebra isomorphism described in Theorem 5.1.6.

Definition 6.5.3 (Generalized reflection functors). For a k-mutable QP (Q, W), we define two functors between module categories:

$$F_k^\pm := R \circ \widetilde{F}_k^\pm\colon \operatorname{mod}\mathcal{P}(Q, W) \to \operatorname{mod}\mathcal{P}(\mu_k(Q, W)).$$

These are what we call the *generalized reflection functors* or *mutations of representations*.

In what follows, we work with SPs instead of QPs. Note that two mutations $\mu_k^\pm$ of SPs give the same QP:

$$\mu_k(Q(\mathbf{s}), W) \cong (Q(\mu_k^\pm \mathbf{s}), \mu_k W).$$

For an SP $(\mathbf{s}, W)$, denote the module category $\operatorname{mod} \mathcal{P}(Q(\mathbf{s}), W)$ by $\operatorname{mod} \mathcal{P}(\mathbf{s}, W)$. As the notation of the generalized reflection functors suggests, we consider functors (with the equivalent target category)

$$(6.5.1) \qquad\qquad F_k^+ : \operatorname{mod} \mathcal{P}(\mathbf{s}, W) \to \operatorname{mod} \mathcal{P}(\mu_k^+(\mathbf{s}, W))$$

and respectively

$$F_k^- : \operatorname{mod} \mathcal{P}(\mathbf{s}, W) \to \operatorname{mod} \mathcal{P}(\mu_k^-(\mathbf{s}, W)).$$

The advantage of working with SPs instead of QPs, which we shall explain below, is that how dimension vectors of particular modules get transformed under reflection functors is already encoded in the mutation of seeds.

For any SP $(\mathbf{s}, W)$, we have the natural identification of the Grothendieck group with the lattice N,

$$K_0(\operatorname{mod} \mathcal{P}(\mathbf{s}, W)) \cong N$$

via $[S_i] = s_i$ (as the only simple modules are S_i's because every finite dimensional $\mathcal{P}$-module is nilpotent). When V is a $\mathcal{P}(\mathbf{s}, W)$-module, we denote its class in N by

$$[V]_\mathbf{s} = \sum_{i=1}^n \dim_K V_i \cdot s_i.$$

This notation is sensitive to the seed $\mathbf{s}$.

Let $(\mathbf{s}', W') = \mu_k^+(\mathbf{s}, W)$ and $(\mathbf{s}'', W'') = \mu_k^-(\mathbf{s}, W)$. Then $K_0(\operatorname{mod} \mathcal{P}(\mathbf{s}', W'))$ is also identified with N via $[S_i']_{\mathbf{s}'} = s_i'$; similarly for $(\mathbf{s}'', W'')$.

Straightforward calculations show that

$$(6.5.2) \qquad\qquad [S_i]_\mathbf{s} = \begin{cases} [F_k^+ S_i]_{\mathbf{s}'} = [F_k^- S_i]_{\mathbf{s}''} & \text{for } i \neq k, \\ -[S_k']_{\mathbf{s}'} = -[S_k'']_{\mathbf{s}''} & \text{for } i = k. \end{cases}$$

This observation can be generalized (see (5) of Theorem 6.5.4). There are actually subcategories whose objects' dimension vectors (their classes in the Grothendieck group) in N are invariant under mutations.

Denote $\operatorname{mod}\mathcal{P}(\mathbf{s}, W)$ and $\operatorname{mod}\mathcal{P}(\mathbf{s}', W')$ by $\mathcal{A}$, $\mathcal{A}'$ respectively. We define the following full subcategories of $\mathcal{A}$ (and of $\mathcal{A}'$ accordingly)

$$\mathcal{A}_{k,-} = {}^{\perp}S_k := \{M \in \mathcal{A} \mid \operatorname{Hom}(M, S_k) = 0\},$$

$$\mathcal{A}_{k,+} = S_k^{\perp} := \{M \in \mathcal{A} \mid \operatorname{Hom}(S_k, M) = 0\}$$

and denote by $\langle S_k \rangle$ the full subcategory of $\mathcal{A}$ consisting of direct sums of S_k.

Note that $\mu_k^{-}(\mathbf{s}', W')$ is right-equivalent to $(\mathbf{s}, W)$. So the functor F_k^{-} can be regarded as from $\mathcal{A}'$ to $\mathcal{A}$.

THEOREM 6.5.4. *The generalized reflection functors* $F_k^{+} : \mathcal{A} \to \mathcal{A}'$ *and* $F_k^{-} : \mathcal{A}' \to \mathcal{A}$ *have the following properties.*

(1) F_k^{+} *is right exact and* F_k^{-} *is left exact. They form an adjoint pair, i.e. there is a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{A}}\left(M, F_k^{-}N\right) \cong \operatorname{Hom}_{\mathcal{A}'}\left(F_k^{+}M, N\right)$$

for any $M \in \mathcal{A}$ *and* $N \in \mathcal{A}'$.

(2) $F_k^{+}S_k = 0$ *and* $F_k^{-}S_k' = 0$.

(3) $F_k^{+}(\mathcal{A}_{k,+}) \subset \mathcal{A}'_{k,-}$ *and* $F_k^{-}(\mathcal{A}'_{k,-}) \subset \mathcal{A}_{k,+}$.

(4) *The restrictions* $F_k^{+} : \mathcal{A}_{k,+} \to \mathcal{A}'_{k,-}$ *and* $F_k^{-} : \mathcal{A}'_{k,-} \to \mathcal{A}_{k,+}$ *are quasi-inverse equivalences. Moreover, these functors preserve short exact sequences.*

(5) *For any* V *in* $\mathcal{A}_{k,+}$, *we have* $[V]_{\mathbf{s}} = [F_k^{+}V]_{\mathbf{s}'}$ *where*

$$\left[F_k^{+}V\right]_{\mathbf{s}'} := \dim_K\left(F_k^{+}V\right)_i s_i' \in N.$$

It follows that for any W *in* $\mathcal{A}'_{k,-}$, *then* $[W]_{\mathbf{s}'} = [F_k^{-}W]_{\mathbf{s}} \in N$.

PROOF. For the first part in (1), it suffices to prove $\widetilde{F}_k^{+}$ is right exact and $\widetilde{F}_k^{-}$ is left exact. Suppose we have an exact sequence

$$U \to V \to W \to 0.$$

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in $\mathcal{A}$. This implies we have an exact sequence of complexes of vector spaces concentrated in degree 1 and 0 (the third and second row of the following diagram) which induces an exact sequence on the H_0's (the first row)

$$
\begin{array}{ccccccc}
(\widetilde{F}_k^+ U)_k & \longrightarrow & (\widetilde{F}_k^+ V)_k & \longrightarrow & (\widetilde{F}_k^+ W)_k & \longrightarrow & 0 \\
\big\uparrow & & \big\uparrow & & \big\uparrow & & \\
U_{\mathrm{out}} & \longrightarrow & V_{\mathrm{out}} & \longrightarrow & W_{\mathrm{out}} & \longrightarrow & 0 \\
\beta_U \big\uparrow & & \beta_V \big\uparrow & & \beta_W \big\uparrow & & \\
U_k & \longrightarrow & V_k & \longrightarrow & W_k & \longrightarrow & 0.
\end{array}
$$

This shows that $\widetilde{F}_k^+$ is right exact, and so is F_k^+. The left exactness of F_k^- is proven similarly.

For the adjointness, let V be in $\mathcal{A}$ and W be in $\mathcal{A}'$. We need to show that there is a natural isomorphism

$$
\mathrm{Hom}_{\mathcal{A}}(V, F_k^- W) \cong \mathrm{Hom}_{\mathcal{A}'}(F_k^+ V, W).
$$

The space $\mathrm{Hom}_{\mathcal{A}}(V, F_k^- W)$ is given by the space of linear maps

$$
(f_i \colon V_i \to (F_k^- W)_i)_{i=1}^n
$$

intertwining with the action of $\mathcal{P}(\mathbf{s}, W)$. However, since the map

$$
r_k \colon (F_k^- W)_k \to W_{\mathrm{in}}
$$

is injective, f_k is uniquely determined by

$$
f \coloneqq \bigoplus_{a \in Q_1 \colon s(a)=k} f_{t(a)} \colon V_{\mathrm{out}} \to W_{\mathrm{in}}
$$

to make the following diagram commute

$$
\begin{array}{ccc}
V_k & \xrightarrow{\ \beta_V\ } & V_{\mathrm{out}} \\
\big\downarrow{f_k} & & \big\downarrow{f} \\
(F_k^- W)_k & \xrightarrow{\ r_k\ } & W_{\mathrm{in}}.
\end{array}
$$

Such an f_k exists if and only if the following composition of maps

$$
V_k \xrightarrow{\ \beta_V\ } V_{\mathrm{out}} \xrightarrow{\ f\ } W_{\mathrm{in}} \xrightarrow{\ \alpha_W\ } W_k
$$

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is zero. Therefore $\mathrm{Hom}_{\mathcal{A}}(V, F_k^- W)$ is the space of maps

$$(f_i \colon V_i \to (F_k^- W)_i)_{i \neq k}$$

intertwining with the action of $\mathcal{P}(\mathbf{s}, W)$ such that $\alpha_W \circ f \circ \beta_V$ is zero. Similarly $\mathrm{Hom}_{\mathcal{A}'}(F_k^+ V, W)$ is the space of maps

$$(g_i \colon (F_k^+ V)_i \to W_i)_{i \neq k}$$

intertwining with the action of $\mathcal{P}(\mu_k^+(\mathbf{s}, W))$ such that $\alpha_W \circ g \circ \beta_V$ is zero.

Note that V_i is canonically identified with $(F_k^+ V)_i$ for $i \neq k$ and so is for W_i and $(F_k^- W)_i$ for $i \neq k$. It is proved in [**DWZ08**, corollary 6.6] that the subalgebras are isomorphic

$$\bigoplus_{i,j \neq k} e_i \mathcal{P}(\mathbf{s}, W) e_j \cong \bigoplus_{i,j \neq k} e_i \mathcal{P}(\mu_k^+(\mathbf{s}, W)) e_j$$

and by the construction of F_k^+ on representations, the actions of these subalgebras on $\bigoplus_{i \neq k} V_i$ and $\bigoplus_{i \neq k}(F_k^+ V)_i$ are also identified via the isomorphism. The same is true for W and $F_k^- W$. It then follows that $\mathrm{Hom}_{\mathcal{A}'}(F_k^+ V, W)$ is naturally identified with $\mathrm{Hom}_{\mathcal{A}}(V, F_k^- W)$ via these isomorphisms, proving the adjointness of F_k^+ and F_k^-.

The properties (2) and (3) follow directly from the constructions of $F_k^{\pm}$.

By the adjointness and (3), to prove the first part of (4), we only need to show that the natural homomorphisms

$$\eta \colon V \to F_k^- F_k^+ V \quad \text{and} \quad \epsilon \colon F_k^+ F_k^- W \to W$$

are isomorphisms for any $V \in \mathcal{A}_{k,+}$ and $W \in \mathcal{A}'_{k,-}$. The restriction of η on V_i for $i \neq k$ is always an isomorphism to $(F_k^- F_k^+ V)_i$ by the constructions of $F_k^{\pm}$. For $V \in \mathcal{A}_{k,+}$, by definition we have that $\mathrm{Hom}(S_k, V) = 0$ and this is equivalent to $\beta_V \colon V_k \to V_{\mathrm{out}}$ being injective. Then we have the following short exact sequence of vector spaces

(6.5.3) $$0 \to V_k \to V_{\mathrm{out}} \to (F_k^+ V)_k \to 0.$$

Thus the natural map

$$\eta|_{V_k} \colon V_k \to (F_k^- F_k^+ V)_k = \ker(V_{\mathrm{out}} \to (F_k^+ V)_k)$$

is an isomorphism, implying that η is too. The proof for ϵ is similar.

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To prove that the exactness is preserved, we use the same diagram in the proof of (1). Suppose that now we have a short exact sequence of objects in $\mathcal{A}_{k,+}$

$$0 \to U \to V \to W \to 0.$$

Note that the exact sequence of complexes induces a long exact sequence involving the H_1's,

$$\cdots \longrightarrow \ker \beta_W \longrightarrow (F_k^+ U)_k \longrightarrow (F_k^+ V)_k \longrightarrow (F_k^+ W)_k \longrightarrow 0.$$

However, $\ker \beta_W$ vanishes by our assumption that $W \in \mathcal{A}_{k,+}$. The exactness follows. The proof for the functor F_k^- is similar.

(5) is a direct computation as in (6.5.2). The short exact sequence (6.5.3) leads to the identity of dimensions

$$\dim_K (F_k^+ V)_k = -\dim_K V_k + \sum_{a:\, s(a)=k} \dim_K V_{t(a)} = -\dim_K V_k + \sum_{i \neq k} [b_{ki}]_+ \dim_K V_i.$$

Recall from Section 3.1.4 that the seed $\mathbf{s}'$ is given by

$$s_i' = \begin{cases} -s_k & \text{for } i = k \\ s_i + [-b_{ik}]_+ s_k & \text{for } i \neq k. \end{cases}$$

Then we have

$$[F_k^+ V]_{\mathbf{s}'} = \left(-\dim_K V_k + \sum_{i \neq k} [b_{ki}]_+ \dim_K V_i \right)(-s_k) + \sum_{i \neq k} \dim_K V_i \cdot (s_i + [-b_{ik}]_+ s_k) = [V]_{\mathbf{s}} \in N.$$

$\square$

Remark 6.5.5. The subcategory $\mathcal{A}_{k,-}$ is closed under taking quotients while the subcategory $\mathcal{A}_{k,+}$ is closed under taking subobjects.

