Cellular Automata with Random Rules

By

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To my parents.

Contents

Abstract

We study one-dimensional cellular automata (CA) whose rules are chosen at random from among r-neighbor rules with a large number n of states. In particular, we are interested in CA evolutions with both temporal and spatial periodicity, the periodic solutions (PS). Our main focus are the properties of PS of a randomly chosen n-state CA rule, for large n. We prove that, when $r = 2$, the limiting probability that a random rule has a PS with given spatial and temporal periods is nontrivial when the periods are confined to a finite range. As a corollary, the shortest temporal period of PS with a given spatial period σ is stochastically bounded. By contrast, the longest temporal period of a PS with a given spatial period σ is of order $n^{\sigma/2}$, for any r and σ such that $\sigma \leq r$. In addition, we also explore PS that exhibit certain robustness, weakly robust periodic solutions (WRPS). We show that the probability of existence of WRPS within a finite range of periods is asymptotically $1/n$, if $r = 2$ and the range satisfies a divisibility condition.

We also study the analogous questions for deterministic rules when $r = 2$. When $\sigma = 2, 3, 4$, or 6, and we restrict the rules to be additive, we show that the longest period can be expressed as the exponent of the multiplicative group of an appropriate ring. We also construct non-additive rules with temporal period on the same order as the trivial upper bound n^{σ}. Additionally, we present a natural extension to the R-algorithm from the fundamental paper [**GG12**] that finds robust periodic solution (RPS) and present a proof of the scarcity of bounded growth.

Experimental results, open problems, and possible extensions of our results are also discussed to each problem that we study.

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CHAPTER 1

Introduction

1.1. Cellular Automata

In an autonomous dynamical system, a closed trajectory is a temporally periodic solution and obtaining information about such trajectories is of fundamental importance in understanding the dynamics [**Rei91**]. If the evolving variable is a spatial configuration, we may impose additional requirements on periodic solutions, such as spatial periodicity. What sort of periodic solutions does a typical dynamical system have? This question is perhaps easiest to pose for temporally and spatially discrete local dynamics of a **cellular automaton** (CA). Indeed, if we fix a neighborhood and a number of states, the number of cellular automata rules is finite, and the notion of a *random* rule straightforward. To date, not much seems to be known about properties of random CA.

To introduce our formal set-up, the set of **sites** is one-dimensional integer lattice $\mathbb{Z}$, and the set of possible states at each site is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, thus a **spatial configuration** is a function $\xi : \mathbb{Z} \to \mathbb{Z}_n$. A CA produces a **trajectory**, that is, a sequence ξ_t of configurations, $t \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, which is determined by the initial configuration ξ_0 and the following local and deterministic update scheme. Fix a finite **neighborhood** $\mathcal{N} \subset \mathbb{Z}$. Then a **rule** is a function $f : \mathbb{Z}_n^{\mathcal{N}} \to \mathbb{Z}_n$ that specifies the evolution as follows: $\xi_{t+1}(x) = f(\xi_t|_{x+\mathcal{N}})$. Throughout, we fix an $r \geq 2$, and consider one-sided rule with the neighborhood $\mathcal{N} = \{-(r-1), -(r-2), \ldots, -1, 0\}$, which results in

$$\xi_{t+1}(x) = f(\xi_t(x-r+1), \ldots, \xi_t(x)), \quad \text{for all } x \in \mathbb{Z}. \tag{1.1}$$

In words, the state at a site at time $t+1$ depends in a translation-invariant fashion on the state at the same site and its left $r-1$ neighbors at time t. We often write $f(a_{-r+1}, \ldots, a_0) = b$ as $a_{-r+1} \cdots \underline{a_0} \mapsto b$.

A rule f is **additive** if it commutes with sitewise addition modulo n, or, equivalently, if there exist $c_0, \ldots, c_{r-1} \in \mathbb{Z}_n$ so that $f(a_{-r+1}, \ldots, a_0) = \sum_{j=0}^{r-1} c_j a_j$. When $r = 2$, we give a rule by listing its values for all pairs in reverse alphabetical order, from $(n-1, n-1)$ to $(0,0)$.

It is convenient to interpret a trajectory as a **space-time configuration**, a mapping $(t, x) \mapsto \xi_t(x)$ from $\mathbb{Z}_+ \times \mathbb{Z}$ to $\mathbb{Z}_n$ that is commonly depicted as a two-dimensional grid of painted cells, in which different states are different colors, as in Figure 1.1. We remark that the one-sided neighborhoods are particularly suitable for studying periodicity and that any two-sided rule can be transformed to a one-sided one by a linear transformation of the space-time configuration [**GG12**].

In Figure 1.1, we have three states, i.e., $n = 3$, and the rule is *021102022*, i.e., $2\underline{2} \mapsto 0, 2\underline{1} \mapsto 2, 2\underline{0} \mapsto 1, 1\underline{2} \mapsto 1, 1\underline{1} \mapsto 0, 1\underline{0} \mapsto 2, 0\underline{2} \mapsto 0, 0\underline{1} \mapsto 2$ and $0\underline{0} \mapsto 2$.

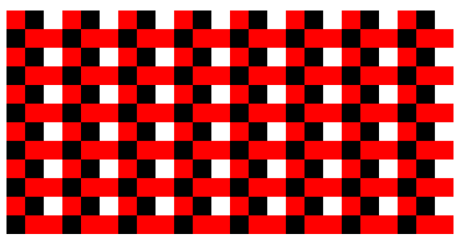


FIGURE 1.1. A piece of the space-time configuration of a 3-state rule. In the space-time configuration, 0, 1 and 2 are represented by white, red and black cells, respectively.

1.2. Periodic Solution and Robustness

The space-time configuration in Figure 1.1 exhibits periodicity in both space and time. In the literature [**BL07**], such a configuration is called doubly or jointly periodic. Since these are the only objects we study, we simply refer to such a configuration as a **periodic solution** (PS). To be precise, start with a periodic spatial configuration ξ_0, such that there is a $\sigma > 0$ satisfying $\xi_0(x) = \xi_0(x + \sigma)$, for all $x \in \mathbb{Z}$. Run a CA rule f starting with ξ_0. If we have $\xi_\tau(x) = \xi_0(x)$, for all $x \in \mathbb{Z}$ and that σ and τ are both minimal, then we have found a PS of **temporal period** τ and **spatial period** σ. Each of the periodic spatial configurations $\xi_0, \ldots, \xi_{\tau-1}$ is called a **PS configuration**. A **tile** is any rectangle with τ rows and σ columns within the resulting space-time configuration. We interpret a tile as a configuration on a discrete torus; we will not distinguish between spatial and temporal translations of a PS, and therefore between either rotations of a tile. The tile of a PS is by definition unique and we will identify a PS with its tile. As an example, in Figure 1.1, we start with the initial configuration $\xi_0 = 120^\infty = \ldots 120120120\ldots$ (we give a configuration as a bi-infinite sequence when the position of the origin is clear or unimportant). After 2 updates, we have $\xi_2(x) = \xi_0(x)$, for all $x \in \mathbb{Z}$, thus the PS has temporal period 2 and spatial period 3. Its tile is

$$\begin{matrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{matrix}\ .$$

PS with expansion properties are of particular interests. For example, Figure 1.2 demonstrates two pieces of the space-time configurations under rule *102222210*. The tile

$$\begin{matrix} 0 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 \end{matrix}$$

characterizes a PS under this rule and for such PS, even if the spatial periodic configuration is replaced by an arbitrary configuration to the right of some site in $\mathbb{Z}$, the periodic configuration will "repair" itself, that is, it will advance to the right with a minimal velocity $v > 0$ as time grows, uniformly over the perturbed environment. We will make more formal definition below.

Such PS are of particular importance, as they are related to stable limit cycles in continuous dynamical systems. Limit cycles, also known as isolated closed trajectories, are such that neighboring trajectories either spiral toward or away from them. In the former case, when a perturbation of a limit cycle converges back, the limit cycle is called stable [**Str15**]. Thus we consider an analogous stability property for CA: after a one-sided perturbation of a periodic configuration, the dynamics make the configuration converge back. In this paper, we keep the terminology from [**GG12**] and refer to such stability as robustness. We remark that the minimal velocity v gives the minimal exponential rate of convergence to the PS in the standard metric, by which the distance between $\xi, \eta \in \mathbb{Z}_n^{\mathbb{Z}}$ is $\mathfrak{m}(\xi, \eta) = 2^{-n}$, where $n = \inf\{|x| : \xi(x) \neq \eta(x)\}$.

To be more precise, let ξ_0 be a PS configuration under rule f and η_0 be any initial configuration that agrees with ξ_0 on all $x \leq y$, for some $y \in \mathbb{Z}$. Such initial configurations, adapting the definition from [**GG12**], are called **proper**. Let ξ_t and η_t be the configurations obtained by running f starting

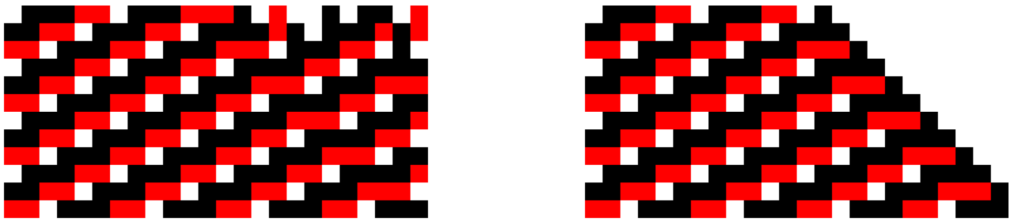


FIGURE 1.2. Two pieces of the space-time configuration of the 3-state rule *102222210*. The underlying PS exhibits weak robustness: the periodicity expands if it is terminated and continued by an arbitrary configuration, for example a random configuration (left) or all 0s (right).

with ξ_0 and η_0, respectively. Let

$$s_t(\eta_0) = \sup\{y : \eta_t(x) = \xi_t(x), \text{ for all } x < y\}$$

be the first location that η_t does not agree with ξ_t at time t. Then the expansion velocity in the initial environment η_0 is

$$v(\eta_0) = \liminf_{t\to\infty} \frac{s_t}{t},$$

which describes the rate at which spacial periodicity expands. If the **expansion velocity**

$$v = \inf\{v(\eta_0) : \eta_0 \text{ is proper for } \xi_0\}$$

is strictly positive, we say that the PS ξ_t is **weakly robust**, in comparative to robustness in [**GG12**].

We will also explore the (non-weak) robustness that was originally investigated in [**GG12**] under a moderately generalized setting. Let f be a fixed rule and $\xi_t^{(0)}$ and ξ_t be two PS under f, with temporal period τ_0, τ and spatial periods σ_0, σ, respectively. We may write each of the two configurations by appending infinitely many finite configurations L_0 and L, where they have length σ_0 and σ, respectively. Let H be another finite configuration of length h and put the leftmost site of H at the origin. Form infinitely many L_0 and L's to the left and right of H, respectively, denoted as $\eta_0 = L_0^\infty H L^\infty$. Run the CA rule f starting from η_0 and call the dynamics η_t. Under this setting, we call H a **handle** if every site to the right of the origin, inclusive, are temporally periodic: there is a $\tau_h > 0$ such that $\eta_t(x) = \eta_{t+\tau_h}(x)$ for all $t \geq 0$ and $x = 0, 1, \ldots, h-1$ and for all $x = h, h+1, \ldots$, we still have $\eta_t(x) = \eta_{t+\tau}(x)$ for all $t \geq 0$. If ξ_t, the original PS is weakly robust, we say that ξ_t is **a robust periodic solution (RPS) with respect to background PS** $\xi_t^{(0)}$ **and handle** H.

We remark here that in the fundamental paper that studies robustness of PS [**GG12**], rules are restricted to satisfy the "edge" condition that $f(0,a) = a$, for all $a \in \mathbb{Z}_n$. Under this setup, $\xi_t^{(0)} = \ldots 000 \ldots$ consistently serves as a background PS with temporal and spatial period 1. In addition, we also remark here that it is clear that an RPS ξ_t itself is weakly robust, while a WRPS ξ_t is always robust with respect to itself if H is selected to be the empty configuration.

Besides the PS, the property of bounded growth is also of interest for rules that satisfy $f(0,0) = 0$. If the initial configuration ξ_0 is restricted to have only finitely many non-zero states, we may investigate the growth velocity of the (right) boundary of ξ_t. We define $s_g(A)$ as the site of the rightmost non-zero value in the configuration A. Hence, for a initial configuration ξ_0, we define its growth velocity to be

$$v_g(\xi_0) = \limsup_{t\to\infty} \frac{s_g(\xi_t)}{t},$$

and the growth velocity of an edge CA is

$$v_g = \sup_{\xi_0} v_g(\xi_0).$$

A CA is said to have bounded growth if there exists an integer $K = K(\xi_0)$ such that $s_g(\xi_t) < K$, for all $t \geq 0$.

CA that exhibit temporally periodic or jointly periodic behavior have been addressed to some extend in the literature.

First, this thesis is primarily motivated by the fundamental work that investigates RPS [**GG12**], in which the authors explore all the 64 one-dimensional binary 3-neighbor, i.e., $r = 3$, edge CA rules and their RPS. Besides this work, the robustness of PS is also explored for the *Exactly 1* rule (elementary CA *Rule 22*) in [**GG11**], together with other types of evolution, e.g., replication and chaos, that this rule exhibits.

The groundwork that studies (non-robust) PS was laid in [**MOW84**], which extensively studies additive CA, but also devotes some attention to non-additive ones. An important observation is the link between periodicity in CA and state transition diagrams, which we find useful in this thesis as well. Successors of [**MOW84**] include [**Jen88a**,**Jen88b**,**Jen86**,**Wol02**,**XSB09**,**Kim09**].

In [**BL07**, **BK99**], the authors take a dynamical systems point of view and explore the density of temporally and spatially periodic (which they call jointly periodic) configurations.

The paper [**CSBK97**], for example, investigates the maximal length of temporal periods of binary CA under null boundary condition, and demonstrates that the maximal length $2^\sigma - 1$ can be obtained by additive rules, for any $\sigma > 0$. In [**AMD18**], the authors address the same question for non-additive CA, and show that the maximal length can also be obtained, if the rule is allowed to be non-uniform among sites. Works that investigate additive rules and their temporal periods also include [**GH86**], [**PTC86**], [**TSL06**], and [**MST06**].

Long temporal periods generated by CA have been of particular interest because of their applications to random number generation [**Wol86**, **CSBK97**, **SRC93**, **Ste99**, **MST06**, **DC10**].

1.3. Summary of Main Results

We now present a formal setting to investigate PS from random rules, which, to our knowledge, have not been explored before.

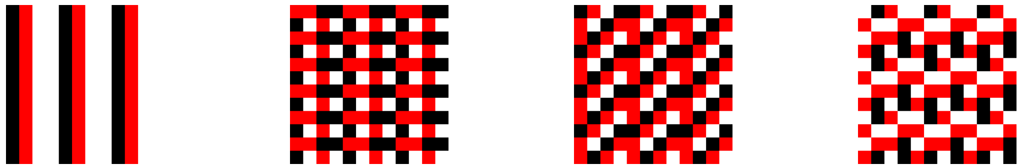
We first remark that in this thesis, except for Theorem 1.3.3, Theorem 1.4.1 and Chapter 3, where we give the proofs to these two theorems, we always assume the simplest nontrivial case that $r = 2$.

For a fixed n and r, the natural probability space is $\Omega_{r,n}$, containing all the n^{n^r} r-neighbor rules, with $\mathbb{P}$ that assigns the uniform probability $\mathbb{P}(\{f\}) = 1/\#\Omega_{r,n} = 1/n^{n^r}$ to every $f \in \Omega_{r,n}$. Let $\mathcal{P}_{\tau,\sigma,n}$ be the random set of PS with temporal period τ and spatial period σ of such a randomly chosen CA rule. In Chapter 2, the main quantity we are interested in is $\lim \mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n} \neq \emptyset\right)$ as $n \to \infty$ for a fixed pair of (τ, σ). In words, our focus is the limiting probability that a random CA rule has a PS with given temporal and spatial periods. In the following theorem, we prove that this limit is nontrivial for any τ and σ. Define

$$\lambda_{\tau,\sigma} = \frac{1}{\tau\sigma} \sum_{d|\gcd(\tau,\sigma)} \varphi(d)d, \tag{1.2}$$

where φ is the Euler totient function.

THEOREM 1.3.1. *For any fixed integers $\tau \geq 1$ and $\sigma \geq 1$, $\mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n} \neq \emptyset\right) \to 1 - \exp\left(-\lambda_{\tau,\sigma}\right)$ as $n \to \infty$.*



(A) *012200210* (B) *021102120* (C) *100112122* (D) *101201021*

FIGURE 1.3. Pieces of PS for $\sigma = 4$ and 3-state rules, *012200210*, *021102120*, *100112122* and *101201021*, with temporal period $\tau = 1, 2, 3$ and 4, respectively. (See the discussion before Corollary 2.4.1.) These temporal periods are the smallest in each case, as verified by Algorithm 2.1.1 in Section 2.1.4. Algorithm 2.1.2 in Section 2.1.5 shows that $\sigma = 4$ is not the minimal spatial period of PS given the corresponding temporal period $\tau = 1, 2$ and 3 in the first three rules, while for the last rule $\sigma = 4$ is also the minimal spatial period of PS for temporal period $\tau = 4$.

We also prove a more general result concerns the number of PS with a range of periods. Assume $\mathcal{T}, \Sigma \subset \mathbb{N} = \{1, 2, \dots\}$, and define $\mathcal{P}_{\mathcal{T},\Sigma,n} = \mathcal{P}_{\mathcal{T},\Sigma,n}(f) = \bigcup_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} \mathcal{P}_{\tau,\sigma,n}$ and

$$\lambda_{\mathcal{T},\Sigma} = \sum_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} \lambda_{\tau,\sigma}. \tag{1.3}$$

THEOREM 1.3.2. *For a finite $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$, $\mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n} \neq \emptyset\right) \to 1 - \exp\left(-\lambda_{\mathcal{T},\Sigma}\right)$ as $n \to \infty$.*

For an n-state rule f and $\sigma \geq 1$, we let $X_{\sigma,n}(f)$ and $Y_{\sigma,n}(f)$ be, respectively, the largest and smallest temporal periods of PS, with spatial period σ, of the rule f. When f is selected uniformly at random, $X_{\sigma,n}$ and $Y_{\sigma,n}$ become random variables. That is

$$X_{\sigma,n} = \max\{\tau : \mathcal{P}_{\tau,\sigma,n} \neq \emptyset\}$$

and

$$Y_{\sigma,n} = \min\{\tau : \mathcal{P}_{\tau,\sigma,n} \neq \emptyset\}.$$

Figure 1.3 provides four examples of rules f, with $Y_{4,3}(f) = 1, 2, 3$ and 4. As a consequence of Theorem 1.3.2, for a given $\sigma > 0$, the random variable $Y_{\sigma,n}$ is stochastically bounded, in the sense of the following corollary.

COROLLARY 1.3.1. *The random variable $Y_{\sigma,n}$ converges weakly to a nontrivial distribution as $n \to \infty$.*

In Chapter 3, we consider the general setting of CA rules with $r \geq 2$ neighbors and the typical size of $X_{\sigma,n}$ when r and σ are fixed and n is large. Our main result covers the case $\sigma \leq r$. The case $\sigma > r$ is much harder, but we expect the same result to hold; see the discussion in Chapter 3.

Theorem 1.3.3. *Fix a number of neighbors r and a spatial period $\sigma \leq r$. Then $X_{\sigma,n}/n^{\sigma/2}$ converges in distribution, as $n \to \infty$, to a nontrivial limit.*

Computations with the limiting distribution are a challenge, so we resort to Monte-Carlo simulations in Section 3.5 to illustrate Theorem 1.3.3.

We also provide empirical evidence that the same result holds when $\sigma > r$, although in that case we do not have a rigorous proof even for $r = 2$. At least for $r = \sigma = 2$, therefore, the shortest temporal period is stochastically bounded while the longest is on the order of n.

In Chapter 4, instead of their typical size, we explore the extremal values of $X_{\sigma,n}(f)$ and $Y_{\sigma,n}(f)$. We again assume the simplest nontrivial case, i.e., $r = 2$. It is clear that $\min_f Y_{\sigma,n}(f) = \min_f X_{\sigma,n}(f) = 1$, as the minima are attained by the identity n-state rule, i.e., the rule f given by $f(c_0, c_1) = c_1$, for all $c_0, c_1 \in \mathbb{Z}_n$. We therefore focus on

$$\max_f Y_{\sigma,n}(f) \text{ and } \max_f X_{\sigma,n}(f), \tag{1.4}$$

the largest among the shortest and longest temporal periods of a PS with spatial period σ and n states. Let $T(\sigma, n)$ be the number of aperiodic length-σ words from alphabet $\mathbb{Z}_n$, that is, words that cannot be written as repetition of a subword. Then it is clear that, for all n state rules f, $1 \leq Y_{\sigma,n}(f) \leq X_{\sigma,n}(f) \leq T(\sigma, n)$. We also have the following counting result.

Lemma 1.3.1. *The number of aperiodic length-σ word from alphabet $\mathbb{Z}_n$ is*

$$T(\sigma, n) = \sum_{d|\sigma} n^d \mu\left(\frac{\sigma}{d}\right) = \begin{cases} n^\sigma - n^{\sigma/2} + o(n^{\sigma/2}), & \text{if } \sigma \text{ is even} \\ n^\sigma + o(n^{\sigma/2}), & \text{if } \sigma \text{ is odd} \end{cases},$$

where $\mu(\cdot)$ is the Möbius function.

Proof. See [**CRS**$^+$**00**]. ∎

For $\sigma = 1$ and any n, it is easy to find a rule f with $Y_{1,n}(f) = X_{1,n}(f) = n = T(1, n)$; for example, any rule f satisfying $f(a, a) = \phi(a)$, where ϕ is any permutation on $\mathbb{Z}_n$ of order n, would do. For $\sigma = 2$, viewing evolution on $\{0, 1\}$ with periodic boundary, a unique CA with temporal period $\binom{n}{2}$ goes through all length-2 configuration ab, with $a < b \in \mathbb{Z}_n$. For instance, when $n = 3$, the evolution

$$\begin{array}{cc} 0 & 1 \\ 0 & 2 \\ 1 & 2 \end{array}$$

defines a rule with $0\underline{1} \mapsto 2$, $1\underline{0} \mapsto 0$, $0\underline{2} \mapsto 2$, $2\underline{0} \mapsto 1$, $1\underline{2} \mapsto 1$ and $2\underline{1} \mapsto 0$. Switching the last two values of f extends the PS to

$$\begin{array}{cc} 0 & 1 \\ 0 & 2 \\ 1 & 2 \\ 1 & 0 \\ 2 & 0 \\ 2 & 1 \end{array},$$

which has temporal period $6 = 3^2 - 3 = T(2, 3)$. It is clear that this construction works for all n and gives $Y_{2,n}(f) = X_{2,n}(f) = n^2 - n = T(2, n)$.

Even for $\sigma \geq 3$, it is not obvious what the extremal values (1.4) are, whether they are equal, or whether the upper bound $T(3, n)$ can always be attained. One of our main results is that $\max_f Y_{\sigma,n}(f) = \Theta(n^\sigma)$, matching the order of $T(\sigma, n)$ given by Lemma 1.3.1.

THEOREM 1.3.4. *Fix an arbitrary $\sigma > 0$. For $n \geq N(\sigma)$, there exists an n-state CA rule f such that $X_{\sigma,n}(f) = Y_{\sigma,n}(f) \geq C(\sigma)n^\sigma$, where $N(\sigma)$ and $C(\sigma)$ are constants depending only on σ.*

To alleviate the difficulties in computing the extremal quantities (1.4), one may try to restrict the set of rules f. The most natural such restriction are the additive rules, which exploit the algebraic structure of the states and enable the use of algebraic tools [**MOW84**,**Jen88a**,**Jen88b**]. We denote by $\mathcal{A}_n$ the set of n-state additive rules and let

$$\pi_\sigma(n) = \max_{f \in \mathcal{A}_n} X_{\sigma,n}(f).$$

It follows from [**MOW84**] that $\pi_\sigma(n) \leq n^{\sigma-1}$ (see Corollary 4.1.1), and therefore by Theorem 1.3.4 the maximal period of additive rules is at least by one power of n smaller than that of non-additive rules. Furthermore, for $\pi_\sigma(n)$ and $\sigma \in \{2,3,4,6\}$, we are able to give an explicit formula for $\pi_\sigma(n)$. Let $\lambda_\sigma(n)$ be the exponents of multiplicative group of $\mathbb{Z}_n$ when $\sigma = 2$, Eisenstein integers modulo n when $\sigma = 3$, and Gaussian integers modulo n when $\sigma = 4$. Then π_σ is related to λ_σ as follows.

Theorem 1.3.5. *For $\sigma = 2, 3$, $\pi_\sigma(n) = \lambda_\sigma(\sigma n)$, for all $n \geq 2$. Moreover, $\pi_4(2) = 4$ and $\pi_4(n) = \lambda_4(n)$, for all $n \geq 3$. Finally, $\pi_6(n) = \lambda_3(6n)$, for all $n \geq 2$.*

This theorem, and Lemmas 4.1.1–4.1.4, give the promised explicit expressions for the four $\pi_\sigma(n)$. It is tempting to conjecture that a variant of Theorem 1.3.5 holds for all σ, with a suitable definition of λ_σ for Kummer ring $\mathbb{Z}_n(\zeta)$, where ζ is the σ'th root of unity. However, this remains unclear as ζ is quadratic only for $\sigma = 3, 4, 6$, and this fact plays a crucial role in our arguments. See the discussion section in Chapter 4 for more details.

We will also study the existence of WRPS of random rules. Let $\mathcal{R}_{\mathcal{T},\Sigma}$ be the set of WRPS of a randomly selected n-state rule f, with temporal period τ and spatial period σ, where $(\tau, \sigma) \in \mathcal{T} \times \Sigma$. While we study the existence of PS on arbitrary finite range $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$ in Theorem 1.3.2, we impose one more restriction on the range to present the result on WRPS.

Theorem 1.3.6. *Let $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$ be fixed and finite. If there exists $(\tau, \sigma) \in \mathcal{T} \times \Sigma$ such that $\sigma \mid \tau$, then $\mathbb{P}(\mathcal{R}_{\mathcal{T},\Sigma} \neq \emptyset) = c(\mathcal{T}, \Sigma)/n + o(1/n)$, where $c(\mathcal{T}, \Sigma)$ is a constant depending only on $\mathcal{T}$ and Σ.*

We also discuss several technique conjectures so that the divisibility condition in the statement can be relaxed. See the discussion in Chapter 5.

In the chapter that investigates RPS, we will gently generalize the R-algorithm proposed in [**GG12**] to fit into our definition.

We will also prove the following result regarding growth velocity.

Theorem 1.3.7. *The probability that an n-state rule with $f(0,0) = 0$ has growth velocity 1 is $1 - 1/n$.*

1.4. Proof Outline and Summary of Chapters

This thesis consists of six chapters, each of which, except for this Chapter 1, corresponds to a research project.

Chapter 2 is based on the paper *Periodic Solutions of One-dimensional Cellular Automata with Random Rules* [**GL19c**]. In this chapter, we first collect our main tools: tiles of PS; circular shifts; oriented graphs induced by a rule; and the Chen-Stein method. We then discuss a class of tiles that plays a central role and prove Theorem 1.3.1 and Theorem 1.3.2 using the Chen-Stein method.

The paper *One-dimensional Cellular Automata with Random Rules: Longest Temporal Period of a Periodic Solution* [**GL19b**] serves as the basis of Chapter 3, where we give the proof of Theorem 1.3.3. We construct a directed graph, similar to the one in Chapter 2, and its use in analysis of PS is spelled out in Section 3.2. The proof of Theorem 1.3.3 is finally given in Section 3.4.

On the way of the proof of Theorem 1.3.3, we prove the following theorem, which may be of independent interest, in which $C_n = C_{\sigma,n}$ is the number of equivalence classes of initial conditions, modulo translations, that are periodic with (minimal) period σ and are such that the CA evolution never reduces the spatial period.

THEOREM 1.4.1. *Assume $\sigma \le r$. If σ is even, then, as $n \to \infty$, $n^{-\sigma} C_n$ converges in distribution to $1 - \tau$, where τ is the hitting time of 0 of the Brownian bridge $\eta(t)$ that starts at $\eta(0) = 1/\sqrt{\sigma}$ and ends at $\eta(1) = 0$. If σ is odd, $n^{-\sigma} C_n \to 1$ in probability.*

See [**AP94**, **AMP04**] for related results on random mappings. To prove Theorem 1.4.1, we present a sequential construction of the random rule that yields a stochastic difference equation whose solution converges to the Brownian bridge. Once Theorem 1.4.1 is established, the remainder of the proof of Theorem 1.3.3 is largely an application of existing results on random mappings and random permutations, which we adapt to our purposes in Section 3.3.

Chapter 4 is based on the paper *Maximal Temporal Period of a Periodic Solution Generated by a One-dimensional Cellular Automaton* [**GL19a**]. In this chapter, we address additive rules and prove Theorem 1.3.4. We relegate a result on multiplicative group structure of Eisenstein numbers modulo n, which is needed for $\sigma = 3, 6$, to the appendix at the end of the thesis. In Section 4.2, we prove Theorem 1.3.5 through explicit construction.

Weakly robustness is studied in Chapter 5, which is based on the paper *Weakly Robust Periodic Solutions of One-dimensional Cellular Automata with Random Rules* [**GL20b**]. In this chapter, we introduce the decidability as the key concept that distinguishes a WRPS from a non-weakly robust one. On the way to the proof of Theorem 1.3.6, we generalize the approaches in [**Sbe90**] to present a result of enumerating of certain types of spanning trees. Based on this counting result and the tools already prove in Chapter 2, we give the proof of Theorem 1.3.6.

In the last Chapter 6, while we do not dig into the RPS generated by random rules, we generalize the R-algorithm proposed in [**GG12**] to fit into our definition. Also, a short proof of Theorem 1.3.7 is given. This chapter is based on a project that is in preparation, *Robust Periodic Solutions of One-dimensional Cellular Automata with Random Rules* [**GL20a**].

We conclude each chapter by presenting related computer simulations and propose a few open problems for future consideration.

CHAPTER 2

Periodic Solutions of a Random Rule

Throughout this chapter, we assume that $r = 2$.

The main quantity we analyze is the asymptotic probability, as $n \to \infty$, that the random rule has a periodic solution with given spatial and temporal periods. To be precise, we will prove Theorem 1.3.1 and Theorem 1.3.2, which is restated here. Define

$$\lambda_{\tau,\sigma} = \frac{1}{\tau\sigma} \sum_{d|\gcd(\tau,\sigma)} \varphi(d)d, \tag{2.1}$$

where φ is the Euler totient function.

Theorem (1.3.1 restated). *For any fixed integers $\tau \geq 1$ and $\sigma \geq 1$, $\mathbb{P}(\mathcal{P}_{\tau,\sigma,n} \neq \emptyset) \to 1 - \exp(-\lambda_{\tau,\sigma})$ as $n \to \infty$.*

Assume $\mathcal{T}, \Sigma \subset \mathbb{N} = \{1, 2, \dots\}$, and define $\mathcal{P}_{\mathcal{T},\Sigma,n} = \mathcal{P}_{\mathcal{T},\Sigma,n}(f) = \bigcup_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} \mathcal{P}_{\tau,\sigma,n}$ and

$$\lambda_{\mathcal{T},\Sigma} = \sum_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} \lambda_{\tau,\sigma}. \tag{2.2}$$

Theorem (1.3.2 restated). *For a finite $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$, $\mathbb{P}(\mathcal{P}_{\mathcal{T},\Sigma,n} \neq \emptyset) \to 1 - \exp(-\lambda_{\mathcal{T},\Sigma})$ as $n \to \infty$.*

Let $Y_{\sigma,n} = \min\{\tau : \mathcal{P}_{\tau,\sigma,n} \neq \emptyset\}$, be the shortest temporal period of a PS with spatial period σ of a randomly chosen rule. Then Corollary 1.3.1 follows immediately.

Corollary (1.3.1 restated). *The random variable $Y_{\sigma,n}$ converges weakly to a nontrivial distribution as $n \to \infty$.*

2.1. Preliminaries

2.1.1. Tiles of a PS. We recall that the spatial and temporal periods σ and τ are assumed to be minimal, so a tile cannot be divided into smaller identical pieces. We now take a closer look into properties of tiles.

If we choose an element in a tile T to be placed at the position $(0,0)$, T may be expressed as a matrix $T = (a_{i,j})_{i=0,\ldots,\tau-1,j=0,\ldots,\sigma-1}$. We always interpret the two subscripts modulo τ and σ. The matrix is determined up to a space-time rotation, but note that two different rotations cannot produce the same matrix due to the minimality of σ and τ. We say that $a_{i,j}$ is an element in T, and write $a_{i,j} \in T$, when we want to refer to the element of the matrix at the position (i,j), and use the notation $\texttt{row}_i$ and $\texttt{col}_j$ to denote the ith row and jth column of a tile T, again after we fix $a_{0,0}$. All the properties we now introduce are independent of the chosen rotation (as they must be, to be meaningful).

Let T_1 and T_2 be two tiles and $a_{i,j}$, $b_{k,m}$ be elements in T_1 and T_2, respectively. We say that T_1 and T_2 are **orthogonal**, and denote this property by $T_1 \perp T_2$, if $(a_{i,j}, a_{i,j+1}) \neq (b_{k,m}, b_{k,m+1})$ for $i,j,k,m \in \mathbb{Z}_+$. It is important to observe that in this case the two assignments $a_{i,j}\underline{a_{i,j+1}} \mapsto a_{i+1,j+1}$ and $b_{k,m}\underline{b_{k,m+1}} \mapsto b_{k+1,m+1}$ occur independently.

We say that T_1 and T_2 are **disjoint**, and denote this property by $T_1 \cap T_2 = \emptyset$, if $a_{i,j} \neq b_{k,m}$, for $i,j,k,m \in \mathbb{Z}_+$. Clearly, every pair of disjoint tiles is orthogonal, but not vice versa.

Let $s(T) = \#\{a_{i,j} : a_{i,j} \in T\}$ be the number of different states in the tile. Furthermore, let $p(T) = \#\{(a_{i,j}, a_{i,j+1}) : a_{i,j}, a_{i,j+1} \in T\}$ be the **assignment number** of T; this is the number of values of the rule f specified by T. Clearly, $p(T) \geq s(T)$, so we define $\ell = \ell(T) = p(T) - s(T)$ to be the **lag** of T. We record a few immediate properties of a tile in the following Lemma.

Lemma 2.1.1. *Let $T = (a_{i,j})_{i=0,\ldots,\tau-1,j=0,\ldots,\sigma-1}$ be the tile of a PS with periods τ and σ. Then T satisfies the following properties:*

(1) *Uniqueness of assignment: if $(a_{i,j}, a_{i,j+1}) = (a_{k,m}, a_{k,m+1})$, then $a_{i+1,j+1} = a_{k+1,m+1}$.*
(2) *Aperiodicity of rows: each row of T cannot be divided into smaller identical pieces.*

Proof. Part 1 is clear since T is generated by a CA rule. Part 2 follows from part 1 and the assumption that the spatial period of T is minimal. $\square$

By contrast, we remark that there *may* exist periodic columns in a tile of a PS. For example, note that the first column in Figure 1.3(d) has period 2 rather than $4 = \tau$.

2.1.2. Circular Shifts. In this section, we introduce circular shifts operation on Z_n^σ (or Z_n^τ), the set of words of length σ (or τ) from the alphabet $\mathbb{Z}_n$. They will be useful in Section 2.2.

DEFINITION 2.1.1. *Let $\mathbb{Z}_n^\sigma$ consist of all length-σ words. A* ***circular shift*** *is a map $\pi : \mathbb{Z}_n^\sigma \to \mathbb{Z}_n^\sigma$, given by an $i \in \mathbb{Z}_+$ as follows: $\pi(a_0a_1 \ldots a_{\tau-1}) = a_ia_{i+1} \ldots a_{i+\sigma-1}$, where the subscripts are modulo σ. The* ***order*** *of a circular shift π is the smallest k such that $\pi^k(A) = A$ for all $A \in \mathbb{Z}_n^\sigma$, and is denoted by* $\operatorname{ord}(\pi)$. *Circular shifts on $\mathbb{Z}_n^\tau$ will also appear in the sequel and are defined in the same way.*

LEMMA 2.1.2. *The following two statements hold:*

(1) *Let π be a circular shift on $\mathbb{Z}_n^\sigma$. Then ord$(\pi) \mid \sigma$;*
(2) *Let $A \in \mathbb{Z}_n^\sigma$ be any aperiodic finite configuration and $d \mid \sigma$. Then*

$$\#\left\{B \in \mathbb{Z}_n^\sigma : A = \pi(B) \text{ for some circular shift } \pi \text{ with } \operatorname{ord}(\pi) = d\right\} = \varphi(d).$$

PROOF. Note that the σ circular shifts form a cyclic group of order σ. Moreover, $\operatorname{ord}(\pi)$ of a circular shift is its order in the group, thus (1) follows. To prove (2), observe that the circular shifts of order d generate a cyclic subgroup and the number of them is $\varphi(d)$. As A is aperiodic, the cardinality in the claim is the same. □

We say that two words A and B of length σ are **equal up to a circular shift** if $B = \pi(A)$ for some circular shift π. For example, words 0123 and 2301 are not equal, but are equal up to a circular shift.

