Pair Dependent Linear Statistics for Circular Random Matrix Ensembles

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Abstract

Let $\theta_1, ..., \theta_N$ be the angles of the eigenvalues of a $N \times N$ matrix sampled from either $C\beta E$, $SO(N)$, or $Sp(N)$. In this dissertation, we study the limiting distribution of the "pair dependent" linear statistic

$$\left(\frac{1}{\sqrt{L_N}}\right) \sum_{1 \leq i \neq j \leq N} f(L_N(\theta_i - \theta_j)),$$

where f is a sufficiently smooth function and L_N is a positive, non-decreasing sequence such that $1 \leq L_N << N$. When $L_N = 1$ (global case), the limiting distribution is an infinite sum of independent random variables, exponential in the case of $C\beta E$ and chi-squared distributed in the cases of $SO(N)$ and $Sp(N)$. When $L_N \to \infty$ (mesoscopic case), we are able to prove central limit theorems for each of the mentioned random matrix ensembles.

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CHAPTER 0

Notations

For convenience, we list the notations that will be commonly used throughout this text.

(1) N: The dimension of a random matrix from the contextually relevant ensemble

(2) CβE: Circular Beta Ensemble

(3) CUE=CβE ($\beta = 2$): Circular Unitary Ensemble

(4) $U(N)$: Ensemble of $N \times N$ unitary matrices.

(5) $SO(N)$: Ensemble of $N \times N$ orthogonal matrices with determinant one

(6) $SO^{-}(N)$: Ensemble of $N \times N$ orthogonal matrices with determinant negative one

(7) $Sp(N)$: Ensemble of $2N \times 2N$ symplectic matrices

(8) $\xrightarrow{\mathcal{D}}$: 'Converges in distribution to'

(9) CLT: Central Limit Theorem

(10) m.g.f. : moment generating function

(11) c.g.f. : cumulant generating function

(12) $\lfloor x \rfloor$: greatest integer less than or equal to x, $\lfloor x \rfloor = k \in \mathbb{Z}$ where $k \leq x < k+1$

(13) $a_N << b_N$: $\left|\frac{a_N}{b_N}\right| \to 0$ as $N \to \infty$

(14) $a_{N,k} = o_N(b_N)$: $\left|\frac{a_{N,k}}{b_N}\right| \to 0$ as $N \to \infty$, independent of k

(15) $a_N = O(b_N)$: $\left|\frac{a_N}{b_N}\right| \leq C$ for sufficiently large N

(16) $t_{N,k}$, t_k: The trace of the k-th power of a random matrix from the contextually relevant ensemble

(17) χ_A: Indicator function on condition A

(18) $\Im(z)$: The imaginary part of z

(19) $\Re(z)$: The real part of z

(20) $\mathbb{T}^n$: The n-torus

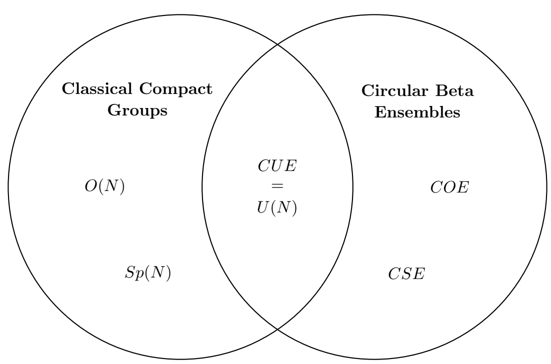
(21) $\mathbb{H}^\alpha(\mathbb{T})$ $(\alpha \geq 0) := \left\{ f \in L^2(\mathbb{T}) \text{ such that } \sum_{k\in\mathbb{Z}} k^{2\alpha} |\hat{f}(k)|^2 < \infty \right\}$.

CHAPTER 1

Introduction

Random Matrix Theory (RMT) can be traced back to the study of random covariance matrices by John Wishart in 1928 [27]. In the 1950s, RMT received a great deal of attention from the Physics community after Eugene Wigner used statistical properties of random matrix ensembles as a means to model the interactions of large systems of nuclear particles [26]. Since then, random matrices have been utilized as a valuable tool for studying a broad range of problems in areas such as Mathematics, Statistics, Physics, and Computer Science.

If M is random matrix, then the operator norm, determinant, trace, eigenvalues, and eigenvectors of M are all examples of random quantities that one might be interested in studying. In this dissertation, we consider statistics related to pairs of eigenvalues from a variety of random matrix ensembles, namely the Circular-Beta Ensembles ($C\beta E$), the ensemble of $N \times N$ orthogonal matrices with determinant one,$SO(N)$, sampled according to Haar measure, and the ensemble of $2N \times 2N$ symplectic matrices, $Sp(N)$ (also sampled with respect to Haar measure). In each of these ensembles, the eigenvalues are random quantities distributed on the unit circle.



The $C\beta E$ was first introduced by F. Dyson as a collection of random matrix ensembles in [9]-[11] for the particular cases $\beta = 1$, $\beta = 2$, and $\beta = 4$ corresponding to the Circular Orthogonal

Ensemble (COE), the Circular Unitary Ensemble (CUE), and the Circular Symplectic Ensemble (CSE), respectively. In all three cases, the N eigenvalues of these random matrix ensembles are of the form $e^{i\theta}$, where $\theta \in [0, 2\pi)$. Moreover, we say that $\theta_1, \ldots, \theta_N$ are distributed according to the CβE if their joint probability density is given by

$$P_{N,\beta}(\theta) = \frac{1}{Z_{N,\beta}} \prod_{1 \leq j < k \leq N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta \tag{1.0.1}$$

$$= \frac{1}{Z_{N,\beta}} \exp\left[\frac{\beta}{2} \sum_{1 \leq j \neq k \leq N} \ln\left(2 \sin\left(\frac{\theta_j - \theta_k}{2} \right) \right) \right], \tag{1.0.2}$$

where $Z_{N,\beta}$ is a normalization constant that can be explicitly written in terms of the Gamma function. In particular,

$$Z_{N,\beta} = \int_{\mathbb{T}^N} \prod_{1 \leq j < k \leq N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta d\theta = (2\pi)^N \cdot \frac{\Gamma\left(1 + \frac{\beta N}{2}\right)}{\left[\Gamma\left(1 + \frac{\beta}{2}\right)\right]^N}$$

which follows from the Selberg integral formula (see [21]). Equivalently, we might say that the spectrum of a random matrix M has $C\beta E$ statistics or is $C\beta E$ distributed if its eigenvalues are distributed according to (1.0.1). Similar distributions are known for $SO(N)$ and $Sp(N)$, which we describe at the start of Chapter 3. Let us first consider the $C\beta E$ in the particular case where $\beta = 2$. The associated β ensemble describes the eigenvalue distribution for the group of $N \times N$ unitary matrices, $U(N)$, sampled according to Haar measure. As stated above, the eigenvalues of a CUE distributed matrix are of the form $\exp(i\theta_1) \ldots \exp(i\theta_N)$, where $\{\theta_i\}_{i=1}^N$ are distributed according to (1.0.1). A matrix with CUE statistics can be obtained by performing a Gram-Schmidt procedure on N N-dimensional vectors with i.i.d. complex standard normal entries. If U is distributed according to the CUE, then $U^T U$ is distributed according to the COE ($\beta = 1$). Similarly, if U is a $2N \times 2N$ CUE distributed matrix, then it is algebraically equivalent to an $N \times N$ quaternion matrix and so has a corresponding quaternion dual, U^D. $U^D U$ is then a matrix distributed according to CSE ($\beta = 4$). For more details, we refer the reader to [21]. In this dissertation, we also consider the more general case $\beta > 0$. For $\beta \neq 1, 2, 4$, there are explicitly defined matrix models with eigenvalues distributed according to (1.0.1), but the constructions are more sophisticated than for COE and

CSE. A general construction for tridiagonal random matrices with eigenvalues jointly distributed according to (1.0.1) is given by R. Killip in [17].

From a Mathematical Physics point of view, one may view the eigenvalues of a CβE matrix as a finite system of repelling unit charges on the unit circle ($\mathbb{T}$) under a logarithmic Coulomb potential (see 1.0.2), where β, corresponding to inverse temperature, indicates the strength of repulsion between particles. Moreover, a straightforward calculation given in [14] shows that the logarithmic potential obtains its maximum on equidistant spaced particle configurations. A particular point of interest to us will be statistics related to the trace of a large $C\beta E$ distributed matrix. Heuristically, if M is an $N \times N$ $C\beta E$ distributed random matrix, then $\text{Tr}(M^k)$ is the sum of many weakly dependent random quantities with a significant amount of cancellation that comes from the uniform behavior of the eigenvalues on the unit circle. As a result, each individual eigenvalue gives a small contribution to the sum and the Central Limit Theorem suggests that $\text{Tr}(M^k)$ ($k \in \mathbb{N}$) should be approximately normally distributed when N is large. It is then reasonable to assume that statistics related to $\text{Tr}(M^k)$ might have close connections to the normal distribution.

Let f be a reasonably well behaved test function and M be an $N \times N$ CβE distributed matrix with eigenvalues $\left\{e^{i\theta_j}\right\}_{j=1}^{N}$. Additionally, let L_N be a positive, non-decreasing sequence satisfying $L_N \leq N$. Then linear statistics of the form

$$\mathcal{T}_N(f(L_N\cdot)) := \sum_{j=1}^{N} f(L_N\theta_j) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\,\text{Tr}\left(M^k\right), \tag{1.0.3}$$

were $\hat{f}(k)$ denotes the k-th Fourier coefficient of $f(L_N\cdot)$, have been studied at length in [14], [7], [6], [15], [25], and [19]. It was proven by K. Johansson in [14](1988) that, when $L_N = 1$, $\mathcal{T}_N(f)$ converges in distribution to a normally distributed random variable. The result was extended for the case $\beta = 2$ and $L_N << N$ by A. Soshnikov in [25](2000). The result was again extended to the case $\beta > 0$ and $L_N << N$ by G. Lambert in [19](2019). The analogous statistic for the $O(N)$ and $Sp(N)$ was studied in [7] and [15] when $L_N = 1$. Central limit theorems were proven in [18] and [12] for the case where $\mathcal{T}_N(f)$ is equal to the eigenvalue counting function, i.e. f is a sum of indicators for the eigenvalues of the $C\beta E$.

In the following chapters, we consider a closely related 'pair counting' or 'pair dependent' statistic, namely

$$S_N(f(L_N\cdot)) := \sum_{1\leq i\neq j\leq N} f(L_N(\theta_i - \theta_j)), \tag{1.0.4}$$

where the θ_js are jointly distributed according to the CβE, $SO(N)$, or $Sp(N)$, L_N is a non-decreasing sequence satisfying $1 \leq L_N << N$, and f is a suitable test function. The case for $L_N = N$ (local statistics) is studied in [2] for the CUE. This work is largely motivated by a classical result stated by H. Montgomery [22]- [23], which connected the behavior of rescaled non-trivial zeros of the Riemann zeta function with the local eigenvalue statistics ($L_N = N$) for the CUE. Assuming the Riemann Hypothesis to be true, suppose that $\{1/2 \pm \gamma_n\}$ are the 'non-trivial' zeroes of the Riemann zeta function and consider the scaling $\overline{\gamma_n} = \frac{\gamma_n}{2\pi}\log(\gamma_n)$, so that the spacing between neighboring, rescaled zeros is on the order of a constant. Montgomery essentially studied the statistic

$$S_T(\alpha) = \frac{1}{T}\sum_{0<\overline{\gamma}_j,\overline{\gamma}_k\leq T} \frac{\exp(i\alpha(\overline{\gamma}_j - \overline{\gamma}_k))}{1 + \left(\frac{\overline{\gamma}_j - \overline{\gamma}_k}{2\log(T)}\right)^2}$$

for real α, and large, real T. Assuming the Riemann Hypothesis, he was able to rigorously show that $S_T(\alpha)$ behaves as

$$\alpha + [T\log(T)]^{-2\alpha}(1 + o(1)) + o(1)$$

for large T and $0 \leq \alpha \leq 1$. When $\alpha > 1$, Montgomery used heuristic arguments to show that $S_T(\alpha) = 1 + o(1)$. Together, this implies that $S_T(\alpha)$ converges in T to $\min(|\alpha|, 1)$, which is the Fourier transform of

$$\delta(x) - \left(\frac{\sin(\pi x)}{\pi x}\right)^2,$$

the limiting pair correlation function for the local CUE eigenvalue statistics. A very natural next step is to consider statistics of the form (1.0.4).

A crucial part of our analysis comes from the fact that $S_N(f(L_N\cdot))$ can be rewritten in terms of the traces of integer powers of appropriately distributed random matrices when the Fourier coefficients of f decay reasonably fast. In particular, if we consider the case where $L_N = 1$ and $f \in \mathbb{H}^1(\mathbb{T})$ is even with $\hat{f}(0) = 0$, then

$$S_N(f) = 2 \sum_{k \in \mathbb{N}} \hat{f}(k) \left(|t_{N,k}|^2 - N \right),$$

where $t_{N,k} = \mathcal{T}_N\left(e^{ik\theta}\right)$ denotes the trace of the k-th power of an appropriately distributed random matrix. For completeness, we present here two particularly influential results regarding the statistical behavior of the traces of random matrices sampled from CβE, O(N), and $Sp(N)$ that will be essential to our analysis of $S_N(f(L_N\cdot))$. The first is due to P. Diaconis and M. Shahshahani:

Proposition 1.0.1. *[7]*

(1) *Let $Z_1, ..., Z_m$ be i.i.d. complex, standard normal random variables and let $\alpha = (\alpha_1, ..., \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$. Then, for all $N \geq \max\left(\sum_{j=1}^m j\alpha_j, \sum_{j=1}^m j\beta_j\right)$,*

$$\mathbb{E}_{U(N)}\left(\prod_{j=1}^m t_{N,j}(t_{N,j})\right) = \delta_{\alpha,\beta} \prod_{j=1}^m j^{\alpha_j}\alpha_j!$$
$$= \delta_{\alpha,\beta}\mathbb{E}\left(\prod_{j=1}^m (\sqrt{j}Z_j)^{\alpha_j}(\sqrt{j}Z_j)^{\beta_j}\right),$$

where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and zero otherwise.

(2) *Let $X_1, ..., X_m$ be i.i.d. real, standard normal random variables and let $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$. Then, for all $N \geq \sum_{j=1}^m j\alpha_j$,*

(a)

$$\mathbb{E}_{O(N)}\left(\prod_{j=1}^m t_{N,j}^{\alpha_j}\right) = \prod_{j=1}^m \mathbb{E}(\sqrt{j}X_j + \eta_j)^{\alpha_j},$$

(b)

$$\mathbb{E}_{Sp(N)}\left(\prod_{j=1}^m t_{N,j}^{\alpha_j}\right) = \prod_{j=1}^m \mathbb{E}(\sqrt{j}X_j - \eta_j)^{\alpha_j},$$

where $\eta_j = 1$ if j is even and zero otherwise.

In short, for any finite collection $\{t_{N,\alpha_1}, \dots, t_{N,\alpha_m}\}$ and fixed l, all of the l-th order joint moments are precisely the same as those for independent Gaussian random variables for large enough N.

Consequently, $\{t_{\alpha_i,N}\}_{i=1}^m$ converges in finite dimensional distribution to a collection of Gaussian random variables as $N \to \infty$.

Remark 1.0.2. While Diaconis and Shahshahani proved an exact formula for the joint moments of a finite collection $\{t_{N,\alpha_1}, \ldots, t_{N,\alpha_m}\}$ for sufficiently large N, it was K. Johansson who first proved the finite dimensional convergence of $\{t_{N,\alpha_1}, \ldots, t_{N,\alpha_m}\}$ to independent Gaussians via analysis of Hankel determinants in [14].

The second result is due to T. Jiang and and S. Matsumoto, which extends the result of Diaconis and Shahshahani for CUE ($\beta = 2$) to the case of general β

Proposition 1.0.3. *[13] Let M be distributed according to $C\beta E$ and $\alpha = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$, with $a_j, b_j \in \{0, 1, \ldots\}$. If $a = b$ and $N \geq K = \sum_{j=1}^M j \cdot a_j$, then*

$$A(N,K,\beta) \prod_{j=1}^k j^{a_j} a_j! \;\leq\; \mathbb{E}\left(\prod_{j=1}^k |t_{N,j}|^{2a_j} \right) \;\leq\; B(N,K,\beta) \prod_{j=1}^k j^{a_j} a_j!,$$

where

$$A(N,K,\beta) = \left(1 - \frac{\left|\frac{2}{\beta} - 1\right|}{N - K + \frac{2}{\beta}} \chi_{(\beta \leq 2)} \right)^K \quad \text{and} \quad B(N,K,\beta) = \left(1 + \frac{\left|\frac{2}{\beta} - 1\right|}{N - K + \frac{2}{\beta}} \chi_{(\beta > 2)} \right)^K.$$

Once again, we can see that the joint moments of any finite collection $t_{N,k_1}, \ldots, t_{N,k_l}$ behave asymptotically like those of independent complex Gaussians up to a well behaved constant multiple.

This dissertation is organized as follows. Chapter Two is dedicated to proving results related to $C\beta E$. We pay special attention CUE ($\beta = 2$), where our main results hold under optimal conditions. The primary tool for proving optimal conditions is an explicit variance calculation using the k-point correlation functions for CUE, which we provide in Section 2.4. Chapter Three is dedicated to proving analogous results for the classical compact groups $SO(N)$ and $Sp(N)$. Again, we are able to obtain optimal conditions via a lengthy variance calculation, this time using joint cumulants. The details of the calculation are given in Section 3.5.

CHAPTER 2

Pair Dependent Linear Statistics for CβE

In this chapter, we consider the limiting distribution of pair counting eigenvalues statistics for the Circular Beta Ensembles (CβE). In particular, let $\theta_1, \dots \theta_N$ be jointly distributed according to CβE (see 1.0.1) and f be a sufficiently smooth, even, test function. Then we study the limiting distribution of

$$S_N(f(L_N\cdot)) := \sum_{1\le i\neq j\le N} f(L_N(\theta_i-\theta_j)) \tag{2.0.1}$$

as $N\to\infty$, i.e. the dimension of our matrix from CβE goes to infinity. Here, L_N is a positive sequence satisfying $L_N << N$. When $L_N = 1$, we determine that the limiting distribution is a sum of independent exponentials under mild conditions on f (depending on β). When $1 << L_N << N$, we prove a central limit theorem. For the case $\beta = 2$, our primary tools are the k-point correlation functions, which allow us to give an exact formula for variance and, consequently, determine optimal conditions for convergence. For the case $\beta \neq 2$, we rely on the results of Jiang and Matsumoto [13] on the joint moments of traces of general CβE matrices.

2.1. Main Results

Denote by

$$\hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)e^{-ik\theta}d\theta$$

the $k-th$ Fourier coefficient of f. The following result holds for the case where $L_N = 1$.

THEOREM 2.1.1. *Let $\{\theta_i\}_{i=1}^N$ be distributed according to CβE and*

$$S_N(f) = \sum_{1\le i\neq j\le N} f(\theta_i-\theta_j), \tag{2.1.1}$$

where f is a real valued, even function on the unit circle such that $f \in H^1(\mathbb{T})$ for $\beta = 2$, $\sum_{k\in\mathbb{Z}} |k| \times |\hat{f}(k)|$ for $\beta < 2$, $\sum_{k\in\mathbb{Z}} |k|^2 |\hat{f}(k)|$ for $\beta > 2$. Then we have the following convergence in distribution as $N \to \infty$:

$$S_N(f) - \mathbb{E}S_N(f) \xrightarrow{\mathcal{D}} \frac{4}{\beta} \sum_{k=1}^{\infty} \hat{f}(k)(\varphi_k - 1), \tag{2.1.2}$$

where $\{\varphi_k\}_{k=1}^{\infty}$ are i.i.d. exponential(1) random variables.

Remark 2.1.2. For the case where $\beta = 4$, corresponding to CSE, we can relax the condition on f to $\sum_{k\in\mathbb{Z}} |\hat{f}(k)||k| \log(|k| + 1) < \infty$.

Remark 2.1.3. For the case where $\beta = 2$, one has

$$\mathbb{E}S_N(f) = N^2 \hat{f}(0) - N f(0) + \sum_{k\in\mathbb{Z}} \min(|k|, N)\hat{f}(k) \tag{2.1.3}$$

and

$$\mathrm{Var}(S_N(f)) = 2 \sum_{k=1}^{N-1} (k\hat{f}(k))^2 + o_N(1).$$

A rigorous calculation is given in Section 2.4. More generally, when $\beta \neq 2$, we can write

$$\mathbb{E}S_N(f) = N^2 \hat{f}(0) - N f(0) + \frac{2}{\beta} \sum_{k\in\mathbb{Z}} |k|\hat{f}(k) + o_N(1). \tag{2.1.4}$$

Remark 2.1.4. If $\sum_{k\in\mathbb{Z}} k^2 \hat{f}^2(k) \to \infty$, then $\mathrm{Var}(S_N(f)) \to \infty$, so it is reasonable to assume that, after renormalization, one might still be able to prove a CLT in certain cases. Although a general proof of such a statement is outside the scope of this dissertation, we prove the special case where $f(\theta) = \frac{1}{2} \ln\left(2 \sin\left(\frac{\theta}{2}\right)\right)$ in Section 2.5.

We now turn our attention to the mesoscopic case, $1 << L_N << N$. Let $f \in C_c^\infty(\mathbb{R})$ be a smooth, compactly supported, even function and consider the random variable

$$S_N(f(L_N \cdot)) = \sum_{1\le i\neq j\le N} f(L_N(\theta_i - \theta_j)) \tag{2.1.5}$$

In this context, denote by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

the Fourier transform of f. The following result holds:

THEOREM 2.1.5. *Assume that $1 << L_N << N$, for $\beta = 2$ and that L_N grows to infinity slower than any power of N for $\beta \neq 2$. Then $(S_N(f(L_N\cdot)) - \mathbb{E}S_N(f(L_N\cdot)))L_N^{-1/2}$ converges in distribution to centered real Gaussian random variable with the variance*

$$\frac{4}{\pi\beta^2} \int_{\mathbb{R}} |\hat{f}(t)|^2 t^2 dt.$$

REMARK 2.1.6. For the case where $\beta = 2$, we can actually weaken the smoothness condition to $f \in C_c^2(\mathbb{R})$.

REMARK 2.1.7. A central limit theorem for the case where $L_N = N$ and $\beta = 2$ is proven in [2] via a combinatorial argument involving joint cumulants. In particular, let $f \in C_c^\infty(\mathbb{R})$ be an even, smooth, compactly supported function. Then

$$\frac{S_N(f(N\cdot)) - \mathbb{E}S_N(f(N\cdot))}{\sqrt{N}} \tag{2.1.6}$$

converges in distribution to a centered real Gaussian variable with variance

$$\frac{1}{\pi} \int_{\mathbb{R}} |\hat{f}(t)|^2 \min(|t|, 1)^2 dt - \frac{1}{\pi} \int_{|s-t|\leq 1, |s|\vee|t|\geq 1} \hat{f}(t)\hat{f}(s)(1 - |s-t|) ds dt - \frac{1}{\pi} \int_{0\leq s,t\leq 1, s+t>1} \hat{f}(s)\hat{f}(t)(s+t-1) ds dt. \tag{2.1.7}$$

The formula for variance follows immediately from the computation given in Section 2.4.

2.2. Proof of Theorem 2.1.1

This section is devoted to the proof of Theorem 2.1.1. We begin with the case where $\beta = 2$. We provide the details for $\beta \neq 2$ at the end of the section. We first show that the case where f is a trigonometric polynomial is in fact an immediate corollary of Theorem 1.1.1. To extend the result to more general test functions, we utilize a variance bound to give an $\epsilon/3$ type argument. To prove the theorem under the optimal condition $\sum_{k\in\mathbb{Z}} k^2|\hat{f}(k)|^2 < \infty$, we require the explicit formula for $\mathrm{Var}(S_N(f))$ given in Section 2.4. The details for $\beta \neq 2$ are given at the end of the section.

Throughout this section we will denote by

$$t_{N,k} := \sum_{j=1}^{N} e^{ik\theta_j} \tag{2.2.1}$$

the trace of the k-th power of an $N \times N$ CβE matrix.

We begin with the case $\beta = 2$. Let f be a real, even, trigonometric polynomial, i.e.

$$f(\theta) = \sum_{|k|\leq m} \hat{f}(k)e^{ik\theta}.$$

We may assume that $\hat{f}(0) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)d\theta = 0$, since

$$\begin{aligned} S_N(f) - \mathbb{E}S_N(f) &= \sum_{1\leq i,j\leq N} f(\theta_i - \theta_j) - (N^2 - N)\hat{f}(0) - 2\sum_{k=1}^{m}(|k| - N)\hat{f}(k) \\ &= \sum_{1\leq i,j\leq N} [f(\theta_i - \theta_j) - \hat{f}(0)] - 2\sum_{k=1}^{m}(|k| - N)\hat{f}(k). \end{aligned} \tag{2.2.2}$$

Moreover, we can rewrite (2.2.2) in terms of traces, $\{t_{N,k}\}_{k\in\mathbb{Z}}$. In particular,

$$\begin{aligned} S_N(f) - \mathbb{E}S_N(f) &= 2\sum_{k=1}^{m} \hat{f}(k)\left(|t_{N,k}|^2 - N - k\right) + Nf(0) \\ &= 2\sum_{k=1}^{m} k\hat{f}(k)\left(\frac{1}{k}|t_{N,k}|^2 - 1\right) \end{aligned}$$

Proposition 1.0.1 implies that $\left\{\frac{t_{N,k}}{\sqrt{k}}\right\}_{k=1}^{m}$ converge, in finite dimensional distribution, to a sequence of i.i.d. complex, standard normal random variables. By the Continuous Mapping theorem [3],

$S_N(f) - \mathbb{E}S_N(f)$ converges in distribution to

$$2\sum_{k=1}^{m} k\hat{f}(k)\left(X_k^2 + Y_k^2 - 1\right),$$

where $\{X_k\}_{k=1}^m, \{Y_k\}_{k=1}^m$ are i.i.d. Real Gaussian random variables withe mean zero and variance $\frac{1}{2}$. Recalling that $X_k^2 + Y_k^2$ has exponential(1) distribution completes the proof of the case where f is trigonometric polynomial.

We now consider the case of more general test functions. Before proceeding with the proof, we present two necessary propositions. The first proposition gives an exact formula for Var $(S_N(f))$.

PROPOSITION 2.2.1. *Let f be a real valued, even function on the unit circle such that $f' \in L^2[\mathbb{T}]$ and let $\beta = 2$. Then*

$$Var(S_N(f)) = 4\left(\sum_{1\le s\le N-1} s^2[\hat{f}(s)]^2 + N^2\sum_{N\le s}[\hat{f}(s)]^2 - N\sum_{N\le s}[\hat{f}(s)]^2\right)$$

$$-4\left(\sum_{\substack{1\le s,t\\ 1\le|s-t|\le N-1\\ N\le \max(s,t)}} (N-|s-t|)\hat{f}(s)\hat{f}(t) + \sum_{\substack{1\le s,t\le N-1\\ N+1\le s+t}} ((s+t)-N)\hat{f}(s)\hat{f}(t)\right).$$

The next proposition allows us to prove Theorem 2.1.1 under the optimal assumptions on the test function f.

PROPOSITION 2.2.2. *Let $\beta = 2$ and f satisfy the conditions of Theorem 2.1.1, i.e. f is an even real function such that $f' \in L^2(\mathbb{T})$. Then*

$$Var(S_N(f)) = 4\sum_{1\le k\le N-1} k^2|\hat{f}(k)|^2 + o_N(1).$$

We provide proofs of Propositions 2.2.1-2.2.2 in Section 2.4.

Now, continuing with the proof, let

$$f_m(\theta) = 2\sum_{k=1}^{m}\hat{f}(k)e^{ik\theta}, \qquad f_{\overline{m}}(\theta) = 2\sum_{k=m+1}^{\infty}\hat{f}(k)e^{ik\theta},$$

$$T_m = 2\sum_{k=1}^{m} \hat{f}(k)(\varphi_k - 1), \qquad \text{and} \qquad T_\infty = 2\sum_{k=1}^{\infty} \hat{f}(k)(\varphi_k - 1).$$

By the previously stated argument, $S_N(f_m) - \mathbb{E}S_N(f_m)$ converges in distribution to T_m. Now

$$\sum_{k=1}^{\infty} \mathrm{Var}\left(k\hat{f}(k)(X_k^2 + Y_k^2 - 1)\right) = \sum_{k=1}^{\infty} k^2(\hat{f}(k))^2$$

converges under the assumptions of the theorem and $\mathbb{E}(T_m) = 0$ for all k, so the Kolmogorov Two Series theorem [8] implies that T_m converges almost surely to T_∞. To complete the proof, one shows that $S_N(f)$ converges to T_∞ with respect to the Lèvy metric, which implies convergence in distribution, by using a standard $\epsilon/3$ type argument combined with a Chebyshev bound for the tail statistic $S_N(f_{\overline{m}})$. The necessary Chebyshev bound follows from Proposition 2.2.2. The precise details of the proof can be found in Appendix A.1.

For the case where $\beta \neq 2$, we apply the same $\epsilon/3$ argument in the case of $\beta = 2$, but replace the Chebyshev bound by the corresponding Markov and apply Proposition 1.0.3. Once again, we direct the reader to Appendix A.1 for the complete details.

2.3. Proof of Theorem 2.1.5

This section is devoted to the proof of Theorem 2.1.5. We use the Lindeberg-Feller condition when $\beta = 2$ and the method of moments when $\beta \neq 2$.

Proof of Theorem 2.1.5

When N is sufficiently large, the support of $f(L_N \cdot)$ is contained in the interval $[-\pi, \pi)$. In particular, $f(L_N \cdot)$ has a Fourier series given by

$$f(L_N \theta) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2\pi} L_N} \hat{f}\left(\frac{k}{L_N}\right) e^{ik\theta}$$

where $\theta \in [-\pi, \pi)$ and

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

is the Fourier transform of f.

Remark 2.3.1.

It follows immediately from the joint probability density for $\{\theta_i\}_{i=1}^N$, (1.0.1), that $\{L_N \theta_i\}_{i=1}^N$ are not truly distributed on the real line, but rather a circle of increasing radius. Accordingly, the above

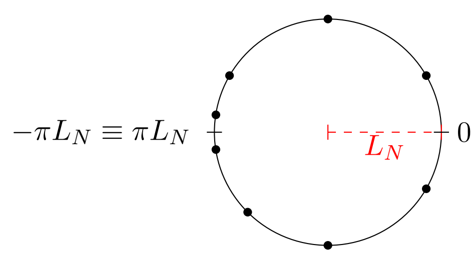


Figure 2.1. Example Configuration

Fourier series is a 2π-periodization of $f(L_N \cdot)$, which correctly preserves the circular relationship between $\{\theta_i\}_{i=1}^N$.

Rewriting $S_N(f(L_N \cdot))$ in terms of the above Fourier series, we have

$$S_N(f(L_N \cdot)) = \sum_{1 \leq j \neq k \leq N} f(L_N(\theta_j - \theta_k)) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2\pi} L_N} \hat{f}\left(\frac{k}{L_N}\right) \left(\left| \sum_{m=1}^N e^{ik\theta_m} \right|^2 - N \right).$$

Consider first the case $\beta = 2$, so $\{\theta_j\}_{j=1}^N$ are distributed according to the CUE. We have

$$S_N(f(L_N\cdot)) - \mathbb{E}(S_N(f(L_N\cdot))) = 2\sum_{k\geq 1} \frac{1}{\sqrt{2\pi}L_N}\hat{f}\left(\frac{k}{L_N}\right)\left(\left|\sum_{m=1}^N e^{ik\theta_m}\right|^2 - \min(k,N)\right). \tag{2.3.1}$$

By inserting the Fourier coefficients for the mesoscopic case into the variance formula given in Proposition 2.2.1, we see that the term which determined variance in the unscaled case, the last term in (2.4.13), becomes

$$L_N\left[\frac{2}{\pi L_N}\sum_{1\leq k\leq N-1}\left(\frac{k}{L_N}\right)^2\left(\hat{f}\left(\frac{k}{L_N}\right)\right)^2\right].$$

The term in the brackets is nearly a Riemann sum which converges to $||f'||_2/\pi$. Thus, the variance of $S_N(f(L_N\cdot))$ is proportional to L_N. We then normalize (2.3.1) by $\sqrt{L_N/(2\pi)}$ and break it up into two pieces:

$$\begin{aligned}\frac{2}{\sqrt{L_N}}\sum_{k=1}^{m_N}\frac{k}{L_N}\hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)}-1) \\ + \frac{2}{\sqrt{L_N}}\sum_{k=m_N+1}^{\infty}\frac{1}{L_N}\hat{f}\left(\frac{k}{L_N}\right)\left(\left|\sum_{m=1}^N e^{ik\theta_m}\right|^2 - \min(k,N)\right),\end{aligned} \tag{2.3.2}$$

where $m_N = \lfloor\sqrt{NL_N}\rfloor$ and

$$\varphi_k^{(N)} := \frac{1}{k}\left|\sum_{m=1}^N e^{ik\theta_m}\right|^2 = \frac{1}{k}|t_{N,k}|^2.$$

We show in Appendix A.2 that the variance of the second sum in (2.3.2) converges to zero by applying Proposition 2.2.1 and analogous arguments from the proof of Proposition 2.2.2. Therefore, it is enough to study the asymptotic distribution of the first sum:

$$\Sigma_N = \frac{2}{\sqrt{L_N}}\sum_{k=1}^{m_N}\frac{k}{L_N}\hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)}-1). \tag{2.3.3}$$

Consider the sequence of random variables $\{\varphi_k^{(N)}\}_{k=1}^{m_N}$ labeled by positive integer k. As $N\to\infty$, this sequence converges, in finite-dimensional distributions, to a sequence of i.i.d. exponential random variables $\{\varphi_k\}_{k=1}^{m_N}$. Moreover, (1.0.1), together with the fact that $L_N << \sqrt{NL_N} << N$, implies that, for any fixed n and sufficiently large N (depending on n), all joint moments up to order n

of random variables $\{\varphi_k^{(N)}\}_{k=1}^{m_N}$ coincide with the corresponding joint moments of i.i.d. exponential random variables $\{\varphi_k\}_{k=1}^{m_N}$. Therefore, it is enough to study the asymptotic distribution of

$$\Sigma_N = \frac{2}{\sqrt{L_N}} \sum_{k=1}^{m_N} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k - 1). \tag{2.3.4}$$

This is done below by routine computation, as we show that the sequence of random variables in the above sum satisfy the Lindeberg-Feller condition [8]. Since the moment generating function for a finite sum of independent exponential random variables is well defined on an interval of positive radius about the origin, one could also proceed by showing that the moment generating function for the truncated sum converges to that of the desired Gaussian distribution. For completeness, we provide the details for the Lindeberg-Feller argument directly below and the details for the moment generating function argument in Appendix A.2.

