A trilogy of (super)crystals: the queer, the set-valued and the hook-valued

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To my late grandfather, Ong Chooi Wan

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Abstract

This dissertation compiles three main results concerning crystals or supercrystals. Firstly, we present a characterization for queer supercrystals introduced by Grantcharov et al. We also construct a type A crystal whose character is the stable Grothendieck polynomials for fully-commutative permutations. Finally, we describe an uncrowding map on hook-valued tableaux which intertwines the crystal operators on hook-valued tableaux to those of set-valued tableaux.

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CHAPTER 1

Introduction

Crystal bases were developed by Kashiwara [**Kas91**] by taking the limit as $q \to 0$ of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ for a classical Lie algebra $\mathfrak{g}$. Lusztig [**Lus90a**] employed a geometric approach to develop the theory of canonical bases, which turn out to yield Kashiwara's crystal bases [**Lus90b**].

A crystal is usually depicted by means of a labeled, directed graph, called a crystal graph. One utility of crystals is that they provide a combinatorial way to compute the character of the associated representations. In addition, crystals behave very nicely under taking the tensor product; indeed, Kashiwara [**Kas91**] provided a tensor product rule to compute this product. This allows for an effective determination of tensor multiplicities of the associated representations, giving the following generalized Littlewood-Richardson rule:

Theorem 1 ([**Nak93**]). *Let λ, μ be dominant, integral weights for a classical Lie algebra $\mathfrak{g}$ and $B(\lambda), B(\mu)$ be the associated crystals with the respective highest weights.*

Then, we have the $\mathcal{U}_q(\mathfrak{g})$-crystal isomorphism

$$B(\lambda) \otimes B(\mu) \cong \bigoplus_{\nu} c^{\nu}_{\lambda\mu} B(\nu),$$

where $c^{\nu}_{\lambda\mu} B(\nu)$ counts the number of highest weight elements within $B(\lambda) \otimes B(\mu)$ of weight ν.

1.0.1. Queer supercrystals. The classification of simple Lie superalgebras was given by Kac [**Kac77**]. One of these Lie superalgebras of particular interest is the queer Lie superalgebra, $\mathfrak{q}(n+1)$, which arises as a super analogue of the Lie algebra $\mathfrak{gl}(n)$. In recent years, crystal bases for the queer Lie superalgebra, $\mathfrak{q}(n+1)$ have received increasing attention and development. Grantcharov et al. [**GJK$^+$10**,**GJKK10**,**GJK$^+$15**] defined a crystal bases for crystals of type $\mathfrak{q}(n+1)$. Additionally, they [**GJK$^+$14**] realized a model for queer supercrystal using semistandard decomposition tableaux and provided a description of those tableaux that are of highest weight. Hiroshima [**Hir19**] and

Assaf and Oguz [**AKO18a**] independently defined odd crystal operators for the type A crystal on shifted prime tableaux defined by Hawkes, Paramonov and Schilling [**HPS17**]. Therefore, the collection of shifted prime tableaux with the structure of a queer supercrystal.

A particular class of problems in the theory of crystal bases is the characterization of crystals. Given a crystal graph, the problem asks whether or not there exists a set of axioms that identifies it as a crystal graph of representation of a classical Lie algebra. Indeed, Stembridge [**Ste03**] provided a set of local axioms that characterize such crystals in the special case when its root system is simply-laced. These local axioms turn out to impose global structure on these crystals, thus enabling a study of crystals of representations from a combinatorial viewpoint. One may enquire if a characterization of queer supercrystals exists. Assaf and Oguz [**AKO18b**] provided a list of local axioms for queer supercrystals involving relations between even crystal operators f_i, where $1 \leq i \leq n$, and the odd operator f_{-1}. They conjectured that these axioms, in addition to Stembridge's local axioms, were sufficient for a characterization.

The first part of the dissertation provides a characterization of queer supercrystals introduced by Grantcharov et al. This is based on joint work with Maria Gillespie, Graham Hawkes and Anne Schilling [**GHPS20**]. This characterization is a combination of an extension of characterization by Stembridge, for crystals whose root system is simply-laced, using local queer axioms introduced by Assaf and Oguz, a graph G on type A_n components associated to a component of queer supercrystal and a set of axioms describing the odd crystal operators for elements near the lowest weight element for each component. We also provide a counterexample to an earlier conjecture by Assaf and Oguz that their axioms uniquely characterizes queer supercrystals. Furthermore, the graph G on type A_n components admits a combinatorial description using explicit combinatorial rules for odd queer operators on certain types of highest weight elements.

1.0.2. Crystal for fully commutative stable Grothendieck polynomials. The Grothendieck polynomials were introduced by Lascoux and Schützenberger [**LS82**, **LS83**] in their study of the Grothendieck ring of the flag manifold. These functions are indexed by permutations and serve as K-theoretic analogues of the Schubert polynomials, which are polynomial representatives of the cohomology classes for the flag manifold. Fomin and Kirillov [**FK94**] generalized

Lascoux-Schützenberger's definition of the Grothendieck polynomials in terms of a parameter β and provided a definition of the stable limit, known as the stable Grothendieck polynomials, $\mathfrak{G}_w$.

In the special case where the permutation is fully-commutative, whose circle diagram is associated to a skew partition, Buch [**Buc02**] provided a combinatorial definition of the stable Grothendieck polynomials in terms of set-valued tableaux (of skew shape):

$$\mathfrak{G}_{\lambda/\mu}(x;\beta) = \sum_{T \in \mathsf{SVT}(\lambda/\mu)} x^{\mathrm{wt}(T)} \beta^{|T|-|\lambda/\mu|}.$$

Here, $\mathsf{SVT}(\lambda/\mu)$ refers to the collection of all set-valued tableaux whose shape is λ/μ, $|T|$ counts the total number of letters within a tableaux T and $|\lambda/\mu|$ counts the number of boxes within the skew shape λ/μ. The collection of set-valued tableaux additionally carries a type A crystal structure; Monical, Pechenik and Scrimshaw [**MPS20**] defined the crystal operators on these tableaux and showed that their crystal is Stembridge using the uncrowding bijection.

On the other hand, the stable Grothendieck polynomials, $\mathfrak{G}_w$, may be viewed as K-theoretic analogues of the stable limit of Schubert polynomials, which are known as Stanley symmetric functions. Morse and Schilling [**MS16**] defined a type A crystal on the set of decreasing factorizations of reduced words in the symmetric group, whose characters are the Stanley symmetric functions. As a consequence, they showed that the Stanley symmetric functions admit a Schur positive expansion. Moreover, they showed that their crystal is isomorphic to a crystal on pairs of semistandard Young tableaux and standard Young tableaux using the Edelman-Greene insertion.

Fomin and Greene [**FG98**] employed the theory of nonsymmetric Schur functions to prove that the stable Grothendieck polynomials admit a Schur positive expansion in the following sense:

$$\mathfrak{G}_w(\mathbf{x};\beta) = \sum_{\lambda} g_{w,\lambda} s_\lambda \beta^{\ell(w)-|\lambda|},$$

where $\ell(w)$ is the length of permutation w and $g_{w,\lambda}$ counts the number of semistandard Young tableaux of shape λ^t whose column reading word is equivalent to w.

The stable Grothendieck polynomials have a combinatorial realization in terms of decreasing factorizations in the 0-Hecke monoid, whose reduced words that are identified with those of the symmetric group. One could ask if there is a type A crystal that could be defined on these decreasing factorizations for each choice of permutation. Towards this direction, Monical, Pechenik and Scrimshaw [**MPS20**] used the inverse of Hecke insertion to obtain an induced crystal on the set of decreasing factorizations for words in the 0-Hecke monoid. However, unlike the crystal operators given by Morse-Schilling, their crystal operators in general do not involve local changes.

Within the second part of this dissertation, we construct a type A crystal, which we call the $\star$-crystal, whose character is the stable Grothendieck polynomials for fully-commutative permutations. This is based on joint work with Jennifer Morse, Jianping Pan and Anne Schilling [**MPPS20**]. This crystal is a K-theoretic generalization of the Morse-Schilling crystal on decreasing factorizations for reduced words in the symmetric group. Using the residue map, we show that this crystal intertwines with the crystal on set-valued tableaux given by Monical, Pechenik and Scrimshaw. We prove that this crystal is isomorphic to that of pairs of semistandard Young tableaux using a newly defined insertion called the $\star$-insertion. Furthermore, the $\star$-insertion has interesting properties in relation to row Hecke insertion and the uncrowding algorithm.

1.0.3. Uncrowding algorithm for hook-valued tableaux. Buch [**Buc02**] described a bijection from the collection of set-valued tableaux to pairs of tableaux (P, Q) of the same shape, where P is semistandard and Q is a flagged, increasing tableau. Since then, this bijection, known as the uncrowding map, has been employed in various settings, including Bandlow and Morse [**BM12**] who obtained G-expansions and g-expansions of symmetric functions that have a tableaux Schur expansion, Reiner, Tenner and Yong [**RTY18**] who enumerated barely set-valued tableaux. More recently, Chan and Pflueger [**CP19**] extended the map on set-valued tableaux of skew shape; their enumeration is related to the algebraic Euler characteristic of the Brill-Noether variety.

As mentioned above, Buch introduced set-valued tableaux to give a combinatorial definition for the stable Grothendieck polynomials for Grasmannian permutations, which is also known as symmetric Grothendieck functions, G_λ. Similar to symmetric Grothendieck functions, Lam and Pylyavskyy [**LP07**] introduced weak symmetric Grothendieck functions, J_λ, that are described combinatorially by multiset-valued tableaux (known as weak set-valued tableaux).

In contrast to the Schur functions, neither the symmetric Grothendieck functions nor the weak symmetric Grothendieck functions are self dual under the involution ω on symmetric functions. Yeliusizzov [**Yel17**] introduced the canonical Grothendieck functions, $\widetilde{G}_\lambda$, as a deformation of both

G_λ and J_λ, that have the same structure constants as G_λ. He also gave a combinatorial definition of $\tilde{G}_\lambda$ in terms of hook-valued tableaux.

Monical, Pechenik and Scrimshaw [**MPS20**] have shown that the uncrowding map on set-valued tableaux intertwines the crystal operators on set-valued tableaux with those on pairs of tableaux (P, Q) of the same shape, where P is semistandard and Q is a flagged, increasing tableau. Meanwhile, Hawkes and Scrimshaw [**HS20**] defined the uncrowding map from the collection of multiset-valued tableaux to pairs of tableaux (P, Q) of the same shape where P is semistandard and Q is a column flagged, increasing tableau. Similar to [**MPS20**], they showed that their uncrowding map intertwines the crystal operators defined on both collections. However, an uncrowding map on hook-valued tableau that intertwines with the relevant crystal operators interpolating both uncrowding maps was not known.

The final part of the dissertation describes a new uncrowding algorithm for hook-valued tableaux, which uncrowds the entries in the arm of the hooks within such a tableau and returns a set-valued tableau paired with a column-flagged increasing tableau. This is based on joint work with Jianping Pan, Joseph Pappe and Anne Schilling [**PPPS20**]. We show that the uncrowding algorithm intertwines with the crystal operators on hook-valued tableaux. Moroever, we provide a brief description of an analogous uncrowding algorithm which uncrowds the entries in the leg rather than those in the arm of the hooks. We also describe a crowding insertion and use it to provide a recursive definition for the image of the uncrowding map. This enables us to provide a definition of the crowding map and show that the constructed crowding map is the inverse to the uncrowding map. Finally, as an application, we obtain various expansions of the canonical Grothendieck polynomials in terms of the stable symmetric Grothendieck polynomials and the dual stable symmetric Grothendieck polynomials.

CHAPTER 2

Characterization of queer supercrystals

This chapter is based on joint work with Maria Gillespie, Graham Hawkes and Anne Schilling published in [**GHPS20**].

2.1. Queer supercrystals

In Section 2.1.1, we review the queer supercrystals constructed in [**GJK$^+$10**,**GJK$^+$14**,**GJK$^+$15**]. In Section 2.1.2, we review some properties of queer supercrystals discovered in [**AKO18a**,**AKO18b**]. In Section 2.1.3, we provide new explicit combinatorial descriptions of f_{-i} and e_{-i} on certain highest weight elements, which will be used in Section 2.3 to construct the graph $G(\mathcal{C})$. In Section 2.1.4, we provide relations between e_{-i} when acting on certain highest weight elements, which will be used in Section 2.3 to deal with "by-pass arrows" in the component graph $G(\mathcal{C})$.

2.1.1. Definition of queer supercrystals. An *(abstract) crystal* of type A_n is a nonempty set B together with the maps

$$
\begin{aligned}
&e_i, f_i \colon B \to B \sqcup \{0\} \qquad \text{for } i \in I,\\
&\mathrm{wt} \colon B \to \Lambda,
\end{aligned}
\tag{2.1.1}
$$

where $\Lambda = \mathbb{Z}^{n+1}_{\geqslant 0}$ is the weight lattice of the root of type A_n and $I = \{1, 2, \ldots, n\}$ is the index set, subject to several conditions. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I$ the simple roots of type A_n, where ϵ_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$. Then we require:

A1. For $b, b' \in B$, we have $f_i b = b'$ if and only if $b = e_i b'$. In this case $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.

For $b \in B$, we also define

$$\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid f_i^k(b) \neq 0\} \qquad \text{and} \qquad \varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid e_i^k(b) \neq 0\}.$$

We remark that the above defines what is usually known as a seminormal crystal of type A_n elsewhere in the literature. For further details, see for example [**BS17**, Definition 2.13].

There is an action of the symmetric group S_n on a type A_n crystal B given by the operators

$$s_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geqslant 0, \\ e_i^{-k}(b) & \text{if } k < 0, \end{cases} \tag{2.1.2}$$

for $b \in B$, where $k = \varphi_i(b) - \varepsilon_i(b)$.

An element $b \in B$ is called *highest weight* if $e_i(b) = 0$ for all $i \in I$. Similarly, b is called *lowest weight* if $f_i(b) = 0$ for all $i \in I$. For a subset $J \subseteq I$, we say that b is J-highest weight if $e_i(b) = 0$ for all $i \in J$ and similarly b is J-lowest weight if $f_i(b) = 0$ for all $i \in J$.

We are now ready to define an abstract queer supercrystal.

Definition 2. *[**GJK$^+$14**, Definition 1.9] An* abstract $\mathfrak{q}(n+1)$-crystal *is a type A_n crystal B together with the maps $e_{-1}, f_{-1} \colon B \to B \sqcup \{0\}$ satisfying the following conditions:*

Q1. $\mathrm{wt}(B) \subset \Lambda$*;*

Q2. $\mathrm{wt}(e_{-1}b) = \mathrm{wt}(b) + \alpha_1$ *and* $\mathrm{wt}(f_{-1}b) = \mathrm{wt}(b) - \alpha_1$*;*

Q3. *for all $b, b' \in B$, $f_{-1}b = b'$ if and only if $b = e_{-1}b'$;*

Q4. *if $3 \leqslant i \leqslant n$, we have*

(a) *the crystal operators e_{-1} and f_{-1} commute with e_i and f_i;*

(b) *if $e_{-1}b \in B$, then $\varepsilon_i(e_{-1}b) = \varepsilon_i(b)$ and $\varphi_i(e_{-1}b) = \varphi_i(b)$.*

Given two $\mathfrak{q}(n+1)$-crystals B_1 and B_2, Grantcharov et al. [**GJK$^+$14**, Theorem 1.8] provide a crystal on the tensor product $B_1 \otimes B_2$, which we state here in reverse convention. It consists of the type A_n tensor product rule (see for example [**BS17**, Section 2.3]) and the *tensor product rule* for $b_1 \otimes b_2 \in B_1 \otimes B_2$

$$\begin{aligned} e_{-1}(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes e_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ e_{-1}b_1 \otimes b_2 & \text{otherwise,} \end{cases} \\ f_{-1}(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes f_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ f_{-1}b_1 \otimes b_2 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1.3}$$

$$\boxed{1} \overset{1}{\underset{-1}{\rightrightarrows}} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n} \boxed{n+1}$$

FIGURE 2.1. $\mathfrak{q}(n+1)$-crystal of letters $\mathcal{B}$

The crystals of interest are the *crystals of words* $\mathcal{B}^{\otimes \ell}$, where $\mathcal{B}$ is the $\mathfrak{q}(n+1)$-crystal of letters depicted in Figure 2.1.

In addition to the queer supercrystal operators $f_{-1}, f_1, \ldots, f_n$ and $e_{-1}, e_1, \ldots, e_n$, we define the crystal operators for $1 < i \leqslant n$

$$f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i} \qquad \text{and} \qquad e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}, \tag{2.1.4}$$

where $s_{w_i} = s_2 \cdots s_i s_1 \cdots s_{i-1}$ and s_i is the reflection along the i-string in the crystal defined in (2.1.2). Furthermore for $i \in I_0 := \{1, 2, \ldots, n\}$

$$f_{-i'} := s_{w_0} e_{-(n+1-i)} s_{w_0} \qquad \text{and} \qquad e_{-i'} := s_{w_0} f_{-(n+1-i)} s_{w_0}, \tag{2.1.5}$$

where w_0 is the longest word in the symmetric group S_{n+1}. By [**GJK$^+$14**, Theorem 1.14], with all operators e_i, f_i for $i \in \{-1, -2, \ldots, -n, 1, 2, \ldots, n\}$ each connected component of $\mathcal{B}^{\otimes \ell}$ has a unique highest weight vector and with all operators e_i, f_i for $i \in \{-1', -2', \ldots, -n', 1, 2, \ldots, n\}$ each connected component of $\mathcal{B}^{\otimes \ell}$ has a unique lowest weight vector.

2.1.2. Properties of queer supercrystals. We now review and prove several properties about the queer supercrystal operators.

LEMMA 3. *For $1 \leqslant i < n$, we have*

$$\begin{aligned} f_{-(i+1)} &= (s_i s_{i+1})\, f_{-i}\, (s_{i+1} s_i), \\ e_{-(i+1)} &= (s_i s_{i+1})\, e_{-i}\, (s_{i+1} s_i). \end{aligned} \tag{2.1.6}$$

PROOF. We use the definition (2.1.4). Note that the following recursion holds

$$s_{w_{i+1}} = (s_2 \cdots s_{i+1})(s_1 \cdots s_i) = (s_2 \cdots s_i)(s_1 \cdots s_{i-1}) s_{i+1} s_i = s_{w_i} s_{i+1} s_i, \tag{2.1.7}$$

which implies the statement. $\square$

Remark 4. *The operators f_i for $i \in I_0$ have an easy combinatorial description on $b \in \mathcal{B}^{\otimes \ell}$ given by the* signature rule*, which can be directly derived from the tensor product rule (see for example [**BS17**, Section 2.4]). One can consider b as a word in the alphabet $\{1, 2, \ldots, n+1\}$. Consider the subword of b consisting only of the letters i and $i+1$. Pair (or bracket) any consecutive letters $i+1, i$ in this order, remove this pair, and repeat. Then f_i changes the rightmost unpaired i to $i+1$; if there is no such letter $f_i(b) = 0$. Similarly, e_i changes the leftmost unpaired $i+1$ to i; if there is no such letter $e_i(b) = 0$.*

Remark 5. *From* (2.1.3)*, one may also derive a simple combinatorial rule for f_{-1} and e_{-1}. Consider the subword v of $b \in \mathcal{B}^{\otimes \ell}$ consisting of the letters* 1 *and* 2*. The crystal operator f_{-1} on b is defined if the leftmost letter of v is a* 1*, in which case it turns it into a* 2*. Otherwise $f_{-1}(b) = 0$. Similarly, e_{-1} on b is defined if the leftmost letter of v is a* 2*, in which case it turns it into a* 1*. Otherwise $e_{-1}(b) = 0$.*

Lemmas 6 and 7 have appeared in [**AKO18a**, **AKO18b**]. We provide proofs for completeness.

Lemma 6. *Let $b \in \mathcal{B}^{\otimes \ell}$. The following holds:*

(1) *If $\varphi_1(b) \geqslant 2$ and $\varphi_{-1}(b) = 1$, we have $\varphi_1(b) = \varphi_1(f_{-1}(b)) + 2$ and $\varepsilon_1(b) = \varepsilon_1(f_{-1}(b))$. If furthermore $\varphi_1(b) > 2$, then*

$$f_1 f_{-1}(b) = f_{-1} f_1(b).$$

(2) *If $\varphi_1(b) = \varphi_{-1}(b) = 1$, we have*

$$f_1(b) = f_{-1}(b).$$

(3) *If $\varepsilon_1(b), \varepsilon_{-1}(b) > 0$ and $e_1(b) \neq e_{-1}(b)$, we have $\varepsilon_1(b) = \varepsilon_1(e_{-1}(b))$, $\varphi_1(b) = \varphi_1(e_{-1}(b)) - 2$, and*

$$e_1 e_{-1}(b) = e_{-1} e_1(b).$$

Proof. Let $p = \varphi_1(b)$ and $q = \varepsilon_1(b)$. Consider the subword v consisting of all letters 1 and 2 in b. After performing 1,2-bracketing onto v according to the signature rule, we have a subword of unbracketed letters in b as

$$v_{i_1}v_{i_2}\dots v_{i_p}v_{j_1}\dots v_{j_q}, \tag{2.1.8}$$

where $v_{i_k}=1$ for all $1\leqslant k\leqslant p$ and $v_{j_k}=2$ for all $1\leqslant k\leqslant q$.

(1) We assume that $\varphi_{-1}(b)>0$, so that $f_{-1}(b)$ is defined. This implies $v_1=1$. Since v_1 is necessarily unbracketed, $i_1=1$ as well. The word $b'=f_{-1}(b)$ is formed by changing the leftmost 1 in b, namely v_{i_1}, into 2. This introduces a new bracketed 1,2-pair formed by $v_1=2$ and $v_{i_2}=1$. The subword of unbracketed letters in b' now becomes

$$v_{i_3}\dots v_{i_p}v_{j_1}\dots v_{j_q}$$

so that $\varphi_1(f_{-1}(b))=p-2=\varphi_1(b)-2$ and $\varepsilon_1(f_{-1}(b))=q=\varepsilon_1(b)$. This establishes the first assertion.

Now, assume in addition that $p=\varphi_1(b)>2$. Using the sequence of unbracketed letters in b as in the preceding paragraph, f_1 changes the rightmost unbracketed 1 in b, namely v_{i_p}, into 2. We still have v_1 to be 1 after the change, so that $f_{-1}(f_1(b))$ is defined and the leftmost 1 in $f_1(b)$, namely v_1, is changed into 2 under f_{-1}. On the other hand, $f_1(f_{-1}(b))$ is defined precisely because $p>2$, and the rightmost unbracketed 1 in $f_{-1}(b)$, namely v_{i_p}, is changed into 2 under f_1. As the changes introduced in b to form $f_{-1}(f_1(b))$ are the same as in those of $f_1(f_{-1}(b))$, we conclude that $f_1(f_{-1}(b))=f_{-1}(f_1(b))$, proving the second assertion.

(2) We assume $\varphi_1(b)=1$, so that (2.1.8) is of the form $v_{i_1}v_{j_1}\dots v_{j_q}$. Furthermore, as $\varphi_{-1}(b)=1$, $f_{-1}(b)$ is defined and $v_1=1$. As v_1 is necessarily unbracketed, $i_1=1$ as well. Therefore, we see that $f_1(b)=f_{-1}(b)$, since the rightmost unbracketed 1 in b and the leftmost 1 in b are the same, namely $v_{i_1}=v_1$.

(3) We assume that $\varepsilon_{-1}(b)>0$, so that $e_{-1}(b)$ is defined. This implies $v_1=2$. However, since $e_{-1}(b)\neq e_1(b)$, e_{-1} and e_1 must change a 2 in b at different locations, so we have $j_1>1$. Consequently v_1 is a bracketed 2 and hence must be paired with some $v_h=1$ where $h<i_1<j_1$ (in case $p=0$, $h<j_1$ still holds). The word $b'=e_{-1}(b)$ is obtained by changing the leftmost 2 in b, namely v_1, to 1. This introduces two new unbracketed 1's,

namely, v_1 and v_h. The subword of unbracketed letters in b' is now

$$v_1 v_h v_{i_1} \dots v_{i_p} v_{j_1} \dots v_{j_q}$$

so that $\varepsilon_1(b) = q = \varepsilon_1(e_{-1}(b))$ and $\varphi_1(e_{-1}(b)) = p + 2 = \varphi_1(b) + 2$. This establishes the first two equalities.

Now, $e_1(e_{-1}(b))$ is the word formed by changing the leftmost unbracketed 2 in $b' = e_{-1}(b)$, namely v_{j_1}, to 1. On the other hand, using the subword of v in b containing unbracketed letters as described in the preceding paragraph, $e_1(b)$ changes the leftmost unbracketed 2 in b, namely v_{j_1}, into a 1. We still have $v_1 = 2$ and $v_h = 1$ after the change, so that $e_{-1}(e_1(b))$ is defined, with the leftmost 2 in $e_1(b)$, namely v_1, being changed into 1 under e_{-1}. As the changes introduced in b to form $e_{-1}(e_1(b))$ are the same as in those of $e_1(e_{-1}(b))$, we conclude that $e_1(e_{-1}(b)) = e_{-1}(e_1(b))$, thereby proving the final relation.

$\square$

LEMMA 7. *Let $b \in \mathcal{B}^{\otimes \ell}$. The following holds:*

(1) *If $\varphi_2(b), \varphi_{-1}(b) > 0$, we have $\varphi_2(b) = \varphi_2(f_{-1}(b)) - 1$, $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$ and*

$$f_2 f_{-1}(b) = f_{-1} f_2(b).$$

(2) *If $\varphi_2(b) = 0$ and $\varphi_{-1}(b) > 0$, we have either*
   (a) *$\varphi_2(f_{-1}(b)) = 1$ and $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$, or*
   (b) *$\varphi_2(f_{-1}(b)) = 0$ and $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b)) + 1$.*

(3) *If $\varepsilon_2(b), \varepsilon_{-1}(b) > 0$, we have either*
   (a) *$\varepsilon_2(e_{-1}(b)) = \varepsilon_2(b) + 1$, $\varphi_2(b) = \varphi_2(e_{-1}(b)) = 0$, or*
   (b) *$\varepsilon_2(e_{-1}(b)) = \varepsilon_2(b)$, $\varphi_2(b) = \varphi_2(e_{-1}(b)) + 1$, and*

$$e_{-1} e_2(b) = e_2 e_{-1}(b).$$

PROOF. We prove each part separately.

(1) Assume that $\varphi_2(b), \varphi_{-1}(b) > 0$, so that $f_2(b)$ and $f_{-1}(b)$ are both nonzero. Let $b' = f_{-1}(b)$ and $b'' = f_2(b)$.

By the signature rule, $\varphi_2(b)$ is the number of unbracketed 2 entries in the $2,3$-bracketing of b. Since $\varphi_2(b) > 0$, there exists a rightmost unbracketed 2, say b_j. As in Remark 5 $b' = f_{-1}(b)$ is formed by changing the leftmost 1, say b_i, to $b'_i = 2$, where b_i is the leftmost of all 1 and 2 entries (so in particular $i < j$).

Since $\varphi_{-1}(b) > 0$, every 2 must be to the right of b_i. Assume that there is a 3 left of b_i bracketed with a 2 to the right of b_i, and let $b_{s_1} \cdots b_{s_r} b_{t_1} \cdots b_{t_r} = 3^r 2^r$ be the subsequence of all 3 and 2 entries bracketed with each other for which $s_k < i$ and $i < t_k$ for all k. Then in b', we have that b'_{s_r} brackets with b'_i rather than b'_{t_1}, and $b'_{s_{r-1}}$ brackets with b'_{t_1}, and so on, leaving b'_{t_r} a new unbracketed 2. Thus we always have $\varphi_2(b') = \varphi_2(b)+1$. Furthermore, since the number of unbracketed 3 entries remains unchanged, we have $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$.

For the commutativity relation, note that since $j > i$, so $b'_j = 2$ is still the rightmost unbracketed 2 in b' and $b''_i = 1$ is the leftmost 1 in b'' without a 2 to the left of b''_i. Thus both $f_2(f_{-1}(b))$ and $f_{-1}(f_2(b))$ are formed by changing b_i to 2 and b_j to 3. Hence

$$f_2(f_{-1}(b)) = f_{-1}(f_2(b))$$

as desired.

(2) Assume $\varphi_2(b) = 0$ and $\varphi_{-1}(b) > 0$, so that $b' = f_{-1}(b)$ is defined but $f_2(b)$ is not. Then there is an entry $b_i = 1$ with no 1 or 2 left of it that changes to 2 to form b'. There are also no unbracketed 2 entries in the $2,3$ bracketing.

We consider two cases. First, suppose that every 3 to the left of b_i in b is bracketed with some 2 to its right. Then in b' with $b'_i = 2$, the bracketed pairs for the entries $b'_{s_i} = 3$ to the left of b'_i shift left as in part (1) above, leaving a new unbracketed 2 and exactly the same number of unbracketed 3 entries. Thus $\varphi_2(b') = 1$ and $\varepsilon_2(b') = \varepsilon_2(b)$ in this case.

If instead there is an unbracketed 3 to the left of b_i, then this 3 becomes bracketed with a 2 (after the same shift in bracketed pairs) and we have $\varphi_2(b') = 0$ and $\varepsilon_2(b') = \varepsilon_2(b) - 1$, as desired.

(3) Suppose $\varepsilon_2(b), \varepsilon_{-1}(b) > 0$. Then the leftmost 1 or 2 in b is $b_i = 2$ for some i, and $b' := e_{-1}(b)$ is formed by changing b_i to 1. Since $e_2(b)$ is defined, there also exists a leftmost unbracketed 3, say $b_j = 3$.

We consider two cases. First suppose $\varphi_2(b) = 0$, meaning that every 2 is bracketed in the $2, 3$-bracketing of b. Then in particular b_i is bracketed; let $b_{s_1} \cdots b_{s_r} b_i b_{t_1} \cdots b_{t_{r-1}} = 3^r 2^r$ be the subsequence consisting of all bracketed 3's (b_{s_i}) to the left of b_i along with the entries they are bracketed with ($b_{t_{r-i}}$ where $t_0 = i$). Then after lowering b_i to 1 to form b', we have that b'_{s_i} brackets with $b'_{t_{r-i+1}}$ for $i \geqslant 2$, and b'_{s_1} is an unbracketed 3. All other bracketed pairs are the same as in b, so there is only one more 3 among the unbracketed letters. It follows that $\varepsilon_2(b') = \varepsilon_2(b) + 1$ and $\varphi_2(b') = \varphi_2(b) = 0$.

For the second case, suppose $\varphi_2(b) > 0$. Then there is some unbracketed 2 in b; let b_k be the leftmost unbracketed 2. Note that $k \geqslant i$ because b_i is the leftmost 2, and note also that $k < j$ because b_j is the leftmost unbracketed 3. Thus $i < j$.

Now, lowering b_i to 1 to form b' results in shifting the bracketing as in the cases above, which makes b'_k be bracketed (and all other bracketings the same). Thus there is one less unbracketed 2 in b' as b, and the same number of unbracketed 3's. It follows that $\varepsilon_2(b') = \varepsilon_2(b)$ and $\varphi_2(b') = \varphi_2(b) - 1$. Furthermore, b'_j is still the leftmost unbracketed 3 in b', and so both $e_{-1}e_2(b)$ and $e_2e_{-1}(b)$ are formed by changing b_i to 1 and b_j to 2. The result follows.

□

2.1.3. Explicit description of f_{-i} and e_{-i}. In this section, we give explicit descriptions of $\varphi_{-i}(b)$, $\varepsilon_{-i}(b)$, $f_{-i}b$, and $e_{-i}b$ for J-highest-weight elements $b \in \mathcal{B}^{\otimes \ell}$ for certain $J \subseteq I_0$ (see Proposition 10 and Theorems 13 and 17). We will need these results in Section 2.3 when we characterize certain graphs on the type A components of the queer supercrystal.

LEMMA 8. *Let $i \in I_0$ and $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i-1\}$-highest weight. If the first letter in the $(i, i+1)$-subword of b is $i+1$, then $\varepsilon_{-i}(b) = 1$.*

PROOF. The statement is true for $i = 1$ by Remark 5. Now suppose that by induction on i the statement of the lemma is true for $1, 2, \ldots, i-1$. By Lemma 3, we have $e_{-i} = s_{i-1}s_i e_{-(i-1)} s_i s_{i-1}$. Let $u = i+1$ be the leftmost $i+1$ in b and $v = i$ be the leftmost i in b. By assumption, u appears to the left of v and hence v is bracketed in the $(i, i+1)$-bracketing. Since by assumption b is $\{1, 2, \ldots, i-1\}$-highest weight, in the $(i-1, i)$-bracketing there are no unbracketed i and s_{i-1}

raises all unbracketed $i-1$ to i. In particular, all $i-1$ to the left of v are raised to i since v is the leftmost i. In turn, s_i acts on unbracketed i and $i+1$ in the $(i, i+1)$-bracketing. Since v is bracketed and there are no $i-1$ to the left of v, the first letter in the $(i-1, i)$-subword of $s_i s_{i-1}(b)$ is i. Also, $s_i s_{i-1}(b)$ is $\{1, 2, \ldots, i-2\}$-highest weight. Hence by induction $\varepsilon_{-(i-1)}(s_i s_{i-1}(b)) = 1$, which proves that $\varepsilon_{-i}(b) = 1$. $\square$

The next definition below will be used heavily throughout this section.

DEFINITION 9. *The initial k-sequence of a word $b = b_1 \ldots b_\ell \in \mathcal{B}^{\otimes \ell}$, if it exists, is the sequence of letters $b_{p_k}, b_{p_{k-1}}, \ldots, b_{p_1}$, where b_{p_k} is the leftmost k and b_{p_j} is the leftmost j to the right of $b_{p_{j+1}}$ for all $1 \leqslant j < k$.*

Let $i \in I_0$ and $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight with $\mathrm{wt}(b)_{i+1} > 0$, where $\mathrm{wt}(b)_{i+1}$ is the $(i+1)$-st entry in $\mathrm{wt}(b) \in \mathbb{Z}_{\geqslant 0}^{n+1}$. Then note that b has an initial $(i+1)$-sequence, say $b_{p_{i+1}}, b_{p_i}, \ldots, b_{p_1}$. Also let $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ be the initial i-sequence of b. Note that $p_{i+1} < p_i < \cdots < p_1$ and $q_i < q_{i-1} < \cdots < q_1$ by the definition of initial sequence. Furthermore either $q_j = p_j$ or $q_j < p_{j+1}$ for all $1 \leqslant j \leqslant i$.

PROPOSITION 10. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$. Then:*

(a) *$\varepsilon_{-i}(b) = 1$ if and only if $\mathrm{wt}(b)_{i+1} > 0$ and $p_j = q_j$ for at least one $j \in \{1, 2, \ldots, i\}$.*

(b) *$\varphi_{-i}(b) = 1$ if and only if $\mathrm{wt}(b)_i > 0$ and either $\mathrm{wt}(b)_{i+1} = 0$ or $p_j \neq q_j$ for all $j \in \{1, 2, \ldots, i\}$.*

EXAMPLE 11. *Take $b = 1331242312111$ and $i = 3$. Then $p_4 = 6, p_3 = 8, p_2 = 10, p_1 = 11$ and $q_3 = 2, q_2 = 5, q_1 = 9$. We indicate the chosen letters p_j by underlines and q_j by overlines: $b = 1\overline{3}31\overline{2}\underline{4}2\underline{3}\overline{1}\underline{2}\underline{1}11$. Since no letter has a both an overline and underline (meaning $p_j \neq q_j$ for all j), we have $\varphi_{-3}(b) = 1$.*

PROOF OF PROPOSITION 10. Let us first prove claim (a) for $i = 1$. If $\mathrm{wt}(b)_2 = 0$, then certainly $\varepsilon_{-1}(b) = 0$ since by definition e_{-1} changes a 2 into a 1. If $\mathrm{wt}(b)_2 > 0$, then q_1 is the position of the leftmost 1, p_2 is the position of the leftmost 2, and p_1 is the position of the first 1 after this 2. If $p_1 = q_1$, there is no 1 to the left of the leftmost 2. By definition in this case $\varepsilon_{-1}(b) = 1$. If on the other hand $q_1 < p_2$, the leftmost 1 is before the leftmost 2 and hence $\varepsilon_{-1}(b) = 0$. This proves the claim.

Now assume by induction that claim (a) is true for up to $i-1$. If $\mathrm{wt}(b)_{i+1} = 0$, then $\varepsilon_{-i}(b) = 0$ since e_{-i} changes the weight by the simple root α_i. Otherwise assume that $\mathrm{wt}(b)_{i+1} > 0$.