2.1.3. Directed Graphs and Random Graphs. A **directed graph** G, or **digraph** in short, is a set of vertices V, along with a set of ordered pairs of vertices A, called **arcs**. Throughout, an arc starting from vertex A and ending at B is denoted as $A \to B$, where A and B are called the endpoints of the arc. A cycle of length k in G is a directed trail containing k vertices in which the only repeated vertices are the first and last, denoted as $A_0 \to A_1 \to \cdots \to A_{n-1}$. A

k-partite digraph is a digraph whose vertices can be partitioned into k different sets, such that no two endpoints of an arc are in the same set.

After fixing a sample space, an arc from vertex A to B may appear with probability p, thus the underlying digraph can be regarded as a random digraph.

2.1.4. Directed Graph on Configurations. Connections between directed graphs on periodic configurations and cycles are well-established [**MOW84**, **Wol02**, **Kim09**, **XSB09**], as they are useful for analysis of PS with a fixed spatial period.

DEFINITION 2.1.2. *Let $A = a_0 \dots a_{\sigma-1}$ and $B = b_0 \dots b_{\sigma-1}$ be two words from alphabet $\mathbb{Z}_n$. We say that A* ***down-extends to*** *B, if $f(a_i, a_{i+1}) = b_{i+1}$, for all $i \geq 0$, where (as usual) the indices are modulo σ, and we write $A \searrow B$.*

If $A \searrow B$, then $\pi(A) \searrow \pi(B)$, for any circular shift π on $\mathbb{Z}_n^\sigma$. We can define, for a fixed σ, the **configuration digraph**, which has an arc from A to B if $A \searrow B$. See Figure 2.1 for the configuration digraph of the 3-state rule *021102022*. For instance, there is an arc from 122 to 210 as $1\underline{2} \mapsto 1$, $2\underline{2} \mapsto 0$ and $2\underline{1} \mapsto 2$. The following algorithm and self-evident proposition determine the PS in Figure 1.1 from the length-2 cycle $120 \leftrightarrow 211$ in Figure 2.1.

ALGORITHM 2.1.1.

---

input : *Configuration digraph $D_{\sigma,f}$ of f with spatial period σ*

Find all the directed cycles in $D_{\tau,f}$

for *each cycle $A_0 \to A_1 \to \cdots \to A_{\tau-1} \to A_0$* **do**

  form the tile T by placing configurations $A_0, A_1, \dots, A_{\sigma-1}$ on successive columns.

  if *both spatial and temporal periods of T are minimal* **then**

    print *T as a PS*

  end

end

PROPOSITION 2.1.1. *All PS of spatial period σ of f are obtained by Algorithm 2.1.1.*

We remark that check of the periodicity is necessary, as, for instance, the cycle $000 \leftrightarrow 222$ in Figure 2.1 results in a PS of spatial period 1 instead of 3. In the same vein, the periods of configurations are non-increasing, and may decrease, along any directed path on the configuration digraph. For example, in Figure 2.1, the configuration $100 \searrow 222$, thus the period is reduced from 3 to 1 and then remains 1. These period reductions play a crucial role in Chapter 3.

$$
\begin{tikzcd}
122 \arrow[r] & 210 \arrow[r] & 022 \arrow[d] & 101 \arrow[l] & 212 \arrow[d] \\
 & & 100 \arrow[d] & & 021 \arrow[d] \\
111 \arrow[r] & 000 \arrow[r, bend right] & 222 \arrow[l, bend right] & 010 \arrow[l] & 202 \arrow[l] \\
 & & 001 \arrow[u] & & \\
221 \arrow[r] & 102 \arrow[r] & 220 \arrow[u] \arrow[r] & 011 & 110 \arrow[uu] \\
 & 002 \arrow[r] & 120 \arrow[r, bend right] & 211 \arrow[l, bend right] & \\
 & 020 \arrow[r] & 201 \arrow[r, bend right] & 112 \arrow[l, bend right] & \\
 & 200 \arrow[r] & 012 \arrow[r, bend right] & 121 \arrow[l, bend right] &
\end{tikzcd}
$$

FIGURE 2.1. Configuration digraph of the 3-state rule *021102022* and spatial period $\sigma = 3$.

2.1.5. Directed Graph on Labels. In this subsection, we fix the temporal period τ, instead of the spatial period σ, and obtain another digraph induced by the rule. The construction below is an adaption of label trees from [**GG12**]. We call such a graph **label digraph**.

DEFINITION 2.1.3. *Let $A = a_0 \dots a_{\tau-1}$ and $B = b_0 \dots b_{\tau-1}$ be two words from alphabet $\mathbb{Z}_n$, which we call* ***labels*** *of length τ. (While it is best to view them as vertical columns, we write them horizontally for reasons of space, as in [**GG12**].) We say that A* ***right-extends to*** *B if $f(a_i, b_i) = b_{i+1}$, for all $i \in \mathbb{Z}_+$, where (as usual) the indices are modulo τ, and we write $A \to B$. We form the* ***label digraph*** *associated with a given τ by forming an arc from a label A to a label B if A right-extends to B.*

A label $A = a_0 \dots a_{\tau-1}$ right-extends to B if and only if we preserve the temporal periodicity from a column A to the column B to its right. This fact is the basis for the Algorithm 2.1.2 below, which gives all the PS with temporal period τ. The label digraph of same rule as in Figure 2.1 and temporal period $\tau = 2$ is presented in Figure 2.2. For example, we have the arc from label 12 to 10 as $1\underline{1} \mapsto 0$, $2\underline{0} \mapsto 1$. Either of the two 3-cycles in the digraph generates the PS in Figure 1.1.

Algorithm 2.1.2.

---

input : *Label digraph $D_{\tau,f}$ of f with temporal period τ*

Find all the directed cycles in $D_{\tau,f}$

for *each cycle $A_0 \to A_1 \to \cdots \to A_{\sigma-1} \to A_0$* **do**

  form the tile T by placing labels $A_0, A_1, \dots, A_{\sigma-1}$ on successive columns.

  if *both spatial and temporal periods of T are minimal* **then**

    print *T as a PS*

  end

end

Proposition 2.1.2. *All PS of temporal period τ of f can be obtained by the Algorithm 2.1.2.*

Again, the check of periodicity is necessary due to the same reason as Section 2.1.4. However, note the differences between the two graphs: the out-degrees in Figure 2.2 are between 0 and 3, and the temporal periods are not necessarily non-decreasing along a directed path. For example, $00 \to 02$. We also note that lifting the label digraph to one on equivalence classes, although possible, makes cycles more obscure and is thus less convenient.

2.1.6. Chen-Stein Method for Poisson Approximation. The main tool we use to prove Poisson convergence is the Chen-Stein method [**BHJ92**]. We denote by Poisson(λ) a Poisson random variable with expectation λ, and by d_{TV} the total variation distance. We need the following setting for our purposes. Let I_i, for $i \in \Gamma$ be indicators of a finite family of events, which is indexed by Γ,

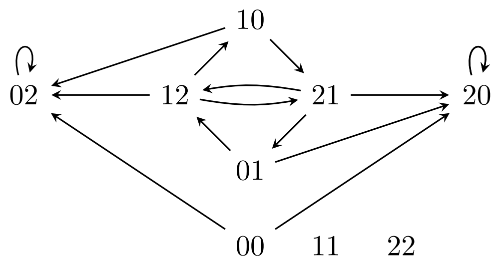


FIGURE 2.2. Label digraph of the 3-state rule *021102022* and temporal period $\tau = 2$.

$p_i = \mathbb{E}(I_i)$, $W = \sum_{i\in\Gamma} I_i$, $\lambda = \sum_{i\in\Gamma} p_i = \mathbb{E}W$, and $\Gamma_i = \{j \in \Gamma : j \neq i,\ I_i \text{ and } I_j \text{ are not independent}\}$. We quote Theorem 4.7 from [**Ros11**].

LEMMA 2.1.3. *We have*

$$d_{TV}(W, \text{Poisson}(\lambda)) \leq \min\left(1, \lambda^{-1}\right)\left[\sum_{i\in\Gamma} p_i^2 + \sum_{i\in\Gamma, j\in\Gamma_i} \left(p_i p_j + \mathbb{E}\left(I_i I_j\right)\right)\right].$$

In our applications of the above lemma, all deterministic and random quantities depend on the number n of states, which we make explicit by the subscripts. In our setting, we prove that $d_{\text{TV}}(W_n, \text{Poisson}(\lambda_n)) = \mathcal{O}(1/n)$ and that $\lambda_n \to \lambda$ as $n \to \infty$, for an explicitly given λ, which implies that W_n converges to Poisson(λ) in distribution. See Theorem 1.3.1 and Theorem 1.3.2.

2.2. Simple Tiles

We call a tile T **simple** if its lag vanishes: $\ell(T) = p(T) - s(T) = 0$. It turns out that in $\mathbb{P}(\mathcal{P}_{\tau,\sigma,n} \neq \emptyset)$, the probability of existence of PS with simple tiles provides the dominant terms, thus this class of tiles is of central importance. For example, consider the tiles

$$T_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ \\ 2 & 3 & 0 & 1 \end{matrix}, \qquad T_2 = \begin{matrix} 0 & 1 & 2 & 1 \\ \\ 2 & 1 & 0 & 1 \end{matrix}.$$

Then T_1 is simple, as $s(T_1) = p(T_1) = 4$, but T_2 is not, as $s(T_2) = 3$ and $p(T_2) = 4$. Naturally, we call a PS simple if its tile is simple.

We denote by $\mathcal{P}^{(\ell)}_{\tau,\sigma,n}$ as the set of PS whose tile T has lag ℓ. Thus the set of simple PS is $\mathcal{P}^{(0)}_{\tau,\sigma,n}$. The following lemma addresses ramifications of repeated states in simple tiles.

LEMMA 2.2.1. *Assume* $T = (a_{i,j})_{i=0,\ldots,\tau-1, j=0,\ldots,\sigma-1}$ *is a simple tile. Then*

(1) *the states on each row of T are distinct;*

(2) *if two rows of T share a state, then they are circular shifts of each other;*

(3) *the states on each column of T are distinct; and*

(4) *if two columns of T share a state, then they are circular shifts of each other.*

PROOF. *Part 1*: When $\sigma = 1$, each row contains only one state, making the claim trivial. Now, assume that $\sigma \geq 2$ and that $a_{i,j} = a_{i,k}$ for some i and $j \neq k$. We must have $a_{i,j+1} = a_{i,k+1}$ in order to avoid $p(T) > s(T)$. Repeating this procedure for the remaining states on $\texttt{row}_i$ shows that this row is periodic, contradicting part 2 of Lemma 2.1.1.

Part 2: If $a_{i,j} = a_{k,m}$, for $i \neq k$, then the states to their right must agree, i.e., $a_{i,j+1} = a_{k,m+1}$, in order to avoid $p(T) > s(T)$. Repeating this observation for the remaining states on $\texttt{row}_i$ and $\texttt{row}_k$ gives the desired result.

Part 3: Assume a column contains repeated state, say $a_{i,j} = a_{k,j}$ for some i, j and k. By part 2, $\texttt{row}_i$ is exactly the same as $\texttt{row}_k$, so that the temporal period of this tile can be reduced, a contradiction.

Part 4: Assume that $a_{i,j} = a_{k,m}$, for $j \neq m$. Then $a_{i,j+1} = a_{k,m+1}$ by parts 1 and 2. So, $a_{i+1,j+1} = a_{k+1,m+1}$ by part 1 in Lemma 2.1.1. So, $a_{i+1,j} = a_{k+1,m}$, again by parts 1 and 2. Now, repeating the previous step for $a_{i+1,j} = a_{k+1,m}$ gives the desired result. $\square$

We revisit the remark following Lemma 2.1.1: a tile may have periodic columns, but such a tile cannot be simple.

Suppose a tile $T = (a_{i,j})_{i=0,\ldots,\tau-1,j=0,\ldots,\sigma-1}$ is simple. We will take a closer look with circular shifts of rows, so we fix a row, say the first row $\texttt{row}_0$. (We could start with any row, but we pick the first one for concreteness.) Let

$$i = \min\{k = 1, 2, \ldots, \tau - 1 : \texttt{row}_k = \pi(\texttt{row}_0),\ \text{for some circular shift } \pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma\}$$

be the smallest i such that $\texttt{row}_i$ is a circular shift of $\texttt{row}_0$, and let $i = 0$ if and only if T does not have circular shifts of $\texttt{row}_0$ other than this row itself. Then this circular shift satisfies $\texttt{row}_{(j+i) \bmod \tau} = \pi(\texttt{row}_j)$, for all $j = 0, \ldots, \tau - 1$ and i is determined by the tile T; we denote this circular shift by π_T^r. We denote by π_T^c the analogous circular shift for columns.

LEMMA 2.2.2. *Let T be a simple tile of a PS, and let $d_1 = ord\ (\pi_T^r)$ and $d_2 = ord\ (\pi_T^c)$. Then d_1 and d_2 are equal and divide* $\gcd(\tau, \sigma)$.

PROOF. Fix an element as $a_{0,0}$. By Lemma 2.2.1, parts 1 and 2, $a_{0,0}$ appears in d_1 rows of T. It also appears in d_2 columns by Lemma 2.2.1, parts 3 and 4. As a consequence, $d_1 = d_2$. The divisibility follows from Lemma 2.1.2. □

LEMMA 2.2.3. *An integer $s \leq n$ is the number of states in a simple tile T of PS if and only if there exists $d \mid \gcd(\tau, \sigma)$, such that $s = \tau\sigma/d$.*

PROOF. Let $T = (a_{i,j})_{i=0,\dots,\tau-1, j=0,\dots,\sigma-1}$. Assume that $s(T) = s$ and let $d = \text{ord}\ (\pi_T^r)$. Then by Lemma 2.2.1, parts 1 and 2, the first τ/d rows of T contain all states that are in T. As a result, $s = \tau\sigma/d$ and $d = \text{ord}\ (\pi_T^r) \mid \gcd(\tau, \sigma)$.

Now assume that $d \mid \gcd(\tau, \sigma)$. Then there exists a circular shift $\pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma$, such that ord $(\pi) = d$. To form a simple tile T with $s(T) = \tau\sigma/d$ states, construct a rectangle of τ/d rows and σ columns using $\tau\sigma/d$ different states in the first τ/d rows of T. Let $\texttt{row}_{\tau/d}$ be defined by $\pi(\texttt{row}_0)$ and the subsequent rows are all automatically defined by the maps that are assigned in the first τ/d rows, by Lemma 2.1.1, part 1. □

The above lemma gives the possible values of $s(T)$ for a simple tile T and the next one enumerates the number of simple tiles of PS containing s different states.

LEMMA 2.2.4. *The number of simple tiles of PS with temporal periods τ and spatial period σ containing s states is $\varphi(d)\binom{n}{s}(s-1)!$, where $d = \tau\sigma/s$.*

PROOF. As in the proof of Lemma 2.2.3, if $s(T) = s = \tau\sigma/d$, then $d = \text{ord}(\pi_T^r)$. Moreover, there are $\binom{n}{s}(s-1)!$ ways to form the first τ/d rows of T. Then, to uniquely determine T, we need to select a circular shift $\pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma$ with ord $(\pi) = d$ and define $\texttt{row}_{\tau/d}$ to be $\pi(\texttt{row}_0)$. By Lemma 2.1.2, there are $\varphi(d)$ ways to do so. □

Consider two different simple tiles T_1 and T_2 under the rule. As the final task of this section, we seek a lower bound on the combined number of values of the rule f assigned by T_1 *and* T_2, in terms of the number of states. If $s(T_1) = s_1$, then $p(T_1) \geq s_1$, i.e., there are at least s_1 values assigned by T_1. If there are s_2' states in T_2 that are not in T_1, then there are at least s_2' additional values to assign. Therefore, a lower bound of the number of values to be assigned in T_1 and T_2 is $s_1 + s_2'$. The next lemma states that we can increase this lower bound by at least 1 when $T_1 \cap T_2 \neq \emptyset$. This fact plays an important role in the proofs of Theorem 1.3.1 and Theorem 1.3.2.

LEMMA 2.2.5. *Let T_1 and T_2 be two different simple tiles for the same rule. If T_1 and T_2 have at least one state in common, then there exist $a_{i,j} \in T_1$ and $b_{k,m} \in T_2$ such that $a_{i,j} = b_{k,m}$ and $a_{i,j+1} \neq b_{k,m+1}$.*

PROOF. As T_1 and T_2 have at least one state in common, we may pick $a_{i,j} \in T_1$ and $b_{k,m} \in T_2$, such that $a_{i,j} = b_{k,m}$. If $a_{i,j+1} \neq b_{k,m+1}$, then we are done. Otherwise, we repeat this procedure for $a_{i,j+1}$ and $b_{k,m+1}$ and see if $a_{i,j+2} = b_{k,m+2}$. We repeat this procedure until we find two pairs such that $a_{i,j+q} = b_{k,m+q}$ and $a_{i,j+q+1} \neq b_{k,m+q+1}$. If we fail to do so, then $\texttt{row}_i$ in T_1 and $\texttt{row}_k$ in T_2 must be equal, up to a circular shift. This implies that T_1 and T_2 must be the same since they are tiles for same rule, a contradiction. $\square$

2.3. Proof of Theorem 1.3.1 and Theorem 1.3.2

We will give a separate proof of Theorem 1.3.1 first, for transparency, and then we show how to adapt the argument to prove the stronger result, Theorem 1.3.2.

PROOF OF THEOREM 1.3.1. We begin with the bounds

$$\mathbb{P}\left(\mathcal{P}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{P}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{\ell=1}^{\tau\sigma} \mathbb{P}\left(\mathcal{P}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right).$$

For $\ell \geq 1$,

$$\mathbb{P}\left(\mathcal{P}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{E}\left(\#\mathcal{P}^{(\ell)}_{\tau,\sigma,n}\right) = \sum_{s=1}^{\tau\sigma} \binom{n}{s} g^{(\ell)}_{\tau,\sigma}(s) \frac{1}{n^{s+\ell}} = \mathcal{O}\left(\frac{1}{n^{\ell}}\right), \tag{2.3}$$

where $g^{(\ell)}_{\tau,\sigma}(s)$ counts the number of $\tau \times \sigma$ tiles that contain s different states and lag is ℓ. Here, $1/n^{s+\ell}$ is the probability of such a tile (determined by a PS), as there are $s + \ell$ assignments to make by a random map, and each assignment occurs independently with probability $1/n$. As $\ell \geq 1$, $\mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n} \neq \emptyset\right) = \mathbb{P}\left(\mathcal{P}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) + \mathcal{O}(1/n)$ as $n \to \infty$.

To find $\mathbb{P}\left(\mathcal{P}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right)$ as $n \to \infty$, let $1 = d_1 < \ldots < d_u = \gcd(\sigma, \tau)$ be the common divisors of τ and σ and $s_j = \tau\sigma/d_j$, for $j = 1, \ldots, u$, be the possible numbers of states in simple tiles. We index the simple tiles that have s_j states in an arbitrary way, so that $T^{(j)}_k$ be the kth simple tile that has s_j states. Here $k = 1, \ldots, N_j$ and $N_j = \varphi(d_j)\binom{n}{s_j}(s_j - 1)!$ is the number of simple tiles with s_j states (by Lemma 2.2.4). Let $I^{(j)}_k$ be the indicator random variable that $T^{(j)}_k$ is a tile determined by a PS. Let $W_n = \displaystyle\sum_{j=1}^{u}\sum_{k=1}^{N_j} I^{(j)}_k$ and

$$\begin{aligned}
\lambda_n = \mathbb{E}W_n = \sum_{j=1}^{u}\sum_{k=1}^{N_j} \mathbb{E}I^{(j)}_k &= \sum_{j=1}^{u} \varphi(d_j)\binom{n}{s_j}(s_j - 1)!\frac{1}{n^{s_j}} \\
&\xrightarrow{n\to\infty} \sum_{j=1}^{u} \varphi(d_j)\frac{1}{s_j} \\
&= \sum_{j=1}^{u} \varphi(d_j)\frac{d_j}{\tau\sigma} = \frac{1}{\tau\sigma}\sum_{d|\gcd(\tau,\sigma)} \varphi(d)d = \lambda_{\tau,\sigma}.
\end{aligned}$$

We next show that $d_{\mathrm{TV}}\left(W_n, \mathrm{Poisson}(\lambda_n)\right) \to 0$ as $n \to \infty$, which will conclude the proof. As orthogonal tiles have independent assignments, Lemma 2.1.3 implies that

$$d_{TV}\left(W_n, \mathrm{Poisson}(\lambda_n)\right) \leq \min(1, \lambda_n^{-1}) \left[\sum_{j,k}\left(\mathbb{E}I^{(j)}_k\right)^2 + \sum_{\substack{j,k,i,m \\ T^{(i)}_m \not\perp T^{(j)}_k}} \left(\mathbb{E}I^{(j)}_k \mathbb{E}I^{(i)}_m + \mathbb{E}I^{(j)}_k I^{(i)}_m\right)\right]. \tag{2.4}$$

To bound $\displaystyle\sum_{j,k}\left(\mathbb{E}I^{(j)}_k\right)^2$, fix a $j \in \{1, \ldots, u\}$ and note that

$$\sum_{k=1}^{N_j}\left(\mathbb{E}I^{(j)}_k\right)^2 = \varphi(d_j)\binom{n}{s_j}(s_j - 1)!\left(\frac{1}{n^{s_j}}\right)^2 = \mathcal{O}\left(\frac{1}{n^{s_j}}\right). \tag{2.5}$$

It follows that $\sum_{j,k} \left(\mathbb{E}I_k^{(j)}\right)^2 = \mathcal{O}\left(1/n^{\mathrm{lcm}(\tau,\sigma)}\right) \to 0$, as $n \to \infty$. It remains to bound the sum over j,k,i,m in 2.4. For a fixed $i,j \in \{1,\ldots,u\}$,

$$\begin{aligned}
&\sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ T_m^{(i)} \not\perp T_k^{(j)}}}^{N_i} \left(\mathbb{E}I_k^{(j)}\mathbb{E}I_m^{(i)} + \mathbb{E}I_k^{(j)} I_m^{(i)}\right) \\
&\leq \sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ T_m^{(i)} \cap T_k^{(j)} \neq \emptyset}}^{N_i} \left(\mathbb{E}I_k^{(j)}\mathbb{E}I_m^{(i)} + \mathbb{E}I_k^{(j)} I_m^{(i)}\right) \qquad (2.6)\\
&= \sum_{k=1}^{N_j} \sum_{h=1}^{\min(s_i,s_j)} \sum_{\substack{m=1 \\ \#\left(T_m^{(i)} \cap T_k^{(j)}\right)=h}}^{N_i} \mathbb{E}I_k^{(j)}\mathbb{E}I_m^{(i)} + \sum_{k=1}^{N_j} \sum_{h=1}^{\min(s_i,s_j)} \sum_{\substack{m=1 \\ \#\left(T_m^{(i)} \cap T_k^{(j)}\right)=h}}^{N_i} \mathbb{E}I_k^{(j)} I_m^{(i)},
\end{aligned}$$

where the inequality holds because two tiles that share an assignment have to share at least one state. Label the two triple sums on the last line of (2.6) $S_{ij}^{(1)}$ and $S_{ij}^{(2)}$. Now, fix also an $h \in \{1,\ldots,\min(s_i,s_j)\}$. We first compute

$$\begin{aligned}
\sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ \#\left(T_m^{(i)} \cap T_k^{(j)}\right)=h}}^{N_i} \mathbb{E}I_k^{(j)}\mathbb{E}I_m^{(i)} &= \varphi(d_j)\binom{n}{s_j}(s_j-1)!\varphi(d_i)\binom{s_j}{h}\binom{n-s_j}{s_i-h}(s_i-1)!\frac{1}{n^{s_j}}\frac{1}{n^{s_i}} \\
&= \mathcal{O}\left(\frac{1}{n^h}\right),
\end{aligned}$$

and therefore $S_{ij}^{(1)} = \mathcal{O}(1/n)$. Next, we estimate

$$\begin{aligned}
\sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ \#\left(T_m^{(i)} \cap T_k^{(j)}\right)=h}}^{N_i} \mathbb{E}I_k^{(j)} I_m^{(i)} &\leq \varphi(d_j)\binom{n}{s_j}(s_j-1)!\varphi(d_i)\binom{s_j}{h}\binom{n-s_j}{s_i-h}(s_i-1)!\frac{1}{n^{s_j}}\frac{1}{n^{s_i-h}}\frac{1}{n} \\
&= \mathcal{O}\left(\frac{1}{n}\right),
\end{aligned}$$

and therefore $S_{ij}^{(2)} = \mathcal{O}(1/n)$. The inequality and the three powers of n above are justified as follows: $1/n^{s_j}$ as there are s_j states in in $T_k^{(j)}$, thus at least as many assignments; $1/n^{s_i-h}$ as there are $s_i - h$ states in $T_m^{(i)}$ that are not in $T_k^{(j)}$, thus at least as many assignments; and $1/n$ by Lemma 2.2.5, as $T_m^{(i)}$ and $T_k^{(j)}$ have $h \geq 1$ states in common and so there is at least one additional assignment. It follows that $d_{TV}(W_n, \mathrm{Poisson}(\lambda_n))$ is bounded above by a constant times

$$\frac{1}{n^{\mathrm{lcm}(\tau,\sigma)}} + \sum_{i,j}\left(S_{ij}^{(1)} + S_{ij}^{(2)}\right) = \mathcal{O}\left(\frac{1}{n}\right),$$

which gives the desired result. □

We now give the proof of Theorem 1.3.2, which mainly adds some notational complexity to the previous proof.

Proof of Theorem 1.3.2. Again, we begin with the bounds

$$\mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n}^{(0)} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n}^{(0)} \neq \emptyset\right) + \sum_{\ell \neq 0} \mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n}^{(\ell)} \neq \emptyset\right),$$

where $\mathcal{P}_{\mathcal{T},\Sigma,n}^{(\ell)}$ is the set of PS with periods $(\tau,\sigma) \in \mathcal{T} \times \Sigma$ whose tile has lag ℓ. Note that the summation is finite since $\mathcal{T}$ and Σ are. For $\ell \geq 1$, as in (2.3),

$$\mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n}^{(\ell)} \neq \emptyset\right) \leq \sum_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} \mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n}^{(\ell)} \neq \emptyset\right) = \mathcal{O}\left(\frac{1}{n^{\ell}}\right).$$

As a consequence, $\mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n} \neq \emptyset\right) = \mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n}^{(0)} \neq \emptyset\right) + \mathcal{O}(1/n)$ as $n \to \infty$.

To find $\mathbb{P}\left(\mathcal{P}_{\mathcal{T},\Sigma,n}^{(0)} \neq \emptyset\right)$ as $n \to \infty$, we adopt the notation u, d_j, s_j, $T_k^{(j)}$ and $I_k^{(j)}$ from the proof of Theorem 1.3.1, for a fixed σ and τ. The dependence of these quantities on σ and τ will be suppressed from the notation, as the periods are taken from a finite range and thus do not affect the computation that follows. Now, $W_n = \sum_{(\tau,\sigma)}\sum_{j=1}^{u}\sum_{k=1}^{N_j} I_k^{(j)}$ and

$$\Lambda_n = \sum_{(\tau,\sigma)}\sum_{j=1}^{u}\sum_{k=1}^{N_j} \mathbb{E}I_k^{(j)} \to \sum_{(\tau,\sigma)\in\mathcal{T}\times\Sigma} \lambda_{\tau,\sigma} = \lambda_{\mathcal{T},\Sigma},$$

as $n \to \infty$. It remains to show that $d_{TV}(W_n, \mathrm{Poisson}(\Lambda_n)) \to 0$ as $n \to \infty$. From Lemma 2.1.3,

$$\begin{aligned} &d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\Lambda_n)) \\ &\leq \min(1, \Lambda_n^{-1}) \left[\sum_{(\tau,\sigma)}\sum_{j,k}\left(\mathbb{E}I_k^{(j)}\right)^2 + \sum_{(\tau,\sigma)}\sum_{j,k}\sum_{(\tau',\sigma')}\sum_{\substack{i,m \\ T_m^{(i)} \not\perp T_k^{(j)}}}\left(\mathbb{E}I_k^{(j)}\mathbb{E}I_m^{(i)} + \mathbb{E}I_k^{(j)}I_m^{(i)}\right)\right]. \end{aligned} \tag{2.7}$$

To bound the double sum in (2.7), observe that, for a fixed (τ, σ), the sum over j, k is $\mathcal{O}\left(1/n^{\mathrm{lcm}(\tau,\sigma)}\right)$ by (2.5). As $\min_{\tau,\sigma} \mathrm{lcm}(\tau, \sigma) \geq 1$, the double sum in (2.7) is $\mathcal{O}(1/n)$.

To bound the quadruple sum in (2.7), fix a (τ, σ) for $I_k^{(j)}$, a (τ', σ') for $I_m^{(i)}$, and $i, j \in \{1, \ldots, u\}$. Then the sum over the remaining indices is bounded by $S_{ij}^{(1)} + S_{ij}^{(2)}$, exactly as in (2.4), except that now $S_{ij}^{(1)}$ and $S_{ij}^{(2)}$ also depend on the periods. The arguments that give $S_{ij}^{(1)} = \mathcal{O}(1/n)$ and $S_{ij}^{(2)} = \mathcal{O}(1/n)$ remain equally valid, and again imply $d_{TV}(W_n, \mathrm{Poisson}(\Lambda_n)) = \mathcal{O}(1/n)$. □

The proof of Corollary 1.3.1 is now straightforward.

Proof of Corollary 1.3.1. Note that

$$\mathbb{P}(Y_{\sigma,n} \leq y) = \mathbb{P}\left(\mathcal{P}_{[1,y],\{\sigma\},n} \neq \emptyset\right) \to 1 - \exp\left(-\lambda_{[1,y],\{\sigma\}}\right),$$

as $n \to \infty$, where $\lambda_{[1,y],\{\sigma\}} = \sum_{\tau=1}^{y} \lambda_{\tau,\sigma}$. □

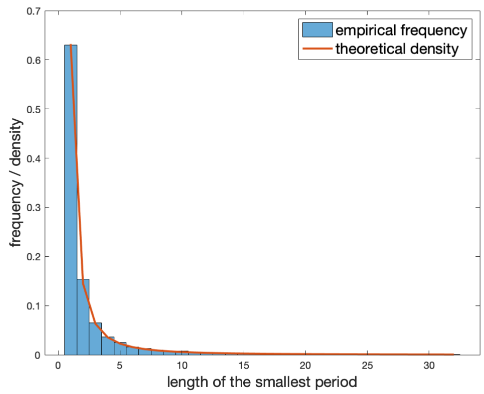
2.4. Computer Simulations and Discussion

For $\sigma = 1, 2, 3$ and 4, the corresponding $\lambda_{\tau,\sigma}$ are

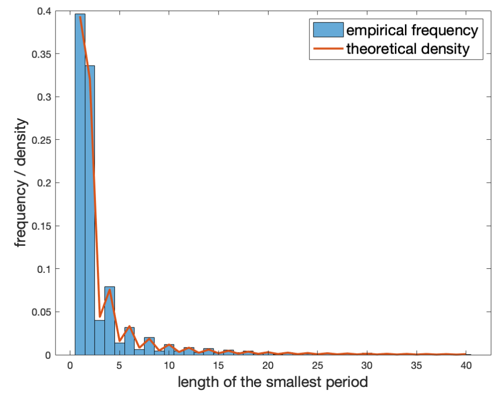
$$\lambda_{\tau,1} = \frac{1}{\tau}, \quad \lambda_{\tau,2} = \begin{cases} \frac{3}{2\tau}, & 2 \mid \tau \\ \frac{1}{2\tau}, & 2 \nmid \tau \end{cases}, \quad \lambda_{\tau,3} = \begin{cases} \frac{7}{3\tau}, & 3 \mid \tau \\ \frac{1}{3\tau}, & 3 \nmid \tau \end{cases}, \quad \lambda_{\tau,4} = \begin{cases} \frac{11}{4\tau}, & \tau = 0 \mod 4 \\ \frac{3}{4\tau}, & \tau = 2 \mod 4 \\ \frac{1}{4\tau}, & \tau = 1, 3 \mod 4 \end{cases}.$$

In Figure 2.3, we present computer simulations to test how close the distribution of $Y_{\sigma,n}$ is to its limit for moderately large n for the above four σ's. To compute $Y_{\sigma,n}(f)$, for every f in the samples, we apply Algorithm 2.1.1.

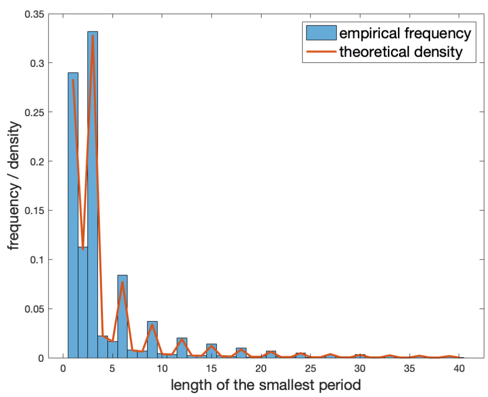
In this chapter, we initiate the study of periodic solutions for one-dimensional CA with random rules. Our main focus is the limiting probability of existence of a PS, when the rule is uniformly selected and the number of states approaches infinity, and we show (Corollary 1.3.1) that the smallest temporal period of PS with a given spatial period σ is stochastically bounded.



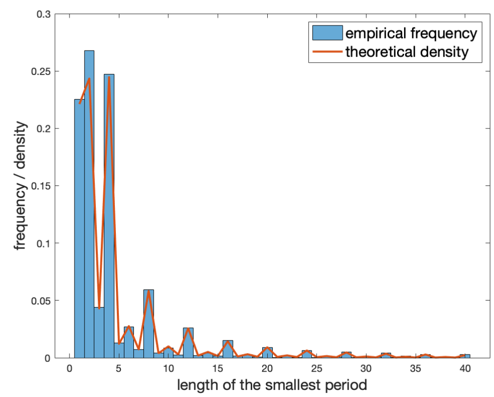
(A) $\sigma = 1$, $n = 100$



(B) $\sigma = 2$, $n = 100$



(C) $\sigma = 3$, $n = 60$



(D) $\sigma = 4$, $n = 20$

FIGURE 2.3. Lengths of the smallest temporal periods of PS with spatial periods $\sigma = 1$ to $\sigma = 4$ and various n. In each case, a histogram from a random sample from 10,000 rules is compared to the theoretical limiting distribution as $n \to \infty$, given by Corollary 1.3.1.

By a similar argument, we can also obtain an analogous result in which we fix the temporal period instead of the spatial period. Define another random variable

$$Y'_{\tau,n} = \min\left\{\sigma : \mathcal{P}_{\tau,\sigma,n} \neq \emptyset\right\},$$

which is the smallest spatial period of a PS given a temporal period τ. For example, for the four rules in Figure 1.3, we may verify that, by Algorithm 2.1.2, $Y'_{1,3}(\mathit{012200210}) = 1$ $(0 \to 0)$, $Y'_{2,3}(\mathit{021102120}) = 2$ $(12 \to 21 \to 12)$, $Y'_{3,3}(\mathit{100112122}) = 3$ $(102 \to 021 \to 210 \to 102)$ and

$Y'_{4,3}(\mathit{101201021}) = 4$ ($0101 \to 2012 \to 1010 \to 0122 \to 0101$), with one cycle that generates the minimal PS given parenthetically for each case.

COROLLARY 2.4.1. *The random variable $Y'_{\tau,n}$ converges to a nontrivial distribution as $n \to \infty$.*

Perhaps the most natural generalization of Theorem 1.3.2 would relax the condition that $\mathcal{T}$ and Σ are finite. The first case to consider surely is when either $\mathcal{T} = \mathbb{N}$ or $\Sigma = \mathbb{N}$. For example, it is clear that $\mathbb{P}(\mathcal{P}_{\mathbb{N},\mathbb{N},n} \neq \emptyset) = \mathbb{P}(\mathcal{P}_{\mathbb{N},\{1\},n} \neq \emptyset) = 1$, as any constant initial configuration eventually generates a PS with spatial period 1.