Let

$$c_{N,k} = \frac{2k}{L_N^{(3/2)}} \hat{f}\left(\frac{k}{L_N}\right), \qquad X_{N,k} = c_{N,k}(\varphi_k - 1).$$

Then $\mathbb{E}(X_k) = 0$, $\mathrm{Var}(X_k) = c_{N,k}^2$, and

$$\Sigma_N = \sum_{k=1}^{m_N} X_{N,k},$$

Denote by s_N^2 the variance of Σ_N, i.e.

$$s_N^2 = \sum_{k=1}^{m_N} c_{N,k}^2.$$

To see that the sequence of random variables (X_k) satisfy the Lindeberg-Feller condition, we check that, given $\epsilon > 0$,

$$\frac{1}{s_N^2} \sum_{k=1}^{m_N} \mathbb{E}(X_k^2 1_{|X_k| > \epsilon s_N}) \to 0.$$

If $|c_{N,k}| = 0$ for some k, then $\mathbb{E}(X_k^2 1_{|X_k| > \epsilon s_N}) = 0$, so, without loss of generality, we will assume that $|c_{N,k}| > 0$ for all k and N. Moreover, s_N is proportional to a constant and, since f' is continuous and bounded, we have $1/|c_{N,K}| \geq C\sqrt{L_N}$ for some positive constant C that is independent of k and

N. It follows that, for large enough N, we can write

$$\begin{aligned}\mathbb{E}(X_k^2 1_{|X_k|>\epsilon s_N}) &= c_{N,k}^2 \mathbb{E}([\varphi_k^2 - 2\varphi_k + 1] 1_{(\varphi_k - 1) > \epsilon s_N / |c_{N,k}|}) + c_{N,k}^2 \mathbb{E}([\varphi_k^2 - 2\varphi_k + 1] 1_{(\varphi_k - 1) < -\epsilon s_N / |c_{N,k}|}) \\ &= c_{N,k}^2 \mathbb{E}([\varphi_k^2 - 2\varphi_k + 1] 1_{(\varphi_k - 1) > \epsilon s_N / |c_{N,k}|}) \\ &= c_{N,k}^2 \int_{x > 1 + (\epsilon s_N / |c_{N,k}|)} (x^2 - 2x + 1) e^{-x} dx.\end{aligned}$$

Once again, for large enough N, we have

$$\begin{aligned}\mathbb{E}(X_k^2 1_{|X_k|>\epsilon s_N}) &\leq c_{N,k}^2 \int_{x > \gamma_N} (x^2 - 2x + 1) e^{-x} dx \\ &= c_{N,k}^2 e^{-\gamma_N} (\gamma_N^2 + 1),\end{aligned}$$

where $\gamma_N = C\sqrt{L_N}[\epsilon\ s_N]$. Clearly $\gamma_N = O(\sqrt{L_N})$ and $e^{-\gamma_N}(\gamma_N^2 + 1)$ goes to zero independent of k. This immediately implies

$$\frac{1}{s_N^2} \sum_{k=1}^{m_N} \mathbb{E}(X_k^2 1_{|X_k|>\epsilon s_N}) \leq \frac{1}{s_N^2} \sum_{k=1}^{m_N} c_{N,k}^2 \cdot o_N(1) = o_N(1),$$

so the Lindeberg-Feller condition is satisfied and we can conclude that

$$\frac{\Sigma_N}{s_N} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \tag{2.3.5}$$

where s_N^2 is a Riemann sum that converges to

$$2 \int_{\mathbb{R}} [x\hat{f}(x)]^2\ dx$$

as $N \to \infty$. This completes the proof of Theorem 2.1.5 when $\beta = 2$.

The proof for the case when $\beta \neq 2$ relies on the results by Jiang and Matsumoto [13] that, in particular, state that for any finitely many positive integers $k_1, k_2, \ldots k_n$, $k_i << N$, $1 \leq i \leq n$, one has

$$\mathbb{E} \prod_{i=1}^{n} \varphi_{k_i}^{(N)} = \left(\mathbb{E} \prod_{i=1}^{n} \varphi_{k_i} \right) \left(1 + O\left(\frac{k_1 + \ldots k_n}{N} \right) \right). \tag{2.3.6}$$

Namely, we proceed as follows. As in the case when $\beta = 2$, we write

$$\frac{S_N(f(L_N\cdot)) - \mathbb{E}(S_N(f(L_N\cdot)))}{\sqrt{L_N}} = \frac{2}{\sqrt{2\pi L_N}} \sum_{k\geq 1} \frac{1}{L_N} \hat{f}\left(\frac{k}{L_N}\right)\left(|t_{N,k}|^2 - \mathbb{E}\,|t_{N,k}|^2\right)$$
$$= \frac{2}{\sqrt{2\pi L_N}} \sum_{k=1}^{\infty} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)}).$$

We then split the last sum into three pieces, namely

$$\frac{2}{\sqrt{L_N}} \sum_{k=1}^{L_N^2} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)}) + \frac{2}{\sqrt{L_N}} \sum_{k=L_N^2+1}^{N/10} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)})$$
$$+ \frac{2}{\sqrt{L_N}} \sum_{k>N/10}^{\infty} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)}),$$

and deal with each term separately. The variance of the second term goes to zero as $N \to \infty$ since the Fourier transform of f decays sufficiently fast for $f \in C_c^\infty(\mathbb{R})$ and $\mathbb{E}|\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)}|^2$ is bounded for $k \leq N/10$. Here, the bound on the variance of $\varphi_k^{(N)}$ follows from (2.3.6).

The variance of the third term goes to zero as well. Indeed, we bound $\mathbb{E}|\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)}|^2$ from above by N^4/k^2 for $k > N/10$ and again use the fast decay of $\hat{f}\left(\frac{k}{L_N}\right)$ to finish the argument.

Now we turn our attention to the first term,

$$\frac{2}{\sqrt{L_N}} \sum_{k=1}^{L_N^2} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k^{(N)} - \mathbb{E}\varphi_k^{(N)}). \tag{2.3.7}$$

It follows from (2.3.6) that, for any positive integer l, the l-th moment of (2.3.7) is equal to the l-th moment of

$$\frac{2}{\sqrt{L_N}} \sum_{k=1}^{L_N^2} \frac{k}{L_N} \hat{f}\left(\frac{k}{L_N}\right)(\varphi_k - 1), \tag{2.3.8}$$

up to a vanishing error term of order $O(L_N^{l/2+2}N^{-1})$. Again, the exponential moment of (2.3.8) converges to that of a Gaussian random variable. Theorem 2.1.5 is proven.

2.4. Variance Calculation ($\beta = 2$)

This section is devoted to the computation and asymptotic analysis of the variance of the pair counting statistic $S_N(f)$ defined in (1.0.4). In particular, we prove Proposition 2.2.1 and Proposition 2.2.2. The covariance function for $\{|t_{N,k}|^2\}_{k=1}^{\infty}$ is presented as a corollary to Proposition 2.2.1 We assume $\beta = 2$ for the rest of the section.

First, we prove Proposition 2.2.1. The proof follows from quite straightforward, but somewhat tedious computations given below. The reader might want to skip the details initially.

Proof of Proposition 2.2.1

We may assume, without loss of generality, that $\hat{f}(0) = 0$. Let $\rho_{N,k}(\bar{\theta})$ be the k-point correlation functions for $\{\theta_j\}_{j=1}^{N}$ distributed according to the CUE. It is well known that CUE point correlation functions have determinantal structure (see e.g. [21]). In particular, if $Q_N(x,y)$ is the kernel of the orthogonal projection on
$Span\{\frac{1}{\sqrt{2\pi}}e^{ikx}, \quad 0 \le k \le N-1\}$, namely

$$Q_N(x,y) = \frac{1}{2\pi}\sum_{k=0}^{N-1} e^{i(x-y)}, \tag{2.4.1}$$

then

$$\rho_{N,k}(\theta_1,\dots,\theta_k) = \det\big(Q_N(\theta_i,\theta_j)\big)_{1\le i,j\le k}. \tag{2.4.2}$$

Using the above determinental structure for $\rho_{N,2}(\theta_1,\theta_2)$, we can see that

$$\begin{aligned}
\mathbb{E}(S_N(f)) &= \mathbb{E}\left(\sum_{1\le i\ne j\le N} f(\theta_i-\theta_j)\right)\\
&= \int_{\mathbb{T}^2} f(\theta_1-\theta_2)\rho_{N,2}(\theta_1,\theta_2)d\theta_1 d\theta_2\\
&= \int_{\mathbb{T}^2} f(\theta_1-\theta_2)[Q_N(\theta_1,\theta_1)Q_N(\theta_2,\theta_2)-Q_N(\theta_1,\theta_2)Q_N(\theta_2,\theta_1)]d\theta_1 d\theta_2
\end{aligned}$$

$$= \left(\frac{N}{2\pi}\right)^2 \int_{\mathbb{T}^2} f(\theta_1-\theta_2)d\theta_1 d\theta_2 - \sum_{0\le j,k\le N-1} \left(\frac{1}{2\pi}\right)^2 \int_{\mathbb{T}^2} f(\theta_1-\theta_2)e^{ij(\theta_1-\theta_2)}e^{ik(\theta_2-\theta_1)}d\theta_1 d\theta_2$$

$$= N^2\hat{f}(0) - \sum_{0\le j,k\le N-1} \left(\hat{f}(j-k)\right)^2$$

$$= -\sum_{|k|\le N-1} (N-|k|)\hat{f}(k).$$

Furthermore, using the fact that f is even, we can see that the variance of $S_N(f)$ is given by

$$\mathbb{E}((S_N(f))^2) - (\mathbb{E}(S_N(f)))^2 = \tag{2.4.3}$$

$$\mathbb{E}\left(2\sum_{1\le i\ne j\le N} f^2(\theta_i-\theta_j)\right) \tag{2.4.4}$$

$$+\,\mathbb{E}\left(4\sum_{1\le i\ne j\ne k\le N} f(\theta_i-\theta_j)f(\theta_k-\theta_j)\right) \tag{2.4.5}$$

$$+\,\mathbb{E}\left(\sum_{1\le i\ne j\ne k\ne l\le N} f(\theta_i-\theta_j)f(\theta_k-\theta_l)\right) - (E_N(S_N(f)))^2 \tag{2.4.6}$$

Rewriting these expectations in terms of the two, three, and four point correlations functions for the CUE, we have

$$\mathrm{Var}\,(S_N(f)) = \tag{2.4.7}$$

$$2\int_{\mathbb{T}^2} f^2(\theta_1-\theta_2)\rho_{N,2}(\theta_1,\theta_2)d\theta_1 d\theta_2 \tag{2.4.8}$$

$$+\,4\int_{\mathbb{T}^3} f(\theta_1-\theta_2)f(\theta_3-\theta_2)\rho_{N,3}(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3 \tag{2.4.9}$$

$$+\int_{\mathbb{T}^4} f(\theta_1-\theta_2)f(\theta_3-\theta_4)\rho_{N,4}(\theta_1,\theta_2,\theta_3,\theta_4)d\theta_1 d\theta_2 d\theta_3 d\theta_4 - (E_N(S_N(f)))^2, \tag{2.4.10}$$

which can be written as

$$(2.4.11) \qquad = \qquad 2N^2\left(\hat{f}^2(0) - \sum_{|k|\le N-1} |\hat{f}(k)|^2\right)$$

$$(2.4.12) \qquad + 4 \sum_{0\le j,k,l\le N-1} \hat{f}(j-k)\hat{f}(k-l) - 2 \sum_{0\le j,k,l,m\le N-1} \hat{f}(j-k)\hat{f}(k-l)\chi_{(j-m=k-l)}$$

$$(2.4.13) \qquad - 2N \sum_{|k|\le N-1} \hat{f}^2(k) + 2 \sum_{|k|\le N-1} |k|\hat{f}^2(k) + 2 \sum_{|k|\le N-1} |k|^2|\hat{f}(k)|^2.$$

To see this, we first observe that our expectation calculation above immediately implies that (2.4.8) is equal to

$$(2.4.14) \qquad 2N^2\hat{f}^2(0) + 2 \sum_{|k|\le N-1} (|k| - N)\hat{f}^2(k).$$

Using the assumption $\hat{f}(0) = 0$ and (2.4.2) to expand $\rho_{N,3}(\overline{\theta})$ in (2.4.9), we observe that the majority of terms will give zero contribution after integrating. Combining the remaining non-negligible terms that give an equal contribution, we can rewrite (2.4.9) as

$$(2.4.15) \qquad - \frac{4N}{2\pi} \int_{\mathbb{T}^3} f(\theta_1-\theta_3)f(\theta_2-\theta_3)Q_N(\theta_1,\theta_2)Q_N(\theta_2,\theta_1)d\theta_1 d\theta_2 d\theta_3$$

$$(2.4.16) \qquad + 4 \int_{\mathbb{T}^3} f(\theta_1-\theta_3)f(\theta_2-\theta_3)Q_N(\theta_1,\theta_2)Q_N(\theta_2,\theta_3)Q_N(\theta_3,\theta_1)d\theta_1 d\theta_2 d\theta_3$$

$$(2.4.17) \qquad + 4 \int_{\mathbb{T}^3} f(\theta_1-\theta_3)f(\theta_2-\theta_3)Q_N(\theta_1,\theta_3)Q_N(\theta_3,\theta_2)Q_N(\theta_2,\theta_1)d\theta_1 d\theta_2 d\theta_3.$$

Term (2.4.15) is equal to

$$(2.4.18) \qquad - \frac{4N}{(2\pi)^3} \sum_{0\le j,k\le N-1} \int_{\mathbb{T}^3} f(\theta_1-\theta_3)f(\theta_2-\theta_3)e^{i(j-k)\theta_1}e^{i(k-j)\theta_2} d\theta_1 d\theta_2 d\theta_3$$

$$(2.4.19) \qquad = - \frac{4N}{2\pi} \sum_{0\le j,k\le N-1} \hat{f}(k-j)\hat{f}(j-k) \int_{\mathbb{T}} 1 \cdot d\theta_3$$

$$(2.4.20) \qquad = -4N \sum_{0\le j,k\le N-1} |\hat{f}(j-k)|^2$$

$$(2.4.21) \qquad = 4N \sum_{|k|\le N-1} (|k| - N)|\hat{f}(k)|^2.$$

Similarly, (2.4.16) can be rewritten as

$$\begin{aligned}
&\frac{4}{(2\pi)^3}\sum_{0\le j,k,l\le N-1}\int_{\mathbb{T}^3} f(\theta_1-\theta_3)f(\theta_2-\theta_3)e^{i(j-l)\theta_1}e^{i(k-j)\theta_2}e^{i(l-k)\theta_3}d\theta_1 d\theta_2 d\theta_3\\
&=\frac{4}{2\pi}\sum_{0\le j,k,l\le N-1}\hat{f}(l-j)\hat{f}(k-j)\int_{\mathbb{T}} e^{i[(j-l)+(k-j)+(l-k)]\theta_3}d\theta_3\\
&=\frac{4}{2\pi}\sum_{0\le j,k,l\le N-1}\hat{f}(l-j)\hat{f}(k-j)\int_{\mathbb{T}} 1\ \ d\theta_3\\
&=4\sum_{0\le j,k,l\le N-1}\hat{f}(k-j)\hat{f}(j-l). \qquad (2.4.22)
\end{aligned}$$

Term (2.4.17) is equal to (2.4.16), so, together, they contribute

$$8\sum_{0\le j,k,l\le N-1}\hat{f}(k-j)\hat{f}(j-l). \qquad (2.4.23)$$

Finally, we turn our attention to (2.4.10). Again, using (2.4.2) with the assumption $\hat{f}(0)=0$ and ignoring the terms that give zero contribution after integrating, we can rewrite (2.4.10) as

$$2\int_{\mathbb{T}^4} f(\theta_1-\theta_2)f(\theta_3-\theta_4)|Q_N(\theta_1,\theta_4)|^2|Q_N(\theta_2,\theta_3)|^2 d\theta_1 d\theta_2 d\theta_3 d\theta_4 \qquad (2.4.24)$$

$$-4\int_{\mathbb{T}^4} f(\theta_1-\theta_2)f(\theta_3-\theta_4)Q_N(\theta_1,\theta_4)Q_N(\theta_4,\theta_3)Q_N(\theta_3,\theta_2)Q_N(\theta_2,\theta_1)d\theta_1 d\theta_2 d\theta_3 d\theta_4 \qquad (2.4.25)$$

$$-2\int_{\mathbb{T}^4} f(\theta_1-\theta_2)f(\theta_3-\theta_4)Q_N(\theta_1,\theta_4)Q_N(\theta_4,\theta_2)Q_N(\theta_2,\theta_3)Q_N(\theta_3,\theta_1)d\theta_1 d\theta_2 d\theta_3 d\theta_4. \qquad (2.4.26)$$

Again using (2.4.1) and "opening the brackets", term (2.4.24) becomes

$$\begin{aligned}
&\frac{2}{(2\pi)^4}\sum_{0\le j,k,l,m\le N-1}\int_{\mathbb{T}^4} f(\theta_1-\theta_2)f(\theta_3-\theta_4)e^{i(j-k)\theta_1}e^{i(l-m)\theta_2}e^{i(m-l)\theta_3}e^{i(k-j)\theta_4}d\theta_1 d\theta_2 d\theta_3 d\theta_4\\
&=\frac{2}{(2\pi)^2}\sum_{0\le j,k,l,m\le N-1}\hat{f}(k-j)\hat{f}(l-m)\int_{\mathbb{T}^2} e^{i[(j-k)+(l-m)]\theta_2}e^{i[(m-l)+(k-j)]\theta_4}d\theta_2 d\theta_4\\
&=2\sum_{0\le j,k,l,m\le N-1}\hat{f}(k-j)\hat{f}(l-m)\chi_{(k-j=l-m)}
\end{aligned}$$

$$= 2 \sum_{0 \le j,k \le N-1} |\hat{f}(k-j)|^2 \left(\sum_{\substack{0 \le l,m \le N-1 \\ l-m=k-j}} 1 \right)$$

$$= 2 \sum_{0 \le j,k \le N-1} (N - |k-j|) |\hat{f}(k-j)|^2$$

$$= 2 \sum_{|k| \le N-1} (N - |k|)^2 |\hat{f}(k)|^2$$

$$= 2N^2 \sum_{|k| \le N-1} |\hat{f}(k)|^2 - 4N \sum_{|k| \le N-1} |k| |\hat{f}(k) + 2 \sum_{|k| \le N-1} k^2 |\hat{f}(k)|^2. \tag{2.4.27}$$

The calculation for (2.4.25) is very similar to (2.4.36). Its contribution is

$$-4 \sum_{0 \le j,k,l \le N-1} \hat{f}(j-k) \hat{f}(k-l). \tag{2.4.28}$$

Lastly, we consider (2.4.26). Using (2.4.1) and "opening the brackets", this term becomes

$$- \frac{2}{(2\pi)^4} \sum_{0 \le j,k,l,m \le N-1} \int_{\mathbb{T}^4} f(\theta_1 - \theta_2) f(\theta_3 - \theta_4) e^{i(j-m)\theta_1} e^{i(l-k)\theta_2} e^{i(m-l)\theta_3} e^{i(k-j)\theta_4} d\theta_1 d\theta_2 d\theta_3 d\theta_4$$

$$= - \frac{2}{(2\pi)^2} \sum_{0 \le j,k,l,m \le N-1} \hat{f}(m-j) \hat{f}(l-m) \int_{\mathbb{T}^2} e^{i[(j-m)+(l-k)]\theta_2} e^{i[(k-j)+(m-l)]\theta_4} d\theta_2 d\theta_4$$

$$= - \frac{2}{(2\pi)} \sum_{0 \le j,k,l,m \le N-1} \hat{f}(m-j) \hat{f}(l-m) \chi_{(j-m=k-l)} \int_{\mathbb{T}} 1 \ d\theta_4$$

$$= -2 \sum_{0 \le j,k,l,m \le N-1} \hat{f}(j-k) \hat{f}(k-l) \chi_{(j-m=k-l)}, \tag{2.4.29}$$

where the last equality comes from using the restriction $j - m = k - l$ and the fact that f is even. By combining (2.4.14), (2.4.21), (2.4.23), (2.4.27), (2.4.28), and (2.4.29), we recover the terms in (2.4.11), (2.4.12), and (2.4.13).

Continuing with our calculation, we use the Plancherel theorem to rewrite (2.4.11) as

$$2N^2 \left(\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2 - \sum_{|k| \le N-1} |\hat{f}(k)|^2 \right) = 4N^2 \sum_{k \ge N} |\hat{f}(k)|^2. \tag{2.4.30}$$

Next, we rewrite the terms in (2.4.12). We start with the first one:

$$4\sum_{0\le j,k,l\le N-1}\hat f(j-k)\hat f(k-l)=4\sum_{|s|,|t|\le N-1}\hat f(s)\hat f(t)\max(0,N-(\max(0,s,t)-\min(0,s,t))) \tag{2.4.31}$$

$$=4\sum_{|s|,|t|\le N-1}\hat f(s)\hat f(t)\max(0,N-L(s,t)), \tag{2.4.32}$$

where

$$L(s,t)=\begin{cases}\max(|s|,|t|), & \text{if } sgn(s)=sgn(t)\\ |s|+|t|, & \text{otherwise.}\end{cases}$$

Splitting up the sum and recalling that $\hat f(s)=\hat f(-s)$, we can further rewrite the first term in (2.4.12) as

$$2\sum_{\substack{|s|,|t|\le N-1\\ |s|+|t|\le N-1}}\hat f(s)\hat f(t)(N-(|s|+|t|))+4\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}\hat f(s)\hat f(t)(N-\max(|s|,|t|)).$$

We rewrite the second term in (2.4.12) as

$$2\sum_{0\le j,k,l,m\le N-1}\hat f(j-k)\hat f(k-l)\chi_{(j-m=k-l)}=2\sum_{\substack{|s|,|t|\le N-1\\ |s|+|t|\le N-1}}\hat f(s)\hat f(t)(N-(|s|+|t|)).$$

Thus, (2.4.12) becomes

$$4\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}\hat f(s)\hat f(t)(N-\max(|s|,|t|)),$$

which can be rewritten as

$$2N\sum_{|s|,|t|\le N-1}\hat f(s)\hat f(t)-2\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}\hat f(s)\hat f(t)(|s-t|+|s+t|). \tag{2.4.33}$$

Combining (2.4.33) with the first two terms of (2.4.13), we have a term proportional to N,

$$2N\left(\sum_{|s|,|t|\le N-1}\hat f(s)\hat f(t)-\sum_{|k|\le N-1}\hat f^2(k)\right), \tag{2.4.34}$$

and a term proportional to a constant,

$$2\left(\sum_{|k|\le N-1}|k|\hat{f}^2(k)-\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s-t|\hat{f}(s)\hat{f}(t)-\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s+t|\hat{f}(s)\hat{f}(t)\right). \tag{2.4.35}$$

The expression (2.4.34) can be rewritten as

$$\begin{aligned}&2N\left(\sum_{|s|,|t|\le N-1}\hat{f}(s)\hat{f}(t)-\sum_{|s+t|\le N-1}\hat{f}(s)\hat{f}(t)\right)\\ =&2N\sum_{\substack{|s|,|t|\le N-1\\ N\le|s+t|}}\hat{f}(s)\hat{f}(t)-2N\sum_{\substack{|s+t|\le N-1\\ N\le\max(|s|,|t|)}}\hat{f}(s)\hat{f}(t).\end{aligned} \tag{2.4.36}$$

Furthermore, (2.4.35) can be rewritten as follows:

$$2\sum_{|s+t|\le N-1}|s+t|\hat{f}(s)\hat{f}(t)-2\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s-t|\hat{f}(s)\hat{f}(t)-2\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s+t|\hat{f}(s)\hat{f}(t).$$

We break up the sum into two parts, namely

$$2\sum_{\substack{|s+t|\le N-1\\ sgn(s)\ne sgn(t)}}|s+t|\hat{f}(s)\hat{f}(t)-2\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s-t|\hat{f}(s)\hat{f}(t) \tag{2.4.37}$$

and

$$2\sum_{\substack{|s+t|\le N-1\\ sgn(s)=sgn(t)}}|s+t|\hat{f}(s)\hat{f}(t)-2\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s+t|\hat{f}(s)\hat{f}(t). \tag{2.4.38}$$

The expression (2.4.37) can be rewritten as

$$2\sum_{\substack{|s-t|\le N-1\\ sgn(s)=sgn(t)}}|s-t|\hat{f}(s)\hat{f}(t)-2\sum_{\substack{|s|,|t|\le N-1\\ sgn(s)=sgn(t)}}|s-t|\hat{f}(s)\hat{f}(t)$$

$$=4\left(\sum_{\substack{|s-t|\le N-1\\ 1\le s,t}}|s-t|\hat{f}(s)\hat{f}(t)-\sum_{1\le s,t\le N-1}|s-t|\hat{f}(s)\hat{f}(t)\right)$$

$$=4\left(\sum_{\substack{|s-t|\le N-1\\ 1\le s,t}}|s-t|\hat{f}(s)\hat{f}(t)-\sum_{\substack{1\le s,t\le N-1\\ |s-t|\le N-1}}|s-t|\hat{f}(s)\hat{f}(t)\right)$$

$$=4\left(\sum_{\substack{|s-t|\le N-1\\ N\le \max(s,t)\\ 1\le s,t}}|s-t|\hat{f}(s)\hat{f}(t)\right). \tag{2.4.39}$$

Rewriting (2.4.38), we have

$$4\sum_{\substack{s+t\le N-1\\ 1\le s,t\le N-1}}(s+t)\hat{f}(s)\hat{f}(t)-4\sum_{1\le s,t\le N-1}(s+t)\hat{f}(s)\hat{f}(t)$$

$$=-4\sum_{\substack{1\le s,t\le N-1\\ N\le s+t}}(s+t)\hat{f}(s)\hat{f}(t). \tag{2.4.40}$$

Combining the last term in (2.4.13) with (2.4.31), (2.4.36), (2.4.39), and (2.4.40) gives

$$\mathrm{Var}(S_N(f))=2\sum_{|s|\le N-1}|s|^2|\hat{f}(s)|^2$$

$$+4N^2\sum_{N\le s}|\hat{f}(s)|^2+2N\sum_{\substack{|s|,|t|\le N-1\\ N\le|s+t|}}\hat{f}(s)\hat{f}(t)-2N\sum_{\substack{|s+t|\le N-1\\ N\le\max(|s|,|t|)}}\hat{f}(s)\hat{f}(t)$$

$$+4\sum_{\substack{|s-t|\le N-1\\ N\le\max(s,t)\\ 1\le s,t}}|s-t|\hat{f}(s)\hat{f}(t)-4\sum_{\substack{1\le s,t\le N-1\\ N\le s+t}}(s+t)\hat{f}(s)\hat{f}(t),$$

which can be rewritten as

$$\mathrm{Var}(S_N(f)) = 4 \sum_{1\le s\le N-1} |s|^2 |\hat{f}(s)|^2$$

$$+4(N^2-N)\sum_{N\le s}|\hat{f}(s)|^2 + 4N \sum_{\substack{1\le s,t\le N-1\\ N\le s+t}} \hat{f}(s)\hat{f}(t) - 4N \sum_{\substack{1\le |s-t|\le N-1\\ N\le \max(s,t)\\ 1\le s,t}} \hat{f}(s)\hat{f}(t)$$

$$+4 \sum_{\substack{1\le |s-t|\le N-1\\ N\le \max(s,t)\\ 1\le s,t}} |s-t|\hat{f}(s)\hat{f}(t) - 4 \sum_{\substack{1\le s,t\le N-1\\ N\le s+t}} (s+t)\hat{f}(s)\hat{f}(t).$$

Combining like sums gives the desired result. Proposition 2.2.1 is proven.

□

As a corollary to Proposition 2.2.1 , we obtain:

Corollary 2.4.1. *Let $s,t\in\mathbb{Z}_{\ge 0}$ and $\{\theta_m\}_{m=1}^N$ be distributed according to the CUE. Then*

$$\mathrm{cov}\left(\left|\sum_{m=1}^N e^{is\theta_m}\right|^2, \left|\sum_{m=1}^N e^{it\theta_m}\right|^2\right) = \begin{cases} s^2, & 1\le s=t\le N-1,\ \ 2s\le N,\\ N+s^2-2s, & 1\le s=t\le N-1,\ \ N+1\le 2s,\\ N(N-1), & N\le s=t,\\ |s-t|-N, & 1\le |s-t|\le N-1,\ \ N\le \max(s,t),\\ N-(s+t), & 1\le s\ne t\le N-1, N+1\le s+t,\\ 0, & \textit{else}. \end{cases}$$

We note that the above formula trivially extends to the case where either s or t is negative, since $\left|\sum e^{is\theta_m}\right| = \left|\sum e^{-is\theta_m}\right|$.

Proof of Corollary 2.4.1

Let $s\in\mathbb{N}$ and $f(\theta)=\cos(s\theta)$. Then

$$S_N(f) - \mathbb{E}(S_N(f)) = |t_{N,s}|^2 .$$

Since $\hat{f}(s) = \hat{f}(-s) = \frac{1}{2}$, Proposition 2.2.1 implies that

$$\text{Var}(S_N(f)) = \text{Var}\left(|t_{N,s}|^2\right) = \begin{cases} s^2, & 1 \le s = t \le N-1, \;\; 2s \le N, \\ N + s^2 - 2s, & 1 \le s = t \le N-1, \;\; N+1 \le 2s, \\ N(N-1), & N \le s. \end{cases} \tag{2.4.41}$$

Similarly, let $(s_1, s_2) \in \mathbb{N}^2$ and $g(\theta) = \cos(s\theta) + \cos(t\theta)$. Then

$$S_N(g) - \mathbb{E}(S_N(g)) = |t_{N,s_1}|^2 + |t_{N,s_2}|^2$$

and

$$\text{Var}(S_N(g)) = \text{Var}\left(|t_{N,s_1}|^2\right) + \text{Var}\left(|t_{N,s_2}|^2\right) + 2\,\text{cov}\left(|t_{N,s_1}|^2, |t_{N,s_2}|^2\right)$$

Applying Proposition 2.2.1 and using (2.4.41) to solve for $\text{cov}\left(|t_{N,s_1}|^2, |t_{N,s_2}|^2\right)$ gives the desired result.

□

For a graphical representation of the covariance function, see the diagram below.

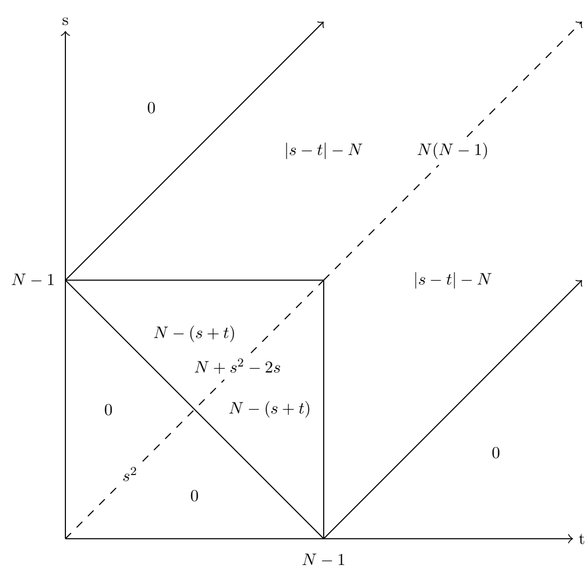


FIGURE 2.2. $\text{cov}(s,t)$

Remark 2.4.2. If we restrict our attention to $(s,t) \in \mathbb{N}^2$ s.t. $s+t \le N/2$ for $s \ne t$ and $2s \le N$ for $s = t$, we recover a special case of Proposition 1.0.1.

Now, we turn our attention to the proof of Proposition 2.2.2. It will follow from Proposition 2.2.1 and the following technical lemma that allows us to control the negligible terms.

Lemma 2.4.3. *Let $f' \in L^2(\mathbb{T})$. Then, as $N \to \infty$, we have*

(i)

$$\sum_{\substack{1\le s,t\le N\\ s+t\ge N+1}} s|\hat{f}(s)| \cdot |\hat{f}(t)| \to 0,$$

(ii)

$$(N+1)\sum_{\substack{s-t\le N\\ s\ge N+1\\ 1\le t\le N}} |\hat{f}(s)| \cdot |\hat{f}(t)| \to 0,$$

(iii)

$$N\sum_{\substack{|s-t|\le N-1\\ s,t\ge N}} |\hat{f}(s)| \cdot |\hat{f}(t)| \to 0.$$

We first quickly prove Proposition 2.2.2 modulo Lemma 2.4.3 and then prove Lemma 2.4.3 at the end of the section.

Proof of Proposition 2.2.2

Recall that $\beta = 2$ and we require that $\sum_{s=1}^{\infty} s^2[\hat{f}(s)]^2 < \infty$, i.e. $f' \in L^2(\mathbb{T})$. We examine the last four sums on the r.h.s. of the formula for $\mathrm{Var}_N(S_N(f))$ in Proposition 2.2.1. Our goal is to show that these four sums go to zero as $N \to \infty$. The analysis of the first two sums is trivial, since

$$0 \le \sum_{s\ge N} N[\hat{f}(s)]^2 \le \sum_{s\ge N} N^2[\hat{f}(s)]^2 \le \sum_{s\ge N} s^2[\hat{f}(s)]^2,$$

which goes to zero under our stated assumptions. The remaining two sums require a little bit more work taken care of by Lemma 2.4.3. We have

$$\left|\sum_{\substack{1\le s,t\le N-1\\ N+1\le s+t}}((s+t)-N)\hat{f}(s)\hat{f}(t)\right| \le 2\sum_{\substack{1\le s,t\le N-1\\ N+1\le s+t}}(s+t)|\hat{f}(s)|\cdot|\hat{f}(t)| = 4\sum_{\substack{1\le s,t\le N-1\\ N+1\le s+t}} s|\hat{f}(s)|\cdot|\hat{f}(t)|.$$

It follows from Lemma 2.4.3(i) that the r.h.s. goes to zero as $N\to\infty$. Finally, we observe that

$$\left|\sum_{\substack{1\le s,t\\ 1\le|s-t|\le N-1\\ N\le\max(s,t)}}(N-|s-t|)\hat{f}(s)\hat{f}(t)\right| \le 2N\sum_{\substack{1\le s,t\\ 1\le|s-t|\le N-1\\ N\le\max(s,t)}}|\hat{f}(s)|\cdot|\hat{f}(t)|$$

$$= 4N\sum_{\substack{|s-t|\le N-1\\ s\ge N\\ 1\le t\le N-1}}|\hat{f}(s)|\cdot|\hat{f}(t)| + 2N\sum_{\substack{1\le|s-t|\le N-1\\ s,t\ge N}}|\hat{f}(s)|\cdot|\hat{f}(t)|$$

$$\le 4N\sum_{\substack{|s-t|\le N-1\\ s\ge N\\ 1\le t\le N-1}}|\hat{f}(s)|\cdot|\hat{f}(t)| + 2N\sum_{\substack{|s-t|\le N-1\\ s,t\ge N}}|\hat{f}(s)|\cdot|\hat{f}(t)|$$

The first term goes to zero by Lemma 2.4.3(ii) and the second term goes to zero by Lemma 2.4.3(iii). This completes the proof of Proposition 2.2.2 modulo Lemma 2.4.3.

