The Inverse Mean Curvature Flow: Singularities, Dynamical Stability, and Applications to Minimal Surfaces

By

BRIAN DONALD HARVIE

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

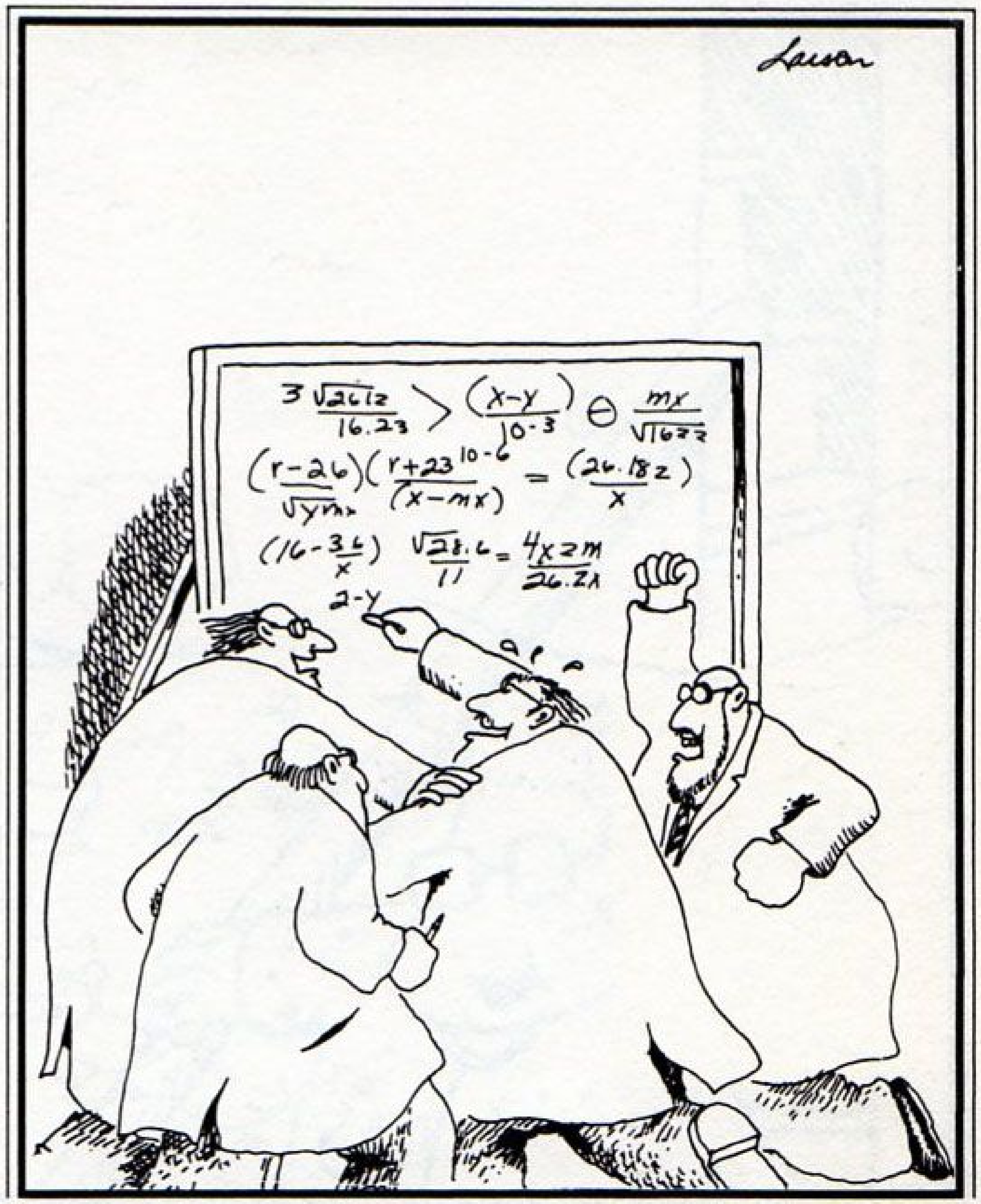
Adam Jacob, Chair

Joel Hass

Andrew Waldron

Committee in Charge

2021



"Go for it, Sidney! You've got it! You've got it! Good hands! Don't choke!"

Source: *The Far Side* by Gary Larson

To my friends, family, and teachers.

Contents

Abstract

This dissertation concerns the Inverse Mean Curvature Flow of closed hypersurfaces in Euclidean Space, and its relationship with minimal surfaces. Inverse Mean Curvature Flow is an extrinsic geometric flow which has become prominent in differential geometry because of its applications to geometric inequalities and general relativity, but deep questions persist about its analytic and geometric structure. The first four chapters of this dissertation focus on singularity formation in the flow, the flow behavior near singularities, and the dynamical stability of round spheres under mean-convex perturbations.

On the topic of singularities, I establish the formation of a singularity for all embedded flow solutions which do not have spherical topology within a prescribed time interval. I later show that mean-convex, rotationally symmetric tori undergo a flow singularity wherein the flow surfaces converge to a limit surface without rescaling, contrasting sharply with the singularities of other extrinsic geometric flows. On the topic of long-time behavior, I show that all flow solutions which exist and remain embedded for some minimal time depending only on initial data must exist for all time and asymptotically converge to round spheres at large times. In the fourth chapter, I utilize this characterization to establish dynamical stability of the round sphere under certain mean-convex, axially symmetric perturbations that are not necessarily star-shaped.

In the last chapter, I relate questions of singularities and dynamical stability for the Inverse Mean Curvature Flow to the mathematics of soap films. Specifically, I show that certain families of solutions to Plateau's problem do not self-intersect and remain contained within a given region of Euclidean space. I accomplish this using a barrier method arising from global embedded solutions of Inverse Mean Curvature Flow. Conversely, I also use minimal disks to establish that a singularity likely forms in the flow of a specific mean-convex embedded sphere.

Acknowledgments

I would like to thank my advisor, Professor Adam Jacob, for his continued support and guidance throughout my time as a graduate student. Early in the program I realized that my mathematical interests diverged from most of the research in the math department at Davis, and I feared that this would make realizing my ambitions difficult or impossible. In a very fortunate occurrence for me, the department hired Adam the same year I began the PhD. program. Through my years as his student, he has consistently provided encouragement and helpful input in our weekly meetings. He has supported my career and my development as a mathematician through using his grant money to fund my conference trips, writing several recommendation letters on my behalf, and providing extensive feedback on all of my research papers. A disproportionate number of the conversations about differential geometry and the Simpsons in my life have been with him, and I am going to miss these.

There are a number of other professors at Davis and elsewhere that I would like to thank. Thank you to Professors Joel Hass and Andrew Waldron for serving on both my qualifying exam committee and my dissertation committee. Professor Hass also wrote a letter of recommendation for me and inspired the last chapter of this dissertation, so I would like to give him a special thanks for this. Thank you to my other letter writers, Professor John Hunter of UC Davis and Professor Eric Woolgar of the University of Alberta, the latter of whom also invited me to the *Workshop on Asymptotically Hyperbolic Manifolds* at the Banff International Research Station.

Several current and former graduate students in this department made the potentially isolating experience of research in mathematics far more manageable for me. I have been blessed over the past six years to meet so many kind and colorful individuals who relieved my stress through difficult times, reinvigorated my enthusiasm for mathematics, and taught me much about myself. I will certainly think back to some of these people the next time I find myself at a Blaze Pizza or a farmer's market. In no particular order, to Michael Ponce, Wencin Poh, Norman Sheu, Jianping Pan, Lily Silverstein, Shanon Rubin, Kyle Chickering, Tonie Scroggin, and Karry Wong: thank you all.

The administration and staff in the mathematics department at UCD have done an outstanding job making the bureaucratic aspects of completing a PhD. much easier for me to navigate. With classes, teaching, and research already being as exhausting as they are, I am grateful to be saved stress on the administrative side of my work. I would like to thank all past and present staff members of the math department for a job well done, especially Tina Denena, Sarah Driver, and Victoria Whistler.

I would like to give a special thanks to the excellent teachers from prior to graduate school who encouraged me to pursue a career in science. In particular, thank you to Thomas Noviello of Leominster High School, whose dedication to physics education has positively affected my life trajectory and the life trajectories of many others, and to Professors Jennie Traschen and David Kastor of the University of Massachusetts Amherst, whose willingness to mentor a naive undergraduate in general relativity paid off eventually.

Last but certainly not least, I give thanks to my family. To my sisters Katie and Julie, we had our face-offs as siblings growing up, but to now understand one another and to have each of you rooting for me means so much. To my parents Patricia and Donald, I am forever grateful for your love and support. I realized in my teenage years that I was interested in the less practical aspects of science, and dad, you were understandably skeptical about the career prospects in this direction. But you believed in me enough to put those concerns aside and fully support my life decision, and I could not have accomplished this without both of you. Throughout many busy weeks crammed with difficult assignments and never-ending edits to papers, I always appreciated being able to call and talk about life during a walk through the campus arboretum. I hope I find as good of a spot in Taipei to chat with each of you. But until that next chapter begins, thank you for everything.

List of Figures

CHAPTER 1

Introduction

Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space.

-Jean-Baptiste Joseph Fourier

1.1. A Brief Overview of Geometric Flows

The heat equation is one of the most thoroughly-studied partial differential equations in mathematics. Beginning as a model for diffusive behavior in physical systems, it has since inspired a large body of mathematical research in analysis and geometry. Given a domain $U \subset \mathbb{R}^n$, a function $u : U \times [0, T) \to \mathbb{R}$ solves the heat equation if for each $(x, t) \in U \times [0, T)$,

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t), \tag{1.1}$$

where $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n}$ is the Laplacian taken with respect to the coordinates of U. Solutions to (1.1) exhibit a number of striking properties. For example, they instantly become smooth after the time $t = 0$, and they diffuse to become more uniform across space as time t progresses.

These behaviors, diffusion especially, inspired differential geometers in the latter half of the twentieth century to investigate analogues of the heat equation that deform non-uniform geometric objects into more uniform ones. For example, a geometric heat equation might deform a surface with non-constant curvature into a round sphere over time. Physical phenomena such as surface tension in liquids and grain boundaries in annealing metals seemed to behave according to heat-like equations for surfaces, and equations like these also appeared to play a role in understanding central objects in geometry such as harmonic maps, minimal surfaces, and canonical metrics. [**Bak05**] and [**Bre11**] thoroughly review the mathematical and physical motivation behind heat equations in geometry.

These geometric heat equations, also called *geometric flows*, have enjoyed great success both in their applications to geometry and topology and to physics ever since. This began with the Harmonic Map Heat Flow introduced in 1964 by Eels and Sampson in [**ES64**] and eventually led to the celebrated proof of the Poincaré Conjecture using Ricci Flow by Perelman in [**Per03**], [**Per06a**], and [**Per06b**]. Ricci Flow is an example of an *intrinsic geometric flow*, where the object being deformed is a Riemannian metric on a smooth manifold. This dissertation will focus on *extrinsic geometric flows*, where the object being deformed is a surface immersed in a Riemannian manifold, and the deformation speed of the surface depends on the extrinsic curvature at each point. We limit our discussion to closed hypersurfaces, and the ambient manifold throughout this dissertation is taken to be Euclidean space $\mathbb{R}^{n+1}$.

The key difficulties in studying any geometric flow typically lie in the analysis. Unlike the standard heat equation, the parabolic evolution equations associated with geometric flows are nonlinear. This allows for certain quantities associated with the evolving surface to become infinitely large in finite time, resulting in a *singularity* in the flow at some time $T_{\max} < +\infty$ beyond which the solution can no longer be continued. These singularities introduce a natural dichotomy to the study of extrinsic flows from the standpoint of the initial value problem: given an initial surface $M_0 \subset \mathbb{R}^{n+1}$, will the corresponding solution $\{M_t\}_{t\in[0,T)}$ to an extrinsic flow exist for all time, or will it develop a finite-time singularity? If the flow does exist for all time, one asks if the flow surfaces M_t will converge, possibly modulo scaling, to some fixed limit surface such as a round sphere. If the flow does not exist for all time, one is interested in characterizing the shape of the surface up to the singular time.

Over the years, mathematicians have developed several powerful analytic and geometric methods to probe the long-time behavior and singularity profile of curvature flows. The most common first line of attack in this analysis is the parabolic maximum principle, which, using only rules of calculus, bounds certain geometric quantities associated with the flow by their initial data. Another effective but often more complicated approach uses energy estimates to obtain an L^∞ estimate on a quantity via an iteration scheme or to establish the existence of a limit surface at a given time via a compactness theorem. If a classical flow solution develops a singularity, one may also seek to construct weak solutions which manage to flow past these singularities. These weak solutions may be useful for extracting information about the classical flow. We apply all of these methods in novel ways to an extrinsic flow known as Inverse Mean Curvature Flow. The end result is a far more complete picture of the singular and convergence properties of this flow. Later in this dissertation, we will also apply our results on Inverse Mean Curvature Flow to gain new insights into the geometry and topology of physical soap films.

1.2. The Contraction and Expansion of Hypersurfaces by their Mean Curvature

Let us begin with a review of extrinsic curvature. Recall for a closed hypersurface $M_0^n \subset \mathbb{R}^{n+1}$ that the *second fundamental form* $A : TM_0 \times TM_0 \to \mathbb{R}$ is a symmetric bilinear form over the tangent bundle of M_0 defined by

$$A(X,Y) = \langle \nabla_X \nu, Y \rangle, \qquad X, Y \in TM_0. \tag{1.2}$$

Here, ν is the outward-pointing unit normal field of M_0, and $\langle \cdot, \cdot \rangle$ and ∇ are the Euclidean metric and associated Levi-Civita connection on $\mathbb{R}^{n+1}$, respectively. The n different *principal curvatures* λ_i of M_0 may then be defined as the eigenvalues of A in a unit-length basis at each point. Geometrically, the λ_i may be thought of as reciprocals of the curvature radii for M_0 at a given point.

The *mean curvature* H is defined as

$$H = \sum_{i=1}^{n} \lambda_i, \tag{1.3}$$

and *total curvature* $|A|$ is

$$|A| = (\sum_{i=1}^{n} \lambda_i^2)^{\frac{1}{2}}. \tag{1.4}$$

In general, the deformation speed of an extrinsic curvature flow can depend on every principal curvature $\lambda_1, \ldots, \lambda_n$, but a natural class of flows to consider are ones where the speed is specifically a function of the mean curvature H. The first such flow we highlight is the Mean Curvature Flow (MCF), which will serve as an important point of reference in our study of the Inverse Mean Curvature Flow.

Definition 1.2.1. *Let M be an oriented, closed smooth manifold. A one-parameter family of immersions $F : M \times [0,T) \to \mathbb{R}^{n+1}$ evolves by **Mean Curvature Flow (MCF)** if*

$$\frac{\partial M_t}{\partial t}(x,t) = -H\nu(x,t), \tag{1.5}$$

where H and ν are the mean curvature and unit normal of $M_t = F_t(M)$.

Therefore, a family of surfaces $\{M_t\}_{0\leq t<T}$ evolves by MCF if the inward deformation speed equals the mean curvature at each point along each surface. MCF is some sense the canonical way to contract a surface by its extrinsic curvature, as it is the L^2 gradient functional for $|M_t|$, the area of M_t. MCF has been thoroughly studied for this reason, for its utility in modelling physical systems, and for its applications to the geometry and topology of surfaces.

Now we introduce Inverse Mean Curvature Flow (IMCF), which by contrast is an expanding geometric flow with an analytic and geometric structure that is not as well understood. Throughout this dissertation, we will use N to denote the underlying manifold here to distinguish from MCF.

Definition 1.2.2. *Let N be a smooth, closed, orientable smooth manifold of dimension n. A one-parameter family of immersions $F : N \times [0,T) \to \mathbb{R}^{n+1}$ solves **Inverse Mean Curvature Flow** if for each $(x,t) \in N \times [0,T)$*

$$\frac{\partial N_t}{\partial t}(x,t) = \frac{1}{H}\nu(x,t), \tag{1.6}$$

where $\nu(x,t)$ and $H > 0$ are the outward unit normal and mean curvature of $N_t = F_t(M)$ respectively.

Therefore, a family of surfaces $\{N_t\}_{0\leq t<T} \subset \mathbb{R}^{n+1}$ evolves by IMCF if the deformation speed in the normal direction equals the reciprocal of mean curvature at each point along each surface. As such, for every t the N_t must each have $H > 0$ everywhere, and the motion is always outward-directed.

Suppose $F_t : M \to \mathbb{R}^{n+1}$ is a family of immersions parametrized by $t \in [0,T)$ which satisfies (1.5). For any other family of immersions $\tilde{F}_t : M \to \mathbb{R}^{n+1}$ which satisfies $\langle \frac{\partial \tilde{F}_t}{\partial t}(x,t), \nu\rangle = -H(x,t)$, i.e. a family of immersions which have the same normal velocity as the F_t but also may move tangentially, one can verify that $M_t = F_t(M) = \tilde{F}_t(M)$. This is true for IMCF as well. We will often identify solutions of (1.5), resp. (1.6), with the images of the F_t in the definition rather than the F_t themselves, but this shows that these immersions are not the unique way to describe the surfaces M_t and N_t. The choice of immersions in the definitions is a scaffolding, or more formally a *gauge choice*, to describe M_t and N_t. Imposing a gauge choice like this is necessary to ensure that the initial value problem for these flows is well-posed. Viewed as an initial value problem, a unique solution to (1.5) is known to always exist, at least for a short time, for any closed C^2 hypersurface $M_0^n = F_0(M) \subset \mathbb{R}^{n+1}$, and a unique short-time solution to (1.6) is known to always exist for a closed C^2, $H > 0$ hypersurface $N_0 = F_0(N)$.

In order to understand the basic properties of these equations in Euclidean space, the initial data one first examines for (1.5) and (1.6) are round spheres. Symmetry arguments allow us to deduce that the evolving solution is also a round sphere in this case, reducing each PDE to an ODE for its time-dependent radius $r(t)$.

Example 1.2.3 (Contracting Spheres under MCF)**.** *Let $M_0 = \mathbb{S}_R(x_0) \subset \mathbb{R}^{n+1}$ be a round sphere, and $\{M_t\}_{t\in[0,T)}$ the corresponding solution to MCF. Then $M_t = \mathbb{S}_{r(t)}(x_0)$, where $r(t) = \sqrt{R^2 - 2nt}$.*

Proof. Let $\{M_t\}_{0\leq t<T}$ be the evolution of $\mathbb{S}_R(x_0)$ by MCF, and let R be a rotation about x_0. Because R is an isometry of $\mathbb{R}^{n+1}$, and so if $\{M_t\}_{0\leq t<T}$ solves (1.6) then so does $\{R(M_t)\}_{0\leq t<T}$. Now, $R(\mathbb{S}_R(x_0)) = \mathbb{S}_R(x_0)$, so by uniqueness of solutions to the initial value problem for MCF, $R(M_t) = M_t$, so the M_t are invariant under R and must therefore also be round spheres $M_t = \mathbb{S}_{r(t)}(x_0)$.

To solve for the time-dependent radius $r(t)$, the mean curvature H of $\mathbb{S}_{r(t)}(x_0)$ is $\frac{n}{r(t)}$. So (1.5) reduces to the ODE

$$r'(t) = -\frac{n}{r(t)}.$$

Applying the condition $r(0) = R$ yields the solution $r(t) = \sqrt{R^2 - 2nt}$. □

Example 1.2.4 (Expanding Spheres under IMCF)**.** *Let $N_0 = \mathbb{S}_R(x_0) \subset \mathbb{R}^{n+1}$ be a round sphere. Then the corresponding solution $\{N_t\}_{0\leq t<T}$ to IMCF is a round sphere $N_t = \mathbb{S}_{r(t)}(x_0)$ with radius $r(t) = Re^{\frac{t}{n}}$.*

Proof. Using the same argument as in Example 1.2.3, we reduce (1.6) to the ODE

$$r'(t) = \frac{1}{n} r(t),$$

with initial value $r(0) = R$, which has as its solution $r(t) = Re^{\frac{t}{n}}$. $\square$

Remark 1.2.5. *The argument showing that M_t is a round sphere may also be used to establish that any symmetry of $M_0 \subset \mathbb{R}^{n+1}$ is maintained by M_t. This will become especially important in chapters 3 and 4.*

Even from this single example, we already observe an interesting contrast between MCF and IMCF: the solution in Example 1.2.3 is only defined for $t \in [0, \frac{R^2}{2n})$, while on the other hand the solution in Example 1.2.4 is defined for any $t \in [0, +\infty)$. Furthermore, these examples are valuable since they indicate how we expect solutions to scale with time. For example, although (1.2.3) shrinks to a point, scaling each M_t up by a factor of $(R^2 - 2nt)^{-\frac{1}{2}}$ about x_0 yields $(R^2 - 2nt)^{-\frac{1}{2}} M_t = \mathbb{S}_R(x_0)$. Likewise, although (1.2.4) expands off to infinity, rescaling N_t down by $e^{-\frac{t}{n}}$ yields a constant sphere. Both of these scale factors become important in analyzing asymptotic limits of (1.5) and (1.6).

1.3. Singularities of MCF and IMCF

The contraction to a point in Example 1.2.3 is an example of a singularity of Mean Curvature Flow. In geometric analysis, understanding the possible singularities of geometric flows is both critically important for applications and a fascinating study in its own right. We first review known results on singularity formation and characterization for MCF before introducing singular solutions to the IMCF. We begin with the precise definition of a singularity for a geometric flow.

Definition 1.3.1. *Given a closed, smooth, orientable manifold M, let the one-parameter family of immersions $F : M \times [0, T) \to \mathbb{R}^{n+1}$ solve a geometric flow of the form*

$$\frac{\partial M_t}{\partial t}(x, t) = f(\lambda_1, \ldots, \lambda_n)\nu(x, t),$$

where $M_t = F_t(M)$ and f is some function of the principal curvartures $\lambda_1, \ldots, \lambda_n$ at (x, t). We say this solution has a ***singularity*** *at time $T < +\infty$ if there does not exist a time $\tilde{T} > T$ and a solution $\tilde{F} : M \times [0, \tilde{T}) \to \mathbb{R}^{n+1}$ which restricts to F over $[0, T)$.*

This definition of a singularity is essentially identical to the one for a solution u to a scalar parabolic PDE. First, we address the question of singularity formation for MCF. Singularities have long been known to never occur in solutions to linear, uniformly parabolic PDE, c.f. chapter 7 of [**Eva10**], but MCF exhibits the opposite behavior: singularities *always* form for closed solutions of MCF in $\mathbb{R}^{n+1}$, and they do so within a prescribed time interval. The tool for proving this is a two-sided avoidance principle.

Proposition 1.3.2 (Two-Sided Avoidance Principle for MCF)**.** *Let $\{M_t\}_{0\leq t<T}$ and $\{\tilde{M}_t\}_{0\leq t<T}$ be two solutions of MCF where M_0 $\tilde{M}_0$ are closed, connected, embedded hypersurfaces disjoint from each other. Then $dist(M_t, \tilde{M}_t)$ is a non-decreasing function of time.*

Proof. The proof utilizes Hamilton's Trick, c.f. Section 2.1 in [**Man11**]. Since $M_0, \tilde{M}_0$ are each connected, each separates $\mathbb{R}^{n+1}$ into two disjoint connected components. The component enclosed by M_0 shall be called E_0, resp. $\tilde{E}_0$. Since embedded solutions of MCF remain embedded, see section 2.2 in [**Man11**], we can compare the domains E_t and $\tilde{E}_t$: let $p \in M_t$ and $q \in \tilde{M}_t$ minimize the distance between M_t and $\tilde{M}_t$. For at least a short time, either $E_t \cap \tilde{E}_t = \varnothing$ or, without loss of generality, $E_t \subset \tilde{E}_t$. Then the shifted set

$$E'_t = E_t + (q-p) = \{x + (q-p) \in \mathbb{R}^{n+1} | x \in E_t\},$$

for which the boundaries M'_t and $\tilde{M}_t$ meet at q, either remains contained in or disjoint from $\tilde{E}_t$. If $E'_t \subset \tilde{E}_t$, then since the tangent planes of M'_t and $\tilde{M}_t$ coincide at q we must have $\lambda'_i \geq \tilde{\lambda}_i$ for each set of principle curvatures $\{\lambda'_i\}_{i=1}^n$, $\{\tilde{\lambda}_i\}_{i=1}^n$ taken with respect to the same outward normal ν. Then $H(p,t) \geq H(q,t)$. In this case, calling $p = F_t(p_0)$ and $q = \tilde{F}_t(q_0)$ we have

$$\begin{aligned} \partial_t |F_t(p_0) - \tilde{F}_t(q_0)|^2 &= 2\langle F_t(p_0) - F_t(q_0), \partial_t F_t(p_0) - \partial_t F_t(q_0)\rangle \\ &= -2|F_t(p_0) - F_t(q_0)|\langle \nu, (H(q_0,t) - H(p_0,t))\nu\rangle \\ &\geq 0. \end{aligned}$$

If $E_t' \cap \tilde{E}_t = \varnothing$, then $\lambda_i' \leq \tilde{\lambda}_i$ when each is taken with respect to the outward unit normal of $\tilde{E}_t$, implying $-H(p,t) \leq H(q,t)$. As the normals of E_t and $\tilde{E}_t$ are anti-parallel at p and q, repeating the above calculation again yields $\partial_t |F_t(p_0) - \tilde{F}_t(q_0)|^2 \geq 0$.

According to Hamilton's Trick, the function $\operatorname{dist}(M_t, \tilde{M}_t) = \min_{(p_0,q_0)\in M\times\tilde{M})} |F_t(p_0) - \tilde{F}_t(q_0)|^2$ is a Lipschitz continuous function of time, and its derivative wherever defined satisfies

$$\frac{d}{dt}\operatorname{dist}(M_t, \tilde{M}_t) = |F_t(p_0) - \tilde{F}_t(q_0)|^2 \geq 0,$$

where p_0, q_0 are a pair of distance-minimizing points for $M_t, \tilde{M}_t$. Therefore, for times $t_1 < t_2$, the Fundamental Theorem of Calculus allows us to write

$$\operatorname{dist}(M_{t_2}, \tilde{M}_{t_2}) - \operatorname{dist}(M_{t_1}, \tilde{M}_{t_1}) = \int_{t_1}^{t_2} \frac{d}{dt}\operatorname{dist}(M_t, \tilde{M}_t) dt \geq 0.$$

$\square$

Coupling this proposition with Example 1.2.3 produces a striking result about singularity formation under MCF: not only do singularities always form for closed initial data M_0, but they do so within a prescribed time interval depending only on the extrinsic diameter of M_0.

Theorem 1.3.3 (Singularity Formation for MCF)**.** *Let M_0 be a closed hypersurface, and $\{M_t\}_{t\in[0,T)}$ the corresponding solution to MCF. Then $T_{\max} < +\infty$. In fact, $T_{\max} < \frac{(diam(M_0))^2}{2n}$, where $diam(M_0) = \sup_{x,y\in M_0} |x-y|$ is the extrinsic diameter of $M_0 \subset \mathbb{R}^{n+1}$.*

Proof. Once again, we denote by E_t the set enclosed by M_t. Let $x_0 \in E_0$ and consider the surfaces $\tilde{M}_t = \mathbb{S}_{\sqrt{\operatorname{diam}(M_0)^2 - 2nt}}(x_0)$, where $t \in [0, \frac{\operatorname{diam}(M_0)^2}{2n})$. Then $\tilde{M}_t$ solves MCF and $E_0 \subset \tilde{E}_0$. If $T_{\max} \geq \frac{\operatorname{diam}(M_0)^2}{2n}$, then $E_t \not\subset \tilde{E}_t$ for some $t \in [0, \frac{\operatorname{diam}(M_0)^2}{2n})$, contradicting the avoidance principle, see Figure 1.1. Conclude $T_{\max} \leq \frac{(\operatorname{diam}(M_0))^2}{2n}$. $\square$

As singularity formation is inevitable for closed solutions to MCF, the nature of these singularities has been thoroughly investigated. What are the characteristics of every singularity of MCF, or equivalently, what conditions are necessary and sufficient to ensure continuation in time is possible for a solution $\{M_t\}_{0\leq t<T}$? Once again, we may gain a reference point for this question

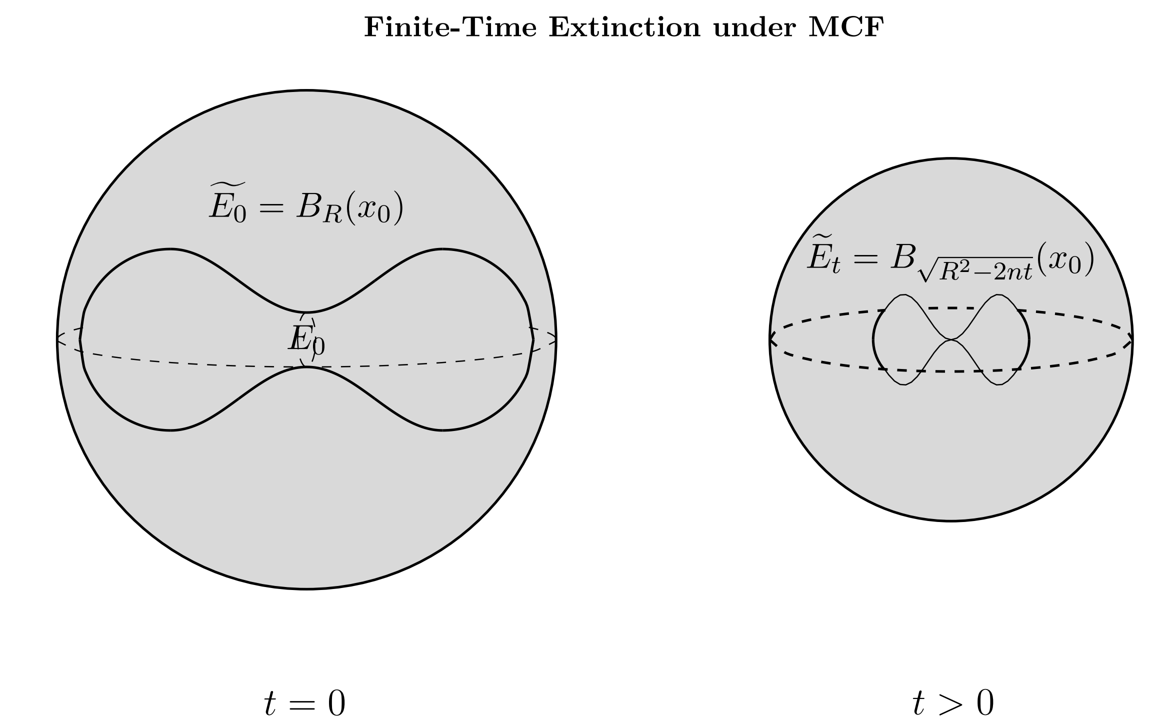


FIGURE 1.1. Comparison of a closed solution of MCF with a sufficiently large sphere establishes the existence of a finite-time singularity.

by considering solutions u_t to scalar parabolic PDE. In this context, the key object in the study of singularities is the Hessian matrix

$$\mathrm{Hess}(u_t)_{ij} = \partial_{ij} u_t.$$

For linear and uniformly parabolic PDE, an L^∞ bound on the norm of the Hessian of u_t as $t \to T$ yields bounds on all higher spatial derivatives of u_t near T. This means for each $i \in \mathbb{N}$ that the functions $\partial_i u_t$ are bounded and equicontinuous in time. The Arzealà-Ascoli Theorem may then be applied to every partial derivative of u_t, implying the existence of a smooth limit function u_T after passing to a subsequence of times. Since continuation from a $C^\infty(U)$ function u_T is always possible by short-time existence, an L^∞ bound on $\mathrm{Hess}(u_t)$ is a necessary and sufficient condition for continuation. In fact, this behavior applies more generally to quasi-linear and fully non-linear parabolic PDE– this singularity characterization for non-linear heat equations dates back to the work of Nash in [**Nas58**], see also [**Kry97**].

Similar to scalar parabolic PDE, a solution to MCF goes extinct at a time T if there does not exist a C^2, immersed limit surface that the M_t's converge to as $t \to T$. The analogue to the Hessian of a scalar function for M_t is the second fundamental form A, which has norm equal to the total curvature $|A|$ given in (1.4). Blow-up in $|A|$ is clearly an obstruction to the existence of a limit surface, and thanks to the parabolic nature of (1.5) this is indeed the only obstruction.

Theorem 1.3.4. *Let $\{M_t\}_{0\leq t<T} \subset \mathbb{R}^{n+1}$ be a solution of MCF. Then $T = T_{\max}$ if and only if $\lim_{t\to T}\max_{M_t}|A| = +\infty$.*

Proof. If $T < T_{\max}$ then $\lim_{t\to T}\max_{M_t}|A| = \max_{M_T}|A| < +\infty$ as this maximum is a Lipschitz function of time, see section 2.1 in [**Man11**]. For the other direction, we must demonstrate that if $\lim_{t\to T}\max_{M_t}|A| = L < +\infty$ then there exists a smooth limit surface M_T from which to continue the flow. To begin, we demonstrate that the F_t's converge in C^1 topology to a limit immersion F_T as $t \to T$ in this case.

From section 2.3 in [**Man11**], given a coordinate patch $(x_1, \ldots, x_n)$ on M, the induced metric of M_t pulled back to M via F_t evolves by the equation

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2HA_{ij}(x,t). \tag{1.7}$$

We proceed with a method first outlined by Hamilton in section 8 of [**Ham95**]. Consider a fixed $x \in M$ and $X \in T_xM$. Writing

$$\frac{d}{dt}\log||X||^2_{g_t} = \frac{1}{||X||^2_{g_t}}\frac{d}{dt}(g_{ij}X^iX^j(x,t)) = \frac{2H}{||X||^2_{g_t}}A_{ij}X^iX^j(x,t),$$

and in view of the inequalities,

$$\begin{aligned} H^2 &\leq n|A|^2, \\ -|A|||X||^2_{g_t} &\leq A_{ij}X^iX^j \leq |A|||X||^2_{g_t}, \end{aligned}$$

we find $|\frac{d}{dt}\log(||X||^2_{g_t})| \leq 2\sqrt{n}L^2$. Integrating from times t_1 to t_2 yields,

$$e^{-2\sqrt{n}L^2}||X||^2_{g_{t_1}} \leq ||X||^2_{g_{t_2}} \leq e^{2\sqrt{n}L^2}||X||^2_{g_{t_1}}$$

so that the inner products on TM through time are uniformly equivalent. The polarization identity then guarantees the existence of a non-degenerate C^1 limit metric g_T over TM at the time T, so that F_T is an immersion.

There remains the question of regularity for M_T. We do not present the full argument here, but [**Hui18**] contains a sketch of the proof for the inductive estimate

$$\sup_{M_t} |\nabla^m A| \leq c(m, M_0) \sup_{M_t} |\nabla^{m-1} A| \tag{1.8}$$

for the higher covariant derivatives ∇^m of A, which one obtains via maximum principles. This yields uniform controls on all higher derivatives of curvature, so that the F_t's converge in C^∞ topology to the map F_T via an Arzeala-Ascoli type theorem. As $M_T = F_T(M)$ is smooth, short-time existence gives a solution to MCF extended to a larger time interval. □

This characterization of singularities is also true for a variety of other geometric flows, including the surface tension flow, the Gauss curvature flow, and the harmonic mean curvature flow, see [**MSS15**] and Section 6 of [**HP99**].

There has been much less research into the singularities of IMCF, which may be due to less prevalence of singularities in this context: for example, $T_{\max} = +\infty$ for the round sphere solution from last section, and as we will soon see this is true for other large families of initial data. There is at least one example of a solution to IMCF which is known to develop a singularity, originally noted by Huisken and Ilmanen in Section 0 of [**HI01**].

Example 1.3.5. *Let $N_0 \subset \mathbb{R}^3$ be a round torus with $H > 0$, and $\{N_t\}_{0 \leq t < T_{\max}}$ the corresponding maximal solution to IMCF. Then $T_{\max} < +\infty$, and $\lim_{t \to T_{\max}} \min_{N_t} H = 0$.*

Proof. As with Example 1.2.3 and Example 1.2.4, IMCF preserves rotational symmetry about the central axis of the torus. Along the ring of N_t closest to the axis of rotation, the negative principal curvature associated with rotation has magnitude u^{-1}, where u is the distance to this axis. Meanwhile, flow speed is bounded below over N_t. Indeed, by a maximum principle that we apply in chapter 3,

$$\max_{N_t} H \leq \max_{N_0} H.$$

Since this ring moves toward the center at some minimum speed, mean curvature along this ring will approach zero in finite time as u continues to decrease. This terminates the flow, as the flow speed H^{-1} becomes infinitely large in finite time, see Figure 1.2. $\square$

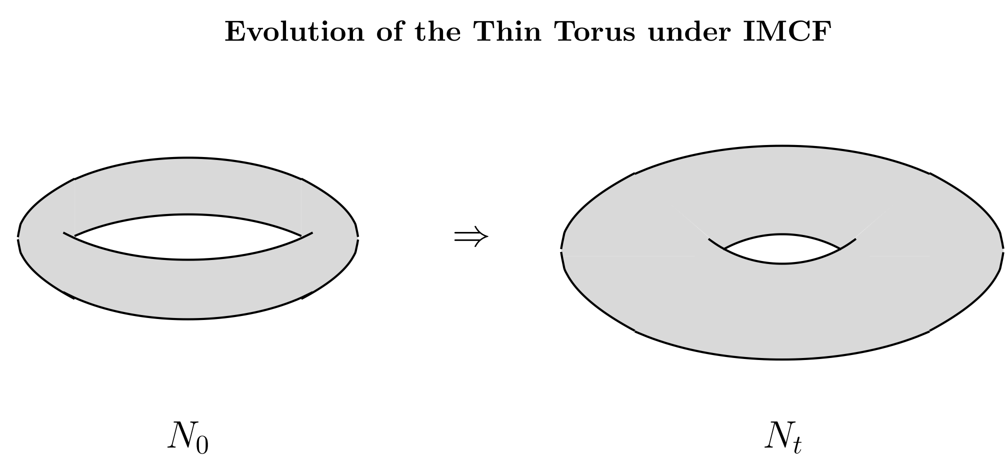


FIGURE 1.2. The thin torus evolved by IMCF will continue to expand until the flow speed along the inner ring becomes infinitely large.