Now, consider a general $\sigma \geq 2$. Let ξ_0 be a periodic configuration of spatial period σ. Under any CA rule f, ξ_1 maintains the spatial periodicity, hence ξ_t eventually enters into a PS, whose spatial period is however a divisor of σ, not necessarily σ itself. For this reason, we cannot reach an immediate conclusion about $\lim \mathbb{P}\left(\mathcal{P}_{\mathbb{N},\{\sigma\},n} \neq \emptyset\right)$, as $n \to \infty$. We also refer the readers to the next chapter, in which the reduction of temporal periods is explored in more detail.

For a fixed temporal period τ, the matter is even less clear as a rule may not have a PS with temporal period that divides τ. For a trivial example with τ odd and $n = 2$, consider the "toggle" rule that always changes the current state and thus $\xi_{t+1} = 1 - \xi_t$ and any initial state results in temporal period 2. Thus we formulate the following intriguing open problem.

QUESTION 2.4.1. *Let $\tau, \sigma \in \mathbb{N}$. What are the behaviors of $\mathbb{P}\left(\mathcal{P}_{\{\tau\},\mathbb{N},n} \neq \emptyset\right)$ and $\mathbb{P}\left(\mathcal{P}_{\mathbb{N},\{\sigma\},n} \neq \emptyset\right)$, as $n \to \infty$?*

Another natural question addresses the case when σ and τ increase with n.

QUESTION 2.4.2. *For positive real numbers $a, b, c, d, \alpha, \beta, \gamma$ and δ, what is the asymptotic behavior of $\mathbb{P}\left(\mathcal{P}_{I_1,I_2,n} \neq \emptyset\right)$, where $I_1 = [an^\alpha, bn^\beta]$ and $I_2 = [cn^\gamma, dn^\delta]$?*

A wider topic for further research is to investigate how different the behavior of the shortest temporal period changes if we choose a random rule from a subset of the set of all rules. There are, of course, many possibilities for such a subset, and we selected two natural ones below. In each case, we denote the resulting random variable with the same letter $Y_{n,\sigma}$.

A rule is **left permutative** if the map $\psi_b : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $\psi_b(a) = f(a, b)$ is a permutation for every $b \in \mathbb{Z}_n$. Permutative rules, such as the famous *Rule 30* [**Wol86**, **Jen86**], are good candidates for generation of long temporal periods.

QUESTION 2.4.3. *Let $\mathcal{L}$ be the set of all $(n!)^n$ permutative rules. Choosing one of these rules uniformly at random from $\mathcal{L}$, what is the asymptotic behavior of $Y_{n,\sigma}$?*

Our final question concerns the most widely studied special class of CA, the additive rules. Such a rule is given by $f(a, b) = \alpha a + \beta b$, for some $\alpha, \beta \in \mathbb{Z}_n$.

QUESTION 2.4.4. *Let $\mathcal{A}$ be the set of all n^2 additive rules. Again, what is the asymptotic behavior of $Y_{n,\sigma}$ if a rule from $\mathcal{A}$ is chosen uniformly at random?*

CHAPTER 3

Maximal Temporal Period of Periodic Solutions of a Random Rule

In this chapter, we assume an arbitrary $r \geq 2$.

The aim of this chapter is to further understanding of temporal periods of a random CA's PS with a fixed period. To this end, the particular random quantity we address in this chapter is the longest temporal period, to complement the work in Chapter 2 on the shortest one.

In particular, we prove Theorem 1.3.3, which is restated as follows.

THEOREM (1.3.3 restated). *Fix a number of neighbors r and a spatial period $\sigma \leq r$. Then $X_{\sigma,n}/n^{\sigma/2}$ converges in distribution, as $n \to \infty$, to a nontrivial limit.*

3.1. The Directed Graph on Equivalence Classes of Configurations

In this section, we introduce a variant of the configuration digraph that we introduced in Chapter 2. First, we generalize Definition 2.1.2 on down-extension.

DEFINITION 3.1.1. *Fix a spatial period $\sigma \geq 1$ and an r-neighbor rule f. Let $A = a_0 \ldots a_{\sigma-1}$ and $B = b_0 \ldots b_{\sigma-1}$ be two configurations. We say that A **down-extends to** B if the rule maps A to B in one update, that is,*

$$f(a_{i-r+1}, \ldots, a_i) = b_i, \quad i = 0, \ldots, \sigma - 1,$$

and we write $A \searrow B$.

For example, if f is the rule with the PS of Figure 1.1, and $\sigma = 3$, then $120 \searrow 211 \searrow 120$, etc.

DEFINITION 3.1.2. *Fix a spatial period σ and suppose σ' is a proper divisor of σ. A configuration $A = a_0 \ldots a_{\sigma-1}$ is **periodic** with **period** σ' if it can be divided into $\sigma/\sigma' > 1$ identical words, and σ' is the smallest such number. If no such σ' exists, A is **aperiodic**.*

In Chapter 2, the relation $A \searrow B$ defines a directed graph on all configurations with length σ. We now define a convenient variant, which we call the **digraph on equivalence classes** (**DEC**) $G_\sigma(f) = (V_\sigma, E_\sigma(f))$, associated with f and σ. As in Chapter 2, we say that A **is equal to** B **up to circular shift**, or in short A **is equivalent to** B, if there is a circular shift $\pi : \mathbb{Z}_n^\sigma \to \mathbb{Z}_n^\sigma$ such that $A = \pi(B)$. Under this equivalence relation, $\mathbb{Z}_n^\sigma$ is partitioned into equivalence classes, which inherit periodicy or aperiodicity from their representatives. Note that the cardinality of an aperiodic equivalence class is σ, while the cardinality of a periodic equivalence class is a proper divisor of σ. We regard each aperiodic equivalence class as a single vertex, called **aperiodic vertex**, of the DEC; thus there are $T(\sigma, n)/\sigma$ aperiodic vertices, where T is defined as in Lemma 1.3.1.

Next, we combine periodic classes together to form vertices called **periodic vertices**, so that, with one possible exception, each vertex contains σ configurations. This can be achieved for a large enough n (certainly for $n \geq \sigma^2$) as follows. For each proper division $\sigma' > 1$ of σ, divide all configurations with period σ' into sets, which all have cardinality σ, except for possibly one set; fill that last set with the necessary number of period-1 configurations to make its cardinality σ. Each of these sets represents a different periodic vertex. At the end, we have $\iota = n^\sigma - T(\sigma, n) - \sigma \lfloor (n^\sigma - T(\sigma, n))/\sigma \rfloor < \sigma$ leftover period-1 configurations, which we combine into the exceptional **initial periodic vertex**, denoted by v_0. We let V_a and V_p be the sets of aperiodic and non-initial periodic vertices, so that the vertex set is $V_\sigma = V_a \cup V_p \cup \{v_0\}$.

Having completed the definition of the vertex set of DEC, we now specify its set $E_\sigma(f)$ of directed edges. An arc $\overrightarrow{uv} \in E_\sigma(f)$ if and only if: 1. $u \in V_a$, $v \in V_\sigma$; and 2. there exist $A \in u$ and $B \in v$ such that $A \searrow B$.

An example of DEC with $\sigma = 2$ of a 5-state rule is given in Figure 3.1. In this example, $V_p = \{\{00, 11\}, \{22, 33\}\}$, $v_0 = \{44\}$ and other vertices are all in V_a. We do not completely specify the rule that generate this DEC, as different CA rules (even a with different range r) may induce the same DEC.

The set of all DEC's generated by r-neighbor n-state rules is denoted by $\mathcal{G}_\sigma = \mathcal{G}_{\sigma,r,n}$. Choosing f at random, we obtain a random DEC denoted by $G_\sigma = (V_\sigma, E_\sigma) \in \mathcal{G}_\sigma$. We now give the resulting distribution of G_σ.

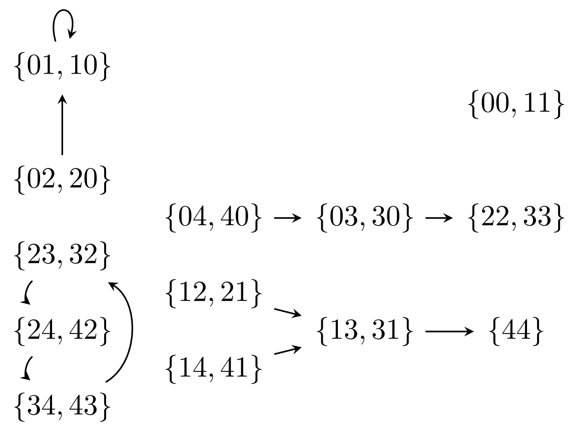


FIGURE 3.1. DEC of a 2-neighbor, 5-state rule.

LEMMA 3.1.1. *For any $u \in V_a$ and $v \in V_\sigma$*

$$\mathbb{P}(\overrightarrow{uv} \in E_\sigma) = \begin{cases} \frac{\sigma}{n^\sigma}, & \textit{if } v \neq v_0 \\ \\ \frac{\iota}{n^\sigma}, & \textit{if } v = v_0 \end{cases}.$$

Moreover, the outgoing edges for different vertices in V_a are independent.

PROOF. For any configurations $A \in u$ and $B \in v$, $\mathbb{P}(A \searrow B) = 1/n^\sigma$. Then $\mathbb{P}(\overrightarrow{uv} \in E_\sigma) = \#v\mathbb{P}(A \searrow B)$, giving the desired result, where $\#v$ denotes the number of elements in v. $\square$

3.2. The Connection between DEC and PS

In a DEC, we call a vertex to be a **cemetery** vertex if it is either a periodic vertex or there is a directed path from it to a periodic vertex (which, we repeat, is a set of configurations with spatial periods less than σ). Otherwise, a vertex is said to be **non-cemetery**. For example, in Figure 3.1, the vertices $\{00, 11\}$, $\{22, 33\}$ and $\{44\}$ are cemetery as they are periodic; $\{03, 30\}$, $\{04, 40\}$, $\{12, 21\}$, $\{14, 41\}$ and $\{13, 31\}$ are also cemetery as there exists a directed path from each of them to a periodic vertex; other five vertices are non-cemetery. The reason that we declare a vertex $C \ni A$ of length σ to be cemetery is that when the CA updates to configuration A, the spatial period is reduced and the dynamics cannot produce a PS of spatial period σ. For example, in the DEC of Figure 3.1, a PS with $\sigma = 2$ cannot contain the configuration 21, as its appearance leads to 44, which has spatial period 1.

It is also important to note that different rules can have the same DEC. In particular, a cycle in a DEC may generate PS with different temporal periods depending on the rule. We illustrate this by the $\sigma = 2$ example in Figure 3.1. First, we locate a directed cycle, say, the one of length 3. Using a configuration from any vertex on the cycle, say 23, as the initial configuration,run the rule starting with 23 until 23 appears again. Now, the temporal period can be either 3 or 6, depending on the rule f. Namely, if the rule assignments result in, say, $23 \searrow 24 \searrow 43 \searrow 23$, then $\tau = 3$, while if they are $23 \searrow 24 \searrow 43 \searrow 32$, then $\tau = 6$. In general, if a cycle in DEC has length ℓ, then the corresponding temporal period of the PS generated by this cycle may have length $d\ell$, where d is any divisor of σ.

For an arbitrary $G \in \mathcal{G}_\sigma$, define $M(G)$ to be the number of directed cycles in G. (For example, $M(G) = 2$ for G in Figure 3.1.) Let $C^{(i)}(G)$ be the ith longest direct cycle of G. Then let $L_i(G)$, $i = 1, 2, \ldots$, be the length of $C^{(i)}(G)$, with $L_i(G) = 0$ for $i > M(G)$. Then, for a rule f, define $M(f) = M(G_\sigma(f))$ and $L_i(f) = L_i(G_\sigma(f))$. Furthermore, if a PS of temporal period $d\ell$ results from a cycle C of length ℓ in $G_\sigma(f)$, we say that C has **expanding number** d under f, and use the notation $E_f(C) = d$. We let $E_i(f) = E_f(C^{(i)}(G_\sigma(f)))$, again defined to be 0 when $C^{(i)}(G_\sigma(f))$ does not exist, i.e., when $i > M(f)$. The following lemma explains how the cycle lengths in DEC and expending numbers determine the longest temporal period.

Lemma 3.2.1. *Let f be a CA rule and $G_\sigma(f)$ be its DEC of period σ. Then we have*

$$X_{\sigma,n}(f) = \max \{L_i(f) \cdot E_i(f) : i = 1, 2, \ldots\}.$$

Moreover, if $C^{(k)}(G_\sigma(f))$ is the longest cycle that is σ-expanded, then

$$X_{\sigma,n}(f) = \max \{L_k(f) \cdot \sigma, L_i(f) \cdot E_i(f) : i = 1, 2, \ldots, k-1\}.$$

Proof. The first part is clear from the definition, and the second part follows as σ is the largest possible expanding number. $\square$

As a consequence of the above lemma, our task is to study the properties of DEC and expanding numbers when a rule is randomly selected. A random DEC is essentially a random mapping, after eliminating cemetery vertices, as we will see. We formulate a lemma on expanding numbers next.

LEMMA 3.2.2. *Let $G \in \mathcal{G}_\sigma$ be a fixed DEC, and $\sigma \leq r$. Select a rule f at random. Then, conditioned on the event $\{G_\sigma(f) = G\}$, the random variables $E_i(f)$, $i = 1, \ldots, M(G)$, are independent. Also*

$$\mathbb{P}\left(E_i(f) = d \,\middle|\, G_\sigma(f) = G\right) = \frac{\varphi(d)}{\sigma},$$

for $i = 1, \ldots, M(G)$ and $d \mid \sigma$.

PROOF. Let a cycle $C^{(i)}(G)$ be $v_1 \to v_2 \to \cdots \to v_k \to v_1$. Let A_j's be configurations of length σ such that $A_j \in v_j$, $j = 1, \ldots, k$. Then there are circular shifts, π_j's, $j = 1, \ldots, k$, such that $A_1 \searrow \pi_2(A_2) \searrow \ldots \searrow \pi_k(A_k) \searrow \pi_1(A_1)$, under rule f. Now, $E_i(f) = d$ if and only if $\mathrm{ord}(\pi_1) = d$, which is independent from other cycles as $\sigma \leq r$ and has the desired probability by Lemma 2.1.2. $\square$

In summary, we may study the probabilistic behavior of $X_{\sigma,n}$ by moving from the sample space $\Omega_{r,n}$ to $\mathcal{G}_\sigma \times \Xi_\sigma^\infty$, where $\Xi_\sigma = \{d \in \mathbb{N} : d \mid \sigma\}$. (However, note that only $M(G) < T(\sigma, n)$ first entries of the second component are used.) The marginal probability distributions on components are independent from each other. The distribution on $\mathcal{G}_\sigma$ is given in Lemma 3.1.1, while the distribution on each component of Ξ_σ^∞ is given by Lemma 3.2.2: $\mathbb{P}(\{w\}) = \varphi(w)/\sigma$, for $w \in \Xi_\sigma$. If the random variables $T_i : \Xi_\sigma \to \Xi_\sigma$ are defined to be identities, then the distribution of $X_{\sigma,n}$ is given by

$$\max\left\{L_i\left(G_\sigma\right) \cdot T_i(w) : i = 1, 2, \ldots\right\} =: \max\left\{L_i \cdot T_i : i = 1, 2, \ldots\right\}.$$

Let $K_\sigma = \min\{i : T_i = \sigma\}$ be a random variable on Ξ_σ^∞, representing the smallest index of T_i's that is equal to σ. Then $\mathbb{P}(K_\sigma = k) = (1 - \varphi(\sigma)/\sigma)^{k-1} (\varphi(\sigma)/\sigma)$ for $k \geq 1$, i.e., K_σ is Geometric $(\varphi(\sigma)/\sigma)$. Then we may write

$$X_{\sigma,n} = \max\left\{L_i \cdot T_i', L_{K_\sigma}\sigma, i = 1, 2, \ldots, K_\sigma - 1\right\},$$

where T_i' are independent (of each other and of L_i and K_σ) random variables with distribution $\mathbb{P}(T_i' = d) = \mathbb{P}(T_i = d \mid T_i \neq \sigma) = \varphi(d)\,(\sigma - \varphi(\sigma))$, for $d \mid \sigma$ and $d \neq \sigma$.

3.3. Random Mappings

In this section, we discuss a result about the cycle structure of random mapping, indicating that the joint distribution of the longest k cycles converges after a proper scaling.

We will consider the function space $\mathcal{R}_N = \{g : \mathbb{Z}_N \to \mathbb{Z}_N\}$ containing all functions from $\mathbb{Z}_N$ into itself. Clearly $\#\mathcal{R}_N = N^N$. A finite sequence $x_0, \ldots, x_{\ell-1} \in \mathbb{Z}_N$ is a **cycle** of length ℓ if $g(x_0) = x_1, g(x_1) = x_2, \ldots, g(x_{\ell-2}) = x_{\ell-1}$ and $g(x_{\ell-1}) = x_0$. We call g a **random mapping** if g is randomly and uniformly selected from $\mathcal{R}_N$. Let $P_N^{(k)}$ be the random variable representing the kth longest cycle length of a random mapping from $\mathcal{R}_N$. More extensively studied function space is $\mathcal{S}_N = \{g : \mathbb{Z}_N \to \mathbb{Z}_N : g \text{ is bijective}\}$ containing all permutations of $\mathbb{Z}_N$. Clearly, $\#\mathcal{S}_N = N!$ and a cycle can be defined in the same way. We call g a **random permutation** if g is randomly and uniformly selected from $\mathcal{S}_N$ and we use $Q_N^{(k)}$ to denote the random variable representing the kth longest cycle length of a random permutation from $\mathcal{S}_N$. The probabilistic properties of $P_N^{(k)}$ and $Q_N^{(k)}$ have been investigated in a number of papers, including [**AT92**, **FO89**, **ABT03**, **HJ02**].

What is relevant to us is the distribution of $\left(P_N^{(1)}, P_N^{(2)}, \ldots, P_N^{(k)}\right)$ as $N \to \infty$, for which we are not aware of a direct reference. We can, however, use the fact that for a random mapping, conditioning on the set of elements that belong to cycles generates a random permutation. To begin, we let M_N be the number of elements from $\mathbb{Z}_N$ that belong to cycles of a random mapping from $\mathcal{R}_N$. The following well-known result provides the distribution of M_N, see [**AT92**] or [**Bol01**].

Lemma 3.3.1. *We have*

$$\mathbb{P}\left(M_N = s\right) = \frac{s}{N} \prod_{j=1}^{s-1} \left(1 - \frac{j}{N}\right), \quad s = 1, \ldots, N.$$

The next result is adapted from Corollary 5.11 in [**ABT03**].

Proposition 3.3.1. *As $N \to \infty$,*

$$\frac{1}{N}\left(Q_N^{(1)}, Q_N^{(2)}, \ldots\right) \to \left(Q^{(1)}, Q^{(2)}, \ldots\right), \text{ in distribution,}$$

in $\Delta = \{(x_1, x_2, \ldots) \subset (0,1)^\infty : \sum_i x_i = 1\}$. Here, for each k, $\left(Q^{(1)}, Q^{(2)}, \ldots, Q^{(k)}\right)$ has density

$$q^{(k)}(x_1, \ldots, x_k) = \frac{1}{x_1 x_2 \cdots x_k}\left(1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{I_j(x)} \frac{dy_1 \cdots dy_j}{y_1 \ldots y_j}\right),$$

on Δ, where $I_j(x)$ is the set of $(y_1, \ldots, y_j)$ that satisfy

$$\min\{y_1, \ldots, y_j\} > x^{-1} \text{ and } y_1 + \cdots + y_j < 1$$

and

$$x = \frac{1 - x_1 - \cdots - x_k}{x_k}.$$

LEMMA 3.3.2. *For a fixed N, let*

$$h_N(x) = \frac{s}{\sqrt{N}} \prod_{j=1}^{s-1} \left(1 - \frac{j}{N}\right),$$

for $x \in \left((s-1)/\sqrt{N}, s/\sqrt{N}\right]$ and $s = 1, 2, \ldots$. Then $h_N(x) \le 4 \max(x, 1) \exp\left(-x^2/2\right)$ for all $x > 0$, which is integrable on $(0, \infty)$. Also, $h_N(x) \to x \exp\left(-x^2/2\right)$, as $N \to \infty$, for all $x \in (0, \infty)$.

PROOF. Since $h_N(x) = 0$ for $x > \sqrt{N}$, it suffices to show the inequality for $x \le \sqrt{N}$, i.e., $s \le N$. Since $1 - j/N < \exp(-j/N)$, it follows that $\prod_{j=1}^{s-1} (1 - j/N) < \exp\left(-s^2/(2N)\right) \exp(s/(2N)) < 2 \exp\left(-s^2/(2N)\right)$, for $s \le N$. So, if $x \in \left((s-1)/\sqrt{N}, s/\sqrt{N}\right]$, then

$$h_N(x) \le 2 \frac{s}{\sqrt{N}} \exp\left(-\frac{x^2}{2}\right).$$

When $s = 1$, $s/\sqrt{N} \le 2$, while for $s \ge 2$, $s/\sqrt{N} \le 2(s-1)/\sqrt{N} \le 2x$, proving the inequality. To prove convergence, observe that

$$\begin{aligned} h_N(x) &= \frac{\lceil \sqrt{N} x \rceil}{\sqrt{N}} \prod_{j=1}^{\lceil \sqrt{N} x \rceil - 1} \left(1 - \frac{j}{N}\right) \\ &= \frac{\lceil \sqrt{N} x \rceil}{\sqrt{N}} \prod_{j=1}^{\lceil \sqrt{N} x \rceil - 1} \exp\left\{-\frac{j}{N} + \mathcal{O}\left(\frac{j^2}{N^2}\right)\right\} \\ &= \frac{\lceil \sqrt{N} x \rceil}{\sqrt{N}} \exp\left[-\frac{\lceil \sqrt{N} x \rceil \left(\lceil \sqrt{N} x \rceil - 1\right)}{2N} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\right] \\ &\to x \exp\left(-x^2/2\right), \end{aligned}$$

as $N \to \infty$. $\square$

THEOREM 3.3.1. *For any $k = 1, 2, \ldots$, let $P_N^{(k)}$ be the kth longest cycle length in a random mapping from $\mathcal{R}_N$. Then*

$$N^{-1/2} \left(P_N^{(1)}, P_N^{(2)}, \ldots, P_N^{(k)}\right)$$

converges to a nontrivial joint distribution, as $N \to \infty$.

Proof. Conditioning on the event that a set $S \subset \mathbb{Z}_N$ is exactly the set of elements of $\mathbb{Z}_N$ that belong to cycles, the random mapping is a random permutation of S. It follows that for any bounded continuous function $\phi : \mathbb{R}^k \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[\phi\left(\frac{P_N^{(1)}}{\sqrt{N}}, \ldots, \frac{P_N^{(k)}}{\sqrt{N}}\right)\right] &= \sum_{s=1}^{N} \mathbb{E}\left[\phi\left(\frac{P_N^{(1)}}{\sqrt{N}}, \ldots, \frac{P_N^{(k)}}{\sqrt{N}}\right) \,\middle|\, M_N = s\right] \mathbb{P}\left(M_N = s\right) \\
&= \sum_{s=1}^{N} \mathbb{E}\left[\phi\left(\frac{Q_s^{(1)}}{\sqrt{N}}, \ldots, \frac{Q_s^{(k)}}{\sqrt{N}}\right)\right] \frac{s}{N} \prod_{j=1}^{s-1}\left(1 - \frac{j}{N}\right) \\
&= \sum_{s=1}^{N} \mathbb{E}\left[\phi\left(\frac{Q_s^{(1)}}{s}\frac{s}{\sqrt{N}}, \ldots, \frac{Q_s^{(k)}}{s}\frac{s}{\sqrt{N}}\right)\right] \frac{s}{N} \prod_{j=1}^{s-1}\left(1 - \frac{j}{N}\right).
\end{aligned}$$

Define $\widetilde{h}_N : \mathbb{R} \to \mathbb{R}$

$$\widetilde{h}_N(x) = \mathbb{E}\left[\phi\left(\frac{Q_s^{(1)}}{s}\frac{s}{\sqrt{N}}, \ldots, \frac{Q_s^{(k)}}{s}\frac{s}{\sqrt{N}}\right)\right] \frac{s}{\sqrt{N}} \prod_{j=1}^{s-1}\left(1 - \frac{j}{N}\right),$$

for $x \in \left((s-1)/\sqrt{N}, s/\sqrt{N}\right]$, $s = 1, 2, \ldots$ By Lemma 3.3.2 and Proposition 3.3.1, $\widetilde{h}_N$ is bounded by an integrable function and, for every fixed $x \geq 0$,

$$\begin{aligned}
\lim_{N\to\infty} \widetilde{h}_N(x) &= \lim_{N\to\infty} \mathbb{E}\left[\phi\left(\frac{Q^{(1)}_{\lceil\sqrt{N}x\rceil}}{\lceil\sqrt{N}x\rceil}\frac{\lceil\sqrt{N}x\rceil}{\sqrt{N}}, \ldots, \frac{Q^{(k)}_{\lceil\sqrt{N}x\rceil}}{\lceil\sqrt{N}x\rceil}\frac{\lceil\sqrt{N}x\rceil}{\sqrt{N}}\right)\right] x \exp\left(-\frac{x^2}{2}\right) \\
&= \mathbb{E}\left[\phi\left(Q^{(1)}x, \ldots, Q^{(k)}x\right)\right] x \exp\left(-\frac{x^2}{2}\right).
\end{aligned}$$

Then,

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\phi\left(\frac{P_N^{(1)}}{\sqrt{N}}, \ldots, \frac{P_N^{(k)}}{\sqrt{N}}\right)\right] \\
&= \lim_{N\to\infty} \int_0^\infty \widetilde{h}_N(x)dx \\
&= \int_0^\infty \mathbb{E}\left[\phi\left(Q^{(1)}x, \ldots, Q^{(k)}x\right)\right] x \exp\left(-\frac{x^2}{2}\right) dx,
\end{aligned}$$

by dominated convergence theorem. □

As a consequence, we obtain the following convergence in distribution.

Lemma 3.3.3. *Let T_j''s, for $j = 1, 2, \ldots,$ be i.i.d. with*

$$\mathbb{P}\left(T_j' = d\right) = \frac{\varphi(d)}{\sigma - \varphi(\sigma)},$$

for all divisors $d \neq \sigma$ of σ, and independent of the random mapping. Let

$$D_N^{(k)} = \max\left\{P_N^{(k)} \cdot \sigma, P_N^{(j)} \cdot T_j' : j = 1, 2, \ldots, k-1\right\}.$$

Then $N^{-1/2} D_N^{(k)}$ converges to a nontrivial distribution, for any k and σ.

Proof. Note that T_j''s do not depend on N. So the vector

$$N^{-1/2}\left(P_N^{(1)} T_1', \ldots, P_N^{(k-1)} T_{k-1}', P_N^{(k)} \sigma\right)$$

converges in distribution as $N \to \infty$. The conclusion follows by continuity. □

In the sequel, we denote by $D^{(k)}$ a generic random variable with the limiting distribution of $N^{-1/2} D_N^{(k)}$.

3.4. Proof of Theorem 1.3.3

3.4.1. The Case $\sigma = 1$. In this case, a DEC does not have cemetery vertices thus our problem simply reduces to a random mapping problem. To be precise,

$$\frac{X_{1,n}}{n^{1/2}} = \frac{L_1}{n^{1/2}} =_d \frac{P_n^{(1)}}{n^{1/2}}, \tag{3.1}$$

which converges in distribution by Theorem 3.3.1. The first equality in (3.1) holds because a cycle in a DEC cannot be expanded when $\sigma = 1$ and the second equality in (3.1) is true because there are no cemetery states for $\sigma = 1$.

For a general σ, the problem may be handled similarly to the case of $\sigma = 1$ only after eliminating the cemetery vertices. As a consequence, we must determine the behavior of $C_n = C_{\sigma,n}$ from Section 1.4, which we may reinterpret as the random variable representing the number of non-cemetery vertices in a DEC of spatial period σ. The strategy is as follows: construct the random DEC via a sequential algorithm that naturally provides a system of stochastic difference equations for the number of non-cemetery classes with C_n related to a hitting time; then show that the solution of the stochastic difference equations, appropriately scaled, converges to a diffusion, giving the asymptotic behavior of C_n.

ALGORITHM 3.4.1.

```
C_A ← V_p ∪ {v_0} or V_p, if v_0 does not exist          // Active cemetery vertices
C_P ← ∅                                                  // Passive cemetery vertices
C_N ← V_a                                                // Non-cemetery vertices
E ← ∅                                                    // Set of arcs
k ← 0
if v_0 ∈ C_A then                                        // If v_0 exists
    C_A ← C_A \ {v_0}
    C_P ← C_P ∪ {v_0}                                    // Make it passive
    Let β_0 ~ Binomial(Y_0, ι/n^σ)
    Pick random v_1, ..., v_{β_0} in C_N   // Select non-cemetery vertices that map to v_0
    for j = 1, ..., β_0 do
        E ← E ∪ {v_j v_0}                                // Add the arcs to the set of arcs
        C_A ← C_A ∪ {v_j}                                // Make the vertices active cemetery
        C_N ← C_N \ {v_j}
    end
    Y_0 ← #C_N              // Update the number of temporary non-cemetery vertices
    Z_0 ← #C_A              // Update the number of active cemetery vertices
    k ← 1
end
while #C_A > 0 do     // When C_A = ∅, the non-cemetery vertices are determined
    Pick an arbitrary v ∈ C_A            // Pick an arbitrary active cemetery vertex v
    C_A ← C_A \ {v}
    C_P ← C_P ∪ {v}                                      // Make v passive
    Let β_k ~ Binomial(Y_k, 1/(Y_k + Z_k))
    Pick random v_1, ..., v_{β_k} in C_N   // Select non-cemetery vertices that map to v
    for j = 1, ..., β_k do
        E ← E ∪ {v_j v}                                  // Add the arcs to the set of arcs
        C_A ← C_A ∪ {v_j}                                // Make the vertices active cemetery
        C_N ← C_N \ {v_j}
    end
    Y_k ← #C_N              // Update the number of non-cemetery vertices
    Z_k ← #C_A              // Update the number of active cemetery vertices
    k ← k + 1
end
for v ∈ C_N do                           // Assign arcs among non-cemetery vertices
    Pick a u uniformly from C_N
    E = E ∪ {vu}
end
```

3.4.2. Construction of a Random DEC and the Difference Equations. Recall the notation from Section 3.1 and Lemma 3.1.1. Algorithm 3.4.1 formally describes a way of generating a random DEC that sequentially adds cemetery vertices until all are gathered. The idea of this algorithm is to start with the set of cemetery vertices, which are essentially the equivalence classes of periodic configurations. Then determine the vertices of DEC that map into those and then iteratively work backwards until the set of all vertices on an oriented path that leads to the periodic configurations is established.

The algorithm specifies the evolution of the set of cemetery vertices, which are separated into active and passive ones, initially all active. After the kth step ($k = 0, 1, \dots$), we let Y_k and Z_k be the number of non-cemetery and active cemetery vertices. In the kth step ($k = 0, 1, \dots$), we pick an active cemetery vertex v, making it passive. We also select β_k non-cemetery vertices that map to v, where $\beta_k \sim \text{Binomial}\,(Y_k, 1/(Y_k + Z_k))$. (If $k = 0$ and v_0 exists, the initial pick is v_0 and the probability changes accordingly.) This distribution is justified by Lemma 3.1.1, i.e., all non-cemetery vertex share the same probability of mapping into a vertex that is not passive cemetery. We make those β_k vertices active cemetery, because each one of them has the ability to "absorb" non-cemetery vertices (thus is active), while itself maps into a periodic class of a lower period along a directed path (thus is cemetery). The above procedure determines all cemetery classes in the **while** loop. In the final **for** loop, we assign a unique target uniformly for each non-cemetery vertex.

We observe $Y_{k+1} = Y_k - \beta_k$ and $Z_{k+1} = Z_k + \beta_k - 1$. To prepare for establishing the convergence to a diffusion, we let $\Delta Y_k = Y_{k+1} - Y_k$, and $\Delta Z_k = Z_{k+1} - Z_k$, we obtain the stochastic difference equation for k such that $Z_k \geq 0$,

$$
\begin{cases}
\Delta Y_k = -1 - \Delta Z_k = -\beta_k \\
\Delta Z_k = \beta_k - 1 = \frac{Y_k}{Y_k+Z_k} - 1 + \Delta B_k \sqrt{\frac{Y_k}{Y_k+Z_k}\left(1 - \frac{1}{Y_k+Z_k}\right)}
\end{cases}
, \tag{3.2}
$$

where β_k's are independent and

$$\beta_k \sim \text{Binomial}\left(Y_k, \frac{1}{Y_k + Z_k}\right),$$

for $k=1,2,\ldots$, thus

$$\Delta B_k = \frac{\beta_k - Y_k/(Y_k+Z_k)}{\sqrt{\frac{Y_k}{Y_k+Z_k}\left(1-\frac{1}{Y_k+Z_k}\right)}}.$$

For the initial condition, we have

$$Y_0 = \begin{cases} -S_0 + \frac{T(\sigma,n)}{\sigma}, & \text{if } \iota = 0 \\ -S_1 + \frac{T(\sigma,n)}{\sigma}, & \text{if } \iota \neq 0 \end{cases},$$

and

$$Z_0 = \begin{cases} S_0 - 1 + \lfloor \frac{n^\sigma - T(\sigma,n)}{\sigma} \rfloor, & \text{if } \iota = 0 \\ S_1 - 1 + \lfloor \frac{n^\sigma - T(\sigma,n)}{\sigma} \rfloor, & \text{if } \iota \neq 0 \end{cases},$$

where $S_0 \sim \text{Binomial}\,(T(\sigma,n)/\sigma, \sigma/n^\sigma)$ and $S_1 \sim \text{Binomial}\,(T(\sigma,n)/\sigma, \iota/n^\sigma)$. To define the processes for *all* $k = 0,1,\ldots,N-1$, we stop Y_k and Z_k once Z_k hits zero.

3.4.3. Convergence to a Diffusion. Let $N = \#V_\sigma = n^\sigma/\sigma + \mathcal{O}(n^{\sigma/2})$ be the total number of vertices. We scale Y_k and Z_k by dividing by N and $\sqrt{N}$, respectively. We do so as Y_k will converge to the time coordinate and Z_k to the space coordinate in the diffusion. To be more precise, consider the 2-dimensional process $\xi_{k,N} = \left(\xi_{k,N}^{(1)}, \xi_{k,N}^{(2)}\right)$, for $k = 0,\ldots,N-1$, where $\xi_{k,N}^{(1)} = Y_k/N$ is the scaled number of non-cemetery states and $\xi_{k,N}^{(2)} = Z_k/\sqrt{N}$ is the scaled number of active cemetery states. For a fixed $\xi_{k,N}$, let $\tau = \tau(\xi_{k,N}) = \inf\{k/N : \xi_{k,N}^{(2)} \le 0\}$ be the hitting time of zero for the second coordinate. We are thus interested in this question: *when the number of active cemetery vertices is zero, what is the limiting distribution of proportion of non-cemetery vertices?* In other words, what is $\lim \mathbb{P}\left(\xi_\tau^{(1)} \le x\right)$, for $x \in (0,1)$, as $N \to \infty$? We will prove the following result, which is a restatement of Theorem 1.4.1.

THEOREM 3.4.1. *As $N \to \infty$, $\xi_\tau^{(1)} \to 1 - \tau(\eta)$ in distribution, where $\tau(\eta) = \inf\{t : \eta(t) = 0\}$ and $\eta(t)$ satisfies*

$$\eta(t) = p(\sigma) - \int_0^t \frac{\eta(s)}{1-s} ds - B_t,$$

where B_t is the standard Brownian motion and $p(\sigma) = 1/\sqrt{\sigma}$ if σ is even and $p(\sigma) = 0$, otherwise. In particular, when σ is even, $\xi_\tau^{(1)}$ converges to a nontrivial limiting distribution, while when σ is odd, $\xi_\tau^{(1)} \to 1$ in probability.