□

The rest of the section is devoted to the proof of Lemma 2.4.3.

Proof of Lemma 2.4.3

Let $x_s = s|\hat{f}(s)|$ for $1\le s\le N$ and $X_N=\{x_s\}_{s=1}^N$. By the assumption of Lemma 4.4 the Euclidean norm of the vector X_N is bounded in N. Note that

(2.4.42)

$$\sum_{\substack{1\le s,t\le N\\ s+t\ge N+1}} s|\hat{f}(s)|\cdot|\hat{f}(t)| = \sum_{t=1}^N x_t\cdot\left(\frac{1}{t}\sum_{s=N-t+1}^N x_s\right) = \sum_{t=1}^N x_t\cdot\left(\frac{1}{t}\sum_{s=1}^t (U_NX_N)_s\right) = \langle X_N, A_NX_N\rangle,$$

with $A_N = B_N U_N$, where U_N is a unitary permutation matrix given by $(U_N)_{s,t} = \mathbb{1}_{(t=N-s+1)}$ and B_N is a lower triangular matrix given by $(B_N)_{s,t} = (1/s)\mathbb{1}_{(t\leq s)}$. In particular,

$$B_N = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \dots & \frac{1}{N} \end{pmatrix}$$

Our goal is to show that the expression in (2.4.42) vanishes in the limit of large N. First we show that the operator norm of the matrix A_N is bounded in N. Indeed, $B_N(B_N)^T = B_N + (B_N)^T - D$, where $D_{s,t} = (1/s)\mathbb{1}_{(s=t)}$. This gives us the bound $||B_N||_{op}^2 \leq 2||B_N||_{op} + 1$, so $||A_N||_{op} = ||B_N||_{op} \leq 3$. The fact that A_N weakly converges to 0 finishes the proof of the Lemma. Indeed,

$$\begin{aligned}
\langle X_N, A_N X_N \rangle &= \left\langle X_N - \sum_{s=1}^{L} \langle e_s, X_N \rangle e_s, \quad A_N X_N \right\rangle + \left\langle \sum_{s=1}^{L} \langle e_s, X_N \rangle e_s, \quad A_N X_N \right\rangle \\
&= \left\langle X_N - \sum_{s=1}^{L} \langle e_s, X_N \rangle e_s, \quad A_N X_N \right\rangle + \sum_{s=1}^{L} \langle \langle e_s, X_N \rangle e_s, \quad A_N X_N \rangle \\
&= \left\langle X_N - \sum_{s=1}^{L} \langle e_s, X_N \rangle e_s, \quad A_N X_N \right\rangle + \sum_{s=1}^{L} x_s (A_N X_N)_s \\
&= \left\langle X_N - \sum_{s=1}^{L} \langle e_s, X_N \rangle e_s, \quad A_N X_N \right\rangle + \sum_{s=1}^{L} x_s \frac{x_{N-s+1} + \dots + x_N}{s}.
\end{aligned}$$

Let $\epsilon > 0$. Then we can choose L sufficiently large such that,

$$\begin{aligned}
|\langle X_N, A_N X_N \rangle| &\leq \left| \left\langle X_N - \sum_{s=1}^{L} \langle e_s, X_N \rangle e_s, \quad A_N X_N \right\rangle \right| + \left| \sum_{s=1}^{L} x_s \frac{x_{N-s+1} + \dots + x_N}{s} \right| \\
&\leq \frac{\epsilon}{2} + \left| \sum_{s=1}^{L} x_s \frac{x_{N-s+1} + \dots + x_N}{s} \right| \\
&\leq \epsilon
\end{aligned}$$

Since this holds for arbitrary ϵ, we can conclude that $\langle X_N, A_N X_N \rangle \to 0$. This completes the proof of Lemma 4.4(i).

To prove part (ii), let B_N be defined as in the proof of part (i). Similarly, let $x_s = s|\hat{f}(s)|$ and $X_N = \{x_s\}_{s=1}^{2N}$. Now, X_N is a $2N$-dimensional vector bounded, uniformly with respect to N, in Euclidean norm. Observe that

$$N \sum_{\substack{s-t\leq N \\ s\geq N+1 \\ 1\leq t\leq N}} |\hat{f}(s)| \cdot |\hat{f}(t)| \leq \sum_{t=1}^{N} x_t \left(\frac{1}{t} \sum_{s=N+1}^{N+t} x_s \right)$$

$$= \langle C_N X_N, M_N X_N \rangle,$$

where

$$C_N = \begin{pmatrix} I_N & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad M_N = \begin{pmatrix} B_N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}.$$

Using the same arguments as in the proof of (i), we can see that $||M_N||_{op} \leq 3$. Clearly, $||C_N||_{op} = 1$.The rest of the proof is similar to that of (i). Indeed, for any $\epsilon > 0$, we can choose L sufficiently large such that

$$\begin{aligned} |\langle C_N X_N, M_N X_N \rangle| &\leq \left| \left\langle C_N X_N - \sum_{k=1}^{L} \langle e_k, C_N X_N \rangle e_k, M_N X_N \right\rangle \right| + \left| \left\langle \sum_{k=1}^{L} \langle e_k, C_N X_N \rangle e_k, M_N X_N \right\rangle \right| \\ &\leq \frac{\epsilon}{2} + \left| \sum_{k=1}^{L} x_k (M_N X_N)_k \right| \\ &= \frac{\epsilon}{2} + \left| \sum_{k=1}^{L} x_k \frac{x_{N+1} + \cdots + x_{N+k}}{k} \right| \\ &\leq \epsilon \end{aligned}$$

In the above inequalities, we assume N is large enough such that we can choose $L \leq N$. This completes the proof of (ii).

To prove (iii), we first observe that

$$N \sum_{\substack{|s-t|\le N-1\\ s,t\ge N}} |\hat{f}(s)|\cdot|\hat{f}(t)| \le \sum_{\substack{|s-t|\le N-1\\ s,t\ge N}} s|\hat{f}(s)|\cdot|\hat{f}(t)| \le \sum_{\substack{|s-t|\le N-1\\ t\ge N}} s|\hat{f}(s)|\cdot|\hat{f}(t)| \tag{2.4.43}$$

Now let $x_s = s|\hat{f}(s)|$ for $s \ge 1$. Then $X = \{x_s\}_{s=1}^{\infty} \in \ell^2(\mathbb{N})$ and the rightmost sum in (2.4.43) can be rewritten as

$$\begin{aligned}\sum_{\substack{t-N+1\le s\le N+t-1\\ t\ge N}} s|\hat{f}(s)|\cdot|\hat{f}(t)| &= \sum_{t=N}^{\infty} x_t \left(\frac{1}{t}\sum_{s=t-N+1}^{N+t-1} x_s\right)\\ &= \langle L^{N-1}X, R_N X\rangle,\end{aligned}$$

where L, R_N are bounded linear operators on $\ell^2(\mathbb{N})$. In particular, L, R_N are infinite dimensional matrices such that $L_{s,t} = \mathbb{1}_{t=s+1}$ and $(R_N)_{s,t} = \frac{1}{N+s-1}\mathbb{1}_{(s\le t\le s+2N-2)}$.

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad R_N = \begin{pmatrix} \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{N+1} & \frac{1}{N+1} & \frac{1}{N+1} & \cdots & \frac{1}{N+1} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{N+2} & \frac{1}{N+2} & \frac{1}{N+2} & \cdots & \frac{1}{N+2} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Clearly $||L||_{op} = 1$ and

$$||R_N||_{op} \le ||R_N||_2 = \sqrt{(2N-1)\sum_{k=N}^{\infty}\frac{1}{k^2}} \le \sqrt{\frac{2N-1}{N-1}} \le \sqrt{3}$$

for $N \ge 2$. Now, by the Cauchy-Schwarz inequality,

$$\begin{aligned}|\langle L^{N-1}X, R_N X\rangle|^2 &\le ||L^{N-1}X||_2^2 \cdot ||R_N||_{op}^2 \cdot ||X||_2^2\\ &\le 3\left(\sum_{k=N}^{\infty} |k|^2|\hat{f}(k)|^2\right)\left(\sum_{k=1}^{\infty} |k|^2|\hat{f}(k)|^2\right)\\ &= o_N(1).\end{aligned}$$

This completes the proof of Lemma 2.4.3.

2.5. The Case of Growing Variance

In this section we briefly touch upon the global case where the smoothness condition on f is relaxed, namely when f' is no longer assumed to be in $L^2(\mathbb{T})$. Proposition 2.2.1 implies that, in this case, $\mathrm{Var}(S_N(f)) \to \infty$ as $N \to \infty$. It is reasonable to assume that, after renormalization, one might still be able to prove a Central Limit Theorem in some cases. We prove a special case below.

PROPOSITION 2.5.1. *Let $f = (1/2)\ln|2\sin(\theta/2)|$ and $\{\theta_1\}_{i=1}^N$ be distributed according to the $C\beta E$ for any $\beta > 0$. We we have the following convergence in distribution as $N \to \infty$:*

$$\frac{S_N(f) - \mathbb{E}(S_N(f))}{\sqrt{N}} \longrightarrow \mathcal{N}\left(0, \frac{2 - \beta\Psi^{(2)}\left(1 + \frac{\beta}{2}\right)}{4\beta}\right),$$

where $\Psi^{(k)}(x) = \left(\frac{d}{dx}\right)^k \log(\Gamma(x))$.

REMARK 2.5.2. $\Psi^{(2)}(z)$ is known as the trigamma function. For the special cases $\beta = 1$ (COE), $\beta = 2$ (CUE), and $\beta = 4$ (CSE), the limiting variances are $\frac{3}{2} - \frac{\pi^2}{8}$, $\frac{1}{2} - \frac{\pi^2}{24}$, and $\frac{7}{16} - \frac{\pi^2}{24}$, respectively.

REMARK 2.5.3. In [1], A. Aguirre and A. Soshnikov proved a CLT for more general functions f under the condition that $\sum_{|k|\le N} k^2|\hat{f}(k)|^2$ is a slowly varying sequence with respect to N for the CUE ($\beta = 2$). Their proof relies on the variance formula given in Proposition 2.2.1.

Our proof is based on the fact that the moment generating function of $S_N(f)$ can be conveniently written in terms of the partition function for $C\beta E$.

Proof of Proposition 2.5.1

It was shown by Selberg [21] that the partition function for Circular β-ensembles is given by

$$Z_{N,\beta} = \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N}\prod_{j<k}\left|e^{i\theta_j} - e^{i\theta_k}\right|^\beta d\bar{\theta}$$

$$= \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N}\exp\left(\beta S_N(f)\right) d\bar{\theta} \tag{2.5.1}$$

$$= \frac{\Gamma\left(1 + \frac{\beta N}{2}\right)}{\left[\Gamma\left(1 + \frac{\beta}{2}\right)\right]^N} \tag{2.5.2}$$

for all $\beta > 0$, where $\Gamma(z)$ is the gamma function. We denote by $M_N(t)$ the moment generating function of $[S_N(f) - \mathbb{E}(S_N(f))]/\sqrt{N}$. It follows immediately from (2.5.1) that

$$M_N(t) = \mathbb{E}\exp\left(t\frac{S_N(f) - E(S_N(f))}{\sqrt{N}}\right) \tag{2.5.3}$$

$$= \mathbb{E}\exp\left(t\frac{S_N(f)}{\sqrt{N}}\right)\exp\left(-t\frac{\mathbb{E}(S_N(f))}{\sqrt{N}}\right) \tag{2.5.4}$$

$$= \left(\frac{1}{Z_{N,\beta}}\int_{\mathbb{T}^N}\exp\left((\beta + \frac{t}{\sqrt{N}})S_N(f)\right)d\bar{\theta}\right)\exp\left(-t\frac{\mathbb{E}(S_N(f))}{\sqrt{N}}\right) \tag{2.5.5}$$

$$= \left(\frac{Z_{N,(\beta+\frac{t}{\sqrt{N}})}}{Z_{N,\beta}}\right)\exp\left(-t\frac{\mathbb{E}(S_N(f))}{\sqrt{N}}\right) \tag{2.5.6}$$

$$= \frac{\left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^N\Gamma\left(1+\frac{\beta N}{2}+\frac{\sqrt{N}t}{2}\right)}{\Gamma\left(1+\frac{\beta N}{2}\right)\left[\Gamma\left(1+\frac{\beta}{2}+\frac{t}{2\sqrt{N}}\right)\right]^N}\exp\left(-t\frac{\mathbb{E}(S_N(f))}{\sqrt{N}}\right) \tag{2.5.7}$$

for all $t \in (-\sqrt{N}\beta, +\infty)$.

To start, we use (2.5.7) to first compute $\mathbb{E}(S_N(f)/\sqrt{N})$ explicitly. The computation is as follows:

$$\mathbb{E}(S_N(f)/\sqrt{N}) = \left[\frac{d}{dt}M_N(t)\right]_{t=0}$$

$$= \frac{\left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^N}{\Gamma\left(1+\frac{\beta N}{2}\right)}\cdot\left(\frac{d}{dt}\cdot\frac{\Gamma\left(1+\frac{\beta N}{2}+\frac{\sqrt{N}t}{2}\right)}{\left[\Gamma\left(1+\frac{\beta}{2}+\frac{t}{2\sqrt{N}}\right)\right]^N}\right)_{t=0}$$

$$= \frac{\left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^N}{\Gamma\left(1+\frac{\beta N}{2}\right)}\cdot\left(\frac{\sqrt{N}}{2}\right)\cdot\left(\frac{\Gamma'(\left(1+\frac{\beta N}{2}\right)}{\left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^N} - \frac{\Gamma\left(1+\frac{\beta N}{2}\right)\Gamma'\left(1+\frac{\beta}{2}\right)}{\left[\Gamma\left(1+\frac{\beta}{2}\right)\right]^{N+1}}\right)$$

$$= \left(\frac{\sqrt{N}}{2}\right)\left(\Psi^{(1)}\left(1+\frac{\beta N}{2}\right) - \Psi^{(1)}\left(1+\frac{\beta}{2}\right)\right)$$

$$= \frac{2+\beta N\left(\Psi^{(1)}\left(\frac{\beta N}{2}\right) - \Psi^{(1)}\left(1+\frac{\beta}{2}\right)\right)}{2\beta}. \tag{2.5.8}$$

We now continue with the proof. Because $\frac{BN}{2}, \frac{BN}{2} + \frac{tN}{2\sqrt{N}} \to \infty$ as $N \to \infty$ for any $t > 0$, we can use Sterling's Approximation for the gamma function to write

$M_N(t) =$

(2.5.9)
$$\exp\left(\frac{-t\mathbb{E}(S_N(f))}{\sqrt{N}}\right)\left(\frac{\sqrt{2\pi\left(\frac{\beta N}{2}+\frac{t\sqrt{N}}{2}\right)}}{\sqrt{2\pi\left(\frac{\beta N}{2}\right)}}\right)\left(\frac{\beta N+\sqrt{N}t}{2e}\right)^{\frac{\beta N}{2}}\left(\frac{2e}{\beta N}\right)^{\frac{\beta N}{2}}\left(\frac{\beta N+\sqrt{N}t}{2e}\right)^{\frac{\sqrt{N}t}{2}}$$
$$\times\left(\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\ \frac{\beta}{2}+\frac{t}{2\sqrt{N}}\right)}\right)^{N}\left(1+O\left(\frac{1}{\beta N+\sqrt{N}t}\right)\right)\left(1+O\left(\frac{1}{\beta N}\right)\right)$$

(2.5.10)
$$=\exp\left(-t\mathbb{E}(S_N(f)/\sqrt{N})\right)\left(1+\frac{t}{\beta\sqrt{N}}\right)^{\frac{\beta N+1}{2}}\left(\frac{\beta N+\sqrt{N}t}{2e}\right)^{\frac{\sqrt{N}t}{2}}\left(\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\ \frac{\beta}{2}+\frac{t}{2\sqrt{N}}\right)}\right)^{N}$$
$$\times\left(1+O\left(\frac{1}{\beta N+\sqrt{N}t}\right)\right)\left(1+O\left(\frac{1}{\beta N}\right)\right).$$

The last two terms in (2.5.9) are error terms for Sterling's Approximation. Instead of dealing with the above product, it will be easier to consider $\log(M_N(t))$, the cumulant generating function of $[S_N(f) - \mathbb{E}S_N(f)]/\sqrt{N}$.

Recall that if X is a random variable with finite moments $\mathbb{E}\left(X^n\right) = m_n$ then its n-th order cumulant, k_n, is explicitly defined in terms of $m_1, \ldots, m_n$. In particular, we have

$$k_n = \sum_{\pi}(|\pi|-1)!(-1)^{|\pi|-1}\prod_{B\in\pi} m_{|B|},$$

where the sum is over all partitions π of $\{1, \ldots, n\}$, B runs through the list of all blocks of the partition π, and $|\pi|$ is the number of blocks in the partition. We note that $k_1 = \mathbb{E}\left(X\right)$ and $k_2 = \mathrm{Var}(X)$. The cumulant generating function (c.g.f.) for X is given by

$$K_X(t) = \log\left(\mathbb{E}e^{tX}\right) = \sum_{n=1}^{\infty} k_n \frac{t^n}{n!}$$

Moreover, if X is a Gaussian random variable with $\mathbb{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, then its distribution uniquely satisfies $k_1 = \mu$, $k_2 = \sigma^2$ and $k_n = 0$ for $n \geq 3$. It follows that

$$K_X(t) = \mu t + \sigma^2 \frac{t^2}{2}.$$

The cumulant generating function for $[S_N(f) - \mathbb{E}S_N(f)]/\sqrt{N}$ is then given by

$$K_N(t) = \log\left(M_N(t)\right) \tag{2.5.11}$$

$$= \left(\frac{\beta N + 1}{2}\right)\log\left(1 + \frac{t}{\beta\sqrt{N}}\right) + \left(\frac{\sqrt{N}t}{2}\right)\log\left(\frac{\beta N}{2} + \frac{\sqrt{N}t}{2}\right) - \frac{\sqrt{N}t}{2} \tag{2.5.12}$$
$$+ N\log\left(\Gamma\left(1 + \frac{\beta}{2}\right)\right) - N\log\left(\Gamma\left(1 + \frac{\beta}{2} + \frac{t}{2\sqrt{N}}\right)\right) - \frac{t}{\sqrt{N}}\mathbb{E}(S_N(f)) + E_N(t),$$

where $E_N(t)$ denotes the corresponding error term that comes from (2.5.9). $\log[\Gamma(1 + x)]$ is an analytic function on $(-1, \infty)$ [28], so we can expand the first, second, and fourth terms of (2.5.12) using Taylor series', assuming that $t \in (-\beta\sqrt{N}, \beta\sqrt{N})$. The first term becomes

$$\left(\frac{\beta N + 1}{2}\right)\left(\frac{t}{\beta\sqrt{N}} - \frac{t^2}{2\beta^2 N} + R_2^{(1)}\left(\frac{t}{\beta\sqrt{N}}\right)\right), \tag{2.5.13}$$

where $R_2^{(1)}(x)$ is the second order error term from Taylor's theorem corresponding to $\log(1 + x)$. Similarly, the second term becomes

$$\frac{\sqrt{N}t}{2}\log\left(\frac{\beta N}{2}\right) + \frac{t^2}{2\beta} - \frac{\sqrt{N}t}{2}R_1^{(2)}\left(\frac{\sqrt{N}t}{2}\right) \tag{2.5.14}$$

and the fourth term becomes

$$-N\log\left(\Gamma\left(1 + \frac{\beta}{2}\right)\right) - \sqrt{N}\Psi^{(1)}\left(1 + \frac{\beta}{2}\right)\frac{t}{2} - \Psi^{(2)}\left(1 + \frac{\beta}{2}\right)\frac{t^2}{8} - N \cdot R_2^{(3)}\left(\frac{t}{2\sqrt{N}}\right), \tag{2.5.15}$$

where $R_1^{(2)}(x)$ and $R_2^{(3)}(x)$ are first and second order error terms for the Taylor expansions of $\log\left(\frac{\beta N}{2} + x\right)$ and $\log\left(\Gamma\left(1 + \frac{\beta}{2} + x\right)\right)$ about $x = 0$, respectively. By combining (2.5.12-2.5.15), we have

$$K_N(t) = \left(\frac{\beta N+1}{2}\right)\left(\frac{t}{\beta\sqrt{N}} - \frac{t^2}{2\beta^2 N}\right) + \frac{\sqrt{N}t}{2}\log\left(\frac{\beta N}{2}\right) + \frac{t^2}{2\beta} - \frac{\sqrt{N}t}{2} \tag{2.5.16}$$

$$- \sqrt{N}\Psi^{(1)}\left(1+\frac{\beta}{2}\right)\frac{t}{2} - \Psi^{(2)}\left(1+\frac{\beta}{2}\right)\frac{t^2}{8} - \frac{t}{\sqrt{N}}\mathbb{E}(S_N(f)) \tag{2.5.17}$$

$$+ E_N(t) + \left(\frac{\beta N+1}{2}\right)R_2^{(1)}\left(\frac{t}{\beta\sqrt{N}}\right) - \frac{\sqrt{N}t}{2}R_1^{(2)}\left(\frac{\sqrt{N}t}{2}\right) - N\cdot R_2^{(3)}\left(\frac{t}{2\sqrt{N}}\right) \tag{2.5.18}$$

The terms in (2.5.18) are all error terms that disappear in the limit $N \to \infty$ for any fixed $t \in \mathbb{R}$. We postpone the details until the end of the proof. Let $C_{N,1}$ and $C_{N,2}$ be the terms in (2.5.16-2.5.17) that are proportional to t and $t^2/2$, respectively. Then

$$C_{N,1} = \frac{1+\beta N\left(\log(\beta N/2) - \Psi^{(1)}\left(1+\frac{\beta}{2}\right)\right)}{2\beta\sqrt{N}} - \frac{\mathbb{E}\left(S_N(f)\right)}{\sqrt{N}}$$

and

$$C_{N,2=}\frac{2-\beta\Psi^{(2)}\left(1+\frac{\beta}{2}\right)}{4\beta} - \frac{1}{2\beta^2 N}.$$

Now, $C_{N,1}$ is equal to the first cumulant of $[S_N(f) - \mathbb{E}\left(S_N(f)\right)]/\sqrt{N}$ up to an asymptotically negligible error term. Since the first cumulant of a random variable is equal to its expectation, it should be the case that $C_{N,1} \to 0$ as $N \to \infty$. Indeed, using (2.5.8), we observe that

$$C_{N,1} - E(S_N(f)/\sqrt{N}) = \frac{\beta N\left(\log\left(\frac{N\beta}{2}\right) - \Psi^{(1)}\left(\frac{N\beta}{2}\right)\right) - 1}{2\beta}$$

$$= \frac{N\beta\left(\frac{1}{N\beta} + 2\int_0^\infty \frac{x}{\left(x^2+\left(\frac{N\beta}{2}\right)^2\right)(e^{2\pi x}-1)}\ dx\right) - 1}{2\beta} \tag{2.5.19}$$

$$= \frac{2N\beta\int_0^\infty \frac{x}{\left(x^2+\left(\frac{N\beta}{2}\right)^2\right)(e^{2\pi x}-1)}\ dx}{2\beta}$$

$$= \frac{\frac{8}{N\beta}\int_0^\infty \frac{x}{\left(\left(\frac{2x}{N\beta}\right)^2+1\right)(e^{2\pi x}-1)}\ dx}{2\beta} \tag{2.5.20}$$

$$= O(N^{-1}), \tag{2.5.21}$$

where (2.5.19) follows from an integral representation of $\Psi^{(1)}(z)$ given by Binet's second integral for the gamma function (Section 12.32 in [28]). (2.5.21) follows from the fact that the integral in (2.5.21) is uniformly bounded above with respect to both N and β. We note that $\Psi^{(1)}(z)$ is the logarithmic derivative of the gamma function, also referred to as the digamma function.

Finally, let us consider the error terms in (2.5.18). It follows from (2.5.9) that, for any fixed t and sufficiently large N, the first term is given by

$$E_N(t) = \log\left(1 + O\left(\frac{1}{\beta N + \sqrt{N}t}\right)\right) + \log\left(1 + O\left(\frac{1}{\beta N}\right)\right) = o_N(1). \tag{2.5.22}$$

For the second term, we observe that, for any fixed t,

$$\left|R_2^{(1)}\left(\frac{t}{\beta\sqrt{N}}\right)\right| = \left|\frac{2}{(1+\alpha)^3}\right| \cdot \frac{|t|^3}{3!\beta^3 N^{3/2}}$$

for some α satisfying $|\alpha| \leq \left|\frac{t}{\beta\sqrt{N}}\right|$. Therefore, when N is sufficiently large, we can write

$$\left|R_2^{(1)}\left(\frac{t}{\beta\sqrt{N}}\right)\right| = O(N^{-3/2})$$

and

$$\left|\left(\frac{\beta N + 1}{2}\right) R_2^{(1)}\left(\frac{t}{\beta\sqrt{N}}\right)\right| = O(N^{-1/2}). \tag{2.5.23}$$

For the third term in (2.5.18), we recall that $R_1^{(2)}(x)$ is the first order error term for the Taylor expansion of $\log\left(\frac{\beta N}{2} + x\right)$ about $x = 0$ (Once again, we are assuming that $\beta > 0$ is fixed and N is large). It follows that

$$\left|R_1^{(2)}\left(\frac{\sqrt{N}t}{2}\right)\right| = \left|\frac{1}{\left(\frac{\beta N}{2} + \alpha\right)^2}\right| \cdot \frac{N|t|^2}{2}$$

where $|\alpha| \leq \left|\frac{\sqrt{N}t}{2}\right|$. Once again, for any fixed t and large enough N, we can write

$$\frac{\sqrt{N}t}{2} R_1^{(2)}\left(\frac{\sqrt{N}t}{2}\right) = O(N^{-1/2}). \tag{2.5.24}$$

For the third term, we recall that $R_2^{(3)}(x)$ is the second order error term for the Taylor expansion of $\log\left(\Gamma\left(1+\frac{\beta}{2}+x\right)\right)$ about $x=0$, so

$$\left|R_2^{(3)}\left(\frac{t}{2\sqrt{N}}\right)\right| = \left|\Psi^{(3)}\left(1+\frac{\beta}{2}+\alpha\right)\right| \cdot \frac{|t|^3}{(3!)(2^3)N^{3/2}},$$

where $|\alpha| \leq \left|\frac{t}{2\sqrt{N}}\right|$. Since $\Psi^{(3)}\left(1+\frac{\beta}{2}+x\right)$ is continuous on $[-1/2,1/2]$, there exists $M>0$ such that $\left|\Psi^{(3)}\left(1+\frac{\beta}{2}+x\right)\right| \leq M$ for all $x \in [-1/2,1/2]$. Consequently, for any fixed t and sufficiently large N, $\left[-\frac{t}{2\sqrt{N}}, \frac{t}{2\sqrt{N}}\right] \subseteq [-1/2,1/2]$, so

$$\left|R_2^{(3)}\left(\frac{t}{2\sqrt{N}}\right)\right| \leq M \cdot \frac{|t|^3}{(3!)(2^3)N^{3/2}}$$

and

$$N \cdot R_2^{(3)}\left(\frac{t}{2\sqrt{N}}\right) = O(N^{-1/2}). \tag{2.5.25}$$

combining (2.5.16-2.5.18) and (2.5.22-2.5.25), we can see that, for any fixed $t \in \mathbb{R}$ and large enough N,

$$K_N(t) = C_{N,2}\frac{t^2}{2} + o_N(1) = \left(\frac{2-\beta\Psi^{(2)}\left(1+\frac{\beta}{2}\right)}{4\beta}\right)\frac{t^2}{2} + o_N(1). \tag{2.5.26}$$

This completes the proof of Proposition 2.5.1.

□

CHAPTER 3

Pair Dependent Linear Statistics for SO(N) & Sp(N)

In this chapter, we prove analogous results to Theorems 2.1.1-2.1.5 for random matrices sampled from two other classical compact groups (according to Haar measure): $N \times N$ orthogonal matrices with determinant one ($SO(N)$), and the $2N \times 2N$ symplectic matrices ($Sp(N)$). We also prove a CLT in the case where $L_N = 1$ and $\mathrm{Var}(S_N(f))$ is allowed to grow very slowly. We break up $SO(N)$ into two cases: even N and odd N, based upon the distinctive behavior of their eigenvalues. Similar to the case of the CUE, the main underlying tools for studying the "pair counting" statistic [2] with respect to these matrix ensembles are the k-point correlation functions, which can be found in [24]. Unlike with the CUE, the three and four point correlations functions are too unwieldy to be applied directly. Instead, we further develop tools introduced in [25] and [2] regarding joint cumulants in order to reformulate the necessary variance computation into something more manageable.

3.1. Distributional Properties for SO(N) & Sp(N)

Like the $N \times N$ unitary matrices (U(N)=CUE), SO(N) and Sp(N) are compact topological groups, so they each have unique Haar probabilitiy measures. Moreover, the distribution of the eigenvalues of a matrix sampled from any of these groups, according to the corresponding Haar measure, has the form of a determinantal random point field with a fixed number of particles on the interval $[0, \pi)$. For more on determinental random point fields and their associated correlation functions, we defer the reader to [24].

If M is sampled from $SO(2N)$, then, with probability one, M has N pairs of eigenvalues $e^{i\theta_1}, e^{-i\theta_1}, \ldots$ $e^{i\theta_N}, e^{-i\theta_N}$, where $\theta_j \in [0, \pi)$ for $1 \leq j \leq N$. The joint probability density of $\{\theta_j\}_{j=1}^N$ is given by

$$P_{SO(2N)}(\theta_1, \ldots, \theta_N) = \left(\frac{1}{\pi}\right)^N \prod_{1 \leq i < j \leq N} (2\cos(\theta_i) - 2\cos(\theta_j))^2 \tag{3.1.1}$$

Like in the case of the CUE, $SO(2N)$ has k-point correlation functions with a determinental structure. Let

$$K_{2N-1}(\theta_i,\theta_j) = \frac{1}{2\pi}\cdot\left(\frac{\sin\left(\frac{(2N-1)(\theta_i-\theta_j)}{2}\right)}{\sin\left(\frac{\theta_i-\theta_j}{2}\right)}\right) = \frac{1}{2\pi}\sum_{k=-(N-1)}^{N-1} e^{ik(\theta_i-\theta_j)}. \tag{3.1.2}$$

Then the k-point correlation function for $SO(2N)$ is given by the following determinental formula:

$$\rho_{2N,k}(\theta_1,\ldots,\theta_N) := \det\left(K^+_{2N-1}(\theta_i,\theta_j)\right)_{1\le i,j\le k}, \tag{3.1.3}$$

where

$$K^+_{2N-1}(\theta_i,\theta_j) = K_{2N-1}(\theta_i,\theta_j) + K_{2N-1}(\theta_i,-\theta_j). \tag{3.1.4}$$

Remark 3.1.1. The integration kernel $K_N(\theta_1,\theta_2)$ is unitarily equivalent to $Q_N(\theta_1,\theta_2)$, the kernel used for CUE(see 2.4.1), in the sense that

$$[K_N(\theta_i,\theta_j)]_{1\le i,j\le k} = U^*\,[Q_N(\theta_i,\theta_j)]_{1\le i,j\le k}\,U,$$

where U is a unitary matrix.

Remark 3.1.2. Suppose $f\in L^2(\mathbb{T})$ is even. Then

$$\begin{aligned}
L_f(\theta_1) &= \int_{\mathbb{T}} f(\theta_2)\left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1-\theta_2}{2}\right)}\right)d\theta_2\\
&= \int_{\mathbb{T}} \frac{f(\theta_2)}{2}\left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1-\theta_2}{2}\right)}\right)d\theta_2 + \int_{\mathbb{T}} \frac{f(-\theta_2)}{2}\left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1-\theta_2}{2}\right)}\right)d\theta_2\\
&= \int_{\mathbb{T}} \frac{f(\theta_2)}{2}\left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1-\theta_2}{2}\right)}\right)d\theta_2 + \int_{\mathbb{T}} \frac{f(\theta_2)}{2}\left(\frac{\sin\left(\frac{N(\theta_1+\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1+\theta_2}{2}\right)}\right)d\theta_2\\
&= \frac{1}{2}\int_{\mathbb{T}} f(\theta_2)\left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1-\theta_2}{2}\right)} + \frac{\sin\left(\frac{N(\theta_1+\theta_2)}{2}\right)}{2\pi\cdot\sin\left(\frac{\theta_1+\theta_2}{2}\right)}\right)d\theta_2,
\end{aligned}$$

so the integration kernel $K^+_N(\theta_1,\theta_2)$ is the restriction of $K_N(\theta_1,\theta_2)$ onto the space of even $L^2(\mathbb{T})$ functions.