6.6. Semistable representations under reflection

As in the last section, we denote by $\mathcal{A}$ the category $\operatorname{mod} \mathcal{P}(\mathbf{s}, W)$. The Grothendieck group $K_0(\mathcal{A})$ is identified with the lattice N in a natural way such that $[S_i] = s_i$. For $m \in M_{\mathbb{R}}$ and $V \in \operatorname{mod} \mathcal{P}(\mathbf{s}, W)$, we denote the natural pairing $m([V])$ simply by $m(V)$.

Definition 6.6.1. Given $m \in M_{\mathbb{R}}$, a module $V \in \mathcal{A}$ is *m-semistable* (resp.-*stable*) if

(1) $m(V) = 0$ and

(2) $m(W) \geq 0$ (resp. > 0) for any non-zero proper submodule $W \subset V$.

Lemma 6.6.2. *Let $m \in M_{\mathbb{R}}$ such that $m(s_k) > 0$ and V be a module in the subcategory $\mathcal{A}_{k,-} \subset \mathcal{A}$. Then V is m-semistable if and only if*

(1) $m(V) = 0$ *and*

(2) *for any $W \in \mathcal{A}_{k,-}$ and $W \subset V$, $m(W) \geq 0$.*

PROOF. The only if part follows from the definition. We prove the if part. Let V be a $\mathcal{P}$-module in $\mathcal{A}_{k,-}$. Then every submodule W of V has a unique maximal submodule W' without any quotient isomorphic to S_k, i.e., $W' \in \mathcal{A}_{k,-}$. Since $m(s_k) > 0$, we have $m(W') \leq m(W)$. Therefore to check the semistability of V, it suffices to examine all the subobjects in $\mathcal{A}_{k,-}$. $\square$

We also have the following analogous lemma for $\mathcal{A}_{k,+}$. The proof is similar.

Lemma 6.6.3. *Let $m \in M_{\mathbb{R}}$ such that $m(s_k) < 0$ and V be a module in the subcategory $\mathcal{A}_{k,+} \subset \mathcal{A}$. Then V is m-semistable if and only if*

(1) $m(V) = 0$ *and*

(2) *for any $W \in \mathcal{A}_{k,+}$ a quotient of V, $m(W) \leq 0$.*

By Remark 6.5.5, one can rephrase above two lemmas in terms of short exact sequences in $\mathcal{A}_{k,+}$ and $\mathcal{A}_{k,-}$ as follows.

Lemma 6.6.4. *Let $m \in M_{\mathbb{R}}$ such that $m(s_k) < 0$ (resp. > 0). Let V be a module in the subcategory $\mathcal{A}_{k,+} \subset \mathcal{A}$ (resp. $\mathcal{A}_{k,-}$). Then V is m-semistable if and only if*

(1) $m(V) = 0$ *and*

(2) *for any short exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathcal{A}_{k,+}$ (reps. $\mathcal{A}_{k,-}$), $m(V') \geq 0$.*

Recall that we put $(\mathbf{s}'', W'') = \mu_k^-(\mathbf{s}, W)$ and denote $\mathcal{P}(\mathbf{s}'', W'')$ by $\mathcal{A}''$. The following is a key proposition that relates semistable modules in $\mathcal{A}$ to the ones in $\mathcal{A}'$ and $\mathcal{A}''$.

Proposition 6.6.5. *Let V be in $\mathcal{A}$ not isomorphic to S_k and $m \in M_{\mathbb{R}}$ such that $m(s_k) < 0$ (resp. > 0). Then we have V is m-semistable if and only if $F_k^+(V)$ in $\mathcal{A}'$ (resp. $F_k^-(V)$ in $\mathcal{A}''$) is m-semistable.*

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PROOF. First note that we only need to prove the case $m(s_k) < 0$ and the other case follows from (4) of Theorem 6.5.4.

Now let V be any m-semistable object in $\mathcal{A}$ for some m such that $m(s_k) > 0$. Immediately, the module V belongs in $\mathcal{A}_{k,+}$ since V has no submodule isomorphic to S_k. By Lemma 6.6.4, V is m-semistable if and only if $m(V') > 0$ for any submodule $V' \subset V$ such that the quotient V/V' also belongs in $\mathcal{A}_{k,+}$. Apply the reflection functor $F_k^+ \colon \mathcal{A}_{k,+} \to \mathcal{A}'_{k,-}$ to the short exact sequence

$$0 \to V' \to V \to V/V' \to 0.$$

Note that F_k^+ sends short exact sequences in $\mathcal{A}_{k,+}$ to $\mathcal{A}'_{k,-}$ and does not change their dimension vectors in N by Theorem 6.5.4. Therefore V is m-semistable if and only if for every short exact sequence $0 \to W' \to F_k^+(V) \to W'' \to 0$ in $\mathcal{A}'_{k,-}$, we have $m(W') \geq 0$, which is equivalent to $F_k^+(V)$ being m-semistable in $\mathcal{A}'$ again by Lemma 6.6.4.

$\square$

Let $\mathcal{A}(m)$ be the full subcategory of all m-semistable modules in $\mathcal{A}$. It is standard that $\mathcal{A}(m)$ is an abelian subcategory of $\mathcal{A}$. Summarizing the results in this section and by (4) of Theorem 6.5.4, we have

Proposition 6.6.6. *For any $m \in M_{\mathbb{R}}$ such that $m(s_k) < 0$ (resp. > 0), the functor*

$$F_k^+ \text{ (resp. } F_k^-) \colon \mathcal{A}(m) \to \mathcal{A}'(m) \text{(resp. } \mathcal{A}''(m))$$

is an equivalence between abelian categories.

6.7. Proof of mutation invariance

In this section, we prove Theorem 6.4.1, the mutation invariance for stability scattering diagrams. We start by describing the mutations of Hall algebra scattering diagrams.

6.7.1. Mutation of Hall algebra scattering diagram. Let $(\mathbf{s}, W)$ be a k-mutable SP for some vertex k and $(\mathbf{s}', W') = \mu_k^+(\mathbf{s}, W)$. Now we study the relation between $\mathfrak{D}_{\mathbf{s},W}^{\text{Hall}}$ and $\mathfrak{D}_{\mathbf{s}',W'}^{\text{Hall}}$.

We denote the motivic Hall algebra $H(\mathbf{s}, W)$ by $H(\mathcal{A})$. There is an open substack $\mathfrak{M}_{\mathbf{s},W}^{k,+} \subset \mathfrak{M}_{\mathbf{s},W}$ that parametrizes the modules in $\mathcal{A}_{k,+}$. Since the subcategory $\mathcal{A}_{k,+}$ is closed under extension, the relative Grothendieck group $K\left(\text{St}/\mathfrak{M}_{\mathbf{s},W}^{k,+}\right)$ admits a convolution product by using the same construction of the product in $H(\mathcal{A})$, making it a motivic Hall algebra. The inclusion $\mathfrak{M}_{\mathbf{s},W}^{k,+} \subset \mathfrak{M}_{\mathbf{s},W}$

induces an inclusion of motivic Hall algebras

$$H(\mathcal{A}_{k,+}) := K\left(\mathrm{St}/\mathfrak{M}^{k,+}_{\mathbf{s},W}\right) \hookrightarrow H(\mathcal{A}).$$

Similarly, there is an motivic Hall subalgebra $H(\mathcal{A}'_{k,-}) \hookrightarrow H(\mathcal{A}')$ for the subcategory

$$\mathcal{A}'_{k,-} \subset \mathcal{A}' = \mathrm{mod}\,\mathcal{P}(\mu^+_k(\mathbf{s}, W)).$$

Proposition 6.7.1. *The equivalence $F^+_k \colon \mathcal{A}_{k,+} \to \mathcal{A}'_{k,-}$ induces a geometric bijection between moduli stacks*

$$f^+_k \colon \mathfrak{M}^{k,+}_{\mathbf{s},W} \to \mathfrak{M}^{k,-}_{\mathbf{s}',W'},$$

which further induces an isomorphism of motivic Hall algebras

$$(f^+_k)_* \colon \widehat{H}\left(\mathcal{A}_{k,+}\right) \longrightarrow \widehat{H}(\mathcal{A}'_{k,-}),$$

such that

$$(f^+_k)_* \left(1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s},W}}\right) = 1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s}',W'}}$$

for any $m \in \mathcal{H}^{k,-}_{\mathbf{s}}$.

We give a proof of this proposition in Appendix B.

THEOREM 6.7.2. *For any $m \in \mathcal{H}^{k,-}_{\mathbf{s}}$, we have that the wall-crossing $\Phi^{\mathrm{Hall}}_{\mathbf{s},W}(m)$ lies in the subalgebra $\widehat{H}(\mathcal{A}_{k,+})$ while $\Phi^{\mathrm{Hall}}_{\mathbf{s}',W'}(m)$ lies in $\widehat{H}\left(\mathcal{A}'_{k,-}\right)$, and*

$$\left(f^+_k\right)_* \left(\Phi^{\mathrm{Hall}}_{\mathbf{s},W}(m)\right) = \Phi^{\mathrm{Hall}}_{\mathbf{s}',W'}(m)$$

PROOF. For any $m \in \mathcal{H}^{k,-}_{\mathbf{s}}$, the wall-crossings are

$$\Phi^{\mathrm{Hall}}_{\mathbf{s},W}(m) = 1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s},W}}, \quad \Phi^{\mathrm{Hall}}_{\mathbf{s}',W'}(m) = 1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s}',W'}}.$$

Then the result follows directly from Proposition 6.7.1. □

6.7.2. Proof of Theorem 6.4.1. Now we prove Theorem 6.4.1.

PROOF. We focus on part (2). First let $\varepsilon = +$. We know that for any $m \in \mathcal{H}^{k,-}_{\mathbf{s}}$,

$$\log\left(\Phi^{\mathrm{Stab}}_{\mathbf{s},W}(m)\right) = \mathcal{I}\left(\log\left(\Phi^{\mathrm{Hall}}_{\mathbf{s},W}(m)\right)\right) = \mathcal{I}\left(\log\left(1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s},W}}\right)\right).$$

Since $\left(f_k^+\right)_*$ is an algebra isomorphism, and by Proposition 6.7.1 and the absence of pole theorem (Theorem 6.3.5), we have

$$\left(f_k^+\right)_*\left(\log\left(1_{\mathfrak{M}_{\mathbf{s},W}^{m\text{-ss}}}\right)\right) = \log\left(1_{\mathfrak{M}_{\mathbf{s}',W'}^{m\text{-ss}}}\right) \in \widehat{H}_{\mathrm{reg}}\left(\mathcal{A}'_{k,-}\right)_{>0}/(\mathbb{L}-1).$$

Note that the isomorphism $\left(f_k^+\right)_*$ is induced by a geometric bijection, so it commutes with the integration maps, i.e., we have the following diagram

$$\widehat{H}_{\mathrm{reg}}(\mathcal{A}_{k,+})_{>0}/(\mathbb{L}-1) \xrightarrow{\left(f_k^+\right)_*} \widehat{H}_{\mathrm{reg}}(\mathcal{A}'_{k,-})_{>0}/(\mathbb{L}-1)$$

$$\downarrow^{\mathcal{I}} \qquad\qquad \searrow^{\mathcal{I}} \qquad\qquad \downarrow^{\mathcal{I}}$$

$$\widehat{\mathfrak{g}}_{\mathbf{s}\cap\mathbf{s}'}$$

Therefore we have

$$\begin{aligned}
\log\left(\Phi_{\mathbf{s},W}^{\mathrm{Stab}}(m)\right) &= \mathcal{I}\left(\log\left(1_{\mathfrak{M}_{\mathbf{s},W}^{m\text{-ss}}}\right)\right) \\
&= \mathcal{I}\left(\left(f_k^+\right)_*\left(\log\left(1_{\mathfrak{M}_{\mathbf{s},W}^{m\text{-ss}}}\right)\right)\right) \\
&= \mathcal{I}\left(\log\left(1_{\mathfrak{M}_{\mathbf{s}',W'}^{m\text{-ss}}}\right)\right) \\
&= \log\left(\Phi_{\mathbf{s}',W'}^{\mathrm{Stab}}(m)\right).
\end{aligned}$$

For $\varepsilon = -$, notice that we have

$$\mu_k^+\left(\mu_k^-(\mathbf{s},W)\right) \cong (\mathbf{s},W).$$

Then the SPs $\mu_k^-(\mathbf{s},W)$ and $(\mathbf{s},W)$ play the role of $(\mathbf{s},W)$ and $(\mathbf{s}',W')$ respectively in the case where $\varepsilon = +$. This completes the proof. $\qquad\square$

6.8. Caldero–Chapoton formula via stability scattering diagrams

In this section, we give a proof of the Caldero–Chapoton formula using stability scattering diagrams. The proof is in the spirit of Nagao's proof in [**Nag13**]. However, we think it is worthwhile to work out the details in the framework of scattering diagrams.