If $p_i = q_i$, the first letter i or $i+1$ is the $i+1$ in position $p_{i+1} < p_i = q_i$. Hence by Lemma 8 we have $\varepsilon_{-i}(b) = 1$.

If $q_i < p_i$ (and hence automatically $q_i < p_{i+1}$), recall that by Lemma 3 we have $e_{-i} = s_{i-1}s_i e_{-(i-1)} s_i s_{i-1}$. The operator s_{i-1} leaves the letter $i-1$ in positions q_{i-1} and p_{i-1} unchanged since these letters are bracketed with i in positions q_i and p_i, respectively. All $i-1$ to the left of position q_{i-1} are unbracketed and since b is $\{1, 2, \ldots, i\}$-highest weight, s_{i-1} changes all of these $i-1$ to i. In $s_{i-1}b$ there are possibly new letters i between positions p_{i+1} and p_i; the $i+1$ in position p_{i+1} brackets with the leftmost of these in position $p_{i+1} < p'_i \leqslant p_i$. The operator s_i on $s_{i-1}b$ changes all letters i to the left of position p'_i to $i+1$. Hence $\mathrm{wt}(s_i s_{i-1} b)_i > 0$, $s_i s_{i-1} b$ is $\{1, 2, \ldots, i-1\}$-highest weight with sequences with respect to $i-1$ given by $p'_i > p_{i-1} > \cdots > p_1$ and $q_{i-1} > q_{i-2} > \cdots > q_1$. Claim (a) now follows by induction on i.

If b is $\{1, 2, \ldots, i\}$-highest weight and $\mathrm{wt}(b)_i > 0$, we must have $\varphi_{-i}(b) + \varepsilon_{-i}(b) = 1$. Hence $\varphi_{-i}(b) = 1$ precisely when $\varepsilon_{-i}(b) = 0$, proving (b). □

Recall that in a queer supercrystal B an element $b \in B$ is *highest-weight* if $e_i(b) = 0$ for all $i \in I_0 \cup I_-$, where $I_0 = \{1, 2, \ldots, n\}$ and $I_- = \{-1, -2, \ldots, -n\}$. Additionally, within the weight lattice $\Lambda = \mathbb{Z}^{n+1}$, we say that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ is a *strict partition* if the following hold:

- $\alpha_i \geqslant \alpha_{i+1}$ for all $1 \leqslant i \leqslant n$ and $\alpha \in \mathbb{Z}^{n+1}_{\geqslant 0}$,
- For all $1 \leqslant i \leqslant n$, either $\alpha_{i+1} = 0$ or $\alpha_i > \alpha_{i+1}$.

Proposition 12. *[**GJK$^+$14**, Proposition 1.13] Let $b \in \mathcal{B}^{\otimes \ell}$ be highest weight. Then* $\mathrm{wt}(b)$ *is a strict partition.*

Proof. Let b be highest weight, so that $\mathrm{wt}(b)_i \geqslant \mathrm{wt}(b)_{i+1}$ for all i and $\mathrm{wt}(b) \in \mathbb{Z}^{n+1}_{\geqslant 0}$. Suppose that $\mathrm{wt}(b)_i = \mathrm{wt}(b)_{i+1} > 0$ for some i, meaning that b contains the same number of letters i and $i+1$. Since all letters i and $i+1$ must be bracketed in the $(i, i+1)$-bracketing, this means that the first letter in the $(i, i+1)$-subword of b is the letter $i+1$. Then by Lemma 8, $\varepsilon_{-i}(b) = 1$, which means that b is not highest weight. Hence, either $\mathrm{wt}(b)_i > \mathrm{wt}(b)_{i+1}$ or that $\mathrm{wt}(b)_{i+1} = 0$ for all i, implying that $\mathrm{wt}(b)$ is a strict partition. $\square$

Next, we provide an explicit description of $f_{-i}(b)$ for $i \in I_0$, when b is $\{1, 2, \ldots, i\}$-highest weight. Recall that the sequence $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ is the leftmost sequence of letters $i, i-1, \ldots, 1$ from left to right. Set $r_1 = q_1$ and recursively define $r_j < r_{j-1}$ for $1 < j \leqslant i$ to be maximal such that $b_{r_j} = j$. Note that by definition $q_j \leqslant r_j$. Let $1 \leqslant k \leqslant i$ be maximal such that $q_k = r_k$.

Theorem 13. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$ and $\varphi_{-i}(b) = 1$ (see Proposition 10). Then $f_{-i}(b)$ is obtained from b by changing $b_{q_j} = j$ to $j-1$ for $j = i, i-1, \ldots, k+1$ and $b_{r_j} = j$ to $j+1$ for $j = i, i-1, \ldots, k$.*

Example 14. *Let us continue Example 11 with $b = 1331242312111$ and $i = 3$. We overline b_{q_j} and underline b_{r_j}, so that $b = 1\overline{3}\underline{3}1\overline{2}4\underline{2}3\overline{\underline{1}}2111$. From this we read off $q_3 = 2, q_2 = 5, q_1 = 9$, $r_3 = 3, r_2 = 7, r_1 = 9$, $k = 1$ and $f_{-3}(b) = 1241143322111$.*

As another example, take $b = 545423321211$ in the $\mathfrak{q}(6)$-crystal $\mathcal{B}^{\otimes 12}$ and $i = 5$. Again, we overline b_{q_j} and underline b_{r_j}, so that $b = \overline{54}\underline{54}2\overline{3}\underline{3}\overline{\underline{21}}211$. This means that $q_5 = 1$, $q_4 = 2$, $q_3 = 6$, $q_2 = 8$, $q_1 = 9$, $r_5 = 3$, $r_4 = 4$, $r_3 = 7$, $r_2 = 8$, $r_1 = 9$, $k = 2$, and $f_{-5}(b) = 436522431211$.

Proof of Theorem 13. We prove the claim by induction on i. For $i = 1$, since by assumption $\varphi_{-1}(b) = 1$, the first letter in the subword of b of letters in $\{1, 2\}$ is a 1. This 1 is in position $q_1 = r_1$ and changes to 2, which proves the claim.

Now assume that the claim is true for $f_{-1}, \ldots, f_{-(i-1)}$. Recall that by Lemma 3 we have $f_{-i} = s_{i-1} s_i f_{-(i-1)} s_i s_{i-1}$. Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight. Applying s_{i-1} to b changes all unbracketed $i-1$ in the $(i-1, i)$-bracketing to i. Subsequently applying s_i changes all unbracketed i in the $(i, i+1)$-bracketing to $i+1$. It is not hard to see that the resulting word is $\{1, \ldots, i-1\}$-highest weight, so we can apply the inductive hypothesis in order to apply $f_{-(i-1)}$.

In the notation for Proposition 10, we have either $\mathrm{wt}(b)_{i+1} = 0$ or $q_i < p_{i+1}$ and $q_{i-1} < p_i$ since $\varphi_{-i}(b) = 1$. In particular this means that if p_{i+1} is defined and $p_{i+1} < q_{i-1}$, no letter i lies between p_{i+1} and q_{i-1} since otherwise $p_i < q_{i-1}$ contradicting the requirement $q_{i-1} < p_i$. This implies that all $i-1$ and i in the positions to the left of position q_{i-1} become $i+1$ when applying $s_i s_{i-1}$. The letter $i-1$ in position q_{i-1} remains $i-1$ under $s_i s_{i-1}$ since it is bracketed with an i. Denote the sequences for $f_{-(i-1)}$ in $s_i s_{i-1} b$ by $q'_{i-1}, \ldots, q'_1$ and $r'_{i-1}, \ldots, r'_1$ and call k' the maximal index such that $q'_{k'} = r'_{k'}$. By the above arguments, we have $q'_{i-1} = q_{i-1}$. We need to distinguish three cases given by $k = i, i-1$ and $k < i-1$.

Case $k = i$: The claim is that the i in position q_i changes to $i+1$. Since $q_i = r_i$ for $k = i$, there is only one i to the left of the $i-1$ in position r_{i-1}. Since $q_{i-1} \leqslant r_{i-1}$, this implies that all $i-1$ between positions q_{i-1} and r_{i-1} (and including r_{i-1}) change to $i+1$ when applying $s_i s_{i-1}$. This means that $k' = i-1$ and by induction $f_{-(i-1)}$ changes the $i-1$ in position q_{i-1} to i. Hence under $s_{i-1} s_i$, the letter in position q_i remains an $i+1$ and all other letters $i+1$ and i return to their original value. This proves the claim.

Case $k = i-1$: In this case, we have at least two i to the left of position $q_{i-1} = r_{i-1}$ and there is no $i-1$ between positions q_{i-1} and $r_{i-2} \geqslant q_{i-2}$. Since $s_i s_{i-1}$ lifts all i to the left of position q_{i-1} to $i+1$, but leaves the $i-1$ in position q_{i-1} and possible $i-2$ in positions q_{i-2} and r_{i-2}, we have $k' = i-1$. Hence by induction $f_{-(i-1)}$ changes the $i-1$ in position $q'_{i-1} = q_{i-1}$ to i. When applying $s_{i-1} s_i$ to $f_{-(i-1)} s_i s_{i-1} b$, the $i+1$ in position r_i remains an $i+1$ since it is now bracketed with the i in position q_{i-1} or an i to its left. In addition, the $i+1$ in position q_i becomes an $i-1$ since the i in position q_{i-1} is now bracketed with the previous bracketing partner of letter in position q_i in b, causing it to drop to $i-1$. This proves the claim for $k = i-1$.

Case $k < i-1$: In this case $q_i < r_i$ and $q_{i-1} < r_{i-1}$, so that there are at least two i to the left of position r_{i-1} and at least two $i-1$ between positions q_i and $r_{i-2} \geqslant q_{i-2}$. By the arguments above, all i to the left of position q_{i-1} become $i+1$ under $s_i s_{i-1}$, the letter $i-1$ in position q_{i-1} remains $i-1$ and $q'_{i-1} = q_{i-1} < r'_{i-1} \leqslant r_{i-1}$. Also, since $s_i s_{i-1}$ leaves all letters $i-2$ and smaller untouched, we have $q'_j = q_j$ and $r'_j = r_j$ for $1 \leqslant j < i-1$. Hence by induction $f_{-(i-1)}$ changes the letter in position $q_{i-1} = q'_{i-1}$ to $i-2$ and the letter in position r'_{i-1} to i, in addition to the letters in positions q_j, r_j for $j < i-1$. Next applying $s_{i-1} s_i$ changes the letter in position r_{i-1} to i since it is now bracketed with the $i-1$ in position r_{i-2}. The letters $i+1$ in positions $r'_{i-1} < p < r_{i-1}$ are changed back to $i-1$ since they are not bracketed. If $r'_{i-1} < r_{i-1}$, then the letter i in position r'_{i-1} changes to $i-1$ since it is also not bracketed. The letter in position $q_{i-1} = q'_{i-1}$ remains $i-2$. The letter $i+1$ in position r_i is bracketed with the i in position r'_{i-1} in $f_{-(i-1)} s_i s_{i-1} b$ and hence remains $i+1$ in $s_{i-1} s_i f_{-(i-1)} s_i s_{i-1} b$. The letters $i+1$ between positions q_i and r_i in $f_{-(i-1)} s_i s_{i-1} b$

return to their original value i under $s_{i-1}s_i$ since they are bracketed with $i-1$ to the right. The letter in position q_i lost its bracketing partner since the $i-1$ in position q_{i-1} became $i-2$. Hence the letter in position q_i becomes $i-1$, proving the claim. $\square$

COROLLARY 15. *Let $b \in \mathcal{B}^{\otimes \ell}$ be J-highest weight for $\{1,2,\ldots,i\} \subseteq J \subseteq I_0$ and $\varphi_{-i}(b) = 1$ for some $i \in I_0$. Then:*

(1) *Either $f_{-i}(b) = f_i(b)$ or $f_{-i}(b)$ is J-highest weight.*
(2) *$f_{-i}(b)$ is I_0-highest weight only if $b = f_{i+1}f_{i+2}\cdots f_{h-1}u$ for some $i < h \leqslant n+1$ and u a I_0-highest weight element.*

PROOF. We begin by proving (1). By Theorem 13, in $f_{-i}(b)$ the letters b_{q_j} are changed from j to $j-1$ for $j = i, i-1, \ldots, k+1$ and b_{r_j} are changed from j to $j+1$ for $j = i, i-1, \ldots, k$. Hence $f_{-i}(b)$ is not J-highest weight if and only if either there is an $i+1$ to the left of position q_i that is no longer bracketed with an i or the letter $k+1$ in position r_k is no longer bracketed with a k.

First assume that $k < i$. Since k is maximal such that $q_k = r_k$, there must be at least two $k+1$ to the left of position q_k in b, one in position q_{k+1} and one in position r_{k+1}. Since b is J-highest weight, both of these $k+1$ must be bracketed with a k to their right in b, which implies that there is a k to the right of position q_k that is bracketed with the $k+1$ in position q_{k+1} in b. In $f_{-i}(b)$, the letter $k+1$ in position q_{k+1} changes to k, and hence the new $k+1$ in position $q_k = r_k$ is bracketed with the k to its right.

Since by assumption $\varphi_{-i}(b) = 1$, we have by Proposition 10 that either $\mathrm{wt}(b)_{i+1} = 0$ (in which case there cannot be an $i+1$ to the left of position q_i in b) or $p_j \neq q_j$ for all $j \in \{1,2,\ldots,i\}$. The condition $p_i \neq q_i$ implies that $q_i < p_{i+1}$, so that there cannot be a letter $i+1$ to the left of position q_i. This proves that $f_{-i}(b)$ is J-highest weight when $k < i$.

Next assume that $k = i$. In this case $f_{-i}(b)$ differs from b by changing the letter i in position q_i to $i+1$. If there is a letter i to the right of position q_i that is not bracketed with a letter $i+1$, then the new $i+1$ in position q_i will bracket with this i in $f_{-i}(b)$ (or to the left of this i) and hence $f_{-i}(b)$ is J-highest weight. Otherwise, there is no letter i to the right of position q_i in b that is not bracketed with an $i+1$ and therefore $f_i(b) = f_{-i}(b)$. This proves claim (1).

The above arguments also show that $f_{-i}(b)$ can only be I_0-highest weight if either b is I_0-highest weight or $\varepsilon_j(b) = 0$ for $j \in I_0 \setminus \{i+1\}$ and the new letter $i+1$ in position r_i in $f_{-i}(b)$ is bracketed with a letter $i+2$ in b. Such a b is precisely of the form $b = f_{i+1}f_{i+2}\cdots f_{h-1}u$ proving claim (2). $\square$

Next, we describe e_{-i} on a $\{1, 2, \ldots, i\}$-highest weight element b. We again use the initial $(i+1)$-sequence $b_{p_{i+1}}, b_{p_i}, \ldots, b_{p_1}$ in b.

We also need the notion of *cyclically scanning leftwards* for a letter t starting at an entry b_j. By this we mean choosing the rightmost t to the left of b_j, if it exists, or else the rightmost t in the entire word (i.e., "wrapping around" the edge of the word).

We define the *k-bracketed entries* of a word b as follows. Every k in b is k-bracketed, and for $j = k-1, k-2, \ldots, 1$, we recursively determine which j's in b are k-bracketed by considering the subword of only the k-bracketed $(j+1)$'s and all j's, and performing an ordinary crystal bracketing on this subword. The j's that are bracketed in this process are the k-bracketed j's.

Example 16. *In the word*

$$142334122311322111,$$

to obtain the 4*-bracketed letters we first mark all* 4*'s as* 4*-bracketed:*

$$1\mathbf{4}233\mathbf{4}122311322111$$

and then bracket these with 3*'s and mark the bracketed* 3*'s as being* 4*-bracketed:*

$$1\mathbf{4}2\mathbf{3}3\mathbf{4}122\mathbf{3}11322111.$$

We then consider only the boldface 3*'s and all the* 2*'s and bracket them to obtain the* 4*-bracketed* 2*'s:*

$$1\mathbf{4}2\mathbf{3}3\mathbf{4}1\mathbf{2}2\mathbf{3}113\mathbf{2}2111$$

Finally we bracket these boldface 2*'s with the* 1*'s to obtain:*

$$1\mathbf{4}2\mathbf{3}3\mathbf{4}1\mathbf{2}2\mathbf{3}\mathbf{1}13\mathbf{2}2\mathbf{1}11$$

The boldface letters above are precisely the 4*-bracketed letters in this word.*

We now have the tools to describe the application of e_{-i} to an $\{1, 2, \ldots, i\}$-highest weight word.

THEOREM 17. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$ and $\varepsilon_{-i}(b) = 1$ (see Proposition 10). Let $b_{p_{i+1}}, \ldots, b_{p_1}$ be the initial $(i+1)$-sequence of b. Then $e_{-i}(b)$ is obtained from b by the following algorithm:*

- *Change b_{p_j} from j to $j-1$ for $j = i+1, i, \ldots, 3, 2$ to form a word $c^{(1)}$.*
- *Cyclically scan left in $c^{(1)}$ starting just to the left of position p_1 for a 1 that is not i-bracketed in $c^{(1)}$. Change that 1 to 2 to form a word $c^{(2)}$. In $c^{(2)}$, continue cyclically scanning from just to the left of the previously changed entry for a 2 that is not i-bracketed in $c^{(2)}$, and change it to 3. Continue this process until an $i-1$ changes into an i; the resulting word $c^{(i)}$ is $e_{-i}(b)$.*

PROOF. We will prove this by induction on i. For $i = 1$ the algorithm simply changes the leftmost 2 to a 1 as required, since the second step is vacuous in this case.

Assume the statement is true for i and let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i+1\}$-highest weight. Recall that $e_{-(i+1)} = s_i s_{i+1} e_{-i} s_{i+1} s_i$ by Lemma 3. We will analyze each step of applying $s_i s_{i+1} e_{-i} s_{i+1} s_i$ to b and show that it matches the desired algorithm.

Let $b_{p_{i+2}}, b_{p_{i+1}}, b_{p_i}, \ldots, b_{p_2}, b_{p_1}$ be the initial $(i+2)$-sequence of b. Since $e_i b = 0$, applying s_i to b simply changes all unbracketed i entries in the $(i, i+1)$-pairing to $i+1$. Note that b_{p_i} itself must be bracketed with an $i+1$ in b, for if it is not then $b_{p_{i+1}}$ is paired with an earlier i to its right, contradicting the definition of b_{p_i}. Thus b_{p_i} is still i in $s_i b$. Note also that $s_i b$ still satisfies $e_{i+1} s_i b = 0$.

Let $b' = s_{i+1} s_i b$. Note that any $i+1$ to the left of $b_{p_{i+2}}$ in $s_i b$ is not bracketed with an $i+2$ since $b_{p_{i+2}}$ is the leftmost $i+2$. Thus every $i+1$ left of $b_{p_{i+2}}$ (including those i's that changed to $i+1$ from b) changes to $i+2$ to form b', along with any other unpaired $i+1$. Let $b_{t_{i+1}}$ be the leftmost $i+1$ between $b_{p_{i+2}}$ and $b_{p_{i+1}}$ in $s_i b$. Then $b_{t_{i+1}}$ is either equal to $b_{p_{i+1}}$ or was an i in b. Furthermore, $b_{t_{i+1}}$ is still $i+1$ in $b' = s_{i+1} s_i b$ since it must be paired with either $b_{p_{i+2}}$ itself or some $i+2$ to the right of $b_{p_{i+2}}$.

Now consider $e_{-i} b'$. By the induction hypothesis, this can be computed by first lowering the entries of the initial $(i+1)$-sequence $b'_{p'_{i+1}}, b'_{p'_i}, \ldots, b'_{p'_1}$ appropriately to form a word $c'^{(1)}$, then cyclically raising some non-i-bracketed entries $1, 2, 3, \ldots, i-1$ in order to form words $c'^{(2)}, \ldots, c'^{(i)}$. We will show that $p'_j = p_j$ for $j \leqslant i$, and that the same entries $1, 2, \ldots, i-1$ are changed as would be changed in the $e_{-(i+1)}$ algorithm applied to b.

For the first claim, it suffices to show that $p'_i = p_i$. Note that $b'_{p'_{i+1}}$ may be to the left of $b_{p_{i+1}}$, but it is to the right of $b_{p_{i+2}}$ by the above analysis. If $p'_{i+1} = p_{i+1}$ we are done, so suppose $p_{i+2} < p'_{i+1} < p_{i+1}$. Assume by contradiction that there is an entry $b'_a = i$ between positions p'_{i+1} and p_i in b'. Then we further have $p'_{i+1} < a < p_{i+1}$ by the definition of b_{p_i} and b'. It follows that b_a is an i in b that is bracketed with an $i+1$, since applying s_i kept it an i. But then by the definition of p_{i+1}, the entry $b_c = i+1$ that brackets with b_a in b is to the left of position p_{i+2}. Thus $b_{p'_{i+1}}$ itself was a bracketed i in b, a contradiction. Thus $p'_i = p_i$.

Let $c^{(j)}$ be the word in the definition of $e_{-(i+1)}$ acting on b and $c'^{(j)}$ the word in the definition of e_{-i} on b'. Similarly, let t_j (resp. t'_j) be the position of the chosen j in $c^{(j)}$ (resp. $c'^{(j)}$) that is raised to $j+1$. We now wish to show that, for any $j \leqslant i-1$, we have $t'_j = t_j$.

We first show this for $j = 1$. Note that since $p_2 = p'_2$ (assuming $i \geqslant 2$, since otherwise we are done) the same entries are equal to 1 in both $c = c^{(1)}$ and $c' = c'^{(1)}$. Moreover, $p_1 = p'_1$, so we start searching cyclically left for a 1 in the same position in both. It therefore suffices to show that an entry $c_x = 1$ is $(i+1)$-bracketed in c if and only if $c'_x = 1$ is i-bracketed in c'. Note that the i's in c that are bracketed with $i+1$'s are precisely either:

- $c_{p'_{i+1}}$, or
- an i that was bracketed with an $i+1$ in b.

But since c' is formed by applying s_i to b (which changes all unbracketed i's to $i+1$'s), then s_{i+1} (which does not change any i's), then lowering certain entries, where $b_{p'_{i+1}}$ is the only one that becomes a new i, the above characterization gives precisely *all* i's in c'. Since the $1, 2, \ldots, i-1$ entries are the same in both c and c', it follows that an entry is $(i+1)$-bracketed in c if and only if it is i-bracketed in c'.

It now follows that $t_1 = t'_1$, and inductively we can conclude that $t_j = t'_j$ for all $j \leqslant i-1$. Thus if we apply $s_i s_{i+1}$ to $c'^{(i)}$ to obtain $e_{-(i+1)}b$, the entries less than or equal to $i-1$ match those of $c^{(i+1)}$, the result of the algorithm applied to b. Furthermore, since s_i, s_{i+1}, and e_{-i} only change letters less than or equal to $i+2$, the entries larger than $i+2$ also match.

It remains to consider the entries equal to i, $i+1$, and $i+2$. For $i+2$, the application of s_{i+1} to $s_i b$ changes all unbracketed $i+1$ entries in $s_i b$ to $i+2$, and e_{-i} changes the single entry $b'_{p'_{i+1}} = i+1$ to i and otherwise does not affect the $i+1$ or $i+2$ entries. In the $(i+1, i+2)$-bracketing in b', $b'_{p_{i+2}}$ is the leftmost bracketed $i+2$, and $b'_{p'_{i+1}}$ is the first $i+1$ after it, so removing $b'_{p'_{i+1}}$ from the $(i+1, i+2)$-subword leaves the $i+2$ in position p_{i+2} unbracketed, with all other bracketed $(i+2)$'s remaining bracketed. It follows that applying s_{i+1} to $e_{-i}s_{i+1}s_i b$ lowers the $i+2$ in position p_{i+2} to $i+1$, along with any $i+2$ that was raised in the first s_{i+1} step. Therefore, the $i+2$ entries in $s_{i+1}e_{-i}b'$, and hence in $s_i s_{i+1} e_{-i} b' = e_{-(i+1)} b$, match those in the output of the algorithm.

Finally, we consider the $(i, i+1)$-subwords of the words in question. We first analyze how the $(i, i+1)$-subword of $w := s_i b$ differs from that of $w' := s_{i+1}e_{-i}s_{i+1}s_i b$. By inspecting the above analysis, we see that w' differs from w in the following four ways:

- $w'_{p_{i+2}} = i+1$ is a new $i+1$ in the $(i, i+1)$-subword in w' whereas $w_{p_{i+2}} = i+2$ was not in the subword in w.
- $w'_{p'_{i+1}} = i$ whereas $w_{p'_{(i+1)}} = i+1$.
- $w'_{p_i} = i-1$ is no longer in the subword whereas $w_{p_i} = i$ was an i in the subword.
- $w'_{t_{i-1}} = i$ is a new i in the subword, whereas $w_{t_{i-1}} = i-1$.

Note that the last two items above may coincide and cancel each other out if $t_{i-1} = p_i$.

We now apply s_i to both subwords, and analyze how $s_i w' = e_{-(i+1)} b$ differs from $s_i w = b$ in the $(i, i+1)$-subword. In particular, we will show it is the same as how $c^{(i+1)}$ differs from b. Note that the $(i, i+1)$-subword in $c^{(i+1)}$ is formed from that of b by making the following changes:

- A new $i+1$ is inserted in position p_{i+2} ($b_{p_{i+2}} = i+2$ whereas $c^{(i+1)}_{p_{i+2}} = i+1$).
- The $i+1$ in position p_{i+1} is lowered to i.
- The i in position p_i is removed.
- An i is inserted in position t_{i-1}.
- In the current subword, look for the first unbracketed i cyclically left of position t_{i-1}; call this position t_i and change this i to $i+1$.

First, note that there are no $i+1$ entries between $w'_{p_{i+2}} = i+1$ and $w'_{p'_{i+1}} = i$ in w', for if there were, this would contradict the definition of $b_{p_{i+1}}$. It follows that $w'_{p_{i+2}} = i+1$ is bracketed with an i to its right in w', so in $s_iw' = e_{-(i+1)}b$, the entry in position p_{i+2} remains $i+1$. So this is one position in which it differs from b, since $b_{p_{i+2}} = i+2$, so it matches $c^{(i+1)}$ in this position.

Note also that in w, all i's are bracketed with $(i+1)$'s. Applying s_i to w simply changes the unbracketed $i+1$'s back to i's to form b. We now consider two cases.

Case 1: Suppose $p'_{i+1} \neq p_{i+1}$.

We know that s_iw and s_iw' match b and $c^{(i+1)}$, respectively, in position p_{i+2} by the above analysis. For position p'_{i+1}, note that it is an unbracketed $i+1$ in w, so it changes to i in s_iw. It is a bracketed i in w' since it was the first unbracketed $i+1$ to the right of position p_{i+1} in w, so it stays i in s_iw'. Thus they are both equal to i in the results, matching b and $c^{(i+1)}$, which do not differ in this entry.

We now wish to show that the $i+1$ in position p_{i+1} is unbracketed in w' unless it is bracketed via the insertion of the i in position t_{i-1}. In other words, if we make all the changes that define w' from w besides the i in position t_{i-1}, we claim that position p_{i+1} is an unbracketed $i+1$. Indeed, before removing i in position p_i, this $i+1$ in position p_{i+1} is the leftmost $i+1$ that is bracketed with an entry weakly right of position p_i, since the position p_{i+2} entry is bracketed with some i weakly left of position p'_{i+1}. It follows that removing the i in position p_i leaves $b_{p_{i+1}}$ unbracketed, and otherwise all other $i+1$'s are bracketed if and only if they are bracketed in w.

Furthermore, the combination of lowering both p_{i+2} and p'_{i+1} to $i+1$ and i and removing the i in position p_i leaves all i's still bracketed, as they are in w.

Finally, when we put back the new i in position t_{i-1} to form w', there are two subcases: first suppose inserting this i makes some unbracketed $i+1$ to its left become bracketed. Then by the above analysis, this must have been the position of the first unbracketed i in $c^{(i)}$ to the left of t_{i-1}, and this is position t_i, which remains $i+1$ in s_iw'. Applying s_i to w' then turns the remaining unbracketed $i+1$ entries back to i and matches $c^{(i+1)}$. Otherwise, if inserting the i in position t_{i-1} does not bracket any $i+1$ to the left, it creates an unbracketed i in the word, and so the rightmost unbracketed $i+1$ also will not change under applying s_i to w'. This corresponds to the first unbracketed i cyclically left of position t_{i-1} in $c^{(i)}$, and we are done as before.

Case 2: Suppose $p'_{i+1} = p_{i+1}$.

In this case, the analysis matches the above except for the following steps: first, since position p_{i+1} contains a bracketed $i{+}1$ in w, lowering it to i may make some i to its right become unbracketed. (The new i in position p_{i+1} itself is bracketed due to the new $i+1$ in position p_{i+2} as before.)

Then, removing the i in position p_i will make all i's bracketed once again, since b_{p_i} was the first i to the right of position p_{i+1} in b and hence in w. So once again, at the step before inserting t_{i-1}, all i's are bracketed, and an $i+1$ in that matches one in w is bracketed if and only if it is bracketed in the modified word. Thus inserting t_{i-1} has the same effect as above, and we are done. $\square$

We now show that the output of e_{-i} on a $\{1, 2, \ldots, i\}$-highest weight element is itself $\{1, 2, \ldots, i\}$-highest weight if and only if there is no "cycling around the edge" in the cycling step of Theorem 17.

Proposition 18. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \ldots, i\}$-highest weight for $i \in I_0$, with $\varepsilon_{-i}(b) = 1$. Let $t_1, \ldots, t_{i-1}$ be the positions of the $1, 2, \ldots, i-1$ that change to $2, 3, \ldots, i$ respectively in the second step of the computation of $e_{-i}(b)$ (see Theorem 17). Also define $t_0 = p_1$. Then $e_{-i}(b)$ is $\{1, 2, \ldots, i\}$-highest weight if and only if $t_{i-1} < t_{i-2} < \cdots < t_1 < t_0$.*

Proof. First, suppose that it is not the case that $t_{i-1} < t_{i-2} < \cdots < t_1$; let $1 \leqslant k < i$ be the smallest index for which $t_{k-1} \leqslant t_k$, where $t_0 = p_1$. Then in the algorithm for computing $e_{-i}(b)$, after changing a $k-1$ to k in position t_{k-1}, we search cyclically left for a k that is not i-bracketed to find position t_k. Since $t_{k-1} \leqslant t_k$, we cycle around the end of the word, so t_k is the position of the rightmost k that is not i-bracketed.

Any k to the right of t_k is i-bracketed, and we claim that the $k+1$'s that they bracket with in the i-bracketing are all to the right of position t_k as well. Indeed, if one such $k+1$ was to the left of t_k then it should bracket with the k in position t_k instead, a contradiction. Thus the suffix starting at position $t_k + 1$ has at least as many $k+1$'s as k's.

In particular, just after changing each b_{p_r} to $r-1$ in the first step of the algorithm, the resulting word c is still highest weight. It follows that, just after raising t_{k-1} to k, the resulting word is still $\{k\}$-highest weight. It follows that the suffix starting at position $t_k + 1$ at this step has exactly as many $k+1$'s as k's.

Now, if $t_{k+1} < t_k$, changing t_k to $k+1$ and then changing t_{k+1} to $k+2$ leaves the suffix starting at t_k being not $\{k\}$-highest weight in the final word. Thus we are done in this case.

Otherwise, suppose t_{k+1} also cycles, so that $t_{k+1} \geqslant t_k$ and t_{k+1} is the new position of the rightmost $k+1$ that is not i-bracketed after changing t_k to $k+1$. Changing t_{k+1} to $k+2$ could potentially make the word $\{k\}$-highest weight again. In fact, suppose for contradiction that, just after changing t_{k-1} to k, there were a $k+1$ between position t_{k-1} and t_k that makes its suffix not $\{k\}$-highest weight. Then some entry $k+1$ in position $p < t_k$ brackets with the k in position t_k, and since position t_k is not i-bracketed, this $k+1$ is not i-bracketed either. Thus after changing t_k to $k+1$, the $k+1$ in position p is still not i-bracketed and it would be picked up in the search for t_{k+1}, a contradiction to the assumption that $t_{k+1} \geqslant t_k$.

We now, however, can repeat the argument with t_{k+1} and the $(k+1, k+2)$-subword, and so on until we either reach the last step or a non-cycling step, say with index ℓ. At this point we conclude that the final word $e_{-i}(b)$ is not $\{\ell\}$-highest weight.

It follows that if $t_{k-1} \leqslant t_k$ for some k, then $e_{-i}(b)$ is not $\{1, 2, \ldots, i\}$-highest weight.

For the converse, we wish to show that if $t_{i-1} < t_{i-2} < \cdots < t_1 < t_0$ then $e_{-i}(b)$ remains highest weight. Notice that by construction we must have $t_{k-1} \leqslant p_k$ for all $k \leqslant i$.

We first show that the $(1, 2)$-subword remains highest weight in $e_{-i}(b)$ if $t_2 < t_1$. If $i = 1$, then the first 2 simply changes to a 1 and so it is still $\{1\}$-highest weight. So suppose $i \geqslant 2$.

The changes that affect the $(1, 2)$-subword are that b_{p_3} changes from 3 to 2, b_{p_2} changes from 2 to 1, b_{t_1} changes from 1 to 2, and (if $i \geqslant 3$) b_{t_2} changes from 2 to 3. Note that after the first two of these changes, any suffix of the word starting between positions p_3 and p_2 has at least two more 1's than 2's (due to the change in b_{p_2} starting from a highest weight word) and any suffix starting weakly before position p_3 has at least one more 1 than 2.

If $i = 2$, b_{t_1} is an unbracketed 1, so the suffixes before it must in fact have at least two more 1's than 2's even if $t_1 < p_3$. Thus changing b_{t_1} to 2 leaves the word highest weight, and we are done in this case.

If $i \geqslant 3$, b_{t_1} is a 1 that is not i-bracketed to the left of b_{p_2}, and b_{t_2} is the first 2 that is not i-bracketed to the left of t_1 (and necessarily to the left of b_{p_3}). It follows that, after changing them to 2 and 3 respectively, the suffixes all have at least as many 1's as 2's except possibly those starting between position t_2 and t_1. Assume to the contrary that there is a suffix with more 2's than 1's starting between t_2 and t_1; the rightmost such starts at another entry $b_a = 2$ between t_2 and t_1, and this 2 must be i-bracketed by the definition of t_2. But then since b_{t_1} is not i-bracketed, b_a must be bracketed with a 1 between b_a and b_{t_1}; hence the suffix starting at b_a cannot have a higher difference between 2's and 1's than the suffix starting at b_{t_1} after its change, a contradiction. It follows that the $(1,2)$-subword remains highest weight.

Now consider the $(k,k+1)$-subword for some $k \leqslant i-1$. This is changed by $b_{p_{k+2}}, b_{p_{k+1}}, b_{p_k}$ changing from $k+2$ to $k+1$, $k+1$ to k, and k to $k-1$ respectively, and then $b_{t_{k-1}}, b_{t_k}, b_{t_{k+1}}$ changing from $k-1$ to k, k to $k+1$, $k+1$ to $k+2$ respectively.

If we first change b_{p_k} to $k-1$, then we have removed a k from the subword, but since there are no k entries between $b_{p_{k+1}}$ and b_{p_k}, the rightmost suffix that may become not highest weight for k starts at $b_{p_{k+1}}$ itself. Thus changing $b_{p_{k+1}}$ from $k+1$ to k afterwards keeps the $(k,k+1)$-subword being $\{k\}$-highest weight, and in fact any suffix starting to the left of $b_{p_{k+1}}$ at this point has at least one more k than $k+1$. Finally if we change $b_{p_{k+2}}$ to $k+1$, this adds a single $k+1$ to any suffix starting left of this position, so again the word remains $\{k\}$-highest weight. Next, we change $b_{t_{k-1}}$ from $k-1$ to k, which means any suffix starting left of t_{k-1} has at least one more k than $k+1$. The argument for what happens after changing t_k and t_{k+1} now is identical to that of the $(1,2)$-subword above.

Finally, consider the $(i,i+1)$-subword. This is only affected by the changes to $b_{p_{i+1}}$, b_{p_i}, and $b_{t_{i-1}}$. The same argument as above shows that it stays $\{i\}$-highest weight after changing $b_{p_{i+1}}$ and b_{p_i}, and then changing $b_{t_{i-1}}$ to i certainly keeps it $\{i\}$-highest weight as well. This completes the proof. $\square$

From the above proof, we immediately obtain the following corollary.