To avoid excessive notation, we let τ stand for the hitting time of 0 in both discrete and continuous cases. Our strategy in proving Theorem 3.4.1 is to verify the conditions in [**Kus74**] for a solution of a stochastic difference equation to converge to a diffusion. However, trying to prove this directly for $\xi_{k,N}$ runs into uniform continuity and boundedness problems, so we need an intermediate process $\widetilde{\xi}_{k,N}$. For a fixed N, we define the stochastic difference equations of $\widetilde{\xi}_{k,N} = \left(\widetilde{\xi}^{(1)}_{k,N}, \widetilde{\xi}^{(2)}_{k,N}\right)$ by giving $\Delta\widetilde{\xi}^{(i)}_{k,N} = \widetilde{\xi}^{(i)}_{k+1,N} - \widetilde{\xi}^{(i)}_{k,N}$, $i = 1, 2$, as follows

$$\begin{cases} \Delta\widetilde{\xi}^{(1)}_{k,N} = -\frac{1}{N} - \Delta\widetilde{\xi}^{(2)}_{k,N}\frac{1}{\sqrt{N}} \\ \Delta\widetilde{\xi}^{(2)}_{k,N} = -\frac{1}{N}\widetilde{\Psi} + \frac{1}{\sqrt{N}}\Delta\widetilde{b}\,\widetilde{\Upsilon} \end{cases}. \tag{3.3}$$

The quantities $\widetilde{\Psi}$, $\widetilde{\Upsilon}$, and $\Delta\widetilde{b}$ depend on additional parameters $\delta \geq 0$ and $M \geq 0$, which are necessary to make $\widetilde{\Psi}$ and $\widetilde{\Upsilon}$ bounded. Define

$$g(x) = \max(x, \delta) \text{ and } h(x) = \min(\max(x, -M), M). \tag{3.4}$$

Then

$$\widetilde{\Psi} = \frac{h\left(\widetilde{\xi}^{(2)}_{k,N}\right)}{g\left(\widetilde{\xi}^{(1)}_{k,N}\right) + h\left(\widetilde{\xi}^{(2)}_{k,N}\right)/\sqrt{N}},$$

$$\widetilde{\Upsilon} = \sqrt{\widetilde{\Phi}\left(1 - \frac{1}{\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor + \sqrt{N}h\left(\widetilde{\xi}^{(2)}_{k,N}\right)}\right)},$$

$$\Delta\widetilde{b} = \frac{\widetilde{\beta}_k - \widetilde{\Phi}}{\widetilde{\Upsilon}},$$

$$\widetilde{\beta}_k \sim \text{Binomial}\left(\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor, \frac{1}{\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor + \sqrt{N}h\left(\widetilde{\xi}^{(2)}_{k,N}\right)}\right),$$

$$\widetilde{\Phi} = \frac{\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor}{\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor + \sqrt{N}h\left(\widetilde{\xi}^{(2)}_{k,N}\right)} = \mathbb{E}\widetilde{\beta}_k.$$

We view the $\widetilde{\Psi}$, $\widetilde{\Upsilon}$, and $\widetilde{\Phi}$ (and their relatives defined later) alternatively as the expressions in $\widetilde{\xi}_{k,N}$ or functions from $\mathbb{R}^2$ to $\mathbb{R}$, which use $\widetilde{\xi}_{k,N}$ as values of their independent arguments. When $N > (M/\delta)^2$, the denominators in the above expressions are positive, and thus the process is automatically defined for $k = 1, \ldots, N-1$. When $\delta = 0$ and $M = \infty$, the difference equation (3.3) is exactly the difference equation for $\left(\xi^{(1)}_{k,N}, \xi^{(2)}_{k,N}\right)$, when $\xi^{(2)}_{k,N} \geq 0$. We assume $\delta > 0$ (but small) and $M < \infty$ (but large) for the rest of this section. The initial conditions for $\widetilde{\xi}_{k,N}$ and $\xi_{k,N}$ agree: $\widetilde{\xi}_{0,N} = \xi_{0,N}$. We now record some immediate consequences of the above definitions.

Lemma 3.4.1. *When $N > (2M/\delta)^2$, the following statements hold:*

(1) *For all k, $0 < \widetilde{\Phi} < 3$.*

(2) *For all k, $0 < \widetilde{\Upsilon} < 2$.*

(3) *For all k, $|\widetilde{\Psi}| \leq 2M/\delta$.*

(4) *For all $\ell, k \geq 0$,*

$$\mathbb{E}\left|\Delta\widetilde{b}_k\widetilde{\Upsilon}\right|^\ell \leq D_\ell,$$

where D_ℓ is a constant depending only on ℓ.

Proof. Parts 1–3 are clear. For part 4, observe that $\mathbb{E}\left(\Delta\widetilde{b}_k\widetilde{\Upsilon}\right)^\ell$ is the centered moment of a Binomial(x, p) random variable with $xp < 3$. Then the desired bound follows from Theorem 2.2 in [**Kno08**] for even ℓ and from Cauchy-Schwarz for odd ℓ. □

We have now arrived at the key result on the way to proving Theorems 1.3.3 and 1.4.1. As usual, the process $\widetilde{\xi}_t$ is the piecewise linear process on $[0,1]$, with values $\widetilde{\xi}_{k,N}$ at k/N. Furthermore, we define $\widetilde{\eta}_t = \left(\widetilde{\eta}^{(1)}_t, \widetilde{\eta}^{(2)}_t\right)$ to be

$$\begin{cases} \widetilde{\eta}^{(1)}_t = 1 - t \\ \widetilde{\eta}^{(2)}_t = p(\sigma) - \displaystyle\int_0^t \frac{h\left(\widetilde{\eta}^{(2)}_s\right)}{g(1-s)} ds - B_t \end{cases}, \tag{3.5}$$

for $t \in [0,1]$, where $p(\sigma) = 1/\sqrt{\sigma}$ if σ is even and $p(\sigma) = 0$, otherwise.

Lemma 3.4.2. *As $N \to \infty$, $\widetilde{\xi}_t \to \widetilde{\eta}_t$ in distribution, in $\mathcal{C}([0,1], \mathbb{R}^2)$.*

Proof. We write

$$\mathbb{E}\left[\Delta\widetilde{\xi}_{k,N} \,\middle|\, \mathcal{F}_k\right] = e_N\left(\widetilde{\xi}_{k,N}\right)\Delta t^N_k,$$

where $\mathcal{F}_k$ is the σ-algebra generated by $\widetilde{\xi}_{0,N}, \ldots, \widetilde{\xi}_{k,N}$, $e_N\left(\widetilde{\xi}_{k,N}\right) = \begin{bmatrix} -1 + \frac{\widetilde{\Psi}}{\sqrt{N}} \\ -\widetilde{\Psi} \end{bmatrix}$ and $\Delta t_k^N = 1/N$. Moreover,

$$\operatorname{Cov}\left[\Delta\widetilde{\xi}_{k,N} \,\middle|\, \mathcal{F}_k\right] = s_N\left(\widetilde{\xi}_{k,N}\right) s_N\left(\widetilde{\xi}_{k,N}\right)^T \Delta t_k^N,$$

where $s_N\left(\widetilde{\xi}_{k,N}\right) = \begin{bmatrix} \frac{\widetilde{\Upsilon}}{\sqrt{N}} \\ -\widetilde{\Upsilon} \end{bmatrix}$ and $s_N\left(\widetilde{\xi}_{k,N}\right)^T$ is its transpose. Now, define

$$e\left(\widetilde{\xi}_{k,N}\right) = \begin{bmatrix} -1 \\ -\overline{\Psi} \end{bmatrix},$$

and

$$s\left(\widetilde{\xi}_{k,N}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

where

$$\overline{\Psi} = \frac{h\left(\widetilde{\xi}_{k,N}^{(2)}\right)}{g\left(\widetilde{\xi}_{k,N}^{(1)}\right)}.$$

In the following steps, we suppress the value $\widetilde{\xi}_{k,N}$ of the independent variables in the functions e, e_N, s, s_N.

Step 1. Denoting the Euclidean norm by $|\cdot|$, we will verify that

$$\mathbb{E}\sum_{k=0}^{N-1}\left[|e_N - e|^2 + |s_N - s|^2\right]\frac{1}{N} \to 0,$$

as $N \to \infty$. We write

$$\mathbb{E}\sum_{k=0}^{N-1}|e_N - e|^2\frac{1}{N} = \mathbb{E}\sum_{k=0}^{N-1}\frac{\widetilde{\Psi}^2}{N^2} + \mathbb{E}\sum_{k=0}^{N-1}\frac{\left|\,\widetilde{\Psi} - \overline{\Psi}\,\right|^2}{N}$$

and

$$\mathbb{E}\sum_{k=0}^{N-1}|s_N - s|^2\frac{1}{N} = \mathbb{E}\sum_{k=0}^{N-1}\frac{\widetilde{\Upsilon}^2}{N^2} + \mathbb{E}\sum_{k=0}^{N-1}\frac{\left|\,1 - \widetilde{\Upsilon}\,\right|^2}{N}.$$

In the next fours steps, we show that the four expressions inside the expectations are bounded by deterministic quantities that go to 0.

Step 2. For the first term,

$$\sum_{k=0}^{N-1} \frac{\widetilde{\Psi}^2}{N^2} \leq \left(\frac{2M}{\delta}\right)^2 \cdot \frac{1}{N},$$

by Lemma 3.4.1 part 3.

Step 3. For the second term, the bounds $g \geq \delta$ and $|h| \leq M$ imply that, for a large enough N

$$\sum_{k=0}^{N-1} \frac{\mid \widetilde{\Psi} - \overline{\Psi} \mid^2}{N} = \sum_{k=0}^{N-1} \left| \frac{h^2\left(\widetilde{\xi}^{(2)}_{k,N}\right)/\sqrt{N}}{\left(g\left(\widetilde{\xi}^{(1)}_{k,N}\right) + h\left(\widetilde{\xi}^{(2)}_{k,N}\right)/\sqrt{N}\right) g\left(\widetilde{\xi}^{(1)}_{k,N}\right)} \right|^2 \frac{1}{N} \leq \left(\frac{2M^2}{\delta^2}\right)^2 \cdot \frac{1}{N}.$$

Step 4. For the third term, by Lemma 3.4.1, part 2,

$$\sum_{k=0}^{N-1} \frac{\widetilde{\Upsilon}^2}{N^2} \leq \frac{4}{N}.$$

Step 5. For the final term, we have, for large enough N, by Lemma 3.4.1, parts 1 and 2,

$$\begin{aligned}
\sum_{k=0}^{N-1} \frac{\mid 1 - \widetilde{\Upsilon} \mid^2}{N} &\leq \sum_{k=0}^{N-1} \frac{|1 - \widetilde{\Upsilon}^2|}{N} \\
&= \sum_{k=0}^{N-1} \left| 1 - \widetilde{\Phi}\left(1 - \frac{1}{\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor + \sqrt{N}h\left(\widetilde{\xi}^{(2)}_{k,N}\right)}\right)\right| \frac{1}{N} \\
&\leq \sum_{k=0}^{N-1} \left[|1 - \widetilde{\Phi}| + \widetilde{\Phi} \cdot \frac{1}{\delta N - 1 - M\sqrt{N}}\right] \frac{1}{N} \\
&\leq \sum_{k=0}^{N-1} \frac{|1 - \widetilde{\Phi}|}{N} + \frac{3}{\delta N - 1 - M\sqrt{N}} \\
&\leq \sum_{k=0}^{N-1} \frac{\sqrt{N} \mid h\left(\widetilde{\xi}^{(2)}_{k,N}\right) \mid}{\lfloor Ng\left(\widetilde{\xi}^{(1)}_{k,N}\right)\rfloor + \sqrt{N}h\left(\widetilde{\xi}^{(2)}_{k,N}\right)} \cdot \frac{1}{N} + \frac{3}{\delta N - 1 - M\sqrt{N}} \\
&\leq \frac{2M}{\delta} \cdot \frac{1}{\sqrt{N}} + \frac{3}{\delta N - 1 - M\sqrt{N}}.
\end{aligned}$$

Steps 2–5 establish the claim in Step 1, and thus condition (1) in [**Kus74**]. To finish the proof, we also need to verify the conditions A1–A6 in Theorem 9.1 in [**Kus74**]. The conditions A1 and A5 hold trivially, and remaining four are handled in the next four steps.

Step 6. For A2, it suffices to observe that e and s are bounded and continuous and e_N and s_N are uniformly bounded on $\mathbb{R}^2$ (and none of them depend on the time variable).

Step 7. For A3, the initial value $\widetilde{\xi}_{0,N}$ converges in probability to $\begin{bmatrix} 1 \\ p(\sigma) \end{bmatrix}$.

Step 8. For A4, we show that

$$\mathbb{E} \sum_{k=0}^{N-1} \left| \Delta \widetilde{\xi}_{k,N} - \frac{e_N}{N} \right|^4 \to 0.$$

Indeed, the expectation equals

$$\mathbb{E} \sum_{k=0}^{N-1} \left| \begin{bmatrix} \frac{2\widetilde{\Psi}}{N^{3/2}} - \frac{\Delta\widetilde{b}\widetilde{\Upsilon}}{N} \\ \frac{\Delta\widetilde{b}\widetilde{\Upsilon}}{\sqrt{N}} \end{bmatrix} \right|^4 = \mathbb{E} \sum_{k=0}^{N-1} \Bigg[\left(\frac{2\widetilde{\Psi}}{N^{3/2}} - \frac{\Delta\widetilde{b}\widetilde{\Upsilon}}{N} \right)^4 +$$

$$2 \left(\frac{2\widetilde{\Psi}}{N^{3/2}} - \frac{\Delta\widetilde{b}\widetilde{\Upsilon}}{N} \right)^2 \left(\frac{\Delta\widetilde{b}\widetilde{\Upsilon}}{\sqrt{N}} \right)^2 + \left(\frac{\Delta\widetilde{b}\widetilde{\Upsilon}}{\sqrt{N}} \right)^4 \Bigg]$$

and goes to 0 as $N \to \infty$, by Lemma 3.4.1, parts 2, 3, and 4.

Step 9. Finally, for A6, we apply the standard theory, e.g., Theorems 2.5 and 2.9 in [**KS98**], to show that equation (3.5) has a unique solution. □

We use the notation $\overline{\xi}_t$ and $\overline{\eta}_t$ for the processes resulting from taking $M = \infty$ in (3.4), so that these processes have the same g but $h(x) = x$. Recall that ξ_t and η_t also have $\delta = 0$, i.e., $g(x) = \max(x, 0)$. We now extend Lemma 3.4.2 to show that $\overline{\xi}_t \to \overline{\eta}_t$ in distribution.

LEMMA 3.4.3. *As $N \to \infty$, $\overline{\xi}_t \to \overline{\eta}_t$ in distribution.*

PROOF. By continuity of $\widetilde{\eta}_t$, for any $\epsilon > 0$, there exists an $M > 0$ such that

$$\mathbb{P}\left(\max |\widetilde{\eta}_t^{(2)}| > M/2\right) < \epsilon.$$

Let $\gamma_M : \mathbb{R} \to [0,1]$ be a continuous function that vanishes outside the interval $[-M, M]$ and is 1 on $[-M/2, M/2]$. For any bounded continuous function $F : \mathcal{C}([0,1], \mathbb{R}^2) \to \mathbb{R}$,

$$\begin{aligned}
\limsup_{N\to\infty} \mathbb{E} F\left(\overline{\xi}_t\right) &\le \limsup_{N\to\infty} \mathbb{E}\left[F\left(\overline{\xi}_t\right) \cdot \gamma_M(\overline{\xi}_t^{(2)})\right] + \limsup_{N\to\infty} \mathbb{E}\left[F\left(\overline{\xi}_t\right) \cdot (1 - \gamma_M(\overline{\xi}_t^{(2)}))\right] \\
&\le \limsup_{N\to\infty} \mathbb{E}\left[F\left(\widetilde{\xi}_t\right) \cdot \gamma_M(\widetilde{\xi}_t^{(2)})\right] + \sup|F| \cdot \limsup_{N\to\infty} \mathbb{P}(\max \mid \widetilde{\xi}_t^{(2)} \mid \ge M/2) \\
&\le \mathbb{E}\left[F\left(\widetilde{\eta}_t\right) \cdot \gamma_M(\widetilde{\eta}_t^{(2)})\right] + \sup|F| \cdot \epsilon \\
&\le \mathbb{E}\left[F\left(\overline{\eta}_t\right)\right] + 2\sup|F| \cdot \epsilon,
\end{aligned}$$

and a matching lower bound on $\liminf \mathbb{E}F\left(\bar{\xi}_t\right)$ is obtained similarly. $\square$

LEMMA 3.4.4. *Let $\delta \in (0,1)$ be fixed and we define the approximate hitting time by $\mathcal{T}$: $\mathcal{C}\left([0,1],\mathbb{R}^2\right) \to [0,1]$,*

$$\mathcal{T}\left(\gamma^{(1)},\gamma^{(2)}\right) = \gamma^{(1)}(\min\{1-\delta, \inf\{t : \gamma_t^{(2)} = 0\}\}).$$

Then $\mathcal{T}$ is a.s. continuous on a path of $\overline{\eta}_t$. As a consequence, $\mathcal{T}(\bar{\xi}_t) \to \mathcal{T}(\overline{\eta}_t)$ in distribution, as $N \to \infty$.

PROOF. Note that $\overline{\eta}_t^{(2)}$ is a Brownian bridge prior to $1-\delta$. Thus the claims follows from well-known facts about the Brownian bridge and standard arguments. $\square$

We can now complete the proof of Theorem 3.4.1, and thus also Theorem 1.4.1.

PROOF OF THEOREM 3.4.1. Fix a $\delta > 0$. By Lemma 3.4.4, $\mathbb{P}\left(\mathcal{T}(\bar{\xi}_t) \le x\right) \to \mathbb{P}\left(\mathcal{T}(\overline{\eta}_t) \le x\right)$, for all $x \in (0, 1-\delta)$, as $N \to \infty$. When $x \in (0,1-\delta)$, we also have that $\mathbb{P}\left(\mathcal{T}(\xi_t) \le x\right) = \mathbb{P}\left(\mathcal{T}(\bar{\xi}_t) \le x\right)$ and $\mathbb{P}\left(\mathcal{T}(\eta_t) \le x\right) = \mathbb{P}\left(\mathcal{T}(\overline{\eta}_t) \le x\right)$. It follows that $\mathbb{P}\left(\mathcal{T}(\xi_t) \le x\right) \to \mathbb{P}\left(\mathcal{T}(\eta_t) \le x\right)$, for all $x \in (0, 1-\delta)$. As $\delta > 0$ is arbitrary, the claim follows. $\square$

The following proposition proves the distribution of hitting time of Brownian bridge.

PROPOSITION 3.4.1. *Fix an $a > 0$. Let η_a be the stochastic process satisfying*

$$\eta_a(t) = a - \int_0^t \frac{\eta_a(s)}{1-s} ds - B_t.$$

Define the hitting time $\tau_a = \inf\{t : \eta_a(t) = 0\}$. Then τ_a has density

$$g_{\tau_a}(x) = \frac{a}{\sqrt{2\pi x^3(1-x)}} \exp\left\{-\frac{a^2(1-x)}{2x}\right\}, \quad x \in (0,1).$$

PROOF. This is well-known and follows from the fact that $\eta_a(t)$ has the same distribution as

$$a(1-t) + (1-t)B_{t/(1-t)},$$

which relates τ_a to a hitting time for the Brownian motion. $\square$

Corollary 3.4.1. *When σ is even, the sequence of random variables $\xi_\tau^{(1)}$ converges in distribution to a random variable with density*

$$g_{1-\tau_{1/\sqrt{\sigma}}}(x) = \frac{1}{\sqrt{2\sigma\pi x(1-x)^3}} \exp\left\{-\frac{x}{2\sigma(1-x)}\right\}, \quad x \in (0,1).$$

Proof. This follows from Theorem 3.4.1 and Proposition 3.4.1. $\square$

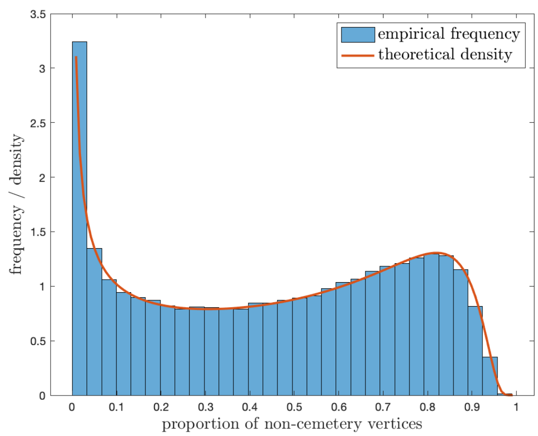


Figure 3.2. Normalized histogram of proportion of non-cemetery vertices in DEC, together with the theoretical limit density.

In Figure 3.2, we compare the empirical distribution of non-cemetery vertices and its limit density given by Corollary 3.4.1. In the simulation, we fix $\sigma = r = 2$ and $n = 100$, and randomly generate 10,000 rules.

3.4.4. Completion of Proof Theorem 1.3.3. We now put together the results from Sections 3.2, 3.3, 3.4.2, and 3.4.3.

Proof of Theorem 1.3.3. Recall the geometric random variable K_σ from Section 3.2. For any $\epsilon > 0$, pick k_ϵ large enough such that $\mathbb{P}(K_\sigma > k_\epsilon) < \epsilon$. Then we have

$$\begin{aligned}
\mathbb{P}\left(\frac{X_{\sigma,n}}{n^{\sigma/2}} \le x\right) &= \mathbb{P}\left(n^{-\sigma/2}\max\{L_i \cdot T_i, i = 1, 2, \dots\} \le x\right) \\
&\le \sum_{k=1}^{k_\epsilon} \mathbb{P}\left(n^{-\sigma/2}\max\{L_i \cdot T_i, i = 1, 2, \dots\} \le x \;\middle|\; K_\sigma = k\right)\mathbb{P}(K_\sigma = k) + \epsilon \\
&= \sum_{k=1}^{k_\epsilon} \mathbb{P}\left(n^{-\sigma/2}\max\left\{L_i \cdot T_i', L_k\sigma, i = 1, 2, \dots, k-1\right\} \le x\right)\mathbb{P}(K_\sigma = k) + \epsilon \\
&= \sum_{k=1}^{k_\epsilon} \mathbb{P}\left(\frac{D_{C_n}^{(k)}}{\sqrt{C_n}} \cdot \sqrt{\frac{C_n}{N}} \cdot \frac{\sqrt{N}}{n^{\sigma/2}} \le x\right)\mathbb{P}(K_\sigma = k) + \epsilon,
\end{aligned}$$

where $D_{C_n}^{(k)}$ is defined in Lemma 3.3.3. Therefore, it suffices to show that

$$\mathbb{P}\left(\frac{D_{C_n}^{(k)}}{\sqrt{C_n}} \cdot \sqrt{\frac{C_n}{N}} \le x\right)$$

converges as $n \to \infty$, for each fixed k. To this end, we partition the interval $(0, 1]$ into M subintervals, and write

$$\begin{aligned}
&\mathbb{P}\left(\frac{D_{C_n}^{(k)}}{\sqrt{C_n}}\sqrt{\frac{C_n}{N}} \le x\right) \\
&= \sum_{i=0}^{M-1} \mathbb{P}\left(\frac{D_{C_n}^{(k)}}{\sqrt{N}} \le x \;\middle|\; \sqrt{\frac{C_n}{N}} \in \left(\frac{i}{M}, \frac{i+1}{M}\right]\right)\mathbb{P}\left(\sqrt{\frac{C_n}{N}} \in \left(\frac{i}{M}, \frac{i+1}{M}\right]\right).
\end{aligned} \tag{3.6}$$

Assume that σ is even and let $a = 1/\sqrt{\sigma}$. By Theorem 1.4.1 and Corollary 3.4.1,

$$\mathbb{P}\left(\sqrt{\frac{C_n}{N}} \in \left(\frac{i}{M}, \frac{i+1}{M}\right]\right) \to \int_{i/M}^{(i+1)/M} g_{\sqrt{1-\tau_a}}(t)\,dt, \tag{3.7}$$

as $n \to \infty$, where $g_{\sqrt{1-\tau_a}}$ is the density of the random variable $\sqrt{1-\tau_a}$. Moreover,

$$\mathbb{P}\left(\frac{D_{C_n}^{(k)}}{\sqrt{N}} \le x \;\middle|\; \sqrt{\frac{C_n}{N}} \in \left(\frac{i}{M}, \frac{i+1}{M}\right]\right) \le \mathbb{P}\left(\frac{D_{\lfloor i^2N/M^2\rfloor}^{(k)}}{\sqrt{N}i/M} \le \frac{x}{i/M}\right). \tag{3.8}$$

It now follows from (3.6)–(3.8), Lemma 3.3.3, and the definition of $D^{(k)}$ after Lemma 3.3.3 that

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}\left(\frac{D^{(k)}_{C_n}}{\sqrt{N}} \le x\right) &\le \sum_{i=0}^{M-1} \mathbb{P}\left(D^{(k)} \le \frac{x}{i/M}\right) \int_{i/M}^{(i+1)/M} g_{\sqrt{1-\tau_a}}(t)\, dt \\ &= \sum_{i=0}^{M-1} \left[\mathbb{P}\left(D^{(k)} \le \frac{x}{i/M}\right) \left(g_{\sqrt{1-\tau_a}}\left(\frac{i}{M}\right)\frac{1}{M} + \mathcal{O}\left(\frac{1}{M^2}\right)\right)\right] \\ &\to \int_0^1 \mathbb{P}\left(D^{(k)} \le \frac{x}{y}\right) g_{\sqrt{1-\tau_a}}(y)\, dy\end{aligned}$$

as $M \to \infty$ and $\mathcal{O}(1/M^2)$ is uniform in i as $g_{\sqrt{1-\tau_a}}$ is differentiable on $[0,1]$. The same lower bound for $\liminf_{n\to\infty} \mathbb{P}\left(D^{(k)}_{C_n}/\sqrt{N} \le x\right)$ is obtained along similar lines. For odd σ, a simpler argument shows that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{D^{(k)}_{C_n}}{\sqrt{N}} \le x\right) = \mathbb{P}\left(D^{(k)} \le x\right),$$

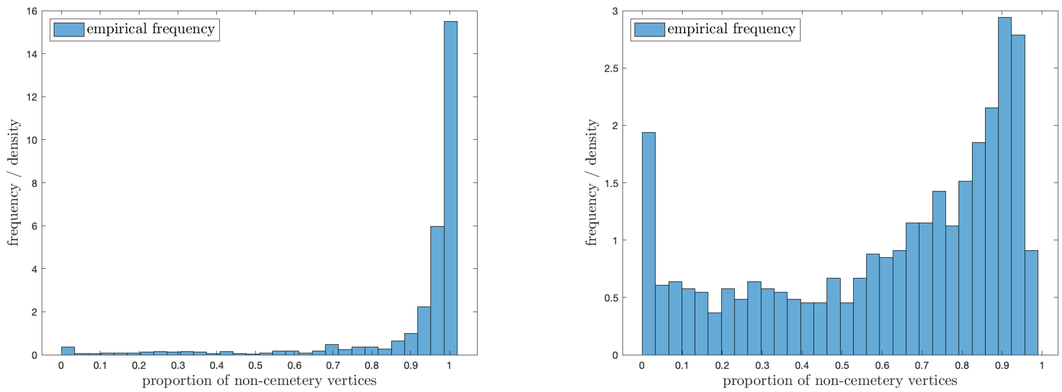
and ends the proof. $\square$

3.5. Computer Simulations and Discussion

In a CA, finding PS of a given temporal period reduces to finding cycles of the corresponding DEC. When a rule is chosen at random, the out-going arcs of different vertices of the DEC are independent from each other, provided that the spatial period is less than the number of neighbors, i.e., if $\sigma \le r$. The problem then reduces to finding the longest of the expanded cycles after the cemetery vertices have been eliminated.

When $\sigma > r$, the independence among arcs in the DEC fails. For example, when $r = 2$ and $\sigma = 3$, the events $\{123 \searrow a_1a_2a_3\}$ and $\{124 \searrow b_1b_2b_3\}$ are dependent (as they cannot occur simultaneously unless $a_2 = b_2$), but they are independent when $r = 3$. Even though rigorous analysis seems elusive in this case, simulations strongly suggest that results very much like Theorems 1.3.3 and 1.4.1 hold. For starters, the random variable C_n and the cemetery vertices in a DEC may be defined in the same manner, and they have the same connection to each other. Figure 3.3 supports the following conjecture.

Conjecture 3.5.1. *Fix arbitrary $\sigma, r \ge 1$, and let $n \to \infty$. If σ is odd, $n^{-\sigma}C_n \to 1$ in probability. If σ is even, then $n^{-\sigma}C_n$ converges in distribution to a nontrivial bimodal distribution.*



(A) $\sigma = 3$, $r = 2$

(B) $\sigma = 4$, $r = 2$

FIGURE 3.3. Empirical proportion of non-cemetery vertices for two examples with $r < \sigma$ and $n = 50$, from 1000 samples.

Turning to the longest periods themselves, we provide the loglog plots for $r = 2$, and $\sigma = 1, 2, 3, 4$ in Figure 3.4. The first two cases are covered by Theorem 1.3.3, while the other two are not. Nevertheless, the average lengths behave with the same regularity, leading to our next conjecture.

CONJECTURE 3.5.2. *Theorem 1.3.3 holds in the same form for $\sigma > r$, i.e., $X_{\sigma,n}/n^{\sigma/2}$ converges in distribution, for any fixed $\sigma \geq 1$ and $r \geq 1$.*

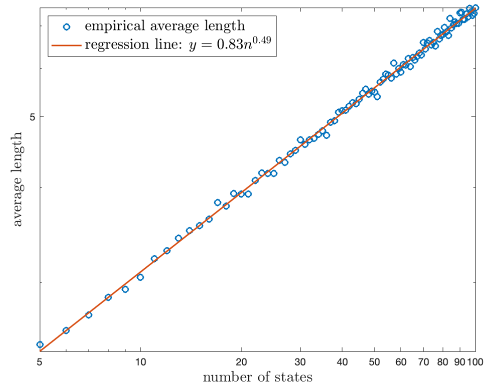
Returning to the case $\sigma \leq r$, one may ask whether our results can be extended to cover other than longest periods. Indeed, as we now sketch, it is possible to show that the length of the jth longest PS of a random rule, again scaled by $n^{\sigma/2}$ converges in distribution. To be more precise, recalling notation from Section 3.2, identify recursively for $\ell \geq 1$ the cycles with largest possible expansion numbers as follows: $K_\sigma^{(0)} = 0$ and

$$K_\sigma^{(\ell)} = \min\left\{k > K_\sigma^{(\ell-1)} : T_k = \sigma\right\}.$$

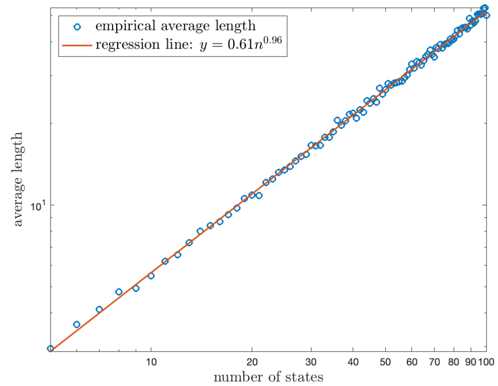
Then the length of jth longest PS is given by

$$X_{\sigma,n}^{(j)} = \max_{(j)}\left\{L_i \cdot T_i', L_{K_\sigma^{(\ell)}}\sigma : i = 1, 2, \ldots, K_\sigma^{(j)} - 1, i \neq K_\sigma^{(\ell)}, \ell = 1, \ldots, j\right\},$$

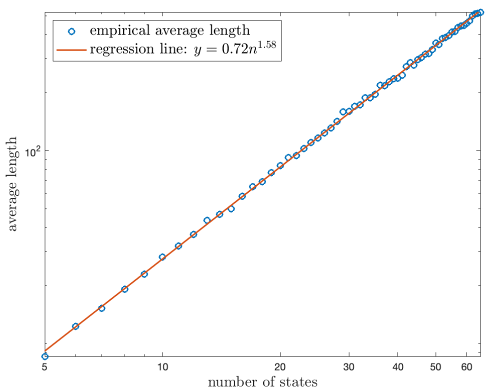
where $\max_{(j)}$ returns the jth largest element of a set. The arguments similar to those in Sections 3.3 and 3.4.4, then show that $X_{\sigma,n}^{(j)}/n^{\sigma/2}$ converges in distribution to a nontrivial limit.



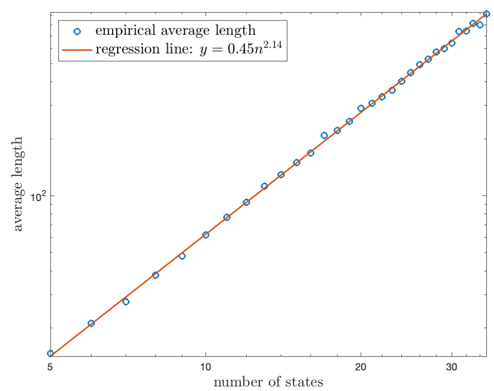
(A) $\sigma = 1$, $r = 2$, $5 \leq n \leq 100$



(B) $\sigma = 2$, $r = 2$, $5 \leq n \leq 100$



(C) $\sigma = 3$, $r = 2$, $5 \leq n \leq 65$



(D) $\sigma = 4$, $r = 2$, $5 \leq n \leq 36$

FIGURE 3.4. Loglog plots of average lengths of longest PS with varied σ, from 1000 samples, with corresponding regression lines.

We conclude with four questions on the extensions of our results in different directions, some of which are analogous to the those posed in 2.

QUESTION 3.5.1. *Assume that n is fixed, but $\sigma, r \to \infty$. What is the asymptotic behavior of the longest temporal period with spatial period σ, depending on the relative sizes of σ and r?*

QUESTION 3.5.2. *For a fixed τ, define the random variable $X'_{\tau,n}$ to be the longest spatial period of a PS with for a given temporal period τ, with $X'_{\tau,n} = 0$ when such a PS does not exist. What is the asymptotic behavior, as $n \to \infty$, of $X'_{\tau,n}$?*

A rule is **left permutative** if the map $\psi_{b_{-r+1},\ldots,b_{-1}} : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $\psi_{b_{-r+1},\ldots,b_{-1}}(a) = f(b_{-r+1},\ldots,b_{-1},a)$ is a permutation for every $(b_{-r+1},\ldots,b_{-1}) \in \mathbb{Z}_n^{r-1}$.

QUESTION 3.5.3. *Let $\mathcal{L}$ be the set of all $(n!)^{n^{r-1}}$ left permutative rules. What is the asymptotic behavior of $X_{\sigma,n}$if a rule from $\mathcal{L}$ is chosen uniformly at random?*

Our final question is on additive rules [**MOW84**], given by $f(b_{-r+1},\ldots,b_0) = \sum_{i=-r+1}^{0} \beta_i b_i$, for some $\beta_i \in \mathbb{Z}_n$.

QUESTION 3.5.4. *Let $\mathcal{A}$ be the set of all n^r additive rules. What is the asymptotic behavior of $X_{\sigma,n}$ if a rule from $\mathcal{A}$ is chosen uniformly at random?*

CHAPTER 4

Maximal Temporal Period of Periodic Solutions

Throughout this chapter, we again assume the simplest nontrivial neighborhood, $r = 2$.

In previous two chapters, we assume a fixed spatial period σ and discuss the temporal periods for randomly selected rules. In this chapter, we instead investigate the analogous extremal questions. As usual, we first restate the theorems to be proved in this chapter.

We denote by $\mathcal{A}_n$ the set of n-state additive rules and let

$$\pi_\sigma(n) = \max_{f \in \mathcal{A}_n} X_{\sigma,n}(f).$$

Let $\lambda_\sigma(n)$ be the exponents of multiplicative group of $\mathbb{Z}_n$ when $\sigma = 2$, Eisenstein integers modulo n when $\sigma = 3$, and Gaussian integers modulo n when $\sigma = 4$. Then the result of the longest temporal periods on additive rules is restated as follows.

Theorem (1.3.5 restated). *For $\sigma = 2, 3$, $\pi_\sigma(n) = \lambda_\sigma(\sigma n)$, for all $n \geq 2$. Moreover, $\pi_4(2) = 4$ and $\pi_4(n) = \lambda_4(n)$, for all $n \geq 3$. Finally, $\pi_6(n) = \lambda_3(6n)$, for all $n \geq 2$.*

Our main result on non-additive rules is as follows.

Theorem (1.3.4 restated). *Fix an arbitrary $\sigma > 0$. For $n \geq N(\sigma)$, there exists an n-state CA rule f such that $X_{\sigma,n}(f) = Y_{\sigma,n}(f) \geq C(\sigma)n^\sigma$, where $N(\sigma)$ and $C(\sigma)$ are constants depending only on σ.*

4.1. Longest Temporal Periods of Additive Rules

In this section, we investigate the longest temporal period that an additive rule is able to generate, for a fixed spatial period σ.