PROOF OF THEOREM 5.3.1. Recall that from Section 4.5, there is an expression of a cluster variable as

$$A_{\mathbf{k},i}^{\mathrm{Prin}} = \mathrm{Ad}_{\widehat{\mathbb{T}}}\,\mathfrak{p}_\gamma^{\mathrm{Cl}}(z^{g_{\mathbf{k},i}}) \in \widehat{\mathbb{T}}$$

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where γ is a path going from a generic point $\theta \in G_{\mathbf{k}}^+$ to the positive chamber. By Theorem 6.4.4, we have

$$\mathfrak{p}_\gamma^{\mathrm{Stab}} := \mathfrak{p}_\gamma(\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}}) = \mathfrak{p}_\gamma^{\mathrm{Cl}} \in \hat{G}.$$

Thus we have

$$A_{\mathbf{k},i}^{\mathrm{Prin}} = \mathrm{Ad}_{\widehat{\mathbb{T}}}\, \mathfrak{p}_\gamma^{\mathrm{Stab}}(z^{g_{\mathbf{k},i}}).$$

In order to compute the action of $\mathfrak{p}_\gamma^{\mathrm{Stab}}$ on the algebra $\widehat{\mathbb{T}}$, we first compute the action of $\mathfrak{p}_\gamma^{\mathrm{Stab}}$ on $\widehat{T} = \mathbb{Q}[M] \otimes \mathbb{Q}[[N^\oplus]]$.

There are several auxiliary algebras we will use. First we define a non-commutative algebra structure on

$$\widehat{B} := \mathbb{Q}[M] \otimes \widehat{H} = \mathbb{Q}[M] \otimes \widehat{H}(\mathbf{s}, W)$$

such that for any $a \in \widehat{H}_n$

$$a \cdot z^m = \mathbb{L}^{m(a)} z^m \cdot a = \mathbb{L}^{m(a)} z^m \otimes a.$$

Then we extend the poisson algebra structure on $\widehat{H}_{\mathrm{reg}}$ to

$$\widehat{B}_{\mathrm{reg}} := \mathbb{Q}[M] \otimes \widehat{H}_{\mathrm{reg}}$$

by setting

$$\{a, z^m\} = \frac{\mathbb{L}^{m(n)} - 1}{\mathbb{L} - 1} z^m \otimes a.$$

So the poisson action $\{a, -\}$ for some $a \in \widehat{H}_{\mathrm{reg}}$ is equal to the action of $a/(\mathbb{L} - 1)$ under the commutator bracket, i.e. for $b \in \widehat{B}_{\mathrm{reg}}$

$$(6.8.1) \qquad\qquad \{a, b\} = [a/(\mathbb{L} - 1), b].$$

Taking exponentials, we have for any $b \in \widehat{B}_{\mathrm{reg}}$ the following lemma.

Lemma 6.8.1. *For any $\delta \in \mathfrak{g}_{\mathrm{reg}} = (\widehat{H}_{\mathrm{reg}})_{>0}/(\mathbb{L} - 1)$, we have*

$$\exp(\delta) \cdot b \cdot \exp(\delta)^{-1} = \exp[\delta, -](b) = \exp\{(\mathbb{L} - 1)\delta, -\}(b) \in \widehat{B}_{\mathrm{reg}}.$$

PROOF. The first equality is by definition of the action $\exp[\delta, -]$ and the second follows from Equation (6.8.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma 6.8.2. *There is a poisson morphism extended from the integration map I (Theorem 6.3.4)*

$$\bar{I} \colon \mathbb{Q}[M] \otimes \widehat{H}_{\mathrm{reg}} \to \mathbb{Q}[M] \otimes \mathbb{Q}[[N^{\oplus}]], \quad \bar{I}(z^m \cdot a) = z^m \cdot I(a).$$

PROOF. One checks directly that $\bar{I}$ respects the poisson structures on $\widehat{B}_{\mathrm{reg}}$ and $\widehat{T}$ since the integration map I is a poisson morphism. $\qquad\square$

We recall that there is also a Lie algebra homomorphism $\mathcal{I} \colon \hat{\mathfrak{g}}_{\mathrm{reg}} \to \hat{\mathfrak{g}}_{\mathbf{s}}$ (Theorem 6.3.4).
The action of $\mathfrak{p}_\gamma^{\mathrm{Stab}}$ on z^m in the algebra $\widehat{T}$ (Section 4.5) is defined to be

$$\mathrm{Ad}\,\mathfrak{p}_\gamma^{\mathrm{Stab}}(z^m) = \exp\left\{ \log\left(\mathfrak{p}_\gamma^{\mathrm{Stab}}\right), - \right\}_{\widehat{T}}(z^m).$$

We also have from Proposition 6.2.5 and Section 6.3 that

$$\mathfrak{p}_\gamma^{\mathrm{Hall}} = 1_{\mathcal{T}(\theta)}^{-1}, \quad \mathcal{I}\left(\log\left(\mathfrak{p}_\gamma^{\mathrm{Hall}}\right)\right) = \log\left(\mathfrak{p}_\gamma^{\mathrm{Stab}}\right).$$

Combine all the ingredients, we have

$$\begin{aligned}
\mathrm{Ad}\,\mathfrak{p}_\gamma^{\mathrm{Stab}}(z^m) &= \exp\left\{ \log\left(\mathfrak{p}_\gamma^{\mathrm{Stab}}\right), - \right\}_{\widehat{T}}(z^m) \\
&= \exp\left\{ \mathcal{I}\left(\log\left(\mathfrak{p}_\gamma^{\mathrm{Hall}}\right)\right), - \right\}_{\widehat{T}}(z^m) \\
&= \exp\left\{ I\left((\mathbb{L} - 1)\log\left(\mathfrak{p}_\gamma^{\mathrm{Hall}}\right)\right), - \right\}_{\widehat{T}}(z^m) \\
&= \bar{I}\left(\exp\left\{ (\mathbb{L} - 1)\log\left(\mathfrak{p}_\gamma^{\mathrm{Hall}}\right), - \right\}_{\widehat{B}_{\mathrm{reg}}}(z^m)\right) \\
&= \bar{I}\left(\exp\left[\log\left(\mathfrak{p}_\gamma^{\mathrm{Hall}}\right), - \right]_{\widehat{B}}(z^m)\right) \\
&= \bar{I}\left(1_{\mathcal{T}(\theta)}^{-1} \cdot z^m \cdot 1_{\mathcal{T}(\theta)}\right) \\
&= \bar{I}\left(1_{\mathcal{T}(\theta)}^{-1} \cdot \left(\sum_{n \in N^\oplus} \mathbb{L}^{-m(n)} 1_{\mathcal{T}(\theta, n)}\right) \cdot z^m\right) \in \widehat{T}.
\end{aligned}$$

It is shown in [**Nag13**] that for $m = g_{\mathbf{k}, i}$, there is an identity in $\widehat{H}_{\mathrm{reg}}$

(6.8.2) $$\left(\sum_{n \in N^\oplus} \mathbb{L}^{-m(n)} 1_{\mathcal{T}(\theta, n)}\right) = 1_{\mathcal{T}(\theta)} \cdot \sum_{n \in N^\oplus} [\mathrm{Gr}(R_{\mathbf{k}, i}, n) \to \mathfrak{M}_n].$$

Thus we have

$$\operatorname{Ad} \mathfrak{p}_\gamma^{\mathrm{Stab}}(z^{g_{\mathbf{k},i}}) = \bar{I}\left(\left(\sum_{n \in N^\oplus} [\operatorname{Gr}(R_{\mathbf{k},i}, n) \to \mathfrak{M}_n]\right) \cdot z^{g_{\mathbf{k},i}}\right)$$

$$= z^{[\Gamma_{\mathbf{k},i}]} \cdot \left(\sum_{n \in N^\oplus} \chi(\operatorname{Gr}(R_{\mathbf{k},i}, n)) x^n\right) \in \widehat{T}.$$

Finally, we apply the isomorphism from $\widehat{T}$ to $\widehat{\mathbb{T}}$ sending x^n to $z^{p^*(n)} x^n$ (see Section 4.5 and Lemma 4.5.4). Thus we conclude,

$$A_{\mathbf{k},i}^{\mathrm{Prin}} = z^{[\Gamma_{\mathbf{k},i}]} \cdot \left(\sum_{n \in N^\oplus} \chi(\operatorname{Gr}(R_{\mathbf{k},i}, n)) z^{p^*(n)} x^n\right).$$

Evaluating x^n at 1, the result on cluster variables without coefficients follows. □

Remark 6.8.3. In [**Bri17**], Bridgeland considered the so-called theta functions for $m \in \mathcal{C}^+ \cap M$

$$\vartheta^m \colon M_\mathbb{R} \longrightarrow \widehat{T}, \quad \theta \mapsto \operatorname{Ad} \mathfrak{p}_{\gamma'}^{\mathrm{Stab}}(z^m),$$

where γ' is a path from m to the point θ, and gave a moduli-theoretic description [**Bri17**, Theorem 1.4]. Our proof explains the link between Bridgeland's description and Nagao's proof of the Caldero–Chapoton formula. We also note that Man-Wai Cheung in the thesis [**Che16**, Section 7.3] gave a proof of the Caldero–Chapoton formula for Dynkin quivers using stability scattering diagrams.

6.9. Initial data of stability scattering diagrams

In this section, we explain how to interpret the initial data (Definition 4.1.3) of stability scattering diagrams in terms of representation-theoretic invariants.

For an SP $(\mathbf{s}, W)$, the wall-crossing of $\mathfrak{D}_{\mathbf{s},W}^{\mathrm{Stab}}$ at any $m \in M_\mathbb{R}$ is

$$\Phi(m) = \exp\left(\sum_{d \in m^\perp} J(m, d) x^d\right)$$

where $J(m, d)$ is the Joyce invariant (see [**Bri17**]), which counts strictly m-semistable modules of dimension d in a sophisticated way.

The initial data of $\mathfrak{D}^{\mathrm{Stab}}_{\mathbf{s},W}$ is given by

$$\left(\exp\left(\sum_{k\in\mathbb{N}} J(p^*(d),kd)x^{kd}\right)\right)_{d\in\mathrm{Prim}(N^+)}.$$

We thus say that an SP $(\mathbf{s},W)$ is *cluster-initial* if for any $k\in\mathbb{N}$ and $d\in\mathrm{Prim}(N^+)\setminus\mathbf{s}$, the Joyce invariant vanishes, i.e.

$$J(p^*(d),kd)=0.$$

The following proposition is straightforward by definition.

Proposition 6.9.1. *The SP $(\mathbf{s},W)$ is cluster-initial if and only if*

$$\mathfrak{D}^{\mathrm{Stab}}_{\mathbf{s},W} = \mathfrak{D}^{\mathrm{Cl}}_{\mathbf{s}}.$$

In general, it is difficult to give a criterion when these Joyce invariants vanish. One sufficient condition (which may be too strict) is that there is no non-trivial $p^*(d)$-semistable modules at all. More interesting cases are that there are only *strictly $p^*(d)$-semistable* modules of dimension d, but we believe this is not enough to be a sufficient condition.

Bridgeland in [**Bri17**] defines *genteel* QPs aiming at the equality between two scattering diagrams in this case. However, according to a recent erratum (see the arXiv preprint of [**Bri17**]), it is unclear whether genteelness is a sufficient condition.

Scattering diagrams of Geiss-Leclerc-Schröer algebras

In this chapter, we study the Hall algebra scattering diagram of the algebra $H(B, D)$ associated to a skew-symmetrizable matrix B with its skew-symmetrizer D. These algebra are defined by Geiss, Leclerc, and Schröer in [**GLS17**]. One of their motivations is to use the module category of $H(B, D)$ to study the corresponding cluster algebra $\mathcal{A}(B)$. This chapter's main result is that the Hall algebra scattering diagram of $H(B, D)$ has the same cluster complex structure as the cluster scattering diagram $\mathfrak{D}_{B,D}^{\mathrm{Cl}}$ from Section 4.2.

7.1. The algebra $H(B, D)$ and its representations

7.1.1. Definitions. Suppose we are given oriented symmetrizable Cartan data (C, D, Ω). As we have pointed out in Section 3.1.1, this is equivalent to a pair (B, D) of a integral skew-symmetrizable matrix B and its left skew-symmetrizer D. Let $g_{ij} = |\gcd(b_{ij}, b_{ji})| > 0$ for $b_{ij} \neq 0$. Let $Q(B) \coloneqq (Q_0, Q_1, s, t)$ be the quiver with the set of vertices $Q_0 \coloneqq \{1, 2, \ldots, n\}$ and with the set of arrows Q_1 described as follows:

(1) If $b_{ji} > 0$, then there are g_{ij} arrows $\alpha_{ij}^{(1)}, \ldots, \alpha_{ij}^{(g_{ij})}$ from j to i.

(2) There is a loop ϵ_i at each vertex $i \in Q_0$.