Corollary 19. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1,2,\ldots,i\}$-highest weight for $i \in I_0$, with $\varepsilon_{-i}(b) = 1$. Let $t_1, \ldots, t_{i-1}$ be the positions of the $1,2,\ldots,i-1$ that change to $2,3,\ldots,i$ respectively in the second step of the computation of $e_{-i}(b)$ (see Theorem 17). Then if $e_{-i}(b)$ is not $\{1,2,\ldots,i\}$-highest weight, the smallest index ℓ for which $e_{-i}(b)$ is not $\{\ell\}$-highest weight is precisely the smallest index for which $t_{\ell-1} \leqslant t_\ell$ and $t_{\ell+1} < t_\ell$ (where the second inequality is assumed to be vacuously true if $\ell = i-1$).*

In other words, ℓ is the smallest index for which one needs to cycle to get from $t_{\ell-1}$ to t_ℓ, but one does not need to cycle to get from t_ℓ to $t_{\ell+1}$.

PROOF. The proof of Lemma 18 shows that $e_{-i}(b)$ is not $\{\ell\}$-highest weight, and that it is $\{k\}$-highest weight for $k < \ell$ if $t_{k-1} \leqslant t_k \leqslant t_{k+1}$ (i.e., if t_k and t_{k+1} both cycle). $\square$

REMARK 20. *For any word $v \in \mathcal{B}^{\otimes \ell}$, we may combine Proposition 10 and Theorem 17 in order to algorithmically determine the highest weight element in the connected component of the queer supercrystal containing v. In particular, we may first apply as many e_i operators as possible to obtain an I_0-highest weight word v', then apply Proposition 10 to determine whether there is an e_{-i} arrow that we may apply. We can then apply e_{-i} to v' using Theorem 17 and repeat this process on the new word, and so on until we have reached a highest weight word w for the queer supercrystal.*

*Since the operators e_{-i} and e_i determine graphs having unique highest weight elements in each connected component [**GJK$^+$14**, Theorem 1.14], this process will always terminate at the highest weight word in a component. In particular, e_{-1} and e_i for $i \in \{1, 2, \ldots, n\}$ were previously the only operators having a known direct combinatorial algorithm, which are not by themselves sufficient to detect the unique highest weight elements. The algorithm in Theorem 17 therefore allows us to bypass the computational difficulty of conjugating e_{-1} by s_{w_i}.*

2.1.4. Relation among e_{-i}. The main result of this section is Proposition 25, which provides relations between e_{-i} that do and do not yield a $\{1, 2, \ldots, i\}$-highest weight element when acting on an I_0-highest weight element. This proposition will be used in Section 2.3 to deal with "by-pass arrows" in the component graph $G(\mathcal{C})$.

We require several technical lemmas about k-bracketed entries and the e_{-i} operation on highest weight words.

LEMMA 21. *Suppose $b \in \mathcal{B}^{\otimes \ell}$ is $\{1, 2, \ldots, i\}$-highest weight and $1 \leqslant k \leqslant i$. If a letter $b_r = a$ in $b = b_1 b_2 \ldots b_\ell$ is k-bracketed, then b_r is j-bracketed for all $a < j \leqslant k$.*

PROOF. We first show that if an entry a in b is $(a+2)$-bracketed, then it is $(a+1)$-bracketed; for simplicity we set $a = 1$. Let v be the subword of b consisting of only the 2's that are bracketed with a 3 along with all the 1's, and let v' be the subword consisting of all the 1's and 2's. Then v'

can be formed from v by inserting some 2 letters. It therefore suffices to show that any 1 that was bracketed in v is still bracketed after inserting a single 2.

Indeed, let $v_s = 2$ and $v_r = 1$ be a bracketed pair in v. Note that by the definition of the ordinary crystal bracketing rule, the subword $v_s \dots v_r$ has exactly the same number of 2's as 1's, all of them bracketed with some other letter in $v_s \dots v_r$. Therefore, if we insert a 2 to the left or right of this pair, then the pair (v_s, v_r) remains bracketed. If instead we insert it between v_s and v_r, then the interval between v_s and v_r contains strictly more 2's than 1's, and so there is some entry v_t between v_s and v_r for which the subword $v_t \cdots v_r$ is tied; in other words, v_r is now bracketed with some 2 to the right of v_s. Thus v_r stays bracketed after inserting a 2, as desired.

Now, if $b_r = a$ is k-bracketed, then by the above reasoning it is also $(k-1)$-bracketed, since there are weakly more $(k-1)$'s available in this bracketing, and hence weakly more $(k-2)$'s available, and so on. The conclusion follows by induction. $\square$

LEMMA 22. *Let $b \in \mathcal{B}^{\otimes \ell}$ be $\{1, 2, \dots, i\}$-highest weight and $\varepsilon_{-i}(b) = 1$. Let $b_{p_{i+1}}, \dots, b_{p_1}$ be the initial $(i+1)$-sequence of b and c the word obtained by changing b_{p_j} from j to $j-1$. Let $k \leqslant i' \leqslant i$. If b contains a sequence of letters $k-1, k-2, \dots, 1$ before position p_1 that is not i'-bracketed, then c contains a sequence of letters $k-1, k-2, \dots, 1$ before position p_1 that is not i'-bracketed.*

PROOF. Suppose that b contains a sequence S of letters $k-1, k-2, \dots, 1$ in positions $s_{k-1}, \dots, s_1$ respectively, before position p_1, that are not i'-bracketed; take S to be the rightmost such sequence in the sense that it contains the rightmost 1 left of p_1 that is not i'-bracketed, then the rightmost 2 that is not i'-bracketed before that, and so on. Note that $s_1 < p_1$ implies that $s_1 < p_2$ by the definition of p_1. Thus $s_2 < s_1 < p_2$ and so $s_2 < p_3$, and so on, showing that $s_j < p_{j+1}$ for all j. Also note that the initial $(i+1)$-sequence $b_{p_{i+1}}, \dots, b_{p_1}$ is $(i+1)$-bracketed, so that the letters $b_{p_k}, \dots, b_{p_1}$ must also be i'-bracketed by Lemma 21. Since $k \leqslant i' \leqslant i$, this means that the initial $(i+1)$-sequence is disjoint from S and hence S remains unchanged in c.

We now form a sequence S' from S that is not i'-bracketed in c as follows. Consider the largest entry $j \leqslant i'$ for which there exists a j between p_{j+2} and p_{j+1}. Then all bracketing with higher letters remains the same in c, but the letter j between positions p_{j+2} and p_{j+1} becomes bracketed with the letter $j+1$ in position p_{j+2} in the i'-bracketing in c, leaving the letter j in position p_{j+1}

to be an i'-unbracketed j. If $s_j < p_{j+2}$ (or otherwise c_{s_j} does not become bracketed) we keep it in S', and if $p_{j+2} < s_j < p_{j+1}$ and it becomes bracketed, we replace s_j with the first i'-unbracketed position s'_j of a j in c to the right of s_j, to choose the j for S'.

We now show that we can choose a $j-1$ after this step to be in S'. If the j on the previous step did not change, then we repeat this process for $j-1$. If it did change, from s_j to an index s'_j, note that if $s_{j-1} < s'_j$ then the previous $j-1$ is now i'-bracketed with s_j in c as well, so we also have to choose the next $j-1$ to the right. Either way we replace s_{j-1} with the next i'-unbracketed $j-1$, in position s'_{j-1}, if the $j-1$ became bracketed, and we see that $s'_j < s'_{j-1}$. Furthermore, $s'_{j-1} \leqslant p_j$ since we know that p_j becomes an i'-unbracketed $j-1$ as in the case of j above. Continuing in this manner we can form a sequence S' of elements of c that are not i'-bracketed, all weakly to the left of p_2 (and hence strictly before p_1). □

Lemma 23. *Let $b \in \mathcal{B}^{\otimes \ell}$ be I_0-highest weight such that $\varepsilon_{-i}(b) > 0$ for some $i \in I_0$ and $e_{-i}(b)$ is not $\{1, 2, \ldots, i\}$-highest weight. Let k be the smallest index for which $t_{k-1} \leqslant t_k$, where $t_0 = p_1$ and t_j for $j = 1, \ldots, i-1$ are the indices that are raised in the second step of the computation of $e_{-i}(b)$ (such a k exists by Proposition 18). Then we have that $\varepsilon_{-k}(b) = 1$ and $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight.*

Proof. Let $b_{p_{i+1}}, b_{p_i}, \ldots, b_{p_1}$ be the initial $(i+1)$-sequence, $b_{q_i}, b_{q_{i-1}}, \ldots, b_{q_1}$ be the initial i-sequence, $b_{p'_{k+1}}, \ldots, b_{p'_1}$ the initial $(k+1)$-sequence, and $b_{q'_k}, \ldots, b_{q'_1}$ the initial k-sequence of b. Also define c and c' respectively to be the words formed by lowering the entries in the sequences $\{b_{p_j}\}$ or $\{b_{p'_j}\}$ by one, respectively.

Since $\varepsilon_{-i}(b) > 0$, we have by Proposition 10 that $q_a = p_a$ for some $1 \leqslant a \leqslant i$. If a is maximal with this property, then in fact $q_j = p_j$ for all $j \leqslant a$ by the definition of the initial sequences. Assume by contradiction that $\varepsilon_{-k}(b) = 0$. Then again by Proposition 10, $q'_j < p'_j$ for all $j \in \{1, \ldots, k\}$. Furthermore, $p'_j \leqslant p_j$ for all $j \leqslant k$ so $q'_j < p_j$ as well.

Suppose that $q'_{a'} = q_{a'}$ for some $1 \leqslant a' \leqslant k$. Then $q'_j = q_j$ for all $j \leqslant a'$ and hence $q'_j = q_j = p_j$ for $j \leqslant \min(a, a')$, contradicting the fact that $q'_j < p_j$ for all j. Hence $q'_j < q_j$ for all $1 \leqslant j \leqslant k$. Thus we also have $q'_j < q_{j+1}$ for all $1 \leqslant j \leqslant k$, for otherwise $b_{q'_j}$ would be the first j after q_{j+1} and we would have $q'_j = q_j$.

The sequence of letters $k, k-1, \ldots, 1$ in positions $q'_k, \ldots, q'_1$ in b is not i-bracketed since the first bracketed $k+1$ in b must be weakly right of position $q_{k+1} > q'_k$. Hence by Lemma 22, the word c also contains a sequence $k, k-1, \ldots, 1$ of letters that are not i-bracketed before position p_1, contradicting the fact that $t_{k-1} \leqslant t_k$. It follows that $\varepsilon_{-k}(b) = 1$.

Next we show that $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight. Note that by the definition of the initial sequences $q'_j \leqslant p'_j \leqslant q_j \leqslant p_j$. Since $\varepsilon_{-i}(b) = 1$ and $\varepsilon_{-k}(b) = 1$, we also have $q'_j = p'_j$ for $j \leqslant a'$ and $q_j = p_j$ for $j \leqslant a$ for some a', a. Suppose $p'_j < q_j$ for all j. Then by a similar argument to that above, in the word c there exists a sequence of positions $t_k < t_{k-1} < \cdots < t_1 < t_0 = p_1$ such that $c_{t_j} = j$ which are not i-bracketed in c. This contradicts the fact that $t_{k-1} \leqslant t_k$. Hence we must have $p'_j = q_j$ for some j and hence $q'_j = p'_j = q_j = p_j$ for $j \leqslant x$ for some $x \geqslant 1$. We claim that $t_j < q'_j$ for all $1 \leqslant j < k$. Indeed, t_1 is to the left of position $p_1 = q'_1$, so that $t_1 < q'_1$. By the definition of p_1 we also cannot have $p_2 < t_1 < p_1$ so in fact $t_1 \leqslant p_2$. The letter in position $q'_j = p_j$ for $1 < j \leqslant x$ in c is $j-1$, so that also $t_j < q'_j$ for $1 < j \leqslant x$. For $j > x$, the letter in position $q'_j < p_j$ in c as well as in b is j. It is k-bracketed in c and b since the first letter k in c and b is in position q'_k. If $t_j \geqslant q'_j$ then since the sequence of entries q'_r for $r \geqslant j$ is k-bracketed but not i-bracketed, we would have $t_k < t_{k-1}$, a contradiction. Thus $t_j < q'_j$.

It follows that the t_j entries are not k-bracketed, so b contains a sequence $k-1, k-2, \ldots, 1$ that is not k-bracketed. By Lemma 22 this means that c' has a sequence $k-1, \ldots, 1$ in positions $t'_{k-1} < \cdots < t'_1$ that is not k-bracketed, proving that $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight by Proposition 18. $\square$

For an element $b \in \mathcal{B}^{\otimes \ell}$, denote by $\uparrow b$ the unique I_0-highest weight element in the same component as b. The next lemma describes the action of $\uparrow$ after an application of e_{-i}.

Lemma 24. *Let $b \in \mathcal{B}^{\otimes \ell}$ be I_0-highest weight such that $\varepsilon_{-i}(b) > 0$ for some $i \in I_0$ and $e_{-i}(b)$ is not $\{1, 2, \ldots, i\}$-highest weight. Let k be as in Lemma 23 and let the sequences p_j and t_j be as in Theorem 17. Then $\uparrow e_{-i}(b)$ can be obtained from b by changing j in position p_j to $j-1$ for $1 < j \leqslant i+1$ and j in position t_j for $1 \leqslant j < k$ to $j+1$, and lowering some letters larger than $i+1$. In particular, the changes in positions t_j for $j \geqslant k$ in $e_{-i}(b)$ are undone by the application of $\uparrow$.*

Proof. By Corollary 19, the smallest index ℓ for which $e_\ell(e_{-i}(b))$ is defined is the first ℓ for which t_ℓ cycled but $t_{\ell+1}$ did not (or does not exist). In particular $\ell \geqslant k$ and all t_j with $k \leqslant j \leqslant \ell$ cycle around the end of the word.

Note that t_ℓ was chosen as the rightmost ℓ that is not i-bracketed (after raising $t_1, \ldots, t_{\ell-1}$). Also recall that the word c formed by lowering the b_{p_j} entries is $\{1, 2, \ldots, i\}$-highest weight, so just before changing t_ℓ the word is still $\{\ell\}$-highest weight. Finally, by assumption t_ℓ is weakly right of $t_{\ell-1}$ (which is the only new ℓ since starting at the word c). Thus, after changing t_ℓ to $\ell+1$, if it bracketed with an ℓ to its right (in the ordinary crystal bracketing) then in fact that ℓ is also not i-bracketed on the previous step, a contradiction since $t_{\ell-1} \leqslant t_\ell$.

Therefore t_ℓ is an unbracketed $\ell+1$ in $e_{-i}(b)$, and since all other $(\ell+1)$'s before it are bracketed with some ℓ, we know that e_ℓ changes it back to an ℓ. After doing so, by the same argument we see that position $t_{\ell-1}$ is now an unbracketed ℓ, so applying $e_{\ell-1}$ changes it back to $\ell - 1$, and so on down to t_k. At this point the resulting word

$$w := e_k \cdots e_{\ell-1} e_\ell(e_{-i} b)$$

is $\{1, 2, \ldots, \ell\}$-highest weight, since t_{k-1} did not cycle and so changing t_k back to k leaves w highest weight at that step.

Now suppose $t_{\ell+1}$ exists (that is, $\ell \leqslant i-2$); then $t_{\ell+1} < t_\ell$, and in w the position t_ℓ is changed back to ℓ. We claim that $e_{\ell+1}$ is defined on w and applying it changes $t_{\ell+1}$ from $\ell+2$ back to $\ell+1$. Indeed, if $t_{\ell+1}$ is bracketed with an $\ell+1$ in w then this $\ell+1$ must be to the right of t_ℓ (since otherwise it would have been a preferred non-i-bracketed choice of $t_{\ell+1}$ in the e_{-i} algorithm). But then this $\ell+1$ is bracketed with an ℓ to its right since w is $\{\ell\}$-highest weight, and then this ℓ similarly contradicts the choice of t_ℓ. Thus $t_{\ell+1}$ is an $\ell+2$ that is not bracketed with an $\ell+1$ after lowering t_ℓ back to ℓ. By the weight changes it must be the only such $\ell+2$ and so applying $e_{\ell+1}$ changes $t_{\ell+1}$ back to $\ell+1$. Continuing in this fashion, we can apply $e_{\ell+2}, e_{\ell+3}$, and so on in that order to change the next entries $t_{\ell+2}$, $t_{\ell+3}$, and so on back to their original values, until some $t_{\ell+r}$ cycles again. Let t_m be the next entry for which t_{m+1} does not cycle (the end of the next block of cycling entries); by the same arguments as above we can now apply e_m, then e_{m-1}, and so on down to $e_{\ell+r}$. Repeating this process on every block of cycling and non-cycling entries yields a $\{1,\ldots,i\}$-highest weight word formed by changing $t_k,\ldots,t_{i-1}$ back to $k,k+1,\ldots,i-1$ respectively. Finally, to finish forming $\uparrow e_{-i}(b)$, only entries larger than $i+1$ may be changed, and the conclusion follows. $\square$

The next proposition will be used in Section 2.3 to deal with "by-pass arrows" in the component graph $G(\mathcal{C})$.

PROPOSITION 25. *Let $b \in \mathcal{B}^{\otimes \ell}$ be I_0-highest weight such that $\varepsilon_{-i}(b) > 0$ for some $i \in I_0$ and $e_{-i}(b)$ is not $\{1,2,\ldots,i\}$-highest weight. Then there exists $1 \leqslant k < i$ such that $\varepsilon_{-k}(b) = 1$, $e_{-k}(b)$ is $\{1,2,\ldots,k\}$-highest weight and*

$$\uparrow e_{-i}(b) = \uparrow e_{-i} \uparrow e_{-k}(b) \quad \text{or} \quad \uparrow e_{-i}(b) = \uparrow e_{-k}(b). \tag{2.1.9}$$

EXAMPLE 26. *Take $b = 343212211 \in \mathcal{B}^{\otimes 9}$, which satisfies $\varepsilon_{-3}(b) > 0$. Then*

$$\uparrow e_{-3}b = e_2e_1e_{-3}b = 332112211 = e_2e_{-3}e_{-1}b = \uparrow e_{-3} \uparrow e_{-1}b.$$

Furthermore, $e_{-1}b = 343112211$ is $\{1\}$-highest weight.

Take $b = 4321321 \in \mathcal{B}^{\otimes 7}$, which satisfies $\varepsilon_{-3}(b) > 0$. Then

$$\uparrow e_{-3}b = e_1e_2e_{-3}b = 3211321 = e_{-3}e_2e_{-1}b = \uparrow e_{-3} \uparrow e_{-1}b.$$

Furthermore, $e_{-1}b = 4311321$ is $\{1\}$-highest weight.

Take $b = 2154321 \in \mathcal{B}^{\otimes 7}$, which satisfies $\varepsilon_{-4}(b) > 0$. Then

$$\uparrow e_{-4}b = e_3e_{-4}b = 3243211 = e_4e_{-3}b = \uparrow e_{-3}b.$$

PROOF OF PROPOSITION 25. Let k be as in Lemma 23. Then the first statements hold for k by Lemma 23 and it only remains to prove (2.1.9). By Lemma 24, $\uparrow e_{-i}b$ changes j in position p_j to $j-1$ for $1 < j \leqslant i+1$ and j in position t_j for $1 \leqslant j < k$ to $j+1$. The changes in positions t_j for $j \geqslant k$ in e_{-i} are undone by $\uparrow$. Some letters bigger than $i+1$ might also be lowered by $\uparrow$.

We use the same notation as in the proof of Lemma 23. There we proved that $t_j < q'_j$ for all $1 \leqslant j < k$. Since $q'_j \leqslant p_j$ and there is no letter j between positions p_{j+1} and p_j in b, it follows that $t_j \leqslant p_{j+1}$ for all $1 \leqslant j < k$. Now suppose that $t_j = p_{j+1}$ for some $1 \leqslant j < k$. We claim that then

$t_{j-1} = p_j$ as well. Let $d-1$ be maximal such that $t_{d-1} = p_d$. Then there has to be a letter $d-1$ in position p in b with $p_{d+1} < p < p_d$, so that the letter $d-1$ in position p_d in c is not i-bracketed. Suppose that there is no letter $d-2$ between positions p and p_{d-1} in b. In this case the letter $d-2$ in position p_{d-1} in c is i-bracketed, so that $t_{d-2} > p_{d-1}$, which contradicts $t_{d-2} \leqslant p_{d-1}$. Continuing this argument, there has to be a sequence of letters $d-1, d-2, \ldots, 1$ between positions p_{d+1} and p_2 that is not i-bracketed. Moreover, letter j in this sequence has to appear before position p_{j+1}. But this means that the letter j in position p_{j+1} for $1 \leqslant j < d$ is not i-bracketed, so that $t_j = p_{j+1}$ for all $1 \leqslant j < d$.

By the arguments above, we have that $t_j = p_{j+1}$ for $1 \leqslant j < d$ for some d and t_j for $j \geqslant d$ is part of a sequence of non k-bracketed letters in b (by the definition of k and the sequence q'_j). Similarly, we have $t'_j = p'_{j+1}$ for $1 \leqslant j < d'$ for some d' and t'_j for $j \geqslant d'$ is part of the same sequence of non k-bracketed letters in b as t_j. Also, $d' \geqslant d$ since $p'_j \leqslant p_j$ for all $1 \leqslant j \leqslant k+1$. In particular, this implies $t_j = t'_j$ for $d' \leqslant j < k$.

Furthermore, before applying the $\uparrow$ operator the entries that change are:

$$
\begin{array}{lll}
\text{In } \uparrow e_{-i}b: & b_{p_j} : j \mapsto j-1 & \text{for } d < j \leqslant i+1 \\[1em]
 & b_{t_j} : j \mapsto j+1 & \text{for } d \leqslant j < i \\[1em]
\text{In } \uparrow e_{-k}b: & b_{p'_j} : j \mapsto j-1 & \text{for } d' < j \leqslant k+1 \\[1em]
 & b_{t'_j} : j \mapsto j+1 & \text{for } d' \leqslant j < k.
\end{array}
$$

Recall also that $p'_j = p_j$ for $1 \leqslant j \leqslant x$ for some $x \geqslant 1$. Denote by $\bar{t}_j$ and $\overline{p}_j$ the selected positions by e_{-i} on the element $\uparrow e_{-k}b$.

First assume that $x = k+1$, so that $p'_j = p_j$ for all $1 \leqslant j \leqslant k+1$. In this case $t'_j = t_j$ for $1 \leqslant j < k$. Furthermore, if in $e_{-k}(b)$ the letter $k+2$ in position p_{k+2} is unbracketed, then in $\uparrow e_{-k}(b)$, the letter $k+2$ in position p_{k+2}, then the letter $k+3$ in position p_{k+3} etc will be lowered. These are the same changes as in $\uparrow e_{-i}(b)$, so that $\uparrow e_{-i}(b) = \uparrow e_{-k}(b)$.

Next assume that $d' < x \leqslant k$ or $x = k+1$ but the letter $k+2$ in position p_{k+2} in $e_{-k}(b)$ is bracketed. We first show that in this case $\overline{p}_j = p_j$ for $x < j \leqslant i+1$. Note that to form $\uparrow e_{-k}(b)$, since $e_{-k}(b)$ is $\{1, 2, \ldots, k\}$-highest weight, we apply $e_{k+1}, e_{k+2}, \ldots, e_r$ in order for some r, so that we lower a $k+2$ to a $k+1$, $k+3$ to $k+2$, and so on until we reach an I_0-highest weight word. Note also that $b_{p'_{k+1}}$ was the entry that lowered from $k+1$ to k, so the $k+2$ that gets lowered, if it exists, is to the left of $p'_{k+1} < p_{k+1}$. Similarly the $k+3$ that gets lowered is left of $p'_{k+2} < p_{k+2}$, and so on, and hence $r < i$ since p_{i+1} is the leftmost $i+1$. It follows that no $i+1$ lowers to an i, and so $\overline{p}_{i+1} = p_{i+1}$. Since the entries lowered by $\uparrow$ are left of p_j for each $j > x$, it follows that $\overline{p}_j = p_j$ for $x < j \leqslant i+1$.

For the sequence $\bar{t}_j$, note that the entries $\overline{p}_j$ that we lower for $j \leqslant x$ cannot be i-bracketed in $\overline{c}$ due to the condition $\overline{p}_{i+1} = p_{i+1}$ shown above, and because $t_{x-1} = t'_{x-1}$, so that t'_{x-1} cannot be between p_{x+1} and p_x. Furthermore, for $x \leqslant j < k$ the letters in positions $\overline{p}_{j+1}$ are all i-bracketed in $\overline{c}$ and $t_j = t'_j < p'_{j+1} < p_{j+1} = \overline{p}_{j+1}$. Also note that $d = d'$ since $p_{j+1} = p'_{j+1} = t'_j$ for $d \leqslant j < d' < x$ and the letter j in position $p_{j+1} = p'_{j+1}$ in c' is not k-bracketed and hence not i-bracketed in c' and c. It follows that

$$\bar{t}_j = \begin{cases} \overline{p}_{j+1} & \text{for } 1 \leqslant j < x, \\ p'_{j+1} & \text{for } x \leqslant j \leqslant k, \end{cases}$$

and for $k < j \leqslant r$, we have that $\bar{t}_j$ is equal to the position of letter $j+1$ that is lowered when applying $\uparrow$ to $e_{-k}(b)$. Hence $\uparrow e_{-i}(b) = \uparrow e_{-i} \uparrow e_{-k}(b)$.

Finally, assume that $x \leqslant d'$. In this case, by a similar argument, we have $\overline{p}_j = p_j$ for $1 \leqslant j \leqslant i+1$ and

$$\bar{t}_j = \begin{cases} \overline{p}_{j+1} & \text{for } 1 \leqslant j < d, \\ t_j & \text{for } d \leqslant j < d', \\ p'_{j+1} & \text{for } d' \leqslant j \leqslant k, \end{cases}$$

and for $k < j \leqslant r$, we have that $\bar{t}_j$ is equal to the position of letter $j+1$ that is lowered when applying $\uparrow$ to $e_{-k}(b)$. Again, we have $\uparrow e_{-i}(b) = \uparrow e_{-i} \uparrow e_{-k}(b)$. $\square$

2.2. Local axioms

In [**AKO18b**, Definition 4.11], Assaf and Oguz give a definition of regular queer supercrystals. In essence, their axioms are rephrased in the following definition, where $\tilde{I} := I_0 \cup \{-1\}$.

Definition 27 (*Local queer axioms*). *Let $\mathcal{C}$ be a graph with labeled directed edges given by f_i for $i \in I_0$ and f_{-1}. If $b' = f_j b$ for $j \in \tilde{I}$ define e_j by $b = e_j b'$.*

LQ1. *The subgraph with all vertices but only edges labeled by $i \in I_0$ is a type A_n Stembridge crystal.*

LQ2. $\varphi_{-1}(b), \varepsilon_{-1}(b) \in \{0, 1\}$ *for all* $b \in \mathcal{C}$.

LQ3. $\varphi_{-1}(b) + \varepsilon_{-1}(b) > 0$ *if* $\mathrm{wt}(b)_1 + \mathrm{wt}(b)_2 > 0$.

LQ4. *Assume $\varphi_{-1}(b) = 1$ for $b \in \mathcal{C}$.*

(a) *If $\varphi_1(b) > 2$, we have*

$$\begin{aligned} f_1 f_{-1}(b) &= f_{-1} f_1(b), \\ \varphi_1(b) &= \varphi_1(f_{-1}(b)) + 2, \\ \varepsilon_1(b) &= \varepsilon_1(f_{-1}(b)). \end{aligned}$$

(b) *If $\varphi_1(b) = 1$, we have*

$$f_1(b) = f_{-1}(b).$$

LQ5. *Assume $\varphi_{-1}(b) = 1$ for $b \in \mathcal{C}$.*

(a) *If $\varphi_2(b) > 0$, we have*

$$\begin{aligned} f_2 f_{-1}(b) &= f_{-1} f_2(b), \\ \varphi_2(b) &= \varphi_2(f_{-1}(b)) - 1, \\ \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)). \end{aligned}$$

(b) *If $\varphi_2(b) = 0$, we have*

$$\begin{aligned} \varphi_2(b) &= \varphi_2(f_{-1}(b)) - 1 = 0, \quad \text{or} \quad & \varphi_2(b) &= \varphi_2(f_{-1}(b)) = 0, \\ \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)), & \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)) + 1. \end{aligned}$$

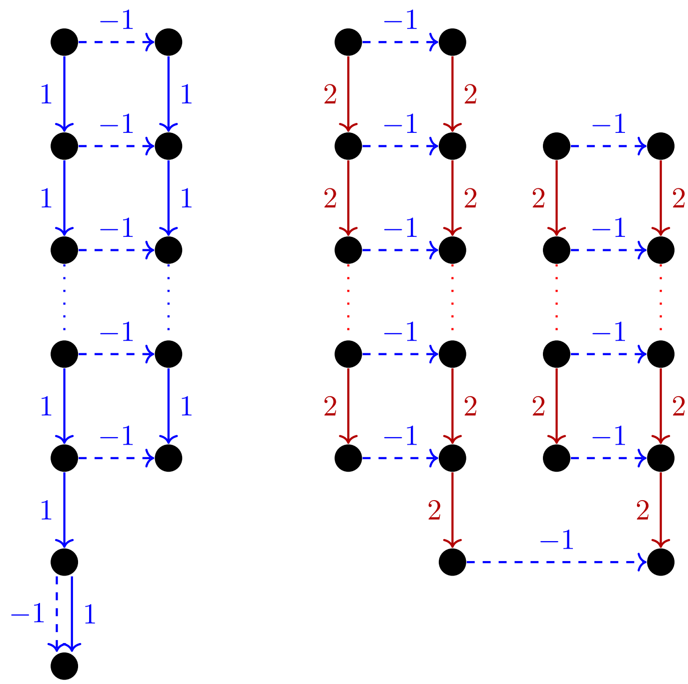


FIGURE 2.2. Illustration of axioms **LQ4** (left) and **LQ5** (right). The (-1)-arrow at the bottom of the right figure might or might not be there.

LQ6. *Assume that* $\varphi_{-1}(b) = 1$ *and* $\varphi_i(b) > 0$ *with* $i \geqslant 3$ *for* $b \in \mathcal{C}$. *Then*

$$f_i f_{-1}(b) = f_{-1} f_i(b),$$

$$\varphi_i(b) = \varphi_i(f_{-1}(b)),$$

$$\varepsilon_i(b) = \varepsilon_i(f_{-1}(b)).$$

Axioms **LQ4** and **LQ5** are illustrated in Figure 2.2.

PROPOSITION 28 ([**AKO18b**]). *The queer supercrystal of words* $\mathcal{B}^{\otimes \ell}$ *satisfies the axioms in Definition 27.*

PROOF. **LQ1** follows by definition. **LQ2** and **LQ3** follow from Remark 5. **LQ4** follows from Lemma 6 and **LQ5** follows from Lemma 7. Finally, **LQ6** is **Q4**. □

In [**AKO18b**, Conjecture 4.16], Assaf and Oguz conjecture that every regular queer supercrystal is a normal queer supercrystal. In other words, every connected graph satisfying the local queer axioms of Definition 27 is isomorphic to a connected component in some $\mathcal{B}^{\otimes \ell}$. We provide a counterexample to this claim in Figure 2.3. In the figure, the I_0-components of the $\mathfrak{q}(3)$-crystal of highest weight $(4,2,0)$ are shown. Some of the f_{-1}-arrows are drawn in green. The remaining arrows can be filled in using the axioms of Figure 2.2 in a consistent manner. If the dashed green arrow from 331131 to 332131 and the dashed green arrow from 331132 to 332132 are replaced by the dashed purple arrow from 331131 to 331231 and the dashed purple arrow from 331132 to 332231, respectively, all axioms of Definition 27 are still satisfied with the remaining f_{-1}-arrows filled in. However, the I_0-component with highest weight element 132121 has become disconnected and hence the two crystals are not isomorphic.

The problem with Axiom **LQ5** illustrated in Figure 2.2 is that the (-1)-arrow at the bottom of the 2-strings is not closed at the top. Hence, as demonstrated by the counterexample in Figure 2.3 switching components with the same I_0-highest weights can cause non-uniqueness. In fact, if $f_{-1}b$ is determined for all $b \in \mathcal{C}$ such that

$$\varphi_i(b) = 0 \quad \text{for all } i \in I_0 \setminus \{1\} \text{ and } \quad \varphi_1(b) = 2, \tag{2.2.1}$$

then, by the relations between f_{-1} and f_i for $i \in I_0$ of Definition 27, f_{-1} is determined on all elements in $\mathcal{C}$. Namely, f_i and f_{-1} commute for $i \neq 1, 2$, so that it is enough to consider $f_{-1}b$ when $\varphi_i(b) = 0$. Similarly, by the right picture in Figure 2.2, once $f_{-1}b$ is determined for b with $\varphi_2(b) = 0$, which are the elements at the bottom of the 2-strings, then $f_{-1}c$ is determined for all c in this picture. And finally, if $f_{-1}b$ is determined for b with $\varphi_1(b) = 2$, which is the element at height 2 in the left picture of Figure 2.2, then f_{-1} is determined on all elements above this b. Furthermore, $f_{-1}(c) = f_1(c)$ when $\varphi_1(c) = 1$. Hence the conditions in (2.2.1) are indeed enough.

LEMMA 29. *Let $v \in \mathcal{B}^{\otimes \ell}$ be an I_0-lowest weight element, that is, $\varphi_i(v) = 0$ for all $i \in I_0$. Then every $b \in \mathcal{B}^{\otimes \ell}$ satisfying* (2.2.1) *is of the form*

$$g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v \qquad \textit{for some } 1 \leqslant j \leqslant k \leqslant n. \tag{2.2.2}$$

Conversely, every $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k \leqslant n$ satisfies (2.2.1).

PROOF. The statement of the lemma is a statement about type A_n crystals and hence can be verified by the tableaux model for type A_n crystals (see for example [**BS17**]). The element v is

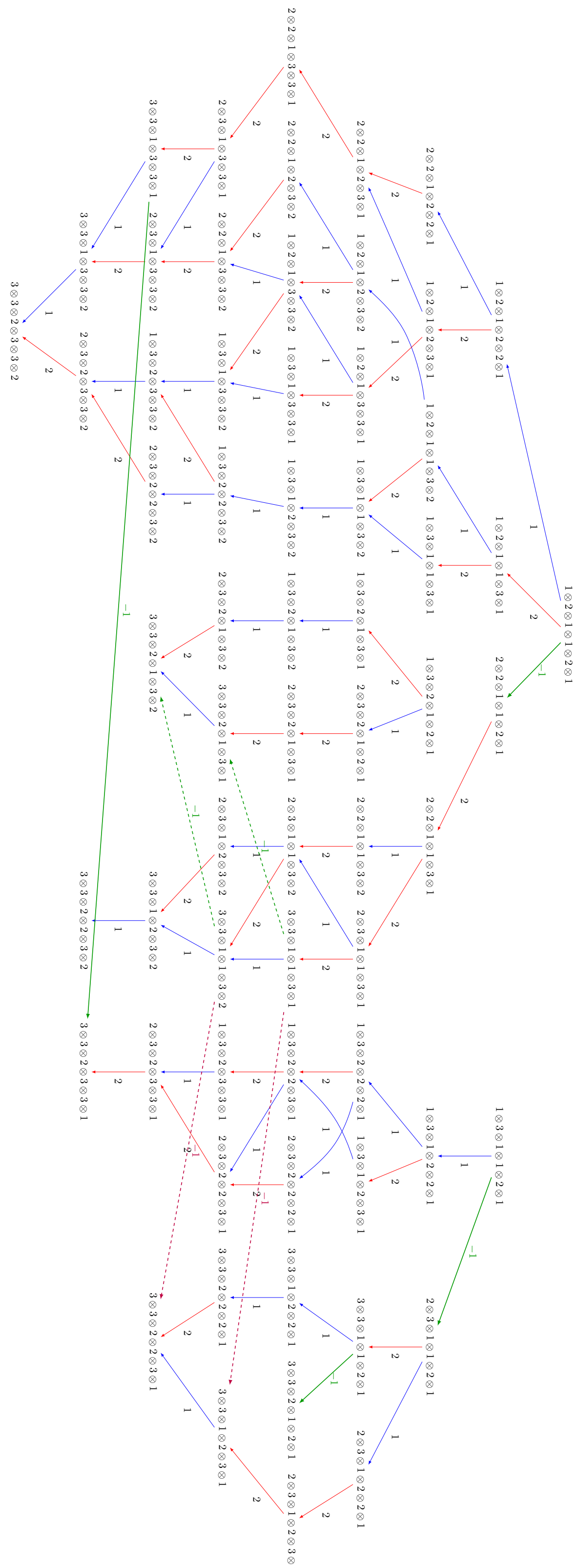


FIGURE 2.3. Counterexample to the unique characterization of the local queer axioms of Definition 27.