4.1.1. Definitions and Preliminary Results. We write a configuration ξ_t on the integer interval $[0, \sigma - 1]$ with periodic boundary as $c_0^{(t)} c_1^{(t)} \ldots c_{\sigma-1}^{(t)}$, where $c_j^{(t)} \in \mathbb{Z}_n$, for $j = 0, 1, \ldots, \sigma - 1$, or, equivalenty, by the polynomial of degree $\sigma - 1$ [**MOW84**]

$$L^{(t)}(x) = \sum_{j=0}^{\sigma-1} c_j^{(t)} x^j.$$

An additive rule f such that $f(c_0, c_1) = bc_0 + ac_1$, for $a, b \in \mathbb{Z}_n$ is characterized by the polynomial $T(x) = a + bx$, and its evolution as polynomial multiplication:

$$L^{(t+1)}(x) = T(x)L^{(t)}(x),$$

in the quotient ring of polynomials $\mathbb{Z}_n[x]$ modulo the ideal generated by the polynomial $x^\sigma - 1$, to implement the periodic boundary condition. In this section, we will use $T(x)$, for some fixed a and b, to specify an additive CA, in place of the rule f.

As a result, a PS generated by the additive rule $T(x) = a + bx$ with temporal period τ and spatial period σ satisfies

$$T^\tau(x)L^{(\ell)}(x) = L^{(\ell)}(x), \text{ in } \mathbb{Z}_n[x]/(x^\sigma - 1).$$

We are interested in the longest temporal period with a fixed spatial period σ. For general CA, this task requires the examination of the longest cycle in the configuration directed graph (Chapter 3) which encapsulates information from all initial configurations. For linear rules, however, the following simple proposition from [**MOW84**] reduces the set of relevant initial configurations to a singleton.

PROPOSITION 4.1.1. *(Lemma 3.4 in [**MOW84**]) Fix an additive CA and a* $\sigma \geq 1$. *The temporal period of any PS with the spatial period* σ *divides the temporal period resulting from the initial configuration* $1\,0^{\sigma-1}$ *(1 followed by* $\sigma - 1$ *0s), represented by the constant polynomial* 1.

Therefore, we may define the longest temporal period $\Pi_\sigma\,(a, b; n)$ of an additive rule $T(x) = a + bx$, as the smallest k, such that

$$(a + bx)^{k+\ell} = (a + bx)^\ell, \text{ in } \mathbb{Z}_n[x]/(x^\sigma - 1),$$

for some $\ell \geq 0$. We will refer to $\Pi_\sigma(a,b;n)$ as simply the **period** of $T(x)$. The largest period is thus

$$\pi_\sigma(n) = \max_{a,b\in\mathbb{Z}_n} \Pi_\sigma(a,b;n).$$

We use the standard notation $\mathbb{Z}_n[i]$ (where $i=\sqrt{-1}$) and $\mathbb{Z}_n[\omega]$ (where $\omega = e^{2\pi i/3}$) for Gaussian integers modulo n and Eisenstein integers modulo n.

For a finite ring R with unity, we denote by $R^\times$ its multiplicative group and, define the (multiplicative) **order** $\mathrm{ord}(x)$ for any $x \in R$ to be the smallest integer k so that $x^k = 1$ if $x \in R^\times$, and let $\mathrm{ord}(x) = 1$ otherwise. Note that this is the standard definition when $x \in R^\times$. Recall that

$$\begin{aligned}
&\mathbb{Z}_n^\times = \{a : \gcd(a,n) = 1\},\\
&\mathbb{Z}_n[i]^\times = \{a + bi : a, b \in \mathbb{Z}_n, \gcd(a^2+b^2, n) = 1\},\\
&\mathbb{Z}_n[\omega]^\times = \{a + b\omega : a, b \in \mathbb{Z}_n, \gcd(a^2+b^2-ab, n) = 1\}.
\end{aligned}$$

Then we define

$$\begin{aligned}
&\Lambda_2(a,b;n) = \mathrm{ord}(a+b) \text{ in } \mathbb{Z}_n,\\
&\Lambda_3(a,b;n) = \mathrm{ord}(a+b\omega) \text{ in } \mathbb{Z}_n[\omega],\\
&\Lambda_4(a,b;n) = \mathrm{ord}(a+bi) \text{ in } \mathbb{Z}_n[i].
\end{aligned} \tag{4.1}$$

Furthermore, we let

$$\lambda_\sigma(n) = \max_{a,b\in\mathbb{Z}_n} \Lambda_\sigma(a,b;n),$$

for $\sigma = 2$, 3, and 4, be the exponents of the multiplicative groups $\mathbb{Z}_n^\times$, $\mathbb{Z}_n[\omega]^\times$, and $\mathbb{Z}_n[i]^\times$.

In Section 4.1.2, we obtain explicit formulas for $\lambda_\sigma(n)$ for these three σ's.

In the sequel, we will use p, and $p_1, p_2 \ldots$ to denote prime numbers; for an arbitrary n, we write its prime decomposition as $n = p_1^{m_1} \ldots p_k^{m_k}$ or as $n = 2^{m_2} 3^{m_3} \ldots p^{m_p}$. When $p \nmid \sigma$, we use $\mathrm{ord}_\sigma(p)$ to denote the order of p in $\mathbb{Z}_\sigma$. We now list several useful results from [**MOW84**].

Proposition 4.1.2. *(Lemma 4.3 in [**MOW84**]) If $p \mid \sigma$, then $\Pi_\sigma(a,b;p) \mid p\Pi_{\sigma/p}(a,b;p)$.*

Proposition 4.1.3. *(Theorem 4.1 and (B.8) in [**MOW84**]) If $p \nmid \sigma$ and $\sigma \geq 2$, then*

$$\Pi_\sigma(a,b;p) \mid \left(p^{\mathrm{ord}_\sigma(p)} - 1\right)$$

and $\mathrm{ord}_\sigma(p) \le \sigma - 1$. *Furthermore,* $\Pi_1(a, b; p) \mid (p-1)$.

PROPOSITION 4.1.4. *(Theorem 4.4 in [**MOW84**]) For* $n = p_1^{m_1} \dots p_k^{m_k}$*, we have*

$$\Pi_\sigma(a, b; n) = \mathrm{lcm}\left(\Pi_\sigma\left(a, b; p_1^{m_1}\right), \dots, \Pi_\sigma\left(a, b; p_k^{m_k}\right)\right).$$

PROPOSITION 4.1.5. *(Theorem 4.5 in [**MOW84**]) Let* $m \ge 2$ *be an integer. Then* $\Pi_\sigma(a, b; p^m)$ *either equals* $p\Pi_\sigma\left(a, b; p^{m-1}\right)$ *or* $\Pi_\sigma\left(a, b; p^{m-1}\right)$.

As a consequence of the above results, we obtain the following upper bound.

COROLLARY 4.1.1. *Let* $\sigma \ge 2$*, then* $\max_{f \in \mathcal{A}_n} X_{\sigma,n}(f) \le n^{\sigma-1}$*, for all* $n \in \mathbb{N}$.

PROOF. Let $n = p_1^{m_1} \dots p_k^{m_k}$ be the prime decomposition of n. For every $j = 1, \dots, k$ write $\sigma = p_j^{n_j}\sigma_j$, where $n_j \ge 0$ and σ_j is such that $p_j \nmid \sigma_j$. Let $\epsilon_j = 1$ if $\sigma_j = 1$, and $\epsilon_j = 0$ otherwise. For any $a, b \in \mathbb{Z}_n$,

$$\begin{aligned}
\Pi_\sigma(a, b; n) &= \mathrm{lcm}\left(\Pi_\sigma(a, b; p_1^{m_1}), \dots, \Pi_\sigma(a, b; p_k^{m_k})\right) \quad \text{(Proposition 4.1.4)} \\
&\le \prod_{j=1}^{k} p_j^{m_j - 1}\Pi_\sigma(a, b; p_j) \quad \text{(Proposition 4.1.5)} \\
&\le \prod_{j=1}^{k} p_j^{m_j + n_j + \sigma_j - 2}(p_j - 1)^{\epsilon_j} \quad \text{(Propositions 4.1.2 and 4.1.3)} \\
&\le \prod_{j=1}^{k} p_j^{m_j(\sigma - 1)} = n^{\sigma - 1},
\end{aligned}$$

provided that the inequality

$$m_j + n_j + \sigma_j - 2 \le m_j\left(p_j^{n_j}\sigma_j - 1\right) \tag{4.2}$$

holds when either $\sigma_j \ge 2$ or $p_j = 2$, and the inequality

$$m_j + n_j + \sigma_j - 1 \le m_j\left(p_j^{n_j}\sigma_j - 1\right) \tag{4.3}$$

holds when $\sigma_j = 1$ and $p_j \ge 3$.

Note that $\sigma_j = 1$ implies that $n_j \geq 1$. Next, observe that $p_j^{n_j} \geq 2^{n_j} \geq n_j + 1$. Assume first that $\sigma_j \geq 2$. Then we have $m_j p_j^{n_j} \sigma_j \geq m_j(n_j+1)\sigma_j \geq n_j\sigma_j + 2m_j$. Moreover, if $n_j \geq 1$, then $n_j\sigma_j - n_j - \sigma_j + 1 = (n_j - 1)(\sigma_j - 1) \geq 0$ and so (4.2) holds. If $n_j = 0$, then (4.2) reduces to $\sigma_j - 2 \leq m_j(\sigma_j - 2)$, which again holds. Next we assume that $\sigma_j = 1$ and $p_j = 2$. Then (4.2) follows from $m_j + n_j - 1 \leq m_j n_j$. Finally, assume that $\sigma_j = 1$ and $p_j \geq 3$. Then the inequality (4.3) follows from $n_j \leq 3^{n_j} - 2$. The equalities (4.2) and (4.3) are thus established and the proof completed. $\square$

4.1.2. Exponents of the Multiplicative Groups. In this section, we find formulas for $\lambda_\sigma(n)$, $\sigma = 2$, 3, and 4, i.e., the exponents of multiplicative groups $\mathbb{Z}_n^\times$, $\mathbb{Z}_n[\omega]^\times$, and $\mathbb{Z}_n[i]^\times$.

LEMMA 4.1.1. *For $\sigma = 2, 3$ and 4,*

$$\lambda_\sigma(n) = \operatorname{lcm}\left(\lambda_\sigma(p_1^{m_1}), \ldots, \lambda_\sigma(p_k^{m_k})\right).$$

PROOF. By the Chinese Remainder Theorem, $\mathbb{Z}_n^\times$ (respectively, $\mathbb{Z}_n[\omega]^\times$, $\mathbb{Z}_n[i]^\times$) is isomorphic to the direct product of the k groups $\mathbb{Z}_{p_j^{m_j}}^\times$ (respectively, $\mathbb{Z}_{p_j^{m_j}}[\omega]^\times$, $\mathbb{Z}_{p_j^{m_j}}[i]^\times$), $j = 1, \ldots, k$. $\square$

To find $\lambda_\sigma(n)$, it therefore suffices to find the formulas for $\lambda_\sigma(p^m)$ for prime p. For $\sigma = 2$, λ_2 is known as the Carmichael function, which is given by the following explicit formula.

LEMMA 4.1.2. *For $m \geq 1$ and p prime,*

$$\lambda_2(p^m) = \begin{cases} 2^{m-1}, & \textit{if } p = 2 \textit{ and } m \leq 2 \\ 2^{m-2}, & \textit{if } p = 2 \textit{ and } m \geq 3 \\ p^{m-1}(p-1), & \textit{if } p > 2 \end{cases} \cdot$$

PROOF. See [**Car10**]. $\square$

The results for λ_3 and λ_4 follow from the classification of the two multiplicative groups. For $\mathbb{Z}_{p^m}[i]^\times$, this task was accomplished in [**ADJ$^+$08**], while for $\mathbb{Z}_{p^m}[\omega]^\times$ we relegate the similar argument to the Appendix.

LEMMA 4.1.3. *For $m \geq 1$ and p prime,*

$$\lambda_3(p^m) = \begin{cases} 6, & \text{if } p = 3 \text{ and } m = 1 \\ 2 \cdot 3^{m-1}, & \text{if } p = 3 \text{ and } m \geq 2 \\ p^{m-1}(p-1), & \text{if } p = 1 \mod 3 \\ p^{m-1}(p^2-1), & \text{if } p = 2 \mod 3 \end{cases}.$$

PROOF. The claim follows from Theorem A.0.1 in the Appendix. □

LEMMA 4.1.4. *For $m \geq 1$ and p prime,*

$$\lambda_4(p^m) = \begin{cases} 2^m, & \text{if } p = 2 \text{ and } m \leq 2 \\ 2^{m-1}, & \text{if } p = 2 \text{ and } m \geq 3 \\ p^{m-1}(p-1), & \text{if } p = 1 \mod 4 \\ p^{m-1}(p^2-1), & \text{if } p = 3 \mod 4 \end{cases}.$$

PROOF. By [**ADJ**$^+$**08**], we have

$$\mathbb{Z}_p[i]^\times \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 2 \\ \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}, & \text{if } p = 1 \mod 4 \\ \mathbb{Z}_{p^2-1}, & \text{if } p = 3 \mod 4 \end{cases}$$

and

$$\mathbb{Z}_{p^m}[i]^\times \cong \begin{cases} \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{m-2}} \times \mathbb{Z}_4, & \text{if } p = 2 \text{ and } m \geq 2 \\ \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_p[i]^\times, & \text{if } p \neq 2 \end{cases}.$$

The claim follows. □

4.1.3. Explicit Formulas for Configurations at Time t. The next lemma makes the connection between the CA evolution and the integer rings apparent.

LEMMA 4.1.5. *For $\sigma = 2$, in $\mathbb{Z}_n[x]/(x^2-1)$,*

$$(a+bx)^t = \frac{1}{2}\left[(a+b)^t + (a-b)^t\right] + \frac{1}{2}\left[(a+b)^t - (a-b)^t\right] x. \tag{4.4}$$

For $\sigma = 3$, *in* $\mathbb{Z}_n[x]/(x^3-1)$,

$$
\begin{aligned}
(a+bx)^t &= \frac{1}{3}\left[(a+b)^t + (a+b\omega)^t + (a+b\omega^2)^t\right] \\
&\quad + \frac{1}{3}\left[(a+b)^t + \omega^2(a+b\omega)^t + \omega(a+b\omega^2)^t\right] x \\
&\quad + \frac{1}{3}\left[(a+b)^t + \omega(a+b\omega)^t + \omega^2(a+b\omega^2)^t\right] x^2.
\end{aligned} \tag{4.5}
$$

For $\sigma = 4$, *in* $\mathbb{Z}_n[x]/(x^4-1)$,

$$
\begin{aligned}
(a+bx)^t &= \frac{1}{4}\left[(a+b)^t + (a-b)^t + (a+bi)^t + (a-bi)^t\right] \\
&\quad + \frac{1}{4}\left[(a+b)^t - (a-b)^t + i(a+bi)^t - i(a-bi)^t\right] x \\
&\quad + \frac{1}{4}\left[(a+b)^t + (a-b)^t - (a+bi)^t - (a-bi)^t\right] x^2 \\
&\quad + \frac{1}{4}\left[(a+b)^t - (a-b)^t - i(a+bi)^t + i(a-bi)^t\right] x^3.
\end{aligned} \tag{4.6}
$$

For $\sigma = 6$, *in* $\mathbb{Z}_n[x]/(x^6-1)$,

(4.7)

$$
\begin{aligned}
(a+bx)^t = \frac{1}{6}&\left[(a+b)^t + (a-b)^t + (a+b\omega)^t + (a+b\omega^2)^t + (a-b\omega)^t + (a-b\omega^2)^t\right] \\
+ \frac{1}{6}&\left[(a+b)^t - (a-b)^t + \omega^2(a+b\omega)^t + \omega(a+b\omega^2)^t - \omega^2(a-b\omega)^t - \omega(a-b\omega^2)^t\right] x \\
+ \frac{1}{6}&\left[(a+b)^t + (a-b)^t + \omega(a+b\omega)^t + \omega^2(a+b\omega^2)^t + \omega(a-b\omega)^t + \omega^2(a-b\omega^2)^t\right] x^2 \\
+ \frac{1}{6}&\left[(a+b)^t - (a-b)^t + (a+b\omega)^t + (a+b\omega^2)^t - (a-b\omega)^t - (a-b\omega^2)^t\right] x^3 \\
+ \frac{1}{6}&\left[(a+b)^t + (a-b)^t + \omega^2(a+b\omega)^t + \omega(a+b\omega^2)^t + \omega^2(a-b\omega)^t + \omega(a-b\omega^2)^t\right] x^4 \\
+ \frac{1}{6}&\left[(a+b)^t - (a-b)^t + \omega(a+b\omega)^t + \omega^2(a+b\omega^2)^t - \omega(a-b\omega)^t - \omega^2(a-b\omega^2)^t\right] x^5.
\end{aligned}
$$

To clarify, say, the formula for $\sigma = 6$, the expression in each square bracket is evaluated in $\mathbb{Z}[\omega]$ first (without the reduction modulo n), then the result, which must be in $6\mathbb{Z}$, is divided by 6, and finally is reduced modulo n.

Proof. This follows from diagonalization of circulant matrices; see, for example, [**Dav70**]. $\square$

4.1.4. The Upper Bounds. In this subsection we prove the upper bounds in Theorem 1.3.5.

LEMMA 4.1.6. *For $n \geq 2$, $\pi_\sigma(n) \leq \lambda_\sigma(\sigma n)$ for $\sigma = 2, 3$ and $\pi_6(n) \leq \lambda_3(6n)$. Moreover, for $n \geq 3$, $\pi_4(n) \leq \lambda_4(n)$.*

PROOF. We will show that, in all cases, $\Pi_\sigma(a, b; n)$ divides the corresponding upper bound for all $a, b \in \mathbb{Z}_n$. Assume that $p \nmid \sigma$, which automatically holds when $p \geq 5$. In this case, we claim that

$$\Pi_\sigma(a, b; p^m) \mid \lambda_\sigma(p^m), \tag{4.8}$$

which is clearly enough. By Propositions 4.1.5 and 4.1.3, $\Pi_\sigma(a, b; p^m) \mid p^{m-1}(p^{\mathrm{ord}_\sigma(p)} - 1)$. As $\mathrm{ord}_2(p) = 1$, $\mathrm{ord}_3(p) = 1$ when $p \mod 3 = 1$ and $\mathrm{ord}_3(p) = 2$ when $p \mod 3 = 2$, and $\mathrm{ord}_4(p) = 1$ when $p \mod 4 = 1$ and $\mathrm{ord}_4(p) = 2$ when $p \mod 4 = 3$, Lemmas 4.1.2–4.1.4 imply (4.8).

We now consider each σ separately. Write $n = 2^{m_2} 3^{m_3} \cdots p^{m_p}$.

We begin with $\sigma = 2$. Note that (4.8) holds for $p = 3$, and we next consider powers of 2. For $m = 1$ and $m = 2$, it can be directly verified that $\Pi_2(a, b; 2^m) \mid 2$. For $m \geq 3$, by Proposition 4.1.5, $\Pi_2(a, b; 2^m) \mid 2^{m-2}\Pi_2(a, b; 2^2)$, and then $\Pi_2(a, b; 2^m) \mid 2^{m-1}$. Therefore

$$\Pi_2(a, b; 2^m) \mid \lambda_2(2^{m+1}),$$

which, together with (4.8) and Proposition 4.1.4, implies that

$$\Pi_2(a, b; n) \mid \mathrm{lcm}\left(\lambda_2(2^{m_2+1}), \ldots, \lambda_2(p^{m_p})\right) = \lambda_2(2n),$$

by Lemma 4.1.1.

We continue with $\sigma = 3$. Now, (4.8) holds for $p = 2$ and we need to consider powers of 3. A direct verification shows that $\Pi_3(a, b; 3) \mid 6$. For $m \geq 2$, $\Pi_3(a, b; 3^m) \mid 3^{m-1}\Pi_3(a, b; 3)$ and so $\Pi_3(a, b; 3^m) \mid 2 \cdot 3^m$. By Lemma 4.1.3,

$$\Pi_3(a, b; 3^m) \mid \lambda_3(3^{m+1})$$

and again (4.8), Proposition 4.1.4, and Lemma 4.1.1 imply that $\Pi_3(a, b; 3^m) \mid \lambda_3(3n)$.

Next in line is $\sigma = 4$. This time, a direct verification (by computer) shows that $\Pi_4(a, b; 2)$, $\Pi_4(a, b; 2^2)$, and $\Pi_4(a, b; 2^3)$ all divide 4. For $m \geq 3$, we then have $\Pi_4(a, b; 2^m) \mid 2^{m-3}\Pi_4(a, b; 2^3)$, thus $\Pi_4(a, b; 2^m) \mid 2^{m-1}$. Now, if $n = 2^{m_2} 3^{m_3} \ldots p^{m_p}$ and $m_2 \geq 2$ or $m_2 = 0$, the result follows similarly as for $\sigma = 2$ or $\sigma = 3$. If $m_2 = 1$,

$$\Pi_4\left(a, b; 2 \cdot 3^{m_3} \dots p^{m_p}\right) \mid \operatorname{lcm}\left(4, \lambda_4(3^{m_3}), \dots, \lambda_4(p^{m_p})\right).$$

But

$$\begin{aligned}\operatorname{lcm}\left(4, \lambda_4(3^{m_3}), \dots, \lambda_4(p^{m_p})\right) &= \operatorname{lcm}\left(2, \lambda_4(3^{m_3}), \dots, \lambda_4(p^{m_p})\right) \\ &= \operatorname{lcm}\left(\lambda_4(2), \lambda_4(3^{m_3}), \dots, \lambda_4(p^{m_p})\right) = \lambda_4(n),\end{aligned}$$

as long as one of the exponents $m_3, \dots, m_p$ is nonzero, i.e., when $n \geq 3$. The desired divisibility therefore holds.

Finally, we deal with $\sigma = 6$. This time, a similar argument shows that $\Pi_6(a, b; 2^{m_2}) \mid 3 \cdot 2^{m_2}$ and $\Pi_6(a, b; 3^{m_3}) \mid 2 \cdot 3^{m_3}$, for all $m_2, m_3 \geq 1$. So, $\Pi_6(a, b; n)$ divides

$$\operatorname{lcm}\left(3 \cdot 2^{m_2}, 2 \cdot 3^{m_3}, \dots, \lambda_3(p^{m_p})\right) = \operatorname{lcm}\left(\lambda_3(2 \cdot 2^{m_2}), \lambda_3(3 \cdot 3^{m_3}), \dots, \lambda_3(p^{m_p})\right) = \lambda_3(6n).$$

The desired divisibility is thus established in all cases. $\square$

4.1.5. The Lower Bounds.

Lemma 4.1.7. *If n has prime decomposition $n = p_1^{m_1} \dots p_k^{m_k}$, then, for any σ,*

$$\operatorname{lcm}\left(\pi_\sigma(p_1^{m_1}), \dots, \pi_\sigma(p_k^{m_k})\right) \leq \pi_\sigma(n). \tag{4.9}$$

Proof. We identify $\mathbb{Z}_n$ by

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{m_1}} \times \dots \times \mathbb{Z}_{p_k^{m_k}}.$$

For the CA rule in the jth coordinate, we find $a_j, b_j \in \mathbb{Z}_{p_j^{m_j}}$ such that $\Pi_\sigma\left(a_j, b_j; p_j^{m_j}\right) = \pi_\sigma\left(p_j^{m_j}\right)$. Then a configuration repeats if and only if all k coordinates simultaneously repeat. $\square$

As a consequence of Lemma 4.1.7, it suffices to consider the cases when $n = p^m$. In each case below, our strategy is to find an $a, b \in \mathbb{Z}_{p^m}$ for which the dynamics never reduces the spatial period and such that $\Pi_\sigma\left(a, b; p^m\right)$ equals the upper bound given by Lemma 4.1.6.

Lemma 4.1.8. *For $\sigma = 2$, we have $\pi_2\left(p^m\right) = \lambda_2\left(2p^m\right)$.*

Proof. We first prove that $a-b \in \mathbb{Z}_{p^m}^{\times}$ implies that the spatial period never reduces. Indeed, such a reduction means that the coefficients of 1 and x in (4.4) agree at some time $t \geq 1$, and then their difference $(a-b)^t$ must vanish in $\mathbb{Z}_{p^m}$, a contradiction.

We now assume that $p \geq 3$. By definition of λ_2, we can select a and b such that $\Lambda_2(a,-b;p^m) = \lambda_2(p^m)$; in particular, $a-b \in \mathbb{Z}_{p^m}^{\times}$. Let $k = \Pi_2(a,-b;p^m)$. Then, for some $\ell \geq 0$, $(a-bx)^{k+\ell} = (a-bx)^{\ell}$ in $\mathbb{Z}_{p^m}[x]/(x^2-1)$. If we replace x by any number $c \in \mathbb{Z}_{p^m}$ that satisfies $c^2-1=0 \mod p^m$, we get an equality in $\mathbb{Z}_{p^m}$, so we can substitute $x=1$ to get $(a-b)^{k+\ell} = (a-b)^{\ell} \mod p^m$. As $a-b$ is invertible in $\mathbb{Z}_{p^m}$, $(a-b)^k = 1 \mod p^m$. We conclude that $\lambda_2(p^m) \leq \Pi_2(a,-b;p^m) \leq \pi_2(p^m)$. As the spatial period does not reduce, the desired conclusion follows from the equality $\lambda_2(p^m) = \lambda_2(2p^m)$ and Lemma 4.1.6.

Finally, we assume that $p=2$. In this case, we need to prove that $\pi_2(2^m) = \lambda_2\left(2^{m+1}\right)$. A direct verification shows that $\pi_2(2) = \pi_2(4) = 2$, so we may assume that $m \geq 3$, in which case $\lambda_2\left(2^{m+1}\right) = 2^{m-1}$. Pick a $c \in \mathbb{Z}_{2^{m+1}}^{\times}$ whose order equals $\lambda_2\left(2^{m+1}\right)$. This is an odd number. Let $b = (c-1)/2$ and $a = b+1$, so that $a+b=c$ and $a-b=1$. Clearly $b \leq 2^m - 1$, but then also $a \leq 2^m-1$, as otherwise $c = 2^{m+1}-1$, which has order 2. It then follows from (4.4) that $(a+bx)^{2^{m-1}} = 1$ in $\mathbb{Z}_{2^m}[x]/(x^2-1)$. Moreover, the coefficient of x in $(a+bx)^{2^{m-2}}$ cannot vanish in $\mathbb{Z}_{2^m}$, as otherwise $c^{2^{m-2}} = 1 \mod 2^{m+1}$. It follows that $\Pi_2(a,b;2^m) = 2^{m-1}$. $\square$

Lemma 4.1.9. *For $\sigma = 3$, we have $\pi_3(p^m) = \lambda_3(3p^m)$.*

Proof. We first show that, provided $a+b\omega \in \mathbb{Z}_{p^m}[\omega]^{\times}$, spatial period does not reduce. Indeed, if the spatial period reduces to 1 at time $t \geq 1$, then from (4.5)

$$\frac{1}{3}\begin{bmatrix} B & A \\ A & B \end{bmatrix} \begin{bmatrix} (a+b\omega)^t \\ (a-b\omega)^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{Z}_{p^m}[\omega],$$

where $A = 1-\omega$ and $B = 1-\omega^2$. This implies that $(a+b\omega)^t = 0$ in $\mathbb{Z}_{p^m}[\omega]$, a contradiction.

This time, we first assume that $p \neq 3$ and select a and b such that $\Lambda_3(a,b;p^m) = \lambda_3(p^m)$. Then, if $k = \Pi_3(a,b;p^m)$, we have $(a+bx)^{k+\ell} = (a+bx)^{\ell}$, in $\mathbb{Z}_{p^m}[x]/(x^3-1)$, for some ℓ. As $\omega^3 = 1$, we may replace x with ω to get $(a+b\omega)^k = 1$ in $\mathbb{Z}_{p^m}[\omega]$. As a result, $\lambda_3(p^m) \leq \Pi_3(a,b;p^m)$. As the spatial period does not reduce, the desired conclusion follows from $\lambda_3(p^m) = \lambda_3(3p^m)$ and Lemma 4.1.6.

It remains to consider $p = 3$. By direct verification, $\pi_3(3) = 6$, and we assume $m \geq 2$ from now on. Select $a = b = 1$. By Proposition 4.1.5, $\Pi_3(1, 1; 3^m) = 2 \cdot 3^{m'}$, for some $m' \in [1, m]$. Also, $(1 + x)^{2 \cdot 3^m} = 1$ in $\mathbb{Z}_{3^m}[x]/(x^3 - 1)$, which can be easily verified by (4.5) using $(1 + \omega)^2 = \omega$, $(1 + \omega^2)^2 = \omega^2$, and the fact, easily verified by induction, that $2^{2 \cdot 3^m} = 1 \mod 3^{m+1}$. So, it suffices to show that $(1 + x)^{2 \cdot 3^{m-1}} \neq 1$ in $\mathbb{Z}_{3^m}[x]/(x^3 - 1)$, and for this we verify that the constant term in (4.5) does not equal 1, that is,

$$(1 + 1)^{2 \cdot 3^{m-1}} + (1 + \omega)^{2 \cdot 3^{m-1}} + (1 + \omega^2)^{2 \cdot 3^{m-1}} \neq 3 \text{ in } \mathbb{Z}_{3^{m+1}}[\omega].$$

Indeed, in $\mathbb{Z}_{3^{m+1}}[\omega]$, $(1+\omega)^{2 \cdot 3^{m-1}} = (1+\omega^2)^{2 \cdot 3^{m-1}} = 1$ and, again by induction, $2^{2 \cdot 3^{m-1}} = 3^m + 1$. $\square$

LEMMA 4.1.10. *For $\sigma = 4$, we have $\pi_4(p^m) = \lambda_4(p^m)$.*

PROOF. For any p, select a and b such that $\Lambda_4(a, b; p^m) = \lambda_4(p^m)$. Then if $k = \Pi_4(a, b; p^m)$, we have $(a + bx)^{k+\ell} = (a + bx)^\ell$, in $\mathbb{Z}_{p^m}[x]/(x^4 - 1)$, for some ℓ. Replacing x with i, we have $(a + bi)^k = 1$ in $\mathbb{Z}_{p^m}[i]$. As a result, $\lambda_4(p^m) \leq \Pi_4(a, b; p^m)$. Thus we only need to verify that the spatial period does not reduce. If it does, then for some t, by (4.6),

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} (a + bi)^t \\ (a - bi)^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{Z}_{p^m}[i],$$

implying that $(a + bi)^t = 0$ in $\mathbb{Z}_{p^m}[i]$, a contradiction with $a + bi \in \mathbb{Z}_{p^m}[i]^\times$. $\square$

LEMMA 4.1.11. *Assume that $\sigma = 6$, $n = p^m$, and that one of these two conditions on a and b is satisfied: $p \neq 3$ and $a + b\omega$ is invertible $\mathbb{Z}_{p^m}[\omega]$; or $p = 3$, $m \geq 2$, $a = 1$ and $b = 2$. Then the spatial period of $(a + bx)^t$ is 6 for all $t \geq 0$.*

PROOF. If the period reduces to 2, then by (4.7),

$$\frac{1}{6}\begin{bmatrix} A & B & A & B \\ B & A & B & A \\ -B & -A & B & A \\ A & B & -A & -B \end{bmatrix}\begin{bmatrix} (a + b\omega)^t \\ (a + b\omega^2)^t \\ (a - b\omega)^t \\ (a - b\omega^2)^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{Z}_{p^m}[\omega],$$

where $A = 1 - \omega$ and $B = 1 - \omega^2$. Multiply rows, in order, by A, $-B$, B, A and add. Using $B^2 - A^2 = 3(2\omega + 1)$, we get that $(1 + 2\omega)(a + b\omega)^t = 0$ in $\mathbb{Z}_{p^m}[\omega]$. Multiplying instead by A, $-B$, $-B$, $-A$ gives $(1+2\omega)(a-b\omega)^t = 0$ in $\mathbb{Z}_{p^m}[\omega]$. If $p \neq 3$, then $1+2\omega \in \mathbb{Z}_{p^m}[\omega]^\times$ and so $(a+b\omega)^t = 0$, a contradiction. Assume now that $p = 3$. Then we use the fact that Eisenstein norm $|1 - 2\omega| = 7$, and so the norm of the product $|(1 + 2\omega)(1 - 2\omega)^t| = 3 \cdot 7^t$, which is not divisible by 3^m if $m \geq 2$, and so$(1 + 2\omega)(1 - 2\omega)^t$ is nonzero in $\mathbb{Z}_{3^m}[\omega]$.

We next show that the spatial period does not reduce to 3. If it does, then by (4.7),

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \begin{bmatrix} (a-b)^t \\ (a-b\omega)^t \\ (a-b\omega^2)^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{Z}_{p^m}[\omega].$$

From this, we get that

$$(a-b)^t = (a-b\omega)^t = (a-b\omega^2)^t = 0 \text{ in } \mathbb{Z}_{p^m}[\omega]. \tag{4.10}$$

Assume $p \neq 3$ first. Then, (4.10) implies that neither $a - b$ nor $a - b\omega$ is invertible in $\mathbb{Z}_{p^m}[\omega]$, and thus p must divide $a - b$ and the norm $a^2 + b^2 + ab$. Then $3ab = (a^2 + b^2 + ab) - (a - b)^2$ is also divisible by p, and then so is ab. This implies that $p \mid (a^2 + b^2 - ab)$, and so $a + b\omega$ is not invertible, a contradiction. If $p = 3$, then (4.10) is not satisfied for $a = 1, b = 2$, as $(a - b)^t$ cannot vanish. $\square$

Lemma 4.1.12. *For $\sigma = 6$, we have $\pi_6(p^m) = \lambda_3(6p^m)$.*

Proof. Assume first that $p \geq 5$. Select any a and b such that $\Lambda_3(a, b; p^m) = \lambda_3(p^m) = \lambda_3(6p^m)$. Then, if $k = \Pi_6(a, b; p^m)$, we have $(a + bx)^{k+\ell} = (a + bx)^\ell$, in $\mathbb{Z}_{p^m}[x]/(x^6 - 1)$, for some ℓ. Replacing x with ω, we have $(a + b\omega)^k = 1$ thus $\lambda_3(p^m) \leq \Pi_6(a, b; p^m)$.

Next in line is $p = 2$. The claim is that $\pi_6(2^m) = 3 \cdot 2^m$. We may assume that $m \geq 3$, after a direct verification for $m = 1, 2$. By Theorem 4.1.5, $\Pi_6(1, 1; 2^m) = 3 \cdot 2^{m'}$, for some $m' \in [1, m]$. Therefore, it suffices to show that there are infinitely many ℓ for which the equality

$$(1 + x)^{3 \cdot 2^{m-1} + \ell} = (1 + x)^\ell, \text{ in } \mathbb{Z}_{2^m}[x]/(x^6 - 1),$$

is *not* satisfied. A necessary condition for this equality is that the constant terms in (4.7) for both sides agree, which yields

$$\frac{1}{6}\Big[2^{\ell}\left(2^{3\cdot 2^{m-1}}-1\right)+(1+\omega)^{\ell}\left((1+\omega)^{3\cdot 2^{m-1}}-1\right)+(1+\omega^2)^{\ell}\left((1+\omega^2)^{3\cdot 2^{m-1}}-1\right)+$$
$$(1-\omega)^{\ell}\left((1-\omega)^{3\cdot 2^{m-1}}-1\right)+(1-\omega^2)^{\ell}\left((1-\omega^2)^{3\cdot 2^{m-1}}-1\right)\Big]=0 \mod 2^m.$$

As $1+\omega=-\omega^2$, $1+\omega^2=-\omega$, the second and third term vanish. The first term vanishes for large enough ℓ. Moreover, as $(1-\omega)^2=-3\omega$ and $(1-\omega^2)^2=-3\omega^2$, $(1-\omega)^{3\cdot 2^{m-1}}=(1-\omega^2)^{3\cdot 2^{m-1}}=3^{3\cdot 2^{m-2}}$, for $m\geq 3$. We obtain the necessary condition

$$(1-\omega)^{\ell}\left[1+(1+\omega)^{\ell}\right]\left(3^{3\cdot 2^{m-2}}-1\right)=0 \mod 3\cdot 2^{m+1}. \tag{4.11}$$

If $\ell=1 \mod 12$, then $(1-\omega)^{\ell}$ is a power of 3 times $(1-\omega)$ and $(1+\omega)^{\ell}=-\omega^2$. By a simple induction argument, $3^{3\cdot 2^{m-2}}-1=2^m \mod 2^{m+1}$. Then, if $\ell=1 \mod 12$, (4.11) reduces to $3^{\ell'}\cdot 2^m=0 \mod 3\cdot 2^{m+1}$, for some $\ell'\geq 1$, which is clearly false. This completes the proof for $p=2$.