Define
$$f_{ij} \coloneqq |b_{ij}|/g_{ij}, \quad k_{ij} \coloneqq \gcd(d_i, d_j).$$

We have
$$g_{ij} = g_{ji}, \quad k_{ij} = k_{ji}, \quad d_i = k_{ij} f_{ji}.$$

Definition 7.1.1. Fix a field K. The algebra $H_K(B, D)$ is defined to be the path algebra KQ modulo the two-sided ideal I generated by the following elements

(1) $\epsilon_i^{d_i}$ for all $i \in Q_0$ and

(2) for all i, j such that $b_{ij} > 0$ and for $k \in \{1, 2, \ldots, g_{ij}\}$,

$$\epsilon_i^{f_{ji}} \alpha_{ij}^{(k)} - \alpha_{ij}^{(k)} \epsilon_j^{f_{ij}}.$$

7.1.2. Representations. Let $H = H_K(B, D)$. Denote by rep H the category of finite dimensional left H-modules. Define

$$H_i := e_i H e_i \cong K[\epsilon_i]/(\epsilon_i^{d_i})$$

where e_i is the idempotent corresponding to the vertex $i \in Q_0$. For any $M \in \text{rep } H$, the subspace $M_i := e_i M$ inherits an H_i-module structure.

Definition 7.1.2. A module $M \in \text{rep } H$ is called *locally free* if M_i is a free H_i-module for every $i \in Q_0$. We denote by $\text{rep}_{\text{l.f.}} H$ the full subcategory of all locally free modules in rep H.

The Grothendieck group $K_0(\text{rep } H)$ is identified with $\mathbb{Z}^I$ where the simple modules give the standard basis $\mathbf{e} = (e_1, e_2, \dots, e_n)$. It is easy to show that the subcategory $\text{rep}_{\text{l.f.}} H$ is an *exact category*. The Grothendieck group $K_0(\text{rep}_{\text{l.f.}} H)$ is embedded in $K_0(\text{rep } H)$ by simply taking the class in rep H of a locally free module. It has a basis

$$\tilde{\mathbf{e}} = (\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n) = D \cdot \mathbf{e}$$

given by H_i as left H-modules.

An H-module can also be represented in the following way. We define an H_i-H_j-bimodule

$$_i H_j := H_i \otimes_K K\langle \alpha_{ij}^{(k)}, 1 \leq k \leq g_{ij} \rangle \otimes_K H_j / I$$

where I is the sub-bimodule generated by elements for $1 \leq k \leq g_{ij}$

$$\epsilon_i^{f_{ji}} \alpha_{ij}^{(k)} - \alpha_{ij}^{(k)} \epsilon_j^{f_{ij}}.$$

An H-module M is then determined by (M_i, M_{ij}) with

$$M_i = e_i M$$

an H_i-module and for i, j, an H_i-morphism

$$M_{ij} \colon {}_i H_j \otimes_j M_j \to M_i, \quad h \otimes m \mapsto hm.$$

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7.2. Reflection functors

Let $k \in I$. Recall that we have the Fomin–Zelevinsky mutation on the level of matrices (Lemma 3.1.4)

$$\mu_k(B, D) := (\mu_k(B), D).$$

If (B, D) is acyclic and k is a sink or a source (as a vertex of the quiver $Q^\circ(B, D)$, defined as the quiver $Q(B, D)$ with loops removed), then the mutation $\mu_k(B, D)$ is again acyclic.

There are reflection functors [**GLS17**] from $\operatorname{rep} H$ to $\operatorname{rep} \mu_k(H) = \operatorname{rep} H(\mu_k(B, D))$ generalizing that of Bernstein–Gelfand–Ponomarev [**BGP73**] for acyclic quivers. In fact, we have two reflection functors for $\varepsilon \in \{+, -\}$ and $k \in I$ either a sink or a source,

$$F_k^\varepsilon \colon \operatorname{rep} H \to \operatorname{rep} \mu_k(H).$$

We briefly explain the construction of F_k^ε below but refer the reader to [**GLS17**] for further details.

Construction of F_k^ε. Let M be an H-module.

For $\varepsilon = +$, consider the following exact sequence

$$0 \longrightarrow \ker M_{k,\text{in}} \longrightarrow \bigoplus_{j \in \Omega(k,-)} {}_kH_j \otimes_{H_j} M_j \xrightarrow{M_{k,\text{in}}} M_k$$

where

$$M_{k,\text{in}} = (M_{kj})_{j \in \Omega(k,-)}$$

and

$$M_{kj} \colon {}_kH_j \otimes_{H_j} M_j \to M_k$$

is the map determined by the H-module structure of M, i.e. for $h \in {}_kH_j$ and $m \in M_j$, we have

$$M_{kj}(h, m) = h \cdot m.$$

Note, for example, that if k is a source, then $\Omega(k, -) = \emptyset$ and thus $\ker M_{k,\text{in}} = 0$. We then define

$$(F_k^+ M)_i = \begin{cases} M_i & \text{if } i \neq k, \\ \ker M_{k,\text{in}} & \text{if } i = k \end{cases}$$

and define

$$(F_k^+ M)_{ik} \colon {}_i(\mu_k H)_k \otimes_{H_k} \ker M_{k,\text{in}} \to M_i$$

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by viewing $_i(\mu_k H)_k$ as $\operatorname{Hom}_{H_i}(_k H_i, H_i)$ (as an H_i-H_k-bimodule) and tensoring it with

$$\ker M_{k,\text{in}} \to {}_k H_i \otimes_{H_i} M_i.$$

One can check that F_k^+ also acts on the space of morphisms naturally. It indeed defines an additive functor.

For $\varepsilon = -$, the construction of F_k^+ is similar. Roughly, one considers outgoing data rather than incomings and takes cokernel instead of kernel [**GLS17**].

Since $\mu_k^2(B,D) = (B,D)$, we use the same notation $F_k^\varepsilon \colon \operatorname{rep} \mu_k(H) \to \operatorname{rep} H$ for functors in the opposite direction.

Define $H_i' \colon = e_i \mu_k(H) e_i \in \operatorname{rep}_{\text{l.f.}} \mu_k(H)$. Denote by $S_i = H_i/(\varepsilon_i) \cong K[\varepsilon_i]/(\varepsilon_i)$, the simples in $\operatorname{rep} H$ and by S_i' the simples in $\operatorname{rep} \mu_k(H)$.

Proposition 7.2.1. *We have that for any $i \neq k$,*

$$\dim F_k^\varepsilon(S_i') = \dim S_i + [\varepsilon b_{ik}]_+ \dim S_k = e_i + [\varepsilon b_{ik}]_+ e_k \in K_0(\operatorname{rep} H)$$

and $F_k^\varepsilon(H_i') \in \operatorname{rep}_{\text{l.f.}} H$ with

$$\dim F_k^\varepsilon(H_i') = \dim H_i + [-\varepsilon b_{ki}]_+ \dim H_k = \tilde{e}_i + [-\varepsilon b_{ki}]_+ \tilde{e}_k \in K_0(\operatorname{rep}_{\text{l.f.}} H).$$

PROOF. Let's first check for the first equation when k is a sink for $\mu_k H$ and $\varepsilon = +$. In this case we consider $i \in I$ such that $b_{ik} > 0$. By construction, we have

$$(F_k^+ S_i')_k = {}_k(\mu_k H)_i \otimes_{H_i} H_i/(\varepsilon_i) \cong {}_k(\mu_k H)_i / {}_k(\mu_k H)_i(\varepsilon_i).$$

Since $_k(\mu_k H)_i = \operatorname{Hom}_K(_i H_k, K)$ is a free right H_i-module of rank $[b_{ik}]_+$. (A way to think about this is to count $\dim_K {}_i H_k = d_i b_{ik} = -b_{ki} d_k$.) Then we have as a K-module

$$_k(\mu_k H)_i / {}_k(\mu_k H)_i(\epsilon_i) \cong H_i^{[b_{ik}]_+}/(\varepsilon_i) \cong K^{[b_{ik}]_+}.$$

For the second equation, we have that

$$(F_k^+ H_i)_k = {}_k(\mu_k H)_i \otimes_{H_i} H_i \cong H_k^{[-b_{ki}]_+}.$$

as a left H_k-module. Thus the module $F_k^+ H_i$ is locally free and the second identity follows.

The remaining cases are checked in similar ways. $\qquad\square$

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Recall that we have defined mutations of seeds in Chapter 3. In the above proposition, the lattice $K_0(\operatorname{rep} H)$ plays the role of N, and the classes of simples S_i form a seed $\mathbf{s}$ and the locally free modules H_i form $\tilde{\mathbf{s}}$. Thus Proposition 7.2.1 gives a way to realize the seeds

$$\mu_k^{\varepsilon} \mathbf{s}, \quad \mu_k^{\varepsilon} \tilde{\mathbf{s}}$$

as the classes modules obtained under reflection functors in cases where k is a sink or source.

7.3. The Hall algebra scattering diagram

We fix the ground field K to be $\mathbb{C}$. Recall that we have defined the motivic Hall algebra $\mathcal{H}(Q, I)$ for any quiver with relations (Q, I) in Chapter 6. Let (B, D) be from acyclic oriented Cartan data (C, D, Ω). The algebra $H(B, D)$, as constructed in Section 7.1, in particular, comes from a quiver with relations $(Q(B), I)$. Thus its category of finite-dimensional representations admits a motivic Hall algebra $\mathcal{H}(B, D) \coloneqq \mathcal{H}(Q(B), I)$.

Recall that in Section 3.1.3 from the data (B, D), we can construct fixed data Γ with an initial seed $\mathbf{s}$. Then the algebra $\mathcal{H}(B, D)$ is naturally graded by $N_{\mathbf{s}}^{\oplus}$. Let $\mathfrak{g}$ be the $N_{\mathbf{s}}^+$-graded Lie algebra $\mathcal{H}(B, D)_{>0}$ with the commutator bracket.

Definition 7.3.1. We define the *Hall algebra scattering diagram* $\mathfrak{D}_{B,D}^{\mathrm{Hall}}$ to be the unique consistent $\mathfrak{g}$-SD determined by the element

$$1_{\mathfrak{M}(B,D)} \in 1 + \widehat{\mathcal{H}}(B, D)_{>0}.$$

Remark 7.3.2. Unlike the case of quivers with potentials, an integration map from (a subalgebra of) the Hall algebra to a simpler algebra is not guaranteed. The lack of integration map makes it much more difficult to study the corresponding cluster algebra from the point of view of scattering diagrams. We leave this question for future studies.

7.4. An example of type B_2

In this section, we illustrate an example of $\mathfrak{D}_{B,D}$ of type B_2, i.e., the Cartan matrix of B is of B_2 type.

Let $B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The quiver $Q(B)$ is

$$1 \xleftarrow{\;\alpha\;} 2 \;\circlearrowright^{\epsilon}$$
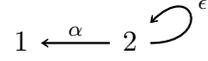
and the only relation is $\epsilon^2 = 0$.

The corresponding Hall algebra scattering diagram $\mathfrak{D}^{\mathrm{Hall}}_{B,D}$ is depicted below, in the basis consisting of e_1^* and e_2^*. The Figure 7.1 shows the underlying canonical cone complex for $\mathfrak{D}^{\mathrm{Hall}}_{B,D}$: it is a rank two complete simplicial cone complex with 6 maximal cones.
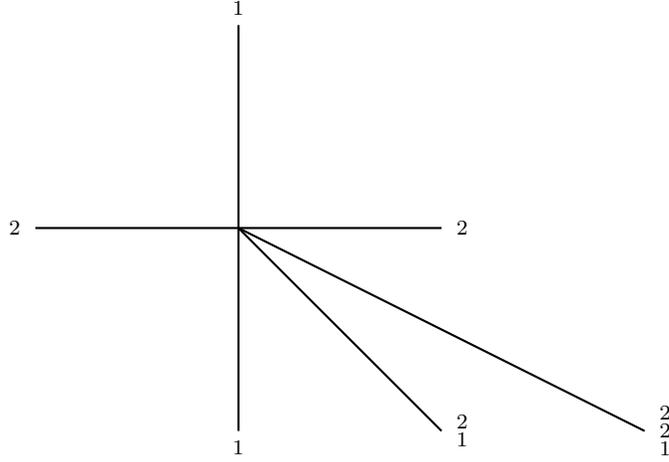
FIGURE 7.1

The rays in the fourth quadrant are orthogonal to normal vectors $e_1 + 2e_2$ and $e_1 + e_2$ respectively. The wall-crossing at each ray is a characteristic stack function of a category of semistable representations. Each ray in the above diagram is labeled by the unique (up to isomorphism) simple object in the corresponding subcategory. For example, the representation $M = \begin{smallmatrix} 2 \\ 2 \\ 1 \end{smallmatrix}$ is given by

$$M_1 = \mathbb{C}, \; M_2 = \mathbb{C}^2, \; \alpha = [\,0 \; 1\,], \; \epsilon = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right].$$

One should compare this scattering diagram with the cluster scattering diagram $\mathfrak{D}^{\mathrm{Cl}}_{B,D}$ in Example 4.2.3 for $b = 1$ and $c = 2$; see Figure 4.2. In fact, they have the same cluster complex structure, that is, their underlying canonical cone complexes both contain the positive and negative cluster complexes Δ_B^+ and Δ_B^-. This turns out to be true for any acyclic skew-symmetrizable matrix B, and we prove this fact in the next section.

7.5. Cluster chamber structure

This section presents the main result of this chapter. We show that the Hall algebra scattering diagram $\mathfrak{D}_{B,D}^{\mathrm{Hall}}$ possesses the same cluster chamber structure as the cluster scattering diagram $\mathfrak{D}_{B,D}^{\mathrm{Cl}}$.