If M is distributed according to $SO(2N+1)$, then, with probability one, 1 is an eigenvalue of M and the remaining eigenvalues of M can once again be arranged in pairs, $e^{i\theta_1}, e^{-i\theta_1}, \ldots e^{i\theta_N}, e^{-i\theta_N}$, where $\theta_j \in (0,\pi)$ for $1 \leq j \leq N$. The corresponding joint probability density function for $\{\theta_j\}_{j=1}^N$ is given by

$$P_{SO(2N+1)}(\theta_1, \ldots, \theta_N) = \left(\frac{2}{\pi}\right)^N \prod_{1\leq i<j\leq N} (2\cos(\theta_i) - 2\cos(\theta_j))^2 \cdot \prod_{i=1}^{N} \sin^2\left(\frac{\theta_i}{2}\right). \tag{3.1.5}$$

The k-point correlation functions are given by

$$\rho_{2N+1,k}(\theta_1, \ldots, \theta_N) = \det\left(K_{2N}^{-}(\theta_i, \theta_j)\right)_{1\leq i,j\leq k}, \tag{3.1.6}$$

where

$$\begin{aligned} K_{2N}^{-}(\theta_i, \theta_j) &= K_{2N}(\theta_i, \theta_j) - K_{2N}(\theta_i, -\theta_j) \\ &= \frac{1}{2\pi} \cdot \left(\frac{\sin\left(N(\theta_i - \theta_j)\right)}{\sin\left(\frac{\theta_i - \theta_j}{2}\right)} - \frac{\sin\left(N(\theta_i + \theta_j)\right)}{\sin\left(\frac{\theta_i + \theta_j}{2}\right)} \right). \end{aligned} \tag{3.1.7}$$

Remark 3.1.3. Suppose $f \in L^2(\mathbb{T})$ is odd. Then

$$\begin{aligned} L_f(\theta_1) &= \int_{\mathbb{T}} f(\theta_2) \left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1-\theta_2}{2}\right)} \right) d\theta_2 \\ &= \int_{\mathbb{T}} \frac{f(\theta_2)}{2} \left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1-\theta_2}{2}\right)} \right) d\theta_2 - \int_{\mathbb{T}} \frac{f(-\theta_2)}{2} \left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1-\theta_2}{2}\right)} \right) d\theta_2 \\ &= \int_{\mathbb{T}} \frac{f(\theta_2)}{2} \left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1-\theta_2}{2}\right)} \right) d\theta_2 - \int_{\mathbb{T}} \frac{f(\theta_2)}{2} \left(\frac{\sin\left(\frac{N(\theta_1+\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1+\theta_2}{2}\right)} \right) d\theta_2 \\ &= \frac{1}{2} \int_{\mathbb{T}} f(\theta_2) \left(\frac{\sin\left(\frac{N(\theta_1-\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1-\theta_2}{2}\right)} - \frac{\sin\left(\frac{N(\theta_1+\theta_2)}{2}\right)}{2\pi \cdot \sin\left(\frac{\theta_1+\theta_2}{2}\right)} \right) d\theta_2, \end{aligned}$$

so the integration kernel $K_N^{-}(\theta_1, \theta_2)$ is the restriction of $K_N(\theta_1, \theta_2)$ onto the space of odd $L^2(\mathbb{T})$ functions.

Finally, if M is sampled from $Sp(N)$, M has $2N$ eigenvalues on the unit circle, which again come in pairs of complex conjugates $e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_N}, e^{-i\theta_N}$. The corresponding joint probability density for $\{\theta_j\}_{j=1}^N$ is given by

$$(3.1.8) \qquad P_{Sp(N)}(\theta_1, \ldots, \theta_N) = \left(\frac{2}{\pi}\right)^N \prod_{1 \leq i < j \leq N} (2\cos(\theta_i) - 2\cos(\theta_j))^2 \cdot \prod_{i=1}^N \sin^2(\theta_i).$$

The k-point correlation functions are given by

$$(3.1.9) \qquad \rho_{N,k}(\theta_1, \ldots, \theta_N) = \det\left(K_{2N+1}^-(\theta_i, \theta_j)\right)_{1 \leq i,j \leq k}.$$

As in the case for the CUE, the k-point correlation functions for SO(2N), SO(2N+1), and Sp(N) play a crucial role in our analysis of the limiting distribution of the "pair counting" statistic defined in (1.0.4).

REMARK 3.1.4. The k-point correlation functions for $SO^-(2N)$ are the same as those for $Sp(N-1)$. As a consequence, every result in this chapter involving $Sp(N)$ holds for $SO^-(2N)$. Similarly, the kernel for $SO^-(2N+1)$ is given by

$$K_{2N}^+(\theta_i, \theta_j) = K_{2N}(\theta_i, \theta_j) + K_2N(\theta_i, -\theta_j).$$

The results for $SO(2N+1)$ given in Section 3.5 hold for $SO^-(2N+1)$ up to a minor sign change that is asymptotically negligible with respect to the main results in the following section.

REMARK 3.1.5. The joint probability density function given in (3.1.1) can be rewritten as a degree $2N-2$ trigonometric polynomial in terms of $e^{i\theta_1}, \ldots, e^{i\theta_N}$. Let $k \in \mathbb{Z}$ such that $|k| \geq 2N-1$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{N}_0$ such that $\sum_{j=1}^N \alpha_i \geq 1$. If $\theta_1, \ldots, \theta_N$ are distributed according to (3.1.1), then

$$\mathbb{E}_{SO(2N)}\left(\prod_{j=1}^N \left(e^{ik\theta_j}\right)^{\alpha_j}\right) = \prod_{j=1}^N \mathbb{E}_{SO(2N)}\left(\left(e^{ik\theta_j}\right)^{\alpha_j}\right) = 0,$$

i.e. $e^{ik\theta_1}, \ldots, e^{ik\theta_N}$ are mutually independent. Analogous statements hold for (3.1.5) and (3.1.8).

3.2. Main Results

For the cases SO(N) and Sp(N), the limiting distribution for (1.0.4), with $L_N = 1$, is a sum of mutually independent $\chi_k^2(\lambda)$ distributed random variables, where $\chi_k^2(\lambda)$ denotes the non-central chi-squared distribution with $k \in \mathbb{N}$ degrees of freedom and non-centrality parameter $\lambda \geq 0$. Recall that if $X_1, \dots, X_k$ are possibly non-central, real, independent, Gaussians with $\mathbb{E}(X_j) = \mu_j$, $\mathrm{Var}(X_j) = 1$, and $Y = \sum_{j=1}^{k} X_j^2$, then Y is said to be $\chi_k^2(\lambda)$ distributed, where

$$\lambda = \sum_{j=1}^{k} \mu_j^2.$$

Moreover,

$$\mathbb{E}(Y) = k + \lambda \quad , \qquad \mathrm{Var}(Y) = 2k + 4\lambda.$$

and the probability density function for Y is given by

$$f_Y(x, k, \lambda) = \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^i}{i!} \left(\frac{x^{(k+2i)/2-1}}{2^{(k+2i)/2} \cdot \Gamma\left(\frac{k+2i}{2}\right)} \right) e^{-\frac{x}{2}} \tag{3.2.1}$$

Remark 3.2.1. When $\lambda = 0$, (3.2.1) is the density function for the central/standard χ_k^2 distribution. If, in addition, we set k=2, (3.2.1) gives the density for the exponential(1/2) distribution. For more details regarding χ^2 distributions, we refer the reader to [5].

Theorem 3.2.2.

Let f be an even function satisfying $\sum_{k=1}^{\infty} k^2 |\hat{f}(k)|^2 < \infty$. If $\theta_1, \dots, \theta_N$ are distributed according to SO(N) or Sp(N), then $S_N(f) - \mathbb{E}\left(S_N(f)\right)$ converges in distribution to

$$2 \sum_{k=1}^{\infty} k \hat{f}(k)(\varphi_k - \lambda_k), \tag{3.2.2}$$

where $\lambda_k = 1 + \frac{1}{k}\left(\frac{1+(-1)^k}{2}\right)$ and $\{\varphi_k\}_{k=1}^{\infty}$ are independent, mean λ_k, variance $2 + \left(\frac{4}{k}\right)\left(\frac{1+(-1)^k}{2}\right)$ random variables. In particular , φ_k is $\chi_1^2(0)$ distributed for odd k and non-central $\chi_1^2\left(\frac{1}{k}\right)$ for even k. We note that, in the case of Sp(N), there are actually $2N$ eigenvalues of the form $e^{i\theta_1}, e^{-i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_N}$.

Remark 3.2.3. Similar to the case of CUE, the key to the proof of Theorem 3.2.2 is an explicit variance calculation, given in Section 3.5, that allows us to extend the result for trigonometric polynomials to more general test functions. Unlike the case of CUE, the three and four point correlation functions for $SO(N)$ and $Sp(N)$ have far too many terms to be used directly. Instead, we further develop tools from Appendix 2 of [2] in order to derive explicit formulas for joint cumulants of traces of powers of $SO(N)$ and $Sp(N)$ distributed random matrices.

For the case $1 << L_N << N$, we once again recover a central limit theorem.

Theorem 3.2.4. ***(Mesoscopic CLT)***
Let f be an even function in $C_c^2(\mathbb{R})$. If $\theta_1, \ldots, \theta_N$ are distributed according to $SO(N)$ or $Sp(N)$ and $1 << L_N << N$, then $L_N^{-1/2}[S_N(f(L_N \cdot)) - \mathbb{E}S_N(f(L_N \cdot))]$ converges in distribution to

$$\mathcal{N}\left(0, \frac{2}{\pi}\int_{\mathbb{R}} t^2|\hat{f}(t)|^2 dt\right). \tag{3.2.3}$$

Here

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

denotes the Fourier transform of f.

Remark 3.2.5. Both Theorem 3.2.2 and Theorem 3.2.4 hold in the case of $SO^-(N)$. The case of $SO^-(2N)$ is identical to $Sp(N-1)$. The case of $SO^-(2N+1)$ is the same as that for $SO(2N+1)$ up to a change of sign in a term with asymptotically negligible magnitude.

Finally, we present a theorem that emerges naturally when we relax the conditions of the Theorem 3.2.2. When f is no longer assumed to be in $\mathbb{H}^1(\mathbb{T})$, Proposition 3.5.4 implies that $\mathrm{Var}(S_N(f))$ diverges to infinity. If the variance grows sufficiently slowly, we are able to prove a central limit theorem after renormalization.

Definition 3.2.6. A positive sequence V_N is said to be slowly varying in the sense of Karamata [4] if

$$\lim_{N\to\infty}\frac{V_{\lfloor \lambda N\rfloor}}{V_N} = 1, \qquad \forall \lambda > 0,$$

where $\lfloor m \rfloor$ denotes the integer part of m.

Remark 3.2.7. It's easy to see from the above definition that any convergent sequence is slowly varying. If V_N is slowly varying, then, for any $\lambda > 0$, so is $V_{\lfloor \lambda N \rfloor}$. Furthermore, $V_N = \log(N)$ is slowly varying while $V_N = N^\alpha$ $(\alpha > 0)$ is not.

Theorem 3.2.8.

Let $f \in L^2(\mathbb{T})$ be a real valued, even function such that $V_N = \sum_{k=1}^{N} |k^2||\hat{f}(k)|^2$ is a slowly varying sequence that diverges to infinity as $N \to \infty$. If $\theta_1, \ldots, \theta_N$ are distributed according to $SO(N)$ or $Sp(N)$, then we have the following convergence in distribution

$$\frac{S_N(f) - \mathbb{E}S_N(f)}{2\sqrt{2V_N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

3.3. Proof of Theorem 3.2.2

This section is devoted to the proof of Theorem 3.2.2. We first consider the case where f is a trigonometric polynomial and then extend the result to more general test functions using Proposition 3.5.4. Throughout the remainder of this chapter we will use t_k to denote the trace of the k-th power of an $SO(N)$ or $Sp(N)$ distributed random matrix, suppressing 'N'. It will also often be convenient to distinguish between $SO(2N)$, $SO(2N+1)$, and $Sp(N)$ by using a triplet of the form (M_N, δ, α). In particular, we will associate $(M_N = 2N-1, \delta = 1, \alpha = 0)$ with $SO(2N)$, $(M_N = 2N, \delta = -1, \alpha = 1)$ with $SO(2N+1)$, and $(M_N = 2N+1, \delta = -1, \alpha = 0)$ with $Sp(N)$.

Proof. (*of Theorem 3.2.2*)

Let us first consider the case where f is an even trigonometric polynomial with $\hat{f}(0) = 0$. In particular, let

$$f(\theta) = \sum_{k=1}^{m} \hat{f}(k) \cos(k\theta).$$

we will assume that N is much larger than m. Then

$$t_k = \alpha + \sum_{j=1}^{N} 2\cos(k\theta_j)$$

and

$$S_N(f) = 2\sum_{k=1}^{m} \hat{f}(k) t_k^2 - (2N+\alpha) f(0).$$

Furthermore, it follows from Corollary 3.5.7 that

$$S_N(f) - \mathbb{E}\left(S_N(f)\right) = 2\sum_{k=1}^{m} k\hat{f}(k) \left(\frac{1}{k}t_k^2 - \left[1 + \frac{1}{k}\left(\frac{1+(-1)^k}{2}\right)\right]\right)$$

when N is sufficiently large. The work of Diaconis and Shahshahani [7] tells us that the joint moments of any finite collection $t_{k_1}, \ldots, t_{k_l}$ are precisely those of real, independent, Gaussians with mean $\left(\frac{1+(-1)^{k_i}}{2}\right)$ and variance k_i. It follows immediately that

$$S_N(f) - \mathbb{E}\left(S_N(f)\right) \xrightarrow{\mathcal{D}} 2\sum_{k=1}^{m} k\hat{f}(k)(\varphi_k - \lambda_k)$$

as $N \to \infty$, where $\lambda_k = 1 + \frac{1}{k}\left(\frac{1+(-1)^k}{2}\right)$ and $\{\varphi_k\}_{k=1}^m$ are independent χ_1^2 (possibly non-central) distributed random variables with mean λ_k and variance $2 + \left(\frac{4}{k}\right)\left(\frac{1+(-1)^k}{2}\right)$. This completes the proof in the case where f is a trigonometric polynomial.

In order to extend the case for trigonometric polynomials to a more general class of functions, we will need to consider $\mathrm{Var}\left(S_N(f)\right)$. Proposition 3.5.4 states that, in all three cases ($SO(2N)$, $SO(2N+1)$, $Sp(N)$),

$$\mathrm{Var}\left(S_N(f)\right) = 4\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor}\left(2k^2 + 4k\left(\frac{1+(-1)^k}{2}\right)\right)|\hat{f}(k)|^2 + o_N(1).$$

Let

$$f_m := 2\sum_{k=1}^{m}\hat{f}(k)\cos(k\theta).$$

Then Proposition 3.5.4 implies that

$$\begin{aligned}\mathrm{Var}\left(S_N(f) - S_N(f_m)\right) &= O\left(\sum_{k=m+1}^{\lfloor \frac{N}{2} \rfloor} k^2|\hat{f}(k)|^2\right) + o_N(1)\\ &= o_m(1) + o_N(1),\end{aligned}$$

where the first term decays uniformly in m, independent of N, and the second term decays uniformly in N, independent of m. The proof of Theorem 3.2.2 can now be completed using a Chebyshev inequality and standard $\epsilon/3$-type argument to show that the desired distributions converges with respect to the Lévy Metric. Since the details for the remainder of the proof are precisely the same as those of the proof of Theorem 2.1.1, but with χ^2 distributed random variable in place of exponentially distributed random variables, we refer the reader to Appendix A.1. $\square$

3.4. Proofs for Theorem 3.2.4 and Theorem 3.2.8

This section is devoted to the proofs of Theorem 3.2.4 and Theorem 3.2.8. In both cases, we show that the moment generating function for the truncated statistic converges to that of the desired Gaussian distribution, while the tail gives a negligible contribution to the limiting distribution.

PROOF. (*of Theorem 3.2.4*)

Let $f \in C_c^2(\mathbb{R})$ be an even, real valued function and $m_N = \lfloor \sqrt{N \cdot L_N} \rfloor$. Then $L_N << m_N << N$ and, for large enough N, we can write

$$\left(\frac{\sqrt{2\pi}}{2}\right) \cdot \frac{S_N(f(L_N\cdot)) - \mathbb{E}\left(S_N(f(L_N\cdot))\right)}{\sqrt{L_N}} =$$

$$\frac{1}{L_N^{3/2}} \sum_{k=1}^{m_N} \hat{f}\left(\frac{k}{L_N}\right)\left(t_k^2 - \mathbb{E}\left(t_k^2\right)\right) + \frac{1}{L_N^{3/2}} \sum_{k=m_N+1}^{\infty} \hat{f}\left(\frac{k}{L_N}\right)\left(t_k^2 - \mathbb{E}\left(t_k^2\right)\right), \tag{3.4.1}$$

where $\hat{f}(\xi)$ denotes the Fourier transform of f.

Similar to the proof of Theorem 2.1.5, Proposition 3.5.4 can be generalized to the mesoscopic case in the exact same way as Proposition 2.2.2 was in the case of CUE (see Appendix A.2). It follows that the variance of the second term in (3.4.1) goes to zero as $N \to \infty$. As a result it suffices to consider the asymptotic distribution of the the first term, which can be rewritten as

$$\frac{1}{L_N^{3/2}} \sum_{k=1}^{m_N} k\hat{f}\left(\frac{k}{L_N}\right)\left(\frac{1}{k}t_k^2 - \lambda_k\right), \tag{3.4.2}$$

where $\lambda_k = 1 + \left(\frac{1+(-1)^k}{2k}\right)$. It follows from Proposition 1.0.1 that, for any $l \in \mathbb{N}$ and sufficiently large N, the l-th order joint moments of $\{t_k^2\}_{k=1}^{m_N}$ are precisely the same as those for independent squared, real Gaussian random variables. For the remainder of the proof, we are then justified in replacing (3.4.2) with a sum of independent random variables, namely

$$\frac{1}{L_N^{3/2}} \sum_{k=1}^{m_N} k\hat{f}\left(\frac{k}{L_N}\right)(\varphi_k - \lambda_k), \tag{3.4.3}$$

where $\{\varphi_k\}_{k=1}^{\infty}$ are $\chi_1^2(0)$ distributed (mean one and variance two) when k is odd and non-central $\chi_1^2\left(\frac{1}{k}\right)$ distributed (mean $1 + \frac{1}{k}$ and variance $2 + \frac{4}{k}$) when k is even.

We complete the proof of Theorem 3.2.4 by showing that the moment generating function (m.g.f.) of (3.4.3) converges pointwise to that of the desired normal distribution. In our case, we will actually show the pointwise convergence of the cumulant generating function (c.g.f.).

Let

$$c_{k,N} = \frac{k}{L_N^{(3/2)}} \hat{f}\left(\frac{k}{L_N}\right) \qquad \text{and} \qquad X_{k,N} = c_{k,N}\varphi_k.$$

Recall that f is assumed to be compactly supported. Since we also require that f is continuously differentiable, we can write $|c_{k,N}| \leq C_f L_N^{-1/2}$, where C_f is positive constant depending only on f. Now, for any fixed t and large enough N, independent of k, the moment generating function for $c_{k,N}X_k$ is given by

$$\mathcal{F}_k(t) = \begin{cases} (1 - 2c_{k,N}t)^{-1/2} \exp\left(-c_{k,N}t\right) & \text{k odd} \\ (1 - 2c_{k,N}t)^{-1/2} \exp\left(-c_{k,N}t - \frac{c_{k,N}t}{k}\right) \exp\left(\frac{c_{k,N}t}{k(1-2c_{k,N}t)}\right) & \text{k even} \end{cases}.$$

(3.4.2) is a finite sum of independent random variables, so its m.g.f. is given by

$$\prod_{k=1}^{m_N} \mathcal{F}_k(t)$$

and c.g.f. by

(3.4.4)
$$\sum_{k=1}^{m_N} \log\left(\mathcal{F}_k(t)\right) = \left[\sum_{k=1}^{\lfloor m_N/2 \rfloor} \frac{c_{2k,N}t}{2k} \frac{1}{1 - 2c_{2k,N}t} - \frac{c_{2k,N}}{2k}t\right] - \left[\sum_{k=1}^{m_N} (1/2)\log(1 - 2c_{k,N}t) + c_{k,N}t\right].$$

Once again, since $|c_{k,N}| \leq C_f L_N^{-1/2}$, we may assume that N is large enough such that $2|c_{k,N}t| \leq 1/2$, independent of k. Applying Taylor's Theorem to each term in the second sum on the r.h.s. of (3.4.4), we have

$$\sum_{k=1}^{m_N} (1/2)\log(1 - 2c_{k,N}t) + c_{k,N}t = -\frac{t^2}{2}\sum_{k=1}^{m_N} 2c_{k,N}^2 + \left(\frac{1}{2}\right)\sum_{k=1}^{m_N} R_2(-2c_{k,N}t), \tag{3.4.5}$$

where $\log(1+x) = x - \frac{x^2}{2} + R_2(x)$, i.e. $R_2(x)$ is the second order error term for $\log(1+x)$ given by Taylor's Theorem. Since

$$\left|\left(\frac{d}{dx}\right)^3 \log(1+x)\right| = \left|\frac{2}{(1+x)^3}\right| \leq 2^4$$

for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we can write

$$|R_2(-2c_{k,N}t)| \leq \frac{2^7}{3!}|c_{k,N}t|^3. \tag{3.4.6}$$

The term proportional to $-t^2/2$ in (3.4.5) is a Riemann sum converging to

$$2\int_0^\infty [x\hat{f}(x)]^2\, dx,$$

whereas (3.4.6) implies that the second term on the r.h.s of (3.4.5) is on the order of $L_N^{-1/2}$ times a Riemann sum converging to

$$\int_0^\infty |x\hat{f}(x)|^3\, dx < \infty.$$

Similarly, using the series expansion for $(1-x)^{-1}$, the first sum on the r.h.s. of (3.4.4) can be written as

$$-t^2 \sum_{k=1}^{\lfloor m_N/2 \rfloor} \frac{c_{2k,N}^2}{k} + O\left(\sum_{k=1}^{\lfloor m_N/2 \rfloor} \frac{|c_{2k,N}t|^3}{k} \cdot |(1-2c_{2k,N}t)^{-1}|\right)$$

$$= t^2 \cdot O(L_N^{-1}) + |t|^3 \cdot O(L_N^{-3/2}), \tag{3.4.7}$$

which goes to zero as $N \to \infty$. By combining (3.4.4), (3.4.5), and (3.4.7), we have the pointwise convergence

$$\sum_{k=1}^{m_N} \log\left(\mathcal{F}_k(t)\right) \to \frac{t^2}{2}\left(\int_{\mathbb{R}} |x\hat{f}(x)|^2\right)$$

for all $t \in \mathbb{R}$. This completes the proof. □

We now proceed to the proof of Theorem 3.2.8, which is easily obtained by generalizing the results of Section 3.5 using the results of [1]. Before we begin, we present the following technical lemma.

Lemma 3.4.1. *Let $\{a_k\}_{k=1}^{\infty}$ be a positive sequence such that $V_N = \sum_{k=1}^{N} a_k^2$ is slowly varying. If $\lim\limits_{N\to\infty} V_N = \infty$, then*

$$\max_{1\le k\le N}(a_k) = o_N\left(\sqrt{V_N}\right).$$

Proof.

Let $A_N = \max\limits_{1\le k\le N} a_k$. A_N is a non-decreasing, integer valued sequence. If A_N converges, then A_N must be constant for sufficiently large N. Since V_N diverges, it then must be the case that $A_N/\sqrt{V_N} \to 0$. Let us then consider the case where A_N diverges. By the definition of A_N, $A_{N+1} = A_N$ or $A_{N+1} = a_{N+1}$ for all $N \in \mathbb{N}$. Let $\{N_\ell\}_{\ell=1}^{\infty}$ be sequence such that $A_{N_{\ell+1}} = a_{N_{\ell+1}}$. Now

$$1 = \frac{V_{N_\ell}}{V_{N_\ell}} = \frac{1}{V_{N_\ell}}\left(\sum_{k=1}^{\lfloor N_\ell/2\rfloor} a_k^2 + \sum_{k=\lfloor N_\ell/2\rfloor}^{N_\ell} a_k^2\right) = \frac{V_{\lfloor V_{N_\ell}/2\rfloor}}{V_{N_\ell}} + \frac{1}{V_{N_\ell}}\sum_{k=\lfloor N_\ell/2\rfloor}^{N_\ell} a_k^2. \tag{3.4.8}$$

Since V_N is slowly varying, it must be the case that

$$\lim_{\ell\to\infty}\frac{V_{\lfloor V_{N_\ell}/2\rfloor}}{V_{N_\ell}} = 1 \quad\text{and}\quad \lim_{\ell\to\infty}\frac{1}{V_{N_\ell}}\sum_{k=\lfloor N_\ell/2\rfloor}^{N_\ell} a_k^2 = 0.$$

Since $A_{N_\ell}^2 = a_{N_\ell}^2$, it must be the case that $\lim\limits_{l\to\infty} A_{N_\ell}/\sqrt{V_{N_\ell}} = 0$. For any $\epsilon > 0$ we can then choose $\ell' \in \mathbb{N}$ such that, for all $\ell \ge \ell'$, $A_{N_\ell}/\sqrt{V_{N_\ell}} < \epsilon$. Suppose $N_\ell \le N < N_{\ell+1}$. Then

$$\frac{A_N}{\sqrt{V_N}} = \frac{A_{N_\ell}}{\sqrt{V_N}} \le \frac{A_{N_\ell}}{\sqrt{V_\ell}}.$$

It follows that, for all $N > N_{\ell'}$, $A_N/\sqrt{V_N} < \epsilon$. This completes the proof. □

Proof. (*of Theorem 3.2.8*)

Let $f \in L^2(\mathbb{T})$ be a real valued, even function and $V_N = \sum_{k=1}^{N} k^2|\hat{f}|^2$ be a slowly varying sequence such that $V_N \to \infty$. Furthermore, let $\{B_N\}_{N=1}^{\infty}$ be a positive, integer-valued sequence that grows to ∞ sufficiently slowly as $N \to \infty$ in such a way that

$$\lim_{N\to\infty}\frac{V_{\lfloor N\cdot B_N\rfloor}}{V_N} = \lim_{N\to\infty}\frac{V_N}{V_{\lfloor N/B_N\rfloor}} = 1. \tag{3.4.9}$$

Such a sequence can be constructed using (3.2.6) and routine subsequence arguments. We now split $[S_N(f) - \mathbb{E}S_N(f)]/(2\sqrt{2V_N})$ into two parts. Namely,

$$\frac{S_N(f) - \mathbb{E}S_N(f)}{2\sqrt{2V_N}} = \frac{1}{\sqrt{2V_N}} \sum_{k=1}^{m_N} \hat{f}(k)(t_k^2 - \mathbb{E}t_k^2) + \frac{1}{\sqrt{2V_N}} \sum_{k=m_N+1}^{\infty} \hat{f}(k)(t_k^2 - \mathbb{E}t_k^2), \tag{3.4.10}$$

where $m_N = \lfloor N/B_N \rfloor$. Replacing Lemma 2.4.3 with Lemma 2.1 from [1] in the proof of Proposition 3.5.4 shows that

$$\mathrm{Var}\left(\frac{S_N(f)}{2\sqrt{2V_N}}\right) =$$

$$\frac{1}{V_N}\sum_{k=1}^{m_N} k^2|\hat{f}(k)|^2 + O\left(\frac{V_{\lfloor N/2\rfloor} - V_{m_N}}{V_N} + \frac{1}{V_N}\sum_{k=1}^{\lfloor N/2\rfloor} k|\hat{f}(k)|^2 + \frac{1}{V_N}\sum_{k=\lfloor N/2\rfloor+1}^{\infty} N^2|\hat{f}(k)|^2\right) + o_N(1)$$

$$= \frac{V_{m_N}}{V_N} + O\left(\frac{1}{V_N}\left(\sum_{k=1}^{\lfloor N/2\rfloor} k^2|\hat{f}(k)|^2\right)^{1/2}\right) + O\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{V_{(k+1)\lfloor N/2\rfloor} - V_{k\lfloor N/2\rfloor}}{V_N}\right)\right) + o_N(1)$$

$$= \frac{V_{m_N}}{V_N} + O\left(\frac{1}{\sqrt{V_N}}\right) + o_N(1)$$

$$= \frac{V_{m_N}}{V_N} + o_N(1)$$

As a result, the variance of the second term on the r.h.s. of (3.4.10) goes to zero as $N \to \infty$, while the variance of the first term converges to one. It then suffices to study the asymptotics of the first sum. It follows from Proposition 1.0.1 that, for $1 \le m < B_N/2$,

$$\mathbb{E}\left(\frac{1}{\sqrt{2V_N}}\sum_{k=1}^{m_N}\hat{f}(k)(t_k^2 - \mathbb{E}t_k^2)\right)^m = \mathbb{E}\left(\frac{1}{\sqrt{2V_N}}\sum_{k=1}^{m_N} k\hat{f}(k)(\varphi_k - \lambda_k)\right)^m, \tag{3.4.11}$$

where $\lambda_k = 1 + \frac{1+(-1)^k}{2k}$ and $\{\varphi_k\}_{k=1}^{\infty}$ are independent χ_1^2 distributed random variables. In particular, φ_k is $\chi_1^2(0)$ distributed (mean one and variance two) when k is odd and non-central $\chi_1^2\left(\frac{1}{k}\right)$ distributed (mean $1 + \frac{1}{k}$ and variance $2 + \frac{4}{k}$) when k is even.

The remainder of the proof follows from a standard moment generating function argument that is nearly identical to the one given in the proof of Theorem 3.2.4. As in the proof of Theorem 3.2.4, (3.4.11) implies that it is sufficient to study the convergence of the moment generating function for

$$\frac{1}{\sqrt{2V_N}}\sum_{k=1}^{m_N} k\hat{f}(k)(\varphi_k-\lambda_k). \tag{3.4.12}$$

Let

$$c_{k,N}=\frac{k\hat{f}(k)}{\sqrt{2V_N}} \quad \text{and} \quad X_{k,N}=c_{k,N}(\varphi_k-\lambda_k).$$

It follows from Lemma 3.4.1 that

$$\max_{1\le k\le N}\{k|\hat{f}(k)|\}=o_N(\sqrt{V_N}). \tag{3.4.13}$$

We may then assume that, for any fixed t and large enough N, the moment generating function for $c_{k,N}X_k$ is given by

$$\mathcal{F}_k(t)=\begin{cases}(1-2c_{k,N}t)^{-1/2}\exp\left(-c_{k,N}t\right) & \text{k odd}\\ (1-2c_{k,N}t)^{-1/2}\exp\left(-c_{k,N}t-\frac{c_{k,N}t}{k}\right)\exp\left(\frac{c_{k,N}t}{k(1-2c_{k,N}t)}\right) & \text{k even}\end{cases}.$$

for $1\le k\le m_N$. (3.4.12) is a finite sum of independent random variables, so its m.g.f. is given by

$$\prod_{k=1}^{m_N}\mathcal{F}_k(t)$$

and c.g.f. by

(3.4.14)
$$\sum_{k=1}^{m_N}\log\left(\mathcal{F}_k(t)\right)=\left[\sum_{k=1}^{\lfloor m_N/2\rfloor}\frac{c_{2k,N}t}{2k}\frac{1}{1-2c_{2k,N}t}-\frac{c_{2k,N}}{2k}t\right]-\left[\sum_{k=1}^{m_N}(1/2)\log(1-2c_{k,N}t)+c_{k,N}t\right].$$

It follows from (3.4.13) that, when N is sufficiently large, $2|c_{k,N}t|\le 1/2$ for $1\le k\le m_N$. Applying Taylor's Theorem to each term in the second sum on the r.h.s. of (3.4.14), we have

$$\sum_{k=1}^{m_N}(1/2)\log(1-2c_{k,N}t)+c_{k,N}t=-\frac{t^2}{2}\sum_{k=1}^{m_N}2c_{k,N}^2+\left(\frac{1}{2}\right)\sum_{k=1}^{m_N}R_2(-2c_{k,N}t), \tag{3.4.15}$$

where $\log(1+x) = x - \frac{x^2}{2} + R_2(x)$, i.e. $R_2(x)$ is the second order error term for $\log(1+x)$ given by Taylor's Theorem. As before, for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we can write

$$|R_2(-2c_{k,N}t)| \leq C|c_{k,N}t|^3, \tag{3.4.16}$$

where $C > 0$ is a constant independent of k, N, and f. The term proportional to $-t^2/2$ in (3.4.15) is the sum

$$\frac{1}{V_N}\sum_{k=1}^{m_N} k^2|\hat{f}(k)|^2,$$

which converges to one. (3.4.16) implies that the second term on the r.h.s of (3.4.15) is equal to

$$o_N(1) \cdot \left(\frac{1}{V_N}\sum_{k=1}^{m_N} k^2|\hat{f}(k)|^2\right) = o_N(1).$$

Similarly, using the series expansion for $(1-x)^{-1}$, the first sum on the r.h.s. of (3.4.14) can be written as

$$-t^2\sum_{k=1}^{\lfloor m_N/2\rfloor}\frac{c_{2k,N}^2}{k} + O\left(\sum_{k=1}^{\lfloor m_N/2\rfloor}\frac{|c_{2k,N}t|^3}{k}\cdot|(1-2c_{2k,N}t)^{-1}|\right)$$

$$= (t^2 + |t|^3)\cdot O(V_N^{-1}), \tag{3.4.17}$$

which goes to zero as $N \to \infty$. By combining (3.4.14), (3.4.15), and (3.4.17), we have the pointwise convergence

$$\sum_{k=1}^{m_N}\log\left(\mathcal{F}_k(t)\right) \to \frac{t^2}{2}$$

for all $t \in \mathbb{R}$. This completes the proof.

□

3.5. Variance Calculation

In this section we prove analogous results regarding $\mathrm{Var}(S_N(f))$ to Corollary 2.4.1 and Proposition 2.2.2 in the case of $SO(N)$ and $Sp(N)$. Assuming that $\sum_{1\le k_1,k_2}\left|\mathrm{Cov}\left(t_{k_1}^2,t_{k_2}^2\right)\right|<\infty$, we can write $\mathrm{Var}\left(S_N(f)\right)$ as

$$4\sum_{1\le k_1,k_2}\mathrm{Cov}\left(t_{k_1}^2,t_{k_2}^2\right)\hat{f}(k_1)\hat{f}(k_2)$$

$$=4\sum_{1\le k}\mathrm{Var}(t_k^2)|\hat{f}(k)|^2+4\sum_{1\le k_1\neq k_2}\mathrm{Cov}\left(t_{k_1}^2,t_{k_2}^2\right)\hat{f}(k_1)\hat{f}(k_2).$$

In the case of $U(N)$, the first sum gives the primary contribution to the variance, whereas the second sum is negligible in the limit. We will show that this is still the case for the $SO(N)$ and $Sp(N)$. As such, the majority of this section is devoted to computing $\mathrm{Cov}\left(t_{k_1}^2,t_{k_2}^2\right)$ and verifying the above assumption. Unlike in Section 2.4, we will use joint cumulants. The Section is organized as follows. We start by giving a short introduction to joint cumulants and some of their proprieties that will be used throughout this section. We then state the main results of the section, namely Propositions 3.5.3-3.5.4, which are proven using several technical lemmas. Proofs of technical lemmas are given at the end of the section.