The above argument is heuristic and utilizes the high degree of symmetry in this example. Nevertheless, this example raises the question of whether singularity formation is a generic feature of solutions to IMCF with topology $\mathbb{T}^n$, or more broadly for any solution to IMCF without spherical topology. Furthermore, singularities of MCF are known to form within a prescribed time interval depending only on the geometry of M_0. Is this also true for the singularities of IMCF? The main theorem of chapter 2 will simultaneously answer both of these questions.

We also aim to understand how to characterize the singularities of IMCF. One of the few known results in this direction thus far is that, as in Example 1.3.5, all singularities are characterized by the mean curvature H degenerating to 0. The following theorem was originally shown by Smoczyk for $n = 2$, and later by Huisken and Ilmanen for arbitrary dimensions:

Theorem 1.3.6 ([**Smo00**], [**HI08**]). *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$ hypersurface, and $\{N_t\}_{0 \leq t < T}$ the corresponding solution to the Inverse Mean Curvature Flow* (1.6) *for some $T < \infty$. Then $T = T_{\max}$ if and only if $\lim_{t \to T_{\max}} \min_{N_t} H = 0$.*

The reverse direction is clear: even if a C^2 limit surface were to exist at the time T, it must satisfy $\min_{N_T} H = 0$, and even a short-time solution to (1.6) with initial data N_T cannot exist in this case. The forward direction can be demonstrated by studying the evolution of a specific tensor that reveals a uniform-in-time bound on the quantity $|A|H$. This tells us that a uniform control on the flow speed H^{-1} over $[0, T)$ also controls the total curvature $|A|$, and from here an estimate similar to (1.8) implies the existence of a smooth, $H > 0$ N_T.

On the other hand, if the total curvature $|A|$ of N_t were to blow up in finite time, this must also be accompanied by blow up in the flow speed H^{-1}. The exact behavior of the second fundamental form near the extinction is difficult to understand both due to a lack of explicit examples of singular solutions and the fully nonlinear character of (1.6). Therefore, analyzing the precise dynamics of the curvature near singularities will be the central focus of chapter 3.

1.4. The Dynamical Stability of Round Spheres under MCF and IMCF

A primary question associated with any evolution equation concerns the dynamical stability of self-similar solutions. That is, given initial data which maintains its shape as it evolves in time, what perturbations of this initial data will converge back to this shape under the evolution? In the context of geometric flows, the dynamical stability of self-similar solutions is another central topic of research that is closely related to singularities. Establishing round spheres as limits of certain solutions modulo translation and scaling is frequently key in proofs of geometric inequalities and in the restricting the possible topologies of manifolds.

The study of dynamical stability for round spheres under Mean Curvature Flow began in the 1980s. One striking early result shown by Matthew Grayson in [**Gra87**] establishes that any closed embedded plane curve $\gamma_0 \subset \mathbb{R}^2$ evolved by curve-shortening flow, the $n = 1$ case of MCF, will converge to a point $x_0 \subset \mathbb{R}^2$ at its extinction time $T_{\max}$. Furthermore, the curves

$\tilde{\gamma}_t = (\sqrt{T_{\max} - t})\gamma_t$ rescaled about x_0 will converge in C^∞ topology to a circle of fixed radius as $t \to T_{\max}$. So round circles are dynamically stable under curve-shortening flow for *any* perturbation in $\mathbb{R}^2$ which preserves embeddedness.

This is notably strong stability for a quasi-linear parabolic PDE, but in higher dimensions this stability is far weaker. Examples of surfaces in $\mathbb{R}^3$ which do not contract to points under MCF go back to [**Gra89**]. However, another early result by Huisken in [**Hui84**] establishes dynamical stability under MCF for convex perturbations of round spheres. Note that in this section's theorems, the M_t converge to a surface in C^∞ topology in the sense that the F_t converge as maps to a limit map in C^∞ topology over M.

Theorem 1.4.1 ([**Hui84**], Convex Stability of Round Spheres under MCF)**.** *Let $M_0 \subset \mathbb{R}^{n+1}$ be a C^2 hypersurface which is strictly convex in the sense that $A(X,X) \geq \lambda g(X,X)$ for a fixed $\lambda \in \mathbb{R}$ and every $X \in TM_0$. Let $\{M_t\}_{0 \leq t < T_{\max}}$ be the corresponding maximal solution to MCF. Then M_t converges to a point $x_0 \in \mathbb{R}^{n+1}$ as $t \to T_{\max}$, and the surfaces $\tilde{M}_t = \sqrt{T_{\max} - t}M_t$ rescaled about x_0 converge in C^∞ topology to some round sphere $\mathbb{S}_R(x_0)$ as $t \to T_{\max}$.*

With this in mind, we would like to compare with the stability properties of IMCF. Unilike the quasi-linear parabolic equation (1.5), equation (1.6) is fully nonlinear. This initially makes one pessimistic about the chances for solutions to exist for all times like in Example 1.2.4 and to become asymptotically round. Remarkably, though, this is indeed what happens for a large family of initial data, namely star-shaped deformations of round spheres.

Definition 1.4.2. *A C^1 hypersurface $N_0 = F_0(N) \subset \mathbb{R}^{n+1}$ is called **star-shaped** with respect to $x_0 \in \mathbb{R}^{n+1}$ if $\langle F_0(x) - x_0, \nu(x)\rangle > 0$ for every $x \in N$, where $\nu(x)$ is the outward unit normal of N at x.*

$\langle F_0(x) - x_0, \nu(x)\rangle$ is often called the *support function of N_0 with respect to x_0*. The literature contains several equivalent definitions of star-shapedness to the above one, but the above one is the most natural one to use in the standard gauge. In [**Ger90**], Claus Gerhardt proved the following for $H > 0$ star-shaped initial data:

Theorem 1.4.3 ([**Ger90**], The Star-Shaped Stability of Round Spheres)**.** *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$ hypersurface which is star-shaped WLOG with respect to $0 \in \mathbb{R}^{n+1}$, and $\{N_t\}_{0 \leq t < T}$ its evolution by IMCF. Then $T_{\max} = +\infty$, and the rescaled surfaces $\tilde{N}_t = e^{-\frac{t}{n}} N_t$ converge in C^2 topology to some round sphere $\mathbb{S}_R(0)$.*

This assumption on N_0 is significantly weaker than the convexity assumption on M_0 in Theorem 1.4.1, and in fact star-shaped M_0 are not known to contract to round points under MCF (although [**Lin15**] and [**Smo98**] establish that blow-up limits of star-shaped MCF are at least weakly convex). Since this first stability result for (1.6), a number of authors have extended this theorem, for example to less regular hypersurfaces in [**HI08**], to larger classes of expanding flows in [**Urb90**], and to warped product manifolds in [**Zho16**].

Gerhardt uses the parabolic maximum principle to prove the global existence statement of Theorem 1.4.3. We give a more in-depth discussion of this technique and our generalization of it in Appendix A1. The key insight here is that the support function $\theta(x,t) = \langle F_t(x), \nu(x) \rangle$ and speed function $H^{-1}(x,t)$ both satisfy the same evolution equation

$$(\partial_t - \frac{1}{H^2}\Delta) f = \frac{|A|^2}{H^2} f, \tag{1.9}$$

meaning that $g(x,t) = H^{-1}\theta^{-1}$ obeys

$$(\partial_t - \frac{1}{H^2}\Delta) g \leq \frac{2\theta^{-1}}{H^2} \langle \nabla g, \nabla \theta^{-1} \rangle.$$

Such a g must be bounded in $N \times [0, T)$ on either side by its initial data. In turn, the flow speed H^{-1} is bounded on either side by the extreme values of $\theta(x,t)$, and these can be controlled via equation (1.9). From Theorem 1.3.6 in the previous section, this implies $T_{\max} = +\infty$. The asymptotic convergence to round spheres likewise uses maximum principle arguments as well as the Krylov-Safanov Theorem, see [**Kry97**].

Given the large body of literature on the Inverse Mean Curvature Flow of star-shaped hypersurfaces, another focus of this dissertation will be to identify in what other ways round spheres are dynamically stable under IMCF. An important fact to be mindful of in this investigation is that IMCF unlike the linear heat equation does not respect a standard parabolic scaling: given a solution $\{N_t\}_{0\leq t<T}$ to IMCF, one defines a new solution via the transformation

$$\begin{aligned} t &\to t, \\ F_t &\to \lambda F_t, \end{aligned}$$

for some fixed $\lambda \in \mathbb{R}$. Therefore, the dynamics of IMCF should not depend on the precise size of the initial curvature $|A|$, since any solution $\{N_t\}_{0\leq t<T}$ may be rescaled in space to have arbitrarily large or small initial curvature. Likewise, dynamical stability should not depend on deviation from a round sphere in a norm sense. The natural perturbations to consider are ones which retain some symmetry of the sphere, or the ones which keep a geometric quantity positive definite.

1.5. Weak Solutions, Outward-Minimizing Sets, and Applications in General Relativity

One of the main original motivations for the study of IMCF arose from general relativity. The flow played a central role in proving the *Riemannian Penrose Inequality* for asymptotically flat Riemannian manifolds with non-negative scalar curvature, which was first proved by Gerhard Huisken and Tom Ilmanen for 3 dimensions in [**HI01**] and then by Hubert Bray for arbitrary dimensions in [**Bra01**].

Theorem 1.5.1 (Riemannian Penrose Inequality, [**Bra01**], [**HI01**])**.** *Let (M^3, g) be an asymptotically flat Riemannian 3-manifold with scalar curvature $R \geq 0$, N_0 its outermost minimal surface, and m its ADM mass. Then*

$$m \geq \sqrt{\frac{|N_0|}{16\pi}}, \tag{1.10}$$

with equality is and only if $(M^3, g) = (\mathbb{R}^3 \setminus B_{\frac{m}{2}}(0), g_s)$. Here, g_s is the Schwarzschild metric $g_s = (1 + \frac{m}{2|x|})^4 \delta$.

Although we do not present precise definitions, asymptotically flat manifolds model isolated gravitating systems in relativity theory, the ADM mass measures total gravitational energy associated with these systems, and outermost minimal surfaces correspond to the boundaries of black holes present in the system. Thus, this inequality bounds the total amount of gravitational energy in an isolated gravitating system below by the size of its event horizon. Both [**Bra02**] and [**BC04**] provide excellent reviews of the history of the Riemannian Penrose Inequality and the role geometric flows played in its resolution.

The key link between the inequality (1.10) and IMCF comes in the form of the *Hawking mass* of a surface $\Sigma^2 \subset M^3$:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}(1 - \frac{1}{16\pi}\int_\Sigma H^2 d\mu). \tag{1.11}$$

Hawking proposed this invariant of Σ as one possible definition for quasi-local mass in general relativity, which is roughly a measure of how much gravitational energy is contained within the region enclosed by Σ. The limit of the Hawking mass evaluated over round spheres at infinity then gives the total ADM mass of an asymptotically flat (M^3, g). The crucial insight, originally noted by Robert Geroch his 1973 paper [**Ger73**], is that for a solution $\{N_t\}_{0\leq t<T}$ of IMCF in (M^3, g) the quantity $m_H(N_t)$ is a monotone function of time. Using the evolution equations from section 0 of [**HI01**]

$$\begin{aligned}
\frac{\partial}{\partial t}H &= \frac{1}{H^2}\Delta H - \frac{|A|^2}{H} - \frac{\text{Ric}(\nu,\nu)}{H}, \\
\frac{\partial}{\partial t}d\mu_t &= d\mu_t,
\end{aligned}$$

one explicitly calculates

$$\frac{d}{dt}m_H(N_t) = \sqrt{\frac{|N_t|}{16\pi}}[\frac{1}{2} + \frac{1}{16\pi}\int_{N_t}(2\frac{|\nabla H|^2}{H^2} + R - 2K + \frac{1}{2}(\lambda_1 - \lambda_2)^2)d\mu_t]. \tag{1.12}$$

Recall that the scalar curvature R is taken to be nonnegative, and for the Gauss curvature K an application of the Gauss-Bonnet Theorem implies that $\int_{N_t} 2K d\mu \leq 8\pi$. Altogether,

$$\frac{d}{dt}m_H(N_t) \geq 0.$$

Thus $m_H(N_t)$ is non-decreasing along IMCF. If N_0 is the outermost minimal surface of (M^3, g), then taking N_0 as initial data for IMCF the corresponding solution N_t would have Hawking mass monotonically approaching the total ADM mass m of (M^3, g) from below as $t \to \infty$, provided the solution exists for all times and asymptotically converges to round spheres at infinity. Since $m_H(N_0) = \sqrt{\frac{|N_0|}{16\pi}}$, the monotonicty of $m_H(N_t)$ would imply the inequality (1.10).

Monotonicity arguments like these are frequently used to prove geometric inequalities for hypersurfaces, but the obvious issue in this case is global existence and convergence for the flow. While one can at least consider IMCF with N_0 being a minimal surface in the sense of backward-in-time convergence, one does not expect a long-time solution that asymptotically converges to a sphere. Asymptotically flat manifolds are a large class of spaces in which understanding the behavior of classical IMCF is intractable, and one does not expect IMCF to avoid singularities in all of these cases as a result.

These difficulties spurred Huisken and Ilmanen to develop a notion of weak solutions of Inverse Mean Curvature Flow in the late 1990s. Their approach was inspired by a level-set approach to MCF by Evans-Spruck in [**ES91**] and Chen-Giga in [**CGG99**]. To introduce this viewpoint, suppose that for a open bounded domain $E_0 \subset M^{n+1}$ of an asymptotically flat Riemannian manifold there exists a C^2 function $u : M \subset E_0 \to \mathbb{R}$ with nonvanishing gradient ∇u satisfying

$$\begin{aligned} \mathrm{div}(\frac{\nabla u}{|\nabla u|}) &= |\nabla u| && \text{on } M \setminus E_0, \\ u &\equiv 0 && \text{on } \partial E_0. \end{aligned} \tag{1.13}$$

The right-hand side of (1.13) corresponds to the mean curvature of a level set of u, while the left-hand side is the reciporical of the flow speed of the level-set flow. This means that the level sets $N_t = \{u = t\}$ form a solution to the classical IMCF. Thus, a solution to the exterior Dirichlet problem (1.13) gives rise to a global solution to (1.6) with initial data $N_0 = \partial E_0$.

Of course, global solutions of IMCF do not exist for certain types of initial data, so solutions to (1.13) should not exist for certain choices of E_0. However, Huisken and Ilmanen developed a theory of variational solutions to (1.13) which are known to always exist provided that (M, g) is asymptotically flat and ∂E_0 is of class $C^{1,\alpha}$. Remarkably, they demonstrated that the Hawking mass is also monotone along the flow surfaces of these weak solutions, and that these flow surfaces converge to round spheres at infinity on asymptotically flat (M^n, g). This overcame the issues with the classical flow to lead to a full proof of the inequality (1.10), and since this publication a number of other authors have utilized this approach to prove more geometric inequalities in relativity, see [**Ang18**], [**BHW14**], [**CLZ17**], [**LdMP20**], [**McC17**], and [**Wei18**].

We now present definition of a variational solution of (1.13). We specifically take the ambient manifold to be $\mathbb{R}^{n+1}$ for our purposes, but this definition generalizes to any background space with a suitable topology. Unlike with genuine C^2 solutions, this variational solution u may have a vanishing gradient somewhere, or fail to be even C^1 at points.

Definition 1.5.2. *Given an open set $U \subset \mathbb{R}^{n+1}$, a function $u \in C^{0,1}_{loc}(U)$ is a **variational solution to IMCF** if for any $K \subset\subset U$ and $v \in C^{0,1}(K)$ with $\{v \neq u\} \subset\subset K$ we have*

$$J_K(u,u) \leq J_K(u,v)$$

where J_K is the functional defined by

$$J_K(u,v) = \int_K |\nabla u| + u|\nabla v|. \tag{1.14}$$

*Furthermore, given an open, bounded subset $E_0 \subset \mathbb{R}^{n+1}$, a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is a **variational solution to IMCF with initial condition $\mathbf{E_0}$** if $E_0 = \{u < 0\}$ and u minimizes* (1.14) *on $U = \mathbb{R}^{n+1} \setminus E_0$.*

This is a variational formulation of (1.13) where the minimization principle corresponds to freezing one argument of the functional J_K. Given a variational solution u with initial condition E_0, one can define "weak flow surfaces" $\tilde{N}_t = \partial\{u < t\}$. The $\tilde{N}_t$ are at least $C^{1,\alpha}$ hypersurfaces even if u itself is only locally Lipschitz, and $\tilde{N}_t = N_t$ for at least a short time, N_t being the classical solution of IMCF beginning from a $C^2, H > 0$ hypersurface N_0.

The $\tilde{N}_t$ may, however, eventually cease to coincide with N_t due to $\tilde{N}_t$ "jumping" beyond N_t in space. These jumps happen precisely when the classical solution N_t fails to be strictly outward minimizing over $\mathbb{R}^{n+1}$.

Definition 1.5.3. *A subset $E \subset \mathbb{R}^n$ is said to be **outward minimizing** if for every F containing E with $F \setminus E \subset\subset \mathbb{R}^n$ we have $|\partial F| \leq |\partial E|$.*

*Furthermore, E is **strictly outward minimizing** if the above inequality is strict for every $F \neq E$.*

Stated more simply, an open set E is outward minimizing if ∂E has smaller or equal area to every surface enclosing it, and strictly outward minimizing if this area is strictly smaller. An application of the divergence theorem shows that the level sets of a solution u to (1.13) are always strictly outward minimizing, and this property carries over to a variational solution to IMCF with initial condition E_0. That is, the $\tilde{N}_t$ are always strictly outward minimizing, and $\tilde{N}_t = N_t$ until N_t itself is no longer strictly outward minimizing.

Weak IMCF of Two Spheres

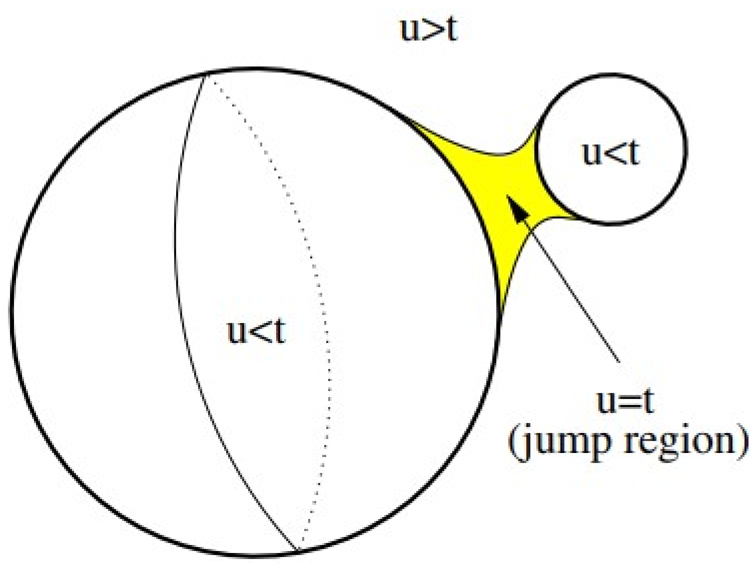


FIGURE 1.3. Two spheres evolved by the classical flow will eventually cease to be strictly outward-minimizing by the time t, triggering a jump across the region $\{u = t\}$ in the weak flow. Image from page 374 of [**HI01**].

An illustrative example of this behavior noted in Chapter 0 of [**HI01**] is two disjoint spheres evolved by weak IMCF, see Figure 1.3. Each sphere will expand under the classical flow, and the spheres eventually become close enough to each other so that they are enclosed by a connected surface with area equal to the sum of their areas. At this point, the $\tilde{N}_t$ jump out beyond this connected surface as the classical N_t continue to expand until the spheres touch. Similarly, the thin torus discussed in the previous section ceases to be strictly outward minimizing before its singularity occurs, resulting in a jump in $\tilde{N}_t$ before the singularity.

One goal of this dissertation is to analyze the regularity of weak flow solutions. That is, can we confine the times at which jumps in the weak flow can occur to a specific time interval, and can we establish that a variational solution to IMCF with initial condition E_0 is smooth and has a non-zero gradient outside of a specific compact set? As we will see, these questions are connected to questions about the behavior of the classical flow. Indeed, this exploration will lead to new results on singularities and the dynamical stability of round spheres under (1.6).

The following chapters of this dissertation are adapted from the author's work in the papers [**Har20a**], [**Har20b**], and [**Har20c**].

CHAPTER 2

The Eventual Star Shape of Classical and Weak Solutions

Too low they build, who build beneath the stars.

-Edward Young

2.1. The Main Theorems

Since singularities may or may not form under IMCF, an obvious first question is for which initial data N_0 does the corresponding solution $\{N_t\}_{0 \leq t < T}$ to the flow equation (1.6) go extinct in finite time. With the singularity profile of MCF in mind, we would also like to know whether or not the maximal time of existence of a singular solution is bounded in terms of the geometry of N_0. Due to the unit-less nature of time in Inverse Mean Curvature Flow, any such bound on $T_{\max}$ must be by a scale-invariant quantity associated with N_0.

This also raises questions about the solutions of IMCF which do not develop singularities. For example, related to the question of singularity formation within a prescribed time interval, are solutions which exist beyond a certain time guaranteed to then exist forever? Also, for those solutions which do exist forever, what are the possible asymptotic limits at large times?

In this chapter, we prove an alternative that simultaneously addresses all of these questions for *embedded* solutions of (1.6). In general, solutions of (1.6) do not necessarily remain embedded. However, this result also identifies the prescribed time interval in which any possible self-intersections must happen.

Theorem 2.1.1 (Singular and Long-Time Behavior of IMCF). *Let $\{N_t\}_{t \in [0, T_{max}]}$ be a solution to (1.6), where N_0 is a connected hypersurface and T_{max} is the maximal time of existence. Furthermore, let R be the inradius of N_0, that is the radius of the largest ball N_0 encloses, and diam(N_0) its extrinsic diameter. Then one of the following alternatives holds:*

(1) $T_{max} = \infty$ *and* N_t *is embedded for every* $t \in [0, \infty)$. *Furthermore,* N_t *is star-shaped for any* $t \geq t^* = n \log (R^{-1} diam(N_0))$.

(2) N_t *goes extinct or self-intersects within the time interval* $[0, 2t^*]$ *for* t^* *defined above.*

This alternative is quite powerful in solving dynamical stability problems. It establishes that a solution $\{N_t\}_{0 \leq t < T_{\max}}$ which exists and does not self-intersect throughout the time interval $[0, 2n \log(R^{-1}\mathrm{diam}(N_0)))$ must become star-shaped. According to Theorem 1.4.3, any star-shaped solution must exist for all time and asymptotically approach a round sphere as $t \to \infty$ after scaling by $e^{-\frac{t}{n}}$. Therefore, as a corollary of this result, existence and embeddedness for a certain minimal time beginning from some perturbation N_0 of a round sphere are sufficient to conclude dynamical stability. We will make use of this in chapter 4.

On the other hand, this links the question of singularity formation with the topology of N_0. For a solution $\{N_t\}_{0 \leq t < T}$ to satisfy the first alternative, N_0 must be homeomorphic to $\mathbb{S}^n$ because N_t is homeomorphic to $\mathbb{S}^n$ at later times and the solution remains embedded. This leads to the following corollary.

Corollary 2.1.2 (Intersections and Singularities in Non-Spherical Topology)**.** *Let* $N_0 \subset \mathbb{R}^{n+1}$ *be an* $H > 0$ *hypersurface without spherical topology. Then for the corresponding maximal solution* $\{N_t\}_{0 \leq t < T_{\max}}$ *to IMCF, either* $T_{\max} < 2t^* = 2n \log (R^{-1} diam(N_0))$, *or* N_t *intersects itself by the time* $2t^*$.

This serves an analogue to Theorem 1.3.3 concerning singularity formation within a prescribed time interval for MCF specifically for non-spheres. Furthermore, this result raises an additional question: IMCF starting from any N_0 without spherical topology must either go extinct or self-intersect within a prescribed time interval. Could this also happen for an $H > 0$ hypersurface N_0 with spherical topology? We demonstrate later that this is indeed possible.

Theorem 2.1.3 (Intersections and Singularities in Spherical Topology)**.** *There exists a* C^2, $H > 0$ $N_0^n \subset \mathbb{R}^{n+1}$ *with spherical topology which either self-intersects or develops a singularity within the time* $T_{max} \leq 2t_*$ *under IMCF for the time* t_* *given in Theorem* 2.2.8.

All of these results come from considering level set solutions of IMCF mentioned in the previous chapter. Theorem 2.1.1 arises from a new regularity theorem for variational weak solutions. Given an open domain $E_0 \subset \mathbb{R}^{n+1}$ with non-star-shaped, C^1 boundary and a variational solution u with initial condition E_0, one asks if there is a time by which the weak flow surfaces $\tilde{N}_t$ must be star-shaped. This question is natural because u must have full regularity over the exterior region of such an $\tilde{N}_t$, and we demonstrate that this must happen by the time $t^* = n \log(R^{-1}\mathrm{diam}(N_0))$. The proof relies on a parabolic version of the moving plane method for constant mean curvature inspired by the one introduced by Chow and Gulliver in [**CG01**].

Theorem 2.1.3 arises from the geometry of level set solutions. Given a C^2 $H > 0$ embedded surface N_0 which fails to be strictly outward minimizing, we will later demonstrate a singularity or self-intersection must always occur within the time $2t^*$. Therefore, we prove Theorem 2.1.3 by constructing a C^2, $H > 0$ hypersurface N_0 with spherical topology that is not strictly outward-minimizing.

The chapter is organized as follows: in Section 2.2, we demonstrate that a variational weak solution with initial condition E_0 respects a reflection property like the one investigated in [**CG01**]. We use this property to conclude that these solutions must be star-shaped by the time they lie entirely outside of the smallest sphere they are initially enclosed by. This implies an upper bound on the "waiting time" for a variational solution to become star-shaped depending only on the diameter and inradius of E_0.

Section 2.3 concerns the applications of this waiting time result to classical solutions. We show that, assuming initial connectedness, a classical solution defines a weak solution in the sense of [**HI01**] if and only if it remains embedded. Using this result, we then establish a correspondence between the global variational solution of IMCF and the one defined by the classical solution. This correspondence serves as the key to proving Theorem 1.

In Section 2.4, we construct a C^2 $H > 0$ hypersurface homeomorphic to $\mathbb{S}^n$ which is not strictly outward minimizing. In particular, the corresponding weak solution must "jump" at the initial time before the classical solution either terminates or self intersects within a fixed time interval. We describe a gluing procedure for two disjoint round spheres which are not strictly outward minimizing to obtain the desired surface. All of these results hold in any dimension.

2.2. An Aleksandrov Reflection Approach to Variational Solutions

The moving plane method for constant mean curvature surfaces dates back to the work of A.D. Aleksandrov in [**Ale56**] and [**Ale57**]. Using an elliptic comparison principle, he showed that any closed, connected, and embedded constant mean curvature surface in $\mathbb{R}^{n+1}$ possesses a high degree of reflection symmetry which implies that it must be a sphere. Critical to Aleksandrov's argument is the embeddedness assumption, for it guarantees that the surface encloses a bounded open set in $\mathbb{R}^{n+1}$.

In [**CG01**], Chow and Gulliver developed a spiritual successor to this moving plane method that applies to viscosity solutions of a large family of geometric flows. Treating the initial data in this context to be the C^2 boundary of a connected open set in $\mathbb{R}^{n+1}$, they establish a reflection property respected by each region which is enclosed by a flow surface. In this section we will first demonstrate that Huisken and Ilmanen's variational solutions to IMCF from [**HI01**] respect a similar type of reflection property, and we will then use this to gain a profile on the shape of the weak flow surfaces.

For the convenience of the reader, we give the definition of a variational weak solution again.

Definition 2.2.1. *Given an open set $U \subset \mathbb{R}^{n+1}$, a function $u \in C^{0,1}_{loc}(U)$ is a* ***variational solution to IMCF*** *if for any $K \subset\subset U$ and $v \in C^{0,1}(K)$ with $\{v \neq u\} \subset\subset K$ we have*

$$J_K(u,u) \leq J_K(u,v)$$

where J_K is the functional defined by

$$J_K(u,v) = \int_K |\nabla u| + u|\nabla v|. \tag{2.1}$$

Furthermore, given an open, bounded subset $E_0 \subset \mathbb{R}^{n+1}$, a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is a ***variational solution to IMCF with initial condition $\mathbf{E_0}$*** *if $E_0 = \{u < 0\}$ and u minimizes* (2.1) *on $U = \mathbb{R}^{n+1} \setminus E_0$.*

Given any solution $\{N_t\}_{0 \leq t < T}$ to (1.6) with the property that $N_{t_1} \cap N_{t_2} = \varnothing$ for $t_1 \neq t_2 \in [0,T)$, one may define a function $u : U \to \mathbb{R}$ over the set $U = \cup_{t \in [0,T)} N_t$ by $u(x) = t$ if $x \in N_t$. This u may be shown to solve the degenerate elliptic Dirichlet problem

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= |\nabla u| \qquad\qquad \text{on } U \\ u_{N_0} &= 0. \end{aligned} \tag{2.2}$$

One can verify that a solution to (2.2) must minimize J_K over U. Conversely, one can verify via appropriate choice of test function that any C^2 function u with nonvanishing gradient that minimizes J_K over a domain $U \subset \mathbb{R}^{n+1}$ must satisfy (2.2). For general variational solutions, however, there may exist points where $\nabla u = 0$, and the presence of regions where the gradient of a solution vanishes also allows for the presence of points where it is not differentiable.

Huisken and Ilmanen nevertheless demonstrated that, given any open set $E_0 \subset \mathbb{R}^{n+1}$ with C^1 boundary there is a unique variational solution u with initial condition E_0 for which the sets $E_t = \{u < t\}$ are precompact for each t (Notice that if $\partial E_0 = N_0$ then this means $u|_{N_0} = 0$), and all of our results in this section apply specifically to these solutions. Our approach utilizes a comparison principle from [**HI01**] which applies to any variational solution u. More specifically, given any locally Lipchitz u and v which minimize J_K over some open U, we know that if $\{u < 0\} \subset \{v < 0\}$ then $\{u < t\} \subset \{v < t\}$ for each $t \in \mathbb{R}$ on U, provided the level sets of v are precompact in U. Let us now give a few more definitions neccessary for our moving plane approach.

Consider the plane $P_{\lambda,\nu} = \{x \in \mathbb{R}^{n+1} | \langle x, \nu\rangle = \lambda\}$ with unit normal vector $\nu \in \mathbb{S}^n$ and upper and lower half-spaces $H^+_{\lambda,\nu} = \{x \in \mathbb{R}^n | \langle x, \nu\rangle > \lambda\}$ and $H^-_{\lambda,\nu} = \{x \in \mathbb{R}^n | \langle x, \nu\rangle < \lambda\}$ respectively. Let $\sigma_{\lambda,\nu} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the reflection about $P_{\lambda,\nu}$.

Definition 2.2.2. *Given a subset $E \subset \mathbb{R}^{n+1}$, we say that $P_{\lambda,\nu}$ is* ***admissible with respect to E*** *if $\sigma_{\lambda,\nu}(E \cap H^-_{\lambda,\nu}) \subset E \cap H^+_{\lambda,\nu}$.*

Our first result concerns the admissibility of the flow surfaces of weak IMCF. Given a plane $P_{\lambda,\nu}$ with corresponding reflection $\sigma_{\lambda,\nu} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, first note that if $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is a variational solution to IMCF, then so is $u^*(x) = u \circ \sigma_{\lambda,\nu}(x)$ since $\sigma_{\lambda,\nu}$ is an isometry of $\mathbb{R}^{n+1}$. Let $E_t = \{x \in \mathbb{R}^{n+1} | u(x) < t\}$ and $E^*_t = \{x \in \mathbb{R}^{n+1} | u^*(x) < t\}$.

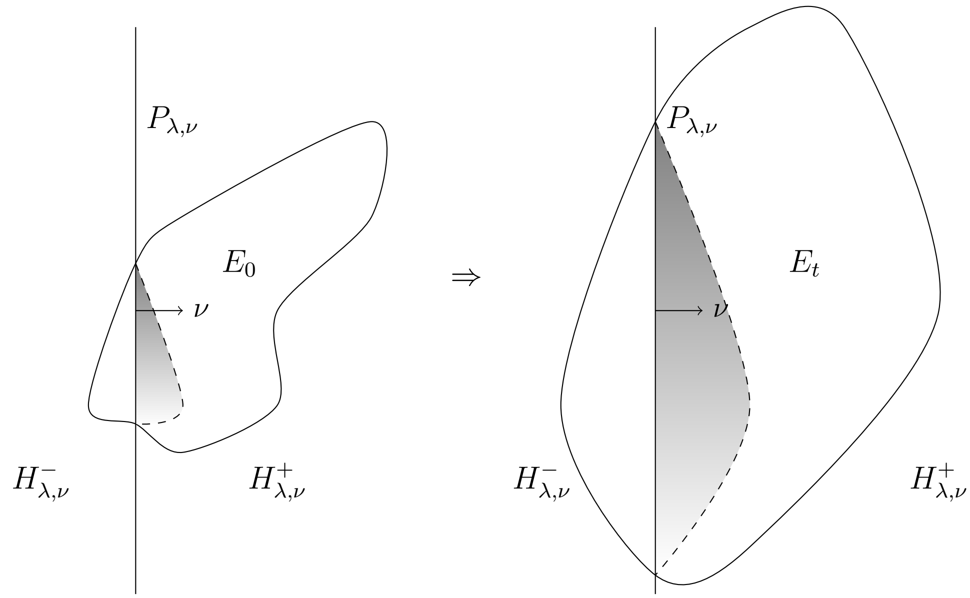


FIGURE 2.1. Given a set E_0 which is initially admissible with respect to some plane, the corresponding solution $\{E_t\}_{0\leq t\leq T}$ to weak IMCF remains admissible for every t.

Proposition 2.2.3. *For some bounded, open $E_0 \subset \mathbb{R}^{n+1}$ with C^1 boundary, let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be the variational solution to IMCF with initial condition E_0 such that $E_t = \{u < t\}$ is precompact for each t. If $E_0^* \cap H^+_{\lambda,\nu} \subset E_0 \cap H^+_{\lambda,\nu}$, then $u^*(x) \geq u(x)$ for every $x \in H^+_{\lambda,\nu}$. In particular, $E_t^* \cap H^+_{\lambda,\nu} \subset E_t \cap H^+_{\lambda,\nu}$ for every $t > 0$.*

Remark 2.2.4. *If N_t is a classical solution to IMCF, then E_t corresponds to the region enclosed by N_t. Then this theorem implies for classical solutions that if for a particular plane the portion of N_0 in the lower half-plane reflected into the upper half-plane lies inside the portion already within the upper half-plane, then this remains true for each N_t.*

PROOF. From Remark 1.18 in [**HI01**], if u minimizes (2.1) over $\mathbb{R}^{n+1}\backslash E_0$, then so does $\min\{u, c\}$ for any constant $c \in \mathbb{R}^+$. Then for some $t \in \mathbb{R}^+$, consider the set $U = (E_t^* \setminus E_0) \cap H^+_{\lambda,\nu}$ and the cut-off solution $u^t = \min\{u, t\}$ to (2.1). We claim that $\{u^t > u^* + \delta\} \subset\subset U$ for every $\delta > 0$.

Observe that $\partial U \subset ((u^*)^{-1}\{t\} \cap H^+_{\lambda,\nu}) \cup (N_0 \cap H^+_{\lambda,\nu}) \cup P_{\lambda,\nu}$. Since $u^t \leq t$, we have $u^t \leq u + \delta$ near $(u^*)^{-1}\{t\} \cap H^+_{\lambda,\nu}$. Since $E_0^* \cap H^+_{\lambda,\nu} \subset E_0 \cap H^+_{\lambda,\nu}$, we have $u^*(x) \geq 0$ and therefore $u^*(x) + \delta \geq$

$\delta \geq u(x) \geq u^t(x)$ near $N_0 \cap H^+_{\lambda,\nu}$. Finally, since $u^*(x) = u(x)$ on $P_{\lambda,\nu}$, we have $u^*(x) + \delta \geq u(x)$ near $P_{\lambda,\nu}$. Then we may conclude that $\{u^t > u^* + \delta\} \subset\subset U$, meaning by Theorem 2.2(i) in [**HI01**] we get $u^t \leq u^* + \delta$ in U, implying $u^t \leq u^*$ in U. But since $u^* < t$ in U we have $u = u^t \leq u^*$ in U. Since $H^+_{\lambda,\nu} \cap U$ is foliated by such W, we may conclude $u^*(x) \geq u(x)$ over $H^+_{\lambda,\nu} \cap U$. Then $E^*_t \cap H^+_{\lambda,\nu} \subset E_t \cap H^+_{\lambda,\nu}$, so $P_{\lambda,\nu}$ is admissible for every E_t. □

Corollary 2.2.5. *Let E_0, u be as in Proposition 1. Suppose $P_{\tilde{\lambda},\nu}$ be admissible with respect to E_0 for every $\tilde{\lambda} \in (-\infty, \lambda)$. Then $u(x)$ is nonincreasing in the ν direction over $H^-_{\lambda,\nu}$.*

Proof. Take $x_1, x_2 \in H^-_{\lambda,\nu}$ which lie on the same line perpendicular to $P_{\lambda,\nu}$, i.e. $x_1 = s_1\nu + y$ and $x_2 = s_2\nu + y$ for $y \in P_{\lambda,\nu}$. Without loss of generality, say $s_2 < s_1 < 0$.