7.5.1. τ-tilting cone complex. To any finite-dimensional algebra A, there is an associated (possibly infinite) simplicial cone complex $\mathfrak{S}(\mathrm{s}\tau\text{-tilt}\,A)$ in $M_{\mathbb{R}}$. A ray in this cone complex is generated by the so-called *g-vector* of an indecomposable τ-rigid pair and a maximal cone corresponds to a support τ-tilting pair. We explain these notions and the construction of this cone complex in this section, starting with τ-rigid modules.

Definition 7.5.1. Let A be a finite-dimensional K-algebra. A finite-dimensional left A-module is said to be τ-rigid if we have

$$\mathrm{Hom}_A(M, \tau M) = 0.$$

Here the functor $\tau\colon \mathrm{mod}\,A \to \mathrm{mod}\,A$ is the *Auslander-Reiten translation.*

We refer the reader to the book [**ARS97**] for the details of AR translation.

Definition 7.5.2. Let A be a finite-dimensional K-algebra.

(1) A *τ-rigid pair* (M, P) is a τ-rigid module M and a projective A-module such that

$$\mathrm{Hom}_A(P, M) = 0.$$

(2) A τ-rigid pair is said to be *support τ-tilting* if $|M| + |P| = |A|$ where $|\cdot|$ counts the number of non-isomorphic indecomposable direct summands of an A-module. Denote by sτ-tiltA the set of isomorphism classes of basic support τ-tilting pairs of A.

(3) A τ-rigid pair is said to be *almost complete support τ-tilting* if $|M| + |P| = |A| - 1$.

Example 7.5.3. By Auslander-Reiten duality, any τ-rigid module T is rigid, i.e.

$$\mathrm{Ext}^1(T, T) = 0.$$

For the path algebra of a Dynkin quiver, the reverse is also true, i.e., every rigid module is also τ-rigid.

For a τ-rigid pair $(M, 0)$, we will simply represent it by the τ-rigid module M. We will denote $(0, P)$ by $P[1]$ for simplicity. It is also clear that both $_AA$ and $_AA[1]$ are support τ-tilting.

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THEOREM 7.5.4 (Adachi-Iyama-Reiten [**AIR14**]). *Let (U, P) be an almost complete support τ-tilting pair. Then it has precisely two completions, i.e., there exist precisely two indecomposable τ-rigid pair, either of the form $(X, 0)$ or $(0, X)$ such that*

$$(U \oplus X, P) \ or \ (U, P \oplus X)$$

is support τ-tilting.

The above theorem provides a way to mutate a basic support τ-tilting pair (T, P). For example, suppose that $T \cong U \oplus X$ (resp. $P \cong V \oplus X$) where X is indecomposable. Then the pair

$$(U, P) \ (resp. \ (T, V))$$

is almost complete support τ-tilting. Then one can complete (U, P) (resp. (T, V)) by the other indecomposable τ-rigid pair guaranteed by Theorem 7.5.4. This is defined to be the mutation of (T, P) at $(X, 0)$ (resp. $(0, X)$).

Let $K_0(\text{proj } A)$ be the Grothendieck group of the category of finitely generated projective modules over A. It is a lattice with a basis consisting of the classes of the indecomposable projective modules

$$P_1, \ P_2, \ldots, \ P_n.$$

The group $K_0(\text{proj } A)$ is naturally dual to the lattice $N = K_0(\text{mod } A)$ (with basis $e_i = [S_i]$) such that $[P_i] = e_i^*$. Thus in our convention, we let $M = K_0(\text{proj } A)$.

For a module $T \in \text{mod } A$, let

$$Q_1 \to Q_0 \to T \to 0$$

be a minimal projective presentation of T. We define the *g-vector* of T to be

$$g^T := [Q_1] - [Q_0] \in M.$$

The *g*-vector of $P[1]$ is defined to be $-g^P = [P]$. Note that our convention is opposite to [**AIR14**] (the difference is superficial). For each τ-rigid pair (T, P), we associate a *g*-vector

$$g^{(T,P)} := g^T - g^P \in M.$$

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THEOREM 7.5.5 (Adachi-Iyama-Reiten [**AIR14**]). *Let A be a finite-dimensional algebra over K of rank n.*

(1) *The g-vectors of indecomposable direct summands of a basic support τ-tilting pair form a basis of M.*

(2) *The map*

$$(T, P) \mapsto g^{(T,P)}$$

defines an injection from the set of isoclasses of τ-rigid pairs to M.

(3) *For a basic support τ-tilting pair (T, P), let $\mathcal{C}_{(T,P)}$ be the rational polyhedral cone generated in $M_{\mathbb{R}}$ by the g-vectors of its indecomposable direct summands. Then the set of cones*

$$\left\{ \mathcal{C}_{(T,P)} \mid (T, P) \in \text{s}\tau\text{-tilt}\, A \right\}$$

and its faces form a simplicial cone complex in $M_{\mathbb{R}}$. Its dual graph is n-regular. We call this cone complex $\mathfrak{S}(\text{s}\tau\text{-tilt}\, A)$.

PROOF. The first two statements are proven in [**AIR14**]. The last statement is a straightforward consequence of the first two combined with Theorem 7.5.4. $\square$

Note that the cone complex $\mathfrak{S}(\text{s}\tau\text{-tilt}\, A)$ may neither be finite nor complete.

7.5.2. A correspondence between stability and τ-tilting. The following theorem is essentially due to Brustle-Smith-Treffinger [**BST19**]. Let A be a finite-dimensional K-algebra as in the last section. Suppose that A comes from a quiver with relations (Q, I). Here we fix $K = \mathbb{C}$.

THEOREM 7.5.6 ([**BST19**]). *The cone complex $\mathfrak{S}(\text{s}\tau\text{-tilt}\, A)$ in $M_{\mathbb{R}}$ is a cone subcomplex of the profinite cone complex $\mathfrak{S}_A^{\text{Hall}} := \mathfrak{S}_{Q,I}^{\text{Hall}}$ of the Hall algebra scattering diagram $\mathfrak{D}_{Q,I}^{\text{Hall}}$.*

Example 7.5.7. We demonstrate the above correspondence by the following example. Let us consider the finite-dimensional algebra $H(B, D)$ defined by a quiver with relations in Section 7.4.

The following Figure 7.2 shows the cone complex $\mathfrak{S}(\text{s}\tau\text{-tilt}\, H)$, depicted in the basis (e_1^*, e_2^*). It is complete with 6 maximal cones. Each ray is labeled by an indecomposable τ-rigid pair, whose g-vector is indicated by the black dot. The cone complex is the same as the one in Figure 7.1.

Each maximal cone corresponds to a support τ-tilting pair. For example, the first quadrant corresponds to $P_1[1] \oplus P_2[1]$ while the second quadrant corresponds to the pair $(S_1, P_2[1])$.

$_1{}^2{}^{}_2{}_1[1]$

1 $1[1]$

$_1{}^2{}_2{}_1$ $\begin{smallmatrix}2\\2\\1\end{smallmatrix}$ $\begin{smallmatrix}2\\2\end{smallmatrix}$
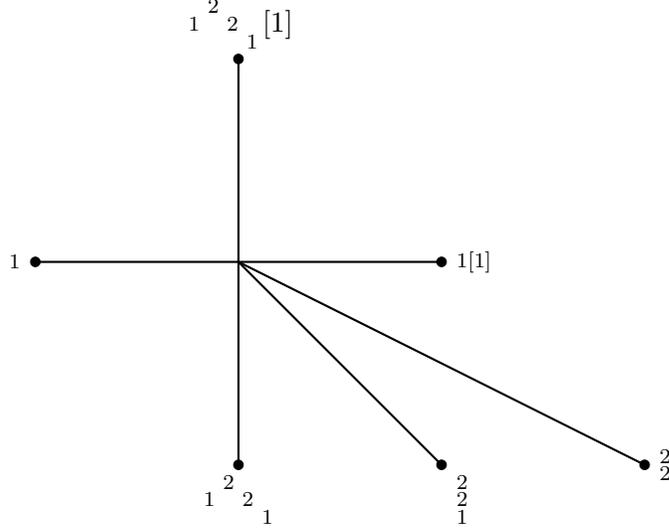
FIGURE 7.2

For example, we have a minimal projective presentation

$$P_1 \to P_2 \to \begin{smallmatrix}2\\2\\1\end{smallmatrix} \to 0.$$

Thus the g-vector for the τ-rigid module $\begin{smallmatrix}2\\2\\1\end{smallmatrix}$ is $e_1^* - e_2^*$.

7.5.3. The cluster complex structure of $H(B, D)$. Let (B, D) be an acyclic skewsymmetrizable matrix with a left symmetrizer. In this section, we apply the construction of the τ-tilting complex and the correspondence of the last section to the finite-dimensional algebra $H = H(B, D)$.

Let us be more precise on the current setting we are at. Given fixed data Γ with an initial seed $\mathbf{s}$ as in Section 3.1.3. We assume that the matrix $B = B(\mathbf{s})$ is acyclic. The algebra H defined in Section 7.1 is associated to (B, D). We have the Hall algebra scattering diagram $\mathfrak{D}_{B,D}^{\mathrm{Hall}}$ as in Definition 7.3.1 with its canonical cone complex denoted by $\mathfrak{S}_{B,D}^{\mathrm{Hall}}$.

On the other hand, the data $(\Gamma, \mathbf{s})$ determines the cluster scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{Cl}}$ (Definition 4.2.1). In some occasions, we have denoted this scattering diagram by $\mathfrak{D}_{B,D}^{\mathrm{Cl}}$ if there is no ambiguity. From Section 4.4, we know that the canonical cone complex $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$ has a subcomplex $\Delta_{\mathbf{s}}^+$.

The following theorem is the main result of this chapter.

THEOREM 7.5.8. *The cluster complex $\Delta_{\mathbf{s}}^+$ is also a cone subcomplex of $\mathfrak{S}_{B,D}^{\mathrm{Hall}}$.*

PROOF. We identify the complex $\Delta_{\mathbf{s}}^+$ with $\mathfrak{S}(\mathrm{s}\tau\text{-tilt}\,H)$. Since we have $\mathfrak{S}(\mathrm{s}\tau\text{-tilt}\,H)$ is a subcomplex of $\mathfrak{S}_{B,D}^{\mathrm{Hall}}$ by Theorem 7.5.6, the theorem follows.

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By theorem 1.1 of [**GLS19**], the dual graph $\mathcal{T}(H)$ of the complex $\mathfrak{S}(\text{s}\tau\text{-tilt }H)$ is isomorphic to the exchange graph of the cluster algebra $\mathcal{A}(B)$, which again is isomorphic to the dual graph of $\Delta_{\mathbf{s}}^+$ (see Section 4.4). We show that via this identification, the cone of a support τ-tilting pair is the same as the cone of the corresponding cluster.

Let (T, P) be a support τ-tilting pair of H and V be a indecomposable direct summand of T. We know from Lemma 6.2 of [**GLS19**] that V is locally free and thus proj. $\dim(V) \leq 1$ by Theorem 1.2 of [**GLS17**]. Then V admits a minimal projective resolution

$$0 \to Q_1 \to Q_0 \to V \to 0$$

and the g-vector of V is computed as

$$g^V = [Q_1] - [Q_0] \in M.$$

For any locally free H-module W, we have

$$0 \to \text{Hom}(V, W) \to \text{Hom}(Q_0, W) \to \text{Hom}(Q_1, W) \to \text{Ext}^1(V, W) \to 0.$$

Thus we have

$$\dim \text{Hom}(V, W) - \dim \text{Ext}^1(V, W) = -([Q_1] - [Q_0])([W]).$$

The left hand side can be computed by a bilinear form defined on N as $\langle [V], [W] \rangle_{C,D,\Omega}$ by Prop. 4.1 of [**GLS17**]. Thus we have

$$g^V = \langle [V], - \rangle_{C,D,\Omega} \in M.$$

We know that V corresponds to a cluster variable A_V via the isomorphism between the aforementioned two exchange graphs. There is a localization functor Loc that sends V to $\text{Loc}(V)$ a module over a hereditary algebra $\widetilde{H}$ (over the field $\mathbb{C}((\epsilon))$) for some central element ϵ in H; see [**GLS19**]). It is also well-known that the vector

$$\langle [V], - \rangle_{C,D,\Omega} = \langle [\text{Loc}(V)], - \rangle_{C,D,\Omega}$$

computes the g-vector for A_V (e.g. see [**Rup15**]). We then conclude that the cones in $\mathfrak{S}(\text{s}\tau\text{-tilt }H)$ and $\mathfrak{S}_{\mathbf{s}}^{\text{Cl}}$ are equal under the identification induced by the the isomorphism between the dual graphs.

$\square$

We point out that this theorem motivates that the category $\mathrm{mod}\, H(B, D)$ might play an important role in categorifying the cluster algebra $\mathcal{A}(B)$.

Recall the notations in Section 3.2. Suppose we have fixed data Γ with an initial seed $\mathbf{s}$ such that $B = B(\mathbf{s})$ is acyclic. For a locally free H-module T, we define $\mathrm{Gr}_{\mathrm{l.f.}}(n, T)$ to be the projective variety of locally free quotients of T with dimension vector n. The following conjecture is due to Geiss-Leclerc-Schröer, which can be considered as a locally free version of Theorem 5.3.1.