Definition 3.5.1. Let $X_1,...X_n$ be random variables. Their joint cumulants are defined in terms of the coefficients of the logarithm of their joint moment generating function in the following manner:

$$K(t_1,...,t_n)=\log\left[\mathbb{E}\left(e^{\sum_{j=1}^n X_j\cdot t_j}\right)\right]$$
$$=\sum_{1\le\ell}\sum_{\substack{m_1+...m_n=\ell\\ m_i\in\mathbb{N}_0}}\kappa_\ell(\underbrace{X_1,...,X_1}_{m_1\text{ times}},\ldots,\underbrace{X_n,...,X_n}_{m_n\text{ times}})\prod_{j=1}^n t_j^{m_j}.$$

$K(t_1,...t_n)$ is referred to as the joint cumulant generating function for $X_1,...,X_n$. [20]

Remark 3.5.2. It easy to see from the above definition that ℓ-th order joint cumulant $\kappa_\ell(X,...,X)$ is equal to the ℓ-th order standard cumulant of X, $c_\ell(X)$. As such $\kappa_1(X)=\mathbb{E}\left(X\right)$ and $\kappa_2(X,X)=\mathrm{Var}\left(X\right)$.

Recall that, for a family of random variables $\{X_1, ..., X_n\}$, we can explicitly express joint cumulants in terms of joint moments in the following manner:

$$\kappa_n(X_{i_1}, \ldots, X_{i_n}) = \sum_{\pi} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E}\left(\prod_{i \in B} X_i\right), \tag{3.5.1}$$

where the sum is over all partitions π of $\{i_1, \ldots, i_n\}$, B runs through the list of all blocks of the partition π, and $|\pi|$ is the number of blocks in the partition. Joint cumulants are symmetric, i.e.

$$\kappa_n(X_1, \ldots, X_n) = \kappa_n(X_{\sigma(1)}, \ldots, X_{\sigma(n)}), \quad \sigma \in S_n, \tag{3.5.2}$$

and have the multilinearity property

$$\kappa_{n-1}(c_1X_1 + c_2X_2, X_3, \ldots, X_n) = c_1\kappa_{n-1}(X_1, X_3, \ldots, X_n) + c_2\kappa_{n-1}(X_2, X_3, \ldots, X_n).$$

It is also the case that joint moments can be expressed in terms of joint cumulants

$$\mathbb{E}(X_1 \cdot \ldots \cdot X_n) := \mathbb{E} \prod_{1 \le i \le n} X_i = \sum_{\pi} \prod_{B \in \pi} \kappa_{|B|}(X_i : i \in B). \tag{3.5.3}$$

Finally, we recall that $\kappa_1(X_1) = \mathbb{E}(X_1)$ and $\kappa_2(X_1, X_2) = \mathrm{Cov}(X_1, X_2)$.

We now proceed to the main results of the section.

Proposition 3.5.3. *Let t_k denote the trace of the k-th power of an $SO(2N)$, $SO(2N+1)$, or $Sp(N)$ distributed random matrix. Then the covariance statistics for t_k^2 are as follows.*

(i) *In the case of $SO(2N)$ ($M_N = 2N - 1, \delta = 1$) or $Sp(N)$ ($M_N = 2N + 1, \delta = -1$),*

$$Var(t_k^2) = \begin{cases} 2k^2 + 4k \cdot \chi_{k\ even} & 1 \le k < (M_N + 1)/4 \\ 2k^2 + 4k \cdot \chi_{k\ even} + \delta & (M_N + 1)/4 \le k < (M_N + 1)/3 \\ 2k^2 + 4(k - \delta) \cdot \chi_{k\ even} + \delta & (M_N + 1)/3 \le k \le (M_N - 1)/2 \\ 2(k + \delta)^2 - 6k + 4k \cdot \chi_{k\ even} + 3(M_N - \delta) & (M_N + 1)/2 \le k \le M_N - 1 \\ 2N(4N - 3) & M_N \le k \end{cases}$$

and

$$Cov(t_{k_1}^2, t_{k_2}^2) =$$

$$\delta\left(4\chi_{\substack{k_1,k_2\ even\\ |k_1-k_2|\le M_N-1\\ M_N+1\le k_1+k_2}} + 2\chi_{\substack{k_1\le (M_N-1)/2\\ (M_N+1)/2\le k_1+k_2\\ |k_1-k_2|\le (M_N-1)/2}} + 2\chi_{\substack{k_2\le (M_N-1)/2\\ (M_N+1)/2\le k_1+k_2\\ |k_1-k_2|\le (M_N-1)/2}} - 3\chi_{\substack{(M_N+1)/2\le k_1+k_2\\ |k_1-k_2|\le (M_N-1)/2}}\right)$$

$$+O\left(N\right)\cdot\left(\chi_{\substack{k_2=2k_1\\ (M_N+1)/4\le k_1\le (M_N-1)/2}} + \chi_{\substack{k_1=2k_2\\ (M_N+1)/4\le k_2\le (M_N-1)/2}}\right)$$

$$-2\delta\left(\chi_{\substack{k_2\ even\\ k_2\le M_N-1\\ |2k_1-k_2|\le M_N-1\\ M_N+1\le 2k_1+k_2}} + \chi_{\substack{k_1\ even\\ k_1\le M_N-1\\ |2k_2-k_1|\le M_N-1\\ M_N+1\le 2k_2+k_1}}\right)$$

$$-2\left((M_N-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le M_N-1)\\ M_N\le \max(k_1,k_2)}} + (k_1+k_2-M_N)\chi_{\substack{k_1,k_2\le M_N-1\\ M_N+1\le k_1+k_2}}\right) + 2\chi_{\substack{k_1,k_2\ even\\ |k_1-k_2|\le M_N-1\\ M_N+1\le k_1+k_2}}$$

for $k_1 \neq k_2$.

(ii) *In the case of* $SO(2N+1)$,

$$Var(t_k^2) = \begin{cases} 2k^2 + 4k\cdot\chi_{k\ even} & 1\le k\le N \\ 2k^2 + 4k\cdot\chi_{k\ even} - 6k + 6N & N+1\le k\le 2N-1 \\ 2N(4N+1) & 2N\le k \end{cases}$$

and

$$Cov(t_{k_1}^2, t_{k_2}^2) = O(1)\cdot\chi_{\substack{|k_1-k_2|\ odd\\ |k_1-k_2|\le 2N-1\\ 2N+1\le k_1+k_2}} + O(N)\cdot\left(\chi_{\substack{2k_1=k_2\\ N/2<k_1\le N-1}} + \cdot\chi_{\substack{2k_2=k_1\\ N/2<k_2\le N-1}}\right)$$

$$-2\left((2N-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le 2N-1)\\ 2N\le \max(k_1,k_2)}} + (k_1+k_2-2N)\chi_{\substack{k_1,k_2\le 2N-1\\ 2N+1\le k_1+k_2}}\right)$$

for $k_1 \neq k_2$.

Proposition 3.5.4. *Let* $\theta_1, ..., \theta_N$ *be distributed according to* $SO(N)$ *or* $Sp(N)$ *and* $f \in \mathbb{H}^1(\mathbb{T})$. *Then*

$$Var(S_N(f)) = 4\sum_{k=1}^{\lfloor N/2\rfloor}\left(2k^2 + 4k\left(\frac{1+(-1)^k}{2}\right)\right)|\hat{f}(k)|^2 + o_N(1).$$

It follows from (3.5.3) that, in order to compute $\mathrm{Cov}(t_{k_1}^2, t_{k_2}^2) = \mathbb{E}\left(t_{k_1}^2 t_{k_2}^2\right) - \mathbb{E}\left(t_{k_1}^2\right)\mathbb{E}\left(t_{k_2}^2\right)$, it is sufficient to understand the behavior of the first, second, third, and fourth order joint cumulants of $t_{k_1}, t_{k_1}, t_{k_2}, t_{k_2}$. We summarize this analysis in the form of the following lemma, which will be proven toward the end of the section.

Lemma 3.5.5.

Let t_k denote the trace of the k-th power of a $SO(2N)$ ($M_N = 2N-1, \alpha = 0, \delta = 1$), $SO(2N+1)$ ($M_N = 2N, \alpha = 1, \delta = -1$), or $Sp(N)$ ($M_N = 2N+1, \alpha = 0, \delta = -1$) distributed random matrix and $\kappa_n(k_{i_1}, ..., k_{i_n})$ denoted the n-th order joint cumulant of $(t_{k_{i_1}} - \alpha), ...(t_{k_{i_n}} - \alpha)$. Then

(i)

$$\kappa_1(k) = \delta \cdot \chi_{\substack{k+\alpha \text{ even} \\ k \le M_N - 1}}$$

(ii)

$$\kappa_2(k_1, k_2) = \min(k, M_N) \cdot \chi_{k=k_1=k_2} + \delta \cdot \chi_{\substack{|k_1-k_2|+\alpha \text{ even} \\ |k_1-k_2| \le M-1 \\ M_N+1 \le k_1+k_2}}$$

(iii)

$$\kappa_3(k_1, k_1, k_2) = \frac{1}{2}(\kappa_3^{'}(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa_3^{'}(t_{-k_1}, t_{-k_1}, t_{k_2})) \cdot \chi_{\substack{2k_1=k_2 \\ 2k_1+k_2 \ge M_N+1}} - \delta \cdot \chi_{\substack{k_2+\alpha \text{ even} \\ k_2 \le M_N-1 \\ 2k_1+k_2 \ge M_N+1 \\ |2k_1-k_2| \le M_N-1}}$$

where $\kappa_3^{'}$ and t_{k_i} denote the third order joint cumulant with respect to $U(M_N)$ and trace of the k_i-th power of a $U(M_N)$ distributed random matrix, respectively.

(iv)

$$\kappa_4(k_1, k_1, k_2, k_2) =$$

$$- \chi_{k_1=k_2} \cdot \left(M_N \cdot \chi_{M_N \le k_1}) + (2k_1 - M_N)\chi_{\substack{k_1 \le M_N-1 \\ M_N+1 \le 2k_1}}\right)$$
$$- 2\left((M_N - |k_1-k_2|)\chi_{\substack{|k_1-k_2| \le M_N-1 \\ M_N \le \max(k_1,k_2)}}) + (k_1+k_2-M_N)\chi_{\substack{k_1,k_2 \le M_N-1 \\ M_N+1 \le k_1+k_2}}\right)$$
$$+ \delta(1-\alpha) \cdot \left(2\chi_{\substack{k_1 \le (M_N-1)/2 \\ (M_N+1)/2 \le k_1+k_2 \\ |k_1-k_2| \le (M_N-1)/2}} + 2\chi_{\substack{k_2 \le (M_N-1)/2 \\ (M_N+1)/2 \le k_1+k_2 \\ |k_1-k_2| \le (M_N-1)/2}} - 3\chi_{\substack{(M_N+1)/2 \le k_1+k_2 \\ |k_1-k_2| \le (M_N-1)/2}}\right).$$

REMARK 3.5.6. As we will later show, the primary contribution to $\text{Var}\left(S_N(f)\right)$ will come from the first and second order joint cumulants. The formulas given above make it clear that the third and fourth order joint cumulants give a vanishing contribution unless (k_1, k_2) is sufficiently far away from $(0,0)$. This observation is the main idea behind the proof of Proposition 3.5.4.

COROLLARY 3.5.7. *(to Lemma 3.5.5) Let* $f \in \mathbb{H}^1(\mathbb{T})$ *and* $\theta_1, \ldots, \theta_N$ *be distributed according to* $SO(2N)$ $(M_N = 2N-1, \alpha = 0, \delta = 1)$*,* $SO(2N+1)$ $(M_N = 2N, \alpha = 1, \delta = -1)$*, or* $Sp(N)$ $(M_N = 2N+1, \alpha = 0, \delta = -1)$*. Then*

$$\mathbb{E}\left(S_N(f)\right) = 2\sum_{k=1}^{M_N-1}\left(k + \chi_{k\ even} + \delta(1-\alpha)\cdot\chi_{(M_N+1)/2\leq k} - (2N+\alpha)\right)\hat{f}(k).$$

PROOF. (of Corollary 3.5.7)

As discussed in Section 3.3,

$$S_N(f) = 2\sum_{1\leq k}\left(t_k^2 - (2N+\alpha)\right)\hat{f}(k).$$

Here we assume that $\hat{f}(0) = 0$. Using Lemma 3.5.5, we can easily compute $\mathbb{E}\left(t_k^2\right)$ for $k \geq 1$. In particular we have

$$\begin{aligned} E\left(t_k^2\right) &= \kappa_2(k,k) + \kappa_1^2(k) \\ &= \min(k, M_N) + \delta\cdot\chi_{(M_N+1)/2\leq k} + \chi_{\substack{k \text{ even} \\ k\leq M_N-1}} \end{aligned}$$

for $SO(2N)/Sp(N)$ and

$$\begin{aligned} E\left(t_k^2\right) &= \kappa_2(k,k) + 2\kappa_1(k) + 1 + \kappa_1^2(k) \\ &= \min(k, 2N) + (1 - \chi_{\substack{k \text{ odd} \\ k\leq 2N-1}}) \\ &= \min(k, 2N) + \chi_{\substack{k \text{ even} \\ k\leq 2N-2}} + \chi_{2N\leq k}. \end{aligned}$$

for $SO(2N+1)$. We remind that reader that, in the above calculation, $\kappa_n(k_{i_1}, ..., k_{i_n})$ denotes the n-th order joint cumulant of $(t_{k_{i_1}} - \alpha), ...(t_{k_{i_n}} - \alpha)$.

It follows that

$$\mathbb{E}\left(t_k^2 - (2N+\alpha)\right) = \begin{cases} k + \chi_{k \text{ even}} + \delta(1-\alpha)\cdot\chi_{(M_N+1)/2\le k} - (2N+\alpha) & \text{for } 1 \le k \le M_N - 1 \\ 0 & \text{for } M_N \le k \end{cases}.$$

Since

$$\sum_{1\le k} \mathbb{E}\left(|(t_k^2 - (2N+\alpha))\hat{f}(k)|\right) < \infty,$$

under the assumption that $f \in \mathbb{H}^1(\mathbb{T})$, this completes the proof. □

The following lemma about joint cumulants for $U(N)$ will be helpful in proving Lemma 3.5.5 and will allow us to control the asymptotic behaviour of the 'unitary part' of the third order joint cumulants in the proof of Proposition 3.5.3. Parts (i)-(iv) are precisely Lemma 5.2 in [2]. The proof of part (v) follows directly from Proposition 2.2.1.

Lemma 3.5.8.

Let t_k denote the k-th power of a $U(N)$ distributed random matrix and $\kappa_n(k_1,\dots,k_n)$ denote the n-th order joint cumulant of $t_{k_1},\dots,t_{k_n}$. Then

(i) $|\kappa_\ell(k_1,\dots,k_n)| \le C_n N$, *where C_n is some universal constant that depends only on n.*
(ii) *Let $n \ge 1$ and $\sum_{i=1}^n k_i \neq 0$. Then $\kappa_n(k_1,\dots,k_n) = 0$.*
(iii) *Let $\sum_{i=1}^n k_i = 0$, $\sum_{i=1}^n |k_i| \le N$ and $n > 2$. Then $\kappa_n(k_1,\dots,k_n) = 0$.*
(iv) *Let $n = 2$ and $k_1 = -k_2$. Then $\kappa_2(k_1,k_2) = \kappa_2(k_1,-k_1) = \min(N,|k_1|)$.*
(v) *Let $n = 4$ Then $\kappa_4(k_1,-k_1,k_2,-k_2)$ is equal to*

$$-\left((N - |k_1-k_2|)\cdot\chi_{\substack{1\le k_1,k_2 \\ |k_1-k_2|\le N-1 \\ N\le \max(k_1,k_2)}} \quad + \quad (k_1+k_2-N)\cdot\chi_{\substack{1\le k_1,k_2\le N-1 \\ N+1\le k_1+k_2}} \right).$$

Proof. (*of Lemma* 3.5.8 (v))

Let $\theta_1,\dots,\theta_N$ be distributed according to $U(N)$ and $t_k = \sum_{j=1}^N e^{ik\theta_j}$. The one point marginal density for $U(N)$ is normalized Lebesgue measure, so $\kappa_1(k) = \mathbb{E}(t_k) = 0$. It follows from (3.5.3) that $\mathrm{Var}(S_N(f))$ can be expressed as a sum of second and fourth order joint cumulants. In particular, we have

$$\left(\frac{1}{4}\right)\operatorname{Var}(S_N(f)) = \sum_{1\le k_1,k_2} \kappa_2(k_1,k_2)\kappa_2(-k_1,-k_2)\hat{f}(k_1)\hat{f}(k_2) + \sum_{1\le k_1,k_2} \kappa_2(k_1,-k_2)\kappa_2(-k_1,k_2)\hat{f}(k_1)\hat{f}(k_2) + \sum_{1\le k_1,k_2} \kappa_4(k_1,-k_1,k_2,-k_2)\hat{f}(k_1)\hat{f}(k_2).$$

Since $k_1, k_2 > 0$, the first sum is equal to zero by part (ii) of Lemma 3.5.8. The terms in the second sum are equal to zero unless $k_1 = k_2$, in which case, $\kappa_2^2(k,k) = \min(k^2, N^2)$. It follows that

$$\left(\frac{1}{4}\right)\operatorname{Var}(S_N(f)) = \sum_{k=1}^{N-1} k^2|\hat{f}(k)|^2 + N^2\sum_{k\ge N}(k_2)|\hat{f}(k)|^2 + \sum_{1\le k_1,k_2} \kappa_4(k_1,-k_1,k_2,-k_2)\hat{f}(k_1)\hat{f}(k_2).$$

Equivalently, Proposition 2.2.1 states that $\left(\frac{1}{4}\right)\operatorname{Var}(S_N(f))$ is equal to

$$\sum_{k=1}^{N-1} k^2|\hat{f}(k)|^2 + N^2\sum_{k\ge N}(k_2)|\hat{f}(k)|^2$$

$$-\left(\sum_{\substack{1\le k_1,k_2\\ |k_1-k_2|\le N-1\\ N\le \max(k_1,k_2)}} (N-|k_1-k_2|)\hat{f}(k_1)\hat{f}(k_2) \quad + \sum_{\substack{1\le k_1,k_2\le N-1\\ N+1\le k_1+k_2}} ((k_1+k_2)-N)\hat{f}(k_1)\hat{f}(k_2)\right).$$

Equating these two expressions gives,

$$\kappa_4(k_1,-k_1,k_2,-k_2) = -\left((N-|k_1-k_2|)\cdot\chi_{\substack{1\le k_1,k_2\\ |k_1-k_2|\le N-1\\ N\le\max(k_1,k_2)}} \quad + \quad (k_1+k_2-N)\cdot\chi_{\substack{1\le k_1,k_2\le N-1\\ N+1\le k_1+k_2}}\right).$$

This completes the proof. □

We now proceed to the proofs of Propositions 3.5.3-3.5.4. In the proof of Proposition 3.5.3, we begin by proving the case of $SO(2N)$ and, by a trivial generalization, $Sp(N)$. We then prove the case of $SO(2N+1)$, which will require some additional modifications. In the proof of 3.5.4 we consider the cases $SO(2N)$ and $Sp(N)$ simultaneously. The proof for the case of $SO(2N+1)$ is simpler and follows immediately from that of $SO(2N)$ and $Sp(N)$.

Proof. (*of Proposition 3.5.3*)

We start our computation of $\mathrm{Cov}\left(t_{k_1}^2, t_{k_2}^2\right)$ by first expressing $\mathbb{E}\left(t_{k_1}^2 t_{k_2}^2\right)$ in terms of joint cumulants using (3.5.3). To simplify notation, we will let $\kappa_n(k_{i_1}, \ldots, k_{i_n})$ denote the joint cumulant of $t_{k_{i_1}}, \ldots, t_{k_{i_n}}$ with respect to $SO(2N)$. As such, we have

$$\begin{aligned}
\mathbb{E}\left(t_{k_1}^2 t_{k_2}^2\right) = {} & \kappa_1^2(k_1)\kappa_1^2(k_2) + \kappa_1^2(k_1)\kappa_2(k_2,k_2) + \kappa_1^2(k_2)\kappa_2(k_1,k_1) + 4\kappa_1(k_1)\kappa_1(k_2)\kappa_2(k_1,k_2) \\
& + \kappa_2(k_1,k_1)\kappa_2(k_2,k_2) + 2\kappa_2(k_1,k_2)\kappa_2(k_1,k_2) \\
& + 2\kappa_1(k_1)\kappa_3(k_1,k_2,k_2) + 2\kappa_1(k_2)\kappa_3(k_1,k_1,k_2) \\
& + \kappa_4(k_1,k_1,k_2,k_2).
\end{aligned}$$

We note that there are 15 terms, including multiplicity, corresponding to the 15 different partitions of $\{1,2,3,4\}$. To obtain the desired formula for covariance, we subtract

$$\mathbb{E}\left(t_{k_1,n}^2\right)\mathbb{E}\left(t_{k_1,n}^2\right) = \left(\kappa_2(k_1,k_1) + \kappa_1^2(k_1)\right)\left(\kappa_2(k_2,k_2) + \kappa_1^2(k_2)\right)$$

to get an expression for covariance in terms of joint cumulants. Namely,

$$\mathrm{Cov}\left(t_{k_1}^2, t_{k_2}^2\right) =$$

$$4\kappa_1(k_1)\kappa_1(k_2)\kappa_2(k_1,k_2) \tag{3.5.4}$$

$$+ 2\kappa_2(k_1,k_2)\kappa_2(k_1,k_2) \tag{3.5.5}$$

$$+ 2\kappa_1(k_2)\kappa_3(k_1,k_1,k_2) + 2\kappa_1(k_1)\kappa_3(k_1,k_2,k_2) \tag{3.5.6}$$

$$+ \kappa_4(k_1,k_1,k_2,k_2). \tag{3.5.7}$$

We will consider two separate cases: $k = k_1 = k_2$, and $k_1 \neq k_2$. When $k = k_1 = k_2$, parts (i) and (ii) of Lemma 3.5.5 imply that (3.5.4) is equal to

$$4\left(\min(k, 2N-1)\cdot \chi_{\substack{k \text{ even} \\ k \leq 2N-2}} + \chi_{\substack{k\text{even} \\ k \geq N}}\right) = \begin{cases} 4k \cdot \chi_{k \text{ even}} & 1 \leq k \leq N-1 \\ 4(k+1)\cdot \chi_{k \text{ even}} & N \leq k \leq 2N-2 \end{cases}. \tag{3.5.8}$$

When $k_1 \neq k_2$, (3.5.4) is equal to

$$4\chi_{\substack{k_1,k_2 \text{ even} \\ k_1,k_2 \leq 2N-2 \\ |k_1-k_2| \leq 2N-2 \\ 2N \leq k_1+k_2}}. \tag{3.5.9}$$

Again, using part (ii) of Lemma 3.5.5, we can see that (3.5.5) is equal to

$$2\left(\min(k,2N-1)+\chi_{N\leq k}\right)^2 = \begin{cases} 2k^2 & 1 \leq k \leq N-1 \\ 2(k+1)^2 & N \leq k \leq 2N-2 \\ 2(2N)^2 & 2N-1 \leq k \end{cases} \tag{3.5.10}$$

when $k = k_1 = k_2$ and

$$2\chi_{\substack{k_1,k_2 \text{ even} \\ |k_1-k_2| \leq 2N-2 \\ 2N \leq k_1+k_2}} \tag{3.5.11}$$

when $k_1 \neq k_2$.

It follows from parts (i) and (iii) of Lemma 3.5.5, that (3.5.6) is equal to

$$-4\chi_{\substack{k \text{ even} \\ 2N/3 \leq k \leq 2N-2}} \tag{3.5.12}$$

when $k = k_1 = k_2$. When $k_1 \neq k_2$, the first term in (3.5.6) is equal to

$$\left(\kappa_3^{'}(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa_3^{'}(t_{-k_1}, t_{-k_1}, t_{k_2})\right)\chi_{\substack{k_2=2k_1 \\ k_2 \leq N-1}} - 2\chi_{\substack{k_2 \text{ even} \\ k_2 \leq 2N-2 \\ |2k_1-k_2| \leq 2N-2 \\ 2N \leq 2k_1+k_2}},$$

which can be rewritten as

$$O(N) \cdot \chi_{\substack{k_2=2k_1 \\ N/2 \leq k_1 \leq N-1}} - 2\chi_{\substack{k_2 \text{ even} \\ k_2 \leq 2N-2 \\ |2k_1-k_2| \leq 2N-2 \\ 2N \leq 2k_1+k_2}}$$

using parts (i) and (iii) of Lemma 3.5.8. Interchanging k_1 and k_2 gives an expression for the second term in (3.5.6). It follows that (3.5.6) is equal to

$$O(N)\cdot\left(\chi_{\substack{k_2=2k_1\\ N/2\le k_1\le N-1}}+\chi_{\substack{k_1=2k_2\\ N/2\le k_2\le N-1}}\right)-2\left(\chi_{\substack{k_2\text{ even}\\ k_2\le 2N-2\\ |2k_1-k_2|\le 2N-2\\ 2N\le 2k_1+k_2}}+\chi_{\substack{k_1\text{ even}\\ k_1\le 2N-2\\ |2k_2-k_1|\le 2N-2\\ 2N\le 2k_2+k_1}}\right). \tag{3.5.13}$$

Finally, we consider the contribution from the fourth order joint cumulant. When $k=k_1=k_2$, part (iv) of Lemma 3.5.5 implies that (3.5.7) is equal to

$$-3\left((2N-1)\chi_{2N-1\le k}+2k\chi_{N\le k\le 2N-2}-(2N-1)\chi_{N\le k\le 2N-2}\right)+\chi_{N/2\le k\le N-1}-3\chi_{N\le k}$$

$$=\begin{cases}0 & 1\le k<N/2\\ 1 & N/2\le k\le N-1\\ -6k+3(2N-2) & N\le k\le 2N-2\\ -3(2N) & 2N-1\le k\end{cases}. \tag{3.5.14}$$

When $k_1\neq k_2$, (3.5.7) is equal to

$$-2\left((2N-1-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le 2N-2\\ 2N-1\le\max(k_1,k_2)}})+(k_1+k_2-2N+1)\chi_{\substack{k_1,k_2\le 2N-2\\ 2N\le k_1+k_2}}\right)$$
$$+2\chi_{\substack{k_1\le N-1\\ N\le k_1+k_2\\ |k_1-k_2|\le N-1}}+2\chi_{\substack{k_2\le N-1\\ N\le k_1+k_2\\ |k_1-k_2|\le N-1}}-3\chi_{\substack{N\le k_1+k_2\\ |k_1-k_2|\le N-1}}. \tag{3.5.15}$$

By combining (3.5.8), (3.5.10),(3.5.12), and (3.5.14), we have

$$\mathrm{Var}(t_k^2)=\mathrm{Cov}(t_k^2,t_k^2)=\begin{cases}2k^2+4k\cdot\chi_{k\text{ even}} & 1\le k<N/2\\ 2k^2+1+4k\cdot\chi_{k\text{ even}} & N/2\le k<2N/3\\ 2k^2+1+4(k-1)\cdot\chi_{k\text{ even}} & 2N/3\le k\le N-1\\ 2(k+1)^2-6k+3(2N-2)+4k\cdot\chi_{k\text{ even}} & N\le k\le 2N-2\\ 2N(4N-3) & 2N-1\le k\end{cases}\cdot$$

REMARK 3.5.9. When $2N-1 \leq k$, $\{\cos(k\theta_i)\}_{i=1}^{N}$ are independent, mean zero, random variables. One can then confirm the formula for variance in this case directly through a somewhat simple computation.

Finally, when $k_1 \neq k_2$, (3.5.9), (3.5.11), (3.5.13), and (3.5.15) imply that $\mathrm{Cov}(t_{k_1}^2, t_{k_2}^2)$ is equal to

$$
\begin{aligned}
&4\chi_{\substack{k_1,k_2 \text{ even} \\ k_1,k_2\leq 2N-2 \\ |k_1-k_2|<2N-1 \\ 2N\leq k_1+k_2}} + 2\chi_{\substack{k_1,k_2 \text{ even} \\ |k_1-k_2|<2N-1 \\ 2N\leq k_1+k_2}} + 2\chi_{\substack{k_1\leq N-1 \\ N\leq k_1+k_2 \\ |k_1-k_2|\leq N-1}} + 2\chi_{\substack{k_2\leq N-1 \\ N\leq k_1+k_2 \\ |k_1-k_2|\leq N-1}} - 3\chi_{\substack{N\leq k_1+k_2 \\ |k_1-k_2|\leq N-1}} \\
&+O(N)\cdot\left(\chi_{\substack{k_2=2k_1 \\ N/2\leq k_1\leq N-1}} + \chi_{\substack{k_1=2k_2 \\ N/2\leq k_2\leq N-1}}\right) - 2\left(\chi_{\substack{k_2 \text{ even} \\ k_2\leq 2N-2 \\ |2k_1-k_2|\leq 2N-2 \\ 2N\leq 2k_1+k_2}} + \chi_{\substack{k_1 \text{ even} \\ k_1\leq 2N-2 \\ |2k_2-k_1|\leq 2N-2 \\ 2N\leq 2k_2+k_1}}\right) \\
&-2\left((2N-1-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\leq 2N-2 \\ 2N-1\leq \max(k_1,k_2)}}) + (k_1+k_2-2N+1)\chi_{\substack{k_1,k_2\leq 2N-2 \\ 2N\leq k_1+k_2}}\right).
\end{aligned}
$$

This completes the proof for $SO(2N)$ and, by extension, $Sp(N)$.

We now turn our attention to the case of $SO(2N+1)$. Recalling that every $SO(2N+1)$ distributed matrix M has one as an eigenvalue, $\mathrm{Tr}\left(M^k\right)$ is given by

$$t_k = 1 + \sum_{j=1}^{N} 2\cos(k\theta_j),$$

where $\theta_1, \ldots, \theta_N \in (0, \pi]$ are distributed according to (3.1.5). Again, we use (3.5.3) to rewrite $\mathrm{Cov}(t_{k_1}^2, t_{k_2}^2)$ as follows:

$$
\begin{aligned}
&\mathbb{E}(t_{k_1}^2 t_{k_2}^2) - \mathbb{E}(t_{k_1}^2)\mathbb{E}(t_{k_2}^2) \\
&= \mathbb{E}\left((t_{k_1}-1)^2(t_{k_2}-1)^2\right) \\
&\qquad + 2\kappa_3(k_1,k_1,k_2) + 2\kappa_3(k_1,k_2,k_2) + 4\kappa_1(k_1)\kappa_2(k_1,k_2) + 4\kappa_1(k_2)\kappa(k_1,k_2) + 4\kappa_2(k_1,k_2) \\
&\qquad - \kappa_1^2(k_1)\kappa_1^2(k_2) - \kappa_1^2(k_1)\kappa(k_2,k_2) - \kappa_1^2(k_2)\kappa(k_1,k_1) - \kappa_2(k_1,k_1)\kappa_2(k_2,k_2),
\end{aligned}
$$

where $\kappa_n(k_{i_1}, ..., k_{i_n})$ denotes the n-th order joint cumulant of $(t_{k_{i_1}} - 1), ...(t_{k_{i_n}} - 1)$. After simplifying, we arrive that the following expression for covariance:

$$\mathrm{Cov}\left(t_{k_1}^2, t_{k_2}^2\right) =$$

$$4(1 + \kappa_1(k_1) + \kappa_1(k_2) + \kappa_1(k_1)\kappa_1(k_2))\kappa_2(k_1, k_2) \tag{3.5.16}$$

$$+ 2\kappa_2(k_1, k_2)\kappa_2(k_1, k_2) \tag{3.5.17}$$

$$+ 2(1 + \kappa_1(k_1))\kappa_3(k_1, k_2, k_2) + 2(1 + \kappa_1(k_2))\kappa_3(k_1, k_1, k_2) \tag{3.5.18}$$

$$+ \kappa_4(k_1, k_1, k_2, k_2), \tag{3.5.19}$$

where $\kappa_n(k_{i_1}, \ldots, k_{i_n})$ denotes the n-th order joint cumulant of $t_{k_{i_1}}, \ldots, t_{k_{i_n}}$.