Let $P_{\tilde{\lambda},\nu}$ be the plane parallel to $P_{\lambda,\nu}$ which bisects x_1 and x_2. Note then that $P_{\tilde{\lambda},\nu}$ is admissible with respect to N_0 since $\tilde{\lambda} < \lambda$. Then by Proposition 2.2.3, we have that $u^*(x) = u \circ \tilde{\sigma}_{\lambda,\nu}(x) \geq u(x)$ for every $x \in \tilde{H}^+_{\lambda,\nu}$. In particular, since $x_1 \in \tilde{H}^+_{\lambda,\nu}$, we must have $u^*(x_1) \geq u(x_1)$. But $u^*(x_1) = u \circ \tilde{\sigma}_{\lambda,\nu}(x_1) = u(x_2)$, so $u(x_2) \geq u(x_1)$. □

Now we may use this result to represent the part of the surface in the lower half-plane as a locally Lipschitz graph. For the purpose of extending these results to weak solutions, we also prove this for the boundary of $E^+_t = \mathrm{Int}(\{x \in \mathbb{R}^n | u(x) \leq t\})$.

Proposition 2.2.6. *Let u, E_0 be as Proposition 1. For a given $\lambda \in \mathbb{R}, \nu \in \mathbb{S}^n$, suppose for some $\epsilon > 0$ that $P_{\tilde{\lambda},\tilde{\nu}}$ is admissible with respect to E_0 for every $\tilde{\lambda} \in (-\infty, \lambda)$ and $\tilde{\nu}$ with $|\tilde{\nu} - \nu| < \epsilon$. Then $\partial E_t \cap H^-_{\lambda,\nu}$ and $\partial E^+_t \cap H^-_{\lambda,\nu}$ are each locally Lipschitz graphs in the ν direction over $P_{\lambda,\nu}$.*

Proof. We prove the result for $\partial E^+_t \cap H^-_{\lambda,\nu}$, as the proof for $\partial E_t \cap H^-_{\lambda,\nu}$ is identical. We begin by noting that u is nonincreasing in the $\tilde{\nu}$ direction over $H_{\lambda,\tilde{\nu}}$ for every $\tilde{\nu}$ with $|\tilde{\nu} - \nu| < \epsilon$ by Corollary 2.2.5.

Fix $x_1, x_2 \in \partial E^+_t \cap H^-_{\lambda,\nu}$. Write $x_1 = s_1\nu + y_1$, $x_2 = s_2\nu + y_2$ for $y_1, y_2 \in P_{\lambda,\nu_0}$, and say without loss of generality that $s_1 \leq s_2 < 0$. There exists $\tilde{\epsilon}$ so that $x_1, x_2 \in H^-_{\lambda,\tilde{\nu}}$ for every unit vector $\tilde{\nu} \in \mathbb{S}^n$ satisfying $|\tilde{\nu} - \nu| < \epsilon$. Define $\hat{\epsilon} = \min\{\epsilon, \tilde{\epsilon}\}$. We will show that

$$|s_1 - s_2| \leq \cot \hat{\epsilon} |y_1 - y_2|. \tag{2.3}$$

To see this, suppose that (2.3) is false. Then for the unit vector

$$\hat{\nu} = \frac{x_2 - x_1}{|x_2 - x_1|} \tag{2.4}$$

we must have

$$\langle \hat{\nu}, \nu \rangle = \frac{\langle y_2 - y_1, \nu \rangle + (s_2 - s_1)}{\sqrt{|y_1 - y_2|^2 + (s_1 - s_2)^2}} > \frac{1}{\sqrt{\tan^2 \hat{\epsilon} + 1}} = \cos \hat{\epsilon}. \tag{2.5}$$

Now, pick $\tilde{x_2}$ with $u(\tilde{x_2}) > t$ sufficiently close to x_2 so that for the vector $\tilde{\nu} = \frac{\tilde{x_2} - x_1}{|\tilde{x_2} - x_1|}$ we have $\langle \tilde{\nu}, \nu \rangle > \cos(\hat{\epsilon})$ and $\langle \tilde{x}_2, \tilde{\nu} \rangle < \lambda$. Note that the first inequality implies $|\tilde{\nu} - \nu| < \hat{\epsilon}$. Then $P_{\lambda,\tilde{\nu}}$ is admissible with respect to N_0, and $x_1, \tilde{x}_2$ lie in $H^-_{\lambda,\tilde{\nu}}$. In fact, we have that $x_1, \tilde{x}_2$ lie on a line perpendicular to $P_{\lambda,\tilde{\nu}}$ with $\mathrm{dist}\{x_1, P_{\lambda,\tilde{\nu}}\} > \mathrm{dist}\{\tilde{x}_2, P_{\lambda,\tilde{\nu}}\}$ by construction. But we also have that $u(x_1) = t$ and $u(\tilde{x}_2) > t$, and this contradicts the nonincreasing property from Corollary 2.2.5.

Thus (2.3) holds, and therefore $y_1 = y_2$ implies $s_1 = s_2$, so $\partial E^+_t \cap H^-_{\lambda,\nu}$, and likewise $\partial E_t \cap H^-_{\lambda,\nu}$, is a graph over $P_{\lambda,\nu}$ (Recall $\partial E_t = N_t$ for classical solutions). Furthermore, the Lipschitz bound $\cot \hat{\epsilon}$ is independent of t. $\square$

Theorem 2.2.7. *Let u, E_0 be as in proposition 1. Then, choosing $0 \in \mathbb{R}^{n+1}$ to be the midpoint of the two furthest points apart on ∂E_0, the region of the surface ∂E_t which lies outside $B_{\frac{diam(N_0)}{2}}(0)$ can be written as a graph $r = r_t(\theta)$ over $\mathbb{S}^n$ in polar coordinates with respect to the origin. Furthermore, this graph satisfies the gradient estimate*

$$|Dr_t| \leq \frac{r_t \Lambda}{\sqrt{r_t^2 - \Lambda^2}} \tag{2.6}$$

for some $\Lambda \leq \frac{diam(N_0)}{2}$.

Proof. We follow the proof of Theorem 4 in [**CG01**]. For a given E_0, take $0 \in \mathbb{R}^{n+1}$ to be the midpoint of the line connecting a pair of distance-maximizing points on ∂E_0. For a given $\nu \in \mathbb{S}^n$, define $\lambda_{\max}$ to be the supremum over all $\lambda \in \mathbb{R}$ such that $P_{\tilde{\lambda},\nu}$ is admissible with respect to E_0 for each $\tilde{\lambda} \in [-\infty, \lambda)$, then define $\Lambda = \sup_{\nu \in \mathbb{S}^n} -\lambda_{\max}$. Then $0 \leq \Lambda \leq \frac{\mathrm{diam}(N_0)}{2}$. Given $x_0 \in \partial E_t$ with $|x_0| = r_0 > \Lambda$, we know $x_0 \in H^-_{\lambda_{\max},\nu}$ for each $\nu \in \mathbb{S}^n$ and associated $\lambda_{\max}$. Write $x_0 = r_0 \frac{\partial}{\partial r}$. Then for $\nu_0 = -\frac{\partial}{\partial r}$ we have $\langle \nu_0, x_0 \rangle = -r_0 < -\Lambda$, so by Proposition 2.2.6 ∂E_t is a

Lipschitz graph $r = r_t(\theta)$ in some neighborhood of x_0. Letting $\frac{\partial}{\partial\theta}$ be a unit tangent over $\mathbb{S}^n$, the vector $\tau = -r\frac{\partial}{\partial\theta} - Dr_t(\theta)\nu_0$ is tangent to ∂E_t. Also by Proposition 2.2.6, τ is transverse to ν for all $\nu \in \mathbb{S}^n$ with $\langle \nu, x_0\rangle < -\Lambda$, so

$$\frac{rDr_t(\theta)}{(r^2 + (Dr_t(\theta))^2)^{\frac{1}{2}}} = \langle \frac{\tau}{|\tau|}, x_0\rangle \geq -\Lambda. \tag{2.7}$$

Rearranging this yields

$$Dr_t(\theta) \leq \frac{r\Lambda}{(r^2 - \Lambda^2)^{\frac{1}{2}}}. \tag{2.8}$$

□

Theorem 2.2.8. *(Waiting Time for Star-shapedness) For bounded, open E_0 with C^2 boundary, suppose $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is the variational solution to IMCF with initial condition E_0 such that the sets $E_t = \{u < t\}$ are precompact for each t. Let R be the inradius of N_0. Then the level sets $N_t = \partial E_t$ of u lie entirely outside $B_{\frac{diam(N_0)}{2}}(0)$ for any $t \geq t^* = n\log(R^{-1}diam(N_0))$. In particular, ∂E_t is star-shaped and hence smooth for every $t > t_*$ and thus u may be extended to all of $\mathbb{R}^{n+1}$.*

Proof. Pick 0 to be the midpoint between the pair of points $x, y \in N_0$ which maximize $|x-y|$. Then $N_0 \subset B_{\frac{\mathrm{diam}(N_0)}{2}}(0)$. By definition of the inradius R, there exists some $x \in E_0$ such that $B_R(x) \subset E_0$. By Theorem 2.2 in [**HI01**], we must have $B_{Re^{\frac{t}{n}}}(x) \subset E_t$ for each $t \in [0, T)$. We must have that $B_{\frac{\mathrm{diam}(N_0)}{2}}(0) \subset B_{Re^{\frac{t*}{n}}}(x) = B_{\mathrm{diam}(N_0)}(x)$. Conclude then that $B_{\frac{\mathrm{diam}(N_0)}{2}}(0) \subset E_t$ and thus ∂E_t is star-shaped for $t \geq t^*$. The smoothness of ∂E_t for $t > t^*$ is a consequence of Theorem 2.5 in [**HI08**], which establishes that weak solutions with precompact level sets and initial condition that is a set whose boundary is C^1 and star-shaped with non-negative weak mean curvature are smooth. □

Remark 2.2.9. *In Remark 2.8(b) of [**HI08**], the authors suggested a similar "waiting time" for star-shapedness of the flow depending on the diameter and area of N_0 if the reflection property was shown to apply to their variational solutions. We were unable to determine how they derived this time, and so we instead include the above one.*

2.3. Consequences for Classical Solutions

In this section, we show that an embedded connected classical solution of (1.6) always gives rise to a variational weak solution. Later, we show that if this solution exists and is embedded beyond the time $2t^*$ defined in Theorem 2.1.1, then its flow surfaces equal the level sets of the variational solution with initial condition E_0 which has E_t precompact. This allows us to apply Theorem 2.2.8 to these classical solutions, establishing star-shapedness beyond the time t^*. The first step toward showing this is a comparison principle for IMCF which is slightly weaker than the well-known two-sided avoidance principle for MCF, although the proof is very similar.

Theorem 2.3.1. *(One-Sided Avoidance Principle) Let $N_0 \subset \mathbb{R}^{n+1}$ be a connected, closed hypersurface, and $\{N_t\}_{0 \leq t < T}$ the corresponding solution to (1.6). Suppose N_t is embedded for each $t \in [0, T)$, and let $E_t \subset \mathbb{R}^{n+1}$ be the open domain enclosed by N_t. Now let $\tilde{N}_0 \subset E_0$ be a closed, connected hypersurface, and $\{\tilde{N}_t\}_{0 \leq t < \tilde{T}}$ the corresponding solution to (1.6) with $\tilde{N}_t$ embedded for each $t \in [0, T)$. Then $\overline{\tilde{E}_t} \subset E_t$ for each $t \in [0, T)$, and dist$\{N_t, \tilde{N}_t\}$ is non-decreasing.*

Proof. Calling $\tilde{N}_t = \tilde{F}_t(\tilde{N})$, $N_t = F_t(N)$ consider the function $f : \tilde{N} \times N \times [0, T) \to \mathbb{R}$ defined by $f(p, q, t) = |\tilde{F}_t(p) - F_t(q)|^2$. Define $\ell : [0, T) \to \mathbb{R}$ by $\ell(t) = \min_{(p,q) \in \tilde{N} \times N} f(p, q, t)$, where $\ell(0) > 0$ by hypothesis. Since f is smooth and $\tilde{N} \times N$ is closed, ℓ is locally Lipschitz in $(0, T)$ according to Lemma 2.1.3 in [**Man11**]. Also by this lemma, for any $t_0 \in [0, T)$ where $\ell(t)$ is differentiable we have

$$\frac{d}{dt}\ell(t_0) = \partial_t f(p_0, q_0, t_0) \tag{2.9}$$

for any pair of points $(p_0, q_0) \in \tilde{N} \times N$ satisfying $\ell(t_0) = f(t_0, p, q)$. We know ℓ is positive at least for small times, so let $\mathcal{A} \subset [0, T)$ be the largest interval containing 0 over which ℓ is strictly positive. Note that $\overline{\tilde{E}_t} \subset E_t$ for $t \in \mathcal{A}$. Take $t_0 \in \mathcal{A}$ where ℓ is differentiable and let $(p_0, q_0) \in \tilde{N} \times N$ be a minimizing pair of points of f at t_0. The outward pointing normals at p_0 and q_0 must be parellel, since the line segment joining $\tilde{F}_{t_0}(p_0)$ and $F_{t_0}(q_0)$ is contained in $\overline{E_t}$ and does not intersect $\tilde{E}_t$. Calling ν_0 the outward unit normal at $\tilde{F}_{t_0}(p_0) \in \tilde{N}_{t_0}$, $\tilde{F}_t(q_0) \in N_{t_0}$, we consider the translated surface N'_{t_0} defined by

$$N'_{t_0} = \{x + \sqrt{\ell(t_0)}\nu_0 | x \in \tilde{N}_{t_0}\}.$$

N'_{t_0} and N_{t_0} share the same tangent plane at $\tilde{F}_{t_0}(p_0) + \sqrt{\ell(t_0)}\nu_0 \in N'_{t_0}$ and $F_{t_0}(q_0) \in N_{t_0}$. Since $\sqrt{\ell(t_0)} = \text{dist}\{N_{t_0}, \tilde{N}_{t_0}\}$, we have the inclusion $E'_{t_0} \subset E_{t_0}$, where E'_{t_0} is the set enclosed by N'_{t_0}. Since $\tilde{F}_{t_0}(p_0) + \sqrt{\ell(t_0)}\nu_0 = F_{t_0}(q_0)$, this inclusion particularly tells us that

$$\lambda_i \leq \lambda'_i, \quad 1 \leq i \leq n$$

where λ_i and λ'_i are the principal curvatures of N_{t_0} and N'_{t_0} at this intersection point respectively. Translating back to $\tilde{N}_{t_0}$, this tells us

$$H(p_0, t_0) \geq H(q_0, t_0). \tag{2.10}$$

Now we compute $\partial_t f(p_0, q_0, t_0)$:

$$\begin{aligned}
\partial_t f(p_0, q_0, t_0) &= \partial_t \langle \tilde{F}_{t_0}(p_0) - F_{t_0}(q_0), \tilde{F}_{t_0}(p_0) - F_{t_0}(q_0) \rangle \\
&= 2\langle \frac{\partial}{\partial t}\tilde{F}_{t_0}(p_0) - \frac{\partial}{\partial t}F_{t_0}(q_0), \tilde{F}_{t_0}(p_0) - F_{t_0}(q_0) \rangle \\
&= 2\langle (\frac{1}{H(p_0, t_0)} - \frac{1}{H(q_0, t_0)})\nu_0, -\sqrt{\ell(t_0)}\nu_0 \rangle \\
&= 2\sqrt{\ell(t_0)}(\frac{1}{H(q_0, t_0)} - \frac{1}{H(p_0, t_0)}) \geq 0.
\end{aligned}$$

So $\frac{d}{dt}\ell(t) \geq 0$ wherever differentiable in $\mathcal{A}$. Taking times $t_1 < t_2$ in $\mathcal{A}$ and using the fact that ℓ has total bounded variation in $[t_1, t_2]$, an application of the Fundamental Theorem of calculus reveals

$$\ell(t_2) = \ell(t_1) + \int_{t_1}^{t_2} \frac{d}{dt}\ell(t)dt \geq \ell(t_1). \tag{2.11}$$

Then if $\tilde{t} = \sup \mathcal{A} < T$, we would obtain the bound $\ell(\tilde{t}) \geq \ell(0) > 0$, which would contradict $\mathcal{A}$ being the largest interval containing 0 over which ℓ is positive. Thus $\mathcal{A} = [0, T)$ and hence $\overline{\tilde{E}_t} \subset E_t$ over $[0, T)$. The non-decreasing property also follows from (2.11). $\square$

Notice that the above argument would not work if the normal vectors at the distance-minimizing point were anti-parallel, which happens in the case that the two disjoint surfaces enclose disjoint subsets. For this same reason, initially embedded solutions to (1.6) need not remain embedded as long as they exist. For example, two initially disjoint spheres, which eventually intersect under IMCF, respect neither a two-sided avoidance principle nor an embeddedness principle. Furthermore, the flow surfaces in this case do not foliate their image. This particularly means that, after a sufficiently long time, the two spheres will not define a weak solution to the flow, even though their classical solution continues. An application of the previous theorem shows, however, that the latter inconvenience cannot happen if the flow surfaces remain embedded.

Theorem 2.3.2. *Let $\{N_t\}_{t\in[0,T)}$ solve (1.6) with N_0 a connected hypersurface. Then the function $u: U = \cup_{0\leq t<T} N_t \subset \mathbb{R}^{n+1} \to \mathbb{R}$ given by $u(x) = t$ if $x \in N_t$ is well-defined and differentiable with nonvanishing gradient if and only if the corresponding F_t are embeddings for every $t \in [0,T)$.*

Proof. $\Rightarrow$ We have by hypothesis that the function u over the region U given by $u(x) = t$ if $x \in N_t$ has nonvanishing gradient. Then the flow surfaces N_t are each level sets of u. Since N_t are the compact level sets of a function with nonvanishing gradient, they are necessarily diffeomorphic to one another, and hence remain embedded. $\Leftarrow$ Since each N_t is a closed, connected, embedded hypersurface, we let E_t be defined as in Theorem 3. In order for the function given by $u(x) = t$ for $x \in N_t$ over the region U to be well-defined, we must have that $N_{t_1} \cap N_{t_2} = \varnothing$ for $t_1 \neq t_2 \in [0,T)$. To show this, first assume T is finite and define $\mathcal{A}$ to be the largest interval of $[0,T)$ containing 0 with the property that $N_{t_1} \cap N_{t_2} = \varnothing$ for any $t_1, t_2 \in \mathcal{A}$. We demonstrate in fact that $\mathcal{A} = [0,T)$. Define $\tilde{t} = \sup \mathcal{A}$. We will argue that $\tilde{t} = T$ by contradiction.

First notice for two times $t_a < t_b$ in $\mathcal{A}$, we have the inclusion $\overline{E}_{t_a} \subset E_{t_b}$. Indeed, for $0 < \delta < t_b - t_a$ small we know $\overline{E_{t_a}} \subset E_t$ for $t \in (t_a, t_a + \delta]$ by the positive outward flow speed. Then if $\overline{E}_{t_a} \not\subset E_{t_b}$, letting t_0 be the first time over $t \in (t_a + \delta, t_b]$ for which $\overline{E}_{t_a} \not\subset E_t$ we would have $\partial E_{t_a} \cap \partial E_{t_0} \neq \varnothing$. But this would contradict the fact that $t_a, t_0 \in \mathcal{A}$ so that N_{t_a} and N_{t_0} cannot intersect. Thus $\overline{E}_{t_a} \subset E_{t_b}$, which also means $N_{t_a} \subset E_{t_b}$.

We claim by contradiction that if $\tilde{t} < T$ then $\mathcal{A}$ is closed. Indeed, if $\mathcal{A} = [0, \tilde{t})$ then $[0, \tilde{t}]$ properly contains $\mathcal{A}$ (assuming $\tilde{t} \neq 0$, in which case $\mathcal{A}$ is automatically closed). Then there are two times $t_1 < t_2$ in $[0, \tilde{t}]$ with $N_{t_2} \cap N_{t_1} \neq \varnothing$. We must have $t_2 = \tilde{t}$ since otherwise $t_1, t_2 \in \mathcal{A}$. On the other hand, the positive outward flow speed tells us that for some small $\delta > 0$, we have $\overline{\tilde{E}}_t \subset E_{\tilde{t}}$ for every $t \in [\tilde{t} - \delta, \tilde{t})$. But for $0 \leq t < \tilde{t} - \delta$ the above nesting result yields $\overline{E}_t \subset E_{\tilde{t}-\delta}$ and so $\overline{E}_t \subset E_{\tilde{t}}$ for each $t \in \mathcal{A}$. This implies $N_{\tilde{t}}$ cannot intersect any N_t with $t \in \mathcal{A}$. So $\mathcal{A} = [0, \tilde{t}]$ for $\tilde{t} < T$.

Now take $\delta < T - \tilde{t}$ and small enough so that $\overline{E}_{\tilde{t}} \subset E_t$ for each $t \in (\tilde{t}, \tilde{t} + \delta)$. Since $\mathcal{A} \subset [0, \tilde{t} + \delta)$, there are two times $t_1 < t_2$ in $[0, \tilde{t} + \delta)$ with $N_{t_1} \cap N_{t_2} \neq \varnothing$. We cannot have $t_1, t_2 \in \mathcal{A}$ by definition, and if $t_1 \in \mathcal{A}, t_2 \notin \mathcal{A}$, we would get $\overline{E}_{t_1} \subset \overline{E}_{\tilde{t}} \subset E_{t_2}$ by nesting in $\mathcal{A}$, meaning $N_{t_1} \cap N_{t_2} = \varnothing$. So $\tilde{t} < t_1 < t_2 < \tilde{t} + \delta$.

Define a new solution $\{\tilde{N}_t\}_{\tilde{t} \leq t < T-(t_2-t_1)}$ to (1.6) by $\tilde{N}_t = N_{t+(t_2-t_1)}$. Then $N_{\tilde{t}} \subset \tilde{E}_{\tilde{t}} = E_{\tilde{t}+(t_2-t_1)}$ since $0 < t_2 - t_1 < \delta$. By the One-Sided Avoidance Principle, this implies $\overline{E}_{t_1} \subset \tilde{E}_{t_1} = E_{t_2}$, but this once again contradicts $N_{t_1} \cap N_{t_2} \neq \varnothing$. Conclude $\mathcal{A} = [0, T)$. According to Lemma 2.3 in [**HI01**], the corresponding u must then minimize (2.1) over U, and since the level sets are smooth hypersurfaces, u must be differentiable with $H = |\nabla u| > 0$. The case $T = \infty$ follows via a continuation argument. $\square$

We would like to establish that if a classical solution N_t to IMCF induces a variational solution u over every $t \in [0, T)$ for sufficiently large T, then N_t must be star-shaped by some time within $[0, T)$. We know this must be true for the flow surfaces of variational solution $\tilde{u} : \mathbb{R}^{n+1} \to \mathbb{R}$ with initial condition E_0 from Theorem 2.2.8, so we seek to establish a correspondence between u and $\tilde{u}$. Recall the sets $\tilde{E}_t = \{\tilde{u} < t\}$ and $\tilde{E}^+_t = \text{Int}(\{\tilde{u} \leq t\})$ from Section 2. First we observe that if $\tilde{E}_{t_1}$ fails to be strictly outward minimizing for some $t_1 \in [0, T)$ (See Definition 3 in the following section), or equivalently that $\tilde{E}^+_{t_1} \neq \tilde{E}^+_{t_1}$, then the classical solution N_t cannot fully escape the minimizing hull $\tilde{E}^+_{t_1}$ of $\tilde{E}_{t_1}$ before the time T without self-intersecting.

Lemma 2.3.3 (No Escape Lemma)**.** *Let $\{N_t\}_{t \in [0,T)}$ be a solution to (1.6) with N_t a connected, embedded hypersurface for each $t \in [0, T)$, and E_t as in Theorem 3. Let $\tilde{u} : \mathbb{R}^{n+1} \to \mathbb{R}$ be the variational solution to IMCF with initial condition E_0 and precompact $\tilde{E}_t$. Suppose there exists a time $t_1 \in [0, T)$ so that $\tilde{E}_{t_1} \neq \tilde{E}^+_{t_1}$. Then there does not exist a time $t_2 > t_1$ in $[0, T)$ so that $\overline{\tilde{E}^+_{t_1}} \subset E_{t_2}$.*

Proof. We proceed by contradiction. Define

$$t_1 = \inf\{t \geq 0 | \tilde{E}_t^+ \neq \tilde{E}_t\}$$

By the Smooth Start Lemma 2.4 and Minimizing Hull Property 1.4 of $\tilde{E}_t^+$ from [**HI01**], we know for the classical solution N_t that $N_t = \partial\tilde{E}_t$ for $t < t_1$. We claim that $\tilde{E}_{t_1} \neq \tilde{E}_{t_1}^+$. By (1.10) from [**HI01**], $\partial\tilde{E}_t = N_t \to \partial\tilde{E}_{t_1} = N_{t_1}$ in $C^{1,\beta}$ as $t \nearrow t_1$. If $\tilde{E}_{t_1} = \tilde{E}_{t_1}^+$, we would have since $H > 0$ on $\partial\tilde{E}_{t_1} = N_{t_1}$ that $\partial\tilde{E}_t = N_t$ over some interval $[t_1, t_1 + \epsilon)$ by the Smooth Start Lemma. This would mean $\tilde{E}_t = \tilde{E}_t^+$ over $[t, t+\epsilon)$ since $N_t = \partial\tilde{E}_t \to N_{t_0} = \partial\tilde{E}_{t_0}^+$ in $C^{1,\beta}$ as $t \searrow t_0$ in $[t_1, t_1+\epsilon)$ by the second part of (1.10). So W.L.O.G. we prove the result for $\tilde{E}_{t_1}^+$, as the $\tilde{E}_t^+$'s are nested in time. $\tilde{E}_{t_1}^+ \setminus \overline{\tilde{E}_{t_1}}$ is open by definition and nonempty by assumption, so it must have positive Hausdorff Measure. Furthermore, according to the Minimizing Hull Property 1.4(iv) and Exponential Growth Lemma 1.6 from [**HI01**], we have

$$|N_{t_1}| = |\partial\tilde{E}_{t_1}| = |\partial\tilde{E}_{t_1}^+| = e^{\frac{t}{n}}|\partial E_0|. \tag{2.12}$$

If there exists a $t_2 \in [0, T)$ with $\overline{\tilde{E}_{t_1}^+} \subset E_{t_2}$, then take the domain $U = E_{t_2} \setminus E_0$. According to Theorem 2.3.2, the classical solution $\{N_t\}_{t\in[0,t_2)}$ induces a variational solution u with nonvanishing gradient over U. If $\partial\tilde{E}_{t_1}^+ \subset U$ we would have, in view of the positivity of $|\nabla u|$, positivity of $|\tilde{E}_{t_1}^+ \setminus \tilde{E}_{t_1}|$, and the Divergence Theorem that

$$\begin{aligned}
0 &< \int_{\tilde{E}_{t_1}^+ \setminus \tilde{E}_{t_1}} |\nabla u| = \int_{\tilde{E}_{t_1}^+ \setminus \tilde{E}_{t_1}} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\
&= \int_{\partial\tilde{E}_{t_1}^+} \frac{\nabla u}{|\nabla u|} \cdot \nu + \int_{N_{t_1}} \frac{\nabla u}{|\nabla u|} \cdot \nu \\
&\leq |\partial\tilde{E}_{t_1}^+| - |N_{t_1}|,
\end{aligned}$$

but this contradicts the equality (2.12). Conclude then that we must have $\overline{\tilde{E}_{t_1}^+} \not\subset E_t$ for any $t \in [0, T)$. □

Remark 2.3.4. *We found this result for weak IMCF in an earlier version of [**AFM19**], where it was shown instead using p-harmonic potentials. However, their proof of this theorem appears to have since been removed from [**AFM19**] for the sake of brevity.*

Next we confine the minimizing hull of some $\tilde{E}_t$ which is not strictly outward minimizing to a ball in $\mathbb{R}^{n+1}$ depending only on initial data.

Lemma 2.3.5. *Let $E_0 \subset \mathbb{R}^{n+1}$ be an open bounded domain with C^2 boundary N_0, and let $\tilde{u} : \mathbb{R}^{n+1} \to \mathbb{R}$ be the variational solution with initial condition E_0 and precompact $\tilde{E}_t$. Choose $0 \in \mathbb{R}^{n+1}$ so that $E_0 \subset B_{\frac{diam(N_0)}{2}}(0)$. Then for each $t \geq 0$, we have $\tilde{E}_t^+ \subset B_{e^{\frac{t}{n}}\frac{diam(N_0)}{2}}(0)$. In particular, if $E_{t_1}^+ \neq E_{t_1}$ for some $t_1 \in \mathbb{R}$, then $E_{t_1}^+ \subset B_{\frac{R-1}{2}(diam(N_0))^2}(0)$, where R is the largest principal curvature of N_0.*

Proof. Observe that the sets $F_t = B_{e^{\frac{t}{n}}\frac{\mathrm{diam}(N_0)}{2}}(0)$ define a variational solution of IMCF with compact level sets and $E_0 \subset F_0$, so $\tilde{E}_t \subset F_t$ by Theorem 2.2(ii) of [**HI01**]. In the case that $\tilde{E}_{t_1} \neq \tilde{E}_{t_1}^+$ for some $t_1 \geq 0$, we show $\tilde{E}_{t_1}^+$ remains contained in F_{t_1}. By definition $\tilde{E}_{t_1}^+ \subset \tilde{E}_t$ for $t > t_1$. Then choosing the sequence

$$\{t_i = t_1 + n\ln(1 + i^{-1})\}_{i=1}^{\infty},$$

we have the inclusion $\tilde{E}_{t_1}^+ \subset F_{t_i} = B_{(1+i^{-1})e^{\frac{t_1}{n}}\frac{\mathrm{diam}(N_0)}{2}}(0)$. Thus

$$\tilde{E}_{t_1}^+ \subset \mathrm{Int}(\cap_{i=1}^{\infty} F_{t_i}) = B_{e^{\frac{t_1}{n}}\frac{\mathrm{diam}(N_0)}{2}}(0)$$

For the second part of the statement, according to Theorem 2.2.8, $\partial\tilde{E}_t$ is star-shaped whenever $t \geq t^*$. Thus $\tilde{u}$ is C^1 with $|\nabla\tilde{u}| \neq 0$ over $\mathbb{R}^{n+1} \setminus \tilde{E}_{t^*}$ by Theorem 0 for star-shaped hypersurfaces in [**Ger90**] and uniqueness. Therefore, we cannot have $\tilde{u} = t_0$ over a positive measure set for $t_0 \geq t^*$, so $\tilde{E}_t = \tilde{E}_t^+$ for these times. So if $\tilde{E}_{t_1}^+ \neq \tilde{E}_{t_1}$ then $t_1 < t^*$, meaning $\tilde{E}_{t_1}^+ \subset F_{t^*} = B_{\frac{R-1}{2}(\mathrm{diam}(N_0))^2}(0)$. $\square$

Combining the previous two lemmas reveals that if the classical solution N_t escapes the ball $B_{\frac{R-1}{2}(\mathrm{diam}(N_0))^2}(0)$ while remaining embedded, then we must have $\tilde{E}_t = \tilde{E}_t^+$ inside this ball. This is sufficient to ensure $N_t = \partial\tilde{E}_t$, making N_t star-shaped beyond the time t^*. For the main theorem of this section, we estimate the time this escape takes to occur.

Proof of Theorem 2.1.1 and Corollary 2.1.2. Let E_0 be the set enclosed by N_0, $\tilde{u} : \mathbb{R}^{n+1} \to \mathbb{R}$ be the variational solution with initial condition E_0 and precompact $\tilde{E}_t$, and $0 \in \mathbb{R}^{n+1}$ be chosen so that $E_0 \subset B_{\frac{\operatorname{diam}(N_0)}{2}}(0)$. Suppose that the classical solution $\{N_t\}_{t\in[0,T)}$ to (1.6) with initial data N_0 exists and is embedded a time $T > 2t^*$. We claim then that the global solution $\tilde{u}$ satisfies $\tilde{E}_t = \tilde{E}_t^+$ for each $t \geq 0$, and we establish this by contradiction. Take a nonnegative time t_1 so that $\tilde{E}_{t_1} \neq \tilde{E}_{t_1}^+$. Lemma 2.3.5 states that $\tilde{E}_{t_1} \subset B_{\frac{R-1}{2}(\operatorname{diam}(N_0))^2}(0)$.

On the other hand, we may take $x \in E_0$ so that $B_R(x) \subset E_0$. The classical solution $\{N_t\}_{0\leq t<T}$ induces a variational solution u over $U = E_T \setminus E_0$, with E_t being as in Theorem 2.3.1. By the Comparison Principle 2.2 of [**HI01**], we must have $B_{Re^{\frac{t}{n}}}(x) \subset E_t$. However, evaluating at $t = 2t^*$ we get $B_{Re^{\frac{2t^*}{n}}}(x) = B_{R^{-1}(\operatorname{diam}(N_0))^2}(x)$. Then we have the containment

$$\overline{\tilde{E}_{t_1}^+} \subset \overline{B_{\frac{R-1}{2}(\operatorname{diam}(N_0))^2}(0)} \subset B_{R^{-1}(\operatorname{diam}(N_0))^2}(x) \subset E_{2t^*}.$$

This contradicts the No-Escape Lemma. Thus we know $\tilde{E}_t = \tilde{E}_t^+$ for $t \geq 0$. Letting $\mathcal{A} \subset [0, T)$ be the largest interval containing 0 over which $\partial\tilde{E}_t = N_t$, we then have $\sup A > 0$ by Lemma 2.4 in [**HI01**]. If $\tilde{t} = \sup A < T$, we would have that $E_{\tilde{t}} = \tilde{E}_{\tilde{t}} = \tilde{E}_{\tilde{t}}^+$ by the above result. Then since $H > 0$ on $\partial\tilde{E}_{\tilde{t}} = N_{\tilde{t}}$, Lemma 2.4 and Property 1.4 would once again imply $N_t = \partial\tilde{E}_t$ over some larger interval $t \in [0, \tilde{t} + \epsilon)$. Conclude then that $\tilde{t} = T$, i.e. that $\partial\tilde{E}_t = N_t$ over $[0, T)$.

$\partial\tilde{E}_t$ is star-shaped for $t \geq t^*$ by Theorem 2.2.8, so by Theorem 0 of [**Ger90**] and continuation, we must have for $T_{\max} = +\infty$ and N_t embedded for all times. The alternative is then that N_t does not exist or remain embedded past the time $2t^*$. For Corollary 2.1.2, solutions which satisfy the first alternative are star-shaped and therefore topological spheres for any $t > 2t^*$, but since they are also embedded for all times $[0, T)$, this implies that N_0 must also be a topological sphere. Thus any initial surface without spherical topology necessarily satisfies the second alternative.

□

Remark 2.3.6. *From [**Ger90**], star-shaped data are known to homothetically converge to spheres, so Theorem* 2.1.1 *shows that the sphere is the unique blow-down limit of all embedded solutions to* (1.6) *which exist at least for the time* $2t^*$.

2.4. Intersections and Singularities in the Spherical Topology

We conclude this chapter with the proof of Theorem 2.1.3, which establishes that Corollary 2.1.2 is not an if-and-only-if. Before proceeding, we remind the reader of the definition of an outward-minimizing set.

Definition 2.4.1. *A subset $E \subset \mathbb{R}^n$ is said to be* ***outward minimizing*** *if for every F containing E with $F \setminus E \subset\subset \mathbb{R}^n$ we have $|\partial F| \leq |\partial E|$.*

Furthermore, E is ***strictly outward minimizing*** *if the above inequality is strict for every $F \neq E$.*

Suppose N_t is a connected classical solution to IMCF which exists for all time and foliates $\mathbb{R}^{n+1}$. This solution induces a variational solution $u : \mathbb{R}^{n+1} \setminus E_0 \to \mathbb{R}$ with $|\nabla u| > 0$ by Theorem 2.3.2. Then given any $t \in [0, \infty)$ and open set F containing N_t, we can perform the same integration as in (2.12) from Lemma 2.3.3 over $N = F \setminus E_t$ using the Divergence Theorem:

$$\begin{aligned} 0 &< \int_N |\nabla u| = \int_N \operatorname{div}(\frac{\nabla u}{|\nabla u|}) \\ &= \int_{\partial F} \frac{\nabla u}{|\nabla u|} \cdot \nu + \int_{N_t} \frac{\nabla u}{|\nabla u|} \cdot \nu \\ &\leq |\partial F| - |N_t|. \end{aligned}$$

Therefore, N_t is strictly outward-minimizing, and so the evolution N_t of a given N_0 can only exist and remain embedded forever if N_0 is strictly outward-minimizing.