Conjecture 7.5.9. *Let (T, P) be a τ-rigid pair with g-vector $g^{(T,P)}$. Then we have*

$$\mathrm{CC}_{\mathrm{l.f.}}(T, P) := z^{g^{(T,P)}} \cdot \sum_{n \in N^{\oplus} \cap N^{\circ}} \chi\left(\mathrm{Gr}_{\mathrm{l.f.}}(n, T)\right) z^{p^*(n)} = A\left(g^{(T,P)}\right)$$

where the right hand side is the unique cluster monomial corresponding to the g-vector $g^{(T,P)}$.

For a skew-symmetrizable matrix B of Dynkin type, Conjecture 7.5.9 has been proven by Geiss-Leclerc-Schröer in [**GLS18**]. To the best of the author's knowledge, the general case remains open.

CHAPTER 8

Cluster scattering diagrams of Chekhov-Shapiro algebras

In this chapter, we develop the theory of cluster scattering diagrams for Chekhov-Shapiro algebras [**CS14**] (see Section 3.3 for definition), generalizing the framework of [**GHKK18**].

8.1. Generalized cluster scattering diagrams

Suppose we are given fixed data Γ with an initial seed $\mathbf{s}$. The Lie algebra $\mathfrak{g}_{\mathbf{s}}$ is the same as the one in Section 4.2. To define a consistent scattering diagram, we need to specify initial data as in (4.1.2). Recall that in Section 3.3, we need some extra data to define a CS algebra. For each $i \in I$, we need a reciprocal monic polynomial

$$\rho_i(v) = \sum_{k=0}^{d_i} c_k v^k.$$

Define the initial data

$$(g_n)_{n \in \mathrm{Prim}(N^+)} \in \prod_{n \in \mathrm{Prim}(N^+)} \exp \mathfrak{g}_n^{\parallel}$$

such that for each $i \in I$, the action of g_{s_i} on the monomial z^m (see Section 4.5) is given by

$$\mathrm{Ad}\, g_{s_i}(z^m) = z^m \cdot \rho_i(x^{s_i})^{m(s_i)} \in \hat{T} \colon\, = \mathbb{Q}[[N^{\oplus}]][M].$$

and for any other $n \in \mathrm{Prim}(N^+)$, $g_n = \mathrm{id} \in \hat{G}$.

This particular initial data defines a consistent $\mathfrak{g}$-SD

$$\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}} = (\mathfrak{S}_{\mathbf{s}}^{\mathrm{CS}}, \Phi_{\mathbf{s}}^{\mathrm{CS}} \colon M_{\mathbb{R}} \to \hat{G})$$

Note that here we omit the choice of $(\rho_i)_{i \in I}$ from the script. In the following sections of this chapter, we will show that this is the right scattering diagram for the corresponding CS algebra.

Example 8.1.1. We present a rank two example. We set

$$\rho_1(v) = 1 + v, \quad \rho_2(v) = 1 + v + v^2.$$

$$1 + x_1$$

$$1 + x_2 + x_2^2$$

$$1 + x_2 + x_2^2$$

$$1 + x_1 x_2^2$$

$$1 + x_1$$

$$1 + x_1 x_2 + x_1^2 x_2^2$$
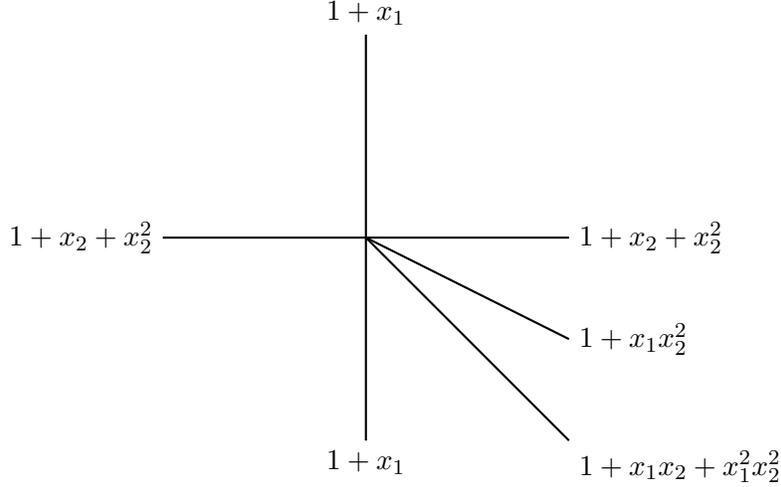
FIGURE 8.1

The initial seed is given by

$$\mathbf{s} = (e_1, e_2),$$

and $\tilde{\mathbf{s}} = (\tilde{e}_1, \tilde{e}_2) = (e_1, 2e_2)$. We also set $\omega(e_1, e_2) = 1$. This is the same fixed data as in Example 4.2.3 when $b = 1$ and $c = 2$.

Let $x_1 = x^{e_1}$ and $x_2 = x^{e_2}$. The generalized cluster scattering diagram for this data is depicted below in Figure 8.1, in the basis (e_1^*, e_2^*). One sees the canonical cone complex $\mathfrak{S}_{\mathbf{s}}^{\mathrm{CS}}$, which is equal to $\mathfrak{S}_{\mathbf{s}}^{\mathrm{Cl}}$. Each ray (also a wall in two dimensions) is labelled by a polynomial that represents the wall-crossing.

Remark 8.1.2. The above scattering diagram has a representation-theoretic interpretation. See the future work of the author joint with Labardini Fragoso [**LFM**].

8.2. Mutation invariance and cluster complex structure

In this section, we study the relationship between the mutation $\mathfrak{D}_{\mu_k^\varepsilon \mathbf{s}}^{\mathrm{CS}}$ and $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$. Notice that the polynomials $(\rho_i)_{i \in I}$ remain unchanged after mutations. The main result is the following theorem.

THEOREM 8.2.1 (Mutation invariance). *Fix the choice of polynomials $\rho = (\rho_i)_{i \in I}$.*

(1) *For $\varepsilon \in \{+, -\}$ and any $m \in \mathcal{H}_k^{-\varepsilon}$, we have*

$$\Phi_{\mu_k^\varepsilon \mathbf{s}}^{\mathrm{CS}}(m) = \Phi_{\mathbf{s}}^{\mathrm{CS}}(m).$$

(2) *At a generic $m \in s_k^\perp$, the wall-crossing of the scattering diagram $\mathfrak{D}_{\mu_k^\varepsilon \mathbf{s}}^{\mathrm{CS}}$ at m is given by the polynomial*

$$\rho_k(x^{-s_k}).$$

PROOF. As in the proof of Theorem 4.3.1, the proof goes exactly the same as the one of mutation invariance in [**GHKK18**]. The *slab* now is the pair

$$\mathfrak{d}_k = (s_k^\perp, \rho_k(x^{s_k})).$$

The identity that leads to consistency is

$$z^m \rho \left(z^{p^*(s_k)} \right)^{-m(s_k)} = z^{m - m(s_k) p^*(\tilde{s}_k)} \rho \left(z^{p^*(-s_k)} \right)^{-m(s_k)}.$$

which follows from the reciprocity of the polynomial ρ_k. □

Just as in Section 4.4, the above theorem will lead to a cluster complex structure of $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$. Additionally, we see immediately from the mutation invariance that the cluster complex we obtain for $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$ is still the cone complex $\Delta_{\mathbf{s}}^+$ as in Section 4.4. We summarize the results regarding the cluster complex structure of $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$ in the following theorem.

THEOREM 8.2.2. *Consider the scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$.*

(1) *The profinite cone complex $\mathfrak{S}_{\mathbf{s}}^{\mathrm{CS}}$ contains $\Delta_{\mathbf{s}}^+$ as a cone subcomplex.*
(2) *Each maximal cone is generated by the g-vectors $(g_{\mathbf{k},i})_{i \in I}$.*
(3) *Let $\mathfrak{d}$ be a facet of a maximal cone in $\Delta_{\mathbf{s}}^+$ with normal vector $c_{\mathbf{s},i}/d_i$, then the wall-crossing is given by the action*

$$\mathrm{Ad}\,\Phi(\mathfrak{d})(z^m) = z^m \cdot \rho_i \left(x^{|c_{\mathbf{s},i}/d_i|} \right)^{m(|c_{\mathbf{s},i}/d_i|)} \in \hat{T}.$$

Moreover, Theorem 4.5.7 extends to CS algebras by using the scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$. As in Section 4.5, we define a complete algebra

$$\hat{\mathbb{T}} := \bigoplus_{m \in M} \mathbb{Q}[[\tilde{p}^*(N^\oplus)]] \cdot z^m$$

with an action of $\hat{G}$ by automorphisms. We denote the CS cluster variables with principal coefficients (resp. without coefficients) by $A_{\mathbf{k},i}^{\mathrm{Prin}}$ (resp. $A_{\mathbf{k},i}$) for a sequence $\mathbf{k}$ of indices and $i \in I$.

THEOREM 8.2.3. *We have*

$$A_{\mathbf{k},i}^{\mathrm{Prin}} = \mathrm{Ad}_{\widehat{\mathbb{T}}} \, \mathfrak{p}_{\gamma}^{\mathrm{CS}}(z^{g_{\mathbf{k},i}}) \in \mathbb{Z}[M \oplus N], \quad A_{\mathbf{k},i} = \mathrm{Ad}_{\widehat{\mathbb{T}}} \, \mathfrak{p}_{\gamma}^{\mathrm{CS}}(z^{g_{\mathbf{k},i}}) \mid_{x^n=1} \in \mathbb{Z}[M]$$

where $\mathfrak{p}_{\gamma}^{\mathrm{CS}}$ denotes the path-ordered product in $\mathfrak{D}_{\mathbf{s}}^{\mathrm{CS}}$ and γ is any path from a point in the interior of the cluster chamber $G_{\mathbf{k}}^{+}$ to the positive chamber.

PROOF. The strategy of the proof is the same as the proof of Theorem 4.5.7. Note that the g-vectors only depend on the data $(\Gamma, \mathbf{s})$, so the algorithm in Proposition 4.4.9 still applies. We prove by induction on the length of $\mathbf{k}$. Recall the notations in the proof of Theorem 4.5.7, and suppose that $i = k_l$. Let $\delta = \mathrm{sgn}(c_{\mathbf{k}',i}) \in \{+, -\}$. Then we have

$$g_{\mathbf{k},i} = -g_{\mathbf{k}',i} + \sum_{j \in I} \left[-\delta \tilde{b}_{ji} \right]_{+} d_i g_{\mathbf{k}',j},$$

where $\tilde{b}_{ji} = \omega(c_{\mathbf{k}',j}/d_j, c_{\mathbf{k}',i}/d_i)$.

The recursive equation in the current situation is then

$$\begin{aligned}
\mathrm{Ad}_{\widehat{\mathbb{T}}} \, \Phi(\mathfrak{d}) \left(z^{g_{\mathbf{k},i}} \right) &= z^{g_{\mathbf{k},i}} \rho_i \left(z^{p^*(\delta c_{\mathbf{k}',i}/d_i)} x^{\delta c_{\mathbf{k}',i}/d_i} \right) \\
&= z^{-g_{\mathbf{k}',i} + \sum_{j \in I} [-\delta \tilde{b}_{ji}]_+ d_i g_{\mathbf{k}',j}} \rho_i \left(z^{p^*(\delta c_{\mathbf{k}',i}/d_i)} x^{\delta c_{\mathbf{k}',i}/d_i} \right) \\
&= z^{-g_{\mathbf{k}',i}} \theta_i \left(\prod_{j \in I} (z^{g_{\mathbf{k}',j}})^{[-\delta \tilde{b}_{ji}]_+}, \, x^{\delta c_{\mathbf{k}',i}/d_i} \prod_{j \in I} (z^{g_{\mathbf{k}',j}})^{[\delta \tilde{b}_{ji}]_+} \right)
\end{aligned}$$

This is exactly the exchange relation of generalized cluster variables in principal coefficients. Note here the vector $c_{\mathbf{k},i}/d_i$ is in N but not in N° in general. The result follows by induction as in the proof of Theorem 4.5.7. $\square$

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Generalized reflection functors from Keller-Yang's derived equivalence

In this appendix, we explain a way to obtain the generalized reflection functors defined in Section 6.5 from Keller–Yang's derived equivalence [**KY11**]. It is suggested to the author by Bernhard Keller. Recall the notations in Section 5.2.

Let (Q, W) be a k-mutable quiver with potential. Let $(\widetilde{Q}, \widetilde{W}) = \widetilde{\mu}_k(Q, W)$ and $(Q', W') = \mu_k(Q, W)$.

Proposition A.0.1. *We have natural isomorphisms*

$$F_k^+ \cong H^0 \circ \mathcal{F}_k^+ \circ \iota \colon \mod \mathcal{P}(Q, W) \to \mod \mathcal{P}(Q', W'),$$

$$F_k^- \cong H^0 \circ \mathcal{F}_k^- \circ \iota \colon \mod \mathcal{P}(Q', W') \to \mod \mathcal{P}(Q, W).$$

We need some preparations before proving this proposition. Let $\mathcal{P} = \mathcal{P}(Q, W)$, $\widetilde{\mathcal{P}} = \mathcal{P}(\widetilde{Q}, \widetilde{W})$ and $\mathcal{P}' = \mathcal{P}(Q', W')$.