Finally, we deal with $p=3$. We aim to prove $\pi_6(3^m)=2\cdot 3^m$, and we will accomplish this by establishing the claim that $\Pi(1,2;3^m)=2\cdot 3^m$. We may, again, assume $m\geq 3$. Similarly to the previous case, it suffices to show that

$$(1+2x)^{2\cdot 3^{m-1}+\ell}=(1+2x)^{\ell},\ \text{in } \mathbb{Z}_{3^m}[x]/(x^6-1), \tag{4.12}$$

fails to hold for infinitely many ℓ, and we will assume that ℓ is large enough and $18\mid\ell$. As before, we show the constant terms in (4.7) do not match. If they do, this expression needs to vanish modulo $2\cdot 3^{m+1}$:

$$\begin{aligned}&(1+2)^{\ell}\left[(1+2)^{2\cdot 3^{m-1}}-1\right]+(1-2)^{\ell}\left[(1-2)^{2\cdot 3^{m-1}}-1\right]\\&+(1+2\omega)^{\ell}\left[(1+2\omega)^{2\cdot 3^{m-1}}-1\right]+(1+2\omega^2)^{\ell}\left[(1+2\omega^2)^{2\cdot 3^{m-1}}-1\right]\\&+(1-2\omega)^{\ell}\left[(1-2\omega)^{2\cdot 3^{m-1}}-1\right]+(1-2\omega^2)^{\ell}\left[(1-2\omega^2)^{2\cdot 3^{m-1}}-1\right].\end{aligned} \tag{4.13}$$

As $(1+2\omega)^2 = (1+2\omega^2)^2 = -3$, the first four terms all vanish when ℓ is large enough. For the fifth and sixth term, we first observe that

$$(1-2\omega)^\ell = [(1-\omega)-\omega]^\ell = (-\omega)^\ell + \sum_{j=1}^{\ell} \binom{\ell}{j} (1-\omega)^j (-\omega)^{\ell-j} = 1 \text{ in } \mathbb{Z}_9[\omega]. \tag{4.14}$$

By a similar calculation, $(1-2\omega^2)^\ell = 1$ in $\mathbb{Z}_9[\omega]$. Next, we have

$$\begin{aligned}
(1-2\omega)^{2\cdot 3^{m-1}} - 1 &= [(1-\omega)-\omega]^{2\cdot 3^{m-1}} - 1 \\
&= -1 + (-\omega)^{2\cdot 3^{m-1}} + 2\cdot 3^{m-1}(1-\omega)(-\omega)^{2\cdot 3^{m-1}-1} \\
&\quad + \frac{2\cdot 3^{m-1}(2\cdot 3^{m-1}-1)}{2}(1-\omega)^2(-\omega)^{2\cdot 3^{m-1}-2} \\
&\quad + \frac{2\cdot 3^{m-1}(2\cdot 3^{m-1}-1)(2\cdot 3^{m-1}-2)}{2\cdot 3}(1-\omega)^3(-\omega)^{2\cdot 3^{m-1}-3} \\
&\quad + \sum_{j=4}^{2\cdot 3^{m-1}} \binom{2\cdot 3^{m-1}}{j}(1-\omega)^j(-\omega)^{2\cdot 3^{m-1}-j} \\
&= 2\cdot 3^{m-1}(1-\omega)(-\omega^2) - 3^m(2\cdot 3^{m-1}-1)\omega^2 \\
&\quad + 3^{m-1}(2\cdot 3^{m-1}-1)(2\cdot 3^{m-1}-2)\omega(1-\omega) \text{ in } \mathbb{Z}_{3^{m+1}}[\omega].
\end{aligned} \tag{4.15}$$

Similarly,

$$\begin{aligned}
(1-2\omega^2)^{2\cdot 3^{m-1}} - 1 &= 2\cdot 3^{m-1}(1-\omega^2)(-\omega) - 3^m(2\cdot 3^{m-1}-1)\omega \\
&\quad + 3^{m-1}(2\cdot 3^{m-1}-1)(2\cdot 3^{m-1}-2)\omega(\omega-1) \text{ in } \mathbb{Z}_{3^{m+1}}[\omega].
\end{aligned} \tag{4.16}$$

Combining (4.14)–(4.16), we conclude that the expression (4.13) equals $3^m \mod 3^{m+1}$. (We need $m \geq 3$ to ensure $3^{m+1} \mid 3^{m-1}\cdot 3^{m-1}$, so that we can ignore products of powers of 3.) Therefore (4.12) does not hold, which concludes the proof for $p = 3$.

We also need that the spatial period is not reduced in considered cases, which are all covered by Lemma 4.1.11. □

Proof of Theorem 1.3.5. The desired claims are established by Lemmas 4.1.6–4.1.10, and Lemma 4.1.12. □

4.2. PS with Long Temporal Periods in Non-additive Rules

In this section, we prove Theorem 1.3.4, by two explicit constructions. Our first rule resembles a car odometer, and is similar to others that have previously appeared in the literature, see [**CY09**]. We view this as the most natural design, which also gives explicit constants $C(\sigma)$ and $N(\sigma)$, although the second construction based on prime partition is much shorter.

4.2.1. The Odometer Rule. For a fixed integer $k \geq 2$, we define the state space

$$\mathcal{S} = \mathbb{Z}_k \times \{\leftarrow, \circ\} \times \{*, \circ\} \times \{E, \circ\},$$

which has cardinality 2^3k. We call these four coordinates the **number**, **particle**, **asterisk**, and **end** coordinate, respectively. In words, each of the symbols $\leftarrow$, $*$, and E can be present at a site in addition to a number, and $\circ$ signifies its absence. We use abbreviations such as $(5, \leftarrow, *, E) = \overleftarrow{{}_E5^*}$, $(5, \leftarrow, \circ, \circ) = \overleftarrow{5}$, and $(5, \circ, \circ, \circ) = 5$. To be consistent with the car odometer interpretation, we construct a *right-sided* rule. That is

$$\xi_{t+1}(x) = f(\xi_t(x), \xi_t(x+1)),$$

or $\underline{\xi_t(x)}\xi_t(x+1) \mapsto \xi_{t+1}(x)$. Clearly, such a rule may be transformed to our standard left-sided one by a vertical reflection.

The rule is described in the following 14 assignments, in which I, J represent numbers in $\mathbb{Z}_k$ and addition is modulo k, i, j represent elements in $\mathbb{Z}_k \setminus \{k-1\}$, and $\diamond$ stands for any state in $\mathcal{S}$:

(1) $\underline{I}\overleftarrow{i^*} \mapsto \overleftarrow{I}$;

(2) $\underline{I}\overleftarrow{J} \mapsto \overleftarrow{I}$;

(3) $\underline{I}\overleftarrow{(k-1)^*} \mapsto \overleftarrow{I^*}$;

(4) $\underline{\overleftarrow{I^*}}\diamond \mapsto (I+1)$;

(5) $\underline{\overleftarrow{I}}\diamond \mapsto I$;

(6) $\underline{I_E}\overleftarrow{(k-1)} \mapsto \overleftarrow{I^*}$;

(7) $\underline{\overleftarrow{{}_Ei}}\diamond \mapsto \overleftarrow{{}_E(i+1)}$;

(8) $\underline{\overleftarrow{{}_E(k-1)}}\diamond \mapsto {}_E0$;

(9) $\underline{{}_EI}\overleftarrow{J^*} \mapsto \overleftarrow{{}_E0}$;

(10) $\underline{{}_EI}\overleftarrow{J} \mapsto \overleftarrow{{}_E0}$;

(11) $\underline{I}J \mapsto I$;

(12) $\underline{{}_EI}J \mapsto {}_EI$;

(13) $\underline{I_E}J \mapsto I$;

(14) $\underline{I_E}\overleftarrow{j} \mapsto I$.

In all cases not covered above, the rule leaves the current state unchanged: $\underline{c_0}c_1 \mapsto c_0$. We view the rule on $[0, \sigma-1]$ with periodic boundary, that is, within one spatial period of the PS.

Our construction simulates the dynamics of an odometer on the number coordinate. The three auxiliary coordinates are needed for the update rule to be a CA. We now give a less formal description. The end position indicator E marks the right end of our interval with periodic boundary. Hence, there has to be exactly one E and it is designed so that it does not appear or disappear (see assignments 7–10 and 12–14). The $\leftarrow$ is a left-moving particle (assignments 1–10), marking the site on which the number coordinate may add 1 in the next step. The number marked by an E adds 1 if its site also contains a particle, i.e., its particle coordinate is an $\leftarrow$ (assignments 7 and 8), and updates to 0 when an $\leftarrow$ is to its right (assignments 9 and 10). The number coordinates not marked by an E add 1 if and only if the asterisk coordinate is $*$ (see assignment 4 and 5). The symbol $*$ plays the role of carry in addition and can appear and disappear: it appears if the E position has number $k-1$, then it moves along with the particle (see assignment 6) if its number coordinate is $k-1$ (see assignment 3), and disappears if there is no carry (see 1) or if it arrives to the E position (see 9).

TABLE 4.1. An odometer PS for $\sigma = 3$, $k = 10$.

0	0	${}_E\overleftarrow{0}$	
0	0	${}_E\overleftarrow{1}$	(11, 14, 7)
	$\vdots$		
0	0	${}_E\overleftarrow{9}$	(11, 14, 7)
0	$\overleftarrow{0}^*$	${}_E0$	(11, 6, 8)
$\overleftarrow{0}$	1	${}_E0$	(1, 4, 12)
0	1	${}_E\overleftarrow{0}$	(5, 13, 10)
0	1	${}_E\overleftarrow{1}$	(11, 14, 7)
	$\vdots$		
0	9	${}_E\overleftarrow{9}$	(11, 14, 7)
0	$\overleftarrow{9}^*$	${}_E0$	(11, 6, 8)
$\overleftarrow{0}^*$	0	${}_E0$	(3, 4, 12)
1	0	${}_E\overleftarrow{0}$	(4, 13, 9)
	$\vdots$		
9	9	${}_E\overleftarrow{9}$	(11, 14, 7)
9	$\overleftarrow{9}^*$	${}_E0$	(11, 6, 8)
$\overleftarrow{9}^*$	0	${}_E0$	(3, 4, 12)
0	0	${}_E\overleftarrow{0}$	(4, 13, 9).

Any rule with the above fourteen **odometer assignments** is called an **odometer CA** and generates a PS of temporal period at least k^σ, called **odometer PS**. This shows that $\max_f X_{\sigma,8k}(f) \geq k^\sigma$. To give an example, let $L = 00\ldots \overleftarrow{{}_E0}$ be the configuration consisting of $(\sigma-1)$ 0's and a $\overleftarrow{{}_E0}$. When $\sigma = 3$, $k = 10$, then the PS is given in Table 4.1, where the relevant assignments are given in the parentheses. The PS has temporal period $1199 > 10^3 = k^\sigma$. We summarize the result of this section, which provides the best lower bound we have on $\max_f X_{\sigma,n}(f)$.

Proposition 4.2.1. *There exists a CA rule f so that $X_{\sigma,n}(f) \geq \lfloor n/8 \rfloor^\sigma$.*

The shortcoming of this construction is that it does not ensure that $Y_{\sigma,n}(f) = \Theta(n^\sigma)$, as the odometer rule, as it stands, has other PS with much shorter temporal periods. For example, in the CA from Table 4.1, the configuration 123 is fixed due to the assignment 11, and so it generates a PS with temporal period 1. We provide the remedy in the next subsection.

4.2.2. The Odometer Rule with Automata. To prevent short temporal periods, we need to extend the state space. The strategy is to introduce a second layer to each state, which encodes two finite automata that determine whether a configuration is legitimate, i.e., either itself or one of its updates is included in the above odometer PS. A legitimate configuration will generate the PS with long temporal period, while an illegitimate one will eventually end up in a spatially constant configuration.

Definition 4.2.1. *Consider the state space $\mathbb{Z}_k \times \{\leftarrow, \circ\} \times \{*, \circ\} \times \{E, \circ\} \times \mathcal{A}$ of the odometer CA, where $\mathcal{A}$ is any finite set. A configuration on $[0, \sigma-1]$ is **legitimate** if the following three conditions are satisfied:*

1. *there is exactly one site that contains an $\leftarrow$;*
2. *there is exactly one site that contains an E;*
3. *if a site contains $*$, then this site contains an $\leftarrow$ but does not contain an E.*

Lemma 4.2.1. *Any odometer rule starting from any legitimate configuration eventually enters the odometer PS.*

Proof. *Case 1.* An inductive argument shows that any legitimate configuration in the form of $a_0 \ldots \overleftarrow{{}_E a_{\sigma-1}}$ generates the odometer PS.

Case 2. Suppose that a legitimate configuration does not contain an $*$ and thus is of the form $a_0 \ldots \overleftarrow{a_j} \ldots_E a_{\sigma-1}$. Then by assignments 2 and 5, the $\leftarrow$ moves left until $\overleftarrow{a_0} \ldots_E a_{\sigma-1}$ and then updates to $a_0 \ldots {}_E\overleftarrow{0}$ because of assignments 5 and 10, reducing to Case 1.

Case 3. A legitimate configuration $a_0 \ldots \overleftarrow{a_j^*} \ldots_E a_{\sigma-1}$, $a_j < k-1$, updates to $a_0 \ldots \overleftarrow{a_{j-1}}(a_j + 1) \ldots_E a_{\sigma-1}$ because of assignments 1 and 4, or to $a_0 \ldots \overleftarrow{{}_E a_{\sigma-1}}$, reducing to either Case 2 or Case 1.

Case 4. A legitimate configuration $a_0 \ldots \overleftarrow{(k-1)^*} \ldots_E a_{\sigma-1}$ (with the $\leftarrow$ at position j) becomes $a_0 \ldots \overleftarrow{a_{j-1}^*} 0 \ldots_E a_{\sigma-1}$, which is reduced to Case 3 when $a_{j-1} < k-1$. If $a_{j-1} = k-1$, repeated updates eventually reduce to Case 3 or Case 1. $\square$

To define the augmented state space, we first introduce the concept of **deterministic finite automata** (DFA) or **automata** in short. An automata is finite-state machine, whose transition of states is determined by a given string. To be precise, an automata is a 5-tuple $(S, \Sigma, \delta, s_0, A)$, where S is the set of finite state; Σ is the alphabet of the input string; $\delta : S \times \Sigma \to S$ is the transition function; $s_0 \in S$ is the initial state and; $A \subset S$ is the set of accept states.

We now define the augmented state space for our two-layer construction of the **odometer rule with automata**:

$$\mathcal{S}_A = (\mathbb{Z}_k \times \{\leftarrow, \circ\} \times \{*, \circ\} \times \{E, \circ\} \times \mathcal{E} \times \mathcal{A}) \cup \{T\},$$

where $\mathcal{E} = \{(0,0), (1,0), \ldots, (\sigma-1, 0), (1,1), (2,1), \ldots, (\sigma-1, 1), T_1\}$ comprises states of a finite automaton, called `END-READER`; $\mathcal{A} = \{0, 1, \ldots, \sigma, T_2\}$ comprises states of another finite automaton, called `ARROW-READER`; and T is the special terminator state that erases the configuartion once it appears. We regard the first four components — those from the odometer rule above — as the first layer of a state, and the two automata components as the second layer.

We proceed to specify the rule. The first layer updates according to the previous odometer assignments. In addition, we include the assignment

- $\underline{(I, \circ, *, \circ)}s \mapsto T$ and $\underline{(I, \circ, *, E)}s \mapsto T$ for all $s \in \mathcal{S}_A$.

That is, if the first layer of a state contains an $*$ but not an $\leftarrow$, the state updates to T. Such an update will happen in any configuration that is illegitimate due to having an $*$ but not an $\leftarrow$.

The next assignment spells out the role of T_1, T_2, and T:

- For any site x, if either x or $x+1$ is in the state T or at least one of the second layers of x, $x+1$ contains a T_1 or a T_2, then x updates its state to T.

A configuration that contains a T_1, a T_2 or a T is called **terminated**. Any terminated configuration will eventually update to the constant configuration consisting of all T's, thus reduce the spatial period to 1.

The transition function δ_E of the finite automaton `END-READER` $= (\mathcal{E}, \{E, \circ\}, \delta_E, (i,j), T_1)$ reads the end coordinate and is given in Figure 4.1; its initial state (i,j) can be any state in $\mathcal{E}$. From time t to time $t+1$, an `END-READER` at position x reads the state on its first layer, updates its state according to δ_E, then "moves" to $x-1$. This left shift of the entire `END-READER` configuration is allowed as we are constructing a right-sided rule. According to the odometer assignments, the E position in a configuration does not appear or disappear and does not move. As a result, the `END-READER` counts the number of E's.

Lemma 4.2.2. *Every configuration with 0 or at least 2 sites containing an E will be terminated for any initial state of the* `END-READER`. *Conversely, starting from a configuration whose first layer is $= 00\ldots\overleftarrow{{}_E0}$, no* `END-READER` *ever reaches T_1 unless it starts there.*

Proof. Start with a configuration with 0 or 2 more states that contain an E. Suppose that it is never terminated by the `END-READER`. Then there is a time t and a position x such that the state of the `END-READER` is $(0,0)$, as it is clear from Figure 4.1. Within σ time steps from t, the `END-READER` transitions to T_1. The converse result is also clear from Figure 4.1. □

We also need to terminate illegitimate configurations with 0 or at least 2 arrows. First, a configuration with 2 or more arrows can be handled by adding the following assignment:

- $\underline{s_1}s_2 \mapsto T$, for all $s_1, s_2 \in \mathcal{S}_A$ such that s_1, s_2 both contain an $\leftarrow$.

Lemma 4.2.3. *Assume $k > \sigma$. Let L be a configuration that is never terminated by the* `END-READER` *and such that at least two states of L contain an $\leftarrow$. Then L will be eventually terminated.*

Proof. Since L is not terminated by the `END-READER`, there is exactly one state of L that contains E. Assume that the two states with $\leftarrow$ are not adjacent, as otherwise the configuration

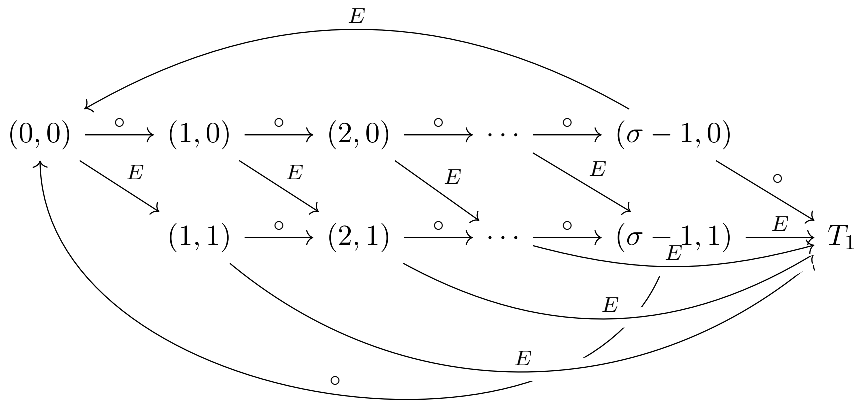


FIGURE 4.1. The transition function δ_E for `END-READER`.

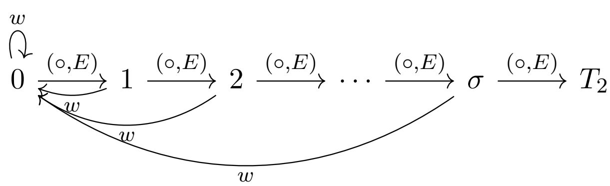


FIGURE 4.2. The transition function δ_A of the `ARROW-READER`. Here w is any symbol in $\{\leftarrow, \circ\} \times \{E, \circ\} \setminus \{(\circ, E)\}$.

is terminated immediately. Note that the arrow at the E position stays there for k updates and other arrows move left at every update. As $k > \sigma$, two arrows will eventually be adjacent. $\square$

Due to Lemma 4.2.3, it suffices to enlist a finite automaton whose mission is to terminate configurations with no $\leftarrow$. This automaton is the `ARROW-READER` that reads the particle and end coordinates and is given by $(\mathcal{A}, \{\leftarrow, \circ\} \times \{E, \circ\}, \delta_A, (i,j), T_2)$, where the transition function δ_A is described in Figure 4.2 and its initial state is any state in $\mathcal{A}$. From time t to time $t+1$, an `ARROW-READER` at site x updates its state according to δ_A and stays at the same position x. According to the odometer assignments, an $\leftarrow$ must appear at the E position within σ updates if there is at least one $\leftarrow$. Hence, the `ARROW-READER` terminates a configuration that fails this condition. The effect of this automaton is summarized in the following lemma.

Lemma 4.2.4. *Every configuration with no $\leftarrow$ is eventually terminated for any initial state of the* ARROW-READER*. Conversely, starting from a configuration whose first layer is $00\ldots\overleftarrow{{}_E0}$, no* ARROW-READER *ever reaches T_2 unless it starts there.*

The next proposition provides our first proof of Theorem 1.3.4.

Proposition 4.2.2. *Let $S(\sigma) = 16\sigma(\sigma+2)$. For the rule f defined in this subsection, we have $X_{\sigma,n}(f) = Y_{\sigma,n}(f) \geq \lfloor n/S(\sigma)\rfloor^\sigma$ for $n \geq (\sigma+2)S(\sigma)+1$.*

Proof. Observe that $\#\mathcal{S}_A = S(\sigma)\cdot k+1$. For a number of states n, let $k = \lfloor (n-1)/S(\sigma)\rfloor$. Encode the odometer rule with automata on $S(\sigma)\cdot k+1$ states, and make any leftover states immediately transition to T. Let $L \in \mathcal{S}_A^\sigma$ be a configuration with its first layer is $00\ldots\overleftarrow{{}_E0}$; on the second layer, the END-READER's are at state $(0,0)$ and the ARROW-READER's are at state 0. Then the configuration is not terminated by either END-READER or ARROW-READER, by Lemmas 4.2.2 and 4.2.4. Then the global configuration restricted on the first layer is the one of odometer CA, which has temporal period at least k^σ. Therefore, $X_{\sigma,n}(f) \geq k^\sigma = \lfloor n/S(\sigma)\rfloor^\sigma$.

Furthermore, note that any illegitimate configuration in $\mathcal{S}_A^\sigma$, as well as any configuration not in $\mathcal{S}_A^\sigma$, will eventually produce the constant configuration of all Ts with spatial period 1, by Lemmas 4.2.2–4.2.4. Furthermore, any legitimate configuration on the first layer will eventually update to a configuration whose first layer is in the odometer PS (by Lemma 4.2.1), and will never be terminated by the second layer that is not already in one of the terminator states (by Lemmas 4.2.2 and 4.2.4). Therefore, $Y_{\sigma,n}(f) = X_{\sigma,n}(f)$. □

4.2.3. The Prime Partition Rule. We begin with a simple consequence of the prime number theorem.

Lemma 4.2.5. *For an arbitrary $\sigma > 0$, and for large enough n, there are σ primes $p_0,\ldots,p_{\sigma-1} \in [(n-1)/(2\sigma),(n-1)/\sigma]$.*

Assume that n is large enough so that Lemma 4.2.5 holds. Find disjoint sets $P_0,\ldots,P_{\sigma-1} \subset \mathbb{Z}_n\setminus\{0\}$ such that $\#P_j = p_j$, for $j = 0,\ldots,\sigma-1$. This can be achieved since $p_0+\cdots+p_{\sigma-1} \leq n-1$. The state $0 \in \mathbb{Z}_n \setminus (P_0\cup\cdots\cup P_{\sigma-1})$ will play the role of the terminator. Let $\phi_j : P_j \to P_j$ be a cyclic permutation of the p_j states. Keeping the right-sided convention from the Section 4.2.2, we define the CA rule f as follows:

$$f(s,s') = \begin{cases} \phi_j(s) & \text{if } s \in P_j \text{ and } s' \in P_{(j+1) \mod \sigma} \text{ for some } j \in \{0,\ldots,\sigma-1\} \\ 0 & \text{otherwise} \end{cases}.$$

PROPOSITION 4.2.3. *For f defined above, we have $X_{\sigma,n}(f) = Y_{\sigma,n}(f)$ and*

$$\liminf_{n\to\infty} n^{-\sigma} Y_{\sigma,n}(f) \geq (2\sigma)^{-\sigma}.$$

PROOF. Call a configuration $s_0 s_1 \ldots s_{\sigma-1}$ *regular* if there exists an ℓ so that $s_j \in P_{(j+\ell) \mod \sigma}$, $j = 0,\ldots,\sigma-1$. To show that $X_{\sigma,n}(f) \geq (n-1)^\sigma/(2\sigma)^\sigma$, run the rule starting from any regular configuration. Such a configuration appears again for the first time after $p_0 p_1 \ldots p_{\sigma-1} \geq (n-1)^\sigma/(2\sigma)^\sigma$ updates. To show that $Y_{\sigma,n}(f) = X_{\sigma,n}(f)$, observe that any non-regular initial configuration eventually ends up in the constant configuration of all 0s. □

4.3. Discussion

In this paper, we continue our study of the shortest and the longest temporal periods of a PS for a fixed spatial period σ. While we are able to construct a rule whose longest temporal period grows as n^σ for large n, more precise results remain elusive even for $\sigma = 3$. We start our discussion with this case.

We call an n-state rule that has a PS with spatial period σ and temporal period $T(\sigma, n)$ as **maximum cycle length** (**MCL**) rule. For $\sigma = 3$, our computations demonstrate that an MCL rule exists for $n \leq 20$. More precisely, the number of MCL rules is 1 for $n = 2$ (out of 2^4 rules), 12 for $n = 3$ (out of 3^9 rules) and 732 for $n = 4$ (out of 4^{16} rules). These numbers match the first three terms of the sequence

$$(-1)^k 7^{2k} E_{2k}\left(\frac{3}{7}\right), k = 0,1,2,3,\ldots = 1, 12, 732, 109332, \ldots, \tag{4.17}$$

where E_n are the Euler polynomials. Unfortunately, it is hard to traverse all of the $5^{25} \approx 2.98 \times 10^{17}$ 5-state rules to count the number of MCL ones, so we merely state an open question.

QUESTION 4.3.1. *Assume $\sigma = 3$. Does there exist an MCL rule for any number of states $n \geq 2$? If so, is the number of MCL rules given by (4.17) for all n, or is the connection just a curious coincidence for $n \leq 4$?*

If $X_{\sigma,n}(f) = T(\sigma, n)$, then automatically $Y_{\sigma,n}(f) = X_{\sigma,n}(f) = T(\sigma, n)$, as the PS goes through all configurations with number of states n and spatial period σ. However, for $\sigma \geq 4$, an MCL may not exist, as demonstrated for $n = 3$ by Table 4.2, and therefore the maxima of $X_{\sigma,n}$ and $Y_{\sigma,n}$ may differ. This motivates our next question.

TABLE 4.2. Maximal temporal period for $n = 3$ and spatial periods $\sigma \leq 10$. We also give N_X, and N_Y, the numbers of rules that realize the respective maxima.

σ	$\max_f X_{\sigma,3}(f)$	N_X	$\max_f Y_{\sigma,3}(f)$	N_Y	$T(\sigma,3)$
1	3	1458	3	1458	3
2	6	216	6	216	6
3	24	12	24	12	24
4	40	12	32	72	72
5	120	2	120	2	240
6	111	6	84	42	696
7	1967	12	546	2	2184
8	904	12	896	24	6480
9	9207	12	1809	12	19656
10	10490	6	410	12	58800

QUESTION 4.3.2. *What is the asymptotic behavior of* $\max_f X_{\sigma,3}(f)$*, as σ grows? Or the asymptotic behavior of* $\max_f X_{\sigma,n}(f)$ *for an arbitrary fixed n? Making n large first, what is the asymptotic behavior of*

$$\liminf_{n\to\infty} n^{-\sigma} \max_f X_{\sigma,n}(f)$$

for large σ? (See Proposition 4.2.1 for an exponentially small lower bound.) The same questions can be posed for $Y_{\sigma,n}$ (for which Propositions 4.2.2 and 4.2.3 provide even smaller lower bounds).

To discuss the relation between $X_{\sigma,n}$ and $Y_{\sigma,n}$ for additive rules, let $\rho_\sigma(n) = \max_{f\in A_n} Y_{\sigma,n}(f)$. As it is clear from Table 4.3, $\pi_\sigma(n)$ and $\rho_\sigma(n)$ may differ, even for $\sigma = 2$ or 3. This suggests our next question.

QUESTION 4.3.3. *Fix a $\sigma \geq 2$. Is there an explicit formula for $\rho_\sigma(n)$, in terms of n, at least for small σ? Can one characterize n for which $\pi_\sigma(n) = \rho_\sigma(n)$?*

TABLE 4.3. Maximum of shortest and longest temporal periods of additive rules, for $\sigma = 2, 3$ and $n = 2, \ldots, 20$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\rho_2(n)$	2	2	2	4	2	6	2	2	4	10	2	12	6	4	2	16	2	18	4
$\pi_2(n)$	2	2	2	4	2	6	4	6	4	10	2	12	6	4	8	16	6	18	4
$\rho_3(n)$	3	6	3	24	6	6	3	6	24	120	6	12	6	24	3	288	6	18	24
$\pi_3(n)$	3	6	6	24	6	6	12	18	24	120	6	12	6	24	24	288	18	18	24

For a prime power p^m, we define the function $\mathtt{ub}_\sigma(p^m)$ to be the upper bound obtained from Propositions 4.1.2, 4.1.3 and 4.1.5. That is, $\mathtt{ub}_1(p) = p - 1$; $\mathtt{ub}_\sigma(p) = p^{\mathrm{ord}_\sigma(p)} - 1$ if $p \nmid \sigma$ and $\sigma \geq 2$; $\mathtt{ub}_\sigma(p) = p^k \cdot \mathtt{ub}_{\sigma/p^k}(p)$ if $k \geq 1$ is the largest power of$p$ dividing σ; and $\mathtt{ub}_\sigma(p^m) = p \cdot \pi_\sigma(p^{m-1})$ if $m \geq 2$. It is common that $\pi_\sigma(p^m) = \mathtt{ub}_\sigma(p^m)$, most notably for $\sigma = 5$.

QUESTION 4.3.4. *Is it true that, for all prime powers p^m, $\pi_5(p^m) = \mathtt{ub}_5(p^m)$?*

We have checked that there are no counterexamples to the "yes" answer on Question 4.3.4 for all p^m such that $p \leq 50$ and $\mathtt{ub}_5(p^m) \leq 10^5$. As counterexamples should be harder to come by for larger p (more a and b to choose from) and for larger m (less chance for $\Pi(a, b; p^m)$ to be equal to $\Pi(a, b; p^{m-1})$), we conjecture that the answer to Question 4.3.4 is indeed affirmative. We also remark, that, if this conjecture holds, there is an explicit formula for $\pi_5(n)$ for all n, due to Lemma 4.1.7 and Proposition 4.1.4.

It is not *always* true that $\pi_\sigma(p^m) = \mathtt{ub}_\sigma(p^m)$. Table 4.4 contains a list of examples of inequality we have found for $\sigma \leq 50$. One hint that the table offers is easy to prove and we do so in the next proposition.

PROPOSITION 4.3.1. *Assume that $\sigma = 2^k$, $k \geq 1$. Then $\pi_\sigma(2^m) = 2^k$ for all $m \leq k + 1$, but $\pi_\sigma(2^{k+2}) = 2^{k+1}$.*

PROOF. When $n = 2$, $(1 + x)^{2^k} = 1 + x^{2^k} = 0$ in $\mathbb{Z}_2[x]/(x^\sigma - 1)$. This implies that, for any m, when a and b are both odd, all states are eventually divisible by 2, and then by additivity

TABLE 4.4. Examples with $\pi(p^m) < \mathtt{ub}(p^m)$. An arrow indicates a range of powers.

σ	2	4	7	8	11	13	14	16
p^m	2^2	$2^{2\to 3}$	3	$2^{2\to 4}$	2	2	3	$2^{2\to 5}$
$\pi_\sigma(p^m)$	2	4	364	8	341	819	364	16
$\mathtt{ub}_\sigma(p^m)$	4	8	728	16	1023	4095	728	32

σ	21	22	26	32	42	44
p^m	3	2	2	$2^{2\to 6}$	3	2
$\pi_\sigma(p^m)$	1092	682	1638	32	1092	1364
$\mathtt{ub}_\sigma(p^m)$	2184	2046	8190	64	2184	4092

$(a + bx)^t = 0$ for large enough t. Clearly the same is true when a and b are both even. If a is odd and b is even,

$$(a + bx)^{2^k} = a^{2^k} = 1 \text{ in } \mathbb{Z}_{2^{k+1}}[x]/(x^\sigma - 1),$$

and the same conclusion holds if a is even and b is odd. This shows that $\pi_\sigma(2^m) \leq 2^k$ for $m \leq k+1$. As clearly $\Pi_\sigma(0, 1; 2^m) = \sigma = 2^k$, we get $\pi_\sigma(2^m) = 2^k$.

By the same argument, $(a + bx)^{2^{k+1}} = 1$ in $\mathbb{Z}_{2^{k+2}}[x]/(x^\sigma - 1)$, for all a and b. Moreover, it is easy to check that $(1 + 2x)^{2^k} = 1 + 2^{k+1}x + 2^{k+1}x^2 \neq 1$ in $\mathbb{Z}_{2^{k+2}}[x]/(x^\sigma - 1)$, proving the last claim. $\square$

Call a prime p **persistent** if $\pi_\sigma(p) < \mathtt{ub}_\sigma(p)$ for infinitely many σ. We conclude with a few questions suggested by Table 4.4.

QUESTION 4.3.5. *(1) Is either 2 or 3 persistent? (2) Are there infinitely many primes p such that is $\pi_\sigma(p) < \mathtt{ub}_\sigma(p)$ for some σ? (3) Is 2 the only prime with $\pi_\sigma(p^m) < \mathtt{ub}_\sigma(p^m)$ for some $m \geq 2$?*

CHAPTER 5

Weakly Robust Periodic Solutions of a Random Rule

Again, $r = 2$ throughout this chapter.

While we study the general PS and its existence of a random rule in the previous three chapters, we will explore the weakly robustness of PS in this chapter.

As usual, we restate the theorem to be proved in this chapter.

THEOREM (1.3.6 restated). *Let $\mathcal{T} \times \Sigma \subset \mathbb{N} \times \mathbb{N}$ be fixed and finite. If there exists $(\tau, \sigma) \in \mathcal{T} \times \Sigma$ such that $\sigma \mid \tau$, then $\mathbb{P}(\mathcal{R}_{\mathcal{T},\Sigma} \neq \emptyset) = c(\mathcal{T}, \Sigma)/n + o(1/n)$, where $c(\mathcal{T}, \Sigma)$ is a constant depending only on $\mathcal{T}$ and Σ.*

5.1. Decidability and WRPS

Recall the definition of right-extension, label digraph and Algorithm 2.1.2 for obtaining PS from label digraph. In order for a PS to be weakly robust, we need one more condition on the directed cycle in the label digraph, which requires that each label *decides* its unique child. To be more accurate, let A and B be two labels. Assume that at a site $k \in \mathbb{Z}$ the temporal evolution of the states, arranged vertically, is the repeated label A: $a_0 \ldots a_{\tau-1} a_0 \ldots a_{\tau-1} \ldots$. Suppose that the states at site $k+1$ eventually "converge" to repetition of B: $b_0 \ldots b_{\tau-1} b_0 \ldots b_{\tau-1} \ldots$, regardless of the initial state at site $k+1$. In this case, we say that A decides B, and then it is clear that A does not decide C for any other length-τ label C that is not equal to B up to a circular shift. We now provide a more formal definition.

DEFINITION 5.1.1. *Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two length-τ labels. We call that label A **decides** B, denoted as $A \Rightarrow B$, if the following two conditions are satisfied:*

1. *label A right-extends to B, i.e., $A \to B$;*
2. *for an arbitrary $c_0 \in \mathbb{Z}_n$, recursively define $c_{j+1} = f(a_{j \bmod \tau}, c_j)$; then there exists a $j \geq 0$ such that $c_{j \bmod \tau} = b_{j \bmod \tau}$.*

The following proposition, analogous to Proposition 2.2 in [**GG12**], provides an algorithm to verify whether a PS is weakly robust.

PROPOSITION 5.1.1. *A tile is a WRPS if and only if each column decides the column to its right.*

PROOF. Assume that a tile $T = (a_{i,j})$ is a WRPS with columns A_j, $j = 0, \ldots, \sigma - 1$. Let η be the initial configuration formed by doubly infinite repetition of $a_{0,0} \ldots a_{0,\sigma-1}$. If $A_j = a_{0,j} \ldots a_{\tau-1,j}$ does not decide $A_{j+1} = a_{0,j+1} \ldots a_{\tau-1,j+1}$, for some $j = 0, \ldots, \tau - 1$, then there exists a $c_0 \in \mathbb{Z}_n$ such that in the position to the right of A_j, the states do not converge to a repetition of A_{j+1}. Now, construct an initial configuration η' by replacing one $a_{0,j+1}$ by c_0 in η. Then η' is proper for η, but the advance of the spatial period is stopped, thus $v(\eta') = 0$ and T cannot be weakly robust.