I_0-lowest weight and hence as a tableau in French notation contains the letter $n+1$ at the top of each column, the letter n in the second to top box in each column, and in general the letter $n+2-i$ in the i-th box from the top in its column. If there is a letter $k+1$ in the first row of v, then $(e_1 \cdots e_k)$ applies to v and $b' = (e_1 \cdots e_k)v$ satisfies $\varphi_i(b') = 0$ for $i \in I_0 \setminus \{1\}$ and $\varphi_1(b') = 1$. The element b' has several changed entries in the first row, and otherwise the entries above the first row all have letter $n+2-i$ in the i-th box from the top in their column. If b' has a letter $j+1$ in the first row with $1 \leqslant j \leqslant k$, then $(e_1 \cdots e_j)$ applies to b' and $b = g_{j,k} = (e_1 \cdots e_j)b'$ satisfies (2.2.1). Note that if $j > k$, then the last e_1 would no longer apply and hence $b = 0$. This proves that $g_{j,k} \neq 0$ as in (2.2.2) satisfies (2.2.1). If conversely b satisfies (2.2.1), then as a tableau it contains two extra 1's in the first row that have a 3 or bigger above them rather than a 2 in their columns, and for entries higher than the first row the i-th box from the top in its column contains $n+2-i$. It is not hard to check that then $(f_k \cdots f_1)(f_j \cdots f_1)b = v$ for some $1 \leqslant j \leqslant k \leqslant n$. Hence b is of the form (2.2.2). □

In the next section, we introduce a new graph just on I_0-highest weight elements and new connectivity axioms (see Definition 33) that uniquely characterizes queer supercrystals (see Theorem 40).

2.3. Graph on type A components

Let $\mathcal{C}$ be an abstract $\mathfrak{q}(n+1)$-crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. In this section, we define a graph for $\mathcal{C}$ labeled by the type A_n components of $\mathcal{C}$. We draw an edge from vertex C_1 to vertex C_2 in this graph if there is an element b_1 in the component C_1 and an element b_2 in the component C_2 such that $f_{-1}b_1 = b_2$. We provide an easy combinatorial way to describe this graph for a queer supercrystal which is a subcrystal of the crystal of words leveraging the explicit actions of f_{-i} described in Theorem 13 and e_{-i} described in Theorem 17, respectively (see Theorem 38). We also provide new axioms in Definition 33 that will be used in Section 2.4 to provide a unique characterization of queer supercrystals.

Definition 30. *Let $\mathcal{C}$ be a crystal with index set $I_0 \cup \{-1\}$ that is a Stembridge crystal of type A_n when restricted to the arrows labeled I_0. We define the* component graph *of $\mathcal{C}$, denoted by $G(\mathcal{C})$,*

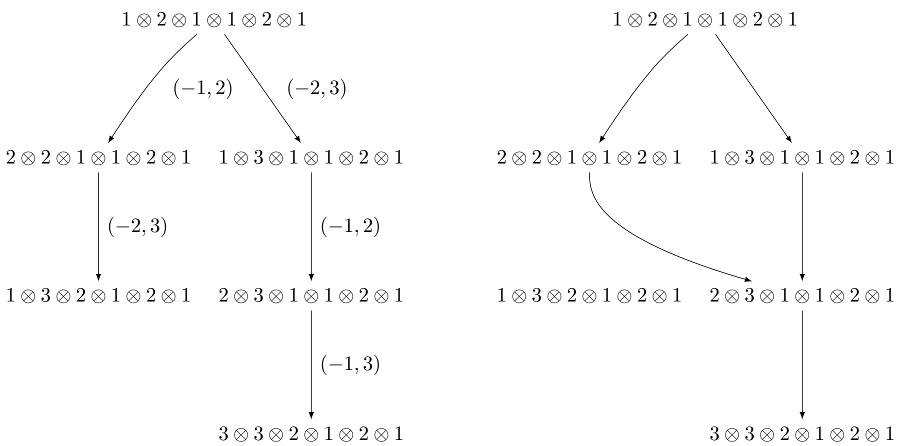


FIGURE 2.4. **Left:** $\overline{G}(\mathcal{C})$. The graph $G(\mathcal{C})$ is obtained from $\overline{G}(\mathcal{C})$ by removing the labels. **Right:** $G(\mathcal{C}')$ for the crystals of Example 31.

as follows. The vertices of $G(\mathcal{C})$ are the type A_n components of $\mathcal{C}$ (typically labeled by their highest weight elements). There is an edge from vertex C_1 to vertex C_2 in this graph, if there is an element b_1 in the component C_1 and an element b_2 in the component C_2 such that

$$f_{-1}b_1 = b_2.$$

EXAMPLE 31. *Let $\mathcal{C}$ be the connected component in the $\mathfrak{q}(3)$-crystal $\mathcal{B}^{\otimes 6}$ with highest weight element $1\otimes 2\otimes 1\otimes 1\otimes 2\otimes 1$ of highest weight $(4,2,0)$. The graph $G(\mathcal{C})$ is given in Figure 2.4 on the left (disregarding the labels on the edges). The graph $G(\mathcal{C}')$ for the counterexample $\mathcal{C}'$ in Figure 2.3 is given in Figure 2.4 on the right. Since the two graphs are not isomorphic as unlabeled graphs, this confirms that the purple dashed arrows in Figure 2.3 do not give the queer supercrystal even though the induced crystal satisfies the axioms in Definition 27.*

EXAMPLE 32. *Let $\mathcal{C}$ be the connected component with highest weight element $1\otimes 1\otimes 2\otimes 1\otimes 2\otimes 1\otimes 3\otimes 2\otimes 1$ in the $\mathfrak{q}(4)$-crystal $\mathcal{B}^{\otimes 9}$. Then the graph $G(\mathcal{C})$ is given in Figure 2.5. One may easily check using Theorem 13 that all arrows in Figure 2.5 are given by the application of f_{-i} for some i except for the arrows that by-pass other arrows, the arrow to the lowest vertex, which is given by $f_{-2}f_3$ (which is also determined by Theorem 13), and the arrow going into $3\otimes 2\otimes 3\otimes 1\otimes 2\otimes 1\otimes 3\otimes 2\otimes 1$, which is given by $f_{-1}f_2$. The result is shown in Figure 2.6.*

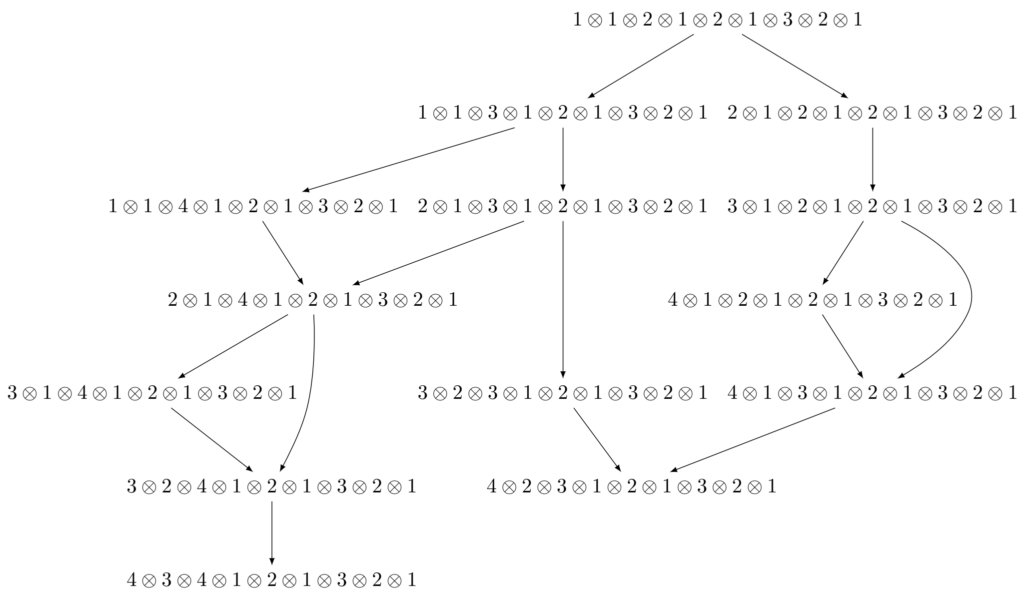


FIGURE 2.5. The graph $G(\mathcal{C})$ for Example 32.

Next we introduce new axioms.

DEFINITION 33 (*Connectivity axioms*). *Let $\mathcal{C}$ be a connected crystal satisfying the local queer axioms of Definition 27. Let $v \in \mathcal{C}$ be an I_0-lowest weight element and $u = \uparrow v$. As in* (2.2.2)*, define $g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v$ for $1 \leqslant j \leqslant k \leqslant n$.*

C0. *$\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.*

C1. *Suppose that $G(\mathcal{C})$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$. For all $h < j \leqslant k$ such that $g_{j,k} \neq 0$, we require that $f_{-1} g_{j,k} \neq 0$ and*

$$f_{-1} g_{j,k} = (e_2 \cdots e_j)(e_1 \cdots e_h)v',$$

where v' is I_0-lowest weight with $\uparrow v' = u'$.

C2. *Suppose that either (a) $G(\mathcal{C})$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$ or (b) no such edge exists in $G(\mathcal{C})$. For all $1 \leqslant j \leqslant h$ in case (a) and all $1 \leqslant j \leqslant k$ in case (b) such that $g_{j,k} \neq 0$ and $f_{-1} g_{j,k} \neq 0$, we require that*

$$f_{-1} g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v.$$

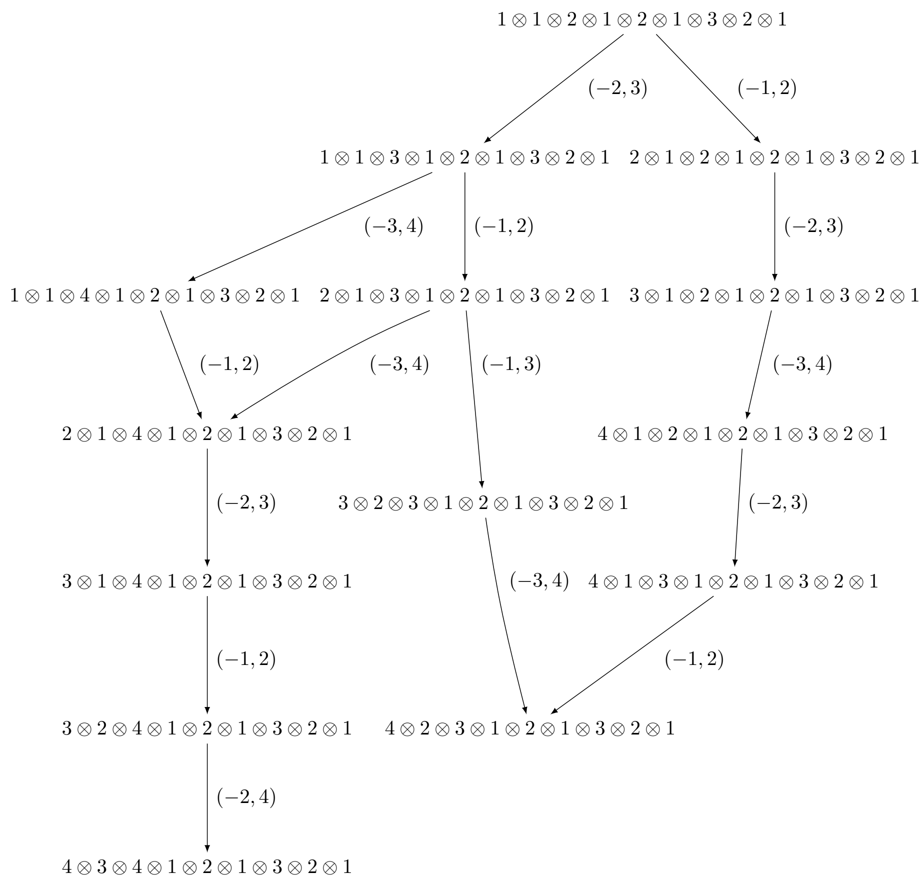


FIGURE 2.6. The graph $\overline{G}(\mathcal{C})$ of Figure 2.5 obtained from $G(\mathcal{C})$ by labeling each edge (except for the by-pass edges) by $(-i, h)$ if $f_{(-i,h)}$ applies.

REMARK 34. *Condition* ***C0*** *can be replaced by the following condition:*

LQ7. *If $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$ for $b \in \mathcal{C}$ with $\varepsilon_2(b) > 0$, then $\varphi_{-1}(b) \leqslant \varphi_{-1}(e_1e_2(b))$.*

This condition indeed implies ***C0****. Suppose $\varphi_{-1}(e_1 \cdots e_k v) = 1$. Then for $b = (e_3 \cdots e_j)(e_1 \cdots e_k)v$, we have $\varphi_{-1}(b) = 1$. However, b satisfies $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$, so the above condition implies that $\varphi_{-1}(e_1e_2(b)) = 1$ as well. But $e_1e_2(b) = g_{j,k}$. Hence $\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.*

Moreover, in $\mathcal{B}^{\otimes \ell}$ the conditions in ***LQ7*** *are satisfied. Namely, the condition $\varepsilon_1(e_2(b)) > \varepsilon_1(b)$ implies that $e_2(b) \neq 0$ and $e_1e_2(b) \neq 0$. Moreover, this condition implies that e_1 acts on $e_2(b)$ in a position weakly to the left of where e_2 acts on b. Thus if $\varphi_{-1}(b) = 1$, it immediately follows that $\varphi_{-1}(e_1e_2(b)) = 1$ which proves the statement.*

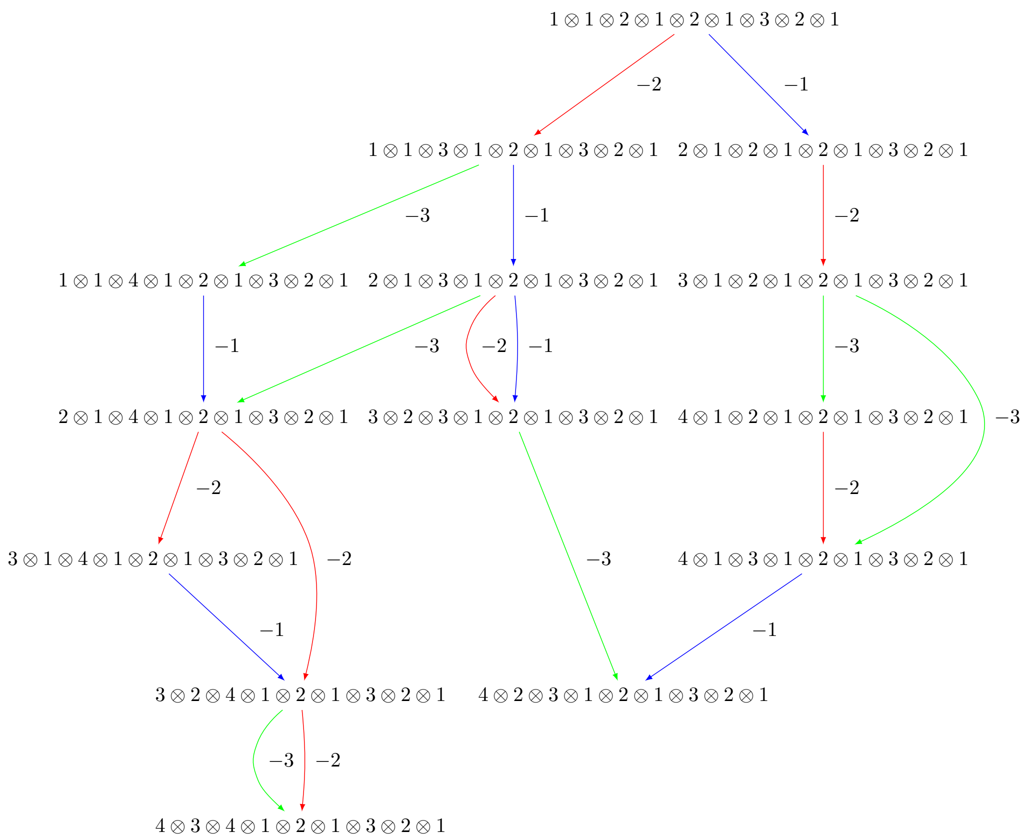


FIGURE 2.7. The graph $\widetilde{G}(\mathcal{C})$ recovered from the graph $\overline{G}(C)$ of Figure 2.6.

THEOREM 35. *The $\mathfrak{q}(n+1)$-crystal $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 33.*

The proof of Theorem 35 is given in the appendix of [**GHPS20**].

Next we show that the arrows in $G(\mathcal{C})$, where $\mathcal{C}$ is a connected component in $\mathcal{B}^{\otimes \ell}$, can be modeled by e_{-i} on type A highest weight elements.

PROPOSITION 36. *Let $\mathcal{C}$ be a connected component in the $\mathfrak{q}(n+1)$-crystal $\mathcal{B}^{\otimes \ell}$. Let C_1 and C_2 be two distinct type A_n components in $\mathcal{C}$ and let u_2 be the I_0-highest weight element in C_2. Then there is an edge from C_1 to C_2 in $G(\mathcal{C})$ if and only if $e_{-i}u_2 \in C_1$ for some $i \in I_0$.*

PROOF. First note that there is an edge from C_1 to C_2 in $G(\mathcal{C})$ if there exists $b_1 \in C_1$ and $b_2 \in C_2$ such that $e_{-1}b_2 = b_1$. Recall that by (2.1.4) we have $e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}$. Hence, if $e_{-i}u_2$ is defined and $e_{-i}u_2 \in C_1$, then $b_1 := e_{-1}b_2$ is defined, where $b_2 := s_{w_i}u_2 \in C_2$ and $b_1 \in C_1$. This proves that there is an edge between C_1 and C_2 in $G(\mathcal{C})$.

Conversely assume that $b_1 = e_{-1}b_2$ for some $b_1 \in C_1$ and $b_2 \in C_2$. We want to show that then $e_{-i}u_2 \in C_1$ for some $i \in I_0$. By the discussion before Lemma 29, we know that the (-1)-arrow on b_1 is induced (using the local queer axioms of Definition 27) by the (-1)-arrow on $g_{j,k} = (e_1 \cdots e_j)(e_1 \cdots e_k)v_1$ for some $j \leqslant k$. By Theorem 35 and Condition **C1** of Definition 33, we must have

$$f_{-1}g_{j,k} = (e_2 \cdots e_j)(e_1 \cdots e_h)v_2 \qquad \text{for some } h < j \leqslant k,$$

where v_2 is the I_0-lowest weight element in the component C_2. In particular, for the edge $u_1 \to u_2$ in $G(\mathcal{C})$, where u_1 is the I_0-highest weight element in the component C_1, the weight $\mathrm{wt}(u_2)$ differs from $\mathrm{wt}(u_1)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $1 \leqslant h < k \leqslant n$. Furthermore, all $g_{j',k} \neq 0$ with $h < j' \leqslant k$ are mapped to component C_2 under f_{-1}.

Claim: Set $b := s_{w_{n-h}}u_2$ and $b' := (e_2 \cdots e_{h+1})(e_1 \cdots e_h)v_2$. If $\mathrm{wt}(b)_2 > 0$, there exist $j_1, \ldots, j_p \in I_0$ such that $b' = f_{j_1} \cdots f_{j_p} b$ and

$$\varphi_2(f_{j_a} \cdots f_{j_p} b) > 0 \quad \text{if } j_a = 2. \tag{2.3.1}$$

The claim is a statement about type A_n crystal operators, hence one may use the tableaux model to verify it. It is straightforward to verify that every column of height $d > n-h$ in the insertion tableau of b contains the letter m in row m; the columns of height $n-h$ contain 1 in the first row and $m+1$ in row $m > 1$; finally the columns of height $d < n-h$ contain the letter $m+2$ in row m. Hence $\mathrm{wt}(b)_2 > 0$ is only satisfied if there is at least one column of height $d > n-h$. Now we start acting with operators f_j on b, where $j \in I_0 \setminus \{2\}$, to make b into a $I_0 \setminus \{2\}$-lowest weight element. This element differs from v_2 only in columns of height $d \geqslant n-h$; columns of height $d > n-h$ contain 1 and 2 in rows 1 and 2, respectively, whereas columns of height $d = n-h$ contain 2 in row 1. Suppose that there are p columns whose height is less than $n+1$ and at least $n-h$. Then we can apply f_2^{p-1} without violating (2.3.1) since each such column contains an unbracketed 2. Then apply again f_j with $j \in I_0 \setminus \{2\}$ to make the tableau into a $I_0 \setminus \{2\}$-lowest weight element, followed by the maximal number of f_2 satisfying (2.3.1), followed by making the result $I_0 \setminus \{2\}$-lowest weight. This tableau is exactly $(e_2 \cdots e_{h+1})(e_1 \cdots e_h)v_2$. This proves the claim.

Now since by assumption $\mathrm{wt}(u_2)$ differs from $\mathrm{wt}(u_1)$ by moving a box from row $n+1-k$ to row $n+1-h$, as a tableau $s_{w_{n-h}}u_2$ indeed has a column of height $d > n-k$, so that $\mathrm{wt}(s_{w_{n-h}}u_2)_2 > 0$. By condition (2.3.1), the (-1)-arrow coming into $s_{w_{n-h}}u_2$ is induced by the (-1)-arrow coming into $(e_2 \cdots e_{h+1})(e_1 \cdots e_h)v_2$ by the local queer axioms of Definition 27. Hence $e_{-(n-h)}u_2 \in C_1$, which proves the proposition where $i = n-h$. $\square$

Example 37. *Let us illustrate the claim in the proof of Proposition 36. Let $n = 5, h = 2$ and consider the type A_5 component C_2 of weight $(4,3,3,2,1)$. Then, using the model for type A crystals in terms of semistandard tableaux (see for example [**BS17**, Chapter 3]), we have*

$$b = s_{w_3}u_2 = \begin{array}{llll} 5 & & & \\ 4 & 4 & & \\ 3 & 3 & 4 & \\ 2 & 2 & 3 & \\ 1 & 1 & 1 & 3 \end{array} \quad . \textit{ This becomes } \begin{array}{llll} 6 & & & \\ 5 & 6 & & \\ 4 & 5 & 6 & \\ 2 & 3 & 5 & \\ 1 & 1 & 3 & 6 \end{array}$$

after making it $\{1,3,4,5\}$-lowest weight and applying f_2^2. Making this element $\{1,3,4,5\}$-lowest weight again, no further f_2 are applicable and we obtain

$$\begin{array}{llll} 6 & & & \\ 5 & 6 & & \\ 4 & 5 & 6 & \\ 2 & 3 & 5 & \\ 1 & 2 & 4 & 6 \end{array} \quad = (e_2e_3)(e_1e_2)v_2.$$

By Proposition 36, there is an edge from component C_1 to component C_2 in $G(\mathcal{C})$ if and only if $e_{-i}u_2 \in C_1$ for some $i \in I_0$, where u_2 is the I_0-highest weight element of C_2. We call the arrow *combinatorial* if $e_{-i}u_2$ is $\{1,2,\ldots,i\}$-highest weight. Otherwise the arrow is called a *by-pass arrow*.

Define $f_{(-i,h)} := f_{-i}f_{i+1}f_{i+2}\cdots f_{h-1}$.

Theorem 38. *Let $\mathcal{C}$ be a connected component in $\mathcal{B}^{\otimes \ell}$. Then each by-pass arrow is the composition of combinatorial arrows. Furthermore, each combinatorial edge in $G(\mathcal{C})$ can be obtained by $f_{(-i,h)}$ for some $i \in I_0$ and $h > i$ minimal such that $f_{(-i,h)}$ applies.*

Proof. Consider a combinatorial arrow from component C_1 to C_2. This means that $e_{-i}u_2$ is defined for some $i \in I_0$ and $e_{-i}u_2$ is $\{1,2,\ldots,i\}$-highest weight. Then by Theorem 13 and Corollary 15 we have $f_{(-i,h)}u_1 = u_2$ for some $h > i$.

If the arrow is a by-pass arrow, then $e_{-i}u_2$ is not $\{1,2,\ldots,i\}$-highest weight. By Proposition 25 and induction, there exists a sequence of indices $1 \leqslant i_1,\ldots,i_a < i$ such that

$$\uparrow e_{-i}u_2 = \uparrow e_{-i} \uparrow e_{-i_1} \cdots \uparrow e_{-i_a}u_2$$

where each partial sequence $e_{-i_j} \uparrow e_{-i_{j+1}} \cdots \uparrow e_{-i_a}u_2$ is $\{1,2,\ldots,i_j\}$-highest weight. This means that each by-pass arrow is the composition of combinatorial arrows. □

Theorem 38 provides a combinatorial description of the graph $G(\mathcal{C})$. Let $\overline{G}(\mathcal{C})$ be the graph $G(\mathcal{C})$ with all by-pass arrows removed and each edge labeled by the tuple $(-i,h)$ for the combinatorial arrow $f_{(-i,h)}u_1 = u_2$, where f_{-i} is given by the combinatorial rules stated in Theorem 13. Hence $\overline{G}(\mathcal{C})$ can be constructed from the $\mathfrak{q}(n+1)$-highest weight element u by the application of combinatorial arrows, see for example Figure 2.6. In particular, the graph $G(\mathcal{C})$ and the graph $\overline{G}(\mathcal{C})$ have the same vertices.

Next we construct a graph $\widetilde{G}(\mathcal{C})$ from $\overline{G}(\mathcal{C})$ by applying $\uparrow e_{-i}$ to each vertex b in the graph $\overline{G}(\mathcal{C})$ (if applicable). This will add additional labeled edges between the vertices in the graph, see Figure 2.7. We would like to emphasize that the construction of $\widetilde{G}(\mathcal{C})$ for a connected component $\mathcal{C}$ of $\mathcal{B}^{\otimes \ell}$ is purely combinatorial, starting with the highest weight element u of a given weight λ, applying $f_{(-i,h)}$ of Theorem 13, and then applying $\uparrow e_{-i}$ to all vertices using Theorem 17. This provides a combinatorial construction of $G(\mathcal{C})$ by dropping the labels in $\widetilde{G}(\mathcal{C})$ (and removing multiple edges between vertices when applicable).

Remark 39. *The Schur P-polynomial $P_\lambda(x_1,\ldots,x_{n+1})$ in $n+1$ variables is the character of a finite-dimensional irreducible representation of the queer Lie superalgebra $\mathfrak{q}(n+1)$ with highest weight λ (up to a power of 2) [**Ser84**]. The above combinatorial construction of the component graph of $\mathcal{C}$ with highest weight λ produces a Schur expansion of the Schur P-polynomial $P_\lambda(x_1,\ldots,x_{n+1})$. This expansion is obtained by counting the multiplicities of highest weights for all type A_n components that are present in $G(\mathcal{C})$. For example, the component graph in Example 31 yields the expansion $P_{42} = s_{42} + s_{33} + s_{411} + 2s_{321} + s_{222}$. This yields an alternative combinatorial description of the Schur expansion of the Schur P-polynomials compared to those given by Stembridge [**Ste89**] and by Choi and Kwon [**CK18**].*

2.4. Characterization of queer supercrystals

Our main theorem gives a characterization of the queer supercrystals. We say that two component graphs $G(\mathcal{C})$ and $G(\mathcal{D})$ are isomorphic if they are isomorphic as graphs and the weights of the vertices are preserved.

THEOREM 40. *Let $\mathcal{C}$ be a connected component of a generic abstract queer supercrystal (see Definition 2). Suppose that $\mathcal{C}$ satisfies the following conditions:*

(1) *$\mathcal{C}$ satisfies the local queer axioms of Definition 27.*

(2) *$\mathcal{C}$ satisfies the connectivity axioms of Definition 33.*

(3) *$G(\mathcal{C})$ is isomorphic to $G(\mathcal{D})$, where $\mathcal{D}$ is some connected component of $\mathcal{B}^{\otimes \ell}$.*

Then the queer supercrystals $\mathcal{C}$ and $\mathcal{D}$ are isomorphic.

Theorem 40 states that the local queer axioms, the connectivity axioms, and the component graph uniquely characterize queer supercrystals.

REMARK 41. *We would like to point out that checking Condition (3) of Theorem 40 is algorithmically straightforward. Each component graph has a unique highest weight vertex. For the isomorphism, the weights of these highest weight vertices need to agree. Then one can recursively compare the edges and weights of adjacent vertices. Condition (3) is similar, albeit more complicated, to the condition by Stembridge [**Ste03**] that for two connected crystal components of a simply-laced crystal to be isomorphic, the highest weights must agree.*

Before we give the proof of Theorem 40, we need the following statement. Recall that $g_{j,k} = (e_1 \cdots e_j)(e_1 \cdots e_k)v$ was defined in (2.2.2), where v is an I_0-lowest weight vector.

LEMMA 42. *In a crystal satisfying the local queer axioms of Definition 27 and **C0** of Definition 33, we have for any $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k$*

$$\varphi_{-1}(g_{j,k}) = 0 \quad \textit{if and only if} \quad \varphi_{-1}(e_1 \cdots e_k v) = 0.$$

PROOF. The condition **C0** requires that $\varphi_{-1}(g_{j,k}) = 0$ implies $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

For the converse direction, note that $\mathrm{wt}(e_1 \cdots e_k v)_1 > 0$. Hence

$$\varphi_{-1}(e_1 \cdots e_k v) = 0 \quad \Leftrightarrow \quad \varepsilon_{-1}(e_1 \cdots e_k v) = 1.$$

By the local queer axioms **LQ6** and **LQ5** of Definition 27 (see also Figure 2.2), we have

$$\varepsilon_{-1}(e_1 \cdots e_k v) = 1 \quad \Leftrightarrow \quad \varepsilon_{-1}((e_3 \cdots e_j)(e_1 \cdots e_k)v) = 1 \quad \Rightarrow \quad \varepsilon_{-1}((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1.$$

It can be easily checked that $\varphi_1((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1$ for $j \leqslant k$ (for example using the tableaux model for type A_n crystals). Hence by the local queer axioms

$$\varepsilon_{-1}((e_2 \cdots e_j)(e_1 \cdots e_k)v) = 1 \quad \Leftrightarrow \quad \varepsilon_{-1}((e_1 \cdots e_j)(e_1 \cdots e_k)v) = 1.$$

This proves that $\varphi_{-1}(e_1 \cdots e_k v) = 0$ implies $\varphi_{-1}(g_{j,k}) = 0$. $\square$

Proof of Theorem 40. By Proposition 28 and Theorem 35, $\mathcal{D}$ satisfies the local queer axioms and the connectivity axioms and hence all conditions of the theorem.

By **LQ1** of the local queer axioms of Definition 27, each type A_n-component of $\mathcal{C}$ is a Stembridge crystal and hence is uniquely characterized by [**Ste03**]. By assumption $G(\mathcal{C}) \cong G(\mathcal{D})$. In particular, the vertices of $G(\mathcal{C})$ and $G(\mathcal{D})$ agree. This proves that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic as A_n crystals.

Next we show that all (-1)-arrows also agree on $\mathcal{C}$ and $\mathcal{D}$. As discussed just before Lemma 29, given the local queer axioms of Definition 27, it suffices to show that f_{-1} acts in the same way in $\mathcal{C}$ and $\mathcal{D}$ on the almost lowest elements satisfying (2.2.1) or equivalently by Lemma 29 on every $g_{j,k} \neq 0$ with $1 \leqslant j \leqslant k \leqslant n$. For the remainder of this proof, fix $g_{j,k} \neq 0$ in the I_0-component u.

Let us first assume that $G(\mathcal{C})$ contains an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ for some $h < k$. If $h < j \leqslant k$, then $f_{-1}g_{j,k}$ is determined by **C1** of Definition 33. If $j \leqslant h$, pick $h < j' \leqslant k$ such that $g_{j',k} \neq 0$. Such a j' must exist since there is an edge $u \to u'$ in $G(\mathcal{C})$. By **C1**, we have $\varphi_{-1}(g_{j',k}) = 1$ and hence by Lemma 42 also $\varphi_{-1}(g_{j,k}) = 1$. Hence $f_{-1}g_{j,k}$ is determined by **C2**(a).

Next assume that $G(\mathcal{C})$ does not contain an edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$.

Claim: If $g_{k,k} \neq 0$, then $f_{-1}g_{j,k} = 0$.

Proof. Suppose $f_{-1}g_{k,k} \neq 0$. By **C2**(b), we have $f_{-1}g_{k,k} = (e_2 \cdots e_k)(e_1 \cdots e_k)v = f_1 g_{k,k}$. But this contradicts the local queer axioms of Definition 27 since $\varphi_1(g_{k,k}) > 1$. Hence $\varphi_{-1}(g_{k,k}) = 0$ and by Lemma 42 also $\varphi_{-1}(g_{j,k}) = 0$, which proves the claim. $\square$

If $g_{k,k} = 0$, we have $j < k$ since by assumption $g_{j,k} \neq 0$.

Claim: Suppose $g_{k,k} = 0$.

(1) Suppose there is an edge $\overline{u} \to u$ in $G(\mathcal{C})$ such that $\mathrm{wt}(u)$ is obtained from $\mathrm{wt}(\overline{u})$ by moving a box from row $n+1-\overline{k}$ to row $n+1-\overline{h}$ such that $\overline{h} < k \leqslant \overline{k}$. Then $f_{-1}g_{j,k} = 0$.

(2) Suppose $G(\mathcal{C})$ does not contain an edge as in (1). Then $f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v$.

Proof. Suppose that the conditions in (1) are satisfied. Then by **C1** there must exist

$$\overline{g}_{\overline{j},\overline{k}} := (e_1 \cdots e_{\overline{j}})(e_1 \cdots e_{\overline{k}})\overline{v} \neq 0,$$

where $\overline{h} < \overline{j} \leqslant \overline{k}$ and $\overline{v}$ is the I_0-lowest weight element in the component of $\overline{u}$, such that

$$f_{-1}\overline{g}_{\overline{j},\overline{k}} = (e_2 \cdots e_{\overline{j}})(e_1 \cdots e_{\overline{h}})v. \tag{2.4.1}$$

Since $g_{j,k} \neq 0$, we have in particular that $(e_1 \cdots e_k)v \neq 0$. Since $\mathrm{wt}(u)$ is obtained from $\mathrm{wt}(\overline{u})$ by moving a box from row $n+1-\overline{k}$ to row $n+1-\overline{h}$, this hence also implies that $\overline{g}_{k,\overline{k}} = (e_1 \cdots e_k)(e_1 \cdots e_{\overline{k}})\overline{v} \neq 0$. Hence by **C1** Equation (2.4.1) holds for $\overline{j} = k$.

If $f_{-1}g_{\overline{h},k} = 0$, we also have $f_{-1}g_{j,k} = 0$ by Lemma 42 as claimed. Hence we may assume that $f_{-1}g_{\overline{h},k} \neq 0$. Then by **C2**(b) we have

$$f_{-1}g_{\overline{h},k} = (e_2 \cdots e_k)(e_1 \cdots e_{\overline{h}})v.$$

But then $f_{-1}\overline{g}_{k,\overline{k}} = f_{-1}g_{\overline{h},k} = (e_2 \cdots e_k)(e_1 \cdots e_{\overline{h}})v$, which contradicts the fact that the crystal operator f_{-1} has a partial inverse since $\overline{g}_{k,\overline{k}} \neq g_{\overline{h},k}$. This proves (1).