Proof of Theorem 2.1.3 Our construction utilizes the fact that by Lemma 2.3.3, an open set E_0 with $\partial E_0 = N_0$ must be strictly outward minimizing for the classical flow N_t to exist longer than $2t^*$. Therefore, we need only construct an $H > 0$ topological sphere which is not strictly outward minimizing to assure that its flow develops a finite-time singularity or intersection.

Consider two disjoint balls $B(p, R)$ and $B(-p, R)$ with centerpoints $p = (p_1, \ldots 0)$ and $-p$ and identical radii R, and take the Hausdorff distance d between the balls to be small enough so that their union is not outward minimizing. Take the minimizing hull E' of $E = B(p, R) \cup B(-p, R)$. We seek first to establish some symmetry for E:

Proposition 2.4.2. *E' is rotationally symmetric about the x_1 axis, and $\partial E' \setminus \partial E$ is a C^∞ minimal hypersurface.*

Proof. Rotations $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ about the x_1 axis are isometries of $\mathbb{R}^{n+1}$, and thus send E' to the minimizing hull of $R(E)$. Since this axis contains that centerpoints of each sphere, we know these R also fix the associated balls, i.e. $R(E) = E$. Then by the uniqueness of strictly minimizing hulls, E' is also the minimizing hull of $R(E)$, implying $R(E') = E'$.

The regularity of $\partial E' \setminus \partial E$ follows from Theorem 5.3(ii) of [**AFM19**] (Also mentioned on page 369 of [**HI01**]: if the singular set $\mathrm{Sing}(\partial E' \setminus \partial E)$ is nonempty, then its Hausdorff dimension is at least $n-1$ by rotational symmetry. But this dimension cannot exceed $n-8$. Thus $\mathrm{Sing}(\partial E' \setminus \partial E) = \varnothing$, and as the surface is smooth outside $\mathrm{Sing}(\partial E' \setminus \partial E)$ we obtain the regularity. Furthermore, $H = 0$ on this surface by (1.15) in [**HI01**].

□

Next, we are going to show that the bridge joining the two spheres does not extend past their equators. This will allow us to glue the spheres together over regions away from $\partial E' \setminus \partial E$, so that the minimizing hull of the resulting surface must still include this part.

Proposition 2.4.3. *The set $E' \setminus E$ is contained within the set $\{x \in \mathbb{R}^{n+1} | |x_1| \leq p_1\}$.*

Proof. We claim first that E' is contained within the cylinder $C_R = \{x \in \mathbb{R}^{n+1} | x_2^2 + \cdots + x_{n+1}^2 \leq R^2$. Suppose not: define the vector field $\hat{w} = \frac{\vec{y}}{|\vec{y}|}$, where $\vec{y}(x_1, \ldots, x_n) = (0, x_2, \ldots x_n)$ points radially away from the x_1 axis, and let $u(\vec{x}) = \langle \hat{w}, \vec{x} \rangle$ be the distance from this axis or "height" of a point $\vec{x} \in \partial E' \setminus \partial E$. Since E is contained within C_R and $a = \sup_{\partial E' \setminus \partial E} u > R$ we must have that this supremum occurs at an interior point x_0 of $\partial E' \setminus \partial E$. The $n-1$ principal curvatures corresponding to rotation all must equal $\frac{1}{a}$ at x_0, and the other principal curvature must be nonnegative since x_0 is a local maximum of the height function u. Thus $H(x_0) > 0$, contradicting the minimality of this complement. Thus $\sup_{\partial E' \setminus \partial E} u \leq R$, and therefore E' lies in C_R.

Now, no connected component of $E' \setminus E$ lies entirely outside $\{x \in \mathbb{R}^{n+1} | |x_1| < p_1\}$ since a single ball is strictly outward minimizing. Thus we can have $E' \setminus E$ intersect $\{|x_1| \geq p_1\}$ only if $(E' \setminus E) \cap \{|x_1| = p_1\} \neq \varnothing$, but this would require $E' \setminus E \not\subset C_R$.

□

Now that we have established that the $H = 0$ part of E' is contained between the equators of the two spheres, we are ready to construct our example. Our surface will be of class C^0 before smoothing.

Begin by attaching a cylinder of radius r for some $r < R$ and finite length about the x_1 axis to the opposite end of the sphere in the $x_1 < 0$ plane (See diagram). Then attach one end of a half torus with small radius r and large radius R^* to the end of this cylinder. Attach another cylinder extending to $x_1 = 0$ to its other end, and reflect this surface about $\{x_1 = 0\}$.

The resulting surface must not be outward minimizing, since the original spheres were not outward minimizing and, by Proposition 4, the new surface does not touch the $H = 0$ part of the original hull E.

It remains to show that one may refine this surface to an $H > 0$ surface which is of class C^2. We require one additional lemma for this purpose.

Lemma 2.4.4. *Let U be any open subset of $\mathbb{R}$ containing 0. Let $f : U \to \mathbb{R}$ be any function of the form*

$$f(x) = \begin{cases} 0 & x \leq 0 \\ g(x) & x > 0 \end{cases} \tag{2.13}$$

for some $g : U \cap \{x > 0\} \to \mathbb{R}$. Then for every $0 < \epsilon < dist\{0, \partial U\}$ there exists a function $p : (0, \epsilon) \to \mathbb{R}$ so that the the function

$$\tilde{f}(x) = \begin{cases} 0 & x \leq 0 \\ p(x) & 0 < x < \epsilon \\ g(x) & x \geq \epsilon \end{cases} \tag{2.14}$$

is in $C^2(U)$.

Proof. Take the polynomial $p(x) = Ax^3 + Bx^4 + Cx^5$ for some constants $A, B, C \in \mathbb{R}$. Clearly $p(0) = p'(0) = p''(0) = 0$. Furthermore, derivatives of p are related to the coefficients A, B, and C by

A Non-Outward-Minimizing, $H > 0$ Sphere

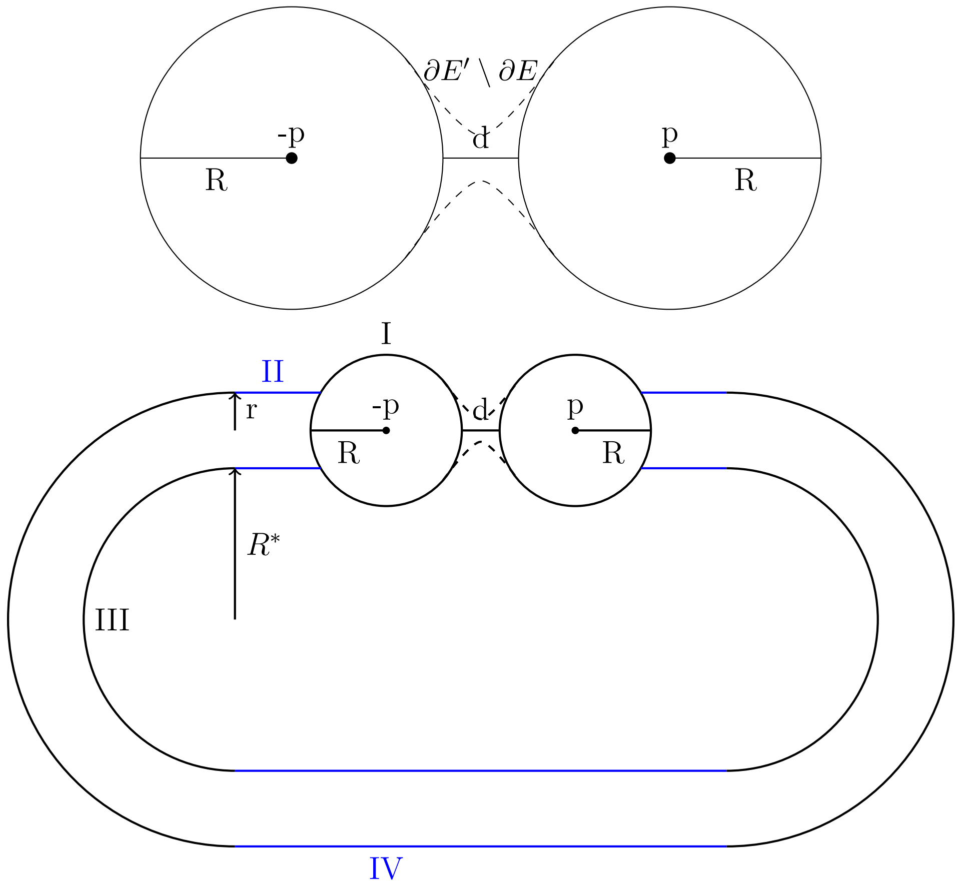


FIGURE 2.2. Given sufficiently close disjoint balls, one can attach handles and glue them together so that the resulting $C^2, H > 0$ surface is not strictly outward minimizing.

$$\begin{pmatrix} p(x) \\ p'(x) \\ p''(x) \end{pmatrix} = \begin{pmatrix} x^3 & x^4 & x^5 \\ 3x^2 & 4x^3 & 5x^4 \\ 6x & 12x^2 & 20x^3 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}. \tag{2.15}$$

One may readily compute for the above matrix M that $\det M = 2x^9 \neq 0$ for any $x \neq 0$. This means that for any triple $(X, Y, Z) \in \mathbb{R}^3$ and any fixed point $x \neq 0$ we may select coefficients A, B, and C so that $(p(x), p'(x), p''(x)) = (X, Y, Z)$. In fact, inverting the above matrix reveals that for a given $(g(\epsilon), g'(\epsilon), g''(\epsilon)) \in \mathbb{R}^3$

$$\begin{aligned} A &= \frac{10}{\epsilon^3}g(\epsilon) - \frac{4}{\epsilon^2}g'(\epsilon) + \frac{1}{2\epsilon}g''(\epsilon) \\ B &= -\frac{15}{\epsilon^4}g(\epsilon) + \frac{7}{\epsilon^3}g'(\epsilon) - \frac{1}{\epsilon^2}g''(\epsilon) \\ C &= \frac{6}{\epsilon^5}g(\epsilon) - \frac{3}{\epsilon^4}g'(\epsilon) + \frac{1}{2\epsilon^3}g''(\epsilon). \end{aligned} \tag{2.16}$$

Then restricting the domain of this p to $(0, \epsilon)$, the first two derivatives of the function

$$\tilde{f}(x) = \begin{cases} 0 & x \leq 0 \\ p(x) & 0 < x < \epsilon \\ g(x) & x \geq \epsilon \end{cases} \tag{2.17}$$

are everywhere continuous. □

Now, we must establish C^2 regularity at the overlap between regions I and II, II and III, and III and IV (See Figure).

Regions I-II The union of these regions is a surface of revolution about the x_1 axis and is therefore given by a graph in the x_1 coordinate. Choose 0 to be the point on the x_1 axis corresponding to the equator of the sphere, and let $\epsilon = \sqrt{R^2 - r^2}$. Then this graph is explicitly $g(x) = f(x) + r$, where

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sqrt{R^2 - x^2} - r & x > 0 \end{cases}. \tag{2.18}$$

Now apply Lemma 1 for this f and ϵ. The resulting function is C^2, and it remains only to show that the corresponding surface of revolution will be mean convex if the tube radius r is sufficiently close to the sphere radius R. Explicitly computing the interpolating polynomial $p(x)$ by inverting the matrix in the proof of Lemma 1, we find

$$
\begin{aligned}
A &= \frac{4}{\epsilon}\frac{1}{r} - \frac{1}{2\epsilon}\frac{R^2}{r^3} \leq \frac{7}{2\epsilon}\frac{1}{r} \\
B &= \frac{-7}{\epsilon^2}\frac{1}{r} + \frac{1}{\epsilon^2}\frac{R^2}{r^3} = \frac{1}{r^3} - \frac{6}{\epsilon^2 r} \\
C &= \frac{3}{\epsilon^3}\frac{1}{r} - \frac{1}{2\epsilon^3}\frac{R^2}{r^3} \leq \frac{5}{2\epsilon^3}\frac{1}{r}.
\end{aligned}
\tag{2.19}
$$

In particular, for every $x \in (0, \epsilon)$ $p(x) = r + Ax^3 + Bx^4 + Cx^5$ and $p''(x) = 6Ax + 12Bx^2 + 20Cx^3$ obey the estimates

$$
p(x) \leq r + A\epsilon^3 + B\epsilon^4 + C\epsilon^5 = r + \frac{\epsilon^2}{r^3} \tag{2.20}
$$

$$
p''(x) \leq 6A\epsilon + 12B\epsilon^2 + 20C\epsilon^3 \leq \frac{12}{r^3}\epsilon^2. \tag{2.21}
$$

Then choosing ϵ small enough to ensure the $H = 0$ part of the original minimizing hull strictly lies in the region $\{x_1 > \epsilon\}$ and that $p(x)p''(x) < 1$, the C^2 surface of revolution is not outward minimizing and has $H = \frac{1}{(1+f'(x)^2)^{\frac{3}{2}}}(1 + f'(x)^2 - f(x)f''(x)) > 0$.

Regions II-III/III-IV: One may apply an identical gluing construction to each of these overlap regions, so we only present the construction for Regions III-IV here. The union of regions III and IV corresponds to a curve which is the union of a semicircle of a line. Parametrizing the lower half of the semicircle and the line as

$$
g(x_1) = \begin{cases} 0 & x < 0 \\ -\sqrt{(R^*)^2 - x^2} + R^* & x \geq 0 \end{cases}. \tag{2.22}
$$

Here we chose the origin to be the point where the arc meets the line. For some sufficiently small $\epsilon > 0$, we apply Lemma 1. It remains only to show that the surface obtained by taking a circle of radius r in each plane normal to the curve at each point is mean convex for R^* sufficiently large. From Lemma 1, the interpolating polynomial $p(x)$ has second derivative given by

$$
\begin{aligned}
p''(x) &= 6\left(\frac{10}{\epsilon^3}(-\sqrt{(R^*)^2-\epsilon^2}+R^*)-\frac{4}{\epsilon^2}\left(\frac{\epsilon}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}\right)+\frac{1}{2\epsilon}\frac{(R^*)^2}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}\right)x \\
&\quad +12\left(\frac{-15}{\epsilon^4}(-\sqrt{(R^*)^2-\epsilon^2}+R^*)+\frac{7}{\epsilon^3}\left(\frac{\epsilon}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}\right)+\frac{-1}{\epsilon^2}\frac{(R^*)^2}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}\right)x^2 \\
&\quad +20\left(\frac{6}{\epsilon^5}(-\sqrt{(R^*)^2-\epsilon^2}+R^*)-\frac{3}{\epsilon^4}\left(\frac{\epsilon}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}\right)+\frac{1}{2\epsilon^3}\frac{(R^*)^2}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}\right)x^3.
\end{aligned}
$$

Since $0 < x < \epsilon$, we have

$$
p''(x) \leq \frac{180}{\epsilon^2}(-\sqrt{(R^*)^2-\epsilon^2}+R^*)+13\frac{(R^*)^2}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}+\frac{84}{((R^*)^2-\epsilon^2)^{\frac{3}{2}}}. \tag{2.23}
$$

For a fixed ϵ each of these terms can be made arbitrarily small by choosing R^* large enough to guarantee that $p''(x) \leq \frac{1}{r}$ for every $x \in [0, \epsilon)$, where $\frac{1}{r}$ is the curvature of the surface in a direction orthogonal to the graph. This in turn guarantees that $H > 0$ in this region, so the entire surface is C^2 and mean convex.

Remark 2.4.5. *Since the spheres in this construction can be chosen to be arbitrarily close to one another without changing the initial flow speed at the closest points, we suspect that this surface develops an intersection rather than a singularity first.*

CHAPTER 3

The Singular Limit of IMCF on a Torus

Mmmm, doughnuts.

-Homer Simpson, The Simpsons

3.1. The Main Theorems

We now know that embedded solutions to the Inverse Mean Curvature Flow (1.6) without spherical topology must always develop finite-time singularities, and so the next step is to characterize the singular behavior. Theorem 1.3.6 tells us that the mean curvature H degenerates to zero somewhere along N_t at the extinction, but in this section we are specifically interested in the behavior of the total curvature $|A|$ near this time.

We reference Example 1.10 where a thin torus $N_0 \subset \mathbb{R}^3$ is evolved by IMCF to model this behavior. Huisken and Ilmanen's heuristic argument tells us that H approaches 0 along the ring of the torus closest to the axis of rotation in finite time, but there is another subtle, fascinating question about the dynamics in this case: do the flow surfaces N_t pinch down at this axis by the time of extinction, or will the flow terminate due to blow-up in the flow speed before this can happen?

If the former possibility is true, then the magnitude of each principal curvature of N_t at the points closest to the axis become infinitely large as their sum H tends to 0. This curvature blow-up is typical behavior in the singularities of extrinsic flows. The latter possibility is more interesting, since it would imply that the magnitude of the principal curvature associated with rotation is uniformly bounded up to the singular time, and this ensures that the other curvature is bounded as well. An L^∞ control on $|A|$ should imply that a limit immersion $F_{T_{\max}}$ exists for a subsequence of the F_t's, possibly after composition with surface diffeomorphisms.

In this chapter, we prove that the $\max_{N_t} |A|$ does indeed remain bounded up to $T_{\max}$ for the IMCF of the thin torus. In fact, our methods apply to a far more general class of topological tori.

Any $H > 0$ torus $N_0 \subset \mathbb{R}^3$ with rotational symmetry about an axis also maintains bounded total curvature throughout its evolution by IMCF.

Theorem 3.1.1 (The Limit of Rotationally Symmetric Tori). *Let $N_0 = F_0(\mathbb{T}^2) \subset \mathbb{R}^3$ be an $H > 0$, rotationally symmetric embedded torus and $F : \mathbb{T}^2 \times [0, T_{\max}) \to \mathbb{R}^3$ the corresponding maximal solution to (1.6). Then $T_{\max} < +\infty$ and $\lim_{t \to T_{\max}} \max_{N_t} |A| \leq L < +\infty$. In particular, there exists a subsequence of times $t_k \nearrow T_{\max}$ and corresponding diffeomorphisms $\alpha_k : \mathbb{T}^2 \to \mathbb{T}^2$ so that the maps $\widetilde{F}_{t_k} = F_{t_k} \circ \alpha_k : \mathbb{T}^2 \to \mathbb{R}^3$ converge in C^1 topology to an immersion $\widetilde{F}_{T_{\max}}$. Furthermore, $\widetilde{F}_{T_{\max}}(\mathbb{T}^2) \subset \mathbb{R}^3$ is a $C^{1,\alpha}$, rotationally symmetric embedded torus for $0 < \alpha < 1$.*

Our argument by contradiction utilizes only elementary properties of the flow and an application of the Gauss-Bonnet Theorem.

This result also imposes the question of how general this type of behavior is for singular solutions of (1.6). As a first step toward answering this, we also prove an L^2 energy estimate on $|A|$ which applies to any solution N_t of (1.6) in $\mathbb{R}^3$.

Theorem 3.1.2 (L^2 Estimate on $|A|$). *Let $\{N_t\}_{0 \leq t < T}$ be a solution to (1.6) in $\mathbb{R}^3$. Then we have the time-independent estimate*

$$\int_{N_t} |A|^2 d\mu \leq 3 \sup_{N_0} H^2 |N_0| - 2\pi\chi(N), \tag{3.1}$$

where $\chi(N)$ is the Euler Characteristic of N.

This estimate hints that a limit immersion likely exists for a subsequence of times modulo surface diffeomorphisms, and may also play a role in establishing an L^∞ control on $|A|$ for other singular solutions.

The chapter is organized as follows: in Section 3.2, we consider the generating curve ρ_0 of a rotationally symmetric embedded torus N_0 in the (x_1, x_2)-plane, and we demonstrate that the generating curve ρ_t of N_t must remain embedded. This also ensures the formation of a singularity for N_0 under (1.6) within a prescribed time interval by Corollary 2.2. In Section 3.3, we obtain a sharp upper bound on the L^1 norm of the Gauss Curvature K over a uniform neighborhood of the ring of the expanding torus closest to the axis of rotation via the Gauss-Bonnet Theorem. This relies on the profile of the generating curve obtained in Section 3.2. We prove in Section 3.4 that N_t cannot reach this axis by the time $T_{\max}$: if it did, we could rescale this neighborhood about this point and obtain convergence to a catenoid, which would contradict the integral bound on K.

In Section 3.5, we apply a compactness theorem from [**Lan85**] to rule out degeneration in the induced metric of N_t near the singular time, so that the un-scaled N_t's approach the embedded $C^{1,\alpha}$ limit surface as $t \to T_{\max}$. Finally, in Section 3.6 we prove Theorem 3.1.2 for general singularities in $\mathbb{R}^3$. the Gauss-Bonnet Theorem plays a central role here as it does in the proof of Theorem 3.1.1. Since $n = 2$, this estimate is scale invariant, and hence may be useful in ruling out blow-up in $\max_{N_t} |A|$ via rescaling arguments.

3.2. The Generating Curve

We are considering an $H > 0$ torus $N_0 \subset \mathbb{R}^3$ which is obtained by revolving a simple closed curve in the upper half of the (x_1, x_2)-plane about the x_1 axis. Let $\{e_1, e_2\}$ be the standard basis in $\mathbb{R}^2$, and ν be the outward normal of ρ_0 in the (x_1, x_2)-plane. We parametrize ρ_0 by arc length, taking $s = 0$ to be any point which minimizes the height $u = \langle \rho_0, e_2 \rangle$. The principal curvatures of N_0 are the same along the ring in $\mathbb{R}^3$ generated by a point $\rho(s)$ on this curve. One of these principal curvatures $p(s)$ corresponds to rotation about the x_1 axis. This curvature is equal to

$$p(s) = \langle \nu, e_2 \rangle u^{-1}(s), \tag{3.2}$$

and the other principal curvature $k(s)$ equals the curvature of ρ_0 in the plane at s. Then we write

$$H(s) = k(s) + p(s) \tag{3.3}$$

for the mean curvature H of N_0. The assumption that H is everywhere positive on N_0 gives us a strong profile for the curve ρ_0. Namely, ρ_0 must be the union of two graphs over the x_1 coordinate which correspond to the "top" and "bottom" of the curve.

Proposition 3.2.1. *Let ρ_0 be the generating curve for an embedded, $H > 0$, rotationally symmetric torus $N_0 \subset \mathbb{R}^3$, and call $a_0 = \min_{x \in \rho_0} \langle \rho_0, e_1 \rangle (x)$ and $b_0 = \max_{x \in \rho_0} \langle \rho_0, e_1 \rangle (x)$. Then ρ_0 is the disjoint union of two graphs for functions $w_0 : (a_0, b_0) \to \mathbb{R}$ and $v_0 : [a_0, b_0] \to \mathbb{R}$ with $w_0(x) < v_0(x)$*

Profile of the Generating Curve

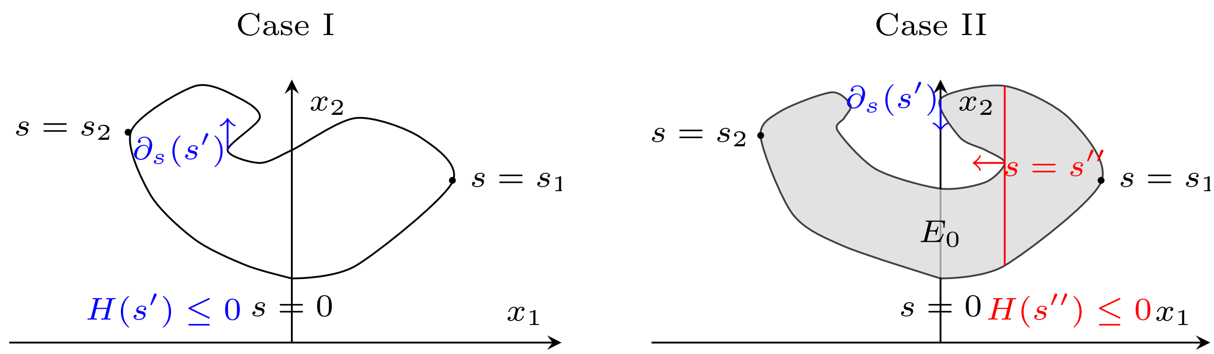


FIGURE 3.1. If $\langle e_2, \nu \rangle = 0$ at some point between s_1 and s_2, one can locate a point where $\langle e_2, \nu \rangle = 0$ and $k \leq 0$.

over (a_0, b_0)*. Furthermore,* $\text{graph}(w_0)$ *is convex, and the outward unit normal* ν *of* ρ_0 *satisfies* $\langle \nu, e_2 \rangle < 0$ *on* $\text{graph}(w_0)$ *and* $\langle \nu, e_2 \rangle > 0$ *on* $\text{Int}(\text{graph}(v_0))$*.*

PROOF. Once again, we parametrize by arc length so that $\rho_0(0) = \rho_0(\ell)$. By taking $\rho_0(0)$ to be a point which minimizes the height u, we know $\langle e_2, \nu \rangle(0) < 0$, and hence this is also true in some neighborhood of this point. Let $[0, s_1)$ and $(s_2, \ell]$ be the largest intervals containing 0 and ℓ respectively over which $\langle e_2, \nu \rangle < 0$. We want to show that $\langle e_2, \nu \rangle(s) > 0$ for each $s \in (s_1, s_2)$. At any point s_0 where $\langle e_2, \nu \rangle$ vanishes, we have in view of the product rule and the fact that $\partial_s(s_0) = \pm e_2$

$$\frac{d}{ds}\langle e_2, \nu \rangle(s_0) = \langle \nabla_s e_2, \nu \rangle + \langle e_2, \nabla_s \nu \rangle = \langle e_2, \nabla_s \nu \rangle = \pm \langle \partial_s, \nabla_s \nu \rangle(s_0) = \pm k(s_0). \tag{3.4}$$

Since $\langle e_2, \nu \rangle(s_1) = 0$, and hence $p(s_1) = 0$, $k(s_1) > 0$ in view of the mean convexity assumption. We then find $\frac{d}{ds}\langle e_2, \nu \rangle(s_1) = +k(s_1) > 0$. Therefore, $\langle e_2, \nu \rangle(s) > 0$ in some right-handed neighborhood of s_1. Let $[s_1, s')$ be the largest such neighborhood over which $\langle e_2, \nu \rangle(s) > 0$. We claim in fact that $s' = s_2$. Suppose not. We know that $\langle e_2, \nu \rangle(s') = 0$ and $\frac{d}{ds}\langle e_2, \nu \rangle(s') \leq 0$. We consider the cases $\partial_s(s') = \pm e_2$ separately, see 3.1.

Case I: $\partial_s(s') = +e_2$: (3.4) implies $k(s') \leq 0$ in this case. But $p(s') = 0$ here, so we would have altogether that $H(s') \leq 0$, a contradiction.

Case II: $\partial_s(s') = -e_2$: If $\frac{d}{ds}\langle e_2, \nu\rangle(s') = 0$, we would have $k(s') = 0$ so that $H(s') = 0$, which is a contradiction. Then assume $\frac{d}{ds}\langle e_2, \nu\rangle(s') < 0$. We also have

$$\begin{aligned} \frac{d}{ds}\rho_1(s') &= 0, \\ \frac{d}{ds}\rho_2(s') &< 0, \\ \frac{d^2}{ds^2}\rho_1(s') &> 0. \end{aligned}$$

So there is some $\widetilde{s} \in (s', s_2)$ with $\rho_1(\widetilde{s}) > \rho_1(s')$, $\rho_2(\widetilde{s}) < \rho_2(s')$. Now, letting E_0 be the open subset which ρ_0 encloses, define for each $s \in (s_1, s_2)$ the line L_s by

$$L_s = \{(x_1, x_2) \in \overline{E}_0 | x_1 = \rho_1(s)\}.$$

Take the smallest s value for which some interior point of L_s intersects ρ_0, which must exist by the above observation and the fact that $k(s_2) > 0$. At the intersection point $\rho(s'')$ we have $\nu(s'') = -e_1$ and $k(s'') \leq 0$, which together imply $H(s'') \leq 0$. Once again by contradiction, we conclude $\langle e_2, \nu\rangle > 0$ on (s_1, s_2).

Now we know that the subsets $\{\langle e_2, \nu\rangle < 0\}$ and $\{\langle e_2, \nu\rangle \geq 0\}$ of ρ_0 are each comprised of a single connected component, and these must each be a graph over the x_1 coordinate. Note also that $k(s) > 0$ wherever $\langle e_2, \nu\rangle(s) < 0$ by the positive mean curvature assumption.

□

We would like to consider the evolution of the torus N_0 generated by ρ_0 by (1.6). Since (1.6) preserves rotational symmetry, we at least know that we can identify the flow surface N_t with the curve ρ_t in the (x_1, x_2)-plane which generates it. As we noted in the previous chapter, a stark contrast between MCF and IMCF is that N_t is not neccessarily embedded even if N_0 is. Our first task is then to rule out the possibility of self-intersections in the curve ρ_t. We may accomplish this using the profile for ρ_0 obtained in the above proposition. Often in the statement of our results, we will identify a solution of (1.6) with the flow surfaces $N_t = F_t(N)$.

Theorem 3.2.2 (Preserving Embeddedness). *Let $N_0 \subset \mathbb{R}^3$ be a rotationally symmetric $H > 0$ embedded torus, and $\{N_t\}_{0\leq t<T_{\max}}$ the corresponding maximal solution to (1.6). Then N_t is embedded for each $t \in [0, T_{\max})$. In particular, each generating curve ρ_t is the disjoint union of two graphs for w_t and v_t satisfying the conditions in Proposition* 3.2.1.

Proof. N_t always remains embedded for short time, so we demonstrate that if $t \nearrow t_0$ and N_t is embedded for each $t < t_0$ then N_{t_0} is necessarily embedded. Each generating curve ρ_t for N_t is the disjoint union of two graphs w_t and v_t with $w_t < v_t$ over the axis of rotation, so we will show this is also true for ρ_{t_0}. Let E_t be the region enclosed by ρ_t. According to Theorem 4 in [**Har20b**], we must have $E_{t_1} \subset E_{t_2}$ for $t_1 < t_2$ in $[0, t_0)$. This implies that $w_{t_2}(x) < w_{t_1}(x)$ and $v_{t_2}(x) > v_{t_1}(x)$ for any $x \in (a_{t_1}, b_{t_1})$. Then for any compact set $K \subset (a_0, b_0)$ and $\widetilde{t} \in [0, t_0)$ so that $K \subset (a_{\widetilde{t}}, b_{\widetilde{t}})$, we have that $w_t|_K$ and $v_t|_K$ are each monotone and bounded over $t \in (\widetilde{t}, t)$. The graphs of the limits $v_{t_0}|_K$ and $w_{t_0}|_K$ must also parametrize part of N_{t_0} by uniqueness of limits and are therefore continuous. Then by Dini's Theorem $w_t \to w_{t_0}$ and $v_t \to v_{t_0}$ in $C^0_{\mathrm{loc}}((a_0, b_0))$ as $t \to t_0$. Note that $v_{t_0}(a_{t_0}) = w_{t_0}(a_{t_0})$ since $w(a_t) = v(a_t)$, and likewise for b_{t_0}, so the union of $\overline{\mathrm{graph}(v_{t_0})}$ and $\mathrm{graph}(w_{t_0})$ forms a closed curve. By uniqueness of limits, this union must equal ρ_{t_0}.

Now, we must have $\langle e_2, \nu\rangle \neq 0$ over the graphs of w_{t_0} and v_{t_0}, since $\langle e_2, \nu\rangle \leq 0$ over w_{t_0} (Resp. ≥ 0 over v_{t_0}), and therefore if $\langle e_2, \nu\rangle(s_0) = 0$ anywhere on $\mathrm{graph}(w_{t_0})$ we would have by (3.4) that

$$k(s_0) = \frac{d}{ds}\langle e_2, \nu\rangle(s_0) = 0.$$

This would leave us with $H(s_0) = 0$ at this point, but we know $H(s) > 0$ over ρ_{t_0}. The same argument yields that $\langle e_2, \nu\rangle > 0$ over the graph of v_{t_0}. By the monotonicty of w_t and v_t noted above, we also have for any $x \in (a_{t_0}, b_{t_0})$ and t sufficiently close to t_0 that

$$w_{t_0}(x) \leq w_t(x) < v_t(x) \leq v_{t_0}(x).$$

Hence these graphs do not intersect over (a_{t_0}, b_{t_0}). Since the graphs are disjoint, ρ_{t_0} must be embedded. $\square$

As mentioned in the above proof, embeddedness implies that N_{t_2} must enclose N_{t_1} whenever $t_2 > t_1$ by Theorem 4 in [**Har20b**]. This, along with the convexity of the bottom graph of ρ_t, will be crucial for ruling out the possibility that $\lim_{t\to T} u_{\min}(t) = 0$.

3.3. An Energy Estimate on Gauss Curvature

We now know by embeddedness and the topology of N_0 that N_t must become singular at some time $T_{\max}$ which occurs within a prescribed time interval, see Corollary 2 in [**Har20b**]. In order to rule out the possibility of pinching, we first estimate the integral of the Gauss curvature around the inner ring of the torus closest to the axis of rotation. To this end, we first establish that $\langle e_2, \nu\rangle$ is bounded away from 0 over some uniform neighborhood of the corresponding point on ρ_t. This neighborhood corresponds in $\mathbb{R}^3$ to the region of N_t between two fixed parallel planes each perpendicular to the axis of rotation.

Proposition 3.3.1. *For a sequence of times $t_n \to T_{\max}$ and corresponding points $x_n \in \rho_{t_n}$ which minimize the height u, W.L.O.G. choose the x_2 axis so that $0 = \lim_n\langle e_1, x_n\rangle$. Then there exists a constant $a > 0$ such that the sets*

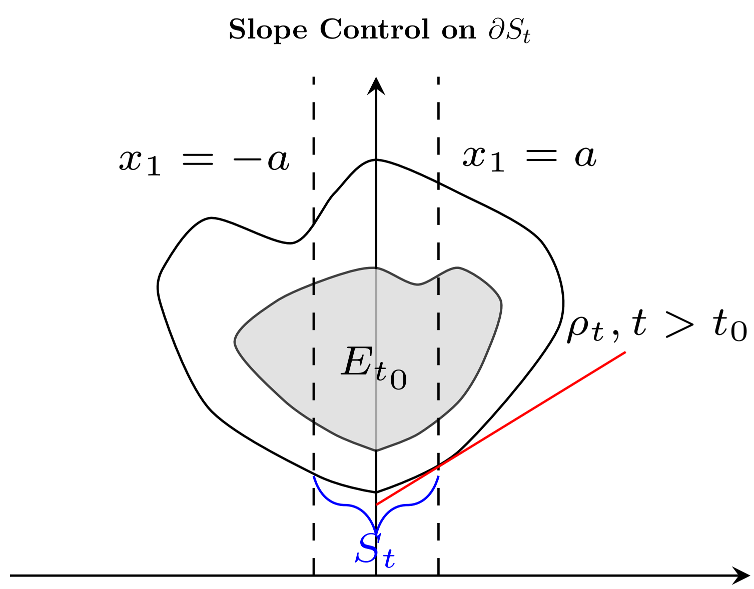


FIGURE 3.2. In order for ρ_t to properly enclose E_{t_0} for $t > t_0$, the tangent line must have a uniformly small slope on ∂S_t.

$$S_t = \{x \in \rho_t | \langle e_2, \nu\rangle(x) < 0 \text{ and } |\langle e_1, \nu\rangle|(x) < a\}$$

are each graphs over $(-a, a)$ *with* $\langle e_2, \nu\rangle|_{\partial S_t} \leq c$ *for some* $c = c(N_0) < 0$.

PROOF. Take a t_0 sufficiently close to $T_{\max}$ so that $\min_{x\in\rho_{t_0}}\langle x, e_1\rangle < 0$, $\max_{x\in\rho_{t_0}}\langle x, e_1\rangle > 0$. Now define $a = \frac{1}{2}\min\{-\min_{x\in\rho_{t_0}}\langle x, e_1\rangle, \max_{x\in\rho_{t_0}}\langle x, e_1\rangle\}$.

For $t > t_0$ sufficiently small, $\overline{S}_t$ may be parametrized by a convex graph $w_t : [-a, a] \to \mathbb{R}$ over the x_1 axis. We claim that $|w_t'(a)| \leq \frac{b}{a}$, where $b = \max_{x\in\rho_{t_0}} u(x)$. Suppose not: then the tangent line L to ρ_t at the point $(a, w_t(a))$ must pass through the region E_{t_0} enclosed by the curve ρ_{t_0}, see Figure 3.2. By Theorem 3.2.2, ρ_t is the union of disjoint graphs with the lower graph convex. This means that ρ_t lies entirely on one side of L, so that ρ_t also intersects E_{t_0}. According to Theorem 4 in [**Har20b**], this cannot happen since the ρ_t are embedded and hence must enclose $\overline{E}_{t_0}$ for $t > t_0$.