For a fixed $k \in I$, define the following $\mathcal{P}$-module

$$U = \bigoplus_{i \in I} U_i$$

where $U_i := P_i = \mathcal{P}e_i$ for $i \neq k$ and $Q_k := \mathrm{coker}\left(P_k \to \bigoplus_{\alpha \colon t(\alpha)=k} P_{s(\alpha)}\right)$, i.e. the module making the following sequence

$$(A.0.1) \qquad P_k \to \bigoplus_{\alpha \colon t(\alpha)=k} P_{s(\alpha)} \to U_k \to 0$$

exact. A component of the above map $P_k \to P_{s(\alpha)}$ is by left multiplication by α.

Lemma A.0.2. *There exists a unique algebra homomorphism*

$$\Phi \colon \widetilde{\mathcal{P}}^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{P}}(U)$$

such that

(1) *for an idempotent e_i, $\Phi(e_i)$ is the projection to direct summand U_i;*

(2) *for $\alpha\colon i \to k$ in Q_1, $\Phi(\alpha^\star)$ is the natural linear map from P_i to U_k in the sequence (A.0.1);*

(3) *for $\beta\colon k \to j$ in Q_1, $\Phi(\beta^\star)$ is the natural map from U_k to P_j as indicated by the following diagram*

$$
\begin{array}{ccccccc}
P_k & \longrightarrow & \displaystyle\bigoplus_{\alpha\colon t(\alpha)=k} P_{s(\alpha)} & \longrightarrow & U_k & \longrightarrow & 0 \\
 & & & \searrow & \downarrow & & \\
 & & & & P_j & &
\end{array}
$$

where a component $P_{s(\alpha)} \to P_j$ is given by multiplying $\partial_{\beta\alpha}W$ and we have that the composition

$$
P_k \longrightarrow \bigoplus_{\alpha\colon t(\alpha)=k} P_{s(\alpha)} \longrightarrow P_j
$$

vanishes;

(4) *for $[\beta\alpha] \in \widetilde{Q}_1$, $\Phi([\beta\alpha])$ is the map from P_j to P_i by right multiplying $\beta\alpha$;*

(5) *for any other $\gamma\colon i \to j \in \widetilde{Q}_1$ not incident to k, $\Phi(\gamma)$ is the map from P_j to P_i by right multiplying γ.*

Moreover, there is a commutative diagram of algebra homomorphisms

(A.0.2)
$$
\begin{array}{ccc}
\widetilde{\Gamma}^{\mathrm{op}} & \longrightarrow & \mathcal{H}om_\Gamma(T,T) \\
\downarrow{\scriptstyle H^0} & & \downarrow{\scriptstyle H^0} \\
\widetilde{\mathcal{P}}^{\mathrm{op}} & \xrightarrow{\ \Phi\ } & \mathrm{Hom}_\mathcal{P}(U,U)
\end{array}
$$

*where the top horizontal map is the dg algebra homomorphism in Section 5.2 (see [**KY11**, Section 3.4]).*

PROOF. In fact, the map Φ is completely determined by the dg algebra homomorphism from $\widetilde{\Gamma}^{\mathrm{op}}$ to $\mathcal{H}om_\Gamma(T,T)$, which is described in details in [**KY11**, Section 3.4]. □

Due to Lemma A.0.2, there is a $\mathcal{P}$-$\widetilde{\mathcal{P}}^{\mathrm{op}}$-bimodule structure on U. Thus we can consider the following functor

$$
\mathrm{Hom}(U,-)\colon \bmod \mathcal{P} \to \bmod \widetilde{\mathcal{P}}.
$$

Lemma A.0.3. *The functor $\mathrm{Hom}(U,-)$ is naturally isomorphic to the generalized reflection functor $\widetilde{F}_k^+$.*

PROOF. As a vector space, we have that

$$\mathrm{Hom}(U, M) = \bigoplus_{i \in I} \mathrm{Hom}(U_i, M).$$

The left $\widetilde{\mathcal{P}}$-structure is described by Lemma A.0.2. For $i \neq k$, the space $\mathrm{Hom}(U_i, M)$ is identified with M_i. When $i = k$, we have the following exact sequence from (A.0.1)

$$0 \to \mathrm{Hom}(U_k, M) \to \bigoplus_{\alpha \,:\, t(\alpha) = k} M_{s(\alpha)} \to M_k.$$

In view of quiver representations, the above identifications induce an isomorphism between $\mathrm{Hom}(U, M)$ and $\widetilde{F}_k^+(M)$ as left $\widetilde{\mathcal{P}}$-modules, which is functorial on M. $\qquad\square$

Recall that $(Q', W') = \mu_k(Q, W)$ is the reduced part of $(\widetilde{Q}, \widetilde{W})$. Thus the Jacobian algebra $\mathcal{P}'$ is isomorphic to $\widetilde{\mathcal{P}}$ induced by the inclusion $Q' \subset \widetilde{Q}$. Then we can view U as a $\mathcal{P}$-$\mathcal{P}'$-bimodule and define the following functor

$$U \otimes_{\mathcal{P}'} - : \ \mathrm{mod}\,\mathcal{P}' \to \mathrm{mod}\,\mathcal{P}.$$

Recall from Section 6.5 that we have defined the generalized reflection functor $F_k^- : \mathrm{mod}\,\mathcal{P}' \to \mathrm{mod}\,\mathcal{P}$.

Lemma A.0.4. *The functor $U \otimes_{\mathcal{P}'} -$ is naturally isomorphic to $F_k^+ : \mathrm{mod}\,\mathcal{P}' \to \mathrm{mod}\,\mathcal{P}$.*

PROOF. We regard $\mathcal{P}'$ as a right module over itself. We define as a right $\mathcal{P}'$-module

$$R := \bigoplus_{i \in I} R_i = \left(\bigoplus_{i \neq k} e_i \mathcal{P}' \right) \oplus \mathrm{coker}\left(e_k \mathcal{P}' \to \bigoplus_{\beta \,:\, s(\beta) = k} e_{t(\beta)} \mathcal{P}' \right).$$

There is an algebra homomorphism from $\widetilde{\mathcal{P}}' := \tilde{\mu}_k(Q', W')$ to $\mathrm{Hom}_{\mathcal{P}'}(R, R)$ so that R becomes a $\widetilde{\mathcal{P}}'$-$\mathcal{P}'$-bimodule. It is straightforward from the definition of R that not hard to check that the functor $R \otimes_{\mathcal{P}'} -$ is naturally isomorphic to $\widetilde{F}_k^- : \mathrm{mod}\,\mathcal{P}' \to \mathrm{mod}\,\widetilde{\mathcal{P}}'$. However, note that $\widetilde{\mathcal{P}}'$ is isomorphic to $\mathcal{P}$, and we have an isomorphism between bimodules

$$_{\mathcal{P}} U_{\mathcal{P}'} \cong \mathcal{P} \otimes_{\widetilde{\mathcal{P}}'} R_{\mathcal{P}'}.$$

The lemma follows. $\qquad\square$

PROOF OF PROPOSITION A.0.1. We compute the functor

$$H^0 \circ \mathcal{R}\mathcal{H}om(T, -) \circ \iota \colon \operatorname{mod}\mathcal{P} \to \operatorname{mod}\widetilde{\mathcal{P}}.$$

Let $M \in \operatorname{mod}\mathcal{P}$. We have

$$H^0 \circ \mathcal{R}\mathcal{H}om(T, M) = \operatorname{Hom}_{\mathcal{D}\Gamma}(T, M) \cong \bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{D}\Gamma}(T_i, M).$$

For $i \neq k$, we have $\operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_i, M) \cong \operatorname{Hom}_{\mathcal{P}}(P_i, M)$. For $i = k$, the exact triangle (5.2.1) implies the following exact sequence (with $\operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_k[1], M) = 0$)

(A.0.3) $\quad \cdots \to \operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_k[1], M) \to \operatorname{Hom}_{\mathcal{D}\Gamma}(T_k, M) \to$

$$\to \bigoplus_{\alpha \in Q_1 \colon t(\alpha)=k} \operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_{s(\alpha)}) \to \operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_k, M) \to \cdots.$$

Thus we have

$$\operatorname{Hom}_{\mathcal{D}\Gamma}(T_k, M) \cong \ker\left(\bigoplus_{\alpha \colon t(\alpha)=k} \operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_{s(\alpha)}, M) \to \operatorname{Hom}_{\mathcal{D}\Gamma}(\Gamma_k, M)\right)$$

(A.0.4)
$$\cong \ker\left(\bigoplus_{\alpha \colon t(\alpha)=k} \operatorname{Hom}_{\mathcal{P}}(P_{s(\alpha)}, M) \to \operatorname{Hom}_{\mathcal{P}}(P_k, M)\right)$$

$$\cong \operatorname{Hom}_{\mathcal{P}}\left(\operatorname{coker}\left(P_k \to \bigoplus_{\alpha \colon t(\alpha)=k} P_{s(\alpha)}\right), M\right)$$

$$\cong \operatorname{Hom}_{\mathcal{P}}(U_k, M).$$

Therefore we have the following functorial isomorphism for any $M \in \operatorname{mod}\mathcal{P}$ (also viewed as a dg Γ-module concentrated in degree 0),

$$H^0 \mathcal{R}\mathcal{H}om(T, M) \cong \operatorname{Hom}_{\mathcal{P}}(U, M).$$

The LHS inherits a left action of $\widetilde{\mathcal{P}} = H^0\widetilde{\Gamma}$ from the right dg $\widetilde{\Gamma}$-module structure on T. It coincides with the action of $\widetilde{\mathcal{P}}$ on the RHS through the right $\widetilde{P}$-module structure on U by the commutative diagram (A.0.2). Combining with Lemma A.0.3, we hence conclude that

$$H^0 \circ \widetilde{\mathcal{F}}_k^+ \circ \iota \cong \widetilde{F}_k^+,$$

which implies $H^0 \circ \mathcal{F}_k^+ \circ \iota \cong F_k^+$. The natural isomorphism between $H^0 \circ \widetilde{\mathcal{F}}_k^- \circ \iota$ and $\widetilde{F}_k^-$ can be proven similarly (using Lemma A.0.4) $\qquad \square$

We remark that the properties of the generalized reflection functors in Theorem 6.5.4 follow easily from Proposition A.0.1. We also note that the functor $\mathrm{Hom}(U, -)$ is also considered in [**Fei19**].

The proof of Proposition 6.7.1

We first introduce the moduli stack $\mathfrak{M}_{\mathbf{s},W}^{k,+} \subset \mathfrak{M} = \mathfrak{M}_{\mathbf{s},W}$ that parametrizes $\mathcal{P}(\mathbf{s}, W)$-modules in the subcategory $\mathcal{A}_{\mathbf{s},W}^{k,+}$.

Fix a dimension vector d. Over a $\mathbb{C}$-scheme S, the groupoid $\mathfrak{M}_d(S)$ consists of

$$\left(\mathcal{V}_i, \rho_\alpha \colon \mathcal{V}_{s(\alpha)} \to \mathcal{V}_{t(\alpha)}\right)_{i \in Q_0(\mathbf{s},W),\ \alpha \in Q_1(\mathbf{s},W)}$$

where each $\mathcal{V}_i$ is a locally free sheaf on S of rank d_i and ρ_α's are morphisms satisfying relations in the ideal $I(N_d)$.

For a vertex $k \in Q_0(\mathbf{s})$, consider the morphism

$$\beta_k := \bigoplus_{s(\alpha)=k} \rho_\alpha \colon \mathcal{V}_k \to \mathcal{V}_{\mathrm{out}} := \bigoplus_{s(\alpha)=k} \mathcal{V}_{t(\alpha)}.$$

To define the substack $\mathfrak{M}_{\mathbf{s},W}^{k,+}$, we consider the subgroupoid $\left(\mathfrak{M}_{\mathbf{s},\mathbf{w}}^{k,+}\right)_d(S) \subset \mathfrak{M}_d(S)$ consists of those such that β_k is injective and $\operatorname{coker}\beta_k$ is also locally free. Note that this condition is stable under pull-back. Thus in particular β_k is injective at every stalk.

We define

$$\mathfrak{M}_{\mathbf{s},W}^{k,+} := \coprod_{d \in N_{\mathbf{s}}^{\oplus}} \left(\mathfrak{M}_{\mathbf{s},W}^{k,+}\right)_d.$$

Recall that we have an affine scheme $(\mathrm{Rep}_{\mathbf{s},\mathbf{w}})_d$ of representations of dimension vector d of the quiver $Q(\mathbf{s})$ satisfying relations in $I(N_d)$. There is an open subscheme $\left(\mathrm{Rep}_{\mathbf{s},\mathbf{w}}^{k,+}\right)_d$ of representations with β_k being injective. It is G_d-invariant.

Lemma B.0.1. *The moduli stack* $\left(\mathfrak{M}_{\mathbf{s},W}^{k,+}\right)_d$ *is equivalent to the quotient stack*

$$\left[\left(\mathrm{Rep}_{\mathbf{s},\mathbf{w}}^{k,+}\right)_d / G_d\right],$$

which is algebraic, of finite type and with affine diagonal.

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PROOF. Over a $\mathbb{C}$-scheme S, the groupoid $[(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}})_d/G_d](S)$ consists of objects that are G-equivariant morphisms from principal G_d-bundles over S to $(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}})_d$. Let

$$f\colon P \to \left(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}}\right)_d$$

be such an object. Then we have locally free sheaves $(P \times_{G_d} \mathbb{C}^{d_i})_{i \in Q_0}$ over S and morphisms

$$\rho_\alpha\colon P \times_{G_d} \mathbb{C}^{d_{s(\alpha)}} \to P \times_{G_d} \mathbb{C}^{d_{t(\alpha)}}$$

such that

$$\rho(p,v) = f(p)_\alpha(v).$$

This data determines an object in $(\mathfrak{M}^{k;+}_{\mathbf{s},W})_d(S)$ and in fact defines a functor from $[(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}})_d/G_d](S)$ to $(\mathfrak{M}^{k;+}_{\mathbf{s},W})_d(S)$.