Now suppose that the conditions in (2) are satisfied. Recall that by assumption $g_{j,k} \neq 0$ with $j < k$. This implies that $y := (e_2 \cdots e_k)(e_1 \cdots e_j)v \neq 0$, $\varphi_i(y) = 0$ for $i \in I_0 \setminus \{2\}$ and $\varphi_2(y) = 1$. By the local queer axioms of Definition 27, this implies that $x := e_{-1}y \neq 0$ with $\varphi_1(x) \in \{1, 2\}$ and $\varphi_i(x) = 0$ for $i \in I_0 \setminus \{1\}$. Thus we may write $x = (e_1 \cdots e_s)(e_1 \cdots e_t)\overline{v}$, where $0 \leqslant s \leqslant t$ and $\overline{v} \in \mathcal{C}$

$$\begin{tikzcd}
1 \otimes 1 \otimes 1 \otimes 1 \arrow[d] \\
2 \otimes 1 \otimes 1 \otimes 1 \arrow[d] \\
3 \otimes 2 \otimes 1 \otimes 1 \arrow[d] \\
4 \otimes 3 \otimes 2 \otimes 1
\end{tikzcd}$$

FIGURE 2.8. The graph $G(\mathcal{C})$ for the example in Remark 43.

is some I_0-lowest weight vector. This yields the equality

$$f_{-1}(e_1 \cdots e_s)(e_1 \cdots e_t)\overline{v} = (e_2 \cdots e_k)(e_1 \cdots e_j)v.$$

If $\overline{v} \neq v$, then by the connectivity axioms of Definition 33 this means that $j < k = s \leqslant t$ and there is an edge in $G(\mathcal{C})$ from $\uparrow \overline{v}$ to $u = \uparrow v$, moving a box from row $n+1-t$ to row $n+1-j$. This contradicts the assumptions of (2). Hence we must have $\overline{v} = v$. By **C2**(b) we have $f_{-1}g_{s,t} = (e_2 \cdots e_t)(e_1 \cdots e_s)v$, so that $k = t$ and $j = s$. This implies $f_{-1}g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j)v$, proving the claim. $\square$

We have now shown that $f_{-1}g_{j,k}$ is determined in all cases, which proves the theorem. $\square$

REMARK 43. *Consider the $\mathfrak{q}(4)$-crystal $\mathcal{B}^{\otimes 4}$. The elements* 4114 *and* 4113 *both lie in the same $\{1,2,3\}$-component of highest weight* $(3,1)$. *The highest (resp. lowest) weight element in this component is $u =$ 2111 (resp. $v =$ 4344). Both* 4114 *and* 4113 *satisfy* (2.2.1). *In fact,* $4114 = (e_1e_2)(e_1e_2e_3)v = g_{2,3}$ *and* $4113 = (e_1e_2e_3)(e_1e_2e_3)v = g_{3,3}$. *In the component of u there is no sequence of crystal operators that would induce the action of f_{-1} on* 4114 *from the action of f_{-1} on* 4113 *using the local queer axioms of Definition 27.*

This suggests that the connectivity axioms of Definition 33 are indeed necessary. However, in this example the graph $G(\mathcal{C})$, where $\mathcal{C}$ is the connected component in $\mathcal{B}^{\otimes 4}$ containing 2111, *is linear and hence forces* 4114 *and* 4113 *to be mapped to the same $\{1,2,3\}$-component by f_{-1}, see Figure 2.8.*

Remark 44. *Consider the connected component $\mathcal{C}$ of* 111212121 *in the* $\mathfrak{q}(6)$*-crystal* $\mathcal{B}^{\otimes 9}$*. The* $\{1,2,3,4,5\}$*-component containing* 321312121 *is connected to the components* 421312121*,* 431312121*, and* 432312121 *in* $G(\mathcal{C})$*. The elements* $g_{4,5} = 651615464$ *and* $g_{3,5} = 651615465$ *in the component of* 321312121 *are mapped to the same component* 432312121 *by* ***C1*** *of Definition 33. However, the element* $g_{4,5}$ *is connected to* 431413131 *in the crystal using only arrows that commute with* f_{-1} *and the element* $g_{3,5}$ *is connected to* 431413143 *in the crystal using only arrows that commute with* f_{-1}*. However, these two components (containing* 431413131 *resp.* 431413143 *using only crystal operators* f_i *and* e_i *with* $i \in I_0$ *that commute with* f_{-1}*) are disjoint. This suggests that* ***C1*** *of Definition 33 is necessary for uniqueness.*

CHAPTER 3

Crystal for fully commutative stable Grothendieck polynomials

This chapter is based on joint work with Jennifer Morse, Jianping Pan and Anne Schilling published in [**MPPS20**].

3.1. The $\star$-crystal

In this section, we define the K-theoretic generalization of the crystal on decreasing factorizations by Morse and Schilling [**MS16**] when the associated word is fully-commutative. The underlying combinatorial objects are decreasing factorizations in the 0-Hecke monoid introduced in Section 3.1.1. The $\star$-crystal on these decreasing factorizations is defined in Section 3.1.2. We review the crystal structure on set-valued tableaux introduced by Monical, Pechenik and Scrimshaw [**MPS20**] in Section 3.1.3. The residue map and the proof that it intertwines the $\star$-crystal and the crystal on set-valued tableaux is given in Section 3.1.4.

3.1.1. Decreasing factorizations in the 0-Hecke monoid. The *symmetric group* $\mathbb{S}_n$ for $n \geqslant 1$ is generated by the simple transpositions $s_1, s_2, \ldots, s_{n-1}$ subject to the relations

$$\begin{aligned} s_i s_j &= s_j s_i, && \text{if } |i-j| > 1,\\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && \text{for } 1 \leqslant i < n-1,\\ s_i^2 &= 1, && \text{for } 1 \leqslant i \leqslant n-1. \end{aligned}$$

A *reduced expression* for an element $w \in \mathbb{S}_n$ is a word $a_1 a_2 \ldots a_\ell$ with $a_i \in [n-1] := \{1, 2, \ldots, n-1\}$ such that

$$w = s_{a_1} \cdots s_{a_\ell} \tag{3.1.1}$$

and ℓ is minimal among all words satisfying (3.1.1). In this case, ℓ is called the length of w.

Definition 45. *The* 0-Hecke monoid $\mathcal{H}_0(n)$*, where* $n \geqslant 1$ *is an integer, is the monoid of finite words generated by positive integers in the alphabet* $[n-1]$ *subject to the relations*

$$\begin{aligned} pq &= qp \qquad \text{if } |p-q| > 1, \\ pqp &= qpq \qquad \text{for all } p, q, \\ pp &= p \qquad \text{for all } p. \end{aligned} \tag{3.1.2}$$

We may form an equivalence relation $\equiv_{\mathcal{H}_0}$ on all words in the alphabet $[n-1]$ based on the relations (3.1.2). The equivalence classes are infinite since the last relation changes the length of the word. We say that a word $a = a_1 a_2 \ldots a_\ell$ is *reduced* if $\ell \geqslant 0$ is the smallest among all words in $\mathcal{H}_0(n)$ equivalent to a. In this case, ℓ is the length of a. Note that $\mathcal{H}_0(n)$ is in bijection with $\mathbb{S}_n$ by identifying the reduced word $a_1 a_2 \ldots a_\ell$ in $\mathcal{H}_0(n)$ with $s_{a_1} s_{a_2} \cdots s_{a_\ell} \in \mathbb{S}_n$. We say $w \in \mathcal{H}_0(n)$ or $\mathbb{S}_n$ is *fully-commutative* or 321*-avoiding* if none of the reduced words equivalent to w contain a consecutive braid subword of the form $i\; i+1\; i$ or $i\; i-1\; i$ for any $i \in [n-1]$.

Remark 46. *Any (not necessarily reduced) word* $w \in \mathcal{H}_0(0)$ *containing a consecutive braid subword is not fully-commutative.*

Definition 47. *A* decreasing factorization *of* $w \in \mathcal{H}_0(n)$ *into* m factors *is a product of the form*

$$\mathbf{h} = h^m \ldots h^2 h^1,$$

where the sequence in each factor

$$h^i = h_1^i h_2^i \ldots h_{\ell_i}^i$$

is either empty (meaning $\ell_i = 0$*) or strictly decreasing (meaning* $h_1^i > h_2^i > \cdots > h_{\ell_i}^i$*) for each* $1 \leqslant i \leqslant m$ *and* $\mathbf{h} \equiv_{\mathcal{H}_0} w$ *in* $\mathcal{H}_0(n)$*.*

The set of all possible decreasing factorizations into m *factors is denoted by* $\mathcal{H}^m$ *or* $\mathcal{H}^m(n)$ *if we want to indicate the value of* n*. We call* $\mathsf{ex}(\mathbf{h}) = \mathsf{len}(\mathbf{h}) - \ell$ *the* excess *of* $\mathbf{h}$*, where* $\mathsf{len}(\mathbf{h})$ *is the number of letters in* $\mathbf{h}$ *and* ℓ *is the length of* w*. We say* $\mathbf{h}$ *is fully-commutative (or* 321*-avoiding) if* w *is fully-commutative.*

3.1.2. The $\star$-crystal.

Let $\mathcal{H}^{m,\star}$ be the set of fully-commutative decreasing factorizations in $\mathcal{H}^m$. We introduce a type A_{m-1} crystal structure on $\mathcal{H}^{m,\star}$, which we call the *$\star$-crystal*. This generalizes the crystal for Stanley symmetric functions [**MS16**] (see also [**Len04**]).

Definition 48. *For any* $\mathbf{h} = h^m \dots h^2 h^1 \in \mathcal{H}^{m,\star}$, *we define crystal operators* $e_i^\star$ *and* $f_i^\star$ *for* $i \in [m-1]$ *and a weight function* $\mathrm{wt}(\mathbf{h})$. *The weight function is determined by the length of the factors*

$$\mathrm{wt}(\mathbf{h}) = (\mathsf{len}(h^1), \mathsf{len}(h^2), \dots, \mathsf{len}(h^m)).$$

To define the crystal operators $e_i^\star$ *and* $f_i^\star$, *we first describe a pairing process:*

- *Start with the largest letter* b *in* h^{i+1}, *pair it with the smallest* $a \geqslant b$ *in* h^i. *If there is no such* a, *then* b *is unpaired.*
- *The pairing proceeds in decreasing order on elements of* h^{i+1} *and with each iteration, previously paired letters of* h^i *are ignored.*

If all letters in h^i *are paired, then* $f_i^\star$ *annihilates* $\mathbf{h}$. *Otherwise, let* x *be the largest unpaired letter in* h^i. *The crystal operator* $f_i^\star$ *acts on* $\mathbf{h}$ *in either of the following ways:*

(1) *If* $x+1 \in h^i \cap h^{i+1}$, *then remove* $x+1$ *from* h^i, *add* x *to* h^{i+1}.
(2) *Otherwise, remove* x *from* h^i *and add* x *to* h^{i+1}.

If all letters in h^{i+1} *are paired, then* $e_i^\star$ *annihilates* $\mathbf{h}$. *Let* y *be the smallest unpaired letter in* h^{i+1}. *The crystal operator* $e_i^\star$ *acts on* $\mathbf{h}$ *in either of the following ways:*

(1) *If* $y-1 \in h^i \cap h^{i+1}$, *then remove* $y-1$ *from* h^{i+1}, *add* y *to* h^i.
(2) *Otherwise, remove* y *from* h^{i+1} *and add* y *to* h^i.

It is not hard to see that $e_i^\star$ *and* $f_i^\star$ *are partial inverses of each other.*

Example 49. *Let* $\mathbf{h} = (7532)(621)(6)$, *then*

$$f_1^\star(\mathbf{h}) = 0, \qquad e_1^\star(\mathbf{h}) = (7532)(62)(61),$$

$$f_2^\star(\mathbf{h}) = (75321)(61)(6), \qquad e_2^\star(\mathbf{h}) = (753)(6321)(6).$$

Remark 50. *Compared to [**MS16**], one pairs a letter* b *in* h^{i+1} *with the smallest letter* $a \geqslant b$ *in* h^i *rather than* $a > b$.

PROPOSITION 51. *Let* $\mathbf{h} = h^m \dots h^1 \in \mathcal{H}^{m,\star}$ *such that* $f_i^\star(\mathbf{h}) \neq 0$. *Then* $f_i^\star(\mathbf{h}) \in \mathcal{H}^{m,\star}$, $f_i^\star(\mathbf{h}) \equiv_{\mathcal{H}_0} \mathbf{h}$, *and* $\mathsf{ex}(f_i^\star(\mathbf{h})) = \mathsf{ex}(\mathbf{h})$. *Furthermore, the* j*-th factor in* $f_i^\star(\mathbf{h})$ *and* $\mathbf{h}$ *agrees for* $j \notin \{i, i+1\}$. *Analogous statements hold for* $e_i^\star$.

PROOF. Suppose $\tilde{\mathbf{h}} := f_i^\star(\mathbf{h}) \neq 0$. Then by definition of $f_i^\star$, $\tilde{\mathbf{h}} = h^m \dots h^{i+2}\tilde{h}^{i+1}\tilde{h}^i h^{i-1} \dots h^1$ and h^j is unchanged for $j \notin \{i, i+1\}$. In addition, the number of factors does not change.

To see $\mathbf{h} \equiv_{\mathcal{H}_0} \tilde{\mathbf{h}}$, it suffices to show that $h^{i+1}h^i \equiv_{\mathcal{H}_0} \tilde{h}^{i+1}\tilde{h}^i$. Let x be the largest unpaired letter in h^i. By the bracketing procedure this implies that $x \notin h^{i+1}$. We can write h^{i+1} as $w_1 w_2$, where w_1 is a word containing only letters greater than x, and w_2 is a word containing only letters smaller than x. We can write h^i as $w_3 x w_4$, where w_3 contains only letters greater than x and w_4 contains only letters smaller than x.

The pairing process will result in one of the two following cases:

(1) If $x+1 \in h^i \cap h^{i+1}$, then obtain $\tilde{h}^i$ by removing $x+1$ from h^i, and $\tilde{h}^{i+1}$ by adding x to h^{i+1}.
(2) Otherwise, obtain $\tilde{h}^i$ by removing x from h^i and obtain $\tilde{h}^{i+1}$ by adding x to h^{i+1}.

We first argue that in either case we must have $x-1 \notin w_2$. Assume $x-1 \in w_2$ and let k be the largest number such that the interval $[x-k, x-1] \subseteq w_2$. By assumption $k \geqslant 1$. In order for x to be the largest unpaired letter in h^i, $[x-k, x-1]$ must be contained in w_4. We can write $w_2 = (x-1)\dots(x-k)w_2'$ and $w_4 = (x-1)\dots(x-k)w_4'$, where all letters in w_2' are smaller than $x-k-1$. When $k=1$, we have the following subword

$$(x-1)w_2'w_3x(x-1) \equiv_{\mathcal{H}_0} w_2'w_3(x-1)x(x-1),$$

which contains a braid $(x-1)x(x-1)$. When $k>1$, we also have the following subword

$$(x-k)w_2'w_3x(x-1)\dots(x-k+1)(x-k) \equiv_{\mathcal{H}_0} w_2'w_3(x-1)\dots(x-k+2)(x-k)(x-k+1)(x-k),$$

which also contains a braid.

Case (1): Let k be the largest letter such that $[x+1, x+k] \subseteq w_3$. Clearly $k \geqslant 1$. Suppose $k>1$, then we can write $w_3 = w_3'(x+k)\dots(x+1)$. Since x is the largest unpaired letter in h^i, everything in $[x+1, x+k] \subseteq w_3$ must be paired. The letter $x+1$ in w_3 is paired with $x+1 \in w_1$, which implies that $x+i$ in w_3 is paired with $x+i \in w_1$ for all $1 \leqslant i \leqslant k$. This implies that $[x+1, x+k] \subseteq w_1$. Then we have the following subword

$$(x+1)w_2w_3'(x+k)\dots(x+2)(x+1) \equiv_{\mathcal{H}_0} w_2w_3'(x+k)\dots(x+1)(x+2)(x+1)$$

which contains a braid. Thus, we must have $k=1$, which implies that $x+2 \notin w_3$. Write $w_1 = w_1'(x+1)$. Then by direct computation

$$\begin{aligned} h^{i+1}h^i &\equiv_{\mathcal{H}_0} w_1'(x+1)w_2w_3'(x+1)xw_4 \equiv_{\mathcal{H}_0} w_1'(x+1)(x+1)w_2w_3'xw_4 \\ &\equiv_{\mathcal{H}_0} w_1'(x+1)w_2w_3'xxw_4 \equiv_{\mathcal{H}_0} \left(w_1'(x+1)xw_2\right)\left(w_3'xw_4\right) = \tilde{h}^{i+1}\tilde{h}^i. \end{aligned}$$

Case (2): We claim that if $x+1 \notin h^{i+1}$, then $x+1 \notin h^i$. Otherwise the $x+1 \in h^i$ must be paired with some $z \in h^{i+1}$, so we have $z \leqslant x+1$. But x is unpaired, which implies $z > x$, that gives us a contradiction. Hence $x+1 \notin w_3$. Recall that $x-1 \notin w_2$. Therefore, by a straightforward computation

$$h^{i+1}h^i = w_1w_2w_3xw_4 \equiv_{\mathcal{H}_0} (w_1xw_2)(w_3w_4) \equiv_{\mathcal{H}_0} \tilde{h}^{i+1}\tilde{h}^i.$$

The above arguments show that $h^{i+1}h^i \equiv_{\mathcal{H}_0} \tilde{h}^{i+1}\tilde{h}^i$, thus $\mathbf{h} \equiv_{\mathcal{H}_0} \tilde{\mathbf{h}}$, and the total length of the decreasing factorization are unchanged under $f_i^\star$. Furthermore, the excess remains unchanged under $f_i^\star$.

Similar arguments hold for $e_i^\star$. □

REMARK 52. *Here we summarize several results from the proof that will be needed later. Namely, if x is the largest unpaired letter in h^i, then*

- $x-1 \notin h^{i+1}$.
- *One and only one of the three statements hold:* $x+1 \in h^{i+1} \cap h^i$, $x+1 \notin h^{i+1} \cup h^i$, *and* $x+1 \in h^{i+1}, x+1 \notin h^i$.

It will be shown in Section 3.1.4 that $\mathcal{H}^{m,\star}$ is indeed a Stembridge crystal of type A_{m-1} (for an introduction to crystal and terminology, see [**BS17**]).

3.1.3. The crystal on set-valued tableaux. In this section, we review the type A crystal structure on set-valued tableaux introduced in [**MPS20**]. In fact, in [**MPS20**] the authors only considered the crystal structure on straight-shaped set-valued tableaux. Here we consider the crystal on skew shapes as well, see Theorem 55.

We use French notation for partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$, that is, in the Ferrers diagram for λ, the largest part λ_1 is at the bottom.

Definition 53 ([**Buc02**]). *A semistandard set-valued tableau T is the filling of a skew shape λ/μ with nonempty subsets of positive integers such that:*

- *for all adjacent cells A, B in the same row with A to the left of B, we have* $\max(A) \leqslant \min(B)$,
- *for all adjacent cells A, C in the same column with A below C, we have* $\max(A) < \min(C)$.

The weight of T, denoted by $\mathrm{wt}(T)$, is the integer vector whose i-th component counts the number of i's that occur in T. The excess of T is defined as $\mathsf{ex}(T) = |\,\mathrm{wt}(T)| - |\lambda|$. We denote the set of all semistandard set-valued tableaux of shape λ/μ by $\mathsf{SVT}(\lambda/\mu)$. Similarly, if the maximum entry is restricted to m, the set is denoted by $\mathsf{SVT}^m(\lambda/\mu)$.

We now review the crystal structure on semistandard set-valued tableaux given in [**MPS20**]. We state the definition on skew shapes rather than just straight shapes.

Definition 54. *Let $T \in \mathsf{SVT}^m(\lambda/\mu)$. We employ the following pairing rule for letters i and $i+1$. Assign $-$ to every column of T containing an i but not an $i+1$. Similarly, assign $+$ to every column of T containing an $i+1$ but not an i. Then, successively pair each $+$ that is to the left of and adjacent to a $-$, removing all paired signs until nothing can be paired.*

The operator f_i changes the i in the rightmost column with an unpaired $-$ (if this exists) to $i+1$, except if the cell b containing that i has a cell to its right, denoted $b^{\rightarrow}$, that contains both i and $i+1$. In that case, f_i removes i from $b^{\rightarrow}$ and adds $i+1$ to b. Finally, if no unpaired $-$ exists, then f_i annihilates T.

Similarly, the operator e_i changes the $i+1$ in the leftmost column with an unpaired $+$ (if this exists) to i, except if the cell b containing that $i+1$ has a cell to its left, denoted $b^{\leftarrow}$, that contains both i and $i+1$. In that case, e_i removes $i+1$ from $b^{\leftarrow}$ and adds i to b. Finally, if no unpaired $+$ exists, then e_i annihilates T.

Based on the pairing procedure above, $\varphi_i(T)$ is the number of unpaired $-$ while $\varepsilon_i(T)$ is the number of unpaired $+$.

One can easily show that the crystal on $\mathsf{SVT}^m(\lambda/\mu)$ of Definition 54 defines a seminormal crystal (for definitions see [**BS17**]). It was proved in [**MPS20**, Theorem 3.9] that the above described operators e_i and f_i define a type A_{m-1} Stembridge crystal structure on $\mathsf{SVT}^m(\lambda)$. We claim that their proof goes through also for skew shapes.

THEOREM 55. *The crystal $\mathsf{SVT}^m(\lambda/\mu)$ of Definition 54 is a Stembridge crystal of type A_{m-1}.*

PROOF. Since the proof is exactly the same as in [**MPS20**, Theorem 3.9], we just state the outline and give a brief description. For details we refer to [**MPS20**].

First note that the signature rule given by column-reading is compatible with the signature rule given by row-reading (top to bottom, left to right, and arrange the letters in the same cell by descending order) by semistandardness. Hence we may consider the crystal to live inside the tensor product of its rows. A single-row semistandard set-valued tableaux of a fixed shape is isomorphic to a Stembridge crystal, as shown in [**MPS20**, Proposition 3.5]:

$$\Phi_s \colon \mathsf{SVT}^m(s\Lambda_1) \to \bigoplus_{k=1}^{m} B((s-1)\Lambda_1 + \Lambda_k),$$

where Λ_k are the fundamental weights of type A_{m-1}.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$ (the last couple μ_i could be zero) be two partitions such that $\mu \subseteq \lambda$. Construct the map below, which is a strict crystal embedding:

$$\Psi \colon \mathsf{SVT}^m(\lambda/\mu) \to \mathsf{SVT}^m((\lambda_1 - \mu_1)\Lambda_1) \otimes \mathsf{SVT}^m((\lambda_2 - \mu_2)\Lambda_1) \otimes \cdots \otimes \mathsf{SVT}^m((\lambda_\ell - \mu_\ell)\Lambda_1).$$

Thus, we have a strict crystal embedding:

$$(\Phi_{\lambda_1-\mu_1} \oplus \cdots \oplus \Phi_{\lambda_\ell-\mu_\ell}) \circ \Psi \colon \mathsf{SVT}^m(\lambda/\mu) \to \bigotimes_{j=1}^{\ell} \left(\bigoplus_{k=1}^{m} B((\lambda_j - \mu_j)\Lambda_1 + \Lambda_k) \right).$$

Since $\mathsf{SVT}^m(\lambda/\mu)$ is a seminormal crystal, we can conclude that it is a Stembridge crystal. $\square$

3.1.4. The residue map. In this section, we define the residue map from set-valued tableaux of skew shape to fully-commutative decreasing factorizations in the 0-Hecke monoid. We then show in Theorem 61 that the residue map intertwines with the crystal operators, proving that $\mathcal{H}^{m,\star}$ is indeed a crystal of type A_{m-1} (see Corollary 62).

Definition 56. *Given $T \in \mathsf{SVT}^m(\lambda/\mu)$, we define the residue map $\mathsf{res} : \mathsf{SVT}^m(\lambda/\mu) \to \mathcal{H}^m$ as follows. Associate to each cell (i,j) in λ/μ its content $\ell(\lambda)+j-i$, where $\ell(\lambda)$ is the number of parts in λ. Produce a decreasing factorization $\mathbf{h} = h^m h^{m-1} \dots h^2 h^1$ by declaring h^k to be the (possibly empty) sequence formed by taking the contents of all cells in T containing the entry k and then arranging the contents in decreasing order. This defines $\mathsf{res}(T) := \mathbf{h}$.*

Example 57. *Let T be the set-valued tableau of skew shape $(2,2)/(1)$*

$$T = \begin{array}{|c|c|}\hline 23 & 3 \\ \hline \multicolumn{1}{c|}{} & 12 \\ \cline{2-2}\end{array}\,.$$

The content of each cell in T is denoted by a subscript as follows:

$$\begin{array}{|c|c|}\hline 23_1 & 3_2 \\ \hline \multicolumn{1}{c|}{} & 12_3 \\ \cline{2-2}\end{array}\,.$$

To read off the third factor, we search for all cells with an entry 3; these cells have contents 1 and 2, so we have 21 in the third factor. Altogether, we obtain $\mathsf{res}(T) = (21)(31)(3) \in \mathcal{H}^3$.

The image of the residue map res is $\mathcal{H}^{m,\star}$, the set of fully-commutative decreasing factorizations into m factors. In fact, res is a bijection from semistandard set-valued skew tableaux on the alphabet $[m]$ to $\mathcal{H}^{m,\star}$ up to shifts in the skew shape.

For this purpose, let us describe the inverse of the residue map. Let $\mathbf{h} = h^m h^{m-1} \dots h^2 h^1 \in \mathcal{H}^{m,\star}$. Begin by filling the diagonals of content that appear in h^m by the entry m. As the resulting T is supposed to be of skew shape, the cells containing m along increasing diagonals need to go weakly down from left to right. If these diagonals are consecutive, then the cells have to be in the same row of T since T is semistandard. Continue the procedure above by putting entry i into the diagonals specified by h^i for all $i = m-1, m-2, \dots, 1$, applying the condition that the resulting filling should be semistandard.

Proposition 58. *If $\mathbf{h} = h^m h^{m-1} \dots h^2 h^1 \in \mathcal{H}^{m,\star}$, then the above algorithm is well-defined up to shifts along diagonals. It produces a skew semistandard set-valued tableau T such that $\mathsf{res}(T) = \mathbf{h}$.*

Proof. We shall show more generally that at any given stage in the algorithm for the inverse of the residue map above, the tableau T produced is of skew shape if and only if $\mathbf{h}$ is fully-commutative.

Assume that T is not of skew shape. Consider the earliest stage in the algorithm when the produced tableau is not of skew shape. Then, either one of the following cases must have occurred for the first time.

Case 1: There are adjacent cells with nonempty sets A and B (where $\max(A) \leqslant \min(B)$) in the same row on diagonals i and $i+1$ respectively with no cells appearing directly below these cells, as illustrated on the left side of Figure 3.1. Moreover, by minimality, we have an integer x with the following properties:

(1) $i+1 \in h^x$ and $x < \min(A)$,

(2) there does not exist a y with $x \leqslant y < \min(B)$ and $i+2 \in h^y$.

By applying semistandardness, a cell containing x is created directly below the cell containing the set A as in the right side of Figure 3.1. Furthermore, by (2), for all $x \leqslant y < \min(B)$, we have that every letter in h^y is either at most $i+1$ or at least $i+3$. It follows that, after possibly applying commutativity ($i+1$ with letters at most $i-1$ or at least $i+3$) and the idempotent relation, $h^{\min(B)} \ldots h^{x+1} h^x$ is equivalent to one containing the braid subword $i+1\ i\ i+1$. This implies that $\mathbf{h}$ is equivalent to a Hecke word containing the same braid subword.

Case 2: There are adjacent cells with nonempty sets A and B in the same column on diagonals $i+1$ and i respectively with no cells appearing directly to the left of these cells, as illustrated on the left side of Figure 3.2. Moreover, by minimality, we have an integer x with the following properties:

(1) $i \in h^x$ and $x \leqslant \min(A)$,

(2) there does not exist a y with $x < y \leqslant \min(B)$ and $i-1 \in h^y$.

By applying semistandardness, a cell containing x is created directly to the left of the cell containing the set A as in the right side of Figure 3.2. Furthermore, by (2), for all $x < y \leqslant \min(B)$, we have that every letter in h^y is either at most $i-2$ or at least i. Similar to the argument in Case 1, $h^{\min(B)} \ldots h^{x+1} h^x$ is equivalent to one containing the braid subword $i\ i+1\ i$. This implies that $\mathbf{h}$ is equivalent to a word in $\mathcal{H}_0(n)$ containing the same braid subword.

$$\begin{array}{|c|c|}\hline A_{i} & B_{i+1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline A_{i} & B_{i+1} \\ \hline x_{i+1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

FIGURE 3.1. A forbidden case while inverting the residue map.

$$\begin{array}{|c|}\hline B_{i} \\ \hline A_{i+1} \\ \hline \end{array} \longrightarrow \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{B_{i}} \\ \hline \multicolumn{1}{|c|}{x_{i}} & \multicolumn{1}{c|}{A_{i+1}} \\ \hline \end{array}$$

FIGURE 3.2. Another forbidden case while inverting the residue map.

The above arguments imply that the image of res is contained in $\mathcal{H}^{m,\star}$. Conversely, if $\mathbf{h}$ is fully-commutative, then the algorithm for res^{-1} does not produce Case 1 or Case 2 above and hence the resulting tableau T is of skew shape which in turn implies that the algorithm is well-defined (up to shifts along the diagonal if a gap of size at least 3 occurs in the labels). $\square$

If the skew shape λ/μ of the tableau T is known, then one may simplify the procedure above noting that the filling of i specified by letters in h^i must occur along a horizontal strip for all $i = m, m-1, \ldots, 1$. In this case, the recovered tableau T is unique and there is no shift ambiguity if a gap of size at least 3 occurs in the labels.

EXAMPLE 59. *Let* $\mathbf{h} = (61)(752)(75)(762)$ *be a decreasing factorization of* $w = 651762$.

In the algorithm for the inverse of the residue map, the entry 4 is placed on diagonal 1 and 6, respectively. Due to semistandardness, the entry 3 in diagonal 2 must be placed below the 4 in diagonal 1, while the 3's in diagonals 5 and 7 are respectively to the left and below the 4 in diagonal 6. Continuing with the remaining fillings, we have two possibilities:

$$T_1 = \begin{array}{cccc} \cline{1-1} \multicolumn{1}{|c|}{4_{1}} & & & \\ \cline{1-1} \multicolumn{1}{|c|}{13_{2}} & & & \\ \cline{1-1} & & \multicolumn{1}{|c|}{23_{5}} & \multicolumn{1}{c|}{4_{6}} \\ \cline{3-4} & & \multicolumn{1}{|c|}{1_{6}} & \multicolumn{1}{c|}{123_{7}} \\ \cline{3-4} \end{array},$$

or

$$T_2 = \begin{array}{ccc} 4_1 & & \\ 13_2 & & \\ & 23_5 & 4_6 \\ & 1_6 & 123_7 \end{array},$$

where $T_1 \in \mathsf{SVT}^4((4,4,1,1)/(2,2))$ *and* $T_2 \in \mathsf{SVT}^4((3,3,1,1,1)/(1,1,1))$. *Note that they indeed just differ by a shift along diagonals as stated in Proposition 58.*

Example 60. *Let* $\mathbf{h} = (8431)(863)(8654)(941)$ *be a decreasing factorization of* $w = 84396541$. *Suppose that* $\mathbf{h} = \mathsf{res}(T)$, *where* $T \in \mathsf{SVT}^4(\lambda/\mu)$ *with* $\lambda/\mu = (5,5,4,2,1)/(4,4,1,1)$.

Then, we fill in 4 *along the diagonals with labels 1, 3, 4, 8 respectively, noting that the* 4 *in diagonal 4 is to the right of the* 4 *in diagonal* 3 *(due to the semistandardness of* T*). Continuing with the remaining fillings, we have*

$$T = \begin{array}{ccccc} 14_1 & & & & \\ & 34_3 & 4_4 & & \\ & 12_4 & 2_5 & 23_6 & \\ & & & & 234_8 \\ & & & & 1_9 \end{array}.$$

Theorem 61. *The crystal on set-valued tableaux* $\mathsf{SVT}^m(\lambda/\mu)$ *and the crystal on decreasing factorizations* $\mathcal{H}^{m,\star}$ *intertwine under the residue map. That is, the following diagrams commute:*

$$\begin{tikzcd} \mathsf{SVT}^m(\lambda/\mu) \arrow[r, "\mathsf{res}"] \arrow[d, "f_k"] & \mathcal{H}^{m,\star} \arrow[d, "f_k^\star"] \\ \mathsf{SVT}^m(\lambda/\mu) \arrow[r, "\mathsf{res}"] & \mathcal{H}^{m,\star} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{SVT}^m(\lambda/\mu) \arrow[r, "\mathsf{res}"] \arrow[d, "e_k"] & \mathcal{H}^{m,\star} \arrow[d, "e_k^\star"] \\ \mathsf{SVT}^m(\lambda/\mu) \arrow[r, "\mathsf{res}"] & \mathcal{H}^{m,\star}. \end{tikzcd}$$

Proof. Let $T \in \mathsf{SVT}^m(\lambda/\mu)$, $\mathbf{h} = \mathsf{res}(T)$ and $\ell = \ell(\lambda)$. We prove the following three statements associated to $f_k(T)$ and $f_k^\star(\mathbf{h})$.

(1) We claim that if there is no unpaired k in T, then $f_k^\star$ annihilates $\mathbf{h}$. Furthermore, if the rightmost unpaired k in cell b of T has content x, then x is also the largest unpaired letter in h^k.

For the proof of (1) it suffices to notice that the signature rule on tableaux is equivalent to the pairing process for decreasing factorizations of $\mathcal{H}_0(n)$. We rephrase the pairing procedure for decreasing factorizations on tableaux:

- At the beginning, no letter is paired.
- Then start with the rightmost column and work westward.
- Successively, for each $k+1$, compute its content a, then pair it with the k of smallest content weakly greater than a that is yet unpaired.

Next, we argue that the signature rule yields the same result on the rightmost unpaired letter. Assume we are looking at cell b containing the current $k+1$ with content a.

Case (a): Suppose there is no unpaired k with content a but at least one unpaired k with strictly greater content(s). Then pair it with the current $k+1$. This is the direct signature rule.

Case (b): Suppose there is no unpaired k with content weakly greater than a, then this $k+1$ is unpaired. This is also the direct signature rule.

Case (c): Suppose there is an unpaired k with content a. Then it must be either in the same cell b, or one row below and one column to the left of b on the diagonal labeled a. If they are in the same cell, then the pairing is the direct signature rule.

Otherwise, there must be cells to the left and below b since the shape is skew. Suppose cell b is in row r. Consider the rightmost entry in cell (r,j) in row r containing a $k+1$, and the leftmost entry in cell $(r-1,q)$ in row $r-1$ containing a k. Considering this as the first of a consecutive occurrence, cell b is cell (r,j), so we have $\ell+j-r=a$. By semistandardness and the condition that the shape is skew, we can partially fill out the involved subtableau of T for rows $r-1,r$ from column q to j:

$k+1\ _{\ell+q-r}$	$k+1\ _{\ell+q+1-r}$	$\dots$	$k+1\ _{\ell+j-1-r}$	$k+1\dots{}_{\ell+j-r}$
$\dots k\ _{\ell+q-r+1}$	$k\ _{\ell+q-r}$	$\dots$	$k\ _{\ell+j-r}$	$k\ _{\ell+j-r+1}$

.

All the cells (s,t) with $q<t<j$ and $s\in\{r,r-1\}$ and the cells (r,q) and $(r-1,j)$ are single-valued by semistandardness as shown in the above figure.

From the $k+1$ in (r,j), we start the pairing process. First, we claim that the k in cell $(r-1,j)$ must be unpaired at this point. Suppose that there is a $k+1$ to the east of cell (r,j) with content smaller or equal to $\ell + j - r + 1$, then it must be cell $(r, j + 1)$, which violates that (r, j) is the rightmost cell in row r containing a $k + 1$. Then the pairing says the $k + 1$ in cell (r, t) pairs with the k in cell $(r-1, t-1)$ for $q < t \leqslant j$. Lastly, the $k+1$ in cell (r, q) has to pair with the previously unpaired k in cell $(r - 1, j)$ since there are no unpaired k with label greater or equal to $\ell + q - r$ and smaller than $\ell + j - r + 1$.

Although the pairing is different than the usual signature rule pairing, which pairs $k + 1, k$ in the same column, the $2(j - q + 1)$ letters end up being paired. Since it will not influence which one will be the rightmost unpaired letter, it is still equivalent to the signature rule.

So in any case, the pairing is equivalent to the signature rule. Thus, the rightmost unpaired k in T corresponds to the largest unpaired letter in h^k.

(2) We claim that if f_k changes the rightmost unpaired k in T to a $k + 1$ (with content x) without moving it, then $f_k^\star$ moves a letter x from h^k to h^{k+1}.

Since f_k does not need to move any letter, it means the cell to the right of b, denoted by $b^\rightarrow$, does not contain a k. It is the only cell with content $x + 1$ that could contain a k. This implies that $x + 1 \notin h^k$. By Definition 48, $f_k^\star$ moves x from h^k to h^{k+1}.