Once again, we consider the two different cases: $k_1 = k_2$ and $k_1 \neq k_2$. When $k_1 = k_2$, it follows from parts (i) and (ii) of Lemma 3.5.5 that (3.5.16) is equal to

$$4(1 - \chi_{\substack{k \text{ odd} \\ k \le 2N-1}}) \min(k, 2N) = \begin{cases} 4k \cdot \chi_{k \text{ even}} & 1 \le k \le 2N - 1 \\ 4(2N) & 2N \le k \end{cases}. \tag{3.5.20}$$

Similarly, (3.5.17) is equal to

$$2\min(k, 2N)^2 = \begin{cases} 2k^2 & 1 \le k \le 2N - 1 \\ 2(2N)^2 & 2N \le k \end{cases}. \tag{3.5.21}$$

It follows from part (iii) of Lemma 3.5.5 that (3.5.18) is equal to

$$4(\chi_{\substack{k \text{ even} \\ k \le 2N-2}} + \chi_{2N \le k}) \cdot \chi_{\substack{k \text{ odd} \\ 2N/3 < k \le 2N-1}} = 0. \tag{3.5.22}$$

It follows from part (iv) of Lemma 3.5.5 that (3.5.19) is equal to

$$-3((2N)\chi_{2N \le k} + (2k - 2N)\chi_{\substack{k \le 2N-1 \\ 2N+1 \le 2k}}) = \begin{cases} 0 & 1 \le k \le N \\ 6N - 6k & N + 1 \le k \le 2N - 1 \\ -6N & 2N \le k \end{cases}. \tag{3.5.23}$$

Combining (3.5.20), (3.5.21), (3.5.22), and (3.5.23), we have

$$\mathrm{Var}(t_k^2) = \begin{cases} 2k^2 + 4k \cdot \chi_{k \text{ even}} & 1 \le k \le N \\ 2k^2 + 4k \cdot \chi_{k \text{ even}} - 6k + 6N & N+1 \le k \le 2N-1 \\ 2N(4N+1) & 2N \le k \end{cases}$$

We now consider the case $1 \le k_1 \ne k_2$. Again, it follows from 3.5.5 that (3.5.16) is equal to

$$-4\left(1 - \chi_{\substack{k_1 \text{ odd} \\ k_1 \le 2N_1}} - \chi_{\substack{k_2 \text{ odd} \\ k_2 \le 2N_1}} + \chi_{\substack{k_1,k_2 \text{ odd} \\ k_1,k_2 \le 2N-1}}\right) \cdot \chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2| \le 2N-1 \\ 2N+1 \le k_1+k_2}} \tag{3.5.24}$$

$$= -4\left(1 - \chi_{\substack{k_1 \text{ odd} \\ k_1 \le 2N_1}} - \chi_{\substack{k_2 \text{ odd} \\ k_2 \le 2N_1}}\right) \cdot \chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2| \le 2N-1 \\ 2N+1 \le k_1+k_2}} \tag{3.5.25}$$

$$= O\left(\chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2| \le 2N-1 \\ 2N+1 \le k_1+k_2}}\right) \tag{3.5.26}$$

and (3.5.17) is equal to

$$2\left(\chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2| \le 2N-1 \\ 2N+1 \le k_1+k_2}}\right)^2 = 2\chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2| \le 2N-1 \\ 2N+1 \le k_1+k_2}} . \tag{3.5.27}$$

(3.5.18) is equal to

$$2\left(\kappa(k_1,k_1,k_2) \cdot \chi_{\substack{k_2 \text{ even} \\ k_2 \le 2N-1}} + \kappa(k_1,k_2,k_2) \cdot \chi_{\substack{k_1 \text{ even} \\ k_1 \le 2N-1}}\right) \tag{3.5.28}$$

$$+ 2\left(\kappa(k_1,k_1,k_2) \cdot \chi_{2N \le k_2} + \kappa(k_1,k_2,k_2) \cdot \chi_{2N \le k_1}\right). \tag{3.5.29}$$

Let us consider the first term in (3.5.28). It follows from part (iii) of Lemma 3.5.5 and part (i) of Lemma 3.5.8 that this term is equal to

$$(\kappa'(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa'(t_{-k_1}, t_{-k_1}, t_{k_2}))\chi_{\substack{2k_1 = k_2 \\ N/2 < k_1 \le N-1}} + 2\chi_{\substack{k_2 \text{ even} \\ k_2 \le 2N-1}} \cdot \chi_{\substack{k_2 \text{ odd} \\ k_2 \le 2N-1 \\ |k_2 - 2k_1| \le 2N-1 \\ 2N \le k_2 + 2k_1}}$$

$$= (\kappa^{'}(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa^{'}(t_{-k_1}, t_{-k_1}, t_{k_2}))\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}}$$

$$= O(N)\cdot\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}}.$$

Interchanging k_1 and k_2 gives an analogous expression for the second term in (3.5.28). (3.5.28) is thus equal to

$$O(N)\cdot\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}} + O(N)\cdot\chi_{\substack{2k_2=k_1 \\ N/2<k_2\le N-1}}. \tag{3.5.30}$$

We again use part (iii) of Lemma 3.5.5 and part (i) of Lemma 3.5.8 to analyze the first term in (3.5.29).

$$\begin{aligned} &(\kappa^{'}(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa^{'}(t_{-k_1}, t_{-k_1}, t_{k_2}))\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}}\cdot\chi_{2N\le k_2} + 2\chi_{2N\le k_2}\cdot\chi_{\substack{k_2 \text{ odd} \\ k_2\le 2N-1}} \\ &= (\kappa^{'}(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa^{'}(t_{-k_1}, t_{-k_1}, t_{k_2}))\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}}\cdot\chi_{N\le k_1} + 2\chi_{2N\le k_2}\cdot\chi_{\substack{k_2 \text{ odd} \\ k_2\le 2N-1}} \\ &= 0. \end{aligned} \tag{3.5.31}$$

By symmetry, we can conclude that (3.5.29) is equal to zero. It follows that (3.5.18) is equal to

$$O(N)\cdot\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}} + O(N)\cdot\chi_{\substack{2k_2=k_1 \\ N/2<k_2\le N-1}}. \tag{3.5.32}$$

Finally, it follows from part (iv) of Lemma 3.5.5 that (3.5.19) is equal to

$$-2\left((2N-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le 2N-1 \\ 2N\le\max(k_1,k_2)}} + (k_1+k_2-2N)\chi_{\substack{k_1,k_2\le 2N-1 \\ 2N+1\le k_1+k_2}}\right).$$

We can now combine (3.5.24), (3.5.27), and (3.5.32), and (3.5.22) to get a final expression for $\mathrm{Cov}(t_{k_1}^2, t_{k_2}^2)$ when $k_1 \neq k_2$. Namely,

$$\begin{aligned} \mathrm{Cov}(t_{k_1}^2, t_{k_2}^2) &= O(1)\cdot\chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2|\le 2N-1 \\ 2N+1\le k_1+k_2}} + O(N)\cdot\left(\chi_{\substack{2k_1=k_2 \\ N/2<k_1\le N-1}} + \cdot\chi_{\substack{2k_2=k_1 \\ N/2<k_2\le N-1}}\right) \\ &\quad -2\left((2N-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le 2N-1 \\ 2N\le\max(k_1,k_2)}} + (k_1+k_2-2N)\chi_{\substack{k_1,k_2\le 2N-1 \\ 2N+1\le k_1+k_2}}\right) \end{aligned}$$

This completes the proof. □

We now turn our attention to the proof of Proposition 3.5.4, which is done via Lemma 2.4.3 and the following technical lemma.

LEMMA 3.5.10.
Let $f \in \mathbb{H}^1(\mathbb{T})$ and $\hat{f}(k)$ denote the k-th Fourier coefficient of f. Then

$$\sum_{\substack{k_1 \ even \\ |k_1 - 2k_2| \leq 2N-2 \\ 2N \leq k_1 + 2k_2}} \hat{f}(k_1)\hat{f}(k_2)$$

goes to zero as $N \to \infty$.

The proof of Lemma 3.5.10 follows immediately from that of Lemma 2.4.3, with some minor alterations. The proof is postponed until the end of the section.

PROOF. (*of Proposition 3.5.4*)
Assuming that $\sum_{1 \leq k_1, k_2} |\mathrm{Cov}(t_{k_1}^2, t_{k_2}^2)\widehat{f}(k_1)\widehat{f}(k_2)| < \infty$, we can write

$$\left(\frac{1}{4}\right) \cdot \mathrm{Var}(S_N(f)) = \sum_{1 \leq k} \mathrm{Var}\left(t_k^2\right) |\hat{f}(k)|^2 + \sum_{\substack{1 \leq k_1 \neq k_2 \\ k_2 \neq 2k_1 \\ k_1 \neq 2k_2}} \mathrm{Cov}(t_{k_1}^2, t_{k_2}^2)\hat{f}(k_1)\hat{f}(k_2). \tag{3.5.33}$$

We first consider the cases of $SO(2N)$ and $Sp(N)$. It follows from part (i) of Proposition 3.5.3 that the first term on the r.h.s. of (3.5.33) is equal to

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \left(2k^2 + 4k\chi_{k \text{ even}}\right) |\hat{f}(k)|^2 + O\left(N^2 \sum_{\lfloor \frac{N}{2} \rfloor + 1 \leq k} |\hat{f}(k)|^2\right)$$

Assuming that $f \in \mathbb{H}^1(\mathbb{T})$, i.e. $\sum_{1 \leq k} k^2 |\hat{f}(k)|^2 < \infty$, the sum on the left is finite and the sum on the right is at most on the order of

$$\sum_{\lfloor \frac{N}{2} \rfloor + 1 \leq k} k^2 |\hat{f}(k)|^2 = o_N(1).$$

Again, using Proposition 3.5.3, we break up the second term in (3.5.33) into 3 parts:

(3.5.34)

$$-2 \sum_{\substack{1\le|k_1-k_2|\le M_N-1\\ M_N\le\max(k_1,k_2)}} (M_N-|k_1-k_2|)\hat f(k_1)\hat f(k_2) - 2\sum_{\substack{k_1,k_2\le M_N-1\\ M_N+1\le k_1+k_2}} (k_1+k_2-M_N)\hat f(k_1)\hat f(k_2),$$

(3.5.35)

$$4\delta \sum_{\substack{k_1,k_2 \text{ even}\\ |k_1-k_2|\le M_N-1\\ M_N+1\le k_1+k_2}} \hat f(k_1)\hat f(k_2) + 4\delta \sum_{\substack{k_1\le(M_N-1)/2\\ (M_N+1)/2\le k_1+k_2\\ |k_1-k_2|\le(M_N-1)/2}} \hat f(k_1)\hat f(k_2) - 3\delta \sum_{\substack{|k_1-k_2|\le(M_N-1)/2\\ (M_N+1)/2\le k_1+k_2}} \hat f(k_1)\hat f(k_2)$$

$$+2 \sum_{\substack{k_1,k_2 \text{ even}\\ |k_1-k_2|\le M_N-1\\ M_N+1\le k_1+k_2}} \hat f(k_1)\hat f(k_2),$$

and

(3.5.36)
$$+O(N) \sum_{\substack{k_2=2k_1\\ \frac{M_N+1}{4}\le k_1\le\frac{M_N-1}{2}}} \hat f(k_1)\hat f(k_2) - 4\delta \sum_{\substack{k_2 \text{ even}\\ k_2\le M_N-1\\ |2k_1-k_2|\le M_N-1\\ M_N+1\le 2k_1+k_2}} \hat f(k_1)\hat f(k_2).$$

The fact that the sums in (3.5.34) converge absolutely and are of the order $o_N(1)$ is precisely the second half of the proof of Proposition 2.2.2. Similarly, the fact that all four sums in (3.5.35) converge absolutely and are of the order $o_N(1)$ follows immediately from Lemma 2.4.3. It thus remains to show that the sums in (3.5.36) converge absolutely and are of the order $o_N(1)$. For the first sum, the Cauchy Schwartz inequality implies that

$$\sum_{\frac{M_N+1}{4}\le k\le\frac{M_N-1}{2}} N|\hat f(k)|\cdot|\hat f(2k)| \le \sum_{\frac{M_N+1}{4}\le k\le\frac{M_N-1}{2}} 2k|\hat f(k)|\cdot|\hat f(2k)|$$

$$\le \left(\sum_{\lfloor\frac{N}{2}\rfloor+1\le k} k^2|\hat f(k)|^2\right)^{1/2} \left(\sum_{\lfloor\frac{N}{2}\rfloor+1\le k} (2k)^2|\hat f(2k)|^2\right)^{1/2},$$

which is equal to $o_N(1)$. Finally, the second sum in (3.5.36) converges absolutely and is equal to $o_N(1)$ by Lemma 3.5.10.

It follows immediately that

$$\mathrm{Var}(S_N(f)) = 4 \sum_{1\le k_1,k_2} \mathrm{Cov}(t_{k_1}^2, t_{k_2}^2) = 4 \sum_{k=1}^{\lfloor N/2 \rfloor} \left(2k^2 + 4k \left(\frac{1+(-1)^k}{2} \right) \right) |\hat{f}(k)|^2 + o_N(1).$$

This completes the proof of Proposition 3.5.4 in the case of $SO(2N)$ and $Sp(N)$. The proof for the case of $SO(2N+1)$ is simpler and follows immediately from the arguments given above. □

A general formula for joint cumulants of linear statistics for determinental random point processes was formulated as part of the proof of Proposition 9.3 in Appendix II of [2]. We restate this formula as Lemma 3.5.11.

LEMMA 3.5.11. *Let $\{\theta_1, \ldots, \theta_N\}$, $\theta_i \in \mathcal{X}$, be a determinental random point process with kernel $K_N(x,y)$. If $f_1, \ldots, f_n$ are functions defined on $\mathcal{X}$, then the n-th order joint cumulant of the linear statistics $\sum f_1(\theta_j), \ldots, \sum f_n(\theta_j)$ is given by*

$$\sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} \sum_{\substack{\text{ordered collections} \\ \text{of subsets } \mathcal{R}=\{R_1,..,R_m\}}} \int_{\mathcal{X}^m} f_{R_1}(\theta_1)...f_{R_m}(\theta_m) \prod_{j=1}^{m} K_N(\theta_j, \theta_{j+1}) d\bar{\theta}, \tag{3.5.37}$$

where $f_{R_i}(\theta) = \prod_{j\in R_i} f_j(\theta)$, $\theta_{m+1} = \theta_1$, and 'Ordered subsets $\mathcal{R} = \{R_1, ..., R_m\}$' refers to all partitions of $\{1, ..., n\}$ with m ordered blocks.

Applying this result to the cases of $SO(N)$ and $Sp(N)$, we can prove Lemma 3.5.12, which allows us to break up joint cumulants for $SO(N)$ and $Sp(N)$ into their 'unitary' and 'non-unitary' parts.

LEMMA 3.5.12. *Let $\theta_1, \ldots, \theta_N$ be distributed according to $SO(2N)$ ($M_N = 2N-1, \alpha = 0, \delta = 1$), $SO(2N+1)$ ($M_N = 2N, \alpha = 1, \delta = -1$), or $Sp(N)$ ($M_N = 2N+1, \alpha = 0, \delta = -1$) and $f_1, \ldots, f_n$ be even functions on $\mathbb{T}$. Then $\kappa_n(\Sigma_j f_1(\theta_j), \ldots, \Sigma_j f_n(\theta_j))$ can be written as*

$$\frac{1}{2}\kappa_n^{'}(\Sigma_j f_1(\theta_j), \ldots, \Sigma_j f_n(\theta_j)) +$$

$$\frac{\delta}{2} \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} \sum_{\substack{\text{ordered} \\ \text{collections} \\ \text{of subsets} \\ \mathcal{R}=\{R_1,..,R_m\}}} \sum_{s_1,\ldots,s_m=-(M_N-1)/2}^{(M_N-1+\alpha)/2} \widehat{f_{R_1}}(s_1+s_m-\alpha)\widehat{f_{R_2}}(s_1-s_2)\ldots\widehat{f_{R_m}}(s_{m-1}-s_m),$$

where $\kappa_n^{'}$ denotes the n-th order joint cumulant with respect to $U(M_N)$.

Proof. (*of Lemma 3.5.12*)

For $SO(2N)$, the integration kernel $K^{+}_{M_N}(x,y) = K_{M_N}(x,y) + K_{M_N}(x,-y)$ where

$$K_{M_N}(x,y) = \frac{\sin\left(\frac{M_N(x-y)}{2}\right)}{2\pi\cdot\sin\left(\frac{(x-y)}{2}\right)} = \frac{1}{2\pi}\left(\sum_{k=-(M_N-1)/2}^{(M_N-1)/2} e^{ik(x-y)}\right). \tag{3.5.38}$$

In the case of $Sp(N)$, the integration kernel is $K^{-}_{M_N} = K_{M_N}(x,y) - K_{M_N}(x,-y)$.

It then follows from Lemma 3.5.11 that

$$\kappa_n\left(\Sigma_j f_1(\theta_j),\ldots,\Sigma_j f_n(\theta_j)\right) \tag{3.5.39}$$

$$= \sum_{m=1}^{n}\frac{(-1)^{m-1}}{m}\sum_{\substack{\text{ordered collections}\\ \text{of subsets } \mathcal{R}=\{R_1,..R_m\}}}\cdots \tag{3.5.40}$$

$$\cdots\sum_{\epsilon_1,\ldots\epsilon_m=\pm 1}\left(\prod_{i=1}^{m}\epsilon_i\right)^{(1-\delta)/2} + \int_{\mathbb{H}^m} f_{R_1}(\theta_1)\ldots f_{R_m}(\theta_m)\prod_{j=1}^{m} K_{M_N}(\theta_j,\epsilon_j\cdot\theta_{j+1})d\bar{\theta}.$$

We can break this expression up into two pieces, one where we sum over all $\epsilon_1,\cdots,\epsilon_m$ such that $\prod_j \epsilon_j = 1$, and one where we sum over all $\epsilon_1,\cdots,\epsilon_m$ such that $\prod_j\epsilon_j = -1$. In either case, the integrals inside the summations each give the same contribution. To see this, we simply observe that $K_N(\epsilon_{j-1}\theta_j,\epsilon_j\theta_{j+1}) = K_N(\theta_j,\epsilon_{j-1}\epsilon_j\theta_{j+1})$. Using the fact that $f_1 \ldots f_n$ are even, the substitution $\Theta_{i+1} = \left(\prod_{j=1}^{i}\epsilon_j\right)\cdot\theta_{i+1}$, $1\le j\le m-1$ allows us to rewrite the integral in the above sum as

$$\Delta^{(1-\delta)/2}\int_{\mathbb{H}^m} f_{R_1}(\theta_1)\ldots f_{R_m}(\theta_m)\left(\prod_{j=1}^{m-1}K_{M_N}(\theta_j,\theta_{j+1})\right)K_{M_N}(\theta_m,\Delta\cdot\theta_1)d\bar{\theta},$$

where $\Delta = \prod_{j=1}^{m}\epsilon_j$. The part of the sum in (3.5.39) corresponding to $\Delta = 1$ is given by

$$\frac{1}{2}\sum_{m=1}^{n}\frac{(-1)^{m-1}}{m}\sum_{\substack{\text{ordered collections}\\ \text{of subsets } \mathcal{R}=\{R_1,..R_m\}}}\int_{\mathbb{T}^m} f_{R_1}(\theta_1)\ldots f_{R_m}(\theta_m)\prod_{j=1}^{m}K_{M_N}(\theta_j,\theta_{j+1})d\bar{\theta}$$

This is precisely $\frac{1}{2}\kappa_n\left(\Sigma f_1,\ldots,\Sigma f_n\right)$ with respect to $U(M_N)$.

We now consider the case when $\Delta = -1$. Using the assumption that $f_1, \ldots, f_n$ are even and (3.5.38), we expand the integrals inside of the summation in (3.5.11) in the following way:

$$\delta \int_{\mathbb{H}^m} f_{R_1}(\theta_1) \ldots f_{R_m}(\theta_m) \prod_{j=1}^{m} K_{M_N}(\theta_j, \epsilon_j \cdot \theta_{j+1}) d\overline{\theta}$$

$$= \frac{\delta}{2^m} \int_{\mathbb{T}^m} f_{R_1}(\theta_1) \ldots f_{R_m}(\theta_m) K_{M_N}(\theta_m, -\theta_1) \prod_{j=1}^{m-1} K_{M_N}(\theta_j, \theta_{j+1}) d\overline{\theta}$$

$$= \frac{\delta}{(4\pi)^m} \int_{\mathbb{T}^m} f_{R_1}(\theta_1) \ldots f_{R_m}(\theta_m) \left(\sum_{s_m=-\frac{M_N-1}{2}}^{\frac{M_N-1}{2}} e^{is_m(\theta_m+\theta_1)} \right) \prod_{j=1}^{m-1} \left(\sum_{s_j=-\frac{M_N-1}{2}}^{\frac{M_N-1}{2}} e^{is_j(\theta_j-\theta_{j+1})} \right) d\overline{\theta}$$

$$(3.5.41) \qquad = \frac{\delta}{2^m} \sum_{s_1=-(N-1)}^{N-1} \cdots \sum_{s_m=-(N-1)}^{N-1} \widehat{f_{R_1}}(\epsilon_m s_1 + s_m) \widehat{f_{R_2}}(s_2 - s_3) \ldots \widehat{f_{R_m}}(s_{m-1} - s_m).$$

It follows that

$$\sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} \sum_{\substack{\text{ordered collections} \\ \text{of subsets} \\ \mathcal{R}=\{R_1,..R_m\}}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_m=\pm 1 \\ \text{s.t. } \prod_j \epsilon_j = -1}} \Delta^{\frac{1-\delta}{2}} \int_{\mathbb{H}^m} f_{R_1}(\theta_1) \ldots f_{R_m}(\theta_m) \prod_{j=1}^{m} K_M(\theta_j, \epsilon_j \cdot \theta_{j+1}) d\overline{\theta}$$

$$= \frac{\delta}{2} \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} \sum_{\substack{\text{ordered collections} \\ \text{of subsets} \\ \mathcal{R}=\{R_1,..R_m\}}} \sum_{s_1,\ldots,s_m=-(N-1)}^{N-1} \widehat{f_{R_1}}(s_1 + s_m) \widehat{f_{R_1}}(s_1 - s_2) \cdots \widehat{f_{R_m}}(s_{m-1} - s_m).$$

The case for $SO(2N+1)$ requires some minor modifications. In particular, the integration kernel is given by $K^-_{M_N}(x, y) = K_{M_N}(x, y) - K_{M_N}(x, -y)$ and

$$K_{M_N}(x, y) = \frac{1}{2\pi} \sum_{\substack{p \text{ odd} \\ |p| \le 2N-1}} e^{ip(\theta_1-\theta_2)/2} = \sum_{s=-(N-1)}^{N} e^{i(s-\frac{1}{2})(\theta_1-\theta_2)}.$$

Again, the joint cumulant can be split up into a unitary and non-unitary part. The unitary parts is the same as before, but the non-unitary part becomes

$$\frac{\delta}{2} \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m} \sum_{\substack{\text{ordered collections} \\ \text{of subsets} \\ \mathcal{R}=\{R_1,..R_m\}}} \sum_{s_1,\ldots s_m=-(N-1)}^{N} \widehat{f_{R_1}}(s_1 + s_m - 1) \widehat{f_{R_1}}(s_1 - s_2) \cdots \widehat{f_{R_m}}(s_{m-1} - s_m).$$

This completes the proof of Lemma 3.5.12. □

We now proceed to the proof of Lemma 3.5.5. We focus our attention on the case $SO(N)$. The case $Sp(N)$ is obtained from that of $SO(2N)$ by making obvious modifications. The proof is done via a tedious, but straightforward, computation that is made significantly easier modulo Lemma 3.5.12.

PROOF. (*of Lemma* 3.5.5)

To see (i), we simply apply Lemma 3.5.12, breaking $\kappa_1(k)$ into two parts:

$$k_1(k) = \frac{1}{2}\kappa_1^{'}(k) + \frac{\delta}{2}\sum_{s=-(N-1)}^{N-1+\alpha} \widehat{f}(2s-\alpha), \tag{3.5.42}$$

where $f = 2\cos(k\theta)$. The first term on the r.h.s. (3.5.42) is equal to

$$\frac{1}{2}\left(\mathbb{E}\left(t_k\right) + \mathbb{E}\left(t_{-k}\right)\right) = 0,$$

which follows from the fact that $t_k = \sum_{j=1}^{M_N} e^{k\theta_j}$ and the one variable marginal density for $U(M_N)$ is Lebesgue measure. $2\cos(k\theta) = e^{ik\theta} + e^{-ik\theta}$, so the second term in (3.5.42) is equal to δ if $k+\alpha$ is even and zero otherwise. This completes the proof of (i).

To see (ii), we again use Lemma 3.5.12 to break up $\kappa_2(k_1, k_2)$ into its unitary and non-unitary parts. We first consider the unitary part. Let t_k denote the trace of the $k-th$ power of a $U(M_N)$ distributed random matrix. Then, using multi-linearity of joint cumulants, we can rewrite $\frac{1}{2}\cdot \kappa_2^{'}(\Sigma_j 2\cos(k_1\theta_j), \Sigma_j 2\cos(k_2\theta_j))$ as

$$\frac{1}{2}\left(\kappa_2(t_{k_1}, t_{k_2}) + \kappa_2(t_{-k_1}, t_{-k_2}) + \kappa_2(t_{k_1}, t_{-k_2}) + \kappa_2(t_{-k_1}, t_{k_2})\right) \tag{3.5.43}$$

By assumption, $k_1, k_2 \geq 1$, so parts (ii) and (iv) of Lemma 3.5.8 imply that the only non-zero contribution to (3.5.43) comes from

$$\frac{1}{2}\left(\kappa_2(t_{k_1}, t_{-k_2}) + \kappa_2(t_{-k_1}, t_{k_2})\right)$$

when $k_1 = k_2$, in which case it is equal to $\min(k_1, M_N)$.

We now consider the non-unitary part. In particular, we compute

$$\frac{1}{2}\sum_{m=1}^{2}\frac{(-1)^{m-1}}{m}\sum_{\substack{\text{ordered collections}\\ \text{of subsets}\\ \mathcal{R}=\{R_1,\ldots,R_m\}}}\sum_{s_1,..,s_m=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_m-\alpha)\cdots\widehat{f_{R_m}}(s_{m-1}-s_m),$$

which can be rewritten as

$$\frac{1}{2}\sum_{s_1=-(N-1)}^{N-1+\alpha}\widehat{f_1f_2}(2s_1-\alpha) \tag{3.5.44}$$

$$-\frac{1}{4}\sum_{\substack{\text{ordered collections}\\ \text{of subsets}\\ \mathcal{R}=\{R_1,R_2\}}}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_2-\alpha)\widehat{f_{R_2}}(s_1-s_2), \tag{3.5.45}$$

where $f_1 = 2\cos(k_1\theta)$ and $f_2 = 2\cos(k_2\theta)$. Starting with (3.5.44), we have

$$f_1f_2(\theta) = e^{i(k_1+k_2)\theta} + e^{-i(k_1+k_2)\theta} + e^{i(k_1-k_2)\theta} + e^{i(k_2-k_1)\theta}.$$

Clearly, the only non-zero contributions from (3.5.44) come from the cases where $|2s_1-\alpha| = k_1+k_2$ or $|2s_1-\alpha| = |k_1-k_2|$ for $-(N-1) \le s_1 \le N-1+\alpha$. (3.5.44) is then equal to

$$\chi_{\substack{k_1+k_2 \text{ even}\\ k_1+k_2\le 2N-2}} + \chi_{\substack{|k_1-k_2| \text{ even}\\ |k_1-k_2|\le 2N-2}}, \tag{3.5.46}$$

in the case of $SO(2N)$, and

$$\chi_{\substack{k_1+k_2 \text{ odd}\\ k_1+k_2\le 2N-1}} + \chi_{\substack{|k_1-k_2| \text{ odd}\\ |k_1-k_2|\le 2N-1}}, \tag{3.5.47}$$

in the case of $SO(2N+1)$.

In order to compute (3.5.45), we first observe that making the substitution $u = \alpha - s_2$ is equivalent to interchanging R_1 and R_2. Since $-(N-1) \le u \le N-1+\alpha$, making this substitution does not alter the value of the inner sum, i.e. we can conclude that the inner sum over s_1, s_2 is invariant under interchanging the role of R_1 and R_2. As a result, we can rewrite (3.5.45) as

$$-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha}\widehat{f_1}(s_1+s_2-\alpha)\widehat{f_2}(s_1-s_2). \tag{3.5.48}$$

The only terms in the above sum that give a non-zero contribution must satisfy the following system of equations:

$$s_1 + s_2 = t_1 + \alpha$$
$$s_1 - s_2 = t_2,$$

where $(t_1, t_2) \in \{k_1, -k_1\} \times \{k_2, -k_2\}$ and $-(N-1) \le s_1, s_2 \le N - 1 + \alpha$. For any given (t_1, t_2), a unique solutions exists, so long as

$$2s_1 - \alpha = t_1 + t_2$$
$$2s_2 - \alpha = t_1 - t_2$$

for some $-(N-1) \le s_1, s_2 \le N - 1 + \alpha$. Each such solution gives a contribution of one. It follows that (3.5.45) equals

$$-2\chi_{\substack{k_1+k_2 \text{ even} \\ k_1+k_2 \le 2N-2 \\ |k_1-k_2| \le 2N-2}} \tag{3.5.49}$$

and

$$-2\chi_{\substack{k_1+k_2 \text{ odd} \\ k_1+k_2 \le 2N-1 \\ |k_1-k_2| \le 2N-1}} \tag{3.5.50}$$

for $SO(2N)$ and $SO(2N+1)$, respectively. By combining (3.5.46) and (3.5.47) we can rewrite the non-unitary part of the joint cumulant as

$$\begin{aligned}
&\chi_{\substack{k_1+k_2 \text{ even} \\ k_1+k_2 \le 2N-2}} + \chi_{\substack{|k_1-k_2| \text{ even} \\ |k_1-k_2| \le 2N-2}} - 2\chi_{\substack{k_1+k_2 \text{ even} \\ k_1+k_2 \le 2N-2 \\ |k_1-k_2| \le 2N-2}} \\
&= \chi_{\substack{k_1+k_2 \text{ even} \\ k_1+k_2 \le 2N-2}} + \chi_{\substack{|k_1-k_2| \text{ even} \\ |k_1-k_2| \le 2N-2}} - 2\chi_{\substack{k_1+k_2 \text{ even} \\ k_1+k_2 \le 2N-2}} \\
&= \chi_{\substack{|k_1-k_2| \text{ even} \\ |k_1-k_2| \le 2N-2}} - \chi_{\substack{k_1+k_2 \text{ even} \\ k_1+k_2 \le 2N-2}} \\
&= \chi_{\substack{|k_1-k_2| \text{ even} \\ k_1+k_2 \text{ even} \\ |k_1-k_2| \le 2N-2 \\ k_1+k_2 \le 2N-2}} + \chi_{\substack{|k_1-k_2| \text{ even} \\ k_1+k_2 \text{ even} \\ |k_1-k_2| \le 2N-2 \\ 2N \le k_1+k_2}} - \chi_{\substack{|k_1-k_2| \text{ even} \\ k_1+k_2 \text{ even} \\ |k_1-k_2| \le 2N-2 \\ k_1+k_2 \le 2N-2}}
\end{aligned}$$

$$= \chi_{\substack{|k_1-k_2| \text{ even} \\ |k_1-k_2|\le 2N-2 \\ 2N\le k_1+k_2}}$$

and

$$\chi_{\substack{|k_1-k_2| \text{ odd} \\ |k_1-k_2|\le 2N-1 \\ 2N+1\le k_1+k_2}}$$

for $SO(2N)$ and $SO(2N+1)$, respectively. This completes the proof of (ii).

To prove (iii), we again use Lemma 3.5.12 to break up $\kappa_3(k_1, k_1, k_2)$ into its unitary and non-unitary parts. Let t_k denote the trace of the $k-th$ power of a $U(M_N)$ distributed random matrix. Then, using multi-linearity of joint cumulants, we can rewrite $\frac{1}{2}\cdot\kappa_3^{'}(\Sigma_j 2\cos(k_1\theta_j), \Sigma_j 2\cos(k_1\theta_j), \Sigma_j 2\cos(k_2\theta_j))$ as

$$\frac{1}{2}\kappa^{'}((t_{k_1}+t_{-k_1}),(t_{k_1}+t_{-k_1}),(t_{k_2}+t_{-k_2})) = \frac{1}{2}\sum_{\epsilon_1=\pm1}\cdots\sum_{\epsilon_3=\pm1}\kappa^{'}(t_{\epsilon_1\cdot k_1}, t_{\epsilon_2\cdot k_1}, t_{\epsilon_3\cdot k_2}) \tag{3.5.51}$$

By assumption, $k_1, k_2 \geq 1$, so Lemma 3.5.8(ii) implies that the only non-zero contribution to (3.5.51) comes from

$$\frac{1}{2}(\kappa^{'}(t_{k_1}, t_{k_1}, t_{-k_2}) + \kappa^{'}(t_{-k_1}, t_{-k_1}, t_{k_2})) \tag{3.5.52}$$

when $2k_1 = k_2$.

We now consider the non-unitary part of the 3rd order joint cumulants from Lemma 3.5.12. In particular, we compute

$$\frac{1}{2}\sum_{m=1}^{3}\frac{(-1)^{m-1}}{m}\sum_{\substack{\text{ordered collections} \\ \text{of subsets } \mathcal{R}=\{R_1,\ldots,R_m\}}}\sum_{s_1=-(N-1)}^{N-1}\cdots\sum_{s_m=-(N-1)}^{N-1}\widehat{f_{R_1}}(s_1+s_m-\alpha)\ldots\widehat{f_{R_m}}(s_{m-1}-s_m)$$

$$= \frac{1}{2}\sum_{s_1=-(N-1)}^{N-1+\alpha}\widehat{f_1 f_2 f_3}(2s_1-\alpha) \tag{3.5.53}$$

$$-\frac{1}{4}\sum_{\substack{\text{ordered collections} \\ \text{of subsets } \mathcal{R}=\{R_1,R_2\}}}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_2-\alpha)\widehat{f_{R_2}}(s_1-s_2) \quad +\cdots \tag{3.5.54}$$

$$+\frac{1}{6}\sum_{\substack{\text{ordered collections}\\ \text{of subsets } \mathcal{R}=\{R_1,R_2,R_3\}}}\sum_{s_1,\ldots,s_3=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_3-\alpha)\widehat{f_{R_2}}(s_1-s_2)\widehat{f_{R_3}}(s_2-s_3), \tag{3.5.55}$$

where $f_1(\theta)=f_2(\theta)=2\cos(k_1\theta)$, $f_3(\theta)=2\cos(k_2\theta)$.

Now

$$\begin{aligned} f_1f_2f_3 &= 8\cos^2(k_1\theta)\cos(k_2\theta)\\ &= 2e^{ik_2x}+2e^{-ik_2x}+e^{i(2k_1+k_2)x}+e^{-i(2k_1+k_2)x}+e^{i(2k_1-k_2)x}+e^{-i(2k_1-k_2)x}, \end{aligned}$$

so the non-zero contributions to (3.5.53) come from the indices where $2s_1-\alpha=k_2$, $2s_1-\alpha=-k_2$, $2s_1-\alpha=2k_1+k_2$, $2s_1-\alpha=-(2k_1+k_2)$, $2s_1-\alpha=2k_1-k_2$, and $2s_1-\alpha=-(2k_1-k_2)$, where $-(N-1)\le s_1\le N-1+\alpha$. As such, (3.5.53) can be expressed as

$$\begin{aligned} &\frac{1}{2}\left(4\chi_{\substack{k_2\text{ even}\\ k_2\le 2N-2}}+2\chi_{\substack{k_2\text{ even}\\ 2k_1+k_2\le 2N-2}}+2\chi_{\substack{k_2\text{ even}\\ |2k_1-k_2|\le 2N-2}}\right)\\ &=2\chi_{\substack{k_2\text{ even}\\ k_2\le 2N-2}}+\chi_{\substack{k_2\text{ even}\\ 2k_1+k_2\le 2N-2}}+\chi_{\substack{k_2\text{ even}\\ |2k_1-k_2|\le 2N-2}}, \end{aligned} \tag{3.5.56}$$

in the case of $SO(2N)$, and

$$2\chi_{\substack{k_2\text{ odd}\\ k_2\le 2N-1}}+\chi_{\substack{k_2\text{ odd}\\ 2k_1+k_2\le 2N-1}}+\chi_{\substack{k_2\text{ odd}\\ |2k_1-k_2|\le 2N-1}}, \tag{3.5.57}$$

in the case of $SO(2N+1)$.