Thus, we have $|w_t'(a)| \leq \frac{b}{a}$ for $t > t_0$ sufficiently small. This also gives a time-independent bound on w_t' over the entire domain by convexity, so $\overline{S_t}$ remains a graph over $[-a, a]$ with this uniform slope estimate on its boundary for each $t \in (t_0, T_{\max})$. The result then follows from the relation $-\langle e_2, \nu\rangle = \frac{1}{(1+|w_t'|^2)^{\frac{1}{2}}}$, see, e.g. [**Hui90**] or [**EH91**]. $\square$

Now, an application of Gauss-Bonnet gives a control on the L^1 norm of the Gauss curvature K over the region of N_t generated by S_t.

Corollary 3.3.2 (Gauss Curvature Estimate)**.** *Let* S_t *be as above, and* S_t' *the surface generated by revolving* S_t *about the* x_1 *axis. Then for some constant* $\epsilon = \epsilon(N_0) > 0$ *we have*

$$\int_{S_t'} |K| d\mu \leq 4\pi(1-\epsilon) \tag{3.5}$$

for each $t \in [0, T_{\max})$.

PROOF. S_t' is an embedded compact surface with boundary $\partial S_t'$ and an Euler Characteristic of 0 in $\mathbb{R}^3$, so Gauss-Bonnet tells us that

$$\int_{S_t'} K d\mu = -\int_{\partial S_t'} k_g ds,$$

where k_g is the geodesic curvature of $\partial S'_t$. Here, $\partial S'_t$ consists of two circles C_1 and C_2, call their radii r_1 and r_2 respectively. The geodesic curvatures over these circles are $k_{g_1} = -\frac{\langle e_1,\nu\rangle(x_0)}{r_1}$ and $k_{g_2} = \frac{\langle e_1,\nu\rangle(y_0)}{r_2}$, where x_0 and y_0 are the left and right endpoints of the curve S_t, respectively.

By Proposition 3.3.1 and the relation $|\langle e_1, \nu\rangle|^2 = 1 - |\langle e_2, \nu\rangle|^2$, we have $|\langle e_1, \nu\rangle| \leq 1 - \epsilon$ for some uniform constant $\epsilon > 0$. Then

$$\int_C k_g ds = \int_{C_1} k_{g_1} ds + \int_{C_2} k_{g_2} ds \leq 4\pi(1-\epsilon).$$

Noting that K is strictly negative over S'_t, the result follows. $\square$

Intuitively, this estimate is a promising sign that $|K|$, and hence $|A|$, remains uniformly bounded in L^∞ norm near $T_{\max}$. We formally prove this in the following section, utilizing both the scale invariance of $\int_{S'_t} |K| d\mu$ as well as the rigidity for rotationally symmetric, complete minimal surfaces in $\mathbb{R}^3$.

3.4. Rescaling the Singularity

In this section, we derive a contradiction if $\lim_{t\to T_{\max}} u_{\min}(t) = 0$ on the evolving curve ρ_t to prove that N_t converges to a smooth limit surface.

Many of the tools developed to analyze the singularities of mean curvature flow do not translate to this setting. For example, the idea of a tangent flow is not applicable here since (1.6) does not obey the standard parabolic scaling. Nevertheless, assuming $\lim_{t\to T} u_{\min}(t) = 0$ in our setting, we may consider the re-scaled surfaces

$$\widetilde{N}_t = \frac{1}{u_{\min}(t)} N_t. \tag{3.6}$$

Crucially, mean curvature remains uniformly bounded above on N_t by initial data under (1.6) (see Section 6), which would imply $H \to 0$ uniformly over $\widetilde{N}_t$. Thus, any limit surface which the $\widetilde{N}_t$ (or some subset of each) converges to will necessarily be minimal. In the setting of a rotationally symmetric torus N_0, the candidate limit is rigid: we should expect convergence to a catenoid.

Lemma 3.4.1. *Let $S_t \subset \rho_t$ be as in Proposition* 3.3.1*, and consider the rescalings $\widetilde{S}_t = \frac{1}{u_{\min}(t)} S_t$ with respect to the origin. Suppose $\lim_{t\to T} u_{\min}(t) = 0$. Then the corresponding graphs $\widetilde{w}_t$ of $\widetilde{S}_t$ converge in $C^2_{loc}(\mathbb{R})$ to the function $\widetilde{w}(x) = \cosh(x)$ as $t \to T_{\max}$.*

Proof. The parabolic maximum principle guarantees for any solution $\{N_t\}_{0\le t<T_{\max}}$ to (1.6) that $\max_{N_t} H \le \max_{N_0} H$ (we give the relevant evolution equation in the proof of Lemma 3.6.1). Therefore, assuming $\lim_{t\to T} u_{\min}(t) = 0$, we also know for the surface $\widetilde{S}'_t$ generated by $\widetilde{S}_t$ that

$$\lim_{t\to T_{\max}} \max_{\widetilde{S}'_t} H = 0. \tag{3.7}$$

Consider $\overline{\widetilde{S}_t}$ to be a graph $\widetilde{w_t} : [-\widetilde{a}_t, \widetilde{a}_t] \to \mathbb{R}$ over the x_1 coordinate, where $\widetilde{a}_t = \frac{1}{u_{\min}(t)} a_t$ for a_t as in (3.3.1). Notice $\widetilde{a}_t \to \infty$ as $t \to T_{\max}$. We verify that a subsequence of the functions $\widetilde{w}_t$ converge in $C^2_{\mathrm{loc}}(\mathbb{R})$ as $t \to T_{\max}$. Fix a compact subset K of $\mathbb{R}$, and pick t_0 sufficiently close to $T_{\max}$ so that $K \subset [-\widetilde{a}_{t_0}, \widetilde{a}_{t_0}]$. For any $t \in (t_0, T_{\max})$, we know also from Proposition 3.3.1 that the function

$$v_t(x) = (1 + |\widetilde{w}'_t|^2)^{\frac{1}{2}}$$

is uniformly bounded in t over $[-\widetilde{a}_t, \widetilde{a}_t]$. This guarantees convergence of a subsequence as $t \to T_{\max}$ at least in $C^0(K)$. Now, we can write the mean curvature of the surfaces generated by $\widetilde{w}_t$ as

$$\widetilde{H}(x) = \frac{\widetilde{w}''}{v^3} - \frac{1}{\widetilde{w}v}. \tag{3.8}$$

Rearranging gives

$$\widetilde{w}''_t(x) = v_t^3(x)\widetilde{H}_t(x) + \frac{v_t^2(x)}{\widetilde{w}_t(x)}. \tag{3.9}$$

Noting that $\widetilde{w}_t \ge 1$, we can immediately see from this that $\widetilde{w}''$ is uniformly bounded in t, yielding precompactness in $C^1(K)$. In fact, we can observe equicontinuity of $\widetilde{w}''_t$ in t: $\widetilde{H}_t(x) \to 0$ uniformly over K as $t \to T_{\max}$ by the uniform bound on H and the assumption that $\lim_{t\to T_{\max}} u_{\min}(t) = 0$. We also have

$$
\begin{aligned}
|(\frac{1}{\widetilde{w}_t(x)})'| &= \frac{1}{\widetilde{w}_t(x)^2}|\widetilde{w}_t'(x)| \leq C(N_0) \\
|v_t'(x)| &= |\frac{\widetilde{w}_t'\widetilde{w}_t''}{((1+|\widetilde{w}_t'|^2)^{\frac{1}{2}})}| \leq C(N_0).
\end{aligned}
$$

in view of the bound from below on $\widetilde{w}_t$ and the bounds from above on the first two derivatives. Therefore, v_t and $\frac{1}{\widetilde{w}_t}$ are each equicontinuous and bounded over K. Since (3.9) gives $\widetilde{w}_t''$ in terms of sums and products of $\widetilde{H}_t$, v_t, and $\frac{1}{\widetilde{w}_t}$, it is also bounded and equicontinuous in t, meaning the $\widetilde{w}_t$ are precompact in $C^{2,0}(K)$.

Pass to a subsequence in t with a $C^2_{\text{loc}}(\mathbb{R})$ limit $\widetilde{w}$ if neccessary. Since (3.8) uniformly approaches 0 over $[-\widetilde{a}_t, \widetilde{a}_t]$, $\widetilde{w}$, $\widetilde{w}$ must satisfy

$$
\widetilde{w}''(x) = \frac{(1+|\widetilde{w}'|^2)}{\widetilde{w}}(x).
$$

The only solution to this differential equation over $\mathbb{R}$ is the catenary

$$
\widetilde{w}(x) = \frac{1}{\gamma}\cosh(\gamma x)
$$

for some $\gamma > 0$. If not, there would exist a complete minimal surface of revolution in $\mathbb{R}^3$ which is not a catenoid. In this context, since convergence to $\widetilde{w}(x)$ is also pointwise, we must have

$$
\widetilde{w}(0) = 1,
$$

implying $\gamma = 1$ (the next theorem will not depend on the precise value of γ, but we scale so that $\gamma = 1$ for the sake of simplicity). □

We can now derive a contradiction using the estimate (3.5).

Theorem 3.4.2. *Let N_0 be an $H > 0$ rotationally symmetric embedded torus, and $\{N_t\}_{0\leq t<T_{\max}}$ the corresponding solution to (1.6). Then $\lim_{t\to T_{\max}} u_{\min}(t) > 0$, and hence*

$$
\lim_{t\to T_{\max}} \max_{N_t} |A| \leq L < +\infty
$$

Proof. Suppose $\lim_{t\to T_{\max}} u_{\min}(t) = 0$. According to the previous lemma, the functions $\widetilde{w}_t$ converge in $C^2_{\text{loc}}(\mathbb{R})$ to $\widetilde{w}(x) = \cosh(x)$. On the one hand, according to the Gauss curvature bound (3.5) which remains invariant under scaling, we should have

$$
\begin{aligned}
\int_{\widetilde{S}'_t} |K| d\mu &= 2\pi \int_{-\widetilde{a}_t}^{\widetilde{a}_t} \Big(\frac{\widetilde{w}''_t}{(1+|\widetilde{w}'_t|^2)^{\frac{3}{2}}}\Big)\Big(\frac{1}{\widetilde{w}_t(1+|\widetilde{w}'_t|^2)^{\frac{1}{2}}}\Big)\widetilde{w}_t(1+|\widetilde{w}'_t|^2)^{\frac{1}{2}} dx \qquad (3.10)\\
&= 2\pi \int_{-\widetilde{a}_t}^{\widetilde{a}_t} \frac{\widetilde{w}''_t}{(1+|\widetilde{w}'_t|^2)^{\frac{3}{2}}} dx \leq 4\pi(1-\epsilon)
\end{aligned}
$$

for some $\epsilon = \epsilon(N_0) > 0$. On the other hand, we may readily compute for $\widetilde{w}(x) = \cosh(x)$ that

$$
2\pi \int_{\mathbb{R}} \frac{\widetilde{w}''}{(1+|\widetilde{w}'|^2)^{\frac{3}{2}}} dx = 2\pi \int_{-\infty}^{\infty} \frac{\cosh(x)}{\cosh^3(x)} dx = 2\pi[\tanh(x)]_{\infty}^{\infty} = 4\pi.
$$

Fix a large enough interval $[-x_0, x_0]$ so that

$$
2\pi \int_{-x_0}^{x_0} \frac{\widetilde{w}''}{(1+|\widetilde{w}'|^2)^{\frac{3}{2}}} dx > 4\pi(1-\epsilon),
$$

for ϵ is as in (3.10). Then we would have $\frac{\widetilde{w}''_t}{(1+|\widetilde{w}'_t|^2)^{\frac{3}{2}}} \to \frac{\widetilde{w}''}{(1+|\widetilde{w}'|^2)^{\frac{3}{2}}}$ in $C^0([-x_0, x_0])$ as $t \to T_{\max}$ but not in $L^1([-x_0, x_0])$. This is a contradiction, so we cannot have the limit of $u_{\min}(t)$ equal to 0. Then we also know for the original surface N_t that

$$
\begin{aligned}
\lim_{t\to T_{\max}} \max_{N_t} p &= \lim_{t\to T_{\max}} \frac{1}{u_{\min}(t)} < +\infty,\\
\lim_{t\to T_{\max}} \max_{N_t} k &\leq \lim_{t\to T_{\max}} \max_{N_t} H + \lim_{t\to T_{\max}} \max_{N_t} p < +\infty.
\end{aligned}
$$

□

A uniform-in-time bound on total curvature leads us to expect the existence of a smooth limit surface at the singular time without rescaling. Establishing this is actually nontrivial specifically in the context of IMCF, as we will explain in the next section.

3.5. Convergence at $T_{\max}$

Typically for extrinsic geometric flows, a uniform control on $\max_{N_t}|A|$ up to a time T would imply uniform controls on all higher derivatives of A as well as on the induced metric g_t of N_t. These together would imply a smooth, non-degenerate limit surface N_T at the time T. In the case of the rotationally symmetric expanding torus, the question of convergence at time $T = T_{\max}$ is more delicate: according to [**Smo00**] and Corollary 2.3 in [**HI08**], singularities of IMCF are always characterized by the mean curvature H of N_t degnerating to 0 somewhere. Therefore, although the total curvature $|A|$ remains bounded near $T_{\max}$ for this family of solutions, the flow speed $\frac{1}{H}$ is neccessarily blowing up. Since $\frac{1}{H}$ appears in the reaction terms of evolution equations for various geometric quantities, one cannot immediately establish C^∞ convergence at the time $T_{\max}$ via a maximum principle.

We can, however, at least obtain C^1 convergence of the embeddings $\widetilde{F}_t$ composed with appropriate diffeomorphisms to a map $\widetilde{F}_{T_{\max}}$ via the bound on $|A|$. Furthermore, according to a result from [**Lan85**], the induced metric g_t cannot degenerate over as $t \to T_{\max}$ for any singular solution $\{N_t\}_{0 \leq t < T_{\max}}$ to (1.6) for which $\lim_{t \to T_{\max}} \max_{N_t}|A| < +\infty$.

Proposition 3.5.1. *Let $F : N \times [0, T) \to \mathbb{R}^3$ be a solution to (1.6) such that $T < +\infty$ and $\sup_{N \times [0,T)}|A| \leq L < +\infty$. Then for a subsequence of times $t_k \nearrow T$ and diffeomorphisms $\alpha_k : N \to N$, the immersions $\widetilde{F}_{t_k} = F_{t_k} \circ \alpha_k$ converge in C^1 topology to an immersion $\widetilde{F}_{T_{\max}}$ as $t \to T_{\max}$.*

Proof. An elementary computation shows that the area under IMCF satisfies

$$\frac{d}{dt}|N_t| = \int_{N_t} \langle \frac{\partial N_t}{\partial t}, H\nu \rangle d\mu = \int_{N_t} d\mu = |N_t|,$$

meaning

$$|N_t| = e^t |N_0|. \tag{3.11}$$

Therefore, the L^p norm of A for $p < +\infty$ is also uniformly bounded in time

$$\|A\|_p = (\int_{N_t} |A|^p d\mu)^{\frac{1}{p}} \leq L^{\frac{1}{p}} e^{\frac{T}{p}} N_0.$$

According to the Compactness Theorem due to Langer from [**Lan85**], the maps $\widetilde{F}_{t_k} = F_{t_k} \circ \alpha_k$ converge in C^1 topology to an immersion $\widetilde{F}_{T_{\max}}$ as $t \to T_{\max}$.

$\square$

In the context of the rotationally symmetric embedded torus, we can deduce that the limit surface should also be embedded.

Proof of Theorem 3.1.1. Consider the generating curve ρ_t of N_t. According to Proposition 3.2.1, for every $t \in [0, T_{\max})$ ρ_t is the union of two disjoint graphs for two functions w_t and v_t which are each respectively monotone in time. Then repeating the same argument as in the proof of Proposition 3.2.1, $w_t \searrow w_{T_{\max}}$ and $v_t \nearrow v_{T_{\max}}$ in $C^0_{\text{loc}}(a_{T_{\max}}, b_{T_{\max}})$. Also by this argument, $w_{T_{\max}}(x) < v_{T_{\max}}(x)$ for $x \in (a_{T_{\max}}, b_{T_{\max}})$, and $\text{graph}(w_{T_{\max}}) \cup \text{graph}(v_{T_{\max}})$ is a closed curve.

By the previous proposition, ρ_t also converges to an immersed, closed C^1 curve $\rho_{T_{\max}}$ as $t \to T_{\max}$, so by uniqueness of limits $\rho_{T_{\max}}$ is embedded. Hence $\rho_{T_{\max}}$ generates an embedded torus in $\mathbb{R}^3$.

$\square$

3.6. The General Case

The natural question that this result raises is whether or not $|A|$ remains bounded and a limit surface will always exist at any singularity of (1.6) without rescaling. Though we cannot currently provide a full answer to this question, Theorem 3.1.2 provides some evidence for a positive answer to both of these questions for solutions of IMCF in $\mathbb{R}^3$. Gauss-Bonnet is a key tool to obtain this energy estimate as it was in Corollary 3.3.2, and the other is a simple L^2 energy estimate on the mean curvature.

Lemma 3.6.1. *Let $\{N_t\}_{0 \le t < T}$ be a solution to (1.6) in $\mathbb{R}^3$. Then $\int_{N_t} H^2 d\mu \le \sup_{N_0} H^2 |N_0|$.*

Proof. From [**HI01**], the evolution equation for H under (1.6) is

$$(\partial_t - \frac{1}{H^2}\Delta)H = -\frac{|A|^2}{H} - 2\frac{|\nabla H|^2}{H^3}.$$

Therefore, for any n we have for the function $f(x,t) = e^{\frac{t}{n}} H(x,t)$ we have that

$$(\partial_t - \frac{1}{H^2}\Delta)f = \frac{1}{n}f - \frac{|A|^2}{H^2}f \leq 0,$$

at any spacetime interior maximum of f in $N \times [0, T)$, implying

$$H(x,t) \leq e^{-\frac{t}{n}} \sup_{N_0} H \tag{3.12}$$

by the parabolic maximum principle. Recalling equation (3.11) we also know $|N_t| = e^t |N_0|$. Then combining this with (3.12) for the case $n = 2$ yields

$$\int_{N_t} H^2 d\mu \leq (\sup_{N_0} H^2)|N_0|.$$

□

Proof of Theorem 3.1.2. At any point $x \in N_t$ where the Gauss curvature K of N_t is positive, the two principal curvatures $\lambda_1(x)$ and $\lambda_2(x)$ of N_t are also positive, since $\lambda_1(x) + \lambda_2(x) = H > 0$ as long as the solution exists. In this case, we know that at these points

$$\lambda_1(x) \leq H(x) \leq e^{\frac{-t}{2}} \sup_{N_0} H$$

in view of (3.12), and likewise for $\lambda_2(x)$. This yields

$$K \leq e^{-t} \sup_{N_0} H^2. \tag{3.13}$$

Now write $K = K_- + K_+$, where $K_- = \min\{K, 0\}$ and $K_+ = \max\{K, 0\}$. According to the above estimate, the area formula (3.11), and Gauss-Bonnet, we have

$$\int_{N_t} K_- d\mu = 2\pi\chi(N_0) - \int_{N_t} K_+ d\mu \geq 2\pi\chi(N) - \sup_{N_0} H^2 |N_0|.$$

We remark that since the Gauss-Bonnet Theorem is intrinsic, see [**Che44**], this estimate applies to immersed solutions rather than exclusively to embedded ones. Finally, in $n = 2$ we can write

$$\int_{N_t} |A|^2 d\mu = \int_{N_t} H^2 d\mu - 2\int_{N_t} K d\mu.$$

The result follows in view of the previous lemma. □

Remark 3.6.2. *Whether $\|A\|_{L^1} < +\infty$ for any solution of MCF in $\mathbb{R}^3$ is still unknown, see* [**Du20**], [**Hea13**], [**GH17**]. *Therefore,* (3.1) *indicates better regularity in general near singularities for IMCF.*

This simple estimate suggests that, at the very least, there is likely no metric degeneration for any singular solution of IMCF in $\mathbb{R}^3$. Indeed, since $n = 2$ this estimate corresponds to the borderline case for the Sobolev inequality, so an upgrading this to some L^p estimate for some $p > 2$ would allow one to conclude the existence of a limit immersion modulo diffeomorphisms by Langer's Compactness Theorem from [**Lan85**].

The estimate also hints that the procedure in section 4 may generalize to establish that $|A|$ may always remain bounded in L^∞ norm. If $\lim_{t \to T_{\max}} \max_{x \in N_t} |A|(x) = +\infty$ then one would expect a sequence of times $t_i \to T_{\max}$ and corresponding scale factors $\lambda_i \to +\infty$ such that the surfaces $\widetilde{N}_{t_i} = \lambda_i N_{t_i}$ converge in C^2 to some non-compact limit $\widetilde{N}_{T_{\max}}$. Since (3.1) is scale-invariant like with (3.5), the estimate would also apply to this limit, and due to the upper bound on $\max_{N_t} H$ the limit $\widetilde{N}_{T_{\max}}$ must be minimal.

Ruling out all candidate blowup limits of $\tilde{N}_t$ for scale factors tending do infinity may be possible, but this would require a careful treatment of convergence and a sharper L^2 estimate than (3.1) (for instance, we needed (3.5) to show the catenoid cannot arise as a limit of the rotationally symmetric torus). This does, however, lead us to the following conjecture.

Conjecture 3.6.3. *Let $\{N_t\}_{0 \le t < T_{\max}}$ be a solution to (1.6) in $\mathbb{R}^3$. Then*

$$\max_{N \times [0, T_{\max})} |A|(x,t) < +\infty.$$

CHAPTER 4

The Flow of Rotationally Symmetric Hypersurfaces into Spheres

Those who flow as life flows know they need no other force.

-Lao-Tzu

4.1. The Main Theorem

Gerhardt's Convergence Theorem 1.4.3 for star-shaped hypersurfaces in $\mathbb{R}^{n+1}$ is interesting because it establishes dynamical stability of the round sphere under IMCF for a very large family of initial perturbations. In fact, star-shaped hypersurfaces evolved by Mean Curvature Flow are not known to always flow into spheres, so in this regard the round sphere is likely more dynamically stable under IMCF than it is under MCF. This is surprising since the system of partial differential equations associated with IMCF is fully nonlinear, and the evolution equations for the important geometric quantities contain reaction terms that are much stronger than even the ones encountered in MCF.

Although many authors have generalized this convergence theorem to other ambient Riemannian manifolds, see [**Ger11**], [**Lu17**], [**CLZ17**], [**Zho16**], [**Wei18**], and [**Ger15**], to our knowledge there has been no previous research on the dynamical stability of non-star-shaped hypersurfaces in $\mathbb{R}^{n+1}$ under IMCF. This question shall be the focus of this chapter. Theorem 2.1.1 aids us greatly here, for it establishes that existence and embeddedness of a solution for some minimal time are sufficient to guarantee global existence and asymptotic roundness at large times.

A natural setting to consider dynamical stability in other than star-shaped perturbations of a sphere is perturbations of a sphere which maintain rotational symmetry about some axis. The Mean Curvature Flow of rotationally symmetric surfaces such as a dumbbell have been studied at least as far back as [**Gra89**], see also see [**AAG95**], [**AV97**], and [**GKS13**]. Surfaces like these serve as models for several types of non-trivial singularities of MCF such as the "neckpinch", see Figure 4.1.

Neckpinching in the Mean Curvature Flow

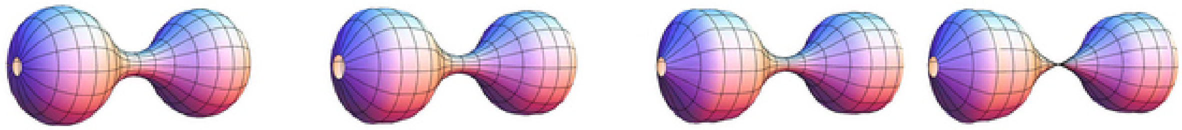


FIGURE 4.1. Given a rotationally symmetric initial surface with a sufficiently thin neck, the flow speed at the neck under MCF will exceed the flow speed at the caps. This may cause the neck to "pinch" at a point along the axis before the surface is able to contract to a round point. Image from [**AMC^{+}13**].

Do singularities also form in rotationally symmetric IMCF? Since a bound from below on mean curvature over a time interval is always sufficient to extend the flow in time, one asks whether saddle points or geodesic points can form near the necks of an evolving dumbbell surface. Once again, this question is difficult because of the strong nonlinearity of the underlying PDE. Unlike with star-shaped IMCF, there is no obvious quantity to combine with the flow speed H^{-1} so that the reaction terms in the corresponding evolution equation vanish. Controlling H^{-1} via maximum principles therefore requires much more care and some extra attention to the geometry of rotationally symmetric surfaces. Also, it is not obvious that embeddedness of the flow surface is preserved for a long-time rotationally symmetric solution to (1.6), and this is required to show asymptotic convergence to spheres.

In this chapter, we show global existence and asymptotic roundness for solutions to IMCF when the initial data N_0 is a rotationally symmetric surface that satisfies an additional curvature condition.

Theorem 4.1.1 (Dynamical Stability for Rotationally Symmetric Initial Data)**.** *Let N_0 be a C^2, $H > 0$ rotationally symmetric embedded sphere which is admissible in the sense of Definition* 4.3.2*, and $\{N_t\}_{0 \leq t < T_{\max}}$ the corresponding maximal solution to (1.6). Then $T_{\max} = +\infty$, and the rescaled surfaces $\tilde{N}_t = e^{-\frac{t}{n}} N_t$ converge in C^∞ topology to some round sphere $\mathbb{S}_R(x_0)$ as $t \to +\infty$.*

Once again, the convergence in the statement of this theorem is in the sense of the rescaled immersions $\tilde{F}_t = e^{\frac{-t}{n}} F_t$. Definition 4.3.2 contains an additional constraint on one of the principal curvatures of N_0. While we postpone explaining precisely this condition, it essentially controls the shape of the necks of a rotationally symmetric N_0 by requiring that these necks to have a certain minimum length and thickness. Admissible N_0 need not be star-shaped, and so Theorem 4.1.1 gives new dynamical stability results for IMCF. The question of whether or not IMCF flows any rotationally symmetric surface into a sphere remains open.

Key to our approach is separating the evolving surface N_t into two "cap" regions which intersect the axis of rotation and their complement that contains all necks of the surface. Doing so is necessary to effectively utilize the different geometric properties that these different regions exhibit. The unions of all caps and of all complements of caps form domains with potentially complicated parabolic boundaries in the spacetime $N \times [0, T)$. In order to apply maximum principles separately to each domain, we must modify the standard parabolic maximum principle to apply over so-called "non-cylindrical" spacetime domains. We further develop a non-cylindrical maximum principle from [**AK12**], [**HK19**], and [**Lum87**] in the context of IMCF.

The chapter is organized as follows: in Section 4.2, we further characterize rotationally symmetric N_0 and show that N_t remains embedded as long as the flow exists. In Section 4.3, we define the separate regions of N_t mentioned above. We then justify that one of these regions gives rise to a domain in $N \times [0, T)$ to which the Non-Cylindrical Maximum Principle is fully applies. It is also in this section that we present the additional curvature assumption on N_0 which will later prove crucial.

After deriving evolution equations and elementary estimates in Sections 4.4 and 4.5 respectively, we obtain an L^∞ estimate on H^{-1} in each of the regions of N_t mentioned above individually. First, in Section 4.6, we inspect the "bridge" region of the surface which lies away from the axis of rotation. We show that H^{-1} can also be controlled over this region via a sharp gradient estimate on the corresponding graph obtained using the curvature assumption. This allows us to in turn control H^{-1} over the cap regions which intersect the axis of rotation in Section 4.7. Altogether, this leads to global existence and convergence for N_t in Section 4.8.

Both sections of the Appendix of this dissertation pertain to this chapter. In Appendix A.1, we prove the Non-Cylindrical Maximum Principle for IMCF. We make note in Section 4.3 of an additional condition to the ones noted in [**AK12**], [**HK19**], and [**Lum87**] on a domain for the second part of the principle to apply, and so our proof is in accordance with this condition. In

Appendix A.2, we construct a non-star-shaped, rotationally symmetric N_0 which is admissible in the sense of Definition 4.3.2 and must therefore flow into a sphere under IMCF.

4.2. Preserving Embeddedness

A hypersurface $N_0 \subset \mathbb{R}^{n+1}$ is rotationally symmetric without loss of generality with respect to the x_1 axis if it is obtained by revolving a generating curve $\rho : [0, \ell] \to \mathbb{R}^2$ in the (x_1, x_2) plane about this axis. The ambient vector fields $\vec{e_1} = (1, 0, \ldots, 0)$ and $\hat{w} = \frac{\vec{x}}{|\vec{x}|}$, where $\vec{x} = (0, x_2, \ldots, x_{n+1}))$, defined over $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1} \setminus \{x_2 = x_3 = \cdots = x_{n+1} = 0\}$ respectively allow us to extract information about such an N_0 which will be useful for estimating geometric quantities along the flow. The "height" u of a point $x \in N_0$ which does not lie on the x_1 axis (where $u = 0$) is defined as

$$u(x) = \langle \hat{w}, \vec{F}(x) \rangle, \tag{4.1}$$

where $\vec{F}$ is the position vector in $\mathbb{R}^{n+1}$ of x. One of the principal curvatures at a point of N_0 will always be equal to the curvature k of the generating curve in the plane at the corresponding point, while the $(n-1)$ others corresponding to rotation are all equal. We call this other curvature p, which is given by

$$p = \langle \hat{w}, \nu \rangle u^{-1}. \tag{4.2}$$

The mean curvature H of N_0 must then equal $(n-1)p + k$. In this paper, we examine a rotationally symmetric N_0 which is a C^2 embedding of $\mathbb{S}^n$ and has positive mean curvature everywhere. We would like to consider the evolution N_t of this N_0 by (1.6). Recall that one of the difficulties in studying the evolution of a hypersurface by IMCF is that, unlike MCF, F_t is not necessarily an embedding for as long as the flow exists, even if F_0 is an embedding. For example, evolving two disjoint spheres by IMCF will eventually result in an intersection as the spheres expand. Our first objective will be to rule out the possibility of self-intersections in this setting, and we begin by obtaining a clearly profile on N_0.

Proposition 4.2.1. *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$, rotationally symmetric embedded sphere. Then for $x \in N_0$ with $u(x) \neq 0$, we have $\langle \hat{w}, \nu \rangle (x) > 0$.*

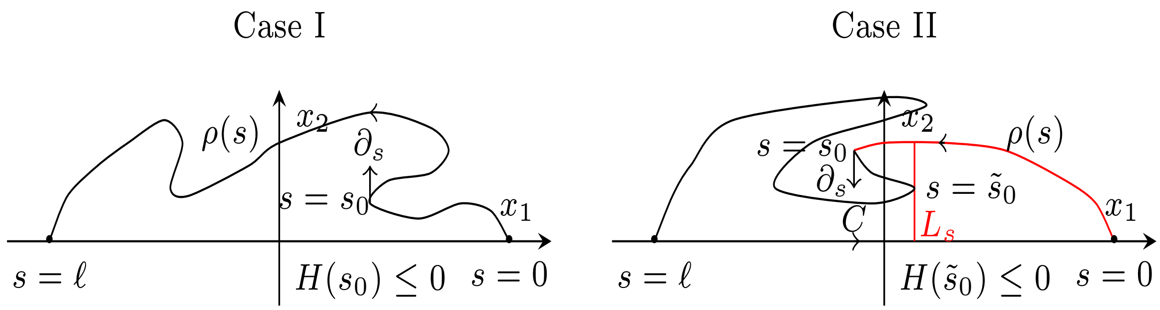


FIGURE 4.2. If $\langle e_2, \nu\rangle = 0$ at some interior point of ρ, one can locate a point where $\langle e_2, \nu\rangle = 0$ and $k \leq 0$.

PROOF. W.L.O.G. taking the axis of rotation to be the x_1 axis, we consider the generating curve $\rho : [0, \ell] \to \mathbb{R}$ parametrized counterclockwise with respect to arc length in the (x_1, x_2) plane. Write $\rho(s) = (\rho_1(s), \rho_2(s))$, and let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. We have $\rho_2(0) = \rho_2(\ell) = 0$, and, in view of embeddedness of N_0, that $\rho_1(\ell) < \rho_1(0)$ and $\rho_2(s) > 0$ for $s \in (0, \ell)$. Furthermore, for N_0 to be C^2, we require $\frac{d}{ds}\rho_1(0) = \frac{d}{ds}\rho_1(\ell) = 0$ (The derivatives at 0 and ℓ are taken to be one-sided), and the mean convexity of N_0 is equivalent to the sum $k(s) + (n-1)p(s)$ being positive for each $s \in [0, \ell]$, where $k(s)$ is the curvature of the plane curve, and $p(s) = \langle \nu, e_2\rangle(s)(\rho_2(s))^{-1}$ for outward normal ν in the plane. The quantity $p(s)$ may be continuously extended to $s = 0, \ell$ by the C^2 condition. We must show that this positivity implies $\langle e_2, \nu\rangle(s) > 0$ for any $s \in (0, \ell)$.

First, observe that for small $\epsilon > 0$, $\langle e_2, \nu\rangle(s) > 0$ over $(0, \epsilon)$. Indeed, if for a sequence $s_n \to 0$ we had $\langle e_2, \nu\rangle(s_n) \leq 0$ then $\lim_{n\to\infty}\langle \nu, e_2\rangle(s_n)(\rho_2(s_n))^{-1} \leq 0$. Now, at any point s_0 with $\langle e_2, \nu\rangle(s_0) = 0$ we have

$$\frac{d}{ds}\langle \nu, e_2\rangle(s_0) = \langle \nabla_s e_2, \nu\rangle + \langle e_2, \nabla_s \nu\rangle = \langle e_2, \nabla_s \nu\rangle(s_0) = \pm\langle \partial_s, \nabla_s \nu\rangle = \pm k(s_0). \tag{4.3}$$

At $s = 0$, we have $e_2 = \partial_s$ and hence $\frac{d}{ds}\langle \nu, e_2\rangle(0) = k(0)$. If $\langle e_2, \nu\rangle(s_n) \leq 0$ and $\langle e_2, \nu\rangle(0) = 0$, then $\frac{d}{ds}\langle e_2, \nu\rangle(0) \leq 0$ and hence $k(0) \leq 0$ as well, contradicting the $H > 0$ condition. So $\langle e_2, \nu\rangle(s) > 0$ over some interval $(0, \epsilon)$. Let s_0 be the first point in $(0, \ell)$ with $\langle e_2, \nu\rangle = 0$. Then $\frac{d}{ds}\langle e_2, \nu\rangle \leq 0$ and $\partial_s = \pm e_2$. We examine these cases separately.

Case I, $\partial_s(s_0) = +e_2$: We have from (4.3) that $\frac{d}{ds}\langle e_2, \nu\rangle = +k(s_0)$, so $k(s_0) \leq 0$. Since $p(s_0) = 0$, this contradicts the $H > 0$ condition.

Case II, $\partial_s(s_0) = -e_2$: If $\frac{d}{ds}\langle e_2, \nu\rangle(s_0) = 0$, then $k(s_0) + (n-1)p(s_0) = 0$ and we are done. So assume $\frac{d}{ds}\langle e_2, \nu\rangle(s_0) < 0$. Note then that we also have

$$\begin{aligned} \frac{d}{ds}\rho_1(s_0) &= 0, \\ \frac{d}{ds}\rho_2(s_0) &< 0, \\ \frac{d^2}{ds^2}\rho_1(s_0) &> 0. \end{aligned}$$

Then there is some $\tilde{s} \in (s_0, \ell)$ with $\rho_1(\tilde{s}) > \rho_1(s_0)$, $\rho_2(\tilde{s}) < \rho_2(s_0)$. Now, calling $\rho_1(0) = a$, $\rho_1(\ell) = b$, take the C^0 piecewise closed curve $\tilde{\rho} : [0, \ell + (a-b)]$ defined by

$$\tilde{\rho}(s) = \begin{cases} \rho(s) & s \in (0, \ell) \\ \left(\frac{a}{a-b}(s-\ell) + \frac{b}{a-b}((a-b)-(s-\ell)), 0\right) & s \in [\ell, \ell + (a-b)]. \end{cases} \tag{4.4}$$

Since N_0 is embedded, ρ is simple, so $\tilde{\rho}$ is also simple and hence separates $\mathbb{R}^2$ into disjoint connected subsets. Let C be the open set enclosed by $\tilde{\rho}$. Now, for each $s \in (0, s_0)$, define the line segment L_s to be the set

$$L_s = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = \rho_1(s), 0 < x_2 < \rho_2(s)\}.$$

Note $L_s \subset C$ for small s. Furthermore, since $\langle e_2, \nu\rangle > 0$ for $s \in (0, s_0)$ the image $\rho((0, s_0))$ is graphical over x_1, and so $L(s)$ does not intersect $\rho((0, s_0)$ for $s \in (0, s_0)$, but must intersect the other part of ρ since $\rho_1(\tilde{s}) > \rho_1(s_0)$, $\rho_2(\tilde{s}) < \rho_2(s_0)$ for some $\tilde{s} \in (s_0, \ell]$, see Figure 4.2. Let $\tilde{s}_0$ be the smallest s value where L_s intersects the other part of the curve ρ. We must have that $L_{\tilde{s}_0} \subset \overline{C}$, meaning that $k \leq 0$ wherever ρ intersects this segment. But $\langle e_2, \nu\rangle = 0$ at these intersection points as well, so we have $k(s) + (n-1)p(s) \leq 0$ there. Once again, this is a contradiction, so we conclude $k(s) > 0$ along interior points of ρ. $\square$

Since (1.6) preserves rotational symmetry, we may identify the evolving surface N_t with a generating curve ρ_t. As we know ρ_0 is graphical, we can demonstrate that this remains true for ρ_t, guaranteeing N_t is embedded for $t \in [0, T_{\max})$.