Conversely, note that if label A_j decides A_{j+1}, then for any $c_0 \in \mathbb{Z}_n$ to the right of $a_{0,j}$, the label converges to A_{j+1} within $n\tau$ iterations. Thus the expansion velocity must be at least $1/(\tau n)$. □

Recall that by Lemma 2.1.1, a tile of a PS does not have periodic rows. The following lemma concludes that a periodic label cannot be a part of WRPS tile, since otherwise the temporal period of the WRPS is reduced.

LEMMA 5.1.1. *If T is a tile of WRPS of period τ, then every column has minimal period τ.*

PROOF. Assume that A is a label of length τ that is formed by concatenating shorter label A' that has length τ'. It is clear that if $A \Rightarrow B = b_0 \ldots b_{\tau-1}$, A also decides the circular shift $b_{\tau'} b_{\tau'-1} \ldots b_\tau b_0 \ldots b_{\tau'-1}$. This implies that $b_0 = b_{\tau'}$, $b_1 = b_{\tau'+1}$, etc. That is, B is also periodic with period τ'. By induction, every label in T is periodic with period τ', thus T is temporally reducible. □

In a label digraph $D_{\tau,f}$, we call an arc $A \to B$ **deciding arc** if $A \Rightarrow B$ and a directed cycle **deciding cycle** if all the arcs contained in this cycle are deciding arcs. The following algorithm finds all WRPS of temporal period τ for rule f.

ALGORITHM 5.1.1.

---

input **:** *Label digraph $D_{\tau,f}$ of f with temporal period τ*

Find all deciding cycles in $D_{\tau,f}$

for *each deciding cycle $A_0 \Rightarrow A_1 \Rightarrow \cdots \Rightarrow A_{\sigma-1} \Rightarrow A_0$* **do**

  form the tile T by placing labels $A_0, A_1, \ldots, A_{\sigma-1}$ on successive columns.

  if *both spatial and temporal periods of T are minimal* **then**

    print *T as a WRPS*

  end

end

5.2. Decidability Probability

We call a label $A = a_0 \ldots a_{\tau-1}$ **simple** if $a_i \neq a_j$ for $i \neq j$. We next prove the main result regarding the probability of the decidability of simple labels.

THEOREM 5.2.1. *Fix a number of states n and a $\tau \leq n$. Let $A = a_0 \ldots a_{\tau-1}$ be a simple label with length τ and $B = b_0 \ldots b_{\tau-1}$ be any other label (not necessarily simple) of length τ. Then*

$$\mathbb{P}(A \Rightarrow B) = \frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau}.$$

The theorem is proved in four lemmas below. The key idea reduces to calculating the probability that a random τ-partite graph is a directed **pseudo-tree**, i.e., a weakly connected directed graph that has at most one directed cycle. To be precise, we construct **label assignment digraph** (LAD) $G_{\tau,n}(f, A)$ of a label A under a rule f in the following manner.

We consider τ-partite digraphs with the ith part denoted by $(i, *) = \{(i, j) : j = 0, \ldots, n-1\}$, $i = 0, \ldots, \tau-1$. The arcs of the digraph $G_{\tau,n}(f, A)$ are determined as follows: for all $i = 0, \ldots, \tau-1$ and $j = 0, \ldots, n-1$, there is an arc $(i, j) \to (i+1, j')$ if $f(a_i, j) = j'$. As usual, we identify $i = \tau$ with $i = 0$, $i = \tau + 1$ with $i = 1$, etc. We next state the conditions for $G_{\tau,n}(f, A)$ that characterize when $A \to B$ and when $A \Rightarrow B$.

Definition 5.2.1. *Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two labels. Consider the following conditions on a τ-partite graph G:*

(1) *G contains the cycle $(0, b_0) \to (1, b_1) \to \cdots \to (\tau-1, b_{\tau-1}) \to (0, b_0)$;*

(2) *there is a directed path in G from (i, j) to $(0, b_0)$ for all $i = 0, \ldots, \tau-1$ and $j = 0, \ldots, n-1$.*

The set $\mathcal{E}(A, B)$ is the set of all τ-partite digraphs G, which satisfy condition (1) and the set $\mathcal{D}(A, B)$ is the set of all such digraphs G that satisfy both conditions (1) and (2).

Lemma 5.2.1. *Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be any two labels. Then $A \to B$ if and only if $G_{\tau,n}(f, A) \in \mathcal{E}(A, B)$ and $A \Rightarrow B$ if and only if $G_{\tau,n}(f, A) \in \mathcal{D}(A, B)$.*

We skip the proof as it follows immediately from the definitions, and instead give two examples for different rules by Figure 5.1. For the reader's convenience, we denote a node $(i[a_i], j)$ instead of (i, j) as in the definition. The two labels are $A = 12$ and $B = 00$ in both cases. Under the rule that generates the left LAD, $A \to B$, but $A \not\Rightarrow B$, i.e., $G_{\tau,n}(f, A) \in \mathcal{E}(A, B) \setminus \mathcal{D}(A, B)$; under the rule that generates the right LAD, $A \Rightarrow B$, i.e., $G_{\tau,n}(f, A) \in \mathcal{D}(A, B)$.

Figure 5.1. Two LADs of label $A = 12$ under two different rules. We use $(i[a_i], j)$ to represent a node for the reader's convenience. In the left one, $A \to 00$ but $A \not\Rightarrow 00$; in the right one, $A \Rightarrow 00$.

Fix a label $A = a_0 \ldots a_{\tau-1}$. The LAD $G_{\tau,n}(f, A)$ becomes a random graph if the rule f is selected randomly and we are interested in $\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{E}(A, B)\right)$ and $\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{D}(A, B)\right)$. The case that A is simple is easier as we can take advantage of independence of assignments of f. To be precise, let A be a simple label with length τ and B be an arbitrary label with the same length. We clearly have that $\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{E}(A, B)\right) = 1/n^{\tau}$, as the assignments on (a_j, b_j)'s are independent.

Next, we find $\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{D}(A, B)\right)$ for simple label A thus complete the proof of Theorem 5.2.1. We start by the following observation.

Lemma 5.2.2. *If A and A' are simple labels with the same length, $\mathbb{P}(A \Rightarrow B) = \mathbb{P}(A' \Rightarrow B)$ for any label B; if B and B' are labels with the same length, $\mathbb{P}(A \Rightarrow B) = \mathbb{P}(A \Rightarrow B')$ for any simple label A.*

To find $\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{D}(A, B)\right)$, we adapt the counting techniques in [**Sbe90**] to enumerate $\mathcal{D}(A, B)$. We start by proving the following combinatorial result.

Lemma 5.2.3. *Let $A_{k,\ell} = \binom{n-1}{k}(\ell+1)^k(n-1-\ell)^{n-1-k}$, and k_{m+1} a non-negative integer. Then*

$$\begin{aligned} S_m &:= \sum_{k_m=0}^{n-1} A_{k_m,k_{m+1}} \dots \left[\sum_{k_2=0}^{n-1} A_{k_2,k_3}\left[\sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2}\right]\right] \\ &= n^{(m+1)(n-2)}\left[P_{m+1} + k_{m+1}(n-1)^m\right], \end{aligned}$$

where $P_m = n^m - (n-1)^m$.

Proof. We use induction on m. Assume $m = 1$. Observe that

$$A_{k,\ell} = n^{n-1}\mathbb{P}\left(\text{Binomial}\left(n-1, \frac{\ell+1}{n}\right) = k\right).$$

Therefore,

$$\begin{aligned} \sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2} &= n^{n-2} \cdot n^{n-1} \cdot \left[1 + (n-1)\frac{k_2+1}{n}\right] \\ &= n^{2(n-2)}\left[P_2 + k_2(n-1)\right]. \end{aligned}$$

Now, by the induction hypothesis

$$\begin{aligned}
S_m &= \sum_{k_m=0}^{n-1} A_{k_m,k_{m+1}} S_{m-1} \\
&= n^{m(n-2)} \sum_{k_m=0}^{n-1} \binom{n-1}{k_m} (k_{m+1}+1)^{k_m} (n-1-k_{m+1})^{n-1-k_m} \left[P_m + k_m(n-1)^{m-1}\right] \\
&= n^{m(n-2)} \left[n^{n-1} P_m + (n-1)^m (k_{m+1}+1) n^{n-2}\right] \\
&= n^{(m+1)(n-2)} \left[n P_m + k_{m+1}(n-1)^m + (n-1)^m\right] \\
&= n^{(m+1)(n-2)} \left[P_{m+1} + k_{m+1}(n-1)^m\right],
\end{aligned}$$

which is the desired result. □

Now, we are ready to prove the key combinatorial result.

Lemma 5.2.4. *Let A and B be labels with length τ and let A be simple. Then $\#\mathcal{D}(A,B) = n^{\tau(n-2)}(n^\tau - (n-1)^\tau)$.*

Proof. The argument we give partly follows the proof of Theorem 1 in [**Sbe90**]. Applying Lemma 5.2.2, we may assume that $B = 0\ldots0$, without loss of generality.

First, choose a $k_{\tau-1} \in \{0,\ldots,n-1\}$, pick $k_{\tau-1}$ nodes in $(\tau-1,*) \setminus \{(\tau-1,0)\}$, and form $k_{\tau-1}$ arcs from those nodes to the node $(0,0)$. There are $\binom{n-1}{k_{\tau-1}}$ choices for a fixed $k_{\tau-1}$. Denote this subset of $(\tau-1,*)$ together with $(\tau-1,0)$ as $(\tau-1,*)'$; thus, $(\tau-1,*)' \subset (\tau-1,*)$ are the nodes in $(\tau-1,*)$ that are mapped to $(0,0)$. Assign the images of the nodes in $(\tau-1,*) \setminus (\tau-1,*)'$ to $(0,*) \setminus \{(0,0)\}$, for which there are $(n-1)^{n-1-k_{\tau-1}}$ choices. So, for a fixed $k_{\tau-1}$ to assign the image of nodes in $(\tau-1,*)$, there are

$$\binom{n-1}{k_{\tau-1}}(n-1)^{n-1-k_{\tau-1}}$$

choices.

Second, we need to assign the image of the nodes in $(\tau-2,*)$ to $(\tau-1,*)$. Choose a $k_{\tau-2} \in \{0,\ldots,n-1\}$, pick $k_{\tau-2}$ nodes in $(\tau-2,*) \setminus (\tau-2,0)$, and form $k_{\tau-2}$ arcs from those nodes to the nodes in $(\tau-1,0)'$. There are $\binom{n-1}{k_{\tau-2}}$ choices to choose those nodes for a fixed $k_{\tau-2}$ and $(k_{\tau-1}+1)^{k_{\tau-2}}$ choices to assign the images. Denote this subset of $(\tau-2,*)$ together with $(\tau-2,0)$

as $(\tau-2,*)'$. Now, the images of the nodes in $(\tau-2,*)\setminus(\tau-2,*)'$ should be in $(\tau-1,*)\setminus(\tau-1,*)'$, for which there are $(n-1-k_{\tau-1})^{n-1-k_{\tau-2}}$ choices. Hence, for fixed $k_{\tau-1}$ and $k_{\tau-2}$, to assign the image of the nodes in $(\tau-2,*)$ to $(\tau-1,*)$, there are

$$\binom{n-1}{k_{\tau-2}}(k_{\tau-1}+1)^{k_{\tau-2}}(n-1-k_{\tau-1})^{n-1-k_{\tau-2}}$$

choices.

Repeat the above steps for $(\tau-3,*)$, $\ldots$, $(1,*)$. To complete the construction, we assign the images of the nodes in $(0,*)\setminus\{(0,0)\}$. We choose a $t\in\{0,\ldots,n-2\}$, and add t arcs from $(0,*)\setminus\{(0,0)\}$ to $(1,*)\setminus(1,*)'$ consecutively as specified below, making sure to avoid creating a cycle that does not include $(0,0)$.

In the evolving digraph, a **component** is a weakly connected component, obtained by ignoring the orientation of edges. First note that there are n components in the current digraph; more precisely, each node of $(0,*)$ belongs to a different component (possibly consisting of a single node).

To select the first arc, pick a $b\in(1,*)\setminus(1,*)'$ ($n-1-k_1$ choices). There is one component that contains $(0,0)$ and one other component containing b. As a result, there are $n-2$ components and among each of them, there is a node in $(0,*)\setminus\{(0,0)\}$ with zero out-degree. Among these $n-2$ nodes, we select one and connect it to b. Therefore, there are $(n-2)(n-1-k_1)$ choices for the first arc. The addition of this arc decreases the number of components by one.

To assign the second arc, again pick a $b\in(1,*)\setminus(1,*)'$ (again $n-1-k_1$ choices). Now there are exactly $n-3$ components, among which there is a node in $(0,*)\setminus\{(0,0)\}$ with zero out-degree. We again select one and connect it with this b, leading to $(n-3)(n-1-k_1)$ choices.

In subsequent steps, we add an arc from a to b, where $b\in(1,*)\setminus(1,*)'$ is arbitrary, while $a\in(0,*)\setminus\{(0,0)\}$ is a unique node with zero out-degree in any component not containing b in the graph already constructed. The algorithm guarantees that the number of components decreases by one after each arc is added, i.e., that a cycle not including $(0,0)$ is never created.

In the above steps we add t arcs, with the number of choices, in order: $(n-2)(n-1-k_1),(n-3)(n-1-k_1)\ldots,(n-t-1)(n-1-k_1)$. As any order in which they are assigned produces the same digraph, there are

$$\frac{(n-2)(n-1-k_1)(n-3)(n-1-k_1)\cdots(n-t-1)(n-1-k_1)}{t!} = \binom{n-2}{t}(n-1-k_1)^t$$

choices. Finally, we assign the remaining $n-1-t$ arcs to $(1,*)'$, for which we have $(k_1+1)^{n-1-t}$ choices. Hence, for a fixed k_1, to assign the arcs originating from $(0,*)\setminus\{(0,0)\}$, there are

$$\sum_{t=0}^{n-2}\binom{n-2}{t}(n-1-k_1)^t(k_1+1)^{n-1-t} = (k_1+1)n^{n-2}$$

choices, in total. Lastly, we use Lemma 5.2.3 to get

$$\begin{aligned}
\#\mathcal{D}(A,B) &= \sum_{k_{\tau-1}=0}^{n-1}\binom{n-1}{k_{\tau-1}}(n-1)^{n-1-k_{\tau-1}} \\
&\quad\cdot\left[\sum_{k_{\tau-2}=0}^{n-1}A_{k_{\tau-2},k_{\tau-1}}\cdots\left[\sum_{k_2=0}^{n-1}A_{k_2,k_3}\left[\sum_{k_1=0}^{n-1}A_{k_1,k_2}(k_1+1)n^{n-2}\right]\right]\cdots\right] \\
&= n^{(\tau-1)(n-2)}\sum_{k_{\tau-1}=0}^{n-1}\binom{n-1}{k_{\tau-1}}(n-1)^{n-1-k_{\tau-1}}[P_{\tau-1}+k_{\tau-1}(n-1)^{\tau-2}] \\
&= n^{(\tau-1)(n-2)}\left[n^{n-1}P_{\tau-1}+(n-1)^{\tau-1}n^{n-2}\right] \\
&= n^{\tau(n-2)}P_\tau,
\end{aligned}$$

as claimed. □

Now, proof of Theorem 5.2.1 is straightforward.

Proof of Theorem 5.2.1. It is clear that the number of LAD $G_{\tau,n}(f,A)$ is $n^{\tau n}$. Then, by Lemma 5.2.4,

$$\mathbb{P}(A\Rightarrow B) = \mathbb{P}(G_{\tau,n}(f,A)\in\mathcal{D}(A,B)) = \frac{n^{\tau(n-2)}[n^\tau-(n-1)^\tau]}{n^{\tau n}} = \frac{n^\tau-(n-1)^\tau}{n^\tau}\cdot\frac{1}{n^\tau},$$

as claimed. □

By Theorem 5.2.1, assuming that A is simple and B is any label of the same length τ, we have

$$\mathbb{P}(A\Rightarrow B \mid A\to B) = \frac{n^\tau-(n-1)^\tau}{n^\tau} = \frac{\tau}{n}+o\left(\frac{1}{n}\right).$$

The case when A is not simple is much harder, since the parts of $G_{\tau,n}(f, A)$ are no longer independent from each other for a random rule f. While it is possible to obtain the deciding probability for a specific label using a similar method as in Theorem 5.2.1, it is hard to find a general formula or even to prove this probability is always $\mathcal{O}(1/n)$. We are, however, able to obtain the following weaker result.

THEOREM 5.2.2. *Let* $A = a_0 \dots a_{\tau-1}$ *and* $B = b_0 \dots b_{\tau-1}$ *be two fixed labels (not necessarily simple) with length* τ. *Then*

$$\mathbb{P}\left(G_{\tau,n}(f, A) \in \mathcal{D}(A, B) \;\middle|\; G_{\tau,n}(f, A) \in \mathcal{E}(A, B)\right) = o(1).$$

Equivalently, we have

$$\mathbb{P}\left(A \Rightarrow B \;\middle|\; A \to B\right) = o(1).$$

PROOF. Again, we assume that $B = 0 \dots 0$. We remark that, unlike Theorem 5.2.1, label B here does affect the deciding probability. However, the case of general B does not significantly alter the proof but it makes it transparent, so we choose this B for readability.

Let $a'_0, \dots, a'_{\ell-1}$ be the different states in A and m_i be the repetition numbers of a_i's, for $i = 0, \dots, \ell - 1$. Clearly, $\sum_{i=0}^{\ell-1} m_i = \tau$. Let ζ be the cycle $(0, 0) \to (1, 0) \to \cdots \to (\tau - 1, 0) \to (0, 0)$. It suffices to show that

$$\mathbb{P}(\text{there are no other cycles in } G_{\tau,n}(f, A) \mid \zeta \in G_{\tau,n}(f, A)) = o(1).$$

To accommodate the conditional probability, our probability space will be a uniform choice of a digraph from $\mathcal{E}(A, B)$ for the remainder of the proof.

Fix an integer $K \geq 1$. Call a cycle $\zeta' = (0, j_0) \to (1, j_1) \to \cdots \to (0, j_0)$ **simple with respect to** ζ if:

(1) ζ' contains no parallel arcs, i.e., if (i, j) and (i', j) are nodes in ζ', then $a_i \neq a_{i'}$; and

(2) if (i, j) is on ζ and (i', j') on ζ', then $(a_{i'}, b_{j'}) \neq (a_i, b_j)$.

Let Y_k be the random number of simple cycles with respect to ζ with length exactly τk and $Z_K = \sum_{k=1}^{K} Y_k$ be the random variable that counts the number of such cycles with length less than or equal to τK. We will show that, for any K, $\lim_{n\to\infty} \mathbb{P}(Z_K \geq 1) = 1 - \exp\left(-\sum_{k=1}^{K} 1/k\right)$, converging to 1 as $K \to \infty$. As a consequence, the LAD has another simple cycle asymptotically almost surely (in n), and this will conclude the proof.

We first compute the expectation of Y_k:

$$\mathbb{E}Y_k = \frac{(n-1)_{m_1 k} \cdots (n-1)_{m_\ell k}}{k} \cdot \frac{1}{n^{\tau k}} \to \frac{1}{k}, \quad \text{as } n \to \infty.$$

Here and in the sequel, we use the falling factorial notation $(x)_n = x(x-1)\cdots(x-n+1)$. The first factor counts the number of simple cycles with respect to ζ and the second factor is the probability that a fixed simple cycle with length τk is formed.

Now, let $\lambda_K = \mathbb{E}Z_K = \sum_{k=1}^{K} \mathbb{E}Y_k$. We use the notation Γ^k to denote the set of all possible simple cycles with length τk and define $\Gamma = \bigcup_{1\leq k\leq K} \Gamma^k$ as set of such cycles with length less than or equal to τK. The set Γ_i consists of cycles in Γ that has at least one node in common with the cycle i. The random variable I_i is the indicator that the cycle $i \in \Gamma$ is formed and $p_i = \mathbb{E}I_i$.

We use Lemma 2.1.3 to find an upper bound for $d_{\mathrm{TV}}(Z_K, \mathrm{Poisson}(\lambda_K))$. For the first term $\sum_{i\in\Gamma} p_i^2$, we have

$$\sum_{i\in\Gamma} p_i^2 = \sum_{k=1}^{K} \frac{(n-1)_{m_1 k} \cdots (n-1)_{m_\ell k}}{k} \frac{1}{n^{2\tau k}} = \mathcal{O}\left(\frac{1}{n^{\tau}}\right).$$

To obtain an upper bound for $\sum_{i\in\Gamma} \sum_{j\in\Gamma_i} p_i p_j$, we note that if i is the index of a simple cycle of length τr, then we may count the number of length-τk simple cycles that have no common vertex with the cycle i, that is

$$\#\left(\Gamma^k \setminus \Gamma_i\right) = \frac{(n-1-r)_{m_1 k} \cdots (n-1-r)_{m_\ell k}}{k}.$$

It immediately follows that,

$$\begin{aligned}
&\#\left(\Gamma^k \cap \Gamma_i\right) \\
&= \frac{(n-1)_{m_1 k} \cdots (n-1)_{m_\ell k} - (n-1-r)_{m_1 k} \cdots (n-1-r)_{m_\ell k}}{k} \\
&= \mathcal{O}\left(n^{\tau k - 1}\right),
\end{aligned}$$

as the highest powers of n in the numerator cancel. Hence, for a fixed r and k, we have

$$\begin{aligned}
&\sum_{i\in\Gamma^r}\sum_{k\in\Gamma_i\cap\Gamma^k} p_ip_j \\
&= \frac{(n-1)_{m_1 r}\cdots(n-1)_{m_\ell r}}{r}\cdot\#\left(\Gamma^k\cap\Gamma_i\right)\cdot\frac{1}{n^{\tau r}}\cdot\frac{1}{n^{\tau k}} \\
&= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore, the total sum

$$\sum_{i\in\Gamma}\sum_{j\in\Gamma_i} p_ip_j = \mathcal{O}\left(\frac{K^2}{n}\right).$$

For the last term in the upper bound in Lemma 2.1.3, we observe that $\mathbb{E}I_iI_j = 0$ if two cycles have shared vertices.

Now, by Lemma 2.1.3,

$$\mathbb{P}\left(Z_K = 0\right) \le e^{-\lambda_K} + \mathcal{O}\left(\frac{K^2}{n}\right) \le \frac{1}{K+1} + \mathcal{O}\left(\frac{K^2}{n}\right).$$

Sending $n\to\infty$ and noting that K is arbitrary conclude the proof. □

5.3. Proof of Theorem 1.3.6

Let T be a tile with τ rows and σ columns. Define the **rank** of T to be the largest x such that there exist x columns of T with distinct $x\tau$ states. We denote the rank of a tile as $\mathrm{rank}(T)$. For example, the tiles

$$T_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ \\ 2 & 3 & 0 & 1 \end{matrix}, \qquad T_2 = \begin{matrix} 0 & 1 & 2 & 1 \\ \\ 2 & 1 & 0 & 1 \end{matrix}.$$

have $\mathrm{rank}(T_1) = 2$ and $\mathrm{rank}(T_2) = 1$.

As in [**GL19c**], we denote by $\mathcal{R}^{(\ell)}_{\tau,\sigma,n}$ as the set of tile of WRPS that has lag ℓ. Thus the set of simple WRPS is $\mathcal{R}^{(0)}_{\tau,\sigma,n}$. We also use the notation $\mathcal{R}^{(0,y)}_{\tau,\sigma,n}\subset\mathcal{R}^{(0)}_{\tau,\sigma,n}$ to denote the set of WRPS whose tile is simple and has rank y. We use $\mathcal{T}_{\tau,\sigma,n}$ to denote the set of all PS tiles; to be more precise, this is the set of all $\tau\times\sigma$ arrays T with state space $\mathbb{Z}_n$ that satisfy properties 1 and 2 in Lemma 2.1.1, so that there exists a CA rule with a PS given by T. We also use $\mathcal{T}^{(0)}_{\tau,\sigma,n}$ and $\mathcal{T}^{(0,y)}_{\tau,\sigma,n}$ to denote the tiles in $\mathcal{T}_{\tau,\sigma,n}$ that are simple, and that are simple with rank y, respectively.

Our first step is to study the probability that $\mathcal{R}^{(0,x)}_{\tau,\sigma,n}$ is not empty, where $x = \sigma/\gcd(\tau,\sigma)$. Before we advance, we state two lemmas on simple tiles.

LEMMA 5.3.1. *Let T be a simple tile. Then*

(1) $\operatorname{rank}(T) \geq \sigma/\gcd(\sigma,\tau)$;

(2) $\operatorname{rank}(T) = y$ *if and only if* $s(T) = \tau y$. *In particular,* $\operatorname{rank}(T) = \sigma/\gcd(\sigma,\tau)$ *if and only if* $s(T) = \tau\sigma/\gcd(\sigma,\tau) = \operatorname{lcm}(\sigma,\tau)$.

PROOF. By Lemma 2.2.1, the states on each column of T are distinct and two columns either share no common states or are circular shifts of each other. As a result, $\operatorname{rank}(T) \geq s(T)/\tau$. Together with Lemma 2.2.3, this proves (1) and implication ($\Longrightarrow$) of (2). The reverse implication in (2) follows from $s(T) \geq \tau \cdot \operatorname{rank}(T)$. $\square$

In the sequel, we write $d = \gcd(\tau,\sigma)$, $k = \operatorname{lcm}(\sigma,\tau)$. By Lemma 5.3.1, k is the number of distinct states in a simple tile with rank $x = \sigma/d$. As before, φ is the Euler totient function. We index the tiles in $\mathcal{T}^{(0,x)}_{\tau,\sigma,n}$ in an arbitrary way. Let

$$\mathfrak{T}_m = \left\{ (T_i,T_j) \subset \mathcal{T}^{(0,x)}_{\tau,\sigma,n} \times \mathcal{T}^{(0,x)}_{\tau,\sigma,n} : i < j \text{ and } T_i, T_j \text{ have } m \text{ states in common} \right\}.$$

The following lemma gives the cardinality of these sets.

LEMMA 5.3.2. *The following enumeration results hold:*

(1) *the set* $\mathcal{T}^{(0,x)}_{\tau,\sigma,n}$ *has cardinality* $\varphi(d)\binom{n}{k}(k-1)!$;

(2) *if* $m < k$, *the set* $\mathfrak{T}_m$ *has cardinality*

$$\frac{1}{2}\varphi(d)\binom{n}{k}(k-1)!\varphi(d)\binom{k}{m}\binom{n-k}{k-m}(k-1)! = \mathcal{O}\left(n^{2k-m}\right);$$

(3) *if* $m = k$, *the set* $\mathfrak{T}_m$ *has cardinality*

$$\frac{1}{2}\varphi(d)\binom{n}{k}(k-1)!\left(\varphi(d)(k-1)!-1\right) = \mathcal{O}\left(n^{k}\right).$$

PROOF. Part (1) follows directly from Lemma 2.2.4. Then, part (2) follows from (1). Part (3) also follows from (1), after we note that once we select T_i, we have all k colors fixed and we are not allowed to select T_j equal to T_i. $\square$

We will also need the following consequence of Theorem 5.2.1.

LEMMA 5.3.3. *Let T be a simple tile and* $\operatorname{rank}(T) = y$. *Let $A_0, \ldots, A_{\sigma-1}$ be the labels in T. Then we have*

$$\mathbb{P}\left(A_i \Rightarrow A_{i+1},\ \textit{for } i = 0, \ldots, \sigma - 1 \,\middle|\, A_i \to A_{i+1},\ \textit{for } i = 0, \ldots, \sigma - 1\right) = \left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^y.$$

PROOF. Assume that the y columns with $y\tau$ states have indices in $I \subset \{0, \ldots, \sigma - 1\}$ and let those columns have labels A_i, $i \in I$. As A_i's do not share any states, the events $\{A_i \to A_{i+1}\}$, $i \in I$ are independent, and so are $\{A_i \Rightarrow A_{i+1}\}$, $i \in I$. We use Lemma 2.2.1 and Theorem 5.2.1 to get

$$\begin{aligned}
&\mathbb{P}\left(A_i \Rightarrow A_{i+1},\ \text{for } i = 0, \ldots, \sigma - 1 \,\middle|\, A_i \to A_{i+1},\ \text{for } i = 0, \ldots, \sigma - 1\right) \\
&= \frac{\mathbb{P}\left(A_i \Rightarrow A_{i+1},\ \text{for } i \in I\right)}{\mathbb{P}\left(A_i \to A_{i+1},\ \text{for } i \in I\right)} \\
&= \frac{\prod_{i \in I} \mathbb{P}\left(A_i \Rightarrow A_{i+1}\right)}{\prod_{i \in I} \mathbb{P}\left(A_i \to A_{i+1}\right)} \\
&= \left(\frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau}\right)^y \Big/ \left(\frac{1}{n^\tau}\right)^y \\
&= \left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^y,
\end{aligned}$$

as desired. ∎

Theorem 1.3.6 will now be established through next three propositions, the first one of which deals with existence of WRPS with zero lag and minimal rank $x = \sigma/d$.

PROPOSITION 5.3.1. *We have*

$$\mathbb{P}\left(\mathcal{R}^{(0,x)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau, \sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

for some constant $c(\tau, \sigma)$.

PROOF. We first find an upper bound by Markov inequality.

By Lemma 5.3.2, we have that $\#\mathcal{T}^{(0,x)}_{\tau,\sigma,n} = \varphi(d)\binom{n}{k}(k-1)!$. The probability that a tile in $\mathcal{T}^{(0,x)}_{\tau,\sigma,n}$ forms a PS is $1/n^k$ and the probability that the desired decidability, thus weak robustness, holds is $(\tau/n + o(1/n))^x$ by Lemma 5.3.3. As a result, we have

$$\mathbb{E}\left(\#\mathcal{R}^{(0,x)}_{\tau,\sigma,n}\right) = \varphi(d)\binom{n}{k}(k-1)!\frac{1}{n^k}\left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^x = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

as an upper bound.

To find an asymptotically matching lower bound, we use the Bonferroni's inequality

$$\mathbb{P}\left(\bigcup_i A_i\right) \geq \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}\left(A_i \cap A_j\right).$$

Here, A_i is the event that $T_i \in \mathcal{T}^{(0,x)}_{\tau,\sigma,n}$ is formed as a simple WRPS, for $i = 1, \ldots, \varphi(d)\binom{n}{k}(k-1)!$. Clearly, $\sum_i \mathbb{P}(A_i) = \mathbb{E}\left(\#\mathcal{R}^{(0,x)}_{\tau,\sigma,n}\right)$. Then it suffices to show that $\sum_{i<j} \mathbb{P}(A_i \cap A_j) = o\left(1/n^x\right)$.

For a pair of tiles $(T_i, T_j) \in \mathfrak{T}_m$, there are $2k-m$ different colors in $T_i \cup T_j$. By Lemma 2.2.5, there is at least one additional restriction on the number of maps. Using this lemma, the enumeration result Lemma 5.3.2, and Lemma 5.3.3, we have

$$\begin{aligned}
\sum_{i<j} \mathbb{P}(A_i \cap A_j) &= \sum_{m=0}^{k} \sum_{i<j} \mathbb{P}\left(A_i \cap A_j \cap \{(T_i, T_j) \in \mathfrak{T}_m\}\right) \\
&= \sum_{m=0}^{k} \mathcal{O}\left(n^{2k-m}\right) \frac{1}{n^{2k-m+1}} \left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^x \\
&= \mathcal{O}\left(\frac{1}{n^{x+1}}\right).
\end{aligned}$$

□

Next, we consider *all* simple tiles and show that among simple tiles, the WRPS with rank x provide the dominant probability.

PROPOSITION 5.3.2. *We have*

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

for the same constant $c(\tau,\sigma)$ as in Proposition 5.3.1.

Proof. First, we note the following bounds for $\mathbb{P}(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset)$,

$$\mathbb{P}\left(\mathcal{R}^{(0,x)}_{\tau,\sigma} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0,x)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{y} \mathbb{P}\left(\mathcal{R}^{(0,y)}_{\tau,\sigma,n} \neq \emptyset\right),$$

where the last sum is over $y = \sigma/d'$ for $d' \mid \gcd(\tau,\sigma)$ and $d < \gcd(\tau,\sigma)$. As $x < y$, we have from Lemmas 5.3.1–5.3.3,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{R}^{(0,y)}_{\tau,\sigma,n} \neq \emptyset\right) &\leq \mathbb{E}\left(\#\mathcal{R}^{(0,y)}_{\tau,\sigma,n}\right) \\
&= \varphi(d_y)\binom{n}{k_y}(k_y-1)!\frac{1}{n^{k_y}}\left(\frac{\tau}{n} + o\left(\frac{1}{n}\right)\right)^{y} \\
&= o\left(\frac{1}{n^x}\right),
\end{aligned}$$

where, $k_y = \tau y$ is the number of states in a tile in $\mathcal{R}^{(0,y)}_{\tau,\sigma,n}$ and $d_y = \sigma/y$. The conclusion now follows from Proposition 5.3.1. □

Lemma 5.3.4. *If $\ell > 0$, then*

$$\mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) = o\left(\frac{1}{n}\right).$$

Proof. For a fixed ℓ, let $g_{\tau,\sigma}(s)$ count the number of tiles with periods τ and σ, and s different *fixed* states. By Theorem 5.2.2,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{E}\left(\#\mathcal{R}^{(\ell)}_{\tau,\sigma,n}\right) &= \sum_{s=1}^{\tau\sigma}\binom{n}{s} g_{\tau,\sigma,\ell}(s)\frac{1}{n^{s+\ell}} \cdot o(1) \\
&= o\left(\frac{1}{n^{\ell}}\right) = o\left(\frac{1}{n}\right).
\end{aligned}$$

□

Next, we extend Proposition 5.3.2 to cover non-simple tiles. It is here that we impose the condition that $\sigma \mid \tau$.

Proposition 5.3.3. *If $\sigma \mid \tau$, then*

$$\mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n} + o\left(\frac{1}{n}\right).$$

PROOF. First, note that $\sigma \mid \tau$ implies that $x = \sigma / \gcd(\tau, \sigma) = 1$ and as a result of Proposition 5.3.2, we have

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n} + o\left(\frac{1}{n}\right).$$

The desired result now follows from the bounds

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) \leq \sum_{\ell=0}^{\tau\sigma} \mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right)$$

and Lemma 5.3.4. □

PROOF OF THEOREM 1.3.6. If $\sigma \nmid \tau$, then $x = \sigma / \gcd(\tau, \sigma) > 1$, and by Proposition 5.3.2 and Lemma 5.3.4,

$$\mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{\ell=1}^{\tau\sigma} \mathbb{P}\left(\mathcal{R}^{(\ell)}_{\tau,\sigma,n} \neq \emptyset\right) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right).$$

These bounds, together with Proposition 5.3.3, now give the desired result:

$$\begin{aligned}
\frac{c(\mathcal{T},\Sigma)}{n} + o\left(\frac{1}{n}\right) &= \sum_{\sigma \mid \tau} \mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) \\
&\leq \mathbb{P}(\mathcal{R}_{\mathcal{T},\Sigma,n} \neq \emptyset) \\
&\leq \sum_{\sigma \mid \tau} \mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) + \sum_{\sigma \nmid \tau} \mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) \leq \frac{c(\mathcal{T},\Sigma)}{n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

□

5.4. Discussion

In this chapter, inspired by [**GG12**], we prove that the probability that a randomly chosen CA has a weakly robust periodic solution with periods in the finite set $\mathcal{T} \times \Sigma$ is asymptotically $c(\mathcal{T},\Sigma)/n$, provided that $\mathcal{T} \times \Sigma$ contains a pair (τ, σ) with $\sigma \mid \tau$. A natural first question is whether the divisibility condition may be removed.

Question 5.4.1. *Let $\mathcal{R}_{\tau,\sigma,n}$ be the set of WRPS with periods τ and σ from a random rule f. Do we have*

$$\mathbb{P}(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset) = \frac{c(\tau,\sigma)}{n^x} + o\left(\frac{1}{n^x}\right),$$

where $x = \sigma/\gcd(\tau,\sigma)$?

A possible strategy to answer Question 5.4.1 affirmatively is through proving the following two conjectures, the first of which provides a lower bound of the rank of a tile. Recall that $x = \sigma/\gcd(\tau,\sigma)$.

Conjecture 5.4.1. *Let T be a tile of period τ and σ and $\ell = p(T) - s(T)$, then $rank(T) \geq \sigma/\gcd(\tau,\sigma) - \ell$.*

We recall that a tile of a WRPS satsifies the properties stated in Lemmas 2.1.1 and 5.1.1. The next conjecture presents an asymptotic property similar to the one in Theorem 5.2.2. In its formulation, we assume validity of Conjecture 5.4.1: for a tile T of a WRPS, we let $I = I(T) \subset \{0, \ldots, \sigma - 1\}$ be the index set with $\#I = x - \ell$, such that the labels indexed by I are the leftmost $x - \ell$ labels without a repeated state.