To compute (3.5.54), we again take advantage of the fact that interchanging the order of R_1 and R_2 does not alter the value of the sum over s_1 and s_2.

$$-\frac{1}{4}\sum_{\substack{\text{Partitions } \mathcal{R}=\{R_1,R_2\}\\ \text{of } \{1,2,3\}}}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_2-\alpha)\widehat{f_{R_2}}(s_1-s_2) \tag{3.5.58}$$

$$=-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1=\alpha}\widehat{f_1f_2}(s_1+s_2-\alpha)\widehat{f_3}(s_1-s_2)\quad -\cdots \tag{3.5.59}$$

(3.5.60) $$\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1=\alpha}\widehat{f_1f_3}(s_1+s_2-\alpha)\widehat{f_2}(s_1-s_2)-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha}\widehat{f_2f_3}(s_1+s_2-\alpha)\widehat{f_1}(s_1-s_2)$$

Starting with (3.5.59), we have

$$\begin{aligned} f_1f_2 &= 4\cos^2(k_1\theta) \\ &= 2+e^{i2k_1\theta}+e^{-i2k_1\theta}. \end{aligned}$$

The only terms in the sum over s_1, s_2 that give a non-zero contribution must satisfy the following system of equations:

$$\begin{aligned} s_1+s_2 &= t_1+\alpha \\ s_1-s_2 &= t_2, \end{aligned}$$

where $(t_1,t_2)\in\{0,2k_1,-2k_1\}\times\{k_2,-k_2\}$. Each of the solutions to

$$\begin{aligned} s_1+s_2 &= \alpha & \qquad s_1+s_2 &= \alpha \\ s_1-s_2 &= k_2 & \qquad s_1-s_2 &= -k_2 \end{aligned}$$

gives a contribution of two, whereas each of the solutions to

$$\begin{aligned} s_1+s_2 &= 2k_1+\alpha & \quad s_1+s_2 &= 2k_1+\alpha & \quad s_1+s_2 &= -2k_1+\alpha & \quad s_1+s_2 &= -2k_1+\alpha \\ s_1-s_2 &= k_2 & \quad s_1-s_2 &= -k_2+ & \quad s_1-s_2 &= k_2+ & \quad s_1-s_2 &= -k_2 \end{aligned}$$

gives a contribution of one. Now, given (t_1,t_2), the corresponding solution must satisfy the relations

$$\begin{aligned} 2s_1-\alpha &= t_1+t_2 \\ 2s_2-\alpha &= t_1-t_2. \end{aligned}$$

Since $-(N-1)\le s_1,s_2\le(N-1+\alpha)$, it follows that (3.5.59) is equal to

$$-\frac{1}{2}\left(4\chi_{\substack{k_2\text{ even}\\ k_2\le 2N-2}}+4\chi_{\substack{k_2\text{ even}\\ 2k_1+k_2\le 2N-2\\ |2k_1-k_2|\le 2N-2}}\right)$$

(3.5.61) $$= -2\chi_{\substack{k_2 \text{ even} \\ k_2 \leq 2N-2}} - 2\chi_{\substack{k_2 \text{ even} \\ 2k_1+k2 \leq 2N-2 \\ |2k_1-k_2| \leq 2N-2}},$$

in the case of $SO(2N)$ and

(3.5.62) $$-2\chi_{\substack{k_2 \text{ odd} \\ k_2 \leq 2N-1}} - 2\chi_{\substack{k_2 \text{ odd} \\ 2k_1+k2 \leq 2N-1 \\ |2k_1-k_2| \leq 2N-1}}$$

in the case of $SO(2N+1)$.

Both sums in (3.5.60) give the same contribution since $f_1 = f_2$. Similar to before, computing

$$\sum_{s_1,s_1=-(N-1)}^{N-1+\alpha} \widehat{f_1 f_3}(s_1+s_2-\alpha)\widehat{f_2}(s_1-s_2)$$

amounts to solving system of linear equations of the form

$$s_1 + s_2 = t_1 + \alpha$$

$$s_1 - s_2 = t_2,$$

where $(t_1,t_2) \in \{k_1+k_2, -(k_1+k_2), k_1-k_2, k_2-k_1\} \times \{k_1, -k_1\}$. The total contribution from (3.5.60) is

(3.5.63) $$-4\chi_{\substack{k_2 \text{ even} \\ k_2 \leq 2N-2 \\ 2k_1+k2 \leq 2N-2}} - 4\chi_{\substack{k_2 \text{ even} \\ k_2 \leq 2N-2 \\ |2k_1-k_2| \leq 2N-2}}$$

and

(3.5.64) $$-4\chi_{\substack{k_2 \text{ odd} \\ k_2 \leq 2N-1 \\ 2k_1+k2 \leq 2N-1}} - 4\chi_{\substack{k_2 \text{ odd} \\ k_2 \leq 2N-1 \\ |2k_1-k_2| \leq 2N-1}},$$

for $SO(2N)$ and $SO(2N+1)$, respectively.

Finally, we consider (3.5.55). Using the same substitution argument as before, it is easy to see that every ordering of R_1, R_2, R_3 gives the same contribution, so

$$\frac{1}{6} \sum_{\substack{\text{ordered collections} \\ \text{of subsets } \mathcal{R}=\{R_1,R_2,R_3\}}} \sum_{s_1,\ldots,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_{R_1}}(s_1+s_3-\alpha)\widehat{f_{R_2}}(s_1-s_2)\widehat{f_{R_3}}(s_2-s_3)$$

$$= \sum_{s_1,\ldots,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_1}(s_1+s_3-\alpha)\widehat{f_2}(s_1-s_2)\widehat{f_3}(s_2-s_3). \tag{3.5.65}$$

As before, computing (3.5.65) is a matter of solving the collection of systems of linear equations of the form

$$s_1 + s_3 = t_1 + \alpha$$

$$s_1 - s_2 = t_2$$

$$s_2 - s_3 = t_3,$$

where $(t_1, t_2, t_3) \in \{k_1, -k_1\} \times \{k_1, -k_1\} \times \{k_2, -k_2\}$ and $-(N-1) \le s_1, s_2, s_3 \le N-1+\alpha$. Given (t_1, t_2, t_3), the corresponding solution must satisfy

$$2s_1 - \alpha = t_1 + t_2 + t_3$$

$$2s_2 - \alpha = t_1 - t_2 + t_3$$

$$2s_3 - \alpha = t_1 - t_2 - t_3.$$

It follows that (3.5.65) is equal to

$$2\chi_{\substack{k_2 \text{ even} \\ k_2 \le 2N-2 \\ 2k_1+k_2 \le 2N-2}} + 2\chi_{\substack{k_2 \text{ even} \\ k_2 \le 2N-2 \\ |2k_1-k_2| \le 2N-2}} + 4\chi_{\substack{k_2 \text{ even} \\ k_2 \le 2N-2 \\ 2k_1+k_2 \le 2N-2 \\ |2k_1-k_2| \le 2N-2}} \tag{3.5.66}$$

and

$$2\chi_{\substack{k_2 \text{ odd} \\ k_2 \le 2N-1 \\ 2k_1+k_2 \le 2N-1}} + 2\chi_{\substack{k_2 \text{ odd} \\ k_2 \le 2N-1 \\ |2k_1-k_2| \le 2N-1}} + 4\chi_{\substack{k_2 \text{ odd} \\ k_2 \le 2N-1 \\ 2k_1+k_2 \le 2N-1 \\ |2k_1-k_2| \le 2N-1}} \tag{3.5.67}$$

for $SO(2N)$ and $SO(2N+1)$, respectively.

Combining (3.5.56), (3.5.61), (3.5.63), and (3.5.66), we can see that

$$\frac{1}{2}\sum_{m=1}^{3} \frac{(-1)^{m-1}}{m} \sum_{\substack{\text{ordered collections} \\ \text{of subsets } \mathcal{R}=\{R_1,..R_m\}}} \sum_{s_1,\ldots,s_m=-(N-1)}^{N-1+\alpha} \widehat{f_{R_1}}(s_1+s_m-\alpha)\ldots\widehat{f_{R_m}}(s_{m-1}-s_m) \tag{3.5.68}$$

is equal to

$$
\begin{aligned}
&2\chi_{\substack{k_2 \text{ even} \\ 2k_1+k_2\le 2N-2 \\ |2k_1-k_2|\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ 2k_1+k_2\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ |2k_1-k_2|\le 2N-2}} \\
&= 2\chi_{\substack{k_2 \text{ even} \\ 2k_1+k_2\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ 2k_1+k_2\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ |2k_1-k_2|\le 2N-2}} \\
&= \chi_{\substack{k_2 \text{ even} \\ 2k_1+k_2\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ |2k_1-k_2|\le 2N-2}} \\
&= \chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ 2k_1+k_2\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ 2k_1+k_2\le 2N-2}} - \chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ |2k_1-k_2|\le 2N-2 \\ 2k_1+k_2\ge 2N}} \\
&= -\chi_{\substack{k_2 \text{ even} \\ k_2\le 2N-2 \\ 2k_1+k_2\ge 2N \\ |2k_1-k_2|\le 2N-2}}
\end{aligned}
$$

in the case of $SO(N)$. Combining (3.5.57), (3.5.62), (3.5.64), and (3.5.67), it must also be the case that (3.5.68) is equal to

$$
-\chi_{\substack{k_2 \text{ odd} \\ k_2\le 2N-1 \\ 2k_1+k_2\ge 2N+1 \\ |2k_1-k_2|\le 2N-1}}
$$

in the case of $SO(2N+1)$.

We note that this term gives a vanishing contribution when $2k_1 + k_2 \le 2N - 1$, which is consistent with the results on third order cumulants for linear statistics in [25] when $k_1 = k_2$. This completes the proof of (iii).

We now proceed to the proof of (iv). The proof is entirely similar to that of (iii), with the addition that we are able to give an explicit formula for the 'unitary part' of the joint cumulant, which follows immediately from the variance computation in Section 2.4.

Using Lemma 3.5.12, we break up $\kappa_4(k_1, k_1, k_2, k_2)$ into its unitary and non-unitary parts. Let t_k denote the trace of the $k-th$ power of a $U(2N-1)$ distributed random matrix. We first consider the unitary part of the joint cumulant. Using multi-linearity of joint cumulants, we can rewrite $\kappa_4'(\Sigma_j 2\cos(k_1\theta_j), \Sigma_j 2\cos(k_1\theta_j), \Sigma_j 2\cos(k_2\theta_j), \Sigma_j 2\cos(k_2\theta_j))$ as

(3.5.69)

$$\frac{1}{2}\kappa_4^{'}(t_{k_1}+t_{-k_1},t_{k_1}+t_{-k_1},t_{k_2}+t_{-k_2},t_{k_2}+t_{-k_2}) = \frac{1}{2}\sum_{\epsilon_1,\dots\epsilon_4=\pm 1}\kappa(\epsilon_1\cdot t_{k_1},\epsilon_2\cdot t_{k_1},\epsilon_3\cdot t_{k_2},\epsilon_4\cdot t_{k_2})$$

Again, assuming that $1\le k_1,k_2$, Lemma 3.5.8 (ii) implies that the only non-zero terms in the above sum are

$$\frac{1}{2}(\kappa(t_{k_1},t_{k_1},t_{-k_2},t_{-k_2})+\kappa(t_{-k_1},t_{-k_1},t_{k_2},t_{k_2})) \tag{3.5.70}$$

and

(3.5.71)

$$\frac{1}{2}(\kappa(t_{k_1},t_{-k_1},t_{k_2},t_{-k_2})+\kappa(t_{k_1},t_{-k_1},t_{-k_2},t_{k_2})+\kappa(t_{-k_1},t_{k_1},t_{-k_2},t_{k_2})+\kappa(t_{-k_1},t_{k_1},t_{k_2},t_{-k_2})).$$

The terms in (3.5.70) are equal to zero if when $k_1\neq k_2$. When this is not the case, both terms are equal and so (3.5.70) simplifies to

$$\chi_{k_1=k_2}\cdot\kappa(t_{k_1},t_{k_1},t_{-k_2},t_{-k_2}). \tag{3.5.72}$$

Since joint cumulants are invariant under permutation (see 3.5.2), the four terms in (3.5.71) are equal, so (3.5.71) can be rewritten as

$$2\cdot\kappa(t_{k_1},t_{-k_1},t_{k_2},t_{-k_2}). \tag{3.5.73}$$

Together, this means that (3.5.69) can be rewritten as

$$\chi_{k_1=k_2}\cdot\kappa(t_{k_1},t_{k_1},t_{-k_2},t_{-k_2})+2\cdot\kappa(t_{k_1},t_{-k_1},t_{k_2},t_{-k_2}). \tag{3.5.74}$$

By part (v) of Lemma 3.5.8,

$$\kappa^{'}(t_{k_1},t_{-k_1},t_{k_2},t_{-k_2}) = -\left((M_N-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le M_N-1\\ M_N\le\max(k_1,k_2)}}+(k_1+k_2-M_N)\chi_{\substack{k_1,k_2\le M_N-1\\ M_N+1\le k_1+k_2}}\right).$$

It follows that (3.5.74) can be rewritten as

(3.5.75)
$$-\chi_{k_1=k_2}\cdot\Big(M_N\cdot\chi_{M_N\le k_1})+(2k_1-M_N)\chi_{\substack{k_1\le M_N-1\\ M_N+1\le 2k_1}}\Big)$$
$$-2\Bigg((M_N-|k_1-k_2|)\chi_{\substack{|k_1-k_2|\le M_N-1\\ M_N\le\max(k_1,k_2)}})+(k_1+k_2-M_N)\chi_{\substack{k_1,k_2\le M_N-1\\ M_N+1\le k_1+k_2}}\Bigg)$$

This completes our analysis of the the unitary part of $\kappa_4(k_1,k_1,k_2,k_2)$.

We now consider the 'non-unitary' part of the 4-th order joint cumulant. In particular, we study

(3.5.76)
$$\frac{1}{2}\sum_{m=1}^{4}\frac{(-1)^{m-1}}{m}\sum_{\substack{\text{ordered collections}\\ \text{of subsets}\\ \mathcal{R}=\{R_1,\ldots,R_m\}}}\sum_{s_1,\ldots,s_m=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_m-\alpha)\widehat{f_{R_2}}(s_1-s_2)\ldots\widehat{f_{R_m}}(s_{m-1}-s_m)$$

(3.5.77)
$$=\frac{1}{2}\sum_{s_1=-(N-1)}^{N-1+\alpha}\widehat{f_1f_2f_3f_4}(2s_1-\alpha)$$

(3.5.78)
$$-\frac{1}{4}\sum_{\substack{\text{ordered collections}\\ \text{of subsets}\\ \mathcal{R}=\{R_1,R_2\}}}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_2-\alpha)\widehat{f_{R_2}}(s_1-s_2)$$

(3.5.79)
$$+\frac{1}{6}\sum_{\substack{\text{ordered collections}\\ \text{of subsets}\\ \mathcal{R}=\{R_1,R_2,R_3\}}}\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_3-\alpha)\widehat{f_{R_2}}(s_1-s_2)\widehat{f_{R_3}}(s_2-s_3)$$

(3.5.80)
$$-\frac{1}{8}\sum_{\substack{\text{ordered collections}\\ \text{of subsets}\\ \mathcal{R}=\{R_1,R_2,R_3,R_4\}}}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha}\widehat{f_{R_1}}(s_1+s_4-\alpha)\widehat{f_{R_2}}(s_1-s_2)\widehat{f_{R_3}}(s_2-s_3)\widehat{f_{R_4}}(s_3-s_4),$$

where $f_1(\theta)=f_2(\theta)=2\cos(k_1\theta)$, $f_3(\theta)=f_4(\theta)=2\cos(k_2\theta)$. Starting with (3.5.77), we follow the same procedure as before. Writing $f_1f_2f_3f_4$ as a trigonometric polynomial, we have

$$
\begin{aligned}
f_1f_2f_3f_4 &= 16\cos^2(k_1\theta)\cos^2(k_2\theta) \\
&= 4 + 2e^{i2k_1\theta} + 2e^{-i2k_1\theta} + 2e^{i2k_2\theta} + 2e^{-i2k_2\theta} \\
&\quad + e^{i(2k_1+2k_2)\theta} + e^{-i(2k_1+2k_2)\theta} + e^{i(2k_1-2k_2)\theta} + e^{-i(2k_1-2k_2)\theta}.
\end{aligned}
$$

The only non-zero terms in (3.5.77) occur when $2s_1 - \alpha = \pm 2k_1, \pm 2k_2, \pm 2(k_1+k_2), \pm 2(k_1-k_2)$, for $-(N-1) \leq s_1 \leq N-1+\alpha$. (3.5.77) is then equal to

$$
\begin{aligned}
&\frac{1}{2}\left(4 + 4\chi_{k_1 \leq N-1} + 4\chi_{k_2 \leq N-1} + 2\chi_{k_1+k_2 \leq N-1} + 2\chi_{|k_1-k_2| \leq N-1}\right) \\
&= 2 + 2\chi_{k_1 \leq N-1} + 2\chi_{k_2 \leq N-1} + \chi_{k_1+k_2 \leq N-1} + \chi_{|k_1-k_2| \leq N-1}
\end{aligned} \tag{3.5.81}
$$

in the case of $SO(2N)$ ($\alpha = 0$) and zero in the case of $SO(2N+1)$ ($\alpha = 1$).

Using the same substitution argument as in the proof of (ii), we can see that each ordering of $R_1, \ldots, R_m$ in (3.5.78) and (3.5.79) gives the same contribution, so we can replace the sum over ordered subsets with a sum over partitions with m blocks. As a result, (3.5.78) becomes

$$
\begin{aligned}
&-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_2f_3}(s_1+s_2+\alpha)\widehat{f_4}(s_1-s_2) - \frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_2f_4}(s_1+s_2+\alpha)\widehat{f_3}(s_1-s_2) \\
&-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_3f_4}(s_1+s_2+\alpha)\widehat{f_2}(s_1-s_2) - \frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_2f_3f_4}(s_1+s_2+\alpha)\widehat{f_1}(s_1-s_2) \\
&-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_3}(s_1+s_2+\alpha)\widehat{f_2f_4}(s_1-s_2) - \frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_4}(s_1+s_2+\alpha)\widehat{f_2f_3}(s_1-s_2) \\
&-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_2}(s_1+s_2+\alpha)\widehat{f_3f_4}(s_1-s_2),
\end{aligned}
$$

which simplifies to

$$
-\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_2f_3}(s_1+s_2-\alpha)\widehat{f_4}(s_1-s_2) \quad \cdots \tag{3.5.82}
$$

$$-\sum_{s_1,s_2=-(N-1)}^{+\alpha} \widehat{f_1f_3f_4}(s_1+s_2-\alpha)\widehat{f_2}(s_1-s_2)) \tag{3.5.83}$$

$$-\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_3}(s_1+s_2-\alpha)\widehat{f_2f_4}(s_1-s_2) \tag{3.5.84}$$

$$-\frac{1}{2}\sum_{s_1,s_2=-(N-1)}^{N-1+\alpha} \widehat{f_1f_2}(s_1+s_2-\alpha)\widehat{f_3f_4}(s_1-s_2), \tag{3.5.85}$$

under the assumption that $f_1 = f_2, f_3 = f_4$. Again, computing the above sums is equivalent to computing solutions to a collection of systems of equations of the form

$$s_1 + s_2 = t_1 + \alpha$$
$$s_1 - s_2 = t_2,$$

where $(t_1, t_2) \in A$. For some finite indexing set A. In the case of (3.5.82),

$$\begin{aligned} f_1f_2f_3 &= 8\cos^2(k_1\theta)\cos(k_2\theta) \\ &= 2e^{ik_2x} + 2e^{-ik_2x} + e^{i(2k_1+k_2)x} + e^{-i(2k_1+k_2)x} + e^{i(2k_1-k_2)x} + e^{-i(2k_1-k_2)x}, \end{aligned}$$

so $A = \{k_2, -k_2, 2k_1+k_2, -(2k_1+k_2), 2k_1-k_2, k_2-2k_1\} \times \{k_2, -k_2\}$. The corresponding contribution is

$$-8\chi_{k_2\leq N-1} - 4\chi_{\substack{k_1\leq N-1 \\ k_1+k_2\leq N-1}} - 4\chi_{\substack{k_1\leq N-1 \\ |k_1-k_2|\leq N-1}} \tag{3.5.86}$$

for $SO(2N)$ and zero for $SO(2N+1)$. The formula for (3.5.83) is obtained by interchanging the role of k_1 and k_2 in the computation of (3.5.82) and is thus equal to

$$-8\chi_{k_1\leq N-1} - 4\chi_{\substack{k_2\leq N-1 \\ k_1+k_2\leq N-1}} - 4\chi_{\substack{k_2\leq N-1 \\ |k_1-k_2|\leq N-1}} \tag{3.5.87}$$

for $SO(2N)$ and zero for $SO(2N+1)$

For (3.5.84), $A = \{k_1 + k_2, -(k_1 + k_2), k_1 - k_2, k_2 - k_1\}^2$. The corresponding contribution is

$$-4\chi_{k_1+k_2 \le N-1} - 4\chi_{|k_1-k_2| \le N-1} - 8\chi_{k_1,k_2 \le N-1} \tag{3.5.88}$$

for $SO(2N)$ and zero for $SO(2N+1)$.

For (3.5.85), $A = \{0, 2k_1, -2k1\} \times \{0, 2k_2, -2k_2\}$. The corresponding contribution is

$$\begin{aligned} &-\frac{1}{2}\left(4 + 4\chi_{k_1 \le N-1} + 4\chi_{k_2 \le N-1} + 4\chi_{\substack{k_1+k_2 \le N-1 \\ |k_1-k_2| \le N-1}}\right) \\ &= -2 - 2\chi_{k_1 \le N-1} - 2\chi_{k_2 \le N-1} - 2\chi_{\substack{k_1+k_2 \le N-1 \\ |k_1-k_2| \le N-1}}. \end{aligned} \tag{3.5.89}$$

for $SO(2N)$ and, again, zero for $SO(2N+1)$.

Combining (3.5.86), (3.5.87), (3.5.88), and (3.5.88), we see that (3.5.78) is equal to

$$\begin{aligned} &-2 - 10\chi_{k_1 \le N-1} - 10\chi_{k_2 \le N-1} - 8\chi_{k_1,k_2 \le N-1} \\ &\quad - 4\chi_{\substack{k_1 \le N-1 \\ k_1+k_2 \le N-1}} - 4\chi_{\substack{k_1 \le N-1 \\ |k_1-k_2| \le N-1}} - 4\chi_{\substack{k_2 \le N-1 \\ k_1+k_2 \le N-1}} - 4\chi_{\substack{k_2 \le N-1 \\ |k_1-k_2| \le N-1}} \\ &\quad - 4\chi_{k_1+k_2 \le N-1} - 4\chi_{|k_1-k_2| \le N-1} - 2\chi_{\substack{k_1+k_2 \le N-1 \\ |k_1-k_2| \le N-1}} \end{aligned} \tag{3.5.90}$$

for $SO(2N)$ and zero for $SO(2N+1)$.

We now consider (3.5.79). This sum is equivalent to

$$\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_1 f_3}(s_1 + s_3 - \alpha)\widehat{f_2}(s_1 - s_2)\widehat{f_4}(s_2 - s_3)$$

$$+ \sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_2 f_3}(s_1 + s_3 - \alpha)\widehat{f_1}(s_1 - s_2)\widehat{f_4}(s_2 - s_3)$$

$$+ \sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_1 f_4}(s_1 + s_3 - \alpha)\widehat{f_2}(s_1 - s_2)\widehat{f_3}(s_2 - s_3) \quad + \cdots$$

$$+\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_2 f_4}(s_1+s_3-\alpha)\widehat{f_1}(s_1-s_2)\widehat{f_3}(s_2-s_3)$$

$$+\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_3 f_4}(s_1+s_3-\alpha)\widehat{f_1}(s_1-s_2)\widehat{f_2}(s_2-s_3)$$

$$+\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_1 f_2}(s_1+s_3-\alpha)\widehat{f_3}(s_1-s_2)\widehat{f_4}(s_2-s_3),$$

which simplifies down to

$$+4\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_1 f_3}(s_1+s_3-\alpha)\widehat{f_2}(s_1-s_2)\widehat{f_4}(s_2-s_3) \tag{3.5.91}$$

$$+\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_1 f_2}(s_1+s_3-\alpha)\widehat{f_3}(s_1-s_2)\widehat{f_4}(s_2-s_3) \tag{3.5.92}$$

$$+\sum_{s_1,s_2,s_3=-(N-1)}^{N-1+\alpha} \widehat{f_3 f_4}(s_1+s_3-\alpha)\widehat{f_1}(s_1-s_2)\widehat{f_2}(s_2-s_3), \tag{3.5.93}$$

under the assumption that $f_1 = f_2, f_3 = f_4$. Again, we must consider the solutions to a collection of systems of linear equations of the form

$$s_1 + s_3 = t_1 + \alpha$$

$$s_1 - s_2 = t_2$$

$$s_2 - s_3 = t_3,$$

where $(t_1, t_2, t_3) \in A$ for some finite indexing set A. In the case of (3.5.91),

$$\begin{aligned} f_1 f_3 &= 4\cos(k_1\theta)\cos(k_2\theta) \\ &= e^{i(k_1+k_2)\theta} + e^{-i(k_1+k_2)\theta} + e^{i(k_1-k_2)\theta} + e^{i(k_2-k_1)\theta}, \end{aligned}$$

so $A=\{k_1+k_2,-(k_1+k_2),k_1-k_2,k_2-k_1\}\times\{k_1,-k_1\}\times\{k_2,-k2\}$. For $SO(2N)$, the corresponding contribution is

$$\begin{aligned}
&4\Big(4\chi_{k_1,k_2\le N-1}+2\chi_{\substack{k_1\le N-1\\ k_1+k_1\le N-1}}+2\chi_{\substack{k_2\le N-1\\ k_1+k_1\le N-1}}+2\chi_{\substack{k_1\le N-1\\ |k_1-k_2|\le N-1}}+2\chi_{\substack{k_2\le N-1\\ |k_1-k_2|\le N-1}}\\
&\quad+2\chi_{\substack{k_1,k_2\le N-1\\ k_1+k_1\le N-1}}+2\chi_{\substack{k_1,k_2\le N-1\\ |k_1-k_2|\le N-1}}\Big)\\
&=16\chi_{k_1,k_2\le N-1}+8\chi_{\substack{k_1\le N-1\\ k_1+k_1\le N-1}}+8\chi_{\substack{k_2\le N-1\\ k_1+k_1\le N-1}}+8\chi_{\substack{k_1\le N-1\\ |k_1-k_2|\le N-1}}+8\chi_{\substack{k_2\le N-1\\ |k_1-k_2|\le N-1}}\\
&\quad+8\chi_{\substack{k_1,k_2\le N-1\\ k_1+k_1\le N-1}}+8\chi_{\substack{k_1,k_2\le N-1\\ |k_1-k_2|\le N-1}}.
\end{aligned}\tag{3.5.94}$$

For $SO(2N+1)$ we observe that, given $(t_1,t_2,t_3)\in A$, there are no integer solutions satisfying $2s_1-1=t_1+t_2+t_3$. This implies that (3.5.91) is equal to zero in the case of $SO(2N+1)$.

For (3.5.92),

$$\begin{aligned}
f_1f_2&=4\cos^2(k_1\theta)\\
&=2+e^{i2k_1}+e^{-i2k_1},
\end{aligned}$$

so $A=\{0,2k_1,-2k_1\}\times\{k_2,-k_2\}\times\{k_2,-k_2\}$. The corresponding contribution is

$$8\chi_{k_2\le N-1}+4\chi_{\substack{k_1\le N-1\\ k_1+k_2\le N-1\\ |k_1-k_2|\le N-1}}+2\chi_{\substack{k_1\le N-1\\ k_1+k_2\le N-1}}+2\chi_{\substack{k_1\le N-1\\ |k_1-k_2|\le N-1}}\tag{3.5.95}$$

for $SO(2N)$ and, by the same reasoning as before, zero for $SO(2N+1)$.

The formula for (3.5.93) is obtained by interchanging k_1 and k_2 in (3.5.95) and is thus equal to

$$8\chi_{k_1\le N-1}+4\chi_{\substack{k_2\le N-1\\ k_1+k_2\le N-1\\ |k_1-k_2|\le N-1}}+2\chi_{\substack{k_2\le N-1\\ k_1+k_2\le N-1}}+2\chi_{\substack{k_2\le N-1\\ |k_1-k_2|\le N-1}}\tag{3.5.96}$$

for $SO(2N)$ and zero for $SO(2N+1)$.

Combining (3.5.94), (3.5.95), and (3.5.96), we have the following expression for (3.5.79) in the case of $SO(2N)$:

$$16\chi_{k_1,k_2\le N-1}+8\chi_{k_1\le N-1}+8\chi_{k_2\le N-1}\quad+\cdots\tag{3.5.97}$$

$$+10\chi_{\substack{k_1\le N-1\\ k_1+k_1\le N-1}} + 10\chi_{\substack{k_2\le N-1\\ k_1+k_1\le N-1}} + 10\chi_{\substack{k_1\le N-1\\ |k_1-k_2|\le N-1}} + 10\chi_{\substack{k_2\le N-1\\ |k_1-k_2|\le N-1}}$$

$$+8\chi_{\substack{k_1,k_2\le N-1\\ k_1+k_1\le N-1}} + 8\chi_{\substack{k_1,k_2\le N-1\\ |k_1-k_2|\le N-1}} + 4\chi_{\substack{k_1\le N-1\\ k_1+k_2\le N-1\\ |k_1-k_2|\le N-1}} + 4\chi_{\substack{k_2\le N-1\\ k_1+k_2\le N-1\\ |k_1-k_2|\le N-1}}.$$

Again, there is no contribution in the case of $SO(2N+1)$.

Finally we consider (3.5.80). Unlike before, the inner sum over s_1, s_2, s_3, s_4 is not invariant under rearranging R_1, R_2, R_3, R_4. Despite this, we can still take advantage of symmetry that comes from the assumption that $f_1 = f_2, f_3 = f_4$ to drastically reduce the total number of necessary computations. In particular, the sum is invariant under swapping f_1 and f_2 or f_3 and f_4. This reduces the 24 possible terms that we need to consider down to just 6.

$$-\frac{1}{8}\sum_{\substack{\text{ordered collections}\\ \text{of subsets } \mathcal{R}=\{R_1,\ldots,R_4\}}}\sum_{s_1,\ldots,s_4=-(N-1+\alpha)}^{N-1}\widehat{f_{R_1}}(s_1+s_4-\alpha)\widehat{f_{R_2}}(s_1-s_2)\widehat{f_{R_3}}(s_2-s_3)\widehat{f_{R_4}}(s_3-s_4)$$

$=$

$$-\frac{1}{2}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha}\widehat{f_1}(s_1+s_4-\alpha)\widehat{f_2}(s_1-s_2)\widehat{f_3}(s_2-s_3)\widehat{f_4}(s_3-s_4) \tag{3.5.98}$$

$$-\frac{1}{2}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha}\widehat{f_3}(s_1+s_4-\alpha)\widehat{f_4}(s_1-s_2)\widehat{f_1}(s_2-s_3)\widehat{f_2}(s_3-s_4) \tag{3.5.99}$$

$$-\frac{1}{2}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha}\widehat{f_1}(s_1+s_4-\alpha)\widehat{f_3}(s_1-s_2)\widehat{f_2}(s_2-s_3)\widehat{f_4}(s_3-s_4) \tag{3.5.100}$$

$$-\frac{1}{2}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha}\widehat{f_3}(s_1+s_4-\alpha)\widehat{f_1}(s_1-s_2)\widehat{f_4}(s_2-s_3)\widehat{f_2}(s_3-s_4) \tag{3.5.101}$$

$$-\frac{1}{2}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha}\widehat{f_3}(s_1+s_4-\alpha)\widehat{f_1}(s_1-s_2)\widehat{f_2}(s_2-s_3)\widehat{f_4}(s_3-s_4) \quad \cdots \tag{3.5.102}$$

$$(3.5.103) \qquad -\frac{1}{2}\sum_{s_1,s_2,s_3,s_4=-(N-1)}^{N-1+\alpha} \widehat{f_1}(s_1+s_4-\alpha)\widehat{f_3}(s_1-s_2)\widehat{f_4}(s_2-s_3)\widehat{f_2}(s_3-s_4).$$

We note that the expressions for (3.5.99), (3.5.101), and (3.5.103) are easily obtained from (3.5.98), (3.5.100), and (3.5.102), respectively, by interchanging k_1 and k_2. Therefore, we need only consider the sums in (3.5.98),(3.5.100), and (3.5.102). As before, evaluating these sums amounts to solving a collection of systems of equations of the form

$$s_1 + s_4 = t_1 + \alpha$$

$$s_1 - s_2 = t_2$$

$$s_2 - s_3 = t_3$$

$$s_3 - s_4 = t_4,$$

where (t_1, t_2, t_3, t_4) belongs to some finite indexing set A. Any $\bar{t} \in A$ will only give a non-zero contribution if there exists $(s_1, s_2, s_3, s_4) \in \mathbb{Z}^4$, $-(N-1) \leq s_i \leq N-1+\alpha$, satisfying

$$2s_1 - \alpha = t_1 + t_2 + t_3 + t_4$$

$$2s_2 - \alpha = t_1 - t_2 + t_3 + t_4$$

$$2s_3 - \alpha = t_1 - t_2 - t_3 + t_4$$

$$2s_4 - \alpha = t_1 - t_2 - t_3 - t_4.$$

. In the case of (3.5.98), $A = \{k_1, -k_1\} \times \{k_1, -k_1\} \times \{k_2, -k_2\} \times \{k_2, -k_2\}$. Since $t_1+t_2+t_3+t_4$ is even for all $\bar{t} \in A$, (3.5.98) is equal to zero in the cae of $SO(2N+1)$. For $SO(2N)$, the corresponding contribution is

$$-2\chi_{k_1,k_2 \leq N-1} - \chi_{\substack{k_1 \leq N-1 \\ k_1+k_2 \leq N-1}} - \chi_{\substack{k_2 \leq N-1 \\ k_1+k_2 \leq N-1}}$$

$$-\chi_{\substack{k_1 \leq N-1 \\ |k_1-k_2| \leq N-1}} - \chi_{\substack{k_2 \leq N-1 \\ |k_1-k_2| \leq N-1}} - 2\chi_{\substack{k_1,k_2 \leq N-1 \\ k_1+k_2 \leq N-1 \\ |k_1-k_2| \leq N-1}}.$$

The above expression is symmetric with respect to k_1 and k_2, so the total contribution from (3.5.98) and (3.5.99) is given by

$$-4\chi_{k_1,k_2\leq N-1} - 2\chi_{\substack{k_1\leq N-1\\ k_1+k_2\leq N-1}} - 2\chi_{\substack{k_2\leq N-1\\ k_1+k_2\leq N-1}} - 2\chi_{\substack{k_1\leq N-1\\ |k_1-k_2|\leq N-1}} - 2\chi_{\substack{k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - 4\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}}. \tag{3.5.104}$$

in the case of $SO(2N)$ and zero in the case of $SO(2N+1)$.