Theorem 4.2.2 (Preserving Embeddedness)**.** *Let N_0 be a C^2, $H > 0$, rotationally symmetric embedded sphere, and $\{N_t\}_{t\in[0,T_{\max})}$ the corresponding maximal solution to (1.6). Then for any $(x,t) \in \mathbb{S}^n \times [0,T)$ with $u(x,t) > 0$, we have $\langle \nu, \hat{w}\rangle(x,t) > 0$. In particular, the N_t are embedded hypersurfaces with $p(x,t) > 0$ whenever $u(x,t) \neq 0$.*

Proof. We proceed by contradiction: suppose $\langle \nu, \hat{w}\rangle(x,t) = 0$ and some point $u(x,t) > 0$. According to Proposition 4.2.1, this cannot be true at $t = 0$, so let $t_0 \in [0, T_{\max})$ be the first time at which this happens, and take the generating curve ρ_{t_0} of N_{t_0} in the (x_1, x_2) plane.

We know $\langle e_2, \nu\rangle(s_0) = 0$ for some interior point $s_0 \in (0, \ell_{t_0})$ of ρ_{t_0}, and that $\langle e_2, \nu\rangle \geq 0$ over ρ_{t_0}. Thus this quantity attains a minimum at s_0, and so $\frac{d}{ds}\langle e_2, \nu\rangle(s_0) = 0$. But from (4.3), this means $k(s_0) = 0$, as well as $p(s_0) = 0$, which contradicts H being positive over N_{t_0}. Thus for any $t \in [0,T)$, $\langle \hat{w}, \nu\rangle(x,t) > 0$ away from the axis of rotation. In particular, the generating curve is a graph over the x_1 coordinate and hence is embedded. □

4.3. Non-Cylindrical Spacetime Domains

The key to the regularity theory for IMCF is a lower bound on the mean curvature H: estimating H from below uniformly over any given finite time interval will guarantee long-time existence and smoothness, see Theorem 2.2 in [**HI08**]. After obtaining such a bound, we will turn our attention to the question of convergence. In Theorem 2.1.1, we showed that a long-time embedded solution to (1.6) becomes star-shaped by the time $2t_* = 2n\log(R^{-1}\mathrm{diam}(N_0))$, where R is the radius of the largest ball enclosed by N_0. Therefore, with Theorem 4.2.2 in hand, we will employ a strategy similar to Grayson in [**Gra87**]: after some prescribed time, the flow surfaces become star-shaped, meaning a continuation argument allows us to apply the results of [**Ger90**], [**HI08**], and [**Urb90**] to obtain more precise estimates as well as rapid convergence to spheres at large times.

On the question of global existence, we will utilize maximum principle techniques to bound the flow speed over $N \times [0,T)$. However, in order to effectively apply these to a rotationally symmetric

$H > 0$ embedded sphere, we will need to impose an additional condition on the principal curvature p in a certain part of the initial surface, and so we first identify different subsets of N_0.

Definition 4.3.1. *Given a C^2, rotationally symmetric, $H > 0$ embedded sphere $N_0 \subset \mathbb{R}^{n+1}$, the* ***right cap*** *$C_0^+$ of N_0 is the connected component of the set $\{x \in N_0 | \langle \nu, e_1 \rangle \geq 0\}$ which intersects the x_1 axis. The* ***left cap*** *C_0^- of N_0 is the connected component of $\{x \in N_0 | \langle \nu, e_1 \rangle \leq 0\}$ which intersects this axis. The* ***bridge*** *L_0 of N_0 is $N_0 \setminus (C_0^+ \cup C_0^-)$.*

Different Regions of N_0

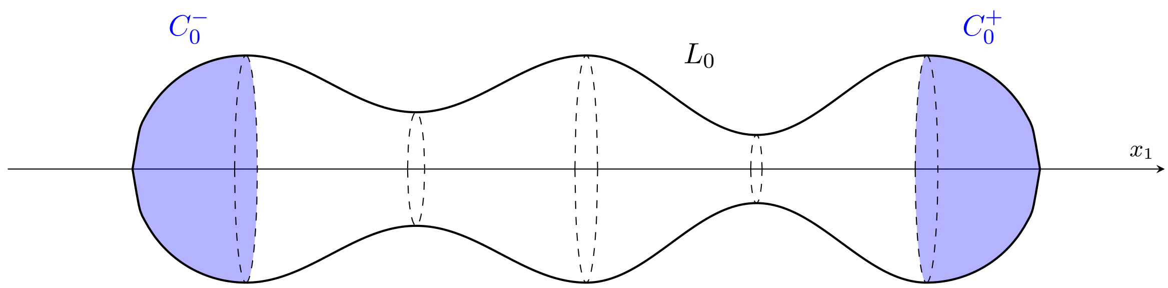


FIGURE 4.3. The caps $C_0^{\pm}$ and bridge L_0 of a rotationally symmetric N_0.

We note that $L_0 = \varnothing$ precisely when the surface contains no "necks", that is, whenever for the generating graph $y_0 : (a, b) \to \mathbb{R}^+$ there is an $x_0 \in (a, b)$ so that y_0 is monotone over (a, x_0) and (x_0, b) respectively. One can easily verify in this case that N_0 will be star-shaped with respect to the point $(x_0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, guaranteeing global existence for the corresponding N_t. Thus we expect the presence of necks on N_0 to be the culprit of possible singularities. If $L_0 \neq \varnothing$, we must impose the additional condition that the ratio of the highest and lowest values of p over $\overline{L}_0$ is no larger than $n^{\frac{n}{2(n-1)}}$ in order to control the mean curvature of the evolving hypersurface.

Definition 4.3.2. *An $H > 0$, C^2, rotationally symmetric embedded sphere $N_0 \subset \mathbb{R}^{n+1}$ is* ***admissible*** *if the principal curvature p of rotation satisfies*

$$\frac{\max_{\overline{L}_0} p}{\min_{\overline{L}_0} p} < n^{\frac{n}{2(n-1)}}. \tag{4.5}$$

Although many of the results throughout this chapter apply more generally to any $C^2, H > 0$, rotationally symmetric embedded sphere, this additional curvature assumption is needed to obtain control on the relevant geometric quantities over the bridge L_t of N_t. Condition (4.5) is equivalent to the requirement that $\max_{\overline{L}_0} uv < n^{\frac{n}{2(n-1)}} \min_{\overline{L}_0} u$, where $v = (\langle \hat{w}, \nu \rangle)^{-1}$. If one takes the generating graph $y_0 : (a,b) \to \mathbb{R}^+$ of N_0, one can verify $v = (1 + |y_0'(x)|^2)^{\frac{1}{2}}$. Thus the additional curvature condition places constraints on the shape of the neck regions of N_0: for example, it implies the ratio of the maximum value of y_0 to its smallest interior minimum is no greater than $n^{\frac{n}{2(n-1)}}$. Conversely, given any $H > 0$ rotationally symmetric surface for which this ratio is smaller than $n^{\frac{n}{2(n-1)}}$, dilating in the x_1 direction by a sufficiently large factor will produce an admissible initial surface. Since star-shaped N_0 are already known to flow to spheres from the results in [**Ger90**], we demonstrate in Appendix A.2 the existence of an admissible N_0 which is not star-shaped.

The solution to (1.6) for a rotationally symmetric sphere N_0 is a one-parameter family of embeddings $F : \mathbb{S}^n \times [0,T) \to \mathbb{R}^{n+1}$. Define the open subsets $C^+ \subset \mathbb{S}^n \times [0,T)$, $C^- \subset \mathbb{S}^n \times [0,T)$, and $L \subset \mathbb{S}^n \times [0,T)$ by

$$\begin{aligned} C^+ &= \mathrm{Int}(\cup_{t \in (0,T)} C_t^+ \times \{t\}), & (4.6)\\ C^- &= \mathrm{Int}(\cup_{t \in (0,T)} C_t^- \times \{t\}), & (4.7)\\ L &= \mathrm{Int}(\cup_{t \in (0,T)} L_t \times \{t\}). & (4.8) \end{aligned}$$

Here $C_t^\pm$ and L_t are the caps and bridges of Definition 4.3.1 for $N_t = F_t(\mathbb{S}^n)$ and $\overline{C_t^\pm}$, $\overline{L_t}$ their closures in N_t. We will apply maximum principles to each of these domains separately, which introduces a boundary to the problem. Moreover, these domains may be non-cylindrical.

Definition 4.3.3. *For a closed manifold N and an open domain $U \subset N \times (0,T)$, let $U_t = U \cap (N \times \{t\})$ for $t \in (0,T)$, $U_0 = \overline{U} \cap (N \times \{0\})$, and $U_T = \overline{U} \cap (N \times \{T\})$. The* ***parabolic boundary*** *$\partial_P U$ of U is $\partial_P U = \partial U \setminus U_T$, where ∂U is the topological boundary of U in $N \times [0,T)$.*

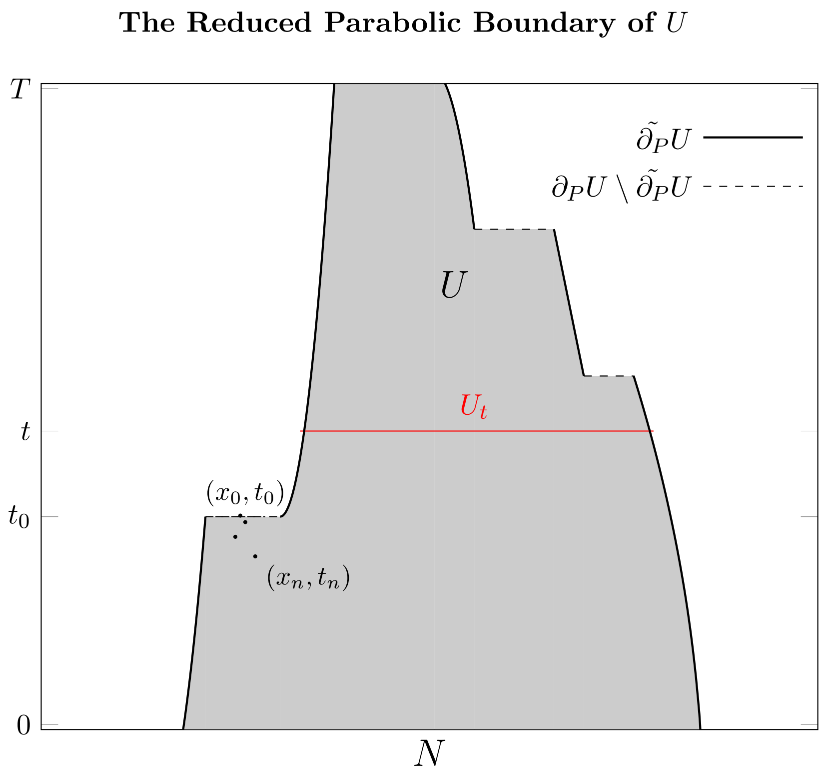


FIGURE 4.4. For a non-cylindrical domain $U \subset N \times [0, T)$, the reduced parabolic boundary $\tilde{\partial}_P U$ does not include the dotted parts of $\partial_P U$. If every point in $\partial_P U \setminus \tilde{\partial}_P U$ can be approached in U from below in time, $\sup_U f$ cannot occur at any of these points.

The ***reduced parabolic boundary*** $\tilde{\partial}_P U$ *of* U *is* $\tilde{\partial}_P U = U_0 \cup (\cup_{0 \le t < T} \partial U_t)$, *where* ∂U_t *is the topological boundary of* U_t *in* $N \times \{t\}$.

In general, $\tilde{\partial}_P U \neq \partial_P U$, see Figure 4.4. In this setting, the domains (4.6)-(4.8) may be non-cylindrical if the generating graph of the evolving surface gains or loses maxima and minima over time. With this possibility in mind, we employ a slightly modified version of the maximum principle over such a domain detailed in [**AK12**], see also [**Lum87**] and [**HK19**]. We also include an additional requirement on a non-cylindrical domain $U \subset N \times [0, T)$ for the second part of this theorem to apply.

Theorem 4.3.4 (Non-Cylindrical Maximum Principle)**.** *Let* $\{N_t\}_{t \in (0,T)}$ *be a solution of the Inverse Mean Curvature Flow (1.6), where* $N_t = F_t(N)$ *for a one-parameter family of embeddings* F_t *over* *a closed manifold N. For $U \subset N \times [0,T)$ and $f \in C^{2,1}(U) \cap C(\overline{U})$, suppose for a smooth vector field η over U we have*

$$(\partial_t - \frac{1}{H^2}\Delta)f \leq \langle \eta, \nabla f \rangle.$$

(Resp. $\geq$ at a minimum) Here Δ and ∇ are the Laplacian and gradient operators over N_t, respectively. Then

$$\sup_U f \leq \sup_{\partial_P U} f$$

(Resp. $\inf_U f \geq \inf_{\partial_P U} f$). Furthermore, suppose that f has a positive supremum over U and that for each $(x_0, t_0) \in \partial_P U \setminus \tilde{\partial}_P U$ there is a sequence of points $(x_n, t_n) \in U$ converging to (x_0, t_0) with $t_n < t_0$. Then

$$\sup_U f \leq \sup_{\tilde{\partial}_P U} f$$

(Resp. $\inf_U f \geq \inf_{\tilde{\partial}_P U} f$ for a positive minimum).

The additional assumption on domain geometry that a sequence from U approaches each point in $\partial_P U \setminus \tilde{\partial}_P U$ from below in time is needed because of the time asymmetry of parabolic equations. We include the proof in the appendix, inspired by a similar argument for the Volume Preserving Mean Curvature Flow in [**AK12**].

Not all of the domains (4.6)-(4.8) meet this extra requirement, but we claim that the domain L does. Roughly speaking, we must show that local maxima of the height u cannot form at interior points of the evolving cap to establish this. As a starting point to demonstrate this rigorously, we investigate the critical points of the generating graph y_t of N_t. In particular, parabolic theory allows us to show that the number of critical points of y_t is finite for $t > 0$.

Proposition 4.3.5. *Let N_0 be an $H > 0$, rotationally symmetric embedded sphere, and $\{N_t\}_{0 \leq t < T_{\max}}$ the corresponding maximal solution to (1.6). Then for any $t > 0$, there are only finitely many values of $a \in \mathbb{R}$ such that $\langle \nu, e_1 \rangle = 0$ over $N_t \cap \{x_1 = a\}$.*

Proof. This statement is equivalent to the generating graph y_t of N_t having only finitely many critical points for $t > 0$. Following Lemma 4.7 of [**AAG95**], see also [**Ang91**], we use the Sturmian

Theorem. The classical version of the Sturmian Theorem states that the number of zeroes of a solution $w : [x_1, x_2] \times [0, T) \subset \mathbb{R}^2 \to \mathbb{R}$ to a linear parabolic equation of the form

$$\partial_t w = a(x,t)\partial_{xx} w + b(x,t)\partial_x w + c(x,t)w$$

is finite and nonincreasing for $t > 0$, provided $y(x_1,t), y(x_2,t) \neq 0$. We apply the Sturmian Theorem to the derivative of generating graph $y_t : (c(t), d(t)) \to \mathbb{R}$, where $c(t) = \inf_{x\in N_t} x_1$ and $d(t) = \sup_{x\in N_t} x_1$. y_t itself satisfies the partial differential equation

$$\partial_t y = \frac{v}{H} \tag{4.9}$$

where $v = (1 + |y_t'|^2)^{\frac{1}{2}}$. Indeed, the normal vector to the graph of a function y in $\mathbb{R}^2$ is

$$\nu(x_1) = \frac{1}{v}(-y'(x_1)\partial_{x_1} + \partial_{x_2}).$$

So the normal component of the velocity vector $\frac{v}{H}\partial_{x_2}$ is

$$\langle \nu, \frac{v}{H}\partial_{x_2} \rangle = \frac{1}{H}.$$

Now, we differentiate (4.9) with respect to x_1 coordinate. Using that the surface generated by the graph of y has mean curvature

$$H(x_1,t) = \frac{n-1}{v y_t(x_1)} + \frac{y_t''(x_1)}{v(x_1)^3},$$

Similar to equation (4.7) in [**AAG95**], the function $w = y_t'$ must satisfy the linear parabolic equation

$$\partial_t w = \frac{1}{H^2 v^2} w_{xx} + (\frac{(n-1)y''}{vH} - \frac{3}{H^2 v^3}|y''|^2 + \frac{(n-1)}{y v^2 H^2} y'' + \frac{(n-1)}{y^2 H^2})w.$$

Now $y'(x,t)$ is defined over the domain $\cup_{t\in[0,T_{\max})}(c(t), d(t)) \times \{t\} \subset \mathbb{R}^2$. We know $\lim_{x\searrow c(t)} y_t'(x) = -\infty$ and $\lim_{x\nearrow d(t)} y_t'(x) = +\infty$. Then we can apply the Sturmian Theorem to conclude for $t > 0$ that $y_t(x) = 0$ for only finitely many $x \in (c(t), d(t))$. $\square$

Proposition 4.3.6. *Let N_0 be an $H > 0$, rotationally symmetric embedded sphere, and $\{N_t\}_{0 \le t < T_{\max}}$ the corresponding maximal solution to (1.6). Call $\partial C_t^- = \{x \in \mathbb{S}^n | \langle F_t(x), e_1 \rangle = a(t)\}$, resp. $\partial C_t^+ = \{x \in \mathbb{S}^n | \langle F_t(x), e_1 \rangle = b(t)\}$. Then for each $t_0 \in [0, T_{\max})$ we have*

$$\limsup_{t \nearrow t_0} a(t) \le \liminf_{t \searrow t_0} a(t), \tag{4.10}$$

resp. $\limsup_{t \searrow t_0} b(t) \le \liminf_{t \nearrow t_0} b(t)$.

Proof. We present the proof for $a(t)$, as the proof for $b(t)$ is identical. Call $a = \limsup_{t \nearrow t_0} a(t)$ and $\tilde{a} = \liminf_{t \searrow t_0} a(t)$. For any $x \in \mathbb{S}^n$ with $\langle F_{t_0}(x), e_1 \rangle < a$, there exists a sequence of points $(x_k, t_k) \in C_{t_k}^-$ with $t_k < t_0$ converging to (x, t_0) by the definition of a. Since $\langle \nu_{t_k}(x_k), e_1 \rangle \le 0$, we know $\langle \nu_{t_0}(x), e_1 \rangle \le 0$ by continuity. Thus $\langle \nu_{t_0}, e_1 \rangle \le 0$ over $N_{t_0} \cap \{x_1 < a\}$.

Now suppose $\tilde{a} < a$. By Proposition 4.3.5, there is some $c \in (\tilde{a}, a)$ with $\langle \nu_{t_0}, e_1 \rangle < 0$ over $N_{t_0} \cap \{x_1 = c\}$. We claim $\langle \nu_t, e_1 \rangle \le 0$ over $N_t \cap \{x_1 \le c\}$ for t chosen in some short-time interval $[t_0, t_0 + \epsilon)$. Clearly $\langle \nu_t, e_1 \rangle < 0$ over $N_{t_0} \cap \{x_1 < c\}$ by what we noted above and over $N_t \cap \{x_1 = c\}$ for some time interval $[t_0, t_0 + \epsilon)$ by continuity in time. We can use a maximum principle to prove the claim.

The evolution equation (4.13) for $\langle \nu, e_1 \rangle$ from Theorem 4.4.1 is

$$(\partial_t - \frac{1}{H^2} \Delta) \langle \nu, e_1 \rangle = \frac{|A|^2}{H^2} \langle \nu, e_1 \rangle. \tag{4.11}$$

The parabolic boundary of $\cup_{t \in [t_0, t_0 + \epsilon)} (N_t \cap \{x_1 < c\})$ is

$$\partial_P (\cup_{t \in [t_0, t_0 + \epsilon)} N_t \cap \{x_1 < c\}) = (N_{t_0} \cap \{x_1 < c\}) \cup (\cup_{t \in [t_0, t_0 + \epsilon)} N_t \cap \{x_1 = c\})$$

From what we noted above $\langle \nu_t, e_1 \rangle \le 0$ over this boundary, so in view of (4.11) and the first part of the non-cylindrical maximum principle this holds true throughout $\cup_{t \in [t_0, t_0 + \epsilon)} N_t \cap \{x_1 < c\}$.

But if $\tilde{a} < c$ there must exist a sequence of times $t_k \searrow t_0$ and points $x_k \in N_{t_k} \cap \{x_1 < c\}$ with $\langle \nu_t(x_k), e_1 \rangle > 0$ (recall that the derivative y'_{t_k} of the generating graph of N_{t_k} has a sign change at $a(t_k)$). This is a contradiction. Conclude then that $a \le \tilde{a}$. □

We are now ready show that L meets the domain geometry requirement of Theorem 4.3.4.

Theorem 4.3.7. *Let N_0 be a C^2, $H > 0$ rotationally symmetric embedded sphere, and $\{N_t\}_{0 \leq t < T_{\max}}$ the corresponding maximal solution to (1.6). Then for any $(x_0, t_0) \in \partial_P L \subset \mathbb{S}^n \times [0, T)$ with $t_0 > 0$, there is a sequence $(x_n, t_n) \in L$ approaching (x_0, t_0) with $t_n < t_0$. In particular, L satisfies the hypothesis in the second part of Theorem* 4.3.4.

Proof. First, $\partial_P L = \partial_P C^+ \cup \partial_P C^-$. We prove the statement when $(x_0, t_0) \in \partial_P C^-$, as the case when $(x_0, t_0) \in \partial_P C^+$ is identical. (x_0, t_0) is approached either above or below in time by a sequence lying in L. If not, for any neighborhood $\mathcal{O}$ of (x_0, t_0) the sets $\mathcal{O} \cap \{t > t_0\}$ and $\mathcal{O} \cap \{t < t_0\}$ are each contained in C^-. One could then use this to show that in fact $(x_0, t_0) \in C^-$, a contradiction. So we need only consider the case where (x_0, t_0) is approached from above in time by a sequence in L.

Suppose (x_0, t_0) is approached by a sequence $(x_n, t_n) \in L$ with $t_n > t_0$, and call $\langle F_{t_0}(x_0), e_1 \rangle = c$. We know $\lim_{t \searrow t_0} \langle F_{t_n}(x_n), e_1 \rangle = c$. Since $(x_0, t_0) \in L_{t_0}$ we know $a(t) < \langle F_{t_n}(x_n), e_1 \rangle$, where $a(t)$ is defined in the previous proposition. Therefore $\liminf_{t \searrow t_0} a(t) \leq c$. Then by the previous proposition $\limsup_{t \nearrow t_0} a(t) \leq c$. We also know that $c \leq \liminf_{t \nearrow t_0} b(t)$ in view of the fact that $(x_0, t_0) \in \partial_P C^-$. Altogether, for any time $t < t_0$ sufficiently close to t_0 we have that

$$\{x \in \mathbb{S}^n | \langle F_t(x), e_1 \rangle = c\} \subset L_t.$$

This allows us to construct a sequence $(x_n, t_n) \in L$ converging to (x_0, t_0) with $t_n < t_0$ (for example, pick a sequence of points each in L_t with $\langle F_{t_n}(x_n), e_1 \rangle = c$ and $\langle F_{t_n}(x_n), \partial_\theta \rangle = \langle F_{t_0}(x_0), \partial_\theta \rangle$). Then either way, (x_0, t_0) is approached from below in time by a sequence in L. Then argument is the same for a point in $\partial_P C_+$. $\square$

4.4. Evolution Equations

In this section, we determine evolution equations for any rotationally symmetric solution of (1.6). Once again, $\hat{w} = \partial_r$ in cylindrical coordinates, and we denote the Second Fundamental Form of N_t by A. We present evolution equations for the quantities H, $u = \langle \hat{w}, \vec{F}(x,t) \rangle$, $v = (\langle \hat{w}, \nu(x,t) \rangle)^{-1}$, and the support function $\langle \vec{F} - \vec{x}_0, \nu \rangle$ with respect to some fixed $x_0 \in \mathbb{R}^{n+1}$.

Theorem 4.4.1 (Evolution Equations for IMCF)**.** *Let N_0 be a C^2, $H > 0$ rotationally symmetric hypersurface, and $\{N_t\}_{0 \leq t < T_{\max}}$ the corresponding maximal solution to (1.6). Then for a fixed vector $\vec{e} \in \mathbb{R}^{n+1}$ and point $\vec{x}_0 \in \mathbb{R}^{n+1}$, the following evolution equations hold*

$$\begin{aligned}
\partial_t \nu &= \frac{1}{H^2} \nabla H; && (4.12)\\
(\partial_t - \frac{1}{H^2}\Delta)\langle \nu, \vec{e} \rangle &= \frac{|A|^2}{H^2}\langle \nu, \vec{e} \rangle && (4.13)\\
(\partial_t - \frac{1}{H^2}\Delta) H &= -\frac{|A|^2}{H^2} H - 2\frac{|\nabla H|^2}{H^3}; && (4.14)\\
(\partial_t - \frac{1}{H^2}\Delta) H^{-1} &= \frac{|A|^2}{H^2} H^{-1}; && (4.15)\\
(\partial_t - \frac{1}{H^2}\Delta)\langle \vec{F} - \vec{x}_0, \nu \rangle &= \frac{|A|^2}{H^2}\langle \vec{F} - \vec{x}_0, \nu \rangle; && (4.16)\\
(\partial_t - \frac{1}{H^2}\Delta) u &= \frac{2p}{H} u - \frac{(n-1)p^2}{H^2} v^2 u; && (4.17)\\
(\partial_t - \frac{1}{H^2}\Delta) v &= -\frac{|A|^2}{H^2} v + \frac{(n-1)p^2}{H^2} v^3 - 2\frac{|\nabla v|^2}{H^2 v}; && (4.18)
\end{aligned}$$

Proof. The first five equations are available in [**HI08**], section 1, and [**CD18**], section 2. For equations (4.17) and (4.18), the Laplacians of the quantities $u = \langle \vec{F}, \hat{w} \rangle$ and $v = \langle \hat{\nu}, \hat{w} \rangle^{-1}$ are shown in, e.g. [**Hui90**], section 5, and [**Ath97**], section 3, to be

$$\begin{aligned}
\Delta_N u &= \frac{n-1}{u} - \frac{H}{v},\\
\Delta_N v &= -v^2 \langle \nabla H, w \rangle + |A|^2 v - \frac{n-1}{u^2} v + 2v^{-1}|\nabla v|^2.
\end{aligned}$$

On the other hand, the time derivatives of these quantities may be computed as

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \langle \hat{w}, \frac{\partial N}{\partial t} \rangle = \frac{1}{Hv},\\
\frac{\partial v}{\partial t} &= -v^2 \langle \hat{w}, \frac{\partial \nu}{\partial t} \rangle = -v^2 \langle \hat{w}, \frac{\nabla H}{H^2} \rangle.
\end{aligned}$$

Noting $\frac{1}{Hv} = \frac{p}{H} u$, $\frac{(n-1)}{u^2} v = (n-1)p^2 v^3$, and $\frac{(n-1)}{u} = (n-1)p^2 u^2 v$, (4.17) and (4.18) follow.

$\square$

4.5. A Priori Height Estimates

In this section, we estimate the position vector $\vec{F}$ of the surface through time. In particular, we establish at most exponential growth on $u = \langle \vec{F}, \hat{w} \rangle$ and $\tilde{u} = \langle \vec{F}, \hat{e}_1 \rangle$ over N_t. We utilize a one-sided version of the well-known avoidance principle for MCF that we proved earlier in this dissertation.

Theorem 4.5.1 (One-Sided Avoidance Principle)**.** *Let $\{N_t\}_{0 \le t < T}$ and $\{\tilde{N}_t\}_{0 \le t < T}$ be two closed, connected solutions to (1.6). For each $t \in [0, T)$, let $E_t \subset \mathbb{R}^{n+1}$ and $\tilde{E}_t \subset \mathbb{R}^{n+1}$ be the bounded, open domains with $N_t = \partial E_t$ and $\tilde{N}_t = \partial \tilde{E}_t$. If $E_0 \subset \tilde{E}_0$ then $E_t \subset \tilde{E}_t$, and dist$(N_t, \tilde{N}_t)$ is non-decreasing.*

This immediately controls the width in the e_1 direction of N_t.

Proposition 4.5.2 (Width Estimate)**.** *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$, rotationally symmetric hypersurface, and $\{N_t\}_{0 \le t < T}$ the corresponding solution to (1.6). Then the function $\tilde{u} = \langle \vec{F}, \hat{e}_1 \rangle$ obeys the estimate*

$$|\tilde{u}(x,t)| \quad \le \quad (\max_{N_0} |\vec{F}|) e^{\frac{t}{n}}. \tag{4.19}$$

Proof. This is a consequence of the One-Sided Avoidance principle. N_0 is enclosed by a sphere of radius $\rho_0 = \max_{N_0} |\vec{F}|$, so comparison with the corresponding spherical solution $\rho(t) = (\max_{N_0} |\vec{F}|) e^{\frac{t}{n}}$ yields (4.19). $\square$

We can also control the height u using Hamilton's trick.

Proposition 4.5.3 (Height Estimate)**.** *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$, rotationally symmetric hypersurface, and $\{N_t\}_{0 \le t < T}$ the corresponding solution to (1.6). Then the function $u = \langle \vec{F}, \hat{w} \rangle$ obeys the estimate*

$$u(x,t) \le (\max_{N_0} u) e^{\frac{t}{n-1}}, \tag{4.20}$$

PROOF. Consider the function $f : [0,T) \to \mathbb{R}$ defined by $f = \max_{N_t} e^{-\frac{t}{n-1}} u$. According to Hamilton's trick, c.f. Section 2.1 of [**Man11**], f is a locally Lipschitz function of time, and where differentiable satisfies

$$f'(t_0) = \partial_t f(x_0, t_0)$$

where $(x_0, t_0) \in \mathbb{S}^n \times [0,T)$ is any point satisfying $f(x_0,t_0) = \max_{N_{t_0}} f$. We know

$$\partial_t f(x,t) = e^{\frac{t}{n-1}} \frac{\langle \nu, \hat{w} \rangle}{H} - \frac{1}{n-1} f \tag{4.21}$$

At (x_0,t_0) we know $\langle \nu, \hat{w} \rangle (x_0,t_0) = 1$ and $k(x_0,t_0) \geq 0$. This means $H(x_0,t_0) \geq \frac{n-1}{u(x_0,t_0)}$. Plugging this into (4.21) yields

$$\partial_t f(x_0, t_0) \leq 0.$$

Therefore, $f'(t_0) \leq 0$ where differentiable. For times $t_1 < t_2$ in $[0,T)$ we use the Fundamental Theorem of Calculus to write

$$f(t_2) = f(t_1) + \int_{t_1}^{t_2} f'(t)dt \leq f(t_1).$$

The estimate follows. $\square$

To conclude this section, we make note of a lower bound on H over the boundary of L_t.

Corollary 4.5.4. *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$, rotationally symmetric embedded sphere, and $\{N_t\}_{0 \leq t < T}$ the corresponding solution to (1.6). Then any $x \in \partial L_t \subset N_t$ is a local maximum of the height function u over N_t. In particular, we have*

$$H|_{\partial L_t} \geq (n-1) \frac{e^{\frac{-t}{n-1}}}{\max_{N_0} u}. \tag{4.22}$$

PROOF. $\partial L_t = \partial C_t^+ \cup \partial C_t^-$, so we present the proof for ∂C_t^+. By rotational symmetry, $\partial C_t^+ = \{x \in N | \langle \vec{F}, \hat{e}_1 \rangle (x,t) = x_0\}$ for some fixed $x_0 \in \mathbb{R}$. Take the generating graph $y : [a,b] \to \mathbb{R}$ of N_t over the axis of rotation. The normal of N_t is given by $\nu(x_1,t) = -\frac{y'(x_1)}{(1+y'(x_1)^2)^{\frac{1}{2}}} \hat{e}_1 + \frac{1}{(1+y'(x_1)^2)^{\frac{1}{2}}} \hat{w}$. Then according to the definition of C_t^+, we have $y'(x_1) \leq 0$ for $x_1 > x_0$ and $y'(x_1) > 0$ for $x_1 < x_0$

sufficiently close to x_0. Thus y has a maximum at x_0, and so u has a maximum at every $x \in \partial C_t^+$. The proof for ∂C_t^- is the same. (4.5.4) follows from noting $k \geq 0$ and so $H \geq (n-1)p$ over $\partial C_t^+ \cup \partial C_t^-$, where we also know $p = \frac{1}{u} \geq e^{\frac{-t}{n-1}} \frac{1}{\max_{N_0} u}$ from (4.20). □

4.6. The Bridge Region

We first consider the region $L \subset \mathbb{S}^n \times [0, T)$, as the geometry of this domain allows us to apply the full non-cylindrical maximum principle. Many of the estimates derived in this section apply generally for any mean-convex, rotationally symmetric embedded sphere, but a crucial sharp bound on the derivative of the generating curve only applies for admissible data. Once again, we must find a uniform-in-time bound on the flow speed over L. We begin by estimating the principal curvature p of rotation.

Theorem 4.6.1 (Rotational Curvature Estimates)**.** *Let N_0 be a C^2, $H > 0$ rotationally symmetric embedded sphere, and $\{N_t\}_{0 \leq t < T}$ the corresponding solution to (1.6). Then the principal curvature $p = (uv)^{-1}$ obeys the estimates*

$$e^{-\frac{t}{n-1}} \min_{\overline{L}_0} p \leq p(x,t) \leq e^{-\frac{t}{n-1}} \max_{\overline{L}_0} p \tag{4.23}$$

over $L = \cup_{0 \leq t < T} L_t \times \{t\}$. In particular, $\frac{\max_{\overline{L}_t} p}{\min_{\overline{L}_t} p}$ is a nonincreasing function of time.

Remark 4.6.2. *By Theorem 4.2.2 for a C^2, $H > 0$, rotationally symmetric embedded sphere, we know $\langle \nu, \hat{w} \rangle(x) > 0$ whenever $u > 0$. It follows that $\min_{\overline{L}_0} p > 0$.*

Proof. We estimate the quantity $f(x,t) = e^{-\frac{t}{n-1}} uv$ from above and below. Combining equations (4.17) and (4.18) yields

$$(\partial_t - \frac{1}{H^2}\Delta) f = (-\frac{|A|^2}{H^2} - \frac{1}{n-1}) f + \frac{2p}{H} f - \frac{2}{H^2 v} \langle \nabla f, \nabla v \rangle. \tag{4.24}$$

We have at a spacetime maximum or minimum (x_0, t_0) of f in L that $\nabla uv(x_0, t_0) = 0$. The formula for ∇uv is given in Section 5 of [**Hui90**] in a local orthonormal frame $v_1, \ldots, v_n$ of $x \in N_t$ with $v_2, \ldots, v_n$ corresponding to angular directions and $\langle v_1, e_1 \rangle > 0$ to be

$$\nabla_i uv = -\delta_{i1} p^3 \langle \nu, e_1 \rangle v (p - k)$$

Then critical points of uv are characterized by either $\langle \nu, e_1 \rangle = 0$ or $k = p$. We consider these cases separately.

Case I: $\langle \nu, e_1 \rangle = 0$: This means that $v(x_0, t_0) = 1$, and that (x_0, t_0) is a critical point of the height function over L_{t_0}. In fact, if (x_0, t_0) is a maximum of uv then since $u(x_0, t_0) = uv(x_0, t_0) \geq uv(x, t) \geq u(x, t)$ for (x, t) in a sufficiently small neighborhood of (x_0, t_0), the point must be a local maximum of the height function over L_{t_0}. This guarantees that $k \geq 0$ and hence $H \geq \frac{(n-1)}{u}$, and furthermore since 1 is an absolute minimum of the function v we know $\partial_t v(x_0, t_0) = 0$. Altogether,

$$\begin{aligned} \partial_t uv(x_0, t_0) &= u(x_0, t_0)\partial_t v(x_0, t_0) + v(x_0, t_0)\partial_t u(x_0, t_0) \\ &= \frac{1}{H(x_0, t_0)} \leq \frac{1}{n-1} u(x_0, t_0). \end{aligned} \tag{4.25}$$

This implies $\partial_t f(x_0, t_0) \leq 0$ at a maximum. The minimum of uv must always occur at the absolute minimum of the height function u over $\overline{L}_t$, and at this point we have

$$\partial_t u(x_0, t_0) = \frac{1}{H(x_0, t_0)} \geq (n-1)u(x_0, t_0),$$

where the inequality follows from the fact that the graph curvature is negative at this point. Therefore $\partial_t f(x_0, t_0) \leq 0$ at a minimum.