(3) We claim the following. If f_k changes a k from $b^\rightarrow$ into a $k + 1$ and moves to cell b, then $f_k^\star$ removes an $x + 1$ from h^k and changes it to an x in h^{k+1}.

That f_k needs to move a number means that k and $k + 1$ are in $b^\rightarrow$, which implies that $x + 1 \in h^k \cap h^{k+1}$. By Definition 48, $f_k^\star$ removes the $x + 1$ from h^k and adds an x to h^{k+1}.

We have proved the three statements and they complete the proof that f_k and $f_k^\star$ intertwine under the residue map. The proof is similar for e_k and $e_k^\star$. $\square$

Corollary 62. *The set $\mathcal{H}^{m,\star}$, together with crystal operators $e_i^\star$ and $f_i^\star$ for $1 \leqslant i < m$ and weight function* wt *defined in Definition 48, is a Stembridge crystal.*

Proof. By Theorem 61 and the fact that the residue map preserves the weight and is invertible, this follows from the fact that $\mathsf{SVT}^m(\lambda/\mu)$ is a Stembridge crystal proven in [**MPS20**, Theorem 3.9] (see also Theorem 55). $\square$

Example 63. *Consider the tableau T (with labels in red) given by*

$$T = \begin{array}{|c|c|} \cline{1-1} 3_{\color{red}1} & \multicolumn{1}{c}{} \\ \hline 1_{\color{red}2} & 123_{\color{red}3} \\ \hline \end{array},$$

with $\mathsf{res}(T) = (31)(3)(32)$.

For the crystal operators on set-valued tableaux we obtain

$$f_1(T) = \begin{array}{|c|c|} \cline{1-1} 3_{\color{red}1} & \multicolumn{1}{c}{} \\ \hline 12_{\color{red}2} & 23_{\color{red}3} \\ \hline \end{array},$$

with $\mathsf{res}\,(f_1(T)) = (31)(32)(2)$. *Then it can be easily checked that the following diagram commutes:*

$$\begin{tikzcd}
T = \begin{array}{|c|c|} \cline{1-1} 3_{\color{red}1} & \multicolumn{1}{c}{} \\ \hline 1_{\color{red}2} & 123_{\color{red}3} \\ \hline \end{array} \arrow[r, "\mathsf{res}"] \arrow[d, "f_1"] & (31)(3)(32) \arrow[d, "f_1^\star"] \\
f_1(T) = \begin{array}{|c|c|} \cline{1-1} 3_{\color{red}1} & \multicolumn{1}{c}{} \\ \hline 12_{\color{red}2} & 23_{\color{red}3} \\ \hline \end{array} \arrow[r, "\mathsf{res}"] & (31)(32)(2).
\end{tikzcd}$$

3.2. Insertion algorithms

In this section, we discuss two insertion algorithms for decreasing factorizations in $\mathcal{H}^m$ (resp. $\mathcal{H}^{m,\star}$). The first is the Hecke insertion introduced by Buch et al. [**BKS$^+$08**], which we review in Section 3.2.1. We prove a relationship between Hecke insertion and the residue map (see Theorem 68). In particular, this proves [**MPS20**, Open Problem 5.8] for fully-commutative permutations. The second insertion is a new insertion, which we call $\star$-insertion, introduced in Section 3.2.2. It goes from fully-commutative decreasing factorizations in the 0-Hecke monoid to pairs of (transposes of) semistandard tableaux of the same shape and is well-behaved with respect to the crystal operators.

3.2.1. Hecke insertion. Hecke insertion was first introduced in [**BKS$^+$08**] as column insertion. Here we state the row insertion version as in [**PP16**]. In this section, we represent a decreasing factorization $\mathbf{h} = h^m h^{m-1} \cdots h^1$, where $h^i = h^i_1 h^i_2 \dots h^i_{\ell_i}$, by a *decreasing Hecke biword*

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} m & \dots & m & \dots & 1 & \dots & 1 \\ h^m_1 & \dots & h^m_{\ell_m} & \dots & h^1_1 & \dots & h^1_{\ell_1} \end{bmatrix}.$$

In addition, we say that $[\mathbf{k}, \mathbf{h}]^t$ is *fully-commutative* if $\mathbf{h}$ is fully-commutative.

EXAMPLE 64. *Consider the decreasing factorization* $\mathbf{h} = (1)(2)(31)(\ )(32)$. *Then the corresponding biword* $[\mathbf{k}, \mathbf{h}]^t$ *is*

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 & 3 & 2 \end{bmatrix}.$$

DEFINITION 65. *Starting with a decreasing Hecke biword* $[\mathbf{k}, \mathbf{h}]^t$*, we define* Hecke row insertion *from the right. The insertion sequence is read from right to left. Suppose there are* n *columns in* $[\mathbf{k}, \mathbf{h}]^t$.

Start the insertion with (P_0, Q_0) *being both empty tableaux. We recursively construct* (P_{i+1}, Q_{i+1}) *from* (P_i, Q_i)*. Suppose the* $(n-i)$*-th column in* $[\mathbf{k}, \mathbf{h}]^t$ *is* $[y, x]^t$.

We describe how to insert x *into* P_i*, denoted* $P_i \leftarrow x$*, by describing how to insert* x *into a row* R*. The insertion may modify the row and may produce an output integer, which will be inserted into the next row. First, we insert* x *into the first row* R *of* P_i *following the rules below:*

(1) *If* $x \geqslant z$ *for all* $z \in R$*, the insertion terminates in either of the following ways:*
   (a) *If we can append* x *to the right of* R *and obtain an increasing tableau, the result* P_{i+1} *is obtained by doing so; form* Q_{i+1} *by adding a box with* y *in the same position where* x *is added to* P_i.
   (b) *Otherwise row* R *remains unchanged. Form* Q_{i+1} *by adding* y *to the existing corner of* Q_i *whose column contains the rightmost box of row* R.
(2) *Otherwise, there exists a smallest* z *in* R *such that* $z > x$.
   (a) *If replacing* z *with* x *results in an increasing tableau, then do so. Let* z *be the output integer to be inserted into the next row.*
   (b) *Otherwise, row* R *remains unchanged. Let* z *be the output integer to be inserted into the next row.*

The entire Hecke insertion terminates at (P_n, Q_n) after we have inserted every letter from the Hecke biword. The resulting insertion tableau P_n is an increasing tableau, meaning that both rows and columns of P_n are strictly increasing. If $\mathbf{k} = (n, n-1, \ldots, 1)$, the recording tableau Q_n is a standard set-valued tableau.

Example 66. *Take $[\mathbf{k}, \mathbf{h}]^t$ from Example 64. Following the Hecke row insertion, we compute its insertion tableau and recording tableau:*

$$\emptyset \to \begin{array}{|c|}\hline 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array} = P,$$

$$\emptyset \to \begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 13 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 13 \\ \hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline 1 & 13 \\ \hline\end{array} = Q.$$

Example 67. *Note that the recording tableau for the Hecke insertion of Definition 65 is not always a semistandard set-valued tableau. For example, for $\mathbf{h} = (21)(41)$ we have*

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix}$$

and

$$P = \begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array} \qquad \text{and} \qquad Q = \begin{array}{|c|c|}\cline{1-1} 22 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}.$$

However, in Theorem 68 below we will see that in certain cases it is.

Theorem 68. *Let $T \in \mathsf{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \mathsf{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q). Then $Q = T$.*

Remark 69. *Combining Theorems 68 and 61 shows that Hecke insertion from right to left (as opposed to left to right in [**PP16**]) intertwines the crystal on set-valued tableaux and the $\star$-crystal, even though in general it is not always well-defined (see Example 67). This resolves [**MPS20**, Open Problem 5.8] when the decreasing factorizations are fully-commutative. Even when $\mathbf{h}$ is fully-commutative, but does not correspond to a straight-shaped tableau under res^{-1} as in Example 67, one can fill the skew part with small enough numbers and apply the Hecke insertion on this tableau.*

In the above example

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 4 & 1 & 3 & 2 \end{bmatrix} \qquad \textit{with} \qquad Q = T = \begin{array}{|c|c|c} \hline 12 & 2 & \multicolumn{1}{c}{} \\ \hline 0 & 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array}.$$

*Note, however, that unlike in [**MPS20**] we use row Hecke insertion from right to left rather than column insertion from left to right (in analogy to [**MS16**] for Edelman–Greene insertion).*

Since $k \in T(i,j)$ if and only if $\ell + j - i \in h^k$ under the residue map, where $\ell = \ell(\lambda)$ and h^k is the k-th factor of $\mathbf{h}$, the statement of Theorem 68 is equivalent to applying Hecke insertion on the entries of T sorted first by ascending order of entries, followed by ascending diagonal content.

Example 70. *Let T be the semistandard set-valued tableau*

$$T = \begin{array}{|c|c|} \hline 2_1 & 4_2 \\ \hline 1_2 & 23_3 \\ \hline \end{array}.$$

The insertion sequence by entry is listed in the table below:

Cell	*(1,1)*	*(2,1)*	*(1,2)*	*(1,2)*	*(2,2)*
Content	*2*	*1*	*3*	*3*	*2*
Entry	*1*	*2*	*2*	*3*	*4*

We will prove Theorem 68 by induction by considering all subtableaux of T, obtained by adding the entries in T one by one in the order above:

$$\emptyset \to \begin{array}{|c|} \hline 1_2 \\ \hline \end{array} \to \begin{array}{|c|} \hline 2_1 \\ \hline 1_2 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2_1 & \multicolumn{1}{c}{} \\ \hline 1_2 & \multicolumn{1}{c|}{2_3} \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2_1 & \multicolumn{1}{c}{} \\ \hline 1_2 & \multicolumn{1}{c|}{23_3} \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 2_1 & 4_2 \\ \hline 1_2 & 23_3 \\ \hline \end{array} = T.$$

In addition, the corresponding sequence of insertion tableaux and recording tableaux is listed here:

$$\emptyset \to \begin{array}{|c|} \hline 2 \\ \hline \end{array} \to \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} = P.$$

$$\emptyset \to \begin{array}{|c|} \hline 1 \\ \hline \end{array} \to \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \to \begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{23} \\ \hline \end{array} \to \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 23 \\ \hline \end{array} = Q.$$

Proof of Theorem 68. We prove the theorem by proving the following more specific statement.

For a given step in the insertion process, suppose that the entries of T that are involved so far form a nonempty subtableau T' of T with shape μ containing cell $(1,1)$, and the insertion tableau and recording tableau at the corresponding step are $P(T')$ and $Q(T')$. Then, they both have shape μ, and the entry of cell (i,j) of $P(T')$ is $\ell + j - \mu'_j + i - 1$, and $Q(T') = T'$, where μ' is the transpose of the partition μ and $\ell := \lambda'_1 = \ell(\lambda)$.

We prove this by induction on subtableaux of T.

Base step: Suppose T' only contains a single cell $(1,1)$ and $T'(1,1) = S$, where S is a subset of $T(1,1)$ with cardinality d. Then $P(T')$ is obtained by inserting d times the number ℓ. So we have $P(T') = \boxed{\ell}$ and $Q(T') = T'$. Here $\mu = (1)$, so for $(i,j) = (1,1)$, we have $\ell + j - \mu'_j + i - 1 = \ell$.

Inductive step: Suppose that the statements hold for some subtableau T' of shape μ. Assume the next insertion step involves adding the entry k in cell (p,q) of T to T' to obtain T''. There are two cases: (1) the cell (p,q) is already in T', or (2) the cell (p,q) is not in T'.

Case (1): We must have (p,q) to be an inner corner of T' (no cell is to its right or above it), so $p = \mu'_q$ and $p > \mu'_{q+1}$. In this case, k is recorded in $Q(T')$. Then by the induction on T', every cell (i,j) of $P(T')$ has value $\ell + j - \mu'_j + i - 1$. To determine the insertion path of $P(T') \leftarrow \ell + q - p$, we compute the columns q and $q+1$ of $P(T')$ as follows:

row number	q-th column	$(q+1)$-st column
p	$\ell + q - 1$	
	$\vdots$	
$\mu'_{q+1} < p$	$\ell + q - p + \mu'_{q+1} - 1$	$\ell + q$
	$\vdots$	$\vdots$
2	$\ell + q - p + 1$	$\ell + q + 2 - \mu'_{q+1}$
1	$\ell + q - p$	$\ell + q + 1 - \mu'_{q+1}$

Following Case 2(b) of Hecke insertion, the insertion path is vertically up column $q+1$. At the top of the column, $\ell + q$ is inserted into row $\mu'_{q+1} + 1$. Furthermore, $\ell + q$ is greater than $\ell + q - p + \mu'_{q+1}$ in cell $(\mu'_{q+1} + 1, q)$ because $p > \mu'_{q+1}$. By Hecke insertion Case 1(b), the insertion ends in row $\mu'_{q+1} + 1$. Also $P(T')$ is unchanged, and k is recorded in cell (p,q) of $Q(T')$ since it is the corner whose column contains the rightmost box of row $\mu'_{q+1}+1$. In this case, we get $Q(T'') = T''$. Since the shape μ is unchanged, we have that $P(T'') = P(T')$ also satisfies the statement.

Case (2): If cell (p,q) is not in T', then it must be an outer corner of T', so $\mu'_q = p-1$ and $\mu'_{q-1} > p-1$. Specifically, two cases can happen: (a) $p = 1$ and $(1, q-1) \in T'$, (b) both $(p-1,q),(p,q-1) \in T'$, or $q = 1$ and $(p-1,1) \in T$.

Case 2(a): The first row of $P(T')$ is $\ell+1-\mu'_1, \ldots, \ell+j-\mu'_j, \ldots, \ell+(q-1)-\mu'_{q-1}$. Since $\ell+q-p = \ell+q-1 > \ell+(q-1)-\mu'_{q-1}$, it is appended to the end of the first row which is the cell $(1,q)$. The letter k is recorded in the same new cell of $Q(T')$. In this case, the only entry in P that is changed is $(1,q)$, and its entry $\ell+q-1$ satisfies the statement. Also $Q(T'')$ equals T''.

Case 2(b): Since entry $(i,q-1)$ of $P(T')$ is $\ell+q-1-\mu'_{q-1}+i-1$ and entry (i,q) of $P(T')$ is $\ell+q-\mu'_q+i-1$, the number $q-p+\ell$ is in-between the two when $i = 1$. So the insertion starts by bumping $(1,q)$. To get the insertion path, we compute columns $q-1$ and q as follows:

row number	$(q-1)$-st column	q-th column
μ'_{q-1}	$\ell+q-2$	
	...	
$p-1$	$\ell+q+p-\mu'_{q-1}-3$	$\ell+q-1$
	...	...
2	$\ell+q-\mu'_{q-1}$	$\ell+q-p+2$
1	$\ell+q-1-\mu'_{q-1}$	$\ell+q-p+1$

By Hecke insertion Case 2(a), $\ell+q-p$ is placed in cell $(1,q)$ and the original column q is shifted one position higher. By Hecke insertion Case 1(a), the insertion terminates at row p and the original entry in cell $(p-1,q)$ is appended at the rightmost box of row p. Thus, μ'_q increases by 1. The updated entries in column q still satisfy the statement. Since the entries in other columns of $P(T')$ are unchanged and μ'_j is unchanged for $j \neq q$, they also satisfy the statement. So we have $P(T'')$ satisfies the statement. The letter k is inserted into the new cell (p,q) of $Q(T')$, which makes $Q(T'') = T''$.

Thus, the statement holds, proving the theorem. $\square$

3.2.2. The $\star$-insertion. We define a new insertion algorithm, which we call $\star$-insertion, from fully-commutative decreasing Hecke biwords $[\mathbf{k},\mathbf{h}]^t$ to pairs of tableaux P and Q, denoted by $\star([\mathbf{k},\mathbf{h}]^t) = (P,Q)$, as follows.

Definition 71. *Fix a fully-commutative decreasing Hecke biword $[\mathbf{k},\mathbf{h}]^t$. The insertion is done by reading the columns of this biword from right to left.*

Begin with (P_0,Q_0) being a pair of empty tableaux. For every integer $i \geqslant 0$, we recursively construct (P_{i+1},Q_{i+1}) from (P_i,Q_i) as follows. Let $[q,x]^t$ be the i-th column (from the right) of $[\mathbf{k},\mathbf{h}]^t$. Suppose that we are inserting x into row R of P_i.

- **Case 1:** *If R is empty or $x > \max(R)$, then form P_{i+1} by appending x to row R and form Q_{i+1} by adding q in the corresponding position to Q_i. Terminate and return (P_{i+1},Q_{i+1}).*
- **Case 2:** *Otherwise, if $x \notin R$, locate the smallest y in R with $y > x$. Bump y with x and insert y into the next row of P_i.*
- **Case 3:** *Otherwise, if $x \in R$, locate the smallest y in R with $y \leqslant x$ and interval $[y,x]$ contained in R. Row R remains unchanged and y is to be inserted into the next row of P_i.*

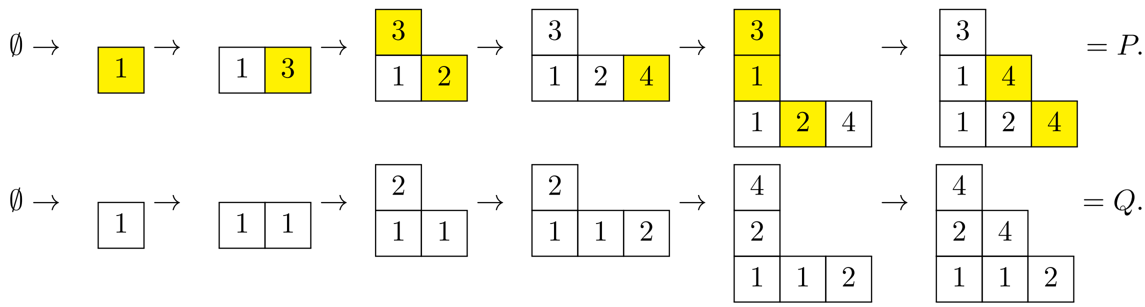
Denote $(P,Q) = (P_\ell,Q_\ell)$ if $[\mathbf{k},\mathbf{h}]^t$ has length ℓ. We define the $\star$-insertion *by $\star([\mathbf{k},\mathbf{h}]^t) = (P,Q)$.*

Furthermore, denote by $P \leftarrow x$ the tableau obtained by inserting x into P. The collection of all cells in $P \leftarrow x$, where insertion or bumping has occurred is called the insertion path *for $P \leftarrow x$. In particular, in Case 1 the newly added cell is in the insertion path, in Case 2 the cell containing the bumped letter y is in the insertion path, and in Case 3 the cell containing the same entry as the inserted letter is in the insertion path.*

Example 72. *Let*

$$\begin{bmatrix}\mathbf{k}\\ \mathbf{h}\end{bmatrix} = \begin{bmatrix}4 & 4 & 2 & 2 & 1 & 1\\ 4 & 2 & 4 & 2 & 3 & 1\end{bmatrix}.$$

The corresponding sequence of insertion tableaux and recording tableaux under the $\star$-insertion is listed here:



Then we have $\star([\mathbf{k},\mathbf{h}]^t) = (P, Q)$*, and the cells in the insertion paths at each step are highlighted in yellow.*

Lemma 73. *Let* $[\mathbf{k},\mathbf{h}]^t$ *be a fully-commutative decreasing Hecke biword. Suppose that* $\star([\mathbf{k},\mathbf{h}]^t) = (P, Q)$*. Then, the following statements hold:*

(1) P^t *is semistandard and* Q *has the same shape as* P*.*

(2) *Let* x *be an integer such that* $x \cdot \mathbf{h}$ *is fully-commutative. Then the insertion path for* $P \leftarrow x$ *goes weakly to the left.*

Proof. We will prove (1) by induction on the number of cells of P. Statement (2) will follow by some results in the proof of statement (1).

Consider the leftmost column $[q, x]^t$ of $[\mathbf{k},\mathbf{h}]^t$ and let $[\mathbf{k}',\mathbf{h}']^t$ be the Hecke biword formed by taking the remaining columns in the same order. If the $\star$-insertion of $[\mathbf{k}',\mathbf{h}']^t$ yields (P', Q'), note that we have $P = P' \leftarrow x$. For all integers $j \geqslant 1$, denote by R_j the (possibly empty) j-th row of P'. Denote by u the entry to be inserted into R_j and B_j as the cell in the insertion path at R_j, where $1 \leqslant j \leqslant k$. Additionally, if bumping occurs at R_j, denote the entry bumped out as y.

(1) We will prove that if $(P')^t$ is semistandard, then the transpose of the updated tableau is semistandard.

Case (a): Suppose that the insertion terminates at R_1. Then Case 1 of the $\star$-insertion has occurred, with a cell containing x appended at the end of the row. If R_1 is nonempty, then $x > \max(R_1)$. Additionally, as $(P')^t$ is semistandard, integers strictly increase along R_1 but weakly increase along the column containing B_1. Hence, the transpose of the resulting tableau P is semistandard.

Case (b): Suppose that insertion terminates at R_k, where $k > 1$. We will show that for all $1 \leqslant j \leqslant k$, the changes introduced at row R_j of P' maintain the property that the transpose of the updated tableau is semistandard.

Case (b)(i): Suppose that $j = k$. In this case, a new cell containing u is appended at the end of R_k and $u > \max(R_k)$ if the row is nonempty, proving that the integers increase strictly along R_k.

If Case 2 occurs at R_{k-1}, then u is the entry bumped out of R_{k-1} with the property that when u' is inserted into R_{k-1}, $u \in R_{k-1}$ is the smallest entry with $u > u'$. Let z be the entry below cell B_k. We claim that $z \leqslant u$. If we assume instead that $z > u$, then the cell containing z is strictly to the right of B_{k-1}. However, the cell above B_{k-1} has value greater than u since $(P')^t$ is semistandard and $u \notin R_k$. This contradicts the minimality of u', as u' is greater than this value, hence proving the claim.

If Case 3 occurs at R_{k-1}, then u is bumped out of R_{k-1} with the property that when u' is inserted into R_{k-1}, $u \in R_{k-1}$ is the smallest entry with $[u, u'] \subseteq R_{k-1}$. Let z be the entry below cell B_k. Then, similar to the argument immediately before, $z \leqslant u'$. Hence, we have established that the integers weakly increase along the column containing B_k after u is appended at the end of R_k.

Case (b)(ii): Suppose that $1 \leqslant j < k$ and Case 2 occurs at R_j. Then y is the entry bumped out of R_j with the property that when u is inserted into R_j, $y \in R_j$ is the smallest entry with $y > u$. Thus, as $u \notin R_j$, for all entries z and z' respectively to the left and to the right of B_j, we have $z < u < y < z'$.

If Case 2 occurs at R_{j-1}, then u is bumped out of R_{j-1} with the property that when u' is inserted into R_{j-1}, $u \in R_{j-1}$ is the smallest entry with $u > u'$. Let z be the entry below cell B_j. Then by repeating the same argument as in the first subcase of in Case (b)(i), we obtain $z \leqslant u$.

If Case 3 occurs at R_{j-1}, then u was bumped out of R_{j-1} with the property that when u' is inserted into R_{j-1}, $u \in R_{j-1}$ is the smallest entry with $[u, u'] \subseteq R_{j-1}$. Let z be the entry below cell B_j. Then by repeating the same argument as in the second subcase of in Case (b)(i), we obtain $z \leqslant u'$.

Hence, we have established that integers increase weakly along the column containing B_j but increase strictly along R_j after u bumps out y.

Case (b)(iii): Suppose that $1 \leqslant j < k$ and Case 3 occurs at R_j. In this case, there are no changes to row R_j after inserting u and bumping y. Hence, it is trivial that integers increase weakly along the column containing B_j but increase strictly along R_j after u bumps out y.

In all cases, we have shown that if $(P')^t$ is semistandard, then the transpose of the updated tableau remains semistandard. Therefore, by induction on the number of added cells, we have proved that the insertion tableau P under $\star$-insertion satisfies the property that P^t is semistandard.

Finally, note that the shape of the recording tableau is modified only when Case 1 of the $\star$-insertion has occurred. In this case, a cell is added to form Q at the same position as the cell added to form P. Since we always begin with a pair of empty tableaux, by inducting on the number of added cells, the shapes of P and Q are the same.

(2) Suppose that the insertion terminates at R_k, where $k \geqslant 1$. We shall prove that B_j is weakly to the left of B_{j-1} for all $1 < j \leqslant k$ by revisiting the cases explored in the proof of part (1) (note that P should replace the role of P').

If Case 2 occurs at R_{j-1}, then u is the entry bumped out of R_{j-1} with the property that when u' is inserted into R_{j-1}, $u \in R_{j-1}$ is the smallest entry with $u > u'$. As in the proof of the first subcase of Case (b)(i) in part (1), we conclude that the entry z of the cell below B_k satisfies $z \leqslant u$, showing that B_j is weakly to the left of B_{j-1}.

If Case 3 occurs at R_{j-1}, then u was bumped out of R_{j-1} with the property that when u' is inserted into R_{j-1}, $u \in R_{j-1}$ is the smallest entry with $[u, u'] \subseteq R_{j-1}$. As in the proof of the second subcase of Case (b)(i) in part (1), we conclude that the entry z of the cell below B_j satisfies $z \leqslant u'$, B_j is weakly to the left of B_{j-1}.

This completes the proof. $\square$

For the following results, given a tableau P with positive integer entries, $\mathsf{row}(P)$ denotes its row reading word, obtained by reading these entries row-by-row starting from the top row (in French notation), reading from left to right. We will consider $\mathsf{row}(P)$ as an element in a fixed 0-Hecke monoid.

LEMMA 74. *Let P be a tableau such that P^t is semistandard and $\mathsf{row}(P)$ is fully-commutative. Let x be an integer such that $\mathsf{row}(P) \cdot x$ is fully-commutative. Then,*

$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(P) \cdot x. \tag{3.2.1}$$

PROOF. To prove (3.2.1), let us first prove the following statements for all row tableaux P:

- With the assumptions in lemma, if insertion terminates at row P while computing $P \leftarrow x$, then
$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(P) \cdot x.$$
- With the assumptions in lemma, if y is bumped from row P and P changes to P' while computing $P \leftarrow x$, then
$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} y \cdot \mathsf{row}(P').$$

Assume that insertion terminates at row P while computing $P \leftarrow x$. Then, Case 1 must have occurred and P changes to P', where P' is P appended by a cell containing x. Hence, we have

$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(P') \equiv_{\mathcal{H}_0} \mathsf{row}(P) \cdot x.$$

Assume that y is bumped from row P and P changes to P' while computing $P \leftarrow x$. Then, either Case 2 or Case 3 must have occurred.

If Case 2 occurs at P, then $x \notin P$ and there is a $y \in P$ with $y > x$; furthermore, y is the smallest value with such property. Write P as AyB, where A and B are the row subtableaux of P formed by entries to the left and to the right of y, respectively. Then, $P \leftarrow x$ is the tableau with row Axb followed by row y. As $x \notin P$, we have $\max(A) < x < y < \min(B)$. Hence by commutativity relations, for all $z \in B$, we have $z \cdot x \equiv_{\mathcal{H}_0} x \cdot z$ and for all $z \in A$, we have $z \cdot y \equiv_{\mathcal{H}_0} y \cdot z$, so that regarding A and B as words in $\mathcal{H}_0(n)$, we obtain

$$A \cdot y \equiv_{\mathcal{H}_0} y \cdot A, \quad B \cdot x \equiv_{\mathcal{H}_0} x \cdot B.$$

It follows that

$$\mathsf{row}(P)\cdot x \equiv_{\mathcal{H}_0} \mathsf{row}(AyB)\cdot x \equiv_{\mathcal{H}_0} A\cdot y\cdot B\cdot x \equiv_{\mathcal{H}_0} y\cdot A\cdot x\cdot B \equiv_{\mathcal{H}_0} y\cdot \mathsf{row}(AxB) \equiv_{\mathcal{H}_0} \mathsf{row}(P \leftarrow x).$$

If Case 3 occurs at P, then $x, y \in P$ with y being the smallest value such that $[y, x] \subseteq P$. Write P as ABC, where $B = [y, x]$, A and C are respectively the row subtableaux of P formed by entries to the left and to the right of B. Then, $P \leftarrow x$ is the tableau with row ABC followed by row y. As $\mathsf{row}(P) \cdot x$ was assumed to be fully-commutative, $x + 1 \notin P$. Furthermore, by minimality of y, $y > \max(A) + 1$. Hence, by commutativity relations, for all $z \in A$, we have $z \cdot y \equiv_{\mathcal{H}_0} y \cdot z$ and for all $z \in C$, we have $x \cdot z \equiv_{\mathcal{H}_0} z \cdot x$, so that

$$A \cdot y \equiv_{\mathcal{H}_0} y \cdot A, \quad C \cdot x \equiv_{\mathcal{H}_0} x \cdot C.$$

Moreover, by using the relations $p-1\ p\ p = p-1\ p-1\ p$, we have $y \cdot B \equiv_{\mathcal{H}_0} B \cdot x$. It follows that

$$\mathsf{row}(P)\cdot x \equiv_{\mathcal{H}_0} \mathsf{row}(ABC)\cdot x \equiv_{\mathcal{H}_0} A\cdot B\cdot C\cdot x \equiv_{\mathcal{H}_0} A\cdot y\cdot B\cdot C \equiv_{\mathcal{H}_0} y\cdot \mathsf{row}(ABC) \equiv_{\mathcal{H}_0} \mathsf{row}(P \leftarrow x).$$

Hence, the two statements above hold for all row tableaux P.

We are now ready to prove (3.2.1) in full generality. The result follows once we prove by induction on the number of rows of P, with the given setup above, that the following statements hold:

- If the insertion terminates within tableau P while computing $P \leftarrow x$, then

$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(P) \cdot x.$$

- If y is bumped from tableau P and P changes to P' while computing $P \leftarrow x$, then

$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} y \cdot \mathsf{row}(P').$$

Indeed, if P is a (possibly empty) row tableau, then we are done by the two previous statements that have been proved. Let $k \geqslant 1$ be an arbitrary integer. Assume that both statements mentioned above hold for all such tableaux P with k rows.

Let P be a tableau with $k+1$ rows with the setup as above. Then, we may consider the subtableau P^* formed from its first k rows and denote the final row as R. Note that $\mathsf{row}(P) = \mathsf{row}(R) \cdot \mathsf{row}(P^*)$ and $\mathsf{row}(R)$ is fully-commutative.

Assume that the changes from P to $P \leftarrow x$ involve at most the first k rows of P. Then $P \leftarrow x$ is the same tableau as $P^* \leftarrow x$ with an extra row R, so that by the inductive hypothesis,

$$\mathsf{row}(P \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(R) \cdot \mathsf{row}(P^* \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(R) \cdot \mathsf{row}(P^*) \cdot x \equiv_{\mathcal{H}_0} \mathsf{row}(P) \cdot x.$$

Now assume that the changes from P to $P \leftarrow x$ involves all $k+1$ rows of P. Let P' be the resulting tableau after performing these changes on P^* and let y be the entry bumped from the final row of P^*. Then, $P \leftarrow x$ is the tableau obtained by concatenating tableau $R \leftarrow y$ after P'.

If the insertion terminates at row R, then by the previous statements for all row tableaux and the inductive hypothesis, we obtain

$$\begin{aligned}\mathsf{row}(P \leftarrow x) &\equiv_{\mathcal{H}_0} \mathsf{row}(R \leftarrow y) \cdot \mathsf{row}(P') \equiv_{\mathcal{H}_0} \mathsf{row}(R) \cdot y \cdot \mathsf{row}(P') \\ &\equiv_{\mathcal{H}_0} \mathsf{row}(R) \cdot \mathsf{row}(P^* \leftarrow x) \equiv_{\mathcal{H}_0} \mathsf{row}(R) \cdot \mathsf{row}(P^*) \cdot x \equiv_{\mathcal{H}_0} \mathsf{row}(P) \cdot x.\end{aligned}$$

Otherwise, if the insertion bumps z from R and R changes to R' while computing $R \leftarrow y$, then it holds that the insertion bumps z from P while computing $P \leftarrow x$. In this case, if we denote P'' as the tableau P' concatenated by row R', then

$$\mathsf{row}(P) \cdot x \equiv_{\mathcal{H}_0} \mathsf{row}(R \leftarrow y) \cdot \mathsf{row}(P') \equiv_{\mathcal{H}_0} z \cdot \mathsf{row}(R') \cdot \mathsf{row}(P') \equiv_{\mathcal{H}_0} z \cdot \mathsf{row}(P'') \equiv_{\mathcal{H}_0} \mathsf{row}(P \leftarrow x).$$

This completes the induction. □

Remark 75. *Observe that the assumption that* $\mathsf{row}(P)$ *is fully-commutative implies that* $\mathsf{row}(R)$ *is fully-commutative for each row* R *of* P*. Moreover, in the proof of Lemma 74, if* x *is to be inserted into row* R *of* P *when computing* $P \leftarrow y$ *and* $x \in R$*, then the extra assumption that* $\mathsf{row}(P) \cdot x$ *is fully-commutative implies that* R *does not contain* $x+1$*.*

Lemma 76. *Let* P *be a tableau such that* P^t *is semistandard and* $\mathsf{row}(P)$ *is fully-commutative. Let* x, x' *be integers such that* $\mathsf{row}(P) \cdot x$ *and* $\mathsf{row}(P) \cdot xx'$ *are fully-commutative.*

Denote the insertion paths of $P \leftarrow x$ and $(P \leftarrow x) \leftarrow x'$ as π and π' respectively. Also, suppose that $P \leftarrow x$ and $(P \leftarrow x) \leftarrow x'$ introduce boxes B and B' respectively. Then the following statements about $\star$-insertion are true:

(1) *If $x < x'$, then π' is strictly to the right of π. Moreover, B' is strictly to the right of and weakly below B.*

(2) *If $x \geqslant x'$, then π' is weakly to the left of π. Moreover, B' is weakly to the left of and strictly above B.*

PROOF. Similar to Fulton's proof [**Ful96**] of the Row Bumping Lemma, we will keep track of the entries as they are bumped from a row. Consider a row R of tableau P and suppose that u and u' are to be inserted into R when computing $P \leftarrow x$ and $(P \leftarrow x) \leftarrow x'$ respectively, where $u < u'$. Denote by C (similarly C') the box in π (similarly π') that is also in R.

Case 1: $x < x'$. We will prove that the following assertions hold for R:

(a) If the insertion terminates at R while computing $P \leftarrow x$, then the insertion terminates at R while computing $(P \leftarrow x) \leftarrow x'$.

(b) C' is strictly to the right of C.

Note that the insertion terminates at R when computing $P \leftarrow x$ precisely when Case 1 of the $\star$-insertion occurs at R. Box C containing u is appended at the end of R. As $u' > u$, Case 1 occurs again at R with box C' containing u' appended to the right of C, so bumping does not occur at R when computing $(P \leftarrow x) \leftarrow x'$. This proves (a) and simultaneously, (b) for this case.

Let us assume that bumping occurs at R with y bumped out when computing $P \leftarrow x$.

Case A: If y is bumped from R because Case 2 occurs, the insertion at row R introduced to box C' occurs strictly to the right of C (containing u) because:

(i) If $u' > \max(R)$, then box C' containing u' is appended to the end of R by Case 1. In particular, C' appears strictly to the right of C.

(ii) Otherwise, since $u' > u$, the letter u' is inserted into a box C' strictly to the right of C with y' bumped out. If $u' \notin R$, Case 2 occurs and $y' > y$ because C' and C originally contained y' and y respectively. Else, $u' \in R$ and Case 3 occurs. Suppose that $[y', u']$ is the longest interval of consecutive integers contained in R. Since box C

that originally contained y is strictly to the left of C', we have $u < y < u'$. Therefore, $[u, u']$ cannot be contained in R, so $y < y'$.

Case B: Otherwise, y is bumped from R because Case 3 occurs when computing $P \leftarrow x$ and $[y, u]$ is the longest interval of consecutive integers contained in R by Remark 75. The insertion at row R introduced to box C' occurs strictly to the right of C (containing u) because:

(i) If either $u' > \max(R)$ or $u' \notin R$, then by similar arguments as in Case A(i) and Case A(ii), C' appears to the right of C. Furthermore, in the latter situation, by a similar argument in Case A(ii), we have $y < y'$.

(ii) Otherwise, $u' \in R$ and Case 3 occurs. As $u' > u$, u' is inserted into box C' strictly to the right of C with y' bumped out. In addition, $[y', u']$ is the longest interval of consecutive integers contained in R. As $\mathsf{row}(R)$ is fully-commutative before computing $P \leftarrow x$, $u + 1 \notin R$. Hence $[u, u']$ cannot be contained in R. It follows that $y \leqslant u < u + 1 < y'$.