For (3.5.100), $A = \{k_1, -k_1\} \times \{k_2, -k_2\} \times \{k_1, -k_1\} \times \{k_2, -k_2\}$. The corresponding contribution is

$$-\frac{1}{2}\left(8\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1}} + 8\chi_{\substack{k_1,k_2\leq N-1\\ |k_1-k_2|\leq N-1}}\right)$$

for $SO(2N)$ and zero for $SO(2N+1)$.

Again, since this expression is symmetric in k_1 and k_2, the total contribution from (3.5.100) and (3.5.101) is

$$-8\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1}} - 8\chi_{\substack{k_1,k_2\leq N-1\\ |k_1-k_2|\leq N-1}}. \tag{3.5.105}$$

in the case of $SO(2N)$ and zero for $SO(2N+1)$.

Finally, for (3.5.102), $A = \{k_1, -k_1\} \times \{k_2, -k_2\} \times \{k_2, -k_2\} \times \{k_1, -k_1\}$. The resulting expression for (3.5.102) is

$$-2\chi_{k_1,k_2\leq N-1} - \chi_{\substack{k_1\leq N-1\\ k_1+k_2\leq N-1}} - \chi_{\substack{k_2\leq N-1\\ k_1+k_2\leq N-1}}$$

$$-\chi_{\substack{k_1\leq N-1\\ |k_1-k_2|\leq N-1}} - \chi_{\substack{k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - 2\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}}$$

for $SO(2N)$ and zero for $SO(2N+1)$.

Again, we notice that the above expression is symmetric with respect to k_1 and k_2 so the total contribution from (3.5.102) and (3.5.103) in the case of of $SO(2N)$ is given by

$$-4\chi_{k_1,k_2\leq N-1} - 2\chi_{\substack{k_1\leq N-1\\ k_1+k_2\leq N-1}} - 2\chi_{\substack{k_2\leq N-1\\ k_1+k_2\leq N-1}} + \cdots$$
$$-4\chi_{\substack{k_1\leq N-1\\ |k_1-k_2|\leq N-1}} - 2\chi_{\substack{k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - 4\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}}. \tag{3.5.106}$$

By combining (3.5.104), (3.5.105), and (3.5.106) , we get the following expression for (3.5.80) in the case of $SO(2N)$:

$$\begin{aligned} &- 8\chi_{k_1,k_2\leq N-1} - 4\chi_{\substack{k_1\leq N-1\\ k_1+k_2\leq N-1}} - 4\chi_{\substack{k_2\leq N-1\\ k_1+k_2\leq N-1}} \\ &- 4\chi_{\substack{k_1\leq N-1\\ |k_1-k_2|\leq N-1}} - 4\chi_{\substack{k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - 8\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}} \\ &- 8\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1}} - 8\chi_{\substack{k_1,k_2\leq N-1\\ |k_1-k_2|\leq N-1}}. \end{aligned} \tag{3.5.107}$$

Finally, we can combine (3.5.81), (3.5.90), (3.5.97), and (3.5.107) to get an expression for the non-unitary part of $\kappa(k_1,k_1,k_2,k_2)$. The non-unitary part is equal to zero in the case of $SO(2N+1)$. In the case of $SO(2N)$, we have

$$\begin{aligned} &2\chi_{\substack{k_1\leq N-1\\ k_1+k_2\leq N-1}} + 2\chi_{\substack{k_2\leq N-1\\ k_1+k_2\leq N-1}} \\ &+ 4\chi_{\substack{k_1\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}} + 4\chi_{\substack{k_2\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - 8\chi_{\substack{k_1,k_2\leq N-1\\ k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}} \\ &+ 2\chi_{\substack{k_1\leq N-1\\ |k_1-k_2|\leq N-1}} + 2\chi_{\substack{k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - 2\chi_{\substack{k_1+k_2\leq N-1\\ |k_1-k_2|\leq N-1}} \\ &- 3\chi_{k_1+k_2\leq N-1} - 3\chi_{|k_1-k_2|\leq N-1}, \end{aligned}$$

Which can be rewritten as

$$2\chi_{\substack{k_1\leq N-1\\ |k_1-k_2|\leq N-1}} + 2\chi_{\substack{k_2\leq N-1\\ |k_1-k_2|\leq N-1}} - \chi_{k_1+k_2\leq N-1} - 3\chi_{|k_1-k_2|\leq N-1}$$

Once again, in the case where $k_1 = k_2$, we note that the above quantity is equal to zero when $k_1 \leq \frac{N-1}{2}$. This is again consistent with the results from [25]. The above expression can further be simplified to

$$2\chi_{\substack{k_1 \leq N-1 \\ N \leq k_1+k_2 \\ |k_1-k_2| \leq N-1}} + 2\chi_{\substack{k_2 \leq N-1 \\ N \leq k_1+k_2 \\ |k_1-k_2| \leq N-1}} - 3\chi_{\substack{N \leq k_1+k_2 \\ |k_1-k_2| \leq N-1}} \tag{3.5.108}$$

This completes the proof. □

Proof. (*of Lemma 3.5.10*)

We start by breaking the sum up into 3 pieces:

$$\sum_{\substack{k_1 \text{ even} \\ 1 \leq k_1 \leq 2N-2 \\ N-1-\frac{k_1}{2} \leq k_2 \leq N-1}} \hat{f}(k_1)\hat{f}(k_2) + \sum_{\substack{k_1 \text{ even} \\ 1 \leq k_1 \leq 2N-2 \\ N \leq k_2 \leq N-1+\frac{k_1}{2}}} \hat{f}(k_1)\hat{f}(k_2) + \sum_{\substack{k1 \text{ even} \\ 2N \leq k_1 \\ -(N-1)+\frac{k_1}{2} \leq k_2 \leq (N-1)+\frac{k_1}{2}}} \hat{f}(k_1)\hat{f}(k_2) \tag{3.5.109}$$

Substituting $k_1 = 2t$ and $k_2 = s$, we can see that the above sums are equal to

$$\sum_{\substack{1 \leq t \leq N-1 \\ N-t \leq s \leq N-1}} \hat{f}(2t)\hat{f}(s) + \sum_{\substack{1 \leq t \leq N-1 \\ N \leq s \leq N-1+t}} \hat{f}(2t)\hat{f}(s) + \sum_{\substack{N \leq t \\ -(N-1)+t \leq s \leq (N-1)+t}} \hat{f}(2t)\hat{f}(s). \tag{3.5.110}$$

For the first sum, let $x_s = |s\hat{f}(s)|, y_t = |2t\hat{f}(2t)|$ for $1 \leq s, t \leq N$ and $X_N = \{x_s\}_{s=1}^N, Y_N = \{y_t\}_{t=1}^N$. By the assumption of Lemma 3.5.10, the Euclidean norms of of both X_N and Y_N are uniformly bounded with respect to N. Note that

$$\left| \sum_{\substack{1 \leq t \leq N-1 \\ N-t \leq s \leq N-1}} \hat{f}(2t)\hat{f}(s) \right| \leq \sum_{t=1}^{N-1} y_t \left(\frac{1}{t} \sum_{s=N-t}^{N-1} x_s \right) = \langle Y_{N-1}, A_{N-1}X_{N-1} \rangle, \tag{3.5.111}$$

where A_N is the $N \times N$ matrix given in the proof of Lemma 2.4.3(i). The weak convergence of A_N is enough to show that (3.5.111) is $o_N(1)$. The full details of the proof are identical to those in that of Lemma 2.4.3.

Similarly, to see that the second sum is $o_N(1)$, we bound absolute value of the second sum from above by

$$\sum_{t=1}^{N-1} y_t \left(\frac{1}{t} \sum_{s=N}^{N+t-1} x_s \right) = \langle C_{N-1} Y_{N-1}, M_{N-1} X_{N-1} \rangle,$$

where $X_N = \{x_s\}_{s=1}^{2N}, Y_N = \{y_t\}_{t=1}^{2N}$, and C_N, M_N are as in the proof of Lemma 2.4.3 (ii). Again, the details of the proof are identical to those of Lemma 2.4.3 (ii).

Finally, we bound the absolute value of the third sum from above by

$$\sum_{\substack{N \leq t \\ -(N-1)+t \leq s \leq (N-1)+t}} 2s|\hat{f}(2t)| \cdot |\hat{f}(s)| \leq \sum_{t=N}^{\infty} y_t \left(\frac{1}{t} \sum_{s=t-N+1}^{N+t-1} x_s \right) = \langle L^{N-1} Y, R_N X \rangle,$$

where $X = \{x_s\}_{s=1}^{\infty}, Y = \{y_t\}_{t=1}^{\infty} \in \ell^2_{\infty}$, and L, R_N are the same operators from the proof of part (iii) of Lemma 2.4.3. Again, the remaining details are identical to those of the proof of Lemma 2.4.3 (iii). $\square$

APPENDIX A

Auxiliary Results

A.1. Details for Proof of Theorem 2.1.1

In this section of the appendix we provide the rigorous details needed to complete the proof of Theorem 2.1.1. We begin by providing some preliminary background information.

Definition A.1.1. (Lèvy Metric)
Let $F, G : \mathbb{R} \to [0,1]$ be two cumulative distribution functions. Define the Lèvy metric to be

$$L(F,G) := \inf\{\epsilon > 0 \mid F(x-\epsilon) - \epsilon \le G(x) \le F(x+\epsilon) + \epsilon \text{ for all } x \in \mathbb{R}\}.$$

The following theorem is a standard result in measure theory.

Theorem A.1.2. *[3] Let F_N be a sequence of cumulative distribution functions. Then $F_N \to F$ at every continuity point of F iff $L(F_N, F) \to 0$.*

We now continue with the proof of Theorem 2.1.1. Let $\{\varphi_k\}_{k=1}^{\infty}$ be independent, exp(1) distributed random variables and recall that

$$f_m(\theta) := 2\sum_{k=1}^{m} \hat{f}(k)e^{ik\theta}, \qquad f_{\overline{m}}(\theta) := 2\sum_{k=m+1}^{\infty} \hat{f}(k)e^{ik\theta},$$

$$T_m := \frac{4}{\beta}\sum_{k=1}^{m} \hat{f}(k)(\varphi_k - 1), \qquad \text{and} \qquad T_\infty := \frac{4}{\beta}\sum_{k=1}^{\infty} \hat{f}(k)(\varphi_k - 1).$$

Note that $\mathbb{E}(T_m) = \mathbb{E}(T_\infty) = 0$. In order to prove that $S_N(f) - \mathbb{E}(S_N(f))$ converges to T_∞ in distribution, we need to verify that, for all $\epsilon > 0$, there exists sufficiently large N such that the following pair of inequalities hold for all $x \in \mathbb{R}$:

$$\mathbb{P}(S_N(f) - \mathbb{E}S_N(f) \leq x) \leq \mathbb{P}(T_\infty \leq x + \epsilon) + \epsilon \tag{A.1.1}$$

$$\mathbb{P}(T_\infty \leq x - \epsilon) - \epsilon \leq \mathbb{P}(S_N(f) - \mathbb{E}S_N(f) \leq x). \tag{A.1.2}$$

To verify (A.1.1), we start by taking advantage of the following trivial probability bound:

$$\begin{aligned}\mathbb{P}(S_N(f) - \mathbb{E}S_N(f) \leq x) &\leq \mathbb{P}(S_N(f_m) - \mathbb{E}S_N(f_m) \leq x + \epsilon/3) + \mathbb{P}(|S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}})| > \epsilon/3) \\ &\leq \mathbb{P}(S_N(f_m) - \mathbb{E}S_N(f_m) \leq x + \epsilon/3) + \frac{9\mathrm{Var}(S_N(f_{\overline{m}})}{\epsilon^2}. \end{aligned} \tag{A.1.3}$$

The last line follows from Chebyshev's inequality. Moreover, Proposition 2.2.2 implies that

$$\frac{9\mathrm{Var}(S_N(f_{\overline{m}}))}{\epsilon^2} \leq \delta_m + \delta_N, \tag{A.1.4}$$

where δ_m goes to zero in m, independent of N, and δ_N goes to zero in N, independent of m. We first choose $N_1, M_1 \in \mathbb{N}$ sufficiently large such that, for all $N \geq N_1$, and $m \geq M_1$, $\delta_N, \delta_m \leq \epsilon/6$. Now, T_m converges in distribution to T_∞, so it must also converge in the Lèvy metric, i.e we can find $M_2 \in \mathbb{N}$ sufficiently large such that the following inequalities hold for any $m \geq M_2$ and $x \in \mathbb{R}$:

$$\mathbb{P}(T_m \leq x) \leq \mathbb{P}(T_\infty \leq x + \epsilon/3) + \epsilon/3, \tag{A.1.5}$$

$$\mathbb{P}(T_\infty \leq x - \epsilon/3) - \epsilon/3 \leq \mathbb{P}(T_m \leq x). \tag{A.1.6}$$

Let $M' = \max(M_1, M_2)$ and note that M' is independent of N and depends only upon ϵ. It is also the case that $S_N(f_m) - \mathbb{E}\left(S_N(f_m)\right)$ converges in distribution to T_m so we can then choose, for any $m \geq M'$, $N_2(m) \in \mathbb{N}$ such that, for all $N \geq N_2(m)$,

$$\mathbb{P}(S_N(f_m) - \mathbb{E}S_N(f_m) \leq x) \leq \mathbb{P}(T_m \leq x + \epsilon/3) + \epsilon/3, \tag{A.1.7}$$

$$\mathbb{P}(T_m \leq x - \epsilon/3) - \epsilon/3 \leq \mathbb{P}(S_N(f_m) - \mathbb{E}S_N(f_m) \leq x) \tag{A.1.8}$$

hold for all $x \in \mathbb{R}$. Finally, let $m \geq M'$ and $N' = \max(N_1, N_2(m))$. Combining ($A.1.3$), ($A.1.4$), ($A.1.5$), and ($A.1.7$), we can conclude that

$$\begin{aligned}\mathbb{P}(S_N(f) - \mathbb{E}S_N(f) \leq x) &\leq \mathbb{P}(S_N(f_m) - \mathbb{E}S_N(f_m) \leq x + \epsilon/3) + \epsilon/3 \\ &\leq \mathbb{P}(T_m \leq x + 2\epsilon/3) + 2\epsilon/3 \\ &\leq \mathbb{P}(T_\infty \leq x + \epsilon) + \epsilon.\end{aligned}$$

To see that ($A.1.2$) holds, we use the same choice of m and N' as above and observe that

$$\begin{aligned}\mathbb{P}(T_\infty \leq x - \epsilon) - \epsilon &\leq \mathbb{P}(T_m \leq x - 2\epsilon/3) - 2\epsilon/3 && \text{(by } A.1.6\text{)} \\ &\leq \mathbb{P}(S_N(f_m) - \mathbb{E}S_N(f_m) \leq x - \epsilon/3) - \epsilon/3 && \text{(by } A.1.8\text{)} \\ &\leq \mathbb{P}(S_N(f) - \mathbb{E}S_N(f) \leq x) + \mathbb{P}\left(|S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}})| > \epsilon/3\right) - \epsilon/3 \\ &\leq \mathbb{P}(S_N(f) - \mathbb{E}S_N(f) \leq x).\end{aligned}$$

The last inequality follows immediately from the bound in (A.1.4). This completes the proof of Theorem 2.1.1 for $\beta = 2$.

If $\beta \neq 2$, then we replace the Chebyshev bound in (A.1.3) with the corresponding Markov bound and apply Proposition 1.0.3. To see this, we will first rewrite the tail as

$$S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}}) = \sum_{m=k+1}^{\infty} \hat{f}(k)\left(|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2\right), \tag{A.1.9}$$

where

$$t_{N,k} = \sum_{j=1}^{N} e^{ik\theta_j}.$$

For $0 < \beta < 2$, the proof of Lemma 4.3 in [13] gives the bound $\mathbb{E}|t_{N,k}|^2 \leq (2/\beta)m$ for all $m \geq 1$ and $N \geq 2$. It follows that

$$\begin{aligned}\Pr\left(|S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}})| \geq \frac{\delta}{3}\right) &\leq \frac{3\mathbb{E}|S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}})|}{\delta} \\ &\leq \frac{3}{\delta}C\left(\sum_{k=m+1}^{\infty} |\hat{f}(k)||k|\right),\end{aligned}$$

where C is a constant independent of N. Applying the condition in Theorem 2.1 for $0 < \beta < 2$, the r.h.s. side of the above inequality vanishes asymptotically, independent of N.

For $\beta = 4$ we break up (A.1.9) into three pieces:

$$S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}}) = \sum_{m=k+1}^{N} (*) + \sum_{m=N+1}^{2N} (*) + \sum_{m=2N+1}^{\infty} (*).$$

Proposition 2 in [13] states that there exist constants C, K, independent of N, such that $\mathbb{E}|t_{N,k}|^2 \leq Ck$ in the first sum, $\mathbb{E}|t_{N,k}|^2 \leq K \cdot k \log(k+1)$ in the second sum, and $\mathbb{E}|t_{N,k}|^2 \leq 2KN$ in the third sum. This gives the following bound for any $\alpha > 0$:

$$\begin{aligned} &|S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}})| \\ &\qquad \leq C' \left(\sum_{k=m+1}^{N} k \cdot |\hat{f}(k)| + \sum_{k=N+1}^{2N} k \log(k+1) |\hat{f}(k)| + \sum_{k=2N+1}^{\infty} k \cdot |\hat{f}(k)| \right), \end{aligned}$$

where C' is a constant independent of m and N. Applying the condition in Theorem 2.1 for $\beta = 4$, the first sum goes to zero in m independent of N and the last two sums go to zero in N independent of m.

When $2 < \beta \neq 4$, we break the tail as follows:

$$\begin{aligned} &S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}}) \\ &\qquad = \sum_{k=m+1}^{N/2} \hat{f}(k) \left(|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2 \right) + \sum_{k=N/2+1}^{\infty} \hat{f}(k) \left(|t_{N,k}|^2 - \mathbb{E}|t_{N,k}|^2 \right) \end{aligned}$$

In the first sum, $\mathbb{E}|t_{N,k}(\theta_n)|^2 \leq Ck$ where $C = 2/\beta$ for $0 < \beta < 2$ and $C = e^{1-2/\beta}$ for $\beta > 2$. For the second sum, we use the trivial bound $\mathbb{E}|t_{N,k}|^2 \leq N^2$ to get

$$\mathbb{E}|S_N(f_{\overline{m}}) - \mathbb{E}S_N(f_{\overline{m}})| \leq \left(2C \sum_{k=m+1}^{N/2} |\hat{f}(k)||k| + 8 \sum_{k=N/2+1}^{\infty} |\hat{f}(k)||k|^2 \right).$$

Once again, the first sum goes to zero in m independent of N and the second sum goes to zero in N independent of m.

A.2. Modifications for Mesoscopic Case: CUE, SO(N), Sp(N)

This section of the appendix provides the modifications to Proposition 2.2.1 and Lemma 2.4.3 that are necessary to adapt the proof of Proposition 2.2.2 to the mesoscopic case. In addition, we prove a version of Proposition 3.5.4 for the mesoscopic case, which we state as Lemma A.2.3.

LEMMA A.2.1. *(Extension of Proposition 2.2.1)*
If $\beta = 2$ and $f \in C_c^2(\mathbb{R})$ is an even, smooth, compactly supported function on the real line, then, for sufficiently large N,

$$\left(\frac{\pi}{2}\right) Var\left(\frac{S_N(f(L_N\cdot))}{\sqrt{L_N}}\right) =$$
$$\frac{1}{L_N}\sum_{1\leq s\leq N-1}\left(\frac{s}{L_N}\right)^2\left[\hat{f}\left(\frac{s}{L_N}\right)\right]^2 + \frac{N^2}{L_N^3}\sum_{N\leq s}\left[\hat{f}\left(\frac{s}{L_N}\right)\right]^2$$
$$-\frac{N}{L_N^3}\sum_{N\leq s}\left[\hat{f}\left(\frac{s}{L_N}\right)\right]^2 - \left(\frac{1}{L_N}\right)^2 \sum_{\substack{1\leq s,t\\ 1\leq |s-t|\leq N-1\\ N\leq \max(s,t)}}\left(\frac{N-|s-t|}{L_N}\right)\hat{f}\left(\frac{s}{L_N}\right)\hat{f}\left(\frac{t}{L_N}\right)$$
$$-\left(\frac{1}{L_N}\right)^2\sum_{\substack{1\leq s,t\leq N-1\\ N+1\leq s+t}}\left(\frac{(s+t)-N}{L_N}\right)\hat{f}\left(\frac{s}{L_N}\right)\hat{f}\left(\frac{t}{L_N}\right).$$

PROOF.
If $f \in C_c^2(\mathbb{R})$, then we may assume N large enough so that the support of $f(L_N\cdot)$ is contained on $[-\pi,\pi)$. We can immediately express $f(L_N\cdot)$ as a Fourier series with coefficients determined by the Fourier transform of f. Lemma A.2.1 is then an immediate corollary to Proposition 2.2.1. ☐

LEMMA A.2.2. *(Extension of Lemma 2.4.3)*
Let $f \in C_c^2(\mathbb{R})$. Then

(i)

$$\left(\frac{1}{L_N}\right)^2\sum_{\substack{1\leq s,t\leq N\\ s+t\geq N+1}}\left(\frac{s}{L_N}\right)\left|\hat{f}\left(\frac{s}{L_N}\right)\right|\cdot\left|\hat{f}\left(\frac{t}{L_N}\right)\right| \to 0;$$

(ii)

$$\frac{N+1}{L_N^3} \sum_{\substack{s-t\le N\\ s\ge N+1\\ 1\le t\le N}} \left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\hat{f}\left(\frac{t}{L_N}\right)\right| \to 0;$$

(iii)

$$\frac{N}{L_N^3} \sum_{\substack{|s-t|\le N-1\\ s,t\ge N}} \left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\hat{f}\left(\frac{t}{L_N}\right)\right| \to 0.$$

Proof.

To see (i), we replace the Fourier coefficients in the the proof of Lemma 4.4(i) with the corresponding coefficients for the scaled case to get

$$\left(\frac{1}{L_N}\right)^2 \sum_{\substack{1\le s,t\le N\\ s+t\ge N+1}} \left(\frac{s}{L_N}\right) \left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\hat{f}\left(\frac{t}{L_N}\right)\right|$$

$$\le 3\left[\frac{1}{L_N}\sum_{s=kL_N+1}^{\infty}\left(\frac{s}{L_N}\right)^2\left|\hat{f}\left(\frac{s}{L_N}\right)\right|^2\right]^{1/2}\left[\frac{1}{L_N}\sum_{s=1}^{\infty}\left(\frac{s}{L_N}\right)^2\left|\hat{f}\left(\frac{s}{L_N}\right)\right|^2\right]^{1/2} \tag{A.2.1}$$

$$+\frac{1}{L_N}\sum_{s=1}^{kL_N}\left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\frac{x_{N-s+1}+\cdots+x_N}{L_N}\right|,$$

where $k \in \mathbb{N}$. The first term in (A.2.1) contains, in brackets, two Riemann Sums and consequently converges to

$$3\left(\int_k^{\infty}[x\hat{f}(x)]^2dx\right)^{1/2}\left(\int_0^{\infty}[x\hat{f}(x)]^2dx\right)^{1/2} = o_k(1).$$

Since $f \in C_c^2(\mathbb{R})$, we can write $|\hat{f}(x)| \le C'/x^2$ for some positive constant C' depending only on f, i.e. independent of k and N. It follows that

$$\frac{1}{L_N}\sum_{s=1}^{kL_N}\left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\frac{x_{N-s+1}+\cdots+x_N}{L_N}\right| \le \frac{C'}{L_N}\sum_{s=1}^{kL_N}\left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left(\frac{1}{N-s+1}+\cdots+\frac{1}{N}\right)$$

$$\le \left(C'\frac{kL_N}{N-kL_N}\right)\left(\frac{1}{L_N}\sum_{s=1}^{kL_N}\left|\hat{f}\left(\frac{s}{L_N}\right)\right|\right).$$

For any fixed k, the term on the left is $O\left(\frac{L_N}{N}\right)$ while the term on the right is a Riemann Sum converging to

$$\int_0^k |\hat{f}(x)| \ dx \leq ||\hat{f}||_1 < \infty$$

as $N \to \infty$. It follows immediately that, for any $\epsilon > 0$, we can choose k and N large enough so that both terms in (A.2.1) are at most $\epsilon/2$. This gives the desired result.

To see (ii), we observe that, in the same way as in the proof of (i), the proof of Lemma 4.4(ii) immediately implies

$$\frac{N+1}{L_N^3} \sum_{\substack{s-t \leq N \\ s \geq N+1 \\ 1 \leq t \leq N}} \left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\hat{f}\left(\frac{t}{L_N}\right)\right|$$

$$\leq 3\left(\frac{1}{L_N} \sum_{s=kL_N+1}^{\infty} \left(\frac{s}{L_N}\right)^2 \left[\hat{f}\left(\frac{s}{L_N}\right)\right]^2\right)^{1/2} \left(\frac{1}{L_N} \sum_{s=1}^{\infty} \left(\frac{s}{L_N}\right)^2 \left[\hat{f}\left(\frac{s}{L_N}\right)\right]^2\right)^{1/2} + \frac{1}{L_N}\left|\sum_{s=1}^{kL_N} \hat{f}\left(\frac{s}{L_N}\right) \frac{x_{N+1} + \cdots + x_{N+s}}{L_N}\right| \tag{A.2.2}$$

The first term in (A.2.2) is the same as the first term in (A.2.1). Similarly, we observe that the second term is bounded above by

$$\left(C' \frac{L_N}{N}\right)\left(\frac{1}{L_N} \sum_{s=1}^{kL_N} \left|\hat{f}\left(\frac{s}{L_N}\right)\right|\right) = O\left(\frac{L_N}{N}\right) \to 0.$$

This completes the proof of (ii).

To see (iii), we once again follow the same argument as in the proof of Lemma 4.4(iii). In particular, bound the sum from above by

$$\left[\frac{N}{L_N^3} \sum_{\substack{t-N+1 \leq s \leq N+t-1 \\ t \geq N}} \left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\hat{f}\left(\frac{t}{L_N}\right)\right|\right] \leq \left[\frac{1}{L_N^3} \sum_{\substack{t-N+1 \leq s \leq N+t-1 \\ t \geq N}} s\left|\hat{f}\left(\frac{s}{L_N}\right)\right| \cdot \left|\hat{f}\left(\frac{t}{L_N}\right)\right|\right].$$

The proof of Lemma 4.4(iii) implies that the sum on the r.h.s. is bounded above by

$$3\left(\frac{1}{L_N}\sum_{s=N}^{\infty}\left[\frac{s}{L_N}\hat{f}\left(\frac{s}{L_N}\right)\right]^2\right)\left(\frac{1}{L_N}\sum_{s=1}^{\infty}\left[\frac{s}{L_N}\hat{f}\left(\frac{s}{L_N}\right)\right]^2\right).$$

The term on the r.h.s. is a Riemann sum that converges to

$$\int_0^\infty [x\hat{f}(x)]^2 dx < \infty$$

as $N \to \infty$, while the term on the l.h.s. is, at most, on the order of

$$\int_k^\infty [x\hat{f}(x)]^2 dx$$

for any $k \in \mathbb{N}$, i.e. goes to zero as $N \to \infty$. This completes the proof of Lemma A.2.2.

□

We can immediately modify the proof of Proposition 2.2.2 by replacing Proposition 2.2.1 with Lemma A.2.1 and Lemma 2.4.3 with Lemma A.2.2 to get the desired analogous result for the mesoscopic/scaled case.

We now consider the case of $SO(2N)$ and $Sp(N)$.

LEMMA A.2.3. *(Extension of Proposition 3.5.4)*
Let $\theta_1, \ldots, \theta_N$ be distributed according to $SO(N)$ or $Sp(N)$ and $f \in C_c^2(\mathbb{R})$ be an even, smooth, compactly supported function on the real line. Then, for sufficiently large N,

$$\left(\frac{\pi}{2}\right)\cdot Var\left(\frac{S_N(f(L_N\cdot))}{\sqrt{L_N}}\right) = \frac{2}{L_N}\sum_{k=1}^{\lfloor \frac{N}{2}\rfloor}\left(\frac{k}{L_N}\right)^2\left|\hat{f}\left(\frac{k}{L_N}\right)\right|^2 + o_N(1).$$

PROOF.

As in the proof of Proposition 3.5.4, we need only consider the following sum:

$$\frac{1}{L_N^3}\sum_{1\le k}\mathrm{Var}\left(t_k^2\right)\left|\hat{f}\left(\frac{k}{L_N}\right)\right|^2 + \frac{1}{L_N^3}\sum_{1\le k_1\neq k_2}\mathrm{Cov}(t_{k_1}^2, t_{k_2}^2)\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right). \tag{A.2.3}$$

We will focus our attention on the cases of $SO(2N)$ and $Sp(N)$. As before, the case of $SO(2N+1)$ is simpler and follows from the same arguments. Up to a constant multiple, the first term in (A.2.3) is equal to

$$\frac{2}{L_N}\sum_{k=1}^{\lfloor \frac{N}{2}\rfloor}\left(\frac{k}{L_N}\right)^2\left|\hat{f}\left(\frac{k}{L_N}\right)\right|^2 \quad +$$

$$\frac{1}{L_N}\left(\frac{4}{L_N}\sum_{k=1}^{\lfloor \frac{N}{2}\rfloor}\left(\frac{k}{L_N}\right)\left|\hat{f}\left(\frac{k}{L_N}\right)\right|^2\cdot\chi_{k\text{ even}}\right)+O\left(\frac{1}{L_N}\sum_{\lfloor \frac{N}{2}\rfloor+1\leq k}\left(\frac{k}{L_N}\right)^2\left|\hat{f}\left(\frac{k}{L_N}\right)\right|^2\right)$$

Assuming that $f\in C_c^2(\mathbb{R})$, the first term is a finite Riemann sum converging to $2\pi\cdot||f'||_2$. The second term is at most on the order of L_N^{-1} times a Riemann sum converging to

$$\int_{\mathbb{R}}|x\hat{f}(x)|^2\,dx<+\infty$$

and so goes to zero as $N\to\infty$. For any fixed $n\in\mathbb{N}$ and sufficiently large N, the third term is at most on the order of

$$\int_n^{\infty}|x\hat{f}(x)|^2\,dx<+\infty$$

and so must also go to zero as $N\to\infty$.

Again using Proposition 3.5.3, the second term in (A.2.3) is equal to

$$-\frac{2}{L_N^2}\sum_{\substack{1\leq|k_1-k_2|\leq M_N-1\\ M_N\leq\max(k_1,k_2)}}\left(\frac{M_N-|k_1-k_2|}{L_N}\right)\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right) \tag{A.2.4}$$

$$-\frac{2}{L_N^2}\sum_{\substack{k_1,k_2\leq M_N-1\\ M_N+1\leq k_1+k_2}}\left(\frac{k_1+k_2-M_N}{L_N}\right)\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right)$$

$$+\delta\frac{4}{L_N^3}\sum_{\substack{k_1,k_2\text{ even}\\ |k_1-k_2|\leq M_N-1\\ M_N+1\leq k_1+k_2}}\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right)+\delta\frac{4}{L_N^3}\sum_{\substack{k_1\leq(M_N-1)/2\\ (M_N+1)/2\leq k_1+k_2\\ |k_1-k_2|\leq(M_N-1)/2}}\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right) \tag{A.2.5}$$

$$-\delta\frac{3}{L_N^3}\sum_{\substack{|k_1-k_2|\leq(M_N-1)/2\\ (M_N+1)/2\leq k_1+k_2}}\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right)+\frac{2}{L_N^3}\sum_{\substack{k_1,k_2\text{ even}\\ |k_1-k_2|\leq M_N-1\\ M_N+1\leq k_1+k_2}}\hat{f}\left(\frac{k_1}{L_N}\right)\hat{f}\left(\frac{k_2}{L_N}\right)+ \tag{A.2.6}$$

$$\text{(A.2.7)} \qquad \frac{O(N)}{L_N^3} \sum_{\frac{M_N+1}{4} \leq k \leq \frac{M_N-1}{2}} \hat{f}\left(\frac{k}{L_N}\right) \hat{f}\left(\frac{2k}{L_N}\right) - \delta \frac{4}{L_N^3} \sum_{\substack{k_2 \text{ even} \\ k_2 \leq M_N - 1 \\ |2k_1 - k_2| \leq M_N - 1 \\ M_N + 1 \leq 2k_1 + k_2}} \hat{f}\left(\frac{k_1}{L_N}\right) \hat{f}\left(\frac{k_2}{L_N}\right).$$

The fact that the sums in (A.2.4-A.2.6) converge absolutely and go to zero as $N \to \infty$ follows from Lemma A.2.2. Again, we can use the Cauchy Schwartz inequality to see that the first term in (A.2.7) is on the order of

$$\frac{1}{L_N} \sum_{\frac{M_N+1}{4} \leq k \leq \frac{M_N-1}{2}} \frac{N}{L_N^2} \left| \hat{f}\left(\frac{k}{L_N}\right) \hat{f}\left(\frac{2k}{L_N}\right) \right| \leq \frac{1}{L_N} \sum_{\frac{M_N+1}{4} \leq k \leq \frac{M_N-1}{2}} \frac{2k}{L_N^2} \left| \hat{f}\left(\frac{k}{L_N}\right) \hat{f}\left(\frac{2k}{L_N}\right) \right|$$

$$\leq \left(\frac{1}{L_N} \sum_{\lfloor \frac{N}{2} \rfloor + 1 \leq k} \left(\frac{k}{L_N}\right)^2 \hat{f}^2\left(\frac{k}{L_N}\right) \right)^{1/2} \left(\frac{1}{L_N} \sum_{\lfloor \frac{N}{2} \rfloor + 1 \leq k} \left(\frac{2k}{L_N}\right)^2 \hat{f}^2\left(\frac{2k}{L_N}\right) \right)^{1/2},$$

which is equal to $o_N(1)$. Finally, the second term in (A.2.7) converges absolutely and is equal to $o_N(1)$ by Lemma A.2.2, modulo the arguments given in the proof of Lemma 3.5.10. This completes the proof of Lemma A.2.3. □

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