Case II: $k = p$: $k(x_0, t_0) = p(x_0, t_0)$ N_{t_0} is umbilic at this point with each principal curvature equalling p and $H^2 = \frac{1}{n}|A|^2$. Thus at a maximum (4.24) satisfies

$$(\partial_t - \frac{1}{H^2}\Delta) f(x_0, t_0) = (-\frac{1}{n} - \frac{1}{n-1}) f(x_0, t_0) + \frac{2}{n} f(x_0, t_0) \leq 0.$$

(Resp. ≥ 0 at a minimum). Since $\partial_t f(x_0, t_0) \leq 0$ at any spacetime maximum (resp. ≥ 0 at a minimum) in L, the non-cylindrical maximum principle yields

$$\inf_{\tilde{\partial}_P L} f \leq f(x, t) \leq \sup_{\tilde{\partial}_P L} f.$$

In fact, $uv|_{\partial L_t} = u|_{\partial L_t} \leq e^{\frac{t}{n-1}} \max_{L_0} u$ from Proposition 4.5.3, and so $f|_{\partial L_t} \leq \max_{L_0} u \leq \max_{L_0} f$. Likewise, the minimum of f at the time t cannot occur on ∂L_t, so the infimum over the reduced parabolic boundary corresponds to $t = 0$. Altogether,

$$e^{\frac{t}{n-1}} \inf_{L_0} uv \leq uv \leq e^{\frac{t}{n-1}} \sup_{L_0} uv.$$

This immediately implies that

$$h(t) = \frac{\max_{\overline{L_t}} p}{\min_{\overline{L_t}} p} \tag{4.26}$$

is a nonincreasing function of time.

□

Remark 4.6.3. *It can be shown for an $H > 0$ rotationally symmetric embedded sphere that the curvature of rotation must be bounded both above and away from 0 over the entire surface, so this result also holds on $N \times [0, T)$ instead of only L. We focus on L because we would like to combine estimates specifically over L with the admissibility assumption.*

Remark 4.6.4. *The umbilicity of the N_t at critical points of p makes the reaction terms in its evolution equation much more tractable compared to the evolution equation under MCF for the same quantity.*

Theorem 4.6.1 provides a sharp interior gradient estimate, as one can show that the quantity v is bounded by the ratio of the highest and lowest values of p at time t. If we use the admissibility condition and an integration trick from [**AAG95**], we can obtain for admissible data that v is specifically bounded away from $\sqrt{n}$.

Corollary 4.6.5. *Let N_0 be a rotationally symmetric admissible hypersurface, and $\{N_t\}_{0 \leq t < T}$ the corresponding solution to (1.6). Then over the region $L \subset \mathbb{S}^n \times [0, T)$ we have*

$$\max_{\overline{L}} v < \sqrt{n} \tag{4.27}$$

Proof. To prove this statement we consider two cases separately. Taken a point $(x, t) \in L$, then either $u(x, t) \geq n^{\frac{1}{2(n-1)}} \min_{L_t} u$ or $u(x, t) < n^{\frac{1}{2(n-1)}} \min_{L_t} u$.

Case I, $u(x, t) \geq n^{\frac{1}{2(n-1)}} \min_{L_t} u$ We know $(\max_{L_t} p)^{-1} = \min_{L_t} u$. Then

$$v(x, t) \leq \frac{\max_{L_t} uv}{u(x, t)} \leq \frac{1}{n^{\frac{1}{2(n-1)}}} \frac{\max_{L_t} p}{\min_{L_t} p} \leq \frac{1}{n^{\frac{1}{2(n-1)}}} \frac{\max_{L_0} p}{\min_{L_0} p} \leq c < \sqrt{n}$$

for some $c < \sqrt{n}$, where we used from Theorem 4.6.1 that $\frac{\max_{L_t} p}{\min_{L_t} p}$ is a nonincreasing function of time.

Case II, $u(x,t) \leq n^{\frac{1}{2(n-1)}} \min_{L_t} u$ We derive an estimate like the one in Lemma 5.13 of [**AAG95**]. We consider the generating graph of N_t which we will denote by y_t. Let a_0 be an interior minimum of this graph. The mean convexity of N_t is equivalent to the condition that

$$\frac{y_t''}{1+|y_t'|^2} < \frac{n-1}{y_t}.$$

Multiplying both sides by y_t' and integrating starting from a_0 yields

$$\ln(1+|y_t'(x)|^2) < 2(n-1)\ln(\frac{y_t(x)}{y_t(a)}) \tag{4.28}$$

Therefore if

$$\frac{y_t(x)}{y_t(a)} < n^{\frac{1}{2(n-1)}}$$

We have $v = (1+|y_t'|^2)^{\frac{1}{2}}$ at the corresponding points in L is smaller than $\sqrt{n}$. □

We are now ready to estimate $\frac{1}{H}$ over the region L. Corollary 4.5.4 ensures that H is bounded below over ∂L_t and hence the entire reduced parabolic boundary of L. Due to the positive term of evolution equation (4.15) for $\frac{1}{H}$, one seeks another well-behaved quantity to combine with the flow speed in order to use a maximum principle.

Equation (4.18) suggests that v is the most natural quantity to combine with the speed function, but due to an extra positive term one finds $(\partial_t - \frac{1}{H^2}\Delta)\frac{v}{H} \leq \frac{n-1}{H^2u^2}\frac{v}{H}$ at an interior maximum, meaning the RHS cannot be immediately controlled. In view of the estimate (4.27) on v, one can compensate for this term using the function $\varphi(r) = \frac{r}{1-\lambda r}$ from the proof of Theorem 3.1 in [**EH91**] (see also Proposition 5 in [**Ath97**] and Theorem A.5 in [**CD18**]). The lower bound on p from Theorem 4.6.1 will also be important in the proof.

Theorem 4.6.6 (Speed Estimate over L)**.** *Let N_0 be an admissible surface, and $\{N_t\}_{0\leq t<T}$ the corresponding solution to (1.6). Then if $T<\infty$ there is a constant $C = C(T, N_0, n) < \infty$ so that*

$$\sup_L \frac{1}{H} \leq C. \tag{4.29}$$

Proof. Consider the function $\varphi(v) = \frac{v}{(1-\lambda v)}$ for λ to be chosen later. From equation (4.18), one finds

$$(\partial_t - \frac{1}{H^2}\Delta)\varphi(v) = -\frac{|A|^2}{H^2}\varphi'(v)v + \frac{(n-1)p^2}{H^2}\varphi'(v)v^3 - (2\frac{\varphi'(v)}{v} + \varphi''(v))\frac{|\nabla v|^2}{H^2}.$$

Define $g = H^{-1}\varphi(v)$. Using the relations

$$\begin{aligned} v\varphi'(v) + \varphi(v) &= -\lambda|\varphi(v)|^2, \\ \varphi'(v)v^2 &= |\varphi(v)|^2, \\ 2\frac{\varphi'(v)}{v} + \varphi''(v) &= 2\frac{|\varphi'(v)|^2}{\varphi(v)}, \end{aligned}$$

we compute

$$\begin{aligned} (\partial - \frac{1}{H^2}\Delta)g &= -|A|^2H^{-3}\varphi'(v)v + (n-1)p^2H^{-3}\varphi'(v)v^3 - (2\frac{\varphi'(v)}{v} + \varphi''(v))H^{-3}|\nabla v|^2 \\ &\quad +|A|^2H^{-3}\varphi(v) - 2H^{-2}\langle\nabla\varphi(v), \nabla H^{-1}\rangle \\ &= (-\lambda|A|^2 + v(n-1)p^2)H^{-1}g^2 - 2\frac{|\varphi'(v)|^2}{\varphi(v)}H^{-3}|\nabla v|^2 \qquad (4.30) \\ &\quad -2H^{-2}\langle\nabla\varphi(v), \nabla H^{-1}\rangle \\ &= (-\lambda|A|^2 + v(n-1)p^2)H^{-1}g^2 - 2(H^2\varphi(v))^{-1}\langle\nabla\varphi(v), \nabla g\rangle. \end{aligned}$$

As $\max_{\overline{L}} v < \sqrt{n}$ for admissible N_0, let $\frac{1}{\sqrt{n}} < \lambda < \frac{1}{\max_{\overline{L}} v}$. Since the corresponding $\varphi(v)$ is bounded over L, $H = (n-1)p + k$ must be near zero when g is large enough. By the lower bound on p of Theorem 4.6.1, $-k \to (n-1)p$ and therefore $|A|^2 \to n(n-1)p^2$ as $H \to 0$. Thus $\lambda|A|^2 \geq \sqrt{n}(n-1)p^2$ for sufficiently large g, and so once again in view of the bound on v the first term in the last line of (4.30) will be non-positive when this happens.

Since this term is clearly bounded for small g, take $\tilde{g} = g - Ct$ for some constant $C = C(n, N_0)$ chosen so that $(-\lambda|A|^2 + v(n-1)p^2)H^{-1}g^2 - C$ is strictly non-positive. $\tilde{g}$ satisfies

$$(\partial_t - \frac{1}{H^2}\Delta)\tilde{g} \leq \langle \eta, \nabla \tilde{g} \rangle$$

for $\eta = 2(H^2\varphi(v))^{-1}\nabla\varphi(v)$ over $L \subset N \times [0, T)$. As mean curvature is bounded below by $(n-1)e^{-\frac{T}{n-1}}(\min_{L_0} u)^{-1}$ over $\tilde{\partial}_P L$ according to Proposition 4.5.3, $\tilde{g} \leq \sup_{\tilde{\partial}_P} \tilde{g} \leq C(N_0, n)e^{\frac{T}{n-1}}$ by the non-cylindrical maximum principle. This bounds the growth of $H^{-1} = (\tilde{g} + Ct)(\varphi(v))^{-1}$ to linear plus exponential, i.e.

$$\sup_L H^{-1} \leq C_1 e^{\frac{T}{n-1}} + C_2 T,$$

for constants C_1, C_2.

$\square$

Remark 4.6.7. *The second condition in Definition* 4.3.2 *is necessary to ensure* $\sup_L v^2 < n$*, allowing us to define* g *in such a way that it is controlled using the non-cylindrical maximum principle. There is still a time-independent bound on* v *over this region for non-admissible data, but not by* $\sqrt{n}$*. Thus it is unclear if one can bound* $\frac{1}{H}$ *over such a surface.*

4.7. The Cap Region

As a result of the previous section, H^{-1} is uniformly controlled over $\partial_P L = \partial_P C^+ \cup \partial_P C^-$ for admisible initial data. This means that we apply the first part of the non-cylindrical maximum principle in order to control this quantity over C^+ and C^-. The maximum principle used in this section applies for any C^2, $H > 0$ rotationally symmetric embedded sphere, but it is only for admissible data that we can control the relevant quantity on the parabolic boundary.

Like in the last section, we require a positive, bounded quantity to combine with the flow speed in order to obtain a useful evolution equation. $\langle \nu, \hat{e}_1 \rangle$ is nonnegative over the right cap C_t^+ (Respectively nonpositive over C_t^-) according to Definition 4.3.1. This allows us to fix an appropriate point on the axis of rotation away from the evolving flow surfaces such that the support function with respect to this point will be strictly positive over each cap.

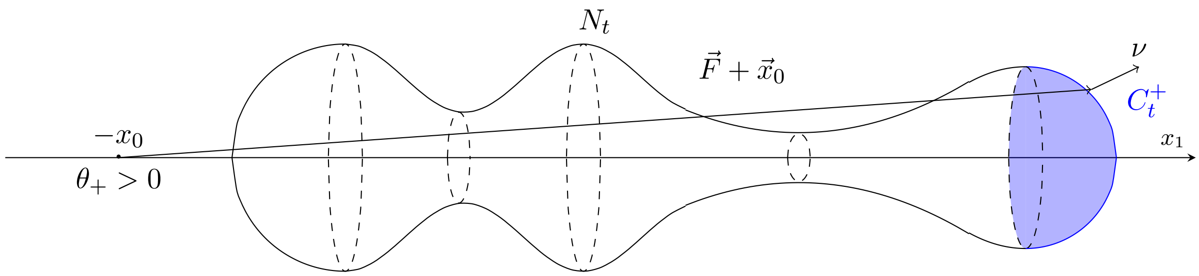


FIGURE 4.5. By picking a point on the axis away from the flow surfaces for $t \in [0,T)$, we ensure the support function is positive over the right cap.

Definition 4.7.1. *Let $N_0^n \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$, rotationally symmetric embedded sphere, and $\{N_t\}_{0\le t<T}$ the corresponding solution to (1.6). For a fixed time interval $[0,T)$, consider the point $x_0 \in \mathbb{R}^{n+1}$ on the axis of rotation x_1 given by*

$$x_0 = (\max_{N_0} |\vec{F}| e^{\frac{T}{n}}, 0, \ldots, 0).$$

The ***right support function*** *$\theta_+ : \mathbb{S}^n \times [0,T) \to \mathbb{R}$ and* ***left support function*** *$\theta_- : \mathbb{S}^n \times [0,T) \to \mathbb{R}$ are defined as*

$$\begin{aligned}\theta_+(x,t) &= \langle \vec{F}(x,t) + \vec{x}_0, \nu\rangle,\\ \theta_-(x,t) &= \langle \vec{F}(x,t) - \vec{x}_0, \nu\rangle.\end{aligned}$$

This particular choice of x_0 ensures that $\theta_\pm$ remains positive over each respective cap, see Figure 4.5.

Proposition 4.7.2. *For any $t \in [0,T)$, the functions θ_+ and θ_- are positive over $\overline{C^+} \subset \mathbb{S}^n \times [0,T)$ and $\overline{C^-} \subset \mathbb{S}^n \times [0,T)$, respectively.*

PROOF. We prove this for right cap first. Write $\vec{F} + \vec{x}_0 = (\tilde{u} + e^{\frac{T}{n}})\hat{e}_1 + u\hat{w}$ and $\nu = \langle \nu, \hat{e}_1\rangle \hat{e}_1 + \langle \nu, \hat{w}\rangle \hat{w}$. For $x \in \overline{C_t^+}$, we know $\tilde{u}(x,t) > \min_{N_t} \tilde{u}(x,t) \ge -(\max_{N_0} |\vec{F}|)e^{\frac{T}{n}}$ by (4.19), so $\langle \vec{F}+\vec{x}_0, \hat{e}_1\rangle > 0$, and $\langle \hat{\nu}, \hat{e}_1\rangle \ge 0$ over $\overline{C_t^+}$ by definition. Now, if $u(x,t) \neq 0$ we have $\langle \hat{w}, \nu\rangle > 0\rangle$ by Theorem 4.2.2, and so $\theta_+(x,t) \ge u\langle \nu, \hat{w}\rangle > 0$. If $u(x,t) = 0$, we have $\theta_+(x,t) = \tilde{u}(x,t) + e^{\frac{T}{n}} \max_{N_0} |\vec{F}| > 0$.

Altogether θ_+ is positive over $\overline{C_t^+}$ for any $t \in [0,T)$, so θ_+ is positive over $\overline{C^+} = \cup_{0\le t<T}\overline{C_t^+} \times \{t\}$. The argument is the same for the left cap, since $\langle \hat{\nu}, \hat{e}_1\rangle \le 0$ and $\langle \vec{F} - x_0, \hat{e}_1\rangle = \tilde{u} - e^{\frac{T}{n}} < 0$ over $\overline{C_t^-}$. □

We now consider the functions $f_\pm(x,t) = (\theta_\pm H)^{-1}$. f_+ and f_- must be well-defined and positive over C^+ and C^- respectively. Furthermore, from Theorem 4.4.1, $\theta_\pm$ and H^{-1} satisfy the same evolution equation. Thus the maximum principle applied to f_+ over C^+ (Resp. f_- over C^-) yields a upper bound on H^{-1}.

Theorem 4.7.3 (Speed Estimate over $C^\pm$)**.** *Let $N_0 \subset \mathbb{R}^{n+1}$ be a C^2, $H > 0$, rotationally symmetric embedded sphere, and $\{N_t\}_{0\le t<T}$ the corresponding solution to (1.6). For the functions $f_+ : C^+ \to \mathbb{R}$ and $f_- : C^- \to \mathbb{R}$ defined by $f_+(x,t) = (\theta_+(x,t)H(x,t))^{-1}$ and $f_-(x,t) = (\theta_-(x,t)H(x,t))^{-1}$*

$$\sup_{C^\pm} f_\pm = \sup_{\partial_P C^\pm} f_\pm.$$

Furthermore, if N_0 is admissible we have $\sup_{\partial_P C^+} f \le c_1$ and $\sup_{\partial_P C^-} f \le c_2$ for some constants $c_1 = c_1(N_0,T)$ and $c_2 = c_2(N_0,T)$. In this case, we have for some constant $C = C(N_0,T)$ that

$$\sup_{C^+\cup C^-} \frac{1}{H} = C. \tag{4.31}$$

Proof. We present the proof for f_+, finding the evolution equation first. From equations (4.14) and (4.16) of Theorem 4.4.1, one can compute

$$(\partial_t - \frac{1}{H^2}\Delta)f_+ \quad = \quad -\frac{2}{H^2}f_+^{-1}|\nabla f_+|^2 - \frac{2}{H^3}\langle \nabla H, \nabla f_+\rangle.$$

Calling $\eta = -\frac{2}{H^2}f_+^{-1}\nabla f_+ - \frac{2}{H^3}\nabla H$, the maximum principle implies

$$\sup_{C^+} f_+ \le \max_{\partial_P C^+} f_+.$$

For the second part of the theorem, we have that θ_+ is uniformly bounded away from 0 over C^+, and as $\partial_P C^+ \subset \partial_P L$, $\sup_{\partial_P C^+} H^{-1} \le C(T,N_0)$ due to Theorem 4.6.6. This yields $\sup_{\partial P C^+} f_+ \le C(T,N_0)$, and in turn

$$\sup_{C^+} H^{-1} \leq C(T, N_0).$$

The proof is the same for C^-. $\square$

4.8. Global Existence and Convergence for Admissible Data

With $\min_{\mathbb{S}^n \times [0,T)} H$ bounded away from 0 for $T < +\infty$ and N_t embedded for admissible initial data, we are able to conclude with a proof of Theorem 4.1.1.

PROOF OF THEOREM 4.1.1. For a solution $\{N_t\}_{0,T}$ of (1.6) with $T < +\infty$ and N_0 admissible, we have from Theorem 4.6.6 that $\sup_{L_t} H^{-1} \leq C_1(T, N_0)$ and from Theorem 4.7.3 that $\sup_{C_t^+ \cup C_t^-} H^{-1} \leq C_2(T, N_0)$, so altogether $H^{-1} \leq \max\{C_1, C_2\}$ over $\mathbb{S}^n \times [0, T)$. According to Corollary 2.3 in [**HI08**], we obtain a smooth, $H > 0$ limit surface N_T at the time T, and hence by parabolicity of (1.6) there exists a solution in short time starting from N_T. Conclude by continuation that $T_{\max} = +\infty$ when N_0 is admissible.

On the question of convergence, Theorem 4.2.2 shows that N_t is embedded for $t \in [0, +\infty)$. We conclude then that $\{N_t\}_{0 \leq t < T_{\max}}$ satisfies the first alternative in Theorem 2.1.1, so that N_t is star-shaped for $t \geq t_*$, where $t_* = n \log(R^{-1} \mathrm{diam}(N_0))$. Theorem 0.1 in [**Ger90**] then implies C^2 convergence to spheres for the $\tilde{N}_t$, and Theorem 0.1 in [**Urb90**] upgrades the strength of this convergence to C^∞. $\square$

CHAPTER 5

An Application of IMCF to Plateau's Problem, and Vice Versa

The mountains of madness have many little plateaux of sanity.

-Terry Pratchett

5.1. Statement of the Problem

To conclude this dissertation, we present a new connection between IMCF and Plateau's problem, one of the oldest and most well-known problems in differential geometry. In 1760, Joseph Lagrange asked if for every closed, embedded curve $\gamma \subset \mathbb{R}^3$ there exists a disk $D \subset \mathbb{R}^3$ with $\partial D = \gamma$ which minimizes the area functional among all disks with boundary γ. This problem is more than a mathematical abstraction, for minimal disks model soap films clinging to closed loops of wire. Joseph Plateau conducted extensive experiments with soap films in the early nineteenth century with this in mind, but a proof of existence took until the following century to arrive. In 1931, Jesse Douglas gave a full solution to the problem in [**Dou31**], for which he was later awarded the first Fields Medal in 1936.

How does one characterize the shape of a soap film? One important property of area minimizers comes from the first variation formula of the area functional. Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a C^2 immersed surface with boundary $\partial\Sigma$ and $U \subset \mathbb{R}^{n+1}$ an open set containing Σ. Let $X \in (TU)_c$ be a vector field with $X|_{\Sigma}^{T} \in (T\Sigma)_c$ (here, $(TU)_c$ and $(T\Sigma)_c$ denote compactly-supported vector fields over U and Σ, respectively, and $X|_{\Sigma}^{T}$ is the projection of X onto $T\Sigma$). For the diffeomorphism flow $f_t : U \to U$ of the vector field X, we consider the deformed surfaces $\Sigma_t = f_t(\Sigma)$. The area functional $|\Sigma_t|$ satisfies

$$\frac{d}{dt}\Big|_{t=0} |\Sigma_t| = \int_{\Sigma} H X|_{\Sigma}^{\perp} d\mu, \tag{5.1}$$

where $X|_{\Sigma}^{\perp}$ is the normal component of X over Σ. An appropriate choice of X then shows that a C^2 immersed surface Σ is a critical point of the area functional, also called a *minimal surface*, if and only if the mean curvature H is identically 0 over Σ. In general, there do exist critical points of

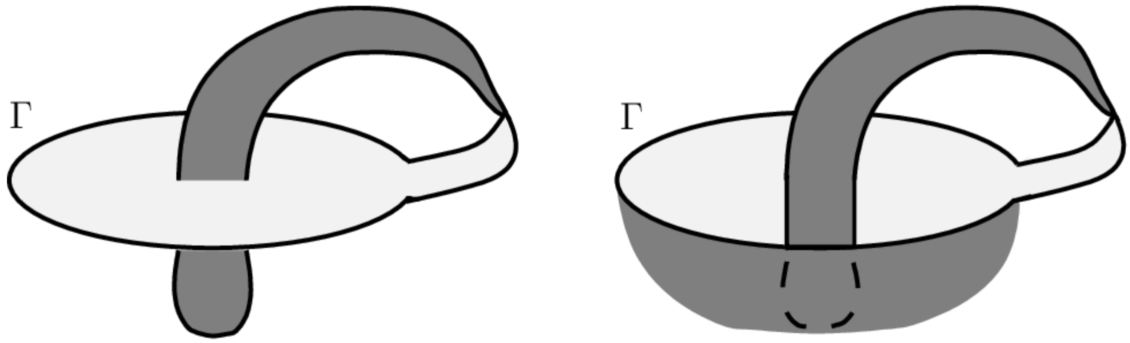


FIGURE 5.1. A curve with a self-intersecting soap film surface. Image from [**Cos12**].

the area functional that are not everywhere C^2 due to the presence of singular or "branch" points, but Osserman shows in [**Oss70**] that branch points do not occur at least for the solutions to the classical Plateau problem in $\mathbb{R}^3$.

The condition of vanishing mean curvature sharply characterizes the geometry of soap films on a point-wise level, as it implies that the principal curvatures must cancel each other out at every point. However, this does not address other questions on the topology and global geometry of area minimizers, at least a priori. One natural topological question on solutions of Plateau's problem is whether or not they self-intersect.

Question 5.1.1. *For which closed, embedded curves $\gamma \subset \mathbb{R}^3$ is the corresponding solution D to Plateau's problem an embedded surface?*

Indeed, there are closed, embedded curves in $\mathbb{R}^3$ for which the minimizer D is not embedded. Figure 5.1 from [**Cos12**] shows one example of this. For this reason, Question 5.1.1 has been a long-standing question in the literature on minimal surfaces. Almgren showed in [**AS79**] that for any curve γ confined to the convex boundary of a bounded, open set that D is embedded, and Ekholm, White, and Wienholtz showed the same in [**EWW02**] for any γ of length less than 4π.

Another question surrounding solutions to Plateau's problem, or more generally any minimal disk, concerns its global shape.

Question 5.1.2. *Let $E \subset \mathbb{R}^3$ be a bounded, open set with ∂E a C^2, $H > 0$ connected hypersurface. Let $\gamma \subset \partial E$ be a closed, embedded curve, and let D be any C^2 immersed minimal disk with boundary γ. For which choice of E is it always true that $D \subset E$?*

In other words, if a curve is confined to a connected, mean convex surface, can a minimal disk with the boundary equalling this curve go outside of this surface? This question was posed by William Meeks in [**Mee78**], and, like with Question 5.1.1, this will not be true for every choice of E. We discuss an example showing this in Section 5.3.

In their 1982 paper [**MY82**], Meeks and S.T. Yau consider Plateau's problem in 3-manifolds with mean-convex boundary, and this viewpoint provides a connection between Question 5.1.1 and Question 5.1.2. Theorem 1 in [**MY82**] guarantees for any curve γ on the closed mean-convex boundary of an open set E the existence of an embedded minimal disk with boundary γ contained with E.

Theorem 5.1.3 ([**MY82**]). *Let $E \subset \mathbb{R}^3$ be a bounded, open domain with ∂E a C^2, $H > 0$ connected hypersurface. Then for every curve $\gamma \subset \partial E$ there exists a unique stable embedded minimal disk D with $\partial D = \gamma$ contained within E that minimizes area among all surfaces contained within E with boundary γ.*

The disk D in this theorem may not be the solution to Plateau's problem in all of $\mathbb{R}^3$ for the curve γ, since there may exist a stable minimal surface of smaller area not contained within E. However, if the answer to Question 5.1.2 is positive for some domain E, this possibility is ruled out, so that D is indeed the global minimizer of area. Since D is embedded, this in turn answers Question 5.1.1 for curves that lie on ∂E.

The main objective is this chapter is to use IMCF together with Theorem 5.1.3 to provide answers to Question 5.1.2 and Question 5.1.1. Additionally, we will later discuss the relationship between minimal disks and singularities of IMCF.

5.2. A Barrier Method via IMCF

We first discuss the conditions on the set E in Question 5.1.2 which imply that $D \subset E$. One rather restrictive requirement which guarantees this is convexity.

Proposition 5.2.1. *Let $E \subset \mathbb{R}^3$ be a bounded, open set with ∂E a C^2 hypersurface, and F the convex hull of E. Then for any closed, embedded curve $\gamma \subset \partial E$ and any C^2, immersed minimal disk D with $\partial D = \gamma$, we have $D \subset F$.*

This is actually a corollary of the more general fact that every compact C^2 minimal surface lies within the convex hull of its boundary, see [**Whi16**]. IMCF allows us to significantly weaken this assumption on E to ensure that $D \subset E$. First, we show that a mean convex foliation of the exterior region of E, at least up to the convex hull of E, is a sufficient condition to guarantee this.

Theorem 5.2.2. *Let $E \subset \mathbb{R}^3$ be a bounded, open set with ∂E a C^2, $H > 0$ connected hypersurface, and $\tilde{E}$ its convex hull. Suppose there exist a family of bounded, open domains $\{E_t\}_{0 \leq t < T}$ in $\mathbb{R}^3$ with the following properties.*

(1) *$E_0 = E$ and $\overline{E_{t_1}} \subset E_{t_2}$ for $t_1 < t_2$.*
(2) *$\tilde{E} \setminus E \subset \cup_{0 \leq t < T} \partial E_t$.*
(3) *∂E_t is an embedded C^2 hypersurface with $H > 0$ for each $t \in [0, T)$.*

Then for any closed, embedded curve $\gamma \subset \overline{E}$ and any C^2 immersed minimal disk D with boundary γ, we have $D \subset E$.

Remark 5.2.3. *This theorem applies not only to the solution of Plateau's problem but to any minimal disk D with boundary γ, regardless of, e.g. its embeddedness or stability.*

Proof. Suppose $D \not\subset E$. Since $D \subset \tilde{E}$, define

$$t_0 = \inf\{t \in [0, T) | D \subset E_t\}.$$

Property (1) implies that $\overline{E_{t_0}} \subset \cap_{t_0 < t \leq T} E_t$, and in fact Property (2) yields equality. Indeed, if $x \in (\cap_{t_0 < t \leq T} E_t) \setminus \overline{E_{t_0}}$, then $x \notin \partial E_{t_1}$ for any $t_1 \in (t_0, T)$ because $x \in E_t$ for $t_0 < t < t_1$ and $E_t \cap \partial E_{t_1} = \varnothing$ for these t. But $x \notin \partial E_{t_0}$ either, contradicting property (2). Since $D \subset E_t$ for each $t \in (t_0, T)$, we have $D \subset \overline{E_{t_0}}$.

Next we claim $D \cap \partial E_{t_0} \neq \varnothing$. If $t_0 = 0$ and $D \not\subset E = E_0$, then by definition $D \cap \partial E_{t_0} \neq \varnothing$. Otherwise, if $D \subset E_{t_0}$ one could pick $\delta > 0$ small enough so that $D \subset \{x \in E_{t_0} | \text{dist}(x, \partial E_{t_0}) > \delta\}$ (such a δ exists by closedness of $D \cup \gamma$) and $t_1 < t_0$ large enough so that $\{x \in E_{t_0} | \text{dist}(x, \partial E_{t_0}) >$

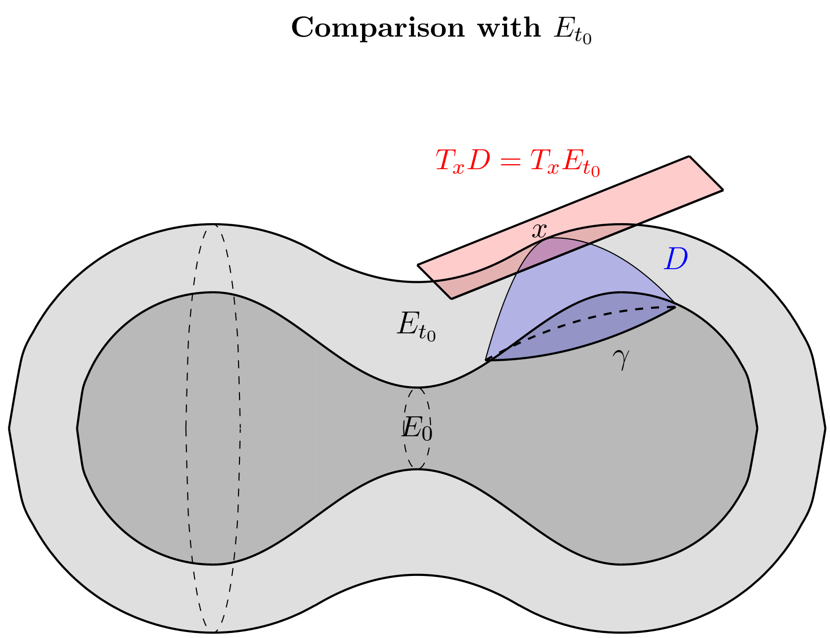


FIGURE 5.2. If the minimal disk $D \not\subset E_0$, then $H_D(x) \geq H_{E_{t_0}}(x) > 0$ by the comparison principle, yielding a contradiction.

$\delta\} \subset E_{t_1}$, again by Property (2). This would contradict the definition of t_0, so conclude $D \subset \overline{E}_{t_0}$ with $D \cap \partial E_{t_0} \neq \varnothing$.

To prove the statement, we utilize a comparison principle. For any $x \in \partial E_{t_0} \cap D$ the tangent planes T_xD and $T_x(\partial E_{t_0})$ are parallel, since if not $D \setminus \overline{E_{t_0}}$ would be nonempty. Calling the principal curvatures of ∂E_{t_0} and D at x $\{\lambda_i\}_{1\leq i\leq n}$ and $\{\lambda_i'\}_{1\leq i\leq n}$ respectively, we must have

$$\lambda_i' \geq \lambda_i, \qquad 1 \leq i \leq n,$$

in view of the inclusion $D \subset \overline{E_{t_0}}$, see Figure 5.2. Property (3) would then yield $H > 0$ at $x \in D$, and this contradicts the minimality assumption. Conclude then that $D \subset E$. $\square$

Given a bounded, open domain $E \subset \mathbb{R}^3$ with C^2, $H > 0$ connected boundary ∂E, examining the solution to IMCF with initial data ∂E allows us to determine if E meets the hypotheses of Theorem 5.2.2. In particular, if the N_t provide a suitable foliation of the exterior region of E, then

E satisfies these hypotheses. We showed in chapter 2 that this happens if and only if the solution exists forever and remains embedded, meaning there is a natural link between global existence problems for (1.6) and Question 5.1.2.

Gerhardt's Theorem 1.4.3 for star-shaped IMCF then indicates that the convexity assumption of E may be replaced by the weaker condition of star-shaped. Furthermore, in the previous chapter we found that this behavior also happens for non-star-shaped, rotationally symmetric families of initial data. As a direct consequence of this, we can answer Question 5.1.2 for E with boundary ∂E lieing in one of these families.

Corollary 5.2.4. *Let $E \subset \mathbb{R}^3$ be a bounded, open domain with ∂E be a C^2, $H > 0$ connected hypersurface. Suppose the Inverse Mean Curvature evolution $\{N_t\}_{0 \leq t < T}$ of $N_0 = \partial E$ satisfies the first alternative in Theorem* 2.1.1*. Then for any closed, embedded curve $\gamma \subset \overline{E}$, any C^2 immersed minimal disk D with $\partial D = \gamma$ satisfies $D \subset E$.*

In particular, if ∂E is either star-shaped or admissibly rotationally symmetric in the sense of Theorem 4.1.1*, then $D \subset E$.*

Proof. According to Theorem 2.3.1 and Theorem 2.3.2, if the flow surfaces N_t of the solution to IMCF satisfy the first alternative in Theorem 2.1, they must foliate $\mathbb{R}^{n+1} \setminus E$. Furthermore, the domains E_t with $N_t = \partial E_t$ satisfy $\overline{E}_{t_1} \subset E_{t_2}$ for $t_1 < t_2$ by these theorems, and therefore satisfy all three criteria in Theorem 5.2.2. Conclude then that for any minimal surface D with $\partial D \subset \overline{E}$, we have $D \subset E$. □

To conclude this section, we apply Corollary 5.2.4 to produce new results on Question 5.1.1.

Corollary 5.2.5 (The Embeddedness of Soap Films)**.** *Let $\gamma \subset \mathbb{R}^3$ be a closed, embedded curve. Suppose there exists a bounded, open domain $E \subset \mathbb{R}^3$ such that ∂E is mean-convex, either star-shaped or admissibly rotationally symmetric, and contains γ. Then the solution D of Plateau's Problem with boundary γ is an embedded minimal disk.*

Proof. First, we claim that the solution D to Plateau's problem for γ is contained within E. According to the regularity result of [**Oss70**], D contains no branch points, i.e. it is C^2 and immersed, and so Corollary 5.2.4 implies that $D \subset E$. Then D must be the unique stable minimal surface along all surfaces contained in E with boundary γ of Theorem 5.1.3. Also by Theorem 5.1.3, D is embedded. $\square$

5.3. Singularity Detection using Minimal Disks

Explicit examples of singularities of IMCF are sparse in the literature. In Section 2.4, we examined a mean-convex topological sphere which failed to be strictly outward-minimizing. This implies an eventual singularity or self-intersection must occur in this surface's evolution by Inverse Mean Curvature Flow, but we conjecture that the latter possibility occurs rather than the former. Minimal disks give an alternative way of establishing singularity formation or self-intersection for mean-convex surfaces with spherical topology, and in this section we use this to establish a likely singularity for one of these surfaces.

To accomplish this, we first construct an explicit example of a bounded, open domain $E \subset \mathbb{R}^3$ with ∂E a C^2, $H > 0$ topological sphere, a curve $\gamma \subset \partial E$, and a C^2 immersed minimal disk D with $\partial D = \gamma$ but $D \not\subset E$. We know that the Inverse Mean Curvature Flow of ∂E must either go extinct or self-intersect in finite time. If not, Corollary 5.2.4 would imply that $D \subset E$.

Our construction is based on the Scherk minimal surface, originally discovered by H.B. Scherk in 1835 in [**Sch**]. This is a graphical minimal surface parametrized by the function $u : (-\frac{\pi}{2a}, \frac{\pi}{2a}) \times (-\frac{\pi}{2a}, \frac{\pi}{2a}) \to \mathbb{R}$ defined by

$$u_a(x_1, x_2) = \frac{1}{a}(\ln(\cos(ax_1)) - \ln(\cos(ax_2))) \tag{5.2}$$

for some parameter $a \in \mathbb{R} \setminus \{0\}$. Actually, we can say more about this construction: we may construct an E so that the solution of Plateau's problem for a curve on ∂E is not contained within E. The solution of Plateau's problem for this curve may or may not be embedded.

Example 5.3.1. *There exists a bounded, open domain $E \subset \mathbb{R}^3$ with ∂E a C^2, $H > 0$ embedded sphere and a curve $\gamma \subset \partial E$ so that the corresponding solution to Plateau's problem does not lie entirely in E.*

Proof. Let $u(x_1, x_2) = u_a(x_1, x_2)$ and $v(x_1, x_2) = 1 + u_b(x_1, x_2)$ for u_a defined in 5.2 and $b < a$. The graphs of u and v intersect on a curve lieing within the set $\{x \in \mathbb{R}^3 | (x_1, x_2) \in (-\frac{\pi}{2a}, \frac{\pi}{2a}) \times$

$(-\frac{\pi}{2a}, \frac{\pi}{2a})\}$. Let γ be their curve of intersection, and let $U \subset \{(x_1, x_2) \in (-\frac{\pi}{2a}, \frac{\pi}{2a}) \times (-\frac{\pi}{2a}, \frac{\pi}{2a})\}$ be the open set in $\mathbb{R}^2$ with $\text{graph}(u|_{\partial U}) = \text{graph}(v|_{\partial U}) = \gamma$. Then the graphs of u and v restricted to U satisfy $u < v$ in view of the fact that $u(0) < v(0)$ and each have γ as their boundary curve. Figure 5.3 illustrates this pair.