Note that in the arguments above, we have also shown that if y and y' are bumped from R when computing $P \leftarrow x$ and $(P \leftarrow x) \leftarrow x'$ respectively, then $y < y'$. It follows that we may apply similar arguments in the rows following R. Since assertion (b) now holds for all rows, we conclude that π' is strictly to the right of π. In addition, π' cannot continue after π ends because of assertion (a). Considering that π' goes weakly left by Lemma 73, we conclude that box B' is strictly to the right of and weakly below B.

Case 2: $x \geqslant x'$. We will prove that the following assertions hold for R:

(1) If the insertion terminates at R while computing $P \leftarrow x$, then bumping occurs at R while computing $(P \leftarrow x) \leftarrow x'$.

(2) C' is weakly to the left of C.

If the insertion terminates at row R when computing $P \leftarrow x$, then Case 1 occurs and box C containing u is appended at the end of R. If $u' \in R$, Case 3 occurs at R with $y' \leqslant u' \leqslant u$ bumped out. Furthermore, box C' containing u' is weakly to the left of C. If $u' \notin R$, Case 2 occurs at R with $y' > u'$ bumped out and $u' < u$. We have $y' \leqslant u$ by minimality of y', so that box C' is weakly to the left of C. In either of the subcases, bumping occurs at R when computing $(P \leftarrow x) \leftarrow x'$. This proves (a) and simultaneously, (b) for this case.

Let us assume that bumping occurs at R with y bumped out when computing $P \leftarrow x$.

Case A: If y is bumped from R because Case 2 occurs when computing $P \leftarrow x$, the insertion at row R introduced to box C' occurs weakly to the left of C (containing u) because:

(i) If $u' \notin R$, then u' is inserted into box C' containing y' by Case 2, while bumping out this y'. As $u' < u$, we have $y' \leqslant u < y$ and that C' appears weakly to the left of C.

(ii) Otherwise, $u' \in R$ and Case 3 occurs. The letter u' is inserted into box C' weakly to the left of C as $u' \leqslant u$. In addition, if $[y', u']$ is the longest interval of consecutive integers in R, then y' is bumped out. Furthermore, we have $y' < y$ as C, which originally contained y before computing $P \leftarrow x$, is to the right of the box containing y'.

Case B: Otherwise, y is bumped from R because Case 3 occurs when computing $P \leftarrow x$. Let $[y, u]$ be the longest interval of consecutive integers that is contained in R. The insertion at row R introduced to box C' occurs weakly to the left of C (containing u) because:

(i) If $u' \notin R$, then $u' < u$, u' is inserted into box C' containing y' and y' is bumped out by Case 2. As $\mathsf{row}(P) \cdot x$ is fully-commutative, in particular $\mathsf{row}(R)$ is fully-commutative. Hence $u' < y$, so that C' is weakly to the left of box containing y (hence also weakly to the left of C). Furthermore, we have $y' \leqslant y$ by the minimality of y'.

(ii) If $u' \in R$, then either $u' = u$ or $u < u'$. The former case is easy as Case 3 occurs again with u' inserted into $C' = C$ and $y' = y$ is bumped out. If $u < u'$, then as $\mathsf{row}(P) \cdot x$ is fully-commutative, $\mathsf{row}(R)$ is fully-commutative, so that $u' < y - 1$. It follows that C' is strictly to the left of box containing y (hence also strictly to the left of C). Furthermore, we have $y' \leqslant u' < y - 1 < y$.

Note that in the arguments above, we have also shown that if y and y' are bumped from R when computing $P \leftarrow x$ and $(P \leftarrow x) \leftarrow x'$ respectively, then $y \geqslant y'$. It follows that we may apply similar arguments in the rows following R. Since assertion (b) now holds for all rows, we conclude that π' is weakly to the left of π. In addition, π' must continue after π ends because of assertion (a). Considering that π' goes weakly left by Lemma 73, we conclude that box B' is weakly to the left of and strictly above B. $\square$

Let U be a tableau such that U^t is semistandard and $\mathsf{row}(U)$ is fully-commutative. We describe the *reverse row bumping* for $\star$-insertion of U as follows. Locate an inner corner of U and remove entry y from that row. Perform the following operations until an entry is bumped out of the bottommost row. Suppose that we are reverse bumping y into a row R. If $y \notin R$, find the largest $x \in R$ with $x < y$; insert y and bump out x. Otherwise, $y \in R$, so find the largest $x \in R$ such that $[y, x]$ is the longest interval of consecutive integers. In this case, row R remains unchanged but x is bumped out. Then reverse bump x into the next row below unless there is no further row below. In this case, terminate and return the resulting tableau as T along with the bumped entry x. It is straightforward to see that reverse row bumping specified above reverses the bumping process specified by the $\star$-insertion.

Example 77. *Let U be the tableau*

$$U \;=\; \begin{array}{|c|ccc}
\cline{1-1}
5 & & & \\
\cline{1-1}
2 & & & \\
\cline{1-2}
2 & \multicolumn{1}{c|}{5} & & \\
\cline{1-3}
2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \\
\cline{1-3}
1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \\
\cline{1-3}
\end{array}.$$

By performing reverse row bumping on the topmost 5 in U, we obtain

$$T \;=\; \begin{array}{|c|ccc}
\cline{1-1}
5 & & & \\
\cline{1-2}
2 & \multicolumn{1}{c|}{5} & & \\
\cline{1-3}
2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \\
\cline{1-3}
1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \\
\cline{1-3}
\end{array}$$

and entry 2. It is also straightforward to check that $U = T \leftarrow 2$.

Corollary 78. *Let T be a tableau of shape λ such that T^t is semistandard and row(T) is fully-commutative. Let k be a positive integer.*

Let $x_1 < x_2 < \cdots < x_k$ (similarly $x_k \leqslant \cdots \leqslant x_2 \leqslant x_1$) be integers such that $\mathsf{row}(T) \cdot x_1 x_2 \ldots x_i$ is fully-commutative for all $1 \leqslant i \leqslant k$. Then, the collection of boxes added to T to form the tableau

$$U = ((T \leftarrow x_1) \leftarrow x_2) \cdots \leftarrow x_k$$

has the property that no two boxes are in the same column (similarly row).

Conversely, if U is a tableau of shape μ such that $\lambda \subseteq \mu$ and μ/λ consists of k boxes with no two boxes in the same column, i.e, a horizontal strip of size k (similarly row, i.e., a vertical strip of size k), then there is a unique tableau T of shape λ and unique integers $x_1 < x_2 < \cdots < x_k$ (similarly $x_k \leqslant \cdots \leqslant x_2 \leqslant x_1$) such that

$$U = ((T \leftarrow x_1) \leftarrow x_2) \cdots \leftarrow x_k.$$

In particular, if $(P, Q) = \star([\mathbf{k}, \mathbf{h}]^t)$, where $[\mathbf{k}, \mathbf{h}]^t$ is a fully-commutative decreasing Hecke biword, then Q is semistandard.

Proof. Assume that $x_1 < x_2 < \cdots < x_k$. By statement (1) of Lemma 76, the sequence of added boxes in $U = ((T \leftarrow x_1) \leftarrow x_2) \cdots \leftarrow x_k$ moves weakly below and strictly to the right when computing U. In particular, no two of the added boxes can be in the same column.

To recover the required tableau T and integers $x_1 < x_2 < \cdots < x_k$, perform reverse row bumping on the boxes specified by the shape μ/λ within U starting from the rightmost box, working from right to left. The tableau T and the integers $x_1, x_2, \ldots, x_k$ are uniquely determined by the operations. Moreover, by Lemma 76, the integers $x_k, x_{k-1}, \ldots, x_1$ obtained in the given order of operations satisfy $x_1 < x_2 < \cdots < x_k$.

Now assume $x_k \leqslant \cdots \leqslant x_2 \leqslant x_1$. By statement (2) of Lemma 76, the sequence of added boxes moves strictly above and weakly to the right when computing U. In particular, no two of the added boxes can be in the same row.

Similarly, one may perform reverse row bumping on the boxes specified by the shape μ/λ within U starting from the topmost box, working from top to bottom. Again, the operations uniquely determine the tableau T and the integers $x_1, x_2, \ldots, x_k$. Moreover, by Lemma 76, the integers $x_k, x_{k-1}, \ldots, x_1$ obtained in the given order of operations satisfy $x_k \leqslant \cdots \leqslant x_2 \leqslant x_1$.

Finally, note that in a decreasing Hecke biword $[\mathbf{k}, \mathbf{h}]^t$, where $\mathbf{h} = h^m \ldots h^2 h^1$, entries within a fixed a^i are inserted in increasing order. It follows that the collection of all boxes with label i form a horizontal strip within the tableau Q. Collecting all these horizontal strips with values i from m to i in order by using the converse recovers Q, implying that Q is semistandard. $\square$

THEOREM 79. *The $\star$-insertion is a bijection from the set of all fully-commutative decreasing Hecke biwords to the set of all pairs of tableaux (P, Q) of the same shape, where both P^t and Q are semistandard and* $\mathsf{row}(P)$ *is fully-commutative.*

PROOF. By successive applications of Lemma 74, if $(P, Q) = \star([\mathbf{k}, \mathbf{h}]^t)$, then as $\mathbf{h}$ is fully-commutative, $\mathsf{row}(P)$ is also fully-commutative. Hence, using Lemma 73 and Corollary 78, $\star$-insertion is a well-defined map from the set of all fully-commutative decreasing Hecke biwords to the set of all pairs of tableaux (P, Q) of the same shape with both P^t, Q semistandard and $\mathsf{row}(P)$ being fully-commutative.

It remains to show that the $\star$-insertion is an invertible map. Assume that P and Q are tableaux of the same shape with both P^t, Q semistandard and $\mathsf{row}(P)$ being fully-commutative. Since Q is semistandard, the collection of boxes with the same entry form a horizontal strip. Starting with the largest such entry m, perform reverse row bumping with the boxes in the strip from right to left. By Lemma 76, this recovers the entries in h^m in decreasing order. Repeating this procedure in decreasing order of entries recovers $\mathbf{h} = h^m \ldots h^2 h^1$, which automatically yields a decreasing Hecke biword $[\mathbf{k}, \mathbf{h}]^t$. Furthermore, by repeated applications of Lemma 74, since $\mathsf{row}(P)$ was fully-commutative, then the reverse word of $\mathbf{h}$ is fully-commutative, so that $\mathbf{h}$ is fully-commutative too. Finally, by repeated applications of the converse stated in Corollary 78, the recovered decreasing Hecke biword $[\mathbf{k}, \mathbf{h}]^t$ is unique. $\square$

3.3. Properties of the $\star$-insertion

In this section, we show that the $\star$-insertion intertwines with the crystal operators. More precisely, the insertion tableau remains invariant on connected crystal components under the $\star$-insertion as shown in Section 3.3.1 by introducing certain micro-moves. In Section 3.3.2, it is shown that the $\star$-crystal on $\mathcal{H}^{m,\star}$ intertwines with the usual crystal operators on semistandard tableaux on the recording tableaux under the $\star$-insertion. In Section 3.3.3, we relate the $\star$-insertion to the uncrowding operation.

3.3.1. Micro-moves and invariance of the insertion tableaux. In this section, we introduce certain equivalence relations of the $\star$-insertion in order to establish its relation with the $\star$-crystal. From now on we are focusing on the sequence in the insertion order. Since each decreasing factorization $\mathbf{h}$ is inserted from right to left, we look at $\mathbf{h}$ read from right to left.

Definition 80. *We define an equivalence relation through micro-moves on fully-commutative words in $\mathcal{H}_0(n)$.*

(1) Knuth moves, *for $x < z < y$:*
   (I1) $xyz \sim yxz$
   (I2) $zxy \sim zyx$
(2) Weak Knuth moves, *for $y > x + 1$:*
   (II1) $xyy \sim yxy$
   (II2) $xxy \sim xyx$
(3) Hecke move, *for $y = x + 1$:*
   (III) $xxy \sim xyy$

Note that the micro-moves preserve the relation $\equiv_{\mathcal{H}_0}$.

Similar relations have appeared in [**FG98**, Eq. (1.2)].

Example 81. *The* $13242 \in \mathcal{H}_0(5)$ *is equivalent to* 31242*,* 13422*,* 13224*,* 31224*, and itself.*

Next, we use the following notation on $\star$-insertion tableaux. For a single-row increasing tableau R, let R^x denote the first row of the tableau $R \leftarrow x$ and let $R(x)$ denote the output of the $\star$-insertion from the first row. If the $\star$-insertion outputs a letter, then denote it by $R(x)$; if x is appended to the end of the row R, then the output $R(x)$ is 0, which can be ignored. We always have $x \cdot 0 \sim x \sim 0 \cdot x$.

Example 82. *Let* $R = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 & 8 \\ \hline\end{array}$*, then the first row of $R \leftarrow 7$ is*

$$R^7 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 & 8 \\ \hline\end{array}$$

and $R(7) = 6$. Furthermore, the first row of $R^7 \leftarrow 9$ is $R^{7,9} =$ $\boxed{1 \mid 3 \mid 4 \mid 6 \mid 7 \mid 8 \mid 9}$ *and $R^8(9) = 0$.*

LEMMA 83. *Let R be a single-row increasing tableau, and x, y, z be letters such that $\mathsf{row}(R) \cdot x \cdot y \cdot z$ is fully-commutative. Let x', y', z' be letters such that $xyz \sim x'y'z'$. Following the above notation, we have*

$$R^{xyz} = R^{x'y'z'} \qquad \text{and} \qquad R(x)R^x(y)R^{xy}(z) \sim R(x')R^{x'}(y')R^{x'y'}(z').$$

PROOF. Let R be a single-row increasing tableau and M be the largest letter in R. First note that if $a \in R$ and $\mathsf{row}(R) \cdot a$ is fully-commutative, then $a + 1 \notin R$, see also Remark 75.

There are five types of equivalence triples, so we discuss them in 3 groups.

1. Cases (I1) and (II1): We have $x < z < y$, or $x < z = y$ and $y > x + 1$. In both cases $x' = y, y' = x, z' = z$.

Case (1A): $M < x < z \leqslant y$. In this case, the first resulting tableau is $R^{xyz} = \boxed{R \mid x \mid z}$ and the outputs are $R(x) = R^x(y) = 0$ and $R^{xy}(z) = y$. The second resulting tableau is $R^{yxz} = \boxed{R \mid x \mid z}$ and the outputs are $R(y) = 0 = R^{yx}(z)$ and $R^y(x) = y$. So we have $R^{xyz} = R^{yxz}$ and also $0 \cdot 0 \cdot y \sim 0 \cdot y \cdot 0$.

Case (1B): $x \leqslant M < z \leqslant y$. In this case, we have $R^{xy} = R^{yx}$ and $R(x) = R^y(x)$ since y is just appended to the end of R and does not influence how x is inserted. This gives $R^{xyz} = R^{yxz}$. The related outputs are $R^x(y) = R(y) = 0$, $R^{xy}(z) = R^{yx}(z) = y$. Thus, $R(x) \cdot 0 \cdot y \sim 0 \cdot R(x) \cdot y$.

Case (1C): $x < z \leqslant M < y$. In this case, we also have that $R^{xy} = R^{yx}$ and $R(x) = R^y(x)$, for the same reason as case (1B). Thus, we have $R^{xyz} = R^{yxz}$ and $R^{xy}(z) = R^{yx}(z)$. Since we have $R^x(y) = R(y) = 0$, $R(x) \cdot 0 \cdot R^{xy}(z) \sim 0 \cdot R^y(x) \cdot R^{yx}(z)$.

Case (1D): $x < z \leqslant y \leqslant M$. If x is the maximal letter in R^x, then it follows as case (1B). Otherwise, this case needs further separation into subcases.

Case 1D-(i): $x, y \notin R$. Then $x < R(x), y < R(y)$ and $R(x) \neq y$.
(1) If $R(x) < y$, then $R^x(y) = R(y)$ and $R^y(x) = R(x)$, which implies $R^{xy} = R^{yx}$, thus $R^{xyz} = R^{yxz}$ and $R^{xy}(z) = R^{yx}(z)$. Hence $R(x)R^x(y)R^{xy}(z) = R^y(x)R(y)R^{yx}(z)$. Since $R(x) < R^{xy}(z) \leqslant y <$

$R(y)$, we have $R(y) > R^y(x) + 1$ and for the outputs $R(x)R^x(y)R^{xy}(z) = R^y(x)R(y)R^{yx}(z) \sim R(y)R^y(x)R(y)R^{yx}(z)$ by move type (I1) or (II1).

(2) If $R(x) > y$, let the letter to the right of $R(x)$ in R be $R(x)^{\rightarrow}$. Then both R^{xyz} and R^{yxz} are obtained by replacing $R(x)$ with x and $R(x)^{\rightarrow}$ with z. For the output, we have $R(x) = R(y)$, $R^x(y) > R(y)$, $R^y(x) = y$, $R^{yx}(z) = R^x(y)$ and $R^{xy}(z) = y$. Since $y < R(x) < R^x(y)$, we have that $R^x(y) = R(x)^{\rightarrow} > y + 1$. Hence the outputs $R(x)R^x(y)R^{xy}(z) = R(x)R(x)^{\rightarrow}y \sim R(x)yR(x)^{\rightarrow} = R(y)R^y(x)R^{yx}(z)$ by move of type (I2).

Case 1D-(ii): $x \in R, y \notin R$. Then $R(x) \leqslant x, R(y) > y$ and $x + 1 \notin R$. In this case, we have $R^x(y) = R(y)$ and $R^y(x) = R(x)$, thus $R^{xy} = R^{yx}$, $R^{xy}(z) = R^{yx}(z)$ and $R^{xyz} = R^{yxz}$. Since $x + 1 \notin R$, we have $R^{xy}(z) > x + 1$. This implies $R(x) \leqslant x < R^{xy}(z) \leqslant y < R(y)$, thus $R(x)R^x(y)R^{xy}(z) \sim R(y)R^y(x)R^{yx}(z)$ as it is a type (I1) move.

Case 1D-(iii): $x \notin R, y \in R$. Then $x < R(x)$, $y \geqslant R(y)$, $y + 1 \notin R$, $R(x) - 1 \notin R$, $R(x) \leqslant y$, $R(y) \leqslant R^x(y)$ and $R^y = R$.

(1) If $R(x) = y$, denote the box to the right of y in y as $y^{\rightarrow}$. Note that $y^{\rightarrow} > y + 1$. Then $R^x(y) = y^{\rightarrow}$, $R^{xy}(z) = y$, $R^y(x) = y$ and $R^{yx}(z) = y^{\rightarrow}$. Note $y - 1 \notin R$, otherwise $R(x) \leqslant y - 1$. Thus, $R(y) = y$. Both R^{xyz} and R^{yxz} are obtained by replacing $y \in R$ with x and $y^{\rightarrow}$ with z, so $R^{xyz} = R^{yxz}$. The outputs $R(x)R^x(y)R^{xy}(z) = yy^{\rightarrow}y \sim yyy^{\rightarrow} = R(y)R^y(x)R^{yx}(z)$ as it is a type (II2) move.

(2) Suppose $R(x) < y$ and $R(x) = R(y)$. Then $[R(x), y] \subset R$ and $R^x(y) = R(x) + 1$. Since $R^y = R$ and $R^{xy} = R^x$, we have that both R^{xy} and R^{yx} equal R^x and furthermore $R^y(x) = R(x)$. Note that z can either be equal to y or $z < R^x(y)$, otherwise $z \in R^{xy}$ and $z + 1 \in R^{xy}$, which will give us a braid from $\mathsf{row}(R^{xy}) \cdot z$. Thus, we have $R^{xy}(z) = R^{yx}(z) = R(x) + 1$. In either case, the outputs are $R(x)R^x(y)R^{xy}(z) = R(x)(R(x) + 1)(R(x) + 1) \sim R(x)R(x)(R(x) + 1) = R(y)R^y(x)R^{yx}(z)$ as they are type (III) moves.

(3) Suppose $R(x) < y$ and $R(x) < R(y)$. Then $R(y) > R(x) + 1$ and $R^x(y) = R(y)$. Similar to the previous case, both R^{xy} and R^{yx} are equal to R^x, and z is either y or $z < R(y)$. In either case, $R^{xy}(z) \leqslant R(y)$.

Then the outputs are $R(x)R^x(y)R^{xy}(z) = R(x)R(y)R^x(z) \sim R(y)R(x)R^x(z) = R(y)R^y(x)R^{yx}(z)$ as they are type (I1) or (II1) moves.

Case 1D-(iv): $x, y \in R$. In this case $x \geqslant R(x)$, $y \geqslant R(y)$, $x+1 \notin R$ and $y+1 \notin R$. Since $x+1 \notin R$, $[x, y]$ is not contained in R and hence $R(y) > x+1 > x \geqslant R(x)$.

Then $R^x(y) = R(y)$, $R^y(x) = R(x)$ and $R^{xy} = R^{yx} = R$. Since $z > x$ and $x+1 \notin R$, we have $R(z) > x+1 \geqslant R(x)+1$. By similar reasons to the previous two subcases of Case 1D-(iii), z can either be y or $z < R(y)$ in order to avoid a braid in $\mathsf{row}(R^{xy})z$. So, we have $R^{xy}(z) \leqslant R(y)$. Then the outputs are $R(x)R^x(y)R^{xy}(z) = R(x)R(y)R(z) \sim R(y)R(x)R(z) = R(y)R^y(x)R^{yx}(z)$ as they are type (I1) moves.

2. Cases (I2) and (II2): We have $z < x < y$, or $z = x < y$ and $y > z+1$. In both cases $x' = x, y' = z, z' = y$. By definition, $x \in R^x$.

Case (2A): $M < x < y$, then $R(x) = R^x(y) = 0$. $R^{xy} = \begin{array}{|c|c|c|}\hline R & x & y \\ \hline\end{array}$ is obtained by appending x and y to the end of R. Since $x \in R^x$ and $z \leqslant x < y$, we have $R^{xy}(z) = R^x(z)$. Moreover, R^{xzy} is obtained by appending y to the end of R^{xz} and hence $R^{xyz} = R^{xzy}$. The outputs are $R(x)R^x(y)R^{xy}(z) = 00R^x(z) \sim 0R^x(z)0 = R(x)R^x(z)R^{xz}(y)$.

Case (2B): $z \leqslant x \leqslant M < y$, then $R^x(y) = R^{xz}(y) = 0$. Since $R^{xy} = \begin{array}{|c|c|}\hline R^x & y \\ \hline\end{array}$, $x \in R^x$ and $z \leqslant x$, we have $R^{xy}(z) = R^x(z)$, thus $R^{xyz} = \begin{array}{|c|c|}\hline R^{xz} & y \\ \hline\end{array} = R^{xzy}$. The output $R(x)R^x(y)R^{xy}(z) = R(x)0R^x(z) \sim R(x)R^x(z)0 = R(x)R^x(z)R^{xz}(y)$.

Case (2C): $z \leqslant x < y \leqslant M$, then we have $R^x(z) \leqslant x$. We discuss the following subcases.

Case 2C-(i): $x, y \notin R$, then we have $R(x) > x$ and $R^x(y) > y$. Since $y > x$ and x replaces $R(x)$ in R, we have $R^x(y) > R(x)$ from row strictness. Since $R^x(y) > R(x)$ and $R^x(z) \leqslant x$, we have $R^{xy}(z) = R^x(z)$ and $R^{xz}(y) = R^x(y)$. Furthermore, $R^{xyz} = R^{xzy}$. Moreover, we have $R^x(z) \leqslant x < R(x) < R^x(y)$, which implies $R^x(y) > R^x(z)+1$. Hence $R(x)R^x(y)R^{xy}(z) = R(x)R^x(y)R^x(z) \sim R(x)R^x(z)R^x(y) = R(x)R^x(z)R^{xz}(y)$ by type (I2) moves.

Case 2C-(ii): $x \in R, y \notin R$. Then $R^x = R$, $R(x) \leqslant x$, $R^x(y) > y$. Since $z \leqslant x$ and $[R(x), x] \subset R^x$, we have that $R^x(z) \leqslant R(x)$. Since $R^x(y) > y > x$ and $R^x(z) \leqslant R(x)$, we have that $R^{xy}(z) = R^x(z)$ and $R^{xz}(y) = R^x(y)$, thus $R^{xyz} = R^{xzy}$. Since $R^x(z) \leqslant R(x) \leqslant x < y < R^x(y)$, we have $R^x(y) > R^x(z)+1$. The outputs are $R(x)R^x(y)R^{xy}(z) = R(x)R^x(y)R^x(z) \sim R(x)R^x(z)R^x(y) = R(x)R^x(z)R^{xz}(y)$ by type (I2) or (II2) moves.

Case 2C-(iii): $x \notin R, y \in R$. Then $R^{xy} = R^x$, $R(x) > x$ and $R^x(y) \leqslant y$. Let the letter to the right of $R(x)$ in R be $R(x)^{\to}$. Then $R(x)^{\to} > R(x) > x$ implies $R(x)^{\to} > x+1$. This also shows that $x+1 \notin R^x$ and thus $R^x(y) > R(x)$. Since $R^{xy} = R^x$ and $R^{xzy} = R^{xz}$, we have $R^{xyz} = R^{xz} = R^{xzy}$. Since $R^x(z) \leqslant x < R(x) < R^x(y)$, we have $R^x(y) > R^x(z)+1$. Since $z \leqslant x$, we also have that $R^x(y) = R^{xz}(y)$. Thus, the outputs are $R(x)R^x(y)R^{xy}(z) = R(x)R^x(y)R^x(z) \sim R(x)R^x(z)R^x(y) = R(x)R^x(z)R^{xz}(y)$ by a type (I2) move.

Case 2C-(iv): $x \in R, y \in R$. Then $R^x = R$, $R^{xy} = R$, $R(x) \leqslant x$, $R^x(y) \leqslant y$, $x+1 \notin R$ and $y+1 \notin R$. Thus, $R^x(y) > x+1$. Since $z \leqslant x$ and $[R(x), x] \subset R^x$, we have that $R^x(z) \leqslant R(x)$. Since $R^{xy} = R$, $R^{xyz} = R^z$. Since $R^x(z) \leqslant x$, $R^{xz}(y) = R^z(y) = R(y)$ and thus $R^{xzy} = R^z$. This implies $R^{xyz} = R^{xzy}$. Now we have $R(z) \leqslant R(x) \leqslant x < x+1 < R(y)$. Therefore, the outputs are $R(x)R^x(y)R^{xy}(z) = R(x)R(y)R(z) \sim R(x)R(z)R(y) = R(x)R^x(z)R^{xz}(y)$ by type (I2) or (II2) moves.

3. Case (III): We have $y = x$, $z = x+1$ and hence $x' = x$, $y' = x+1$ and $z' = x+1$.

Case (3A): $x > M$. Then R^x is obtained by appending x to the end of R and $R(x) = 0$. Also $R^{xx} = R^x$ with output $R^x(x)$. Note $R^{x,x+1}(x+1) = R^x(x)$. Both $R^{xx,x+1}$ and $R^{x,x+1,x+1}$ are obtained by appending $x+1$ to the end of R^x, thus they are the same. The outputs are $R(x)R^x(x)R^{xx}(x+1) = 0R^x(x)0 \sim 00R^x(x) = R(x)R^x(x+1)R^{x,x+1}(x+1)$.

Case (3B): $x \leqslant M, x+1 > M$. Both $R^{xx,x+1}$ and $R^{x,x+1,x+1}$ are obtained by appending $x+1$ to the end of R^x, so they are equal. Since $x \in R^x$, we have $R^{x,x+1}(x+1) = R^x(x)$. Thus, the outputs are $R(x)R^x(x)R^{xx}(x+1) = R(x)R^x(x)0 \sim R(x)0R^{x,x+1}(x+1)$.

Case (3C): $x+1 \leqslant M$. It is clear that $x \in R^x$. If x is the maximal letter in R^x, then the rest follows as case (3B).

Otherwise, let $x^{\to}$ be the letter to the right of x in R^x. Since $x \in R^x$, we must have $x+1 \notin R^x$, thus $x^{\to} > x+1$. Moreover, we have $R^{xx} = R^x$, $R^x(x+1) = R^{xx}(x+1) = x^{\to}$. Since $R^{x,x+1}$ is obtained from R^x by replacing $x^{\to}$ with $x+1$ and $x, x+1 \in R^{x,x+1}$, we have $R^{x,x+1}(x+1) = R^x(x)$. Both $R^{xx,x+1}$ and $R^{x,x+1,x+1}$ are obtained from R^x by replacing $x^{\to}$ with $x+1$, thus they are the same. Furthermore, since $R^x(x) \leqslant x$ and $x^{\to} > x+1$, we have that $R(x)R^x(x)R^{xx}(x+1) = R(x)R^x(x)x^{\to} \sim R(x)x^{\to}R^x(x) = R(x)R^x(x+1)R^{x,x+1}(x+1)$ by a type (I2) or (II2) move. $\square$

Proposition 84. *If two words in $\mathcal{H}_0(n)$ have the property that their reverse words are equivalent according to Definition 80, then they have the same insertion tableau under $\star$-insertion (inserted from right to left).*

Proof. Let P be a $\star$-insertion tableau. By Lemma 73, P^t is a semistandard tableau. Let the rows of P be $R_1, \ldots, R_\ell$. Then each row is strictly increasing. The row R_j is considered to be empty for $j > \ell$.

Let x_1, y_1, z_1 and x'_1, y'_1, z'_1 be letters such that $x_1 y_1 z_1 \sim x'_1 y'_1 z'_1$ and $\mathsf{row}(P) \cdot x_1 \cdot y_1 \cdot z_1$ is fully-commutative. Let the output of the $\star$-insertion algorithm of $P \leftarrow x_1 \leftarrow y_1 \rightarrow z_1$ (resp. $P \leftarrow x'_1 \leftarrow y'_1 \leftarrow z'_1$) from the row i be $x_{i+1}, y_{i+1}, z_{i+1}$ (resp. $x'_{i+1}, y'_{i+1}, z'_{i+1}$). That is:

- $R_i^{x_i y_i z_i}$ is the first row of $[(R_i \leftarrow x_i) \leftarrow y_i] \leftarrow z_i$ and the outputs in order are $x_{i+1}, y_{i+1}, z_{i+1}$.
- $R_i^{x'_i y'_i z'_i}$ is the first row of $[(R_i \leftarrow x'_i) \leftarrow y'_i] \leftarrow z'_i$ and outputs in order are $x'_{i+1}, y'_{i+1}, z'_{i+1}$.

By Lemma 83, we have that $R_i^{x_i y_i z_i} = R_i^{x'_i y'_i z'_i}$ and $x_{i+1} y_{i+1} z_{i+1} \sim x'_{i+1} y'_{i+1} z'_{i+1}$ for all i (possibly some extra rows exceeding ℓ). Thus, we have the desired result. $\square$

Example 85. *The four words in $\mathcal{H}_0(5)$ of Example 81 all have the same $\star$-insertion tableau:*

$$\begin{array}{|c|cc}
\hline
3 \\
\hline
1 \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\
\hline
\end{array}\ .$$

In the next couple of lemmas, we prove that the crystal operators $f_k^\star$ act by a composition of micro-moves as given in Definition 80. More precisely, for a fully-commutative decreasing factorization $\mathbf{h}$, we have $\mathbf{h}^{\mathsf{rev}} \sim f_k^\star(\mathbf{h})^{\mathsf{rev}}$ as long as $f_k^\star(\mathbf{h}) \neq 0$, where $\mathbf{h}^{\mathsf{rev}}$ is the reverse of $\mathbf{h}$.

Remark 86. *By Definition 48 and Remark 52, there are two cases for the k-th and $(k+1)$-st factors under the crystal operator $f_k^\star$, where x is the largest unpaired letter in the k-th factor, $w_i, v_i > x$ and $u_i, b_i < x$:*

(1) $(w_1 \ldots w_p u_1 \ldots u_q)(v_1 \ldots v_s x b_1 \ldots b_t) \xrightarrow{f_k^\star} (w_1 \ldots w_p x u_1 \ldots u_q)(v_1 \ldots v_s b_1 \ldots b_t)$,
where $v_s \neq x+1$.

(2) $(w_1 \ldots w_p u_1 \ldots u_q)(v_1 \ldots v_s x b_1 \ldots b_t) \xrightarrow{f_k^\star} (w_1 \ldots w_p x u_1 \ldots u_q)(v_1 \ldots v_{s-1} x b_1 \ldots b_t)$,
where $v_s = w_p = x+1$.

In both cases, $u_i < x-1$ since if $u_1 = x-1$ then $b_1 = x-1$ due to the fact that x is unbracketed; but this would mean that the word is not fully-commutative. We also notice that since all u_i are paired with some b_j, we have that $t \geqslant q$ and $b_i \geqslant u_i$. Similarly, all v_i are paired with some w_j, so we have that $p \geqslant s$ and $v_i \geqslant w_{p-s+i}$. Let u denote the sequence $u_1 \dots u_q$ and let b denote the sequence $b_1 \dots b_t$.

LEMMA 87.

(1) *For $2 \leqslant i \leqslant q$, $b_{i-1} > u_i + 1$.*

(2) *For $1 \leqslant i < s$, $v_i > w_{p-s+i+1} + 1$.*

PROOF. **(1):** When $b_{i-1} > b_i + 1$ or $u_i < b_i$, the result follows directly.

Consider the case that $u_i = b_i = a$ and $b_{i-1} = b_i + 1 = a + 1$ for some letter a. Since $a = u_i < u_{i-1} \leqslant b_{i-1} = a + 1$, we must have $u_{i-1} = a + 1$. Let c be the largest letter such that $[a, c] \subseteq b$. Then $c \geqslant a + 1$ and $c + 1 \notin b$. Moreover, since all u_i are paired, $u_i \leqslant b_i$ and $u_{j-1} > u_j$, it is not hard to see that $[a, c] \subseteq u$ and $c, c-1 \in u$. Since $c + 1 \notin b$, we can use commutativity to move $c \in b$ to the left and obtain a subword $c(c-1)c$, which contradicts that the original word is fully-commutative.

(2): The proof is almost identical to the first part. When $w_{p-s+i} > w_{p-s+i+1} + 1$ or $v_i > w_{p-s+i}$, the result follows.

Consider the case $w_{p-s+i} = w_{p-s+i+1} + 1 = a + 1$ and $v_i = w_{p-s+i} = a + 1$ for some letter a. Since $a = w_{p-s+i+1} \leqslant v_{i+1} < v_i = a + 1$, we must that $v_{i+1} = a$. Let c be the smallest letter such that $[c, a+1] \subseteq w$. Then $c \leqslant a$ and $c - 1 \notin w$. Moreover, since all v_i are paired, $v_j \geqslant w_{p-s+j}$ and $v_{j+1} < v_j$, we can see that $[c, a+1] \subseteq v$ and $c, c+1 \in v$. Since $c - 1 \notin w$, we can use commutativity to move $c \in w$ to the right and form a subword $c(c+1)c$, which contradicts that the original word is fully-commutative. $\square$

We now summarize several observations that will be used later.

REMARK 88. *For both types of actions of $f_k^\star$ as in Remark 86, we have the following equivalence relations:*

(1) *For $1 \leqslant i \leqslant q, 1 \leqslant j \leqslant s-1$, $v_{j+1}v_j u_i \sim v_{j+1}u_i v_j$, since $u_i < v_{j+1} < v_j$.*

(2) *For* $1 \leqslant i \leqslant q$, $xv_su_i \sim xu_iv_s$, *since* $u_i < x < v_s$.

(3) *For* $1 \leqslant i \leqslant q$, $b_1xu_i \sim b_1u_ix$, *since* $u_i \leqslant u_1 \leqslant b_1 < x$, *and* $u_i < x - 1$.

(4) *For* $1 \leqslant j < i - 1$, $1 \leqslant i \leqslant q$, $b_{j+1}b_ju_i \sim b_{j+1}u_ib_j$, *since* $u_i \leqslant b_i < b_{j+1} < b_j$.

(5) *For* $2 \leqslant i \leqslant q$, $b_ib_{i-1}u_i \sim b_iu_ib_{i-1}$, *since* $u_i \leqslant b_i < b_{i-1}$ *and* $b_{i-1} > u_i + 1$ *by Lemma 87.*

(6) *For* $1 \leqslant i \leqslant s$, $p - s + i - 1 \leqslant j \leqslant p - 1$, $w_{j+1}v_iw_j \sim v_iw_{j+1}w_j$, *since* $w_{j+1} < w_j < w_{p-s+i} \leqslant v_i$.