Now we construct a functor from $(\mathfrak{M}^{k;+}_{\mathbf{s},W})_d(S)$ to $[(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}})_d/G_d](S)$. For an object

$$(\mathcal{V}_i, \rho_\alpha\colon \mathcal{V}_{s(\alpha)} \to \mathcal{V}_{t(\alpha)})_{i \in Q_0(\mathbf{s},W),\ \alpha \in Q_1(\mathbf{s},W)}$$

in $(\mathfrak{M}^{k;+}_{\mathbf{s},W})_d(S)$, consider the frame bundle $\mathcal{P}$ of the structure group G_d of this data over S. The pull-backs of $(\mathcal{V}_i)_{i \in Q_0}$ from S to $\mathcal{P}$ can be trivialized over $\mathcal{P}$. Hence the pull-backs of $(\rho_\alpha)_{\alpha \in Q_1}$ together define a G_d-equivariant morphism from $\mathcal{P}$ to $(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}})_d$. In this way, we have defined a functor from $(\mathfrak{M}^{k;+}_{\mathbf{s},W})_d(S)$ to $(\mathrm{Rep}^{k;+}_{\mathbf{s},\mathbf{w}})_d(S)$.

These two functors respect pull-backs and are in fact quasi-inverse equivalent to each other. Therefore the two stacks are equivalent. $\qquad\square$

We define the stack

$$\mathfrak{M}^{k,-}_{\mathbf{s},W} = \coprod_{d \in N^{\oplus}_{\mathbf{s}}} (\mathfrak{M}^{k,-}_{\mathbf{s},W})_d$$

in a similar way. Here we require that the objects in $\mathfrak{M}^{k,-}_{\mathbf{s},W}(S)$ satisfy that the morphism

$$\bigoplus_{\alpha\colon s(\alpha)} \rho_\alpha\colon \mathcal{V}_k \to \bigoplus_{\alpha\colon s(\alpha)=k} \mathcal{V}_{t(\alpha)}$$

is surjective between locally free sheaves.

Let $(\mathbf{s}', \mathbf{w}') = \mu^+_k(\mathbf{s},\mathbf{w})$. We define a morphism

$$f = f^+_k\colon \mathfrak{M}^{k;+}_{\mathbf{s},W} \to \mathfrak{M}^{k;-}_{\mathbf{s}',W'}$$

as follows. For an object $(\mathcal{V}_i, \rho_\alpha)$ in $\mathfrak{M}^{k,+}_{\mathbf{s},W}(S)$, the map $f(S)$ sends this object to a representation $(\mathcal{V}'_i, \rho'_\gamma)$ of $Q(\mathbf{s}')$ where $\mathcal{V}'_i = \mathcal{V}_i$ for $i \neq k$ and $\mathcal{V}_k = \operatorname{coker} \beta_k$, and ρ'_γ are defined as in Section 6.5.

PROOF OF PROPOSITION 6.7.1. The $\mathbb{C}$-points of $\mathfrak{M}^{k,+}_{\mathbf{s},W}$ are exactly representations of $(Q, I(N_d))$ (for various dimension d) in $\mathcal{A}_{k,+}$ while the $\mathbb{C}$-points of $\mathfrak{M}^{k,-}_{\mathbf{s}',W'}$ correspond to representations in $\mathcal{A}'_{k,-}$. By the equivalence between $\mathcal{A}_{k,+}$ and $\mathcal{A}'_{k,-}$ (Theorem 6.5.4), the morphism $f \colon \mathfrak{M}^{k,+}_{\mathbf{s},W} \to \mathfrak{M}^{k,-}_{\mathbf{s}',W'}$ induces an equivalence between groupoids

$$f(\mathbb{C}) \colon \mathfrak{M}^{k,+}_{\mathbf{s},W}(\mathbb{C}) \to \mathfrak{M}^{k,-}_{\mathbf{s}',W'}(\mathbb{C}),$$

proving that f is a geometric bijection.

A geometric bijection between stacks induces an isomorphism between relative Grothendieck groups; see [**Bri12**]. Thus we have an induced isomorphism

$$f_* \colon K\left(\mathrm{St}/\mathfrak{M}^{k,+}_{\mathbf{s},W}\right) \to K\left(\mathrm{St}/\mathfrak{M}^{k,-}_{\mathbf{s}',W}\right)$$

of $K(\mathrm{St}/\mathbb{C})$-modules. It is not hard to check that f_* respects the convolution product, thus becoming an algebra homomorphism between Hall algebras.

Finally, note that we have the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{M}^{m\text{-ss}}_{\mathbf{s},W} & \longrightarrow & \mathfrak{M}^{m\text{-ss}}_{\mathbf{s}',W'} \\
\downarrow & & \downarrow \\
\mathfrak{M}^{k,+}_{\mathbf{s},W} & \longrightarrow & \mathfrak{M}^{k,-}_{\mathbf{s}',W'}
\end{array}
$$

where the horizontal maps are geometric bijections induced by the functor F^+_k. The vertical maps are inclusions. Then we have

$$f_*\left(1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s},W}}\right) = 1_{\mathfrak{M}^{m\text{-ss}}_{\mathbf{s},W}}.$$

$\square$

Bibliography

[AIR14]   T. Adachi, O. Iyama, and I. Reiten, *τ-tilting theory*, Compositio Mathematica **150** (2014), no. 3, 415–452, arXiv:1210.1036.

[Ami09]   C. Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Annales de l'institut Fourier, vol. 59, 2009, pp. 2525–2590, arXiv:0805.1035.

[ARS97]   M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, vol. 36, Cambridge university press, 1997.

[BGP73]   I. N. Bernstein, I. M. Gel'fand, and V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Russian Mathematical Surveys **28** (1973), no. 2, 17–32.

[BMR⁺06]   A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204** (2006), no. 2, 572–618, arXiv:math/0402054.

[Bri12]   T. Bridgeland, *An introduction to motivic Hall algebras*, Advances in Mathematics **229** (2012), no. 1, 102–138, arXiv:1002.4372.

[Bri17]   ———, *Scattering diagrams, Hall algebras and stability conditions*, Algebraic Geometry **4** (2017), no. 5, 523–561, arXiv:1603.00416.

[BST19]   T. Brüstle, D. Smith, and H. Treffinger, *Wall and chamber structure for finite-dimensional algebras*, Advances in Mathematics **354** (2019), 106746, arXiv:1805.01880.

[CC06]   P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Commentarii Mathematici Helvetici **81** (2006), no. 3, 595–616, arXiv:math/0410187.

[Che16]   M.-W. Cheung, *Tropical techniques in cluster theory and enumerative geometry*, Ph.D. thesis, University of California, San Diego, 2016, available at https://escholarship.org/uc/item/1dm0g430.

[CK06]   P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*, Annales Scientifiques de l' Ecole Normale Supérieure, vol. 39(6), Elsevier, 2006, pp. 983–1009, arXiv:math/0510251.

[CK08]   ———, *From triangulated categories to cluster algebras*, Inventiones mathematicae **172** (2008), no. 1, 169–211, arXiv:math/0506018.

[CPS10]   M. Carl, M. Pumperla, and B. Siebert, *A tropical view of Landau–Ginzburg models*, preprint (2010), available at https://www.math.uni-hamburg.de/home/siebert/preprints/LGtrop.pdf.

[CS14]   L. Chekhov and M. Shapiro, *Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables*, International Mathematics Research Notices **2014** (2014), no. 10, 2746–2772, arXiv:1111.3963v3.

[Dem11]　L. Demonet, *Categorification of skew-symmetrizable cluster algebras*, Algebras and Representation Theory **14** (2011), no. 6, 1087–1162, arXiv:0909.1633.

[DWZ08]　H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Mathematica **14** (2008), no. 1, 59–119, arXiv:0704.0649.

[DWZ10]　———, *Quivers with potentials and their representations II: Applications to cluster algebras*, Journal of the American Mathematical Society **23** (2010), no. 3, 749–790, arXiv:0904.0676.

[Fei19]　J. Fei, *Counting using Hall algebras III: Quivers with potentials*, Pacific Journal of Mathematics **300** (2019), no. 2, 347–373, arXiv:1307.2667.

[FZ02]　S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, Journal of the American Mathematical Society **15** (2002), no. 2, 497–529, arXiv:math/0104151.

[FZ07]　———, *Cluster algebras IV: Coefficients*, Compositio Mathematica **143** (2007), no. 1, 112–164, arXiv:math/0602259.

[GHK13]　M. Gross, P. Hacking, and S. Keel, *Birational geometry of cluster algebras*, arXiv preprint arXiv:1309.2573 (2013).

[GHKK18]　M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, Journal of the American Mathematical Society **31** (2018), no. 2, 497–608, arXiv:1411.1394.

[Gin06]　V. Ginzburg, *Calabi-Yau algebras*, arXiv preprint arXiv:math/0612139 (2006).

[GLS17]　C. Geiß, B. Leclerc, and J. Schröer, *Quivers with relations for symmetrizable Cartan matrices I: Foundations*, Inventiones mathematicae **209** (2017), no. 1, 61–158, arXiv:1410.1403.

[GLS18]　———, *Quivers with relations for symmetrizable Cartan matrices V: Caldero–Chapoton formulas*, Proceedings of the London Mathematical Society **117** (2018), no. 1, 125–148, arXiv:1704.06438.

[GLS19]　———, *Rigid modules and Schur roots*, Mathematische Zeitschrift (2019), 1–33, arXiv:1812.09663.

[GPS10]　M. Gross, R. Pandharipande, and B. Siebert, *The tropical vertex*, Duke Mathematical Journal **153** (2010), no. 2, 297–362, arXiv:0902.0779.

[GS11]　M. Gross and B. Siebert, *From real affine geometry to complex geometry*, Annals of Mathematics (2011), 1301–1428, arXiv:math/0703822.

[Joy07]　D. Joyce, *Configurations in abelian categories. III. Stability conditions and identities*, Advances in Mathematics **215** (2007), no. 1, 153–219, arXiv:math/0410267.

[JS11]　D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, Memoirs of the American Mathematical Society **217** (2011), no. 1020, arXiv:0810.5645.

[Kel94]　B. Keller, *Deriving DG categories*, Annales scientifiques de l'Ecole normale supérieure, vol. 27, 1994, pp. 63–102, available at https://webusers.imj-prg.fr/~bernhard.keller/publ/ddc.pdf.

[Kel08]　———, *Cluster algebras, quiver representations and triangulated categories*, arXiv preprint arXiv:0807.1960 (2008).

[Kin94]　A. D. King, *Moduli of representations of finite dimensional algebras*, The Quarterly Journal of Mathematics **45** (1994), no. 4, 515–530.

[KS06]   M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, The Unity of Mathematics: In Honor of the Ninetieth Birthday of I. M. Gelfand, Birkhäuser Boston, 2006, pp. 321–385, arXiv:math/0406564.

[KS08]   _____, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv preprint arXiv:0811.2435 (2008).

[KS14]   _____, *Wall-crossing structures in Donaldson–Thomas invariants, integrable systems and mirror symmetry*, Homological mirror symmetry and tropical geometry, Springer, 2014, pp. 197–308, arXiv:1303.3253.

[KY11]   B. Keller and D. Yang, *Derived equivalences from mutations of quivers with potential*, Advances in Mathematics **226** (2011), no. 3, 2118–2168, arXiv:0906.0761.

[LFM]   D. Labardini Fragoso and L. Mou, *On the Caldero–Chapoton algebras of marked surfaces with orbifold points*, In preparation.

[Mou19]   L. Mou, *Scattering diagrams of quivers with potentials and mutations*, arXiv preprint arXiv:1910.13714 (2019).

[Mul16]   G. Muller, *The existence of a maximal green sequence is not invariant under quiver mutation*, The Electronic Journal of Combinatorics **23** (2016), no. 2, P2–47, arXiv:1503.04675.

[Nag13]   K. Nagao, *Donaldson–Thomas theory and cluster algebras*, Duke Mathematical Journal **162** (2013), no. 7, 1313–1367, arXiv:1002.4884.

[NZ12]   T. Nakanishi and A. Zelevinsky, *On tropical dualities in cluster algebras*, Algebraic Groups and Quantum Groups, vol. 565, Contemporary Mathematics, 2012, pp. 217–226, arXiv:0805.1035.

[Pal08]   Y. Palu, *Cluster characters for 2-Calabi–Yau triangulated categories*, Annales de l'institut Fourier, vol. 58(6), 2008, pp. 2221–2248, arXiv:math/0703540.

[Pla11]   P.-G. Plamondon, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, Advances in Mathematics **227** (2011), no. 1, 1–39, arXiv:1002.4956.

[Qin19]   F. Qin, *Bases for upper cluster algebras and tropical points*, arXiv preprint arXiv:1902.09507 (2019).

[Rup15]   D. Rupel, *Quantum cluster characters for valued quivers*, Transactions of the American Mathematical Society **367** (2015), no. 10, 7061–7102, arXiv:1109.6694.