Stable Minimal Disks with the Same Boundary

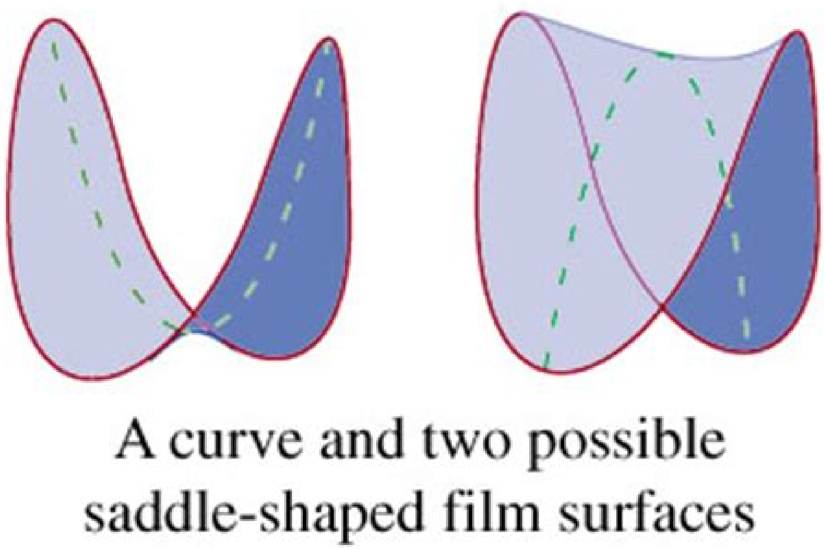


FIGURE 5.3. The two Scherk surfaces are each stable and intersect one another over a closed, embedded curve in $\mathbb{R}^3$. Image from [**Mea**].

The Scherk surface is known to be stable, see [**MP12**]. This allows us to construct a tubular neighborhood E of $\text{graph}(u|_U)$ with ∂E a C^2, $H > 0$ embedded sphere containing $\partial(\text{graph}(u|_U)) = \gamma$ and arbitrarily small width via the Implicit Function Theorem, see [**CM19**]. Choosing E to be sufficiently thin, we can ensure $\text{graph}(v|_U) \not\subset E$.

By the uniqueness statement of Theorem 5.1.3 for stable minimal disks contained in E for a given curve on ∂E, we must have that $\text{graph}(u|_U)$ is the only stable minimal disk with boundary γ contained entirely inside E. $\text{graph}(u|_U)$, however, does not solve Plateau's problem for γ. The area element of (5.2) in graph coordinates can be readily computed as

$$\sqrt{|g|} = \frac{\sqrt{1+\sin^2(ax)}}{\cos(ax)}.$$

If we choose $b < a$ then we see that this quantity is smaller at each $x \in U \setminus \{0\}$ for v compared to u, and so

$$\mathrm{Area}(\mathrm{graph}(v|_U) < \mathrm{Area}(\mathrm{graph}(u|_U).$$

Therefore, the solution to Plateau's problem for γ cannot be contained within E. $\square$

Corollary 5.3.2. *Let E be as in the previous example, and $\{N_t\}_{0\le t<T_{\max}}$ the corresponding maximal solution to IMCF with $N_0 = \partial E$. Then either $T_{\max} < +\infty$ or there is some $t \in [0, T_{\max})$ so that N_t is not an embedded hypersurface.*

PROOF. If $T_{\max} = +\infty$ and the N_t are embedded for $t \in [0, +\infty)$, the $N_0 = \partial E$ must satisfy the first alternative in Theorem 2.1 for IMCF. This means that for the curve $\gamma \subset \partial E$ from the previous example that the solution D to Plateau's problem must be contained within E according to Corollary 5.2.4, as D is a C^2 immersed surface. This is a contradiction. Conclude that either a singularity or intersection occurs. $\square$

We strongly suspect that a singularity occurs for this solution of IMCF.

Conjecture 5.3.3. *Let E be as in the previous example, and $\{N_t\}_{0\le t<T_{\max}}$ the corresponding maximal solution to IMCF with $N_0 = \partial E$. Then $T_{\max} < +\infty$.*

Since we do not have a strong characterization of ∂E, a self-intersection may occur in the flow N_t of $N_0 = \partial E$. However, even if this were the case, one might still be able to make a comparison argument like the one in the previous section to establish singularity formation for N_t. We could compare with the graph of the function $v : U \to \mathbb{R}$ from the previous example, since it is C^2 and embedded regardless of whether it is the global area minimizer. Define

$$F = \{x \in \mathbb{R}^3 | (x_1, x_2) \in U, x_3 > v(x_1, x_2)\},$$

If one can show that for a large enough time a connected component of the set $N_t \cap (U \times \mathbb{R})$ must be contained entirely within F, one could take t_0 to be the infimum of all times for which this is true. N_{t_0} would need to intersect graph(v_U) at some point x_0 in such a way so that

$$H_{\mathrm{graph}(v_U)}(x_0) \geq H_{N_{t_0}}(x_0) > 0,$$

which would contradict graph(v_U) being minimal and establishing a singularity for $\{N_t\}_{0 \leq t < T}$ before this could happen.

APPENDIX A

Non-Cylindrical Maximum Principle

Sometimes you get to what you thought was the end, and you find it's a whole new beginning.

-Anne Tyler

In this section, we prove the non-cylindrical maximum principle for IMCF used in chapter 4.

Theorem A.0.1 (Non-Cylindrical Maximum Principle)**.** *Let $\{N_t\}_{t\in(0,T)}$ be a solution of the Inverse Mean Curvature Flow*

$$\frac{\partial N_t}{\partial t}(x,t) = \frac{1}{H}\nu(x,t) \qquad (x,t) \in N \times [0,T), \tag{A.1}$$

where $N_t = F_t(N)$ for a one-parameter family of embeddings F_t over a closed manifold N. For $U \subset N \times [0,T)$ and $f \in C^{2,1}(U) \cap C(\overline{U})$, suppose for a smooth vector field η over U we have

$$(\partial_t - \frac{1}{H^2}\Delta)f \leq \langle \eta, \nabla f \rangle.$$

(Resp. $\geq$ at a minimum) Here Δ and ∇ are the Laplacian and gradient operators over N_t, respectively. Then

$$\sup_U f \leq \sup_{\partial_P U} f \tag{A.2}$$

(Resp. $\inf_U f \geq \inf_{\partial_P U} f$). Furthermore, suppose that f has a positive supremum over U and that for each $(x_0,t_0) \in \partial_P U \setminus \tilde{\partial}_P U$ there is a sequence of points $(x_n,t_n) \in U$ converging to (x_0,t_0) with $t_n < t_0$.

$$\sup_U f \leq \sup_{\tilde{\partial}_P U} f \tag{A.3}$$

(Resp. $\inf_U f \geq \inf_{\tilde{\partial}_P U} f$ for a positive minimum).

Proof. We modify the proof in the Appendix of [**AK12**]. We begin with the first part, namely that $\sup_U f \le \sup_{\partial_P U} f$. For a given smooth vector field η over U we have by hypothesis

$$(\partial_t - \frac{1}{H^2}\Delta)f \le \langle \eta, \nabla f\rangle.$$

We argue by contradiction: define the function $\tilde{f}(x,t) = f(x,t) - \epsilon t$ for some $\epsilon > 0$. Then

$$\partial_t \tilde{f} = \partial_t f - \epsilon, \quad \partial_i \tilde{f} = \partial_i f, \quad \partial_{ij}\tilde{f} = \partial_{ij} f.$$

The operator over $\tilde{f}$ must then obey

$$(\partial_t - \frac{1}{H^2}\Delta - \eta\cdot\nabla)\tilde{f} < 0. \tag{A.4}$$

On the other hand, at any interior maximum $(x_0, t_0) \in U$ of $\tilde{f}$, the criteria for a local maximum dictate that at (x_0, t_0)

$$\partial_t \tilde{f} \ge 0, \quad \partial_i \tilde{f} = 0, \quad \partial_{ij}\tilde{f} \le 0,$$

where the last inequality is in the operator-theoretic sense for the symmetric matrix $\partial_{ij}\tilde{f}$. Writing

$$\Delta \tilde{f} = g^{ij}(\partial_{ij}\tilde{f} - \Gamma^k_{ij}\partial_k \tilde{f}), \quad \nabla \tilde{f} = g^{ij}\partial_j \tilde{f}\partial_i \vec{F}$$

in view of the positivity of g_{ij}, we see $\Delta\tilde{f} \le 0$ and $\nabla f = 0$. Hence

$$(\partial_t - \frac{1}{H^2}\Delta - \eta\cdot\nabla)\tilde{f}(x_0, t_0) \ge 0,$$

contradicting (A.4). So $\tilde{f}$ has no interior maximum and thus

$$\sup_U f - \epsilon T \le \sup_U \tilde{f} \le \sup_{\partial_P U} f.$$

Then $\sup_U f \le \sup_{\partial_P U} f + \epsilon T$. For $T < \infty$, letting $\epsilon \to 0$ yields the result. To prove the statement for the infimum, take $\tilde{f} = f - \epsilon t$, $\epsilon > 0$, and repeat this argument for a minimum.

For the second part, we show $\sup_U f \leq \sup_{\tilde{\partial}_P U} f$ if $\sup_U f > 0$. Define $Z = \partial_P U \setminus \tilde{\partial}_P U$, and for each $t \in [0, T)$ let $Z_t = Z \cap \{t\}$ be the cross sections of Z. We argue by contradiction: suppose (A.3) does not hold. Then the maximum of f does not occur on $\partial_{\tilde{P}} U$ nor does it occur at an interior point of U, so it must occur on the set Z. Call $Z_{\max}$ the union of Z_t's on which the maximum is achieved, and let $t_* > 0$ be the first time at which $\sup_U f$ is achieved on $Z_{\max}$. Pick $\beta > 0$ such that

$$\sup_U f > \beta > \sup_{\partial_P U \setminus Z_{\max}} f,$$

and define

$$Y_\beta = \{(x,t) \in U | f(x,t) > \beta\}.$$

Since $\beta < \sup_U f$, the set $\overline{Y_\beta}$ must intersect Z_{t_*}. Consider $(x_0, t_*) \in Z_{t_*} \cap \overline{Y_\beta}$. From the additional assumption in the proposition there is a sequence of points $(x_n, t_n) \in U$ with $t_n < t_*$ converging to (x_0, t_*). By continuity of f, $f(x_n, t_n) > \beta$ for large enough n. This means that the set

$$X_\beta = \{t < t_*\} \cap Y_\beta \tag{A.5}$$

is nonempty and open. By openness, we pick a time $t_1 < t_*$ so that the set $X_\beta \cap \{t \leq t_1\} \neq \varnothing$.

Fix a time $t_2 \in (t_1, t_*)$ and choose a cutoff function $\phi : [0, T] \to [0, 1]$ such that $\phi(t) = 1$ when $t \in [0, t_1]$, $\phi'(t) < 0$ when $t \in (t_1, t_2)$, and $\phi(t) = 0$ when $t \in [t_2, T]$, see Figure A.1. Since $(\phi f)(x, t_1) = f(x, t_1)$, and $f(x, t_1) > \beta$ for $(x, t_1) \in X_\beta$, we know $\sup_U \phi f > \beta > 0$.

We calculate

$$\partial_t(\phi f) = \phi \partial_t f + \phi' f, \qquad \Delta \phi f = \phi \Delta f, \qquad \nabla(\phi f) = \phi \nabla f,$$

so

$$(\partial_t - \frac{1}{H^2}\Delta - \eta \cdot \nabla)(\phi f) = \phi(\partial_t - \frac{1}{H^2}\Delta - \eta \cdot \nabla) f + \phi' f. \tag{A.6}$$

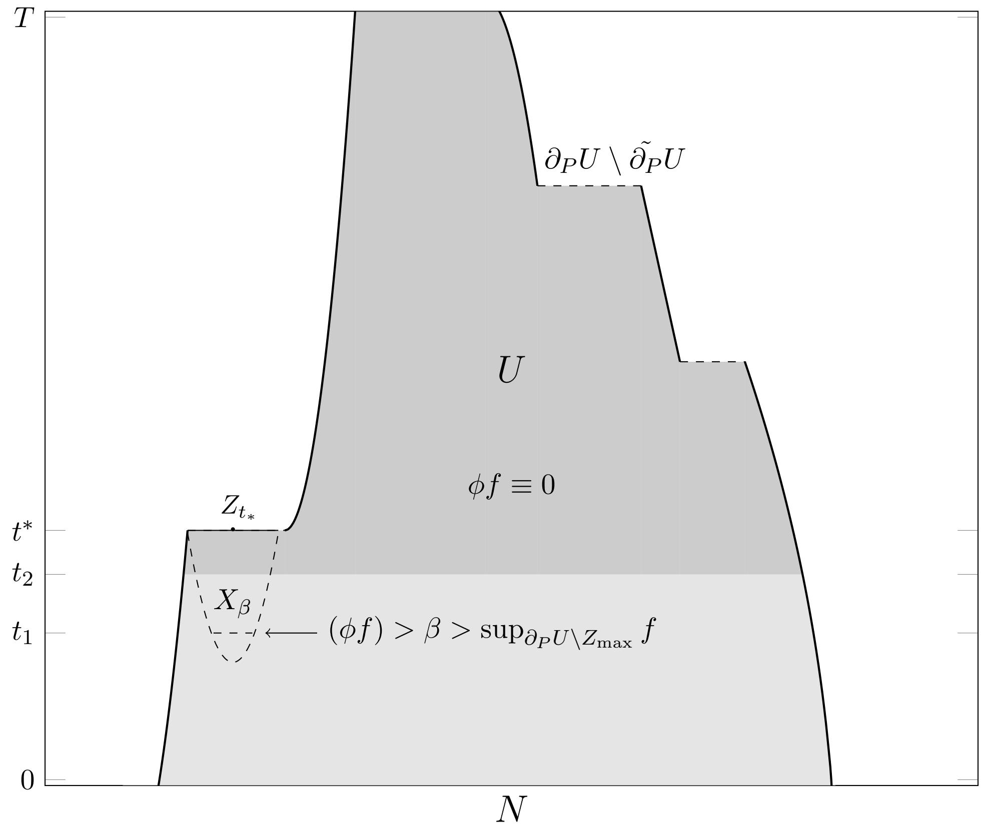


FIGURE A.1. The cutoff function ϕ is 1 for times less than t_1 and 0 for times greater than t_2. This guarantees that the supremum of ϕf occurs at an interior point of X'_β.

By hypothesis we have $(\partial_t - \frac{1}{H^2}\Delta - \eta \cdot \nabla) f(x,t) \leq 0$ and $\phi' \leq 0$. Since $\sup_U \phi f > 0$, any interior point $(x_0, t_0) \in U$ at which ϕf achieves this supremum would need to satisfy

$$(\partial_t - \frac{1}{H^2}\Delta - \eta \cdot \nabla)(\phi f)(x_0, t_0) < 0.$$

By the same argument used for the proof of the first part of Theorem $A.0.1$, this is impossible, and so $\sup_U(\phi f) = \sup_{\partial_P U}(\phi f)$. However, we also know $\sup_U(\phi f) > \beta$. $\partial_P U \cap \{t < t_2\} \subset \partial_P U \setminus Z_{\max}$ since $t_2 < t_*$, and $f < \beta$ over $\partial_P U \setminus Z_{\max}$. Since $\phi f \leq f$, the supremum cannot occur at in $\partial_P U \cap \{t < t_2\}$. However, $\phi f \equiv 0$ on $\partial_P U \cap \{t \geq t_2\}$ since $\phi \equiv 0$ for these times, and so we altogether get

$$\sup_{\partial_P U} \phi f < \beta < \sup_U \phi f.$$

This would contradict the first part of the non-cylindrical maximum principle. Conclude then that

$$f(x,t) \leq \sup_{\tilde{\partial}_P U} f$$

on U. The statement may be shown for a minimum by choosing $\beta > 0$ with $\inf_U f < \beta < \inf_{\partial_P U \setminus Z_{\max}} f$. □

Remark A.0.2. *The version of this principle used in [**AK12**] and [**HK19**] does not include the hypothesis that U approaches Z_{t_*} from below in time. However, if U only touches Z_{t_*} from above in time, $Y_\beta \cap \{t < t_*\}$ may be empty. The corresponding cutoff function would then need to be chosen to increase with t, so that the last term in* (A.6) *is possibly non-negative. Therefore, this additional hypothesis seems to be necessary.*

APPENDIX B

Non-Star-Shaped Admissible Initial Data

It ain't over 'til it's over.

-Yogi Berra

Since star-shaped N_0 are already known to flow to spheres from the results in [**Ger90**], we now demonstrate the existence of a non-star-shaped N_0 satisfying the admissibility condition.

Proposition B.0.1. *For any $n \geq 2$, there is an admissible surface $N_0^n \subset \mathbb{R}^{n+1}$ which is not star-shaped.*

Proof. We begin our construction of a non-star-shaped admissible surface by considering a C^0 surface of revolution: take two disjoint spheres each of radius 1 with centers at $(x_1, x_2, \ldots, x_{n+1}) = (-(\ell+1), 0, \ldots, 0)$ and $(x_1, x_2, \ldots, x_{n+1}) = (\ell+1, 0, \ldots, 0)$ respectively, and connect them by a tube of radius c, where $1 > c > \frac{1}{2}$. We show by taking ℓ sufficiently large that this surface may be refined into a C^2 admissible surface N_0 in any given dimension.

Writing $x = x_1$, the generating curve of the resulting C^0 surface is a graph $x_2 = g(x)$, where $g : (-\ell-2, \ell+2) \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} c & 0 \leq |x| \leq \ell \\ \sqrt{1-x^2} & \ell < |x| \leq \ell+2. \end{cases}$$

We seek an interpolating function f such that the function $y : (-\ell-2, \ell+2) \to \mathbb{R}$ defined by

$$y(x) = \begin{cases} g(x) & 0 \leq |x| \leq (\ell+1)c \\ f(x) & (\ell+1)c < |x| \leq (\ell+1) \\ g(x) & (\ell+1) < |x| \leq (\ell+2) \end{cases}$$

is C^2 (It will become clear later why we chose the point $(\ell+1)c$ to begin the interpolating function). Furthermore, defining $v(x) = (1 + |u'(x)|^2)^{\frac{1}{2}}$, we want that y satisfies $\max_{x\in[-\ell-1,\ell+1]} yv(x) < 2\min_{x\in[-\ell-1,\ell+1]} y(x)$. This would imply that $\frac{\max_{\overline{L_0}} p}{\min_{\overline{L_0}} p} < 2$, which guarantees N_0^n is admissible for any $n \geq 2$. Also, we need that $y''(x) < \frac{v^2(x)}{y(x)}$ everywhere to ensure $H > 0$ for the corresponding surface. We may fulfill these requirements by fixing c and choosing an f depending on the length parameter ℓ which is nondecreasing in $|x|$ has $|f'(x)| \to 0$ and $\sup_{x\in(-\ell-2,\ell+2)} f''(x) \to 0$ as $\ell \to \infty$.

To construct this f, begin with the function

$$h(x) = \frac{1}{\ell^4}\cos(\ell^2(x - (\ell+1))) - \frac{1}{\ell^4} + (1-c),$$

which regardless of choice of ℓ satisfies

$$\begin{aligned} h(\ell+1) &= 1-c, \\ h'(\ell+1) &= 0, \\ h''(\ell+1) &= -1. \end{aligned}$$

For large enough ℓ and fixed c the value $x_0 = -\frac{\pi}{2\ell^2} + (\ell+1)$ (for which the argument of cosine in h is $-\frac{\pi}{2}$) is greater than $(\ell+1)c$. The tangent line $T(x)$ to $h(x)$ at x_0 is

$$T(x) = \frac{1}{\ell^2}(x - x_0) + (1-c) - \frac{1}{\ell^4}.$$

For $x \geq (\ell+1)c$ we know

$$T(x) \geq -\frac{1}{\ell^2}(\ell+1)(1-c) + (1-c) - \frac{1}{\ell^4},$$

meaning that $T(x) \geq 0$ for $x \geq (\ell+1)c$ when ℓ is large enough. Now, we also know at x_0 that $h''(x_0) = 0$, so the piecewise function $\tilde{h} : [(\ell+1)c, \ell+1] \to \mathbb{R}$

$$\tilde{h}(x) = \begin{cases} T(x) & (\ell+1)c \leq x \leq x_0 \\ h(x) & x_0 \leq x \leq \ell+1 \end{cases}$$

is C^2 and nonnegative. Next, consider a cutoff function $\phi : [0,1] \to [0,1]$ with

$$\begin{aligned}\phi(x) &= 0, \quad x \leq \frac{1}{4}, \\ \phi(x) &= 1, \quad x \geq \frac{1}{2}, \\ \phi'(x) &\geq 0.\end{aligned}$$

Take $\tilde{\phi}(x) = \phi(\frac{1}{(\ell+1)(1-c)}(x-(\ell+1)c))$ over $[(\ell+1)c, \ell+1]$, where we also note that $\tilde{\phi}(x) = 1$ when $x \geq x_0$ for any sufficiently large ℓ. We may finally define the f mentioned above for $(\ell+1)c \leq |x| \leq \ell+1$ as

$$f(x) = \tilde{\phi}(|x|)\tilde{h}(|x|) + c. \tag{B.1}$$

f satisfies at the endpoints

$$\begin{aligned}f(\pm(\ell+1)c) &= c & f(\pm(\ell+1)) &= 1, \\ f'(\pm(\ell+1)c) &= 0 & f'(\pm(\ell+1)) &= 0, \\ f''(\pm(\ell+1)c) &= 0 & f''(\pm(\ell+1)) &= -1,\end{aligned}$$

so that the resulting y is C^2. The derivatives of $\tilde{\phi}$ and $\tilde{h}$ obey

$$0 \leq \tilde{\phi}'(x) \leq \frac{1}{\ell+1}\max_{x\in[0,1]}\phi(x), \tag{B.2}$$

$$0 \leq \tilde{h}'(x) \leq \frac{1}{\ell^2}. \tag{B.3}$$

Thus $f(x)$ is monotone for x positive, so since $f(\ell+1) = 1$ we have $\max f = \max y = 1$. Both (B.2) and (B.3) approach 0 uniformly as $\ell \to \infty$. This means that for large enough ℓ and $\frac{1}{2} < c < 1$ we can obtain $(1+|f'(x)|^2)^{\frac{1}{2}} f < 2c$. Then altogether

$$\max_{x\in[-\ell-1,\ell+1]} yv(x) < 2c = 2 \min_{x\in[-\ell-1,\ell+1]} y(x). \tag{B.4}$$

This means that the surface generated by y satisfies the condition $\frac{\max_{\overline{L}_0} p}{\min_{\overline{L}_0} p} < 2$ from Definition 4.3.2. We also confirm the $H > 0$ condition may be met by the surface generated by y: y is convex for $|x| \geq |x_0|$ and for $|x| \leq |x_0|$ we can compute

$$f''(x) \leq \frac{1}{(\ell+1)^2} \sup_{x\in[0,1]} \phi''(x) + \frac{2}{\ell^2(\ell+1)} \sup_{x\in[0,1]} \phi'(x).$$

We see then that the positive supremum of $y''(x)$ approaches 0 as $\ell \to \infty$. In view of the fact that the principal curvature corresponding to rotation is bounded below by, say, $\frac{1}{2c}$ over $(-\ell-1, \ell+1)$ and the curvature of the graph is positive outside this region, the corresponding surface may be chosen to be mean convex.

We claim that the admissible surface N_0^n generated by y is not star-shaped, and it is sufficient to show that the generating curve is not star-shaped. First, take the line segment $L(x) = \frac{1}{\ell+1}x$

A Non-Star-Shaped Admissible N_0

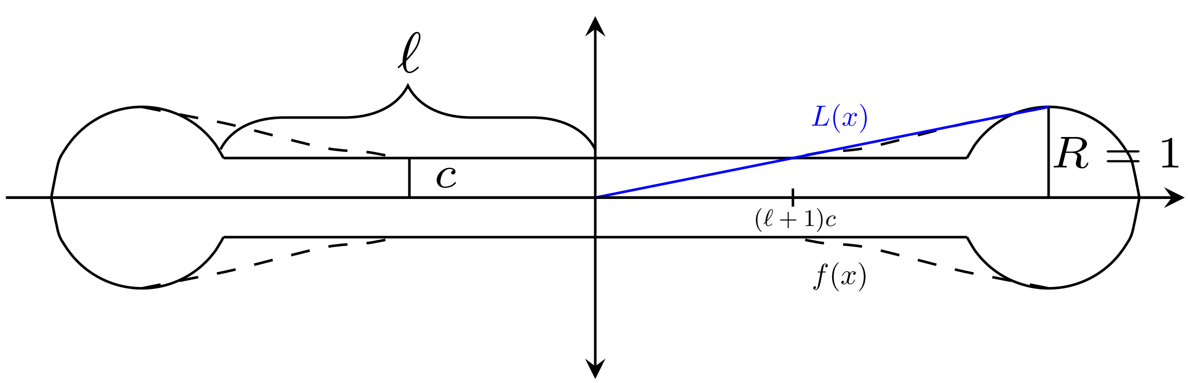


FIGURE B.1. Smoothing spheres attached to a sufficiently long neck produces a non-star-shaped, admissible surface of revolution.

extending from the origin to the point $(x, z) = (\ell + 1, 1)$, where it intersects the generating curve. We find

$$L((\ell + 1)c) = c = y((\ell + 1)c), \tag{B.5}$$

meaning that L also intersects the generating curve at $(x, z) = ((\ell + 1)c, c)$. This means that the curve is not star-shaped with respect to the origin. Indeed, by the symmetry of this curve it cannot be star-shaped with respect to any other point which it encloses: consider the point $(\tilde{x}, \tilde{z})$ with $-(\ell + 2) < \tilde{x} < 0, 0 \leq \tilde{z} \leq y(\tilde{x})$. The line segment $\tilde{L}(x)$ extending from this point to the point $(x, z) = (\ell + 1, 1)$ on the curve will satisfy

$$\tilde{L}((\ell + 1)c) \geq L((\ell + 1)c) = c = y((\ell + 1)c),$$

implying $\tilde{L}$ intersects the curve at two different points. By reflection symmetry of the curve, this must also be true when $\tilde{x} > 0$. Conclude then that the C^2 hypersurface $N_0^n \subset \mathbb{R}^{n+1}$ generated by y is admissible but not star-shaped. $\square$

Bibliography

[AAG95] Steven Altschuler, Sigurd Angenent, and Yoshikazu Giga. Mean curvature flow through singularities for surfaces of rotation. *The Journal of Geometric Analysis*, 5:293–358, 1995.

[AFM19] Virginia Agostiniani, Mattia Fogagnolo, and Lorenzo Mazzieri. Minkowski inequalities via nonlinear potential theory. *arXiv: Analysis of PDEs*, 2019.

[AK12] Maria Athanassenas and Sevvandi Kandanaarachchi. Singularities of axially symmetric volume preserving mean curvature flow. 2012.

[Ale56] A.D. Alexandrov. Uniqueness theorems for surfaces in the large i. *Vestnik Leningrad. Univ. 11*, (19):5–17, 1956.

[Ale57] A.D. Alexandrov. Uniqueness theorems for surfaces in the large ii. *Vestnik Leningrad. Univ. 12*, (7):15–44, 1957.

[AMC+13] Paul M. Alsing, Warner A. Miller, Matthew Corne, Xianfeng Gu, Jonathan R. McDonald, Shannon Ray, Chris Tison, and Shing-Tung Yau. Simplicial Ricci Flow: An Example of a Neck Pinch Singularity in 3D. 8 2013.

[Ang91] S. Angenent. On the formation of singularities in the curve shortening flow. *Journal of Differential Geometry*, 33:601–633, 1991.

[Ang18] Pablo Anglada. Penrose-like inequality with angular momentum for minimal surfaces. *Classical and Quantum Gravity*, 35(4):045018, Jan 2018.

[AS79] F. Almgren and L. Simon. Existence of embedded solutions of plateau's problem. *Annali Della Scuola Normale Superiore Di Pisa-classe Di Scienze*, 6:447–495, 1979.

[Ath97] Maria Athanassenas. Volume-preserving mean curvature flow of rotationally symmetric surfaces. *Comment. Math. Helv.*, 72:52–66, 1997.

[AV97] S.B. Angenent and J.J. Velázquenz. Degenerate neckpinches in mean curvature flow. *Journal für die reine und angewandte Mathematik*, 482:15–66, 1997.

[Bak05] I. Bakas. Geometric flows and (some of) their physical applications. 2005.

[BC04] Hubert L. Bray and Piotr T. Chruściel. The penrose inequality. In Piotr T. Chruściel and Helmut Friedrich, editors, *The Einstein Equations and the Large Scale Behavior of Gravitational Fields*, pages 39–70, Basel, 2004. Birkhäuser Basel.

[BHW14] Simon Brendle, Pei-Ken Hung, and Mu-Tao Wang. A minkowski inequality for hypersurfaces in the anti-de sitter-schwarzschild manifold. *Communications on Pure and Applied Mathematics*, 69, 12 2014.

[Bra01] Hubert L. Bray. Proof of the Riemannian Penrose Inequality Using the Positive Mass Theorem. *Journal of Differential Geometry*, 59(2):177 – 267, 2001.

[Bra02] Hubert L. Bray. Black holes , geometric flows , and the penrose inequality in general relativity. 2002.

[Bre11] S. Brendle. Evolution equations in riemannian geometry, 2011.

[CD18] Beomjun Choi and Panagiota Daskalopolous. Evolution of non-compact hypersurfaces by inverse mean curvature. 12 2018.

[CG01] Bennett Chow and Robert Gulliver. Aleksandrov reflection and the geometric evolution of hypersurfaces. *Communications in Analysis and Geometry*, 9:261–280, 4 2001.

[CGG99] Yun-Gang Chen, Yoshikazu Giga, and Shun'Ichi Goto. *Uniqueness and Existence of Viscosity Solutions of Generalized mean Curvature Flow Equations*, pages 375–412. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[Che44] Shiing-Shen Chern. A simple intrinsic proof of the gauss-bonnet formula for closed riemannian manifolds. *Annals of Mathematics*, 45(4):747–752, 1944.

[CLZ17] Daguang Chen, Haizhong Li, and Tailong Zhou. A penrose type inequaltiy for graphs over reissner-nordström-anti-desitter manifold. *arXiv: Differential Geometry*, 2017.

[CM19] Otis Chodosh and Christos Mantoulidis. Minimal hypersurfaces with arbitrarily large area. *International Mathematics Research Notices*, 7 2019.

[Cos12] Baris Coskunuzer. Embedded plateau problem. *Transactions of the American Mathematical Society*, 364(3):1211–1224, 3 2012.

[Dou31] J. Douglas. Solution of the problem of plateau. *Transactions of the American Mathematical Society*, 33:263–321, 1931.

[Du20] Wenkui Du. Bounded diameter under mean curvature flow, 2020.

[EH91] Klaus Ecker and Gerhard Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Invent Math*, 105:547–569, 1991.

[ES64] James Eells and J.H. Sampson. Harmonic mappings of riemannian manifolds. *American Journal of Mathematics*, 86(1):109–160, 1964.

[ES91] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *Journal of Differential Geometry*, 33(3):635 – 681, 1991.

[Eva10] Lawrence C. Evans. *Partial differential equations.* American Mathematical Society, Providence, R.I., 2010.

[EWW02] T. Ekholm, B. White, and Daniel Wienholtz. Embeddedness of minimal surfaces with total boundary curvature at most 4 pi. *Annals of Mathematics*, 155:209–234, 2002.

[Ger73] Robert Geroch. Energy extraction*. *Annals of the New York Academy of Sciences*, 224(1):108–117, 1973.

[Ger90] Claus Gerhardt. Flow of Nonconvex Hypersurfaces into Spheres. *Journal of Differential Geometry*, 32:299–314, 1990.

[Ger11] Claus Gerhardt. Inverse Curvature Flows in Hyperbolic Space. *J. Differential Geometry*, 89(3):487–527, 2011.

[Ger15] Claus Gerhardt. Curvature flows in the sphere. *Journal of Differential Geometry*, 100(2):301 – 347, 2015.

[GH17] Panagiotis Gianniotis and Robert Haslhofer. Diameter and curvature control under mean curvature flow, 2017.

[GKS13] Zhou Gang, Dan Knopf, and Israel M. Sigal. Neckpinch dynamics for asymmetric surfaces evolving by mean curvature flow. 11 2013.

[Gra87] Matthew Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geometry*, 26(2):285–314, 1987.

[Gra89] Matthew A. Grayson. A short note on the evolution of a surface by its mean curvature. *Duke Math*, 58(3):555–558, 1989.

[Ham95] R.S. Hamilton. The formation of singularities in the ricci flow. *Surveys in Differential Geometry*, 2, 1995.

[Har20a] Brian Harvie. Inverse mean curvature flow of rotationally symmetric hypersurfaces. 2020.

[Har20b] Brian Harvie. Inverse mean curvature flow over non-star-shaped surfaces. *Math. Res. Letters*, 8 2020.

[Har20c] Brian Harvie. The limit of the inverse mean curvature flow on a torus. 2020.

[Hea13] John Head. On the mean curvature evolution of two-convex hypersurfaces. *J. Differential Geom.*, 94(2):241–266, 06 2013.

[HI01] Gerhard Huisken and Tom Illamen. The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality. *J. Differential Geometry*, 59(3):353 – 437, 2001.

[HI08] Gerhard Huisken and Tom Ilmanen. Higher regularity of the inverse mean curvature flow. *Journal of Differential Geometry - J DIFFEREN GEOM*, 80, 11 2008.

[HK19] John Head and Sevvandi Kandanaarachchi. Singularity formation in axially symmetric mean curvature flow with neumann boundary, 2019.

[HP99] Gerhard Huisken and Alexander Polden. *Geometric evolution equations for hypersurfaces*, pages 45–84. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[Hui84] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[Hui18] Gerhard Huisken. Monotonicity formulae and non-collapsing estimates for mean curvature flow with applications – 2. `https://www.youtube.com/watch?v=fHUUBBntV7E&t=1456s`, 2018. Accessed 10/14/20.

[Kry97] Nicolai V. Krylov. Fully nonlinear second order elliptic equations : recent development. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 4, 25(3-4):569–595, 1997.

[Lan85] Joel Langer. A compactness theorem for surfaces with lp-bounded second fundamental form. *Mathematische Annalen*, 270:223–234, 1985.

[LdMP20] B. Leandro, A. de Melo, and H. Pina. A minkowski inequality for the static einstein-maxwell space-time, 2020.

[Lin15] Longzhi Lin. Mean curvature flow of star-shaped hypersurfaces, 2015.

[Lu17] Siyuan Lu. Inverse Curvature Flow in the Anti-De Sitter-Schwarzschild Manifold. *Comm. Anal. Geometry*, 2017.

[Lum87] G. Lumer. Principes du maximum paraboliques pour des domaines (x,t) non-cylindriques. In Francis Hirsch and Gabriel Mokobodzki, editors, *Séminaire de Théorie du Potentiel Paris, No. 8*, pages 105–113, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.

[Man11] Carlo Mantegazza. Lecture notes on mean curvature flow, 2011.

[McC17] Stephen McCormick. On a minkowski-like inequality for asymptotically flat static manifolds. *Proceedings of the American Mathematical Society*, 146, 09 2017.

[Mea] Alex Meadows. MS Windows NT kernel description. `http://faculty.smcm.edu/ammeadows/create/`. Accessed: 2021-05-14.

[Mee78] Meeks. Lectures on plateau's problem. 1978.

[MP12] W. Meeks and J. Pérez. A survey on classical minimal surface theory. 2012.

[MSS15] Luca Mugnai, Christian Seis, and Emanuele Spadaro. Global solutions to the volume-preserving mean-curvature flow, 2015.

[MY82] W. Meeks and S. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Mathematische Zeitschrift*, 179:151–168, 1982.

[Nas58] J. Nash. Continuity of solutions of parabolic and elliptic equations. *American Journal of Mathematics*, 80(4):931–954, 1958.

[Oss70] Robert Osserman. A proof of the regularity everywhere of the classical solution to plateau's problem. *Annals of Mathematics*, 91(3):550–569, 1970.

[Per03] Grisha Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. 7 2003.

[Per06a] Grisha Perelman. Ricci flow with surgery on three-manifolds. 8 2006.

[Per06b] Grisha Perelman. The Entropy formula for the Ricci flow and its geometric applications. 7 2006.

[Sch] H. Scherk. Bemerkungen über die kleinste fläche innerhalb gegebener grenzen. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1835:185 – 208.

[Smo98] Knut Smoczyk. Starshaped hypersurfaces and the mean curvature flow. *Manuscripta mathematica*, 95(2):225–236, 1998.

[Smo00] Knut Smoczyk. Remarks on the Inverse Mean Curvature Flow. *Asian Journal of Mathematics*, 4(2):331–336, 2000.

[Urb90] John I.E. Urbas. On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. *Mathematische Zeitschrift*, 205(3):355–372, 1990.

[Wei18] Yong Wei. On the minkowski-type inequality for outward minimizing hypersurfaces in schwarzschild space. *Calculus of Variations and Partial Differential Equations*, 57, 03 2018.

[Whi16] Brian White. Lectures on minimal surface theory, 2016.

[Zho16] Hengyu Zhou. Inverse mean curvature flows in warped product manifolds. *The Journal of Geometric Analysis*, 28:1749–1772, 2016.

I'm finished.

-Daniel Plainview, There Will Be Blood