(7) *For* $1 \leqslant i \leqslant s - 1$, $w_{p-s+i+1}v_iw_{p-s+i} \sim v_iw_{p-s+i+1}w_{p-s+i}$, *since* $w_{p-s+i+1} < w_{p-s+i} \leqslant v_i$ *and* $v_i > w_{p-s+i+1} + 1$ *by Lemma 87.*

(8) *For all* $1 \leqslant j \leqslant s - 1$, $1 \leqslant i \leqslant q$, $v_{j+1}u_iv_j \sim v_{j+1}v_ju_i$, *since* $u_i < v_{j+1} < v_j$.

(9) *For* $1 < i \leqslant q$, $b_1u_iv_s \sim b_1v_su_i$, *since* $u_i < u_1 \leqslant b_1 < v_s$.

(10) *For* $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant s$, $xu_iv_j \sim xv_ju_i$, *since* $u_i < x < v_j$.

(11) *For* $1 \leqslant j \leqslant s - 1$, $xv_jw_p \sim v_jxw_p$, *since* $x < w_p \leqslant v_s < v_j$.

REMARK 89. *When* $v_s \neq x + 1$, *we have the following equivalence relations:*

(1) $1 \leqslant i \leqslant s$, $xv_iw_p \sim v_ixw_p$, *since* $x < w_p \leqslant v_s$ *and* $v_s > x + 1$.

(2) $b_1u_1v_s \sim b_1v_su_1$, *since* $u_1 \leqslant b_1 < v_s$ *and* $v_s > x + 1 > u_1 + 1$.

LEMMA 90. *We have that* $b_q \dots b_1xv_s \dots v_1u_q \dots u_1$ *is equivalent to* $b_qu_q \dots b_2u_2b_1u_1xv_s \dots v_1$.

PROOF. With the equivalence relations from Remark 88 (1)-(5), we can make the sequences of equivalence moves as follows:

$$b_q \dots b_1xv_s \dots v_2v_1u_qu_{q-1} \dots u_1 \sim b_q \dots b_1xv_s \dots v_2u_qv_1u_{q-1} \dots u_1 \sim$$

$$b_q \dots b_1xv_su_q \dots v_2v_1u_{q-1} \dots u_1 \sim b_q \dots b_1xu_qv_s \dots v_2v_1u_{q-1} \dots u_1 \sim$$

$$b_q \dots b_1u_qxv_s \dots v_2v_1u_{q-1} \dots u_1 \sim b_qu_q \dots b_1xv_s \dots v_2v_1u_{q-1} \dots u_1 \sim$$

$$b_qu_qb_{q-1}u_{q-1} \dots b_1u_1xv_s \dots v_2v_1.$$

□

LEMMA 91. *We have that* $v_s \dots v_1w_p \dots w_{p-s+1}$ *is equivalent to* $v_sw_pv_{s-1}w_{p-1} \dots v_1w_{p-s+1}$.

Proof. With the equivalence relations from Remark 88 (6)-(7), we can make the following equivalence moves:

$$v_s \dots v_2 v_1 w_p w_{p-1} \dots w_{p-s+1} \sim v_s \dots v_2 w_p w_{p-1} \dots w_{p-s+2} v_1 w_{p-s+1} \sim$$

$$v_s \dots v_3 w_p w_{p-1} \dots v_2 w_{p-s+2} v_1 w_{p-s+1} \sim v_s w_p \dots v_1 w_{p-s+1}.$$

□

Lemma 92. *We have*

$$x w_p v_{s-1} w_{p-1} \dots v_2 w_{p-s+2} v_1 w_{p-s+1} \sim v_{s-1} \dots v_1 x w_p \dots w_{p-s+1}.$$

Proof. With the equivalence relations from Remark 88 (6),(7) and (11), we can make the following equivalent moves:

$$x w_p v_{s-1} w_{p-1} \dots v_2 w_{p-s+2} v_1 w_{p-s+1} \sim x v_{s-1} w_p w_{p-1} \dots v_2 w_{p-s+2} v_1 w_{p-s+1} \sim$$

$$v_{s-1} x w_p w_{p-1} \dots v_2 w_{p-s+2} v_1 w_{p-s+1} \sim v_{s-1} \dots v_1 x w_p w_{p-1} \dots w_{p-s+2} w_{p-s+1}.$$

□

Lemma 93. *When $v_s \neq x+1$, we have*

$$x v_s w_p v_{s-1} w_{p-1} \dots v_1 w_{p-s+1} \sim v_s \dots v_1 x w_p \dots w_{p-s+1}.$$

Proof. With the equivalence relations from Remark 88 (6)-(7) and Remark 89 (1), we can make the following equivalence moves:

$$x v_s w_p v_{s-1} w_{p-1} v_{s-2} \dots v_1 w_{p-s+1} \sim v_s x w_p v_{s-1} w_{p-1} v_{s-2} \dots v_1 w_{p-s+1} \sim$$

$$v_s x v_{s-1} w_p w_{p-1} v_{s-2} \dots v_1 w_{p-s+1} \sim v_s v_{s-1} x w_p w_{p-1} v_{s-2} \dots v_1 w_{p-s+1} \sim$$

$$v_s v_{s-1} v_{s-2} \dots v_1 x w_p w_{p-1} \dots w_{p-s+1}.$$

□

Lemma 94. *When $v_s \neq x+1$, we have $b_q u_q \dots b_1 u_1 v_s \dots v_1$ is equivalent to $b_q \dots b_1 v_s \dots v_1 u_q \dots u_1$.*

PROOF. With the equivalence relations from Remark 88 (4), (5), (8)-(9) and Remark 89 (2), we can make the following equivalence moves:

$$b_q u_q \ldots b_1 u_1 v_s \ldots v_1 \sim b_q u_q \ldots b_1 v_s u_1 \ldots v_1 \sim$$

$$b_q u_q \ldots b_1 v_s \ldots v_1 u_1 \sim b_q \ldots b_1 v_s \ldots v_1 u_q \ldots u_1.$$

$\square$

LEMMA 95. *We have $b_q u_q \ldots b_1 u_1 x v_{s-1} \ldots v_1$ is equivalent to $b_q \ldots b_1 x v_{s-1} \ldots v_1 u_q \ldots u_1$.*

PROOF. With the equivalence relations from Remark 88 (1), (3), (5) and (10) we have the following equivalence moves:

$$b_q u_q \ldots b_1 u_1 x v_{s-1} \ldots v_1 \sim b_q u_q \ldots b_1 x u_1 v_{s-1} \ldots v_1 \sim$$

$$b_q u_q \ldots b_1 x v_{s-1} \ldots v_1 u_1 \sim b_q \ldots b_1 x v_{s-1} \ldots v_1 u_q \ldots u_1.$$

$\square$

PROPOSITION 96. *Suppose $\mathbf{h}$ is a fully-commutative decreasing factorization such that $f_k^\star(\mathbf{h}) \neq 0$ (resp. $e_k^\star(\mathbf{h}) \neq 0$). Then $f_k^\star(\mathbf{h})^{\mathsf{rev}} \sim \mathbf{h}^{\mathsf{rev}}$ (resp. $e_k^\star(\mathbf{h})^{\mathsf{rev}} \sim \mathbf{h}^{\mathsf{rev}}$) for the equivalence relation $\sim$ of Definition 80.*

PROOF. We prove the statement for $f_k^\star$. Since $e_k^\star$ is a partial inverse of $f_k^\star$, the result follows.

Let $\mathbf{h} = h^m \ldots h^1 \in \mathcal{H}^{m,\star}$ and define $\widetilde{\mathbf{h}} = f_k^\star(\mathbf{h}) = h^m \ldots \tilde{h}^{k+1}\tilde{h}^k h^{k-1} \ldots h^1$. Specifically, $h^{k+1} = (w_1 \ldots w_p u_1 \ldots u_q)$ and $h^k = (v_1 \ldots v_s x b_1 \ldots b_t)$, where x is the largest unpaired letter in h^k. Then by Lemmas 90 and 91, we have the following sequence of equivalence moves:

$$(b_q \ldots b_1 x v_s \ldots v_1 u_q \ldots u_1) w_p \ldots w_{p-s+1} \sim (b_q u_q \ldots b_1 u_1 x v_s \ldots v_1) w_p \ldots w_{p-s+1}$$

$$b_q u_q \ldots b_1 u_1 x (v_s \ldots v_1 w_p \ldots w_{p-s+1}) \sim b_q u_q \ldots b_1 u_1 x (v_s w_p \ldots v_1 w_{p-s+1}).$$

Case (1): When $v_s \neq x+1$, $\tilde{h}^{k+1} = (w_1 \dots w_p x u_1 \dots u_q)$, $\tilde{h}^k = (v_1 \dots v_s b_1 \dots b_t)$. By Lemmas 93 and 94, we have

$$b_q u_q \dots b_1 u_1 (x v_s w_p \dots v_1 w_{p-s+1}) \sim b_q u_q \dots b_1 u_1 (v_s \dots v_1 x w_p \dots w_{p-s+1})$$

$$(b_q u_q \dots b_1 u_1 v_s \dots v_1) x w_p \dots w_{p-s+1} \sim (b_q \dots b_1 v_s \dots v_1 u_q \dots u_1) x w_p \dots w_{p-s+1}.$$

Thus, we have that

$$b_t \dots b_1 x v_s \dots v_1 u_q \dots u_1 w_p \dots w_1 \sim b_t \dots b_1 x v_s \dots v_1 u_q \dots u_1 x w_p \dots w_1.$$

Case (2): When $v_s = w_p = x+1$, $\tilde{h}^{k+1} = (w_1 \dots w_p x u_1 \dots u_q)$, $\tilde{h}^k = (v_1 \dots v_{s-1} x b_1 \dots b_t)$. Then by Lemmas 92 and 95, we have

$$b_q u_q \dots b_1 u_1 (x v_s w_p) v_{s-1} w_{p-1} \dots v_1 w_{p-s+1} \sim b_q u_q \dots b_1 u_1 (x x w_p) v_{s-1} w_{p-1} \dots v_1 w_{p-s+1}$$

$$b_q u_q \dots b_1 u_1 x (x w_p v_{s-1} w_{p-1} \dots v_1 w_{p-s+1}) \sim b_q u_q \dots b_1 u_1 x (v_{s-1} \dots v_1 x w_p \dots w_{p-s+1})$$

$$(b_q u_q \dots b_1 u_1 x v_{s-1} \dots v_1) x w_p \dots w_{p-s+1} \sim (b_q \dots b_1 x v_{s-1} \dots v_1 u_q \dots u_1) x w_p \dots w_{p-s+1}.$$

Thus, we have that

$$b_t \dots b_1 x v_s \dots v_1 u_q \dots u_1 w_p \dots w_1 \sim b_t \dots b_1 x v_{s-1} \dots v_1 u_q \dots u_1 x w_p \dots w_1.$$

Therefore, we have shown that in both cases, $f_k^\star(\mathbf{h})^{\mathsf{rev}} \sim \mathbf{h}^{\mathsf{rev}}$. □

Proposition 97. *For $\mathbf{h} \in \mathcal{H}^{m,\star}$ such that $f_k^\star(\mathbf{h}) \neq 0$ for some $1 \leqslant k < m$, the $\star$-insertion tableau for $\mathbf{h}$ equals the $\star$-insertion tableau for $f_k^\star(\mathbf{h})$.*

Proof. By Proposition 96, the reverse words for $\mathbf{h}$ and $f_k^\star(\mathbf{h})$ are $\sim$-equivalent. By Proposition 84, the corresponding insertion tableaux are equal. □

Proposition 98. *Let $\mathbf{h} \in \mathcal{H}^{m,\star}$ be a lowest weight element under Definition 48 of weight λ. Then there exists $r \geqslant 1$ where $\lambda_i = 0$ for $i < r$ and $\lambda_{i+1} \geqslant \lambda_i$ for $1 \leqslant i \leqslant m$. Suppose $\mathbf{h} = h^m \cdots h^r = (h^m_{\lambda_m} \dots h^m_1)(h^{m-1}_{\lambda_{m-1}} \cdots h^{m-1}_1) \dots (h^r_{\lambda_r} \dots h^r_1)$, then the i-th row of the $\star$-insertion* *tableau equals* $h_1^{m+1-i}, h_2^{m+1-i}, \ldots, h_{\lambda_{m+1-i}}^{m+1-i}$*, that is,*

$$
(3.3.1) \qquad P^\star(\mathbf{h}) = \begin{array}{|c|c|c|c|c|c|}
\hline
h_1^r & \cdots & h_{\lambda_r}^r \\
\cline{1-4}
\cdots & \cdots & \cdots & \cdots \\
\cline{1-5}
h_1^{m-1} & h_2^{m-1} & \cdots & \cdots & h_{\lambda_{m-1}}^{m-1} \\
\hline
h_1^m & h_2^m & \cdots & \cdots & \cdots & h_{\lambda_m}^m \\
\hline
\end{array} .
$$

Proof. Without loss of generality, we may assume that $r = 1$. We prove the statement by induction on m. The case $m = 1$ is trivial.

Let $m \geqslant 1$ be arbitrary and suppose that the statement holds for this m. We prove the statement for $m + 1$. We need to insert $P^\star(\mathbf{h}) \leftarrow h_1^{m+1} \leftarrow h_2^{m+1} \leftarrow \cdots \leftarrow h_{\lambda_{m+1}}^{m+1}$, where $P^\star(\mathbf{h})$ is as in (3.3.1) with $r = 1$. Note that $h_i^{m+1} \leqslant h_i^m$ for $1 \leqslant i \leqslant \lambda_m$. Specifically, $h_1^{m+1} \leqslant h_1^m$, so its insertion path is vertical along the first column and we obtain

$$
P^\star(\mathbf{h}) \leftarrow h_1^{m+1} = \begin{array}{|c|c|c|c|c|c|}
\cline{1-1}
h_1^1 \\
\cline{1-3}
h_1^2 & \cdots & h_{\lambda_r}^r \\
\cline{1-4}
\cdots & \cdots & \cdots & \cdots \\
\cline{1-5}
h_1^m & h_2^{m-1} & \cdots & \cdots & h_{\lambda_{m-1}}^{m-1} \\
\hline
h_1^{m+1} & h_2^m & \cdots & \cdots & \cdots & h_{\lambda_m}^m \\
\hline
\end{array} .
$$

Since $h_1^{m+1} < h_2^{m+1} \leqslant h_2^m$, the insertion path of h_2^{m+1} is strictly to the right of the insertion path of h_1^{m+1} and weakly left of the second column by Lemma 76, so it is vertical along the second column. Similar arguments show that the insertion path for h_i^{m+1} is just vertical along the i-th column. Thus, the result holds for $m + 1$. □

Remark 99. *For a lowest weight element* $\mathbf{h} \in \mathcal{H}^{m,\star}$ *of weight* $\mathbf{a}$*, the corresponding insertion tableau must have shape* $\mu = \mathsf{sort}(\mathbf{a})$*, which is the partition obtained by reordering* $\mathbf{a}$*.*

Proposition 100. *Let* $T \in \mathsf{SSYT}(\lambda)$ *and* $(P, Q) = \star \circ \mathsf{res}(T)$*. Then* $Q = T$*.*

Proof. The proof is done by induction on subtableaux of T similarly to the proof of Theorem 68.

For a given step in the insertion process, suppose that the entries of T that are involved so far form a nonempty subtableau T' of T with shape μ containing cell $(1,1)$. Furthermore, assume that the insertion and recording tableau at the corresponding step are $P(T')$ and $Q(T')$. Then they both have shape μ, and the entry of cell (i,j) of $P(T')$ is $\ell + j - \mu'_j + i - 1$. In addition, $Q(T') = T'$, where μ' is the conjugate of the partition μ and $\ell := \lambda'_1 = \ell(\lambda)$.

Note that we do not encounter Case (1) in the proof of Theorem 68. All other arguments still hold since for every insertion the letter is not contained in the row it is inserted into, that is, the insertion always bumps the smallest letter that is greater than itself. Thus, we omit the detail of the proof. $\square$

3.3.2. The $\star$-insertion and crystal operators. In this section, we prove that the $\star$-insertion and the crystal operators on fully-commutative decreasing factorizations and semistandard Young tableaux intertwine.

Theorem 101. *Let $\mathbf{h} \in \mathcal{H}^{m,\star}$. Let $(P^\star(\mathbf{h}), Q^\star(\mathbf{h})) = \star(\mathbf{h})$ be the insertion and recording tableaux under the $\star$-insertion of Definition 71. Then*

(1) *$f_i^\star(\mathbf{h})$ is defined if and only if $f_i(Q^\star(\mathbf{h}))$ is defined.*
(2) *If $f_i^\star(\mathbf{h})$ is defined, then $Q^\star(f_i^\star(\mathbf{h})) = f_i Q^\star(\mathbf{h})$.*

In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathsf{SSYT}^m \\ \big\downarrow{\scriptstyle f_i^\star} & & \big\downarrow{\scriptstyle f_i} \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathsf{SSYT}^m. \end{array}$$

Proof. The crystal operator $f_i^\star$ acts only on factors h^{i+1} and h^i. Hence it suffices to prove the statement for $\mathbf{h} = h^{i+1}h^i \dots h^1$ with $i+1$ factors.

Suppose $f_i^\star(\mathbf{h}) \neq 0$. By Proposition 97, $P^\star(\mathbf{h}) = P^\star(f_i^\star(\mathbf{h}))$. Furthermore, by Lemma 73 $P^\star(\mathbf{h})$ and $Q^\star(\mathbf{h})$ have the same shape. Hence in particular, $Q^\star(\mathbf{h})$ and $Q^\star(f_i^\star(\mathbf{h}))$ have the same shape and therefore the letters i and $i+1$ in $Q^\star(\mathbf{h})$ and $Q^\star(f_i^\star(\mathbf{h}))$ occupy the same skew shape.

Recall from Definition 48 that $f_i^\star$ removes precisely one letter from factor $h^i = (h^i_\ell h^i_{\ell-1} \dots h^i_1)$, say h^i_k. By Lemma 76, the insertion paths of $h^i_1, \dots, h^i_\ell$ into $P^\star(h^{i-1} \cdots h^1)$ move strictly to the right and the newly added cells form a horizontal strip. In addition, the letters $h^i_1, \dots, h^i_\ell$ appear in the first row of $P^\star(h^i \cdots h^1)$. Now compare this to the insertion paths for $h^i_1, \dots, \widehat{h^i_k}, \dots, h^i_\ell$ into $P^\star(h^{i-1} \dots h^1)$, where h^i_k is missing. Up to the insertion of h^i_{k-1}, everything agrees. Suppose that h^i_k bumps the letter x in the first row and h^i_{k+1} bumps the letter $y > x$ in the first row by Lemma 76. Then when h^i_{k+1} gets inserted without prior insertion of h^i_k, the letter h^i_{k+1} either still bumps y or h^i_{k+1} bumps x (in which case x and y are adjacent in the first row in $P^\star(h^{i-1} \cdots h^1)$). There are no other choices, since if there are letters between x and y in the first row and h^i_{k+1} bumps one of these, it would have already bumped a letter to the left of y in $P^\star(h^i \cdots h^1)$. If h^i_{k+1} bumps x without prior insertion of h^i_k, then its insertion path is the same as the insertion path of h^i_k previously. If h^i_{k+1} bumps y, then the letter inserted into the second row by similar arguments either bumps the same letter as in the previous insertion path of h^i_{k+1} or h^i_k and so on. The last cell added is hence the same cell added in the previous insertion path of either h^i_k or h^i_{k+1}. Repeating these arguments, exactly one cell containing i in $Q^\star(h^i \cdots h^1)$ is missing in $Q^\star((h^i_\ell \dots \widehat{h^i_k} \dots h^i_1)h^{i-1} \cdots h^1)$ and all other cells containing i are the same. Hence, $Q^\star(f_i^\star(\mathbf{h}))$ is obtained from $Q^\star(\mathbf{h})$ by changing exactly one letter i to $i+1$.

It remains to prove that $f_i^\star(\mathbf{h}) \neq 0$ if and only if $f_i(Q^\star(\mathbf{h})) \neq 0$ and, if $f_i^\star(\mathbf{h}) \neq 0$, then the letter i that is changed to $i+1$ from $Q^\star(\mathbf{h})$ to $Q^\star(f_i^\star(\mathbf{h}))$ is the rightmost unbracketed i in $Q^\star(\mathbf{h})$. First assume that under the bracketing rule for $f_i^\star$, all letters in the factor h^i are bracketed, so that $f_i^\star(\mathbf{h}) = 0$. This means that each letter in h^i is paired with a weakly smaller letter in h^{i+1}. Then by similar arguments as in Lemma 76 (2), for each insertion path for the letters in h^i, there is an insertion path for the letters in h^{i+1} that is weakly to the left and the resulting new cell is weakly to the left and strictly above of the corresponding new cell for the letter in h^i. This means that each i in $Q^\star(\mathbf{h})$ is paired with an $i+1$ and hence $f_i(Q^\star(\mathbf{h})) = 0$.

Now assume that $f_i^\star(\mathbf{h}) \neq 0$. Let us use the same notation as in Remark 86 (with k replaced by i). Since all letters $u_q, \dots, u_1 < x$ are paired with some letters $b_j < x$, their insertion paths (again by similar arguments as in Lemma 76) lie strictly to the left of the insertion path for x. First assume that $v_s \neq x+1$. Recall that by Proposition 97, $P^\star(\mathbf{h}) = P^\star(f_i^\star(\mathbf{h}))$. Also, by the above arguments, moving letter x to factor h^{i+1} under $f_i^\star$, changes one i to $i+1$ (precisely the i that is missing when removing x from h^i). Now the letters $w_p, \ldots, w_1 > x$ are inserted after the letter x in the $(i+1)$-th factor in $f_i^\star(\mathbf{h})$ and by Lemma 76 their insertion paths are strictly to the right of the insertion path of x in $f_i^\star(\mathbf{h})$. But this means that the corresponding $i+1$ in $Q^\star(\mathbf{h})$ cannot bracket with the i that changes to $i+1$ under $f_i^\star$. This proves that $f_i(Q^\star(\mathbf{h})) \neq 0$. Furthermore, each $v_s, \ldots, v_1$ is paired with some w_j and hence the insertion path of this w_j is weakly to the left of the insertion path of the corresponding v_h. Hence all i to the right of the i that changes to an $i+1$ under $f_i^\star$ are bracketed. This proves that this i is the rightmost unbracketed i, proving the claim. The case $v_s = x+1$ is similar. $\square$

Remark 102. *Proposition 98 and Theorem 101 provide another proof via $\star$-insertion, in the case where w is fully-commutative, of the Schur positivity of $\mathfrak{G}_w$ of Fomin and Greene [**FG98**]*

$$\mathfrak{G}_w = \sum_{\mu} \beta^{|\mu|-\ell(w)} g_w^\mu s_\mu,$$

where $g_w^\mu = |\{T \in \mathsf{SSYT}^n(\mu') \mid w_C(T) \equiv w\}|$.

3.3.3. Uncrowding set-valued skew tableaux. Buch [**Buc02**] introduced a bijection from a set-valued tableau of straight shape to a pair (P, Q), where P is a semistandard tableau and Q is a flagged increasing tableau. The map involves the use of a dilation operation [**BM12**, **RTY18**] which can be defined equally to act on set-valued skew tableaux. Chan and Pflueger [**CP19**] recently studied the operation in this more general context. We review here the results needed for our purposes.

Let λ, μ be partitions such that $\lambda \subseteq \mu$ and $\lambda_1 = \mu_1$. A *flagged increasing tableau* (introduced in [**Len00**] and called *elegant fillings* by various authors [**Len00**, **LP07**, **BM12**, **Pat16**]) is a row and column strict filling of the skew shape μ/λ such that the positive integers entries in the i-th row of the tableau are at most $i-1$ for all $1 \leqslant i \leqslant \ell(\mu)$. In particular, the bottom row is empty. Denote the set of all flagged increasing tableaux of shape μ/λ by $\mathcal{F}_{\mu/\lambda}$.

We use *multicell* to refer to a cell in a set-valued tableau with more than one letter.

Definition 103. *For a skew shape λ/μ, the* uncrowding operation *is defined on $T \in \mathsf{SVT}(\lambda/\mu)$ as follows: identify the topmost row r in T containing a multicell. Let x be the largest letter in row*

r which lies in a multicell; delete this x and perform RSK row bumping with x into the rows above. The resulting tableau is the output of this operation. Note that its shape differs from λ/μ by the addition of one cell.

The uncrowding map*, denoted* $\mathsf{uncrowd}$*, is defined as follows. Let $T \in \mathsf{SVT}(\lambda/\mu)$ with $\mathsf{ex}(T) = \ell$.*

- *Start with $\widetilde{P}_0 = T$ and $\widetilde{Q}_0 = F$, where F is the unique flagged increasing tableau of shape λ/λ.*
- *For each $1 \leqslant i \leqslant \ell$, $\widetilde{P}_i$ is obtained from $\widetilde{P}_{i-1}$ by successively applying the uncrowding operation until no multicells remain. Each operation involves the addition of cell C to form $\widetilde{P}_i$ by first deleting an entry in cell B of $\widetilde{P}_{i-1}$; this is recorded by adding a cell with entry k to $\widetilde{Q}_{i-1}$ at the same position as C, where k is the difference in the row indices of cells B and C.*
- *Terminate and return $(\widetilde{P}, \widetilde{Q}) = (\widetilde{P}_\ell, \widetilde{Q}_\ell)$.*

Example 104. *Let T be the semistandard set-valued tableau*

$$T = \begin{array}{|c|c|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-3} 4 & 4 & 5 \\
\cline{1-3} 2 & 23 & 3 \\
\hline 1 & 1 & 1 & 12 & 234 & 5 \\
\hline
\end{array}\,.$$

Perform an uncrowding operation to obtain

$$T' = \begin{array}{|c|c|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-1} 4 \\
\cline{1-3} 3 & 4 & 5 \\
\cline{1-3} 2 & 2 & 3 \\
\hline 1 & 1 & 1 & 12 & 234 & 5 \\
\hline
\end{array}\,.$$

Proceeding with uncrowding the remaining multicells and recording the changes, we have $\mathsf{uncrowd}(T) = (\widetilde{P}, \widetilde{Q})$, *where*

$$\widetilde{P} = \begin{ytableau} 5 & 5 \\ 4 & 4 \\ 3 & 3 & 4 \\ 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 5 \end{ytableau} \qquad \textit{and} \qquad \widetilde{Q} = \begin{ytableau} 3 & 4 \\ *(gray) & 3 \\ *(gray) & *(gray) & *(gray) \\ *(gray) & *(gray) & *(gray) & 1 \\ *(gray) & *(gray) & *(gray) & *(gray) & *(gray) & *(gray) \end{ytableau}\ .$$

LEMMA 105. *For skew shape λ/μ, the crystal operators on $\mathsf{SVT}^m(\lambda/\mu)$ intertwine with those on $\mathsf{SSYT}^m(\nu/\mu)$, for $\lambda \subseteq \nu$, under* $\mathsf{uncrowd}$.

PROOF. Chan and Pflueger [**CP19**] proved that the image of $T \in \mathsf{SVT}(\lambda/\mu)$ under the uncrowding map is a pair (P, Q), where P is a semistandard tableau of shape ν/μ and Q is a flagged increasing tableau of shape ν/λ. Monical, Pechenik and Scrimshaw in [**MPS20**, Theorem 3.12] proved that the crystal operators on $\mathsf{SVT}^m(\lambda)$ intertwine with those on $\mathsf{SSYT}^m(\nu)$ under $\mathsf{uncrowd}$. Since $\mathsf{uncrowd}$ is defined equally on skew shapes, the result follows. $\square$

3.3.4. Compatibility of $\star$-insertion with uncrowding. For a partition μ, let T_μ be the unique tableau of shape μ with μ_i letters i in each row i. Note that $\mathsf{uncrowd}(T_\mu) = (T_\mu, \emptyset)$ since $\mathsf{ex}(T_\mu) = 0$.

LEMMA 106. *For $T \in \mathsf{SVT}^m(\lambda/\mu)$, if $(P, Q) = \star(\mathbf{h}\mathbf{h}')$ where $\mathbf{h} = \mathsf{res}(T)$ and $\mathbf{h}' = \mathsf{res}(T_\mu)$, then T_μ is contained in Q.*

PROOF. For $T \in \mathsf{SVT}^m(\lambda/\mu)$, let $T*$ be the set-valued tableau of shape λ obtained from T by adding $\ell(\mu)$ to each entry and filling in the cells of μ with T_μ. By Proposition 100, we have

$$\star \circ \mathsf{res}(T_\mu) = (P_\mu, T_\mu)\,, \tag{3.3.2}$$

where P_μ is the semistandard tableau specified in the proof of Proposition 100. The claim follows by noting that $\mathsf{res}(T*) = \mathsf{res}(T)\mathsf{res}(T_\mu)$. $\square$

DEFINITION 107. *A modification of $\star$-insertion is defined on $\mathcal{H}^{*,m}$ as follows: for $\mathbf{h} \in \mathcal{H}^{*,m}$, let λ/μ be the shape of $\mathsf{res}^{-1}(\mathbf{h})$ (which is well-defined up to a shift by Proposition 58). For $\mathbf{h}' = \mathsf{res}(T_\mu)$,* *let* $(P*, Q*) = \star(\mathbf{hh}')$*. Define* $\tilde{\star}(\mathbf{h}) = (P, Q)$ *where* P *is obtained from* $P*$ *by deleting all entries in cells of* μ *and* Q *is defined from* $Q*$ *by deleting* T_μ *from it and decreasing all other letters by* $\ell(\mu)$*.*

Note that this is well-defined by Lemma 106 and the fact that each $\mathbf{h} \in \mathcal{H}^{*,m}$ can be associated to a skew shape λ/μ which is the shape of $\mathsf{res}^{-1}(\mathbf{h})$ by Proposition 58. Also note that $\tilde{\star}(\mathbf{h}) = \star(\mathbf{h})$ if $\mu = \emptyset$.

THEOREM 108. *Let* $T \in \mathsf{SVT}^m(\lambda/\mu)$*,* $(\tilde{P}, \tilde{Q}) = \mathsf{uncrowd}(T)$*, and* $(P, Q) = \tilde{\star} \circ \mathsf{res}(T)$*. Then* $Q = \tilde{P}$*.*

PROOF. We start by addressing the straight-shape case; for $T* \in \mathsf{SVT}^m(\lambda)$, consider the following compositions of maps:

$$\begin{tikzcd}
(\tilde{P}, \tilde{Q}) \arrow[d, "f_k"] & T* \arrow[l, "\mathsf{uncrowd}"'] \arrow[r, "\mathsf{res}"] \arrow[d, "f_k"] & \mathbf{h} \arrow[r, "\star"] \arrow[d, "f_k^\star"] & (P, Q) \arrow[d, "f_k"] \\
(f_k(\tilde{P}), \tilde{Q}) & f_k(T*) \arrow[l, "\mathsf{uncrowd}"'] \arrow[r, "\mathsf{res}"] & f_k^\star(\mathbf{h}) \arrow[r, "\star"] & (P, f_k(Q)).
\end{tikzcd}$$

By Lemma 105, the left square commutes. By Theorem 61 the center square commutes. By Proposition 97 and Theorem 101 the right square commutes. Hence it suffices to prove that $Q = \tilde{P}$ when $T*$ is a lowest weight element in the crystal.

Suppose $T* \in \mathsf{SVT}^m(\lambda)$ is of lowest weight with $\mathrm{wt}(T*) = \mathbf{a}$ and $\mathsf{ex}(T*) = \ell$. Then the decreasing factorization $\mathbf{h} \in \mathcal{H}^{m,\star}$ is lowest weight by Theorem 61. By Remark 99, P and hence Q has to be of shape $\nu = \mathsf{sort}(\mathbf{a})$. By Theorem 101, Q is the unique lowest weight element in SSYT^m of shape ν.

Consider the uncrowding operator on $T*$ and record each tableau during the process of uncrowding as in Definition 103 by a sequence of set-valued tableaux $T* = \tilde{P}_0 \to \tilde{P}_1 \to \cdots \to \tilde{P}_\ell = \tilde{P}$. Since $T*$ is of lowest weight, so are all the $\tilde{P}_i$. Furthermore, all $\tilde{P}_i$ have the same weight $\mathbf{a}$. Let $(P_i, Q_i) = \star \circ \mathsf{res}(\tilde{P}_i)$. For all $0 \leqslant i \leqslant \ell$, Q_i is the unique lowest weight element in SSYT^m of shape ν. Hence in particular $Q_i = Q$ for all $0 \leqslant i \leqslant \ell$. By Proposition 100, $Q = Q_\ell = \tilde{P}$, proving the claim for straight shapes.

Now take $T \in \mathsf{SVT}^m(\lambda/\mu)$ and construct $T*$ from T by adding $\ell(\mu)$ to each entry and filling in the cells of μ with T_μ. Note that $T*$ is a set-valued tableaux of shape λ. Let $(P, Q) = \star \circ \mathsf{res}(T*)$ and

$(P*, Q*) = \tilde{\star} \circ \mathsf{res}(T)$. Since $\mathsf{res}(T*) = \mathsf{res}(T)\mathsf{res}(T_\mu)$, Lemma 106 implies that $Q* = Q/T_\mu$. On the other hand, since $T*$ has straight shape, the preceding paragraph gives that $\mathsf{uncrowd}(T*) = (Q, \tilde{Q})$ for some $\tilde{Q}$. We then note that $\mathsf{uncrowd}(T)$ and $\mathsf{uncrowd}(T*)$ are identical on cells of λ/μ up to a shift of the entries by $\ell(\mu)$; in particular, applying $\mathsf{uncrowd}$ to $T*$ does not involve any cell of μ since none of these are multicells and their entries are the smallest $\ell(\mu)$ letters. □

3.4. Results on the non-fully-commutative case

In this section, we discuss some aspects when we generalize to the non-fully-commutative case. In Section 3.4.1, we describe a local crystal on $\mathcal{H}^m(3)$. In Section 3.4.2, we show that under very mild assumptions it is not possible to expect a local crystal for $n > 3$.

3.4.1. The case $n = 3$. We provide a description of a type A_{m-1} crystal structure on $\mathcal{H}^m(3)$.

Definition 109. *Let $\mathbf{h} = h^m h^{m-1} \dots h^2 h^1 \in \mathcal{H}^m(3)$. Fix $1 \leqslant k < m$. Define the pairing process of $\mathbf{h}$ and the number of pairs in $h^{k-1} \dots h^{j+1} h^j$, denoted $p([j, k-1])$, recursively as follows:*

(1) *The empty factorization, denoted $\emptyset$, has no pairs and $p(\emptyset) = 0$.*
(2) *If $p([1, j-1])$ is defined for all $1 \leqslant j \leqslant k$, then we have $p([j, k-1]) = p([1, k-1]) - p([1, j-1])$.*
(3) *If $h^k = (\ )$, then set $p([1, k]) = p([1, k-1])$.*
(4) *Otherwise, if $h^k = (21)$, pair the 2 with the 1 in h^k and set $p([1, k]) = p([1, k-1]) + 1$.*
(5) *Otherwise, if $h^k = (2)$ and $p([1, k-1])$ is even, ignoring all previously paired letters, locate the leftmost unpaired letter in $h^{k-1} \dots h^2 h^1$.*
    (a) *If this letter is in $h^j = (1)$ and $p([j+1, k-1])$ is even, then pair the 2 in h^k with the 1 in h^j and set $p([1, k]) = p([1, k-1]) + 1$.*
    (b) *If this letter is in $h^j = (2)$ and $p([j+1, k-1])$ is odd, then pair the 2 in h^k with the 2 in h^j and set $p([1, k]) = p([1, k-1]) + 1$.*
    (c) *Else, set $p([1, k]) = p([1, k-1])$.*
(6) *Otherwise, if $h^k = (1)$ and $p([1, k-1])$ is odd, ignoring all previously paired letters, locate the leftmost unpaired letter in $h^{k-1} \dots h^2 h^1$.*
    (a) *If this letter is in $h^j = (2)$ and $p([j+1, k-1])$ is even, then pair the 1 in h^k with the 2 in h^j and set $p([1, k]) = p([1, k-1]) + 1$.*

(b) *If this letter is in* $h^j = (1)$ *and* $p([j+1, k-1])$ *is odd, then pair the 1 in* h^k *with the 1 in* h^j *and set* $p([1,k]) = p([1,k-1]) + 1$.

(c) *Else, set* $p([1,k]) = p([1,k-1])$.

(7) *Else, set* $p([1,k]) = p([1,k-1])$.

Example 110. *Let* $m = 8$ *and consider* $\mathbf{h} = (\ )(2)(\ )(21)(1)(1)(2)(21) \in \mathcal{H