We recall that a tile of a WRPS satisfies the properties stated in Lemmas 2.1.1 and 5.1.1. The next conjecture presents an asymptotic property similar to the one in Theorem 5.2.2. In its formulation, we assume validity of Conjecture 5.4.1: for a tile T of a WRPS, we let $I = I(T) \subset \{0, \ldots, \sigma - 1\}$ be the index set with $\#I = x - \ell$, such that the labels indexed by I are the leftmost $x - \ell$ labels without a repeated state.

Conjecture 5.4.2. *Assume that T is a tile of a WRPS. Then there exists a label A_j with index $j \notin I$ so that*

$$\mathbb{P}\left(A_j \Rightarrow A_{j+1} \;\middle|\; \{A_i \Rightarrow A_{i+1} \textit{ for all } i \in I\}\right) = o(1).$$

If there exists a label j that does not share any state with A_i, for any $i \in I$, the conjecture can be proved in the same way as Theorem 5.2.2. To see how Question 5.4.1 is settled in the case that both of the conjectures are satisfied, use again the bounds

$$\mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}_{\tau,\sigma,n} \neq \emptyset\right) \leq \mathbb{P}\left(\mathcal{R}^{(0)}_{\tau,\sigma,n} \neq \emptyset\right) + \sum_{\ell} \mathbb{E}\left(\#\mathcal{R}^{(\ell)}_{\tau,\sigma,n}\right),$$

and then, with $g_{\tau,\sigma}(s)$ as in the proof of Lemma 5.3.4, and using Lemma 5.3.3,

$$\mathbb{E}\left(\#\mathcal{R}^{(\ell)}_{\tau,\sigma,n}\right) = \sum_{s=1}^{\tau\sigma} \binom{n}{s} g_{\tau,\sigma}(s) \frac{1}{n^m} \cdot \mathcal{O}\left(\frac{1}{n^{x-\ell}}\right) \cdot o(1) = o\left(\frac{1}{n^x}\right).$$

To provide some modest evidence for the validity of Conjecture 5.4.1, we prove that it holds when $\sigma = 2$ or $\tau = 2$. Conjecture 5.4.2 remains open even in these cases. We begin by the following lemma.

LEMMA 5.4.1. *Let T be a tile of a WRPS with $\sigma = 2$ and odd τ. Fix an arbitrary row as the 0th row. Let $\mathcal{M}_t = \{$maps up to t th row$\}$, $\mathcal{S}_t = \{$states up to t th row$\}$ and $\ell_t = \#\mathcal{M}_t - \#\mathcal{S}_t$, for* $t = 0, 1, \ldots, \tau - 1$. *Assume the $(t+1)$th row of the tile is ab. Then:*

(1) *if $a \in \mathcal{S}_t$ and $b \in \mathcal{S}_t$, $\ell_{t+1} - \ell_t = 2$;*

(2) *if exactly one of a and b is in $\mathcal{S}_t$, then $\ell_{t+1} - \ell_t = 1$; and*

(3) *if $a \notin \mathcal{S}_t$ and $b \notin \mathcal{S}_t$, $\ell_{t+1} - \ell_t = 0$.*

PROOF. Write $\ell_{t+1} - \ell_t = (\#\mathcal{M}_{t+1} - \#\mathcal{M}_t) - (\#\mathcal{S}_{t+1} - \#\mathcal{S}_t)$. Observe that $a \neq b$, as otherwise the spatial period of the tile is reducible. In addition, $(a, b) \notin \mathcal{M}_t$, as otherwise T is temporally reducible, and $(b, a) \notin \mathcal{M}_t$, as otherwise τ is even. Hence, $\#\mathcal{M}_{t+1} - \#\mathcal{M}_t = 2$, which implies the claim. $\square$

PROOF OF CONJECTURE 5.4.1 WHEN $\sigma = 2$. If τ is even, we need to show that $\text{rank}(T) \geq 1 - \ell$. This is trivial if $\ell \geq 1$, and follows from Lemma 2.2.1 when $\ell = 0$.

If τ is odd, we must show that $\text{rank}(T) \geq 2 - \ell$. We may assume $\ell = 1$ as otherwise this is immediate (as above). Then there exists exactly one $t \in \{0, \ldots, \tau - 1\}$ at which Case 2 of Lemma 5.4.1 happens, and otherwise Case 3 happens. If $a \in \mathcal{S}_t$, then column with b has no repeated state, and vice versa. $\square$

PROOF OF CONJECTURE 5.4.1 WHEN $\tau = 2$. We will prove this for any tile that satisfies the properties stated in Lemmas 2.1.1 and 5.1.1. We assume that no two different labels of T are rotations of each other; otherwise the argument is similar.

We use induction on the lag. If $\ell(T) = 0$, T is simple and Lemma 2.2.1 applies. Suppose now the statement is true for any tile T with $\ell(T) = \ell \geq 0$. Now, consider a tile T with $\ell(T) = \ell + 1$.

As $\ell(T) \geq 1$, there is at least one repeated state, say a. Consider two appearance of a and its neighbors:

$$bac \quad \text{and} \quad b'ac'.$$

As $\tau = 2$ and T has no rotated columns, $b \neq b'$ and $c \neq c'$. Now replace the a in bac by an arbitrary state not represented in T, say z, and denote the new tile by T'. Note that T' also satisfies the properties in Lemmas 2.1.1 and 2.2.1. Moreover, $p(T') = p(T)$ and $s(T') = s(T) + 1$ imply that $\ell(T') = \ell$. By inductive hypothesis, $\text{rank}(T') \geq \sigma / \gcd(\sigma, \tau) - \ell$. Among $\text{rank}(T')$ labels of T' without a repeated state, at most one has the state z. Excluding this label, if necessary, we conclude that $\text{rank}(T) \geq \sigma / \gcd(\sigma, \tau) - (\ell + 1)$. □

Besides the above two special cases, we are also able to prove Conjecture 5.4.1 for a special class of tiles, which may give a hint about the general case. Within T, fix an arbitrary row as the 0th row and find the smallest $\tilde{\tau}$ such that $\mathtt{row}_{\tilde{\tau}}$ is a cyclic permutation of $\mathtt{row}_0$. It is likely that such $\tilde{\tau}$ does not exist, in which case define $\tilde{\tau} = \tau$. We call T **semi-simple** if $p(T) = \tilde{\tau}\sigma$; i.e., within the first $\tilde{\tau}$ rows in T, there are no repeated states. We omit the proof of our last lemma, as it is very similar to the argument above.

Lemma 5.4.2. *A semi-simple tile T has rank at least $\sigma / \gcd(\tau, \sigma) - \ell$.*

CHAPTER 6

Robust Periodic Solutions

In this chapter, we continue the study of RPS from the pioneering paper [**GG12**]. We first present a natural extension of the R-algorithm. We will also give a proof to the property of bounded growth restated as follows.

THEOREM (1.3.7 restated). *The probability that an n-state edge rule with $f(0,0) = 0$ has growth velocity 1 is* $1 - 1/n$.

Again, r is assumed to be 2 in this chapter.

6.1. Algorithms for Finding RPS

In this section, we investigate the algorithm for finding RPS of a given temporal period. The fundamental principle of finding RPS is the idea of R-algorithm proposed in [**GG12**]. To be specific, we take advantage of the label digraph that is the analogy of label tree in [**GG12**] in the multi-state situation. We find cycles in a label digraph to locate PS, while finding deciding cycles defined in Chapter 5 to search for WRPS. Connecting a cycle with a deciding cycle finds an RPS. We state two different algorithms based on this shared principle.

First, we propose an algorithm for finding RPS from a single label digraph. Next we propose the algorithm that finds RPS starting from a fixed background and a given temporal period τ_h of the handle. The latter is the direct analogy of the R-algorithm in [**GG12**]. Throughout this chapter, we use $D_{\tau,f}$ to denote the label digraph of rule f with temporal period τ. We use C (respectively C_d) to represent a cycle (respectively deciding cycle) in $D_{\tau,f}$.

6.1.1. Algorithm 1. The most straightforward algorithm for finding RPS is simply a combination of Algorithm 2.1.2 and Algorithm 5.1.1 from a single label digraph $D_{\tau,f}$. An RPS is obtained once a directed path is found from a cycle to a deciding cycle in $D_{\tau,f}$. We present the pseudo-code in the following algorithm.

ALGORITHM 6.1.1.

---

input **:** τ, f

Generate $D_{\tau,f}$

for *cycles* C *in* $D_{\tau,f}$ **do**

  for *deciding cycles* C_d *in* $D_{\tau,f}$ **do**

    if *there is a path* $H : v \mapsto \cdots \mapsto u$ *such that* $v \in C$ *and* $u \in C_d$ **then**

      print C, H *and* C_d

    end

  end

end

In the above algorithm, the printed C, H and C_d play the role of the background PS, the handle and the RPS, respectively. We remark here that we do not assume H to be minimal. That is, either the background or the RPS may consist part of the handle.

We present an example to illustrate Algorithm 6.1.1. Fix $\tau = 3$ and a 3-state rule f, *220001011*. The partial digraph $D_{\tau,f}$ that contains the cycles of interests is depicted as following in Figure 6.1. There is a non-deciding cycle (the outer larger triangle) and a deciding cycle (the inner smaller triangle), which are connected by the label 001 or its rotations. The non-deciding cycle, label 001 and the deciding cycle in the graph serve as the background PS, the handle and the RPS, respectively. Piece of the RPS $(011)^\infty$, together with its background $(102)^\infty$ and handle 0, is presented in Figure 6.2.

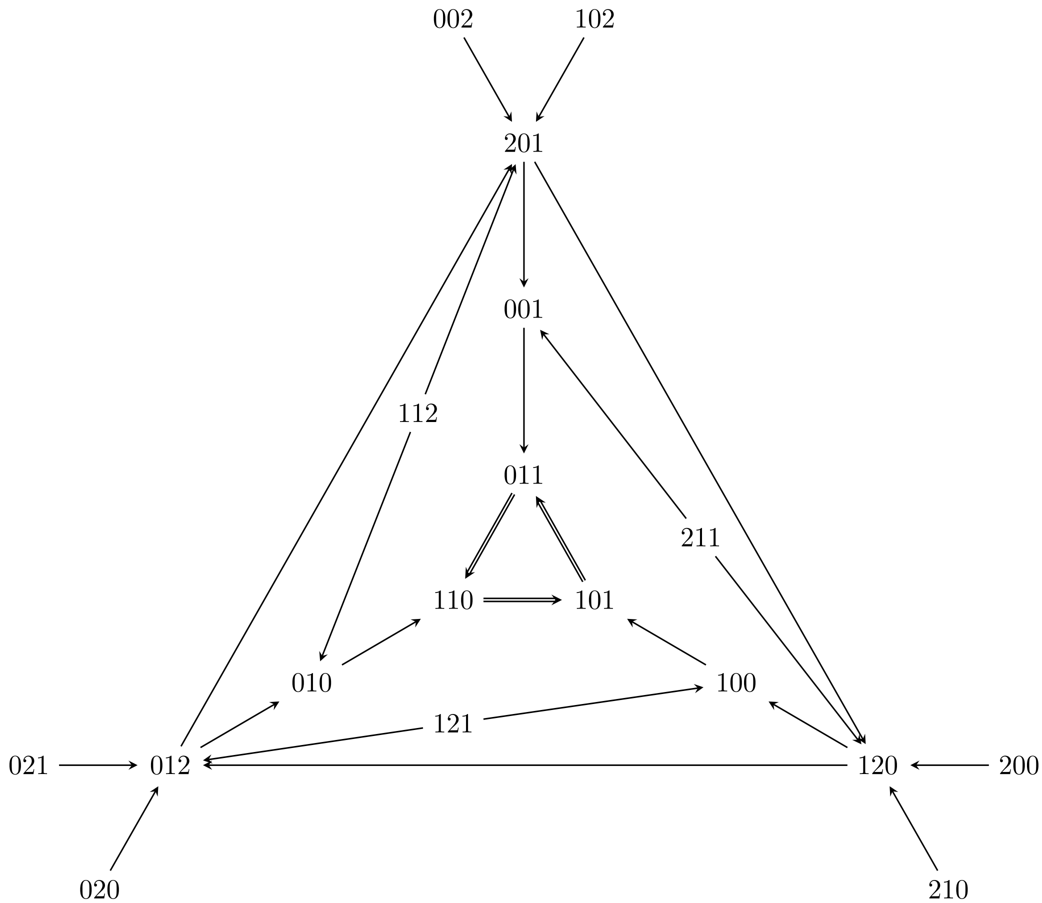


FIGURE 6.1. Part of the label digraph of 3-state rule *220001011* with temporal period $\tau = 3$. Arrows represent the arcs of usual right-extension relation between labels and double-arrows are used to denote deciding arcs.

6.1.2. Algorithm 2. The following algorithm is the direct analogy of the R-algorithm proposed in [**GG12**]. To generalize, we start with an arbitrary PS of a rule. We characterize this PS $\xi_t^{(0)}$ by its tile $T = [T_0, T_1, \ldots, T_{\sigma_0 - 1}]_{\tau_0 \times \sigma_0}$ and also input a τ_h such that $\tau_0 \mid \tau_h$. In this algorithm, we find all RPS of a given rule f with respect to $\xi_t^{(0)}$ such that the handle has temporal period τ_h.

In this algorithm, we use the notation $\texttt{DFS}(D_{\tau,f}, A, [C_0, \ldots, C_{m-1}])$ to represent a black-box function that takes a label digraph, a label and a sequence of (deciding) cycles as the inputs. Starting with the label A, the function initiates a depth first search to find paths leading to deciding cycles in the input sequence.

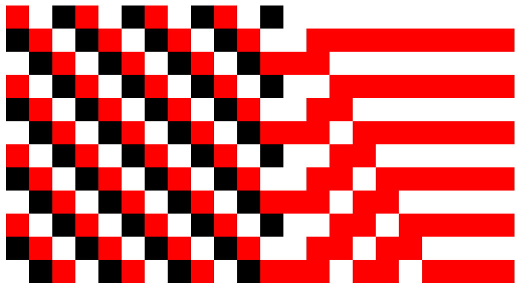


FIGURE 6.2. A piece of the RPS, together with its background and handle, of the 3-state rule *220001011*. This RPS $(011)^\infty$, as well as its background $(102)^\infty$ and handle 0, is located by Algorithm 6.1.1 and the above label digraph Figure 6.1.

We remark here that the two algorithms stated here both depend on the label digraph with a fixed τ of a CA rule f. While the Algorithm 6.1.1 finds the RPS, backgrounds and handles together, in the following Algorithm 6.1.2, we fix a background first and find RPS as well as handles that are with respect to the background.

ALGORITHM 6.1.2.

---

input : f, T, τ_h

Generate $D_{\tau_h,f}$

Find deciding cycles $C_d^{(0)}, \ldots, C_d^{(m-1)}$ *in* $D_{\tau_h,f}$

return : *DFS*$(D_{\tau_h,f}, T_0, [C_d^{(0)}, \ldots, C_d^{(m-1)}])$

An example is also given as an application of Algorithm 6.1.2. We present the RPS along with its background and handle of the 3-state rule *002121011*. The background has temporal period 2 and we input $\tau_h = 4$. The RPS found here has $\tau = 4$, while the minimal temporal period of the label of the handle 1010 is 2. We display piece of the global evolution to illustrate the expansion of the RPS $(21001112)^\infty$, together with its background $(20)^\infty$ and handle 0, while label digraph that generate this picture is not shown due to space limitation. The following proposition is evident.

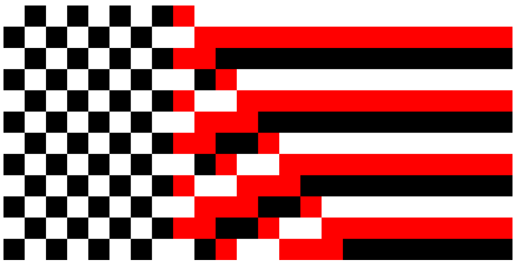


FIGURE 6.3. A piece of the RPS $(21001112)^\infty$, together with its background $(20)^\infty$ and handle 0, of the 3-state rule *002121011*. This RPS, as well as its background and handle, is found by Algorithm 6.1.2. The label digraph of this rule is omitted here.

PROPOSITION 6.1.1. *Let T be the tile of a PS of rule f of temporal period τ_0. Let τ_h be a multiple of τ_0. All possible RPS of temporal period τ, such that $\tau \mid \tau_h$, with respect to T are obtained by Algorithm 6.1.2.*

6.2. Proof of Theorem 1.3.7

In [**GG12**], among all of the 64 one-sided 3-neighbor binary edge CA, 21 are proved to have bounded growth. In our system, we can show that the bounded growth property is rare, in the sense that with high probability, a rule has growth velocity 1.

PROOF OF THEOREM 1.3.7. Fix an n and let $g : \mathbb{Z}_n \to \mathbb{Z}_n$ be defined as $g(a) = b$ if $f(a, 0) = b$. It is clear there are n^{n-1} such functions if we note that $g(0) \equiv 0$.

Also note that a rule has growth velocity < 1 if and only if

$$\text{for any } a \in \mathbb{Z}_n, g^k(a) = 0 \text{ for some } k > 0, \tag{6.1}$$

where the k here may depend on a. Note that there is a 1-1 correspondence between g's satisfying (6.1) and labeled trees with vertices $\mathbb{Z}_n$. For example, the following labeled tree corresponds to g such that $g(1) = 0, g(2) = 1, g(3) = 0, g(4) = 3$ and $g(5) = 1$. There are n^{n-2} such trees by Cayley's formula and thus the result follows.

$\square$

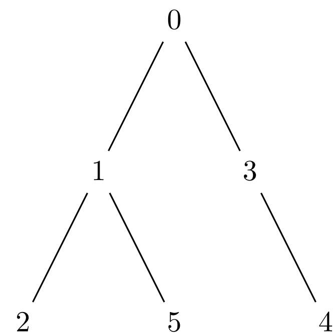


FIGURE 6.4. The labeled tree corresponding to $g(1) = 0, g(2) = 1, g(3) = 0, g(4) = 3$ and $g(5) = 1$.

We also remark here that we may generalize the definition of growth velocity from the point of view of robustness of an arbitrary PS.

Let f be a CA rule and ξ_0 be a PS configuration under f. Also choose any initial configuration η_0 such that there is a $y \in \mathbb{Z}$ and that $\eta_0(x) = \xi_0(x)$ for all $x > y$. We call such initial configuration $\eta_0(x)$ **right proper** for ξ_0. Run f starting from ξ_0 and η_0 to obtain ξ_t and η_t, respectively. Let

$$s_t^e(\eta_0) = \inf\{y : \eta_t(x) = \xi_t(x), \text{ for all } x > y\}$$

be the rightmost site that η_t does not agree with ξ_t at time t. Then the erosion velocity in the initial environment η_0 is

$$v^e(\eta_0) = \limsup_{t \to \infty} \frac{s_t^e}{t},$$

and the **erosion velocity** of the PS ξ_t is

$$v^e = \sup\{v^e(\eta_0) : \eta_0 \text{ is right proper for } \xi_0\}.$$

Clearly, for CA rules f that satisfy $f(0,0) = 0$, the constant configuration $\xi_t = \ldots 000 \ldots$ is a PS of temporal and spatial period 1. As a result, the growth velocity of such a rule defined in the above context can be naturally regarded as the erosion velocity of this constant configuration ξ_t.

Furthermore, recall the expansion velocity defined in Chapter 1, whose positivity characterizes the weak robustness. Then the erosion velocity of a PS and its expansion velocity have a dual relationship that we now discuss.

Let the **mirror** of f be the CA rule, denoted by f^m, such that f^m has the same neighborhood $[-1,0]$, state space $\mathbb{Z}_n$ and $f^m(a,b)=c$ if $f(b,a)=c$, for all $a,b\in\mathbb{Z}_n$. Let the **reflection** of a spatial configuration of ξ_t, denoted by ξ_t^r, be the function $\xi_t:\mathbb{Z}\to\mathbb{Z}_n$ such that $\xi_t^r(x)=\xi_t(-x)$. We then have the following relation between erosion velocity and expansion velocity.

Proposition 6.2.1. *Let ξ_t be a PS of a CA rule f and v be the expansion velocity of ξ_t under f. Let v^e be the erosion velocity of ξ_t^r under the mirror rule f^m. Then we have $v+v^e=1$.*

Proof. We observe that if $\xi_t(x)$ updates to $\xi_{t+1}(x)$ under f, then, under f^m, $\xi_t^r(x)$ updates to $(\xi_{t+1}^r)(x-1)$, i.e., the right shift of the reflection of ξ_{t+1}.

Now, let η_0 be any initial configuration that is proper for ξ_t. Then η_0^r is right proper for ξ_t^r. Update ξ_0 and η_0 to obtain ξ_t and η_t. Also, run f^m from ξ_0^r and η_0^r to obtain ξ_t' and η_t', respectively. Let $s_t(\eta_0)=\sup\{y:\eta_t(x)=\xi_t(x),\text{ for all } x<y\}$ and $s_t^e(\eta_0^r)=\inf\{y:\eta_t'(x)=\xi_t'(x),\text{ for all } x>y\}$. From the above observation, we have $s_t^e(\eta_0^r)=-s_t(\eta_0)+t$.

As a result, $v(\eta_0)=\liminf_{t\to\infty}s_t(\eta_0)/t$ and $v^e(\eta_0^r)=\limsup_{t\to\infty}s_t^e(\eta_0^r)/t$ satisfy $v(\eta_0)+v^e(\eta_0^r)=1$. The desired identity follows from the definition of expansion and erosion velocity of a PS. □

6.3. Discussion

In [**GG12**], the authors investigate the existence of RPS with respect to the 0 PS, among all 64 3-neighbor binary edge rules. By analogy to both [**GG12**] and Chapter 5, we may also study the probability that a random n-state rule has an RPS with constant 0 as the background PS. Let Ω_n^{edge} be the space of n-state edge rules and $\mathcal{R}_{\mathcal{T},\Sigma,n}(f,T)$ be the RPS of rule f with respect to the PS characterized by tile T. Then we may ask the following natural question.

Question 6.3.1. *Let $\mathcal{T}\times\Sigma\subset\mathbb{N}\times\mathbb{N}$. What is the behavior of $\mathbb{P}\left(\mathcal{R}_{\mathcal{T},\Sigma,n}(f,0)\neq\emptyset\right)$, as $n\to\infty$, where f is a random edge rule selected from Ω_n^{edge}?*

While a general answer is not presented for an arbitrary $\mathcal{T}\times\Sigma$, we are able to give a short illustration of the asymptotic behavior of $\mathbb{P}\left(\mathcal{R}_{1,\mathbb{N},n}(f,0)\neq\emptyset\right)$ as follows.

Proposition 6.3.1. *It holds that $\mathbb{P}\left(\mathcal{R}_{1,\mathbb{N},n}(f,0)\neq\emptyset\right)=1/n+o(1/n)$, as $n\to\infty$.*

Proof. Let $f \in \Omega_n^{edge}$ and $D_{1,f}$ be the label digraph of f with temporal period 1. For any label $a \in \mathbb{Z}_n$, we have that $0 \to a$ under rule f. As a result, any deciding cycle formed in $D_{1,f}$ give rise to an RPS with respect to the 0 PS with empty handle.

We may gently modify the proof of Theorem 5.2.1 to show that in the shrunk probability space, it also holds that $\mathbb{P}(a \Rightarrow b) = 1/n^2$, for any label $a \in \mathbb{Z}_n \setminus \{0\}$ and $b \in \mathbb{Z}_n$. That is, in $D_{1,f}$, a deciding arc is formed from any non-zero state to the other with probability $1/n^2$. It thus suffices to compute the probability that this digraph has a deciding cycle under this setting.

To this end, we again apply the Chein-Stein method for Poisson approximation. Let Y_k be the random variable of the number of k-cycles in $D_{1,f}$ and $Z_K = \sum_{k=1}^{K} Y_k$. It is clear that

$$\lambda_K = \mathbb{E}Z_K = \sum_{k=1}^{K} \binom{n-1}{k}(k-1)!\frac{1}{n^{2k}} = \frac{1}{n} + o\left(\frac{1}{n}\right).$$

A routine application of Chein-Stein method as follows gives the bound of the total variation between Z_K and a Poisson random variable with mean λ_K. Notice that the probability that this Poisson random variable is 0 is $\exp\left(-1/n + o(1/n)\right) = 1 - 1/n + o(1/n)$ and this will conclude the proof.

To be specific, let Γ be the cycles of length less than or equal to k, indexed in an arbitrary way. Let Γ^k be the set of cycles with length k and Γ_i be the set of cycles that has at least one node in common with the cycle i. Let $I_i = 1$ if the ith cycle is formed; otherwise, let $I_i = 0$. Also, denote $p_i = \mathbb{E}I_i$. First, we have that

$$\sum_{i \in \Gamma} p_i^2 = \sum_{k=1}^{K} \binom{n-1}{k}(k-1)!\left(\frac{1}{n^{2k}}\right)^2 = \frac{1}{n^3} + o\left(\frac{1}{n^3}\right).$$

Now, for a fixed k and ℓ,

$$\begin{aligned}\sum_{i \in \Gamma^k} \sum_{j \in \Gamma^k \cap \Gamma_i} p_i p_j &= \binom{n-1}{k}(k-1)!\left[\binom{n-1}{\ell}(\ell-1)! - \binom{n-1-k}{\ell}(\ell-1)!\right]\frac{1}{n^{2k}}\frac{1}{n^{2\ell}} \\ &= \frac{1}{n^{k+\ell+1}}.\end{aligned}$$

As a result,

$$\sum_{i \in \Gamma} \sum_{j \in \Gamma_i} p_i p_j = \mathcal{O}\left(\frac{K^2}{n^3}\right).$$

It is also clear that $\mathbb{E} I_j I_i = 0$, if the ith cycle and the jth share nodes. As a consequence, the desired bound follows from Lemma 2.1.3 $\square$

We may also explore the above question under the moderately generalized definition given in Chapter 5. Fix a matrix T with τ rows and σ columns such that T is able to serve as the tile of a PS with temporal and spatial periods τ and σ, respectively. Let Ω_n^T be the set of the n-state rules f such that $T \in \mathcal{P}_{\tau,\sigma,n}(f)$. We may raise the generalized question as follows.

Question 6.3.2. *Let T be a $\tau \times \sigma$ PS tile. What is the behavior of $\mathbb{P}\left(\mathcal{R}_{\mathcal{T},\Sigma,n}(f,T) \neq \emptyset\right)$, as $n \to \infty$, where f is a random rule selected from Ω_n^T?*

APPENDIX A

Multiplicative Group of Eisenstein Integers Modulo n

In this appendix, we determine the structure of the multiplicative group of Eisenstein integers modulo n, that is, the group $\mathbb{Z}_n[\omega]^\times = \{a + b\omega \in \mathbb{Z}_n[\omega] : a^2 + b^2 - ab \in \mathbb{Z}_n^\times\}$, where $\omega = e^{2\pi i/3}$. While our arguments are similar to those in the paper [**ADJ$^+$08**] on Gaussian integers modulo n, we are aware of no reference that directly implies Theorem A.0.1, so we provide a sketch of the proof.

LEMMA A.0.1. (1) *Let $p \geq 3$ be a prime number and a be an integer not divisible by p. Then $x^2 = a \mod p$ either has no solutions or exactly two solutions.*

(2) *Let $p \geq 5$ be a prime number. The number -3 is a quadratic residue modulo p if and only if $p = 1 \mod 6$.*

PROOF. See [**ADJ$^+$08**] for the proof of (1). For (2), see [**LeV96**], Exercise 9 on page 109. □

LEMMA A.0.2. *Let p be a prime.*

(1) *If $p = 3$, then $\mathbb{Z}_p[\omega]^\times \cong \mathbb{Z}_6$.*
(2) *If $p = 1 \mod 6$, then $\mathbb{Z}_p[\omega]^\times \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$.*
(3) *If $p = 5 \mod 6$, then $\mathbb{Z}_p[\omega]^\times \cong \mathbb{Z}_{p^2-1}$.*

PROOF. To prove (1), observe that the group $\mathbb{Z}_3[\omega]^\times$ is abelian, and $\#\mathbb{Z}_3[\omega]^\times = 6$, so $\mathbb{Z}_3[\omega]^\times \cong \mathbb{Z}_6$.

To prove (2), first note that then the equation $x^2 - x + 1 = 0 \mod p$ is equivalent to $(2x-1)^2 = -3 \mod p$. By Lemma A.0.1, the equation $y^2 = -3 \mod p$, where $y = 2x - 1$ has two solutions $y = \pm q$. We next find the cardinality of $\mathbb{Z}_p[\omega]^\times$. Assume that $a + b\omega \notin \mathbb{Z}_p[\omega]^\times$, so that $a^2 + b^2 - ab = 0 \mod p$. If $a \neq 0 \mod p$, then $(a^{-1}b)^2 - (a^{-1}b) = -1 \mod p$ and so $2a^{-1}b - 1 = \pm q \mod p$. So, $b = 2^{-1}a(\pm q + 1)$. In particular, for a fixed non-zero a, there are two possible values for b such that $a + b\omega \notin \mathbb{Z}_p[\omega]^\times$, proving that $\#\mathbb{Z}_p[\omega]^\times = (p-1)^2$.

As $\mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}$, it suffices to show that there is an isomorphism

$$\psi : \mathbb{Z}_p[\omega]^\times \to \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times.$$

It is routine to check that ψ, defined by $\psi(a+b\omega) = (a - 2^{-1}b(q+1), a - 2^{-1}b(-q+1))$, is an injective homomorphism, hence it is an isomorphism by equality of cardinalities.

To prove (3), note that $\mathbb{Z}_p[\omega]$ has p^2 elements, so it suffices to show that $\mathbb{Z}_p[\omega]$ is a field, as the multiplicative group of any field is cyclic. Assume again that $a + b\omega \notin \mathbb{Z}_p[\omega]^\times$, so that $a^2 + b^2 - ab = 0 \mod p$. If $a \neq 0 \mod p$, then $(a^{-1}b)^2 - (a^{-1}b) = -1 \mod p$. By Lemma A.0.1, the equation $x^2 - x + 1 = 0 \mod p$, or equivalently $(2x-1)^2 = -3 \mod p$, has no solution, as $p = 5 \mod 6$. We conclude that $a = 0 \mod p$, and similarly $b = 0 \mod p$, so $\mathbb{Z}_p[\omega]$ is a field. □

LEMMA A.0.3. *For a prime $p \geq 3$ and $m \geq 2$,*

$$\mathbb{Z}_{p^m}[\omega]^\times \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_p[\omega]^\times.$$

PROOF. The proof is analogous to that for Theorem 7 in [**ADJ$^+$08**]. □

LEMMA A.0.4. *For $m \geq 1$, $\mathbb{Z}_{2^m}[\omega]^\times$ is classified as follows: $\mathbb{Z}_2[\omega]^\times \cong \mathbb{Z}_3$, $\mathbb{Z}_{2^2}[\omega]^\times \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and, for $m \geq 3$, $\mathbb{Z}_{2^m}[\omega]^\times \cong \mathbb{Z}_3 \times \mathbb{Z}_{2^{m-1}} \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$.*

PROOF. The multiplicative group $\mathbb{Z}_2[\omega]^\times$ is abelian with 3 elements, so $\mathbb{Z}_2[\omega]^\times \cong \mathbb{Z}_3$. Assume that $m \geq 2$. Write $H = \mathbb{Z}_{2^m}[\omega]^\times$. The elements of the group H are of the form $(1+2k_1) + 2k_2\omega$, $2k_1 + (1+2k_2)\omega$ and $(1+2k_1) + (1+2k_2)\omega$ for $0 \leq k_1, k_2 \leq 2^{m-1} - 1$, so the number of them is $2^{m-1}2^{m-1}3 = 3 \times 2^{2m-2}$. Furthermore (see proof of Theorem 7 in [**ADJ$^+$08**]), each element in H has order at most $3 \cdot 2^{m-1}$, and by verifying that $(1+3\omega)^{3\cdot 2^{m-2}} \neq 1$ in $\mathbb{Z}_{2^m}[\omega]$ and $(1+3\omega)^{2^{m-1}} \neq 1$ in $\mathbb{Z}_{2^m}[\omega]$, we see that there exists an element with order exactly $3 \cdot 2^{m-1}$. As a consequence, $H \cong \mathbb{Z}_3 \times \mathbb{Z}_{2^{m-1}} \times \prod_{j=1}^{r} \mathbb{Z}_{2^{e_j}}$, where $e_j \geq 1$ and $\sum_{j=1}^{r} e_j = m-1$. When $m = 2$, the result follows immediately, so we assume $m \geq 3$ from now on.

We claim that $r = 2$. Since each factor, except $\mathbb{Z}_3$, is cyclic of order at least two, each contains exactly one subgroup of order two. So, H has 2^{r+1} solutions to the equation $(a+b\omega)^2 = 1 \mod 2^m$, which is equivalent to

$$\begin{cases} a^2 - b^2 = 1 \mod 2^m \\ 2ab - b^2 = 0 \mod 2^m \end{cases}.$$

This system has no solution unless a is odd and b is even, so we write $a = 2k_1 + 1$ and $b = 2k_2$ and obtain

$$\begin{cases} k_1^2 + k_1 - k_2^2 = 0 \mod 2^{m-2} \\ (2k_1 + 1 - k_2)k_2 = 0 \mod 2^{m-2} \end{cases}.$$

From the first equation, k_2 is even, so $2k_1 + 1 - k_2$ has an inverse and then $k_2 = 0 \mod 2^{m-2}$, so $k_2 = 0$ or 2^{m-2}. Now $k_1(k_1 + 1) = 0 \mod 2^{m-2}$. If k_1 is odd, then $k_1 + 1 = 0 \mod 2^{m-2}$ implies $a = 2^{m-1} - 1$ or $a = 2^m - 1$; if k_1 is even, then $k_1 = 0 \mod 2^{m-2}$ implies $a = 0$ or $a = 2^{m-1} + 1$. So, the original system has eight solutions, $2^{r+1} = 8$ and $r = 2$.

We now have $H \cong \mathbb{Z}_3 \times \mathbb{Z}_{2^{m-1}} \times \mathbb{Z}_{2^{e_1}} \times \mathbb{Z}_{2^{e_2}}$, where $e_1 + e_2 = m - 1$ and $e_1 \geq e_2$. Now, the result follows for $m = 3$ and 4, so we assume $m \geq 5$. Then, we claim that $e_2 = 1$ and $e_1 = m - 2$. Assume, to the contrary, that $e_2 \geq 2$. Then each factor, except $\mathbb{Z}_3$, has exactly one subgroup of order four, giving $4^3 = 64$ elements of order at most four in the direct product. However, we will show that H has at most 32 solutions to the equation $x^4 = 1$, which will establish our claim and end the proof. To this end, suppose $(a + b\omega)^4 = 1$ for some $a + b\omega \in \mathbb{Z}_{2^m}[\omega]$. Then

$$\begin{cases} a^4 - 6a^2b^2 + 4ab^3 = 1 \mod 2^m \\ b(4a^3 - 6a^2b^2 + b^3) = 0 \mod 2^m \end{cases}.$$

This system has no solutions unless b is even and a is odd, so write $a = 2k_1 + 1$ and $b = 2k_2$, $0 \leq k_1, k_2 \leq 2^{m-1} - 1$. Then the system becomes

$$\begin{cases} k_1(k_1 + 1)(2k_1^2 + 2k_1 + 1) - 3(2k_1 + 1)^2 k_2^2 + 4(2k_2 + 1)k_2 = 0 \mod 2^{m-3} \\ k_2\left[(2k_1 + 1)^3 - 6(2k_1 + 1)^2 k_2^2 + 2k_2^3\right] = 0 \mod 2^{m-3} \end{cases}.$$

The factor in square brackets and $2k_1^2 + 2k_1 + 1$ are odd, reducing the system to

$$\begin{cases} k_1(k_1+1) = 0 \mod 2^{m-3} \\ k_2 = 0 \mod 2^{m-3} \end{cases},$$

which has at most 32 solutions. □

We conclude by summarizing Lemmas A.0.2–A.0.4.

THEOREM A.0.1. *We have*

$$\mathbb{Z}_p[\omega]^\times \cong \begin{cases} \mathbb{Z}_6, & \text{if } p = 3 \\ \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}, & \text{if } p = 1 \mod 3 \\ \mathbb{Z}_{p^2-1} & \text{if } p = 2 \mod 3 \end{cases}$$

and

$$\mathbb{Z}_{p^m}[\omega]^\times \cong \begin{cases} \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{m-2}} \times \mathbb{Z}_6, & \text{if } p = 2 \text{ and } m \geq 2 \\ \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_p[\omega]^\times, & \text{if } p \neq 2 \end{cases}.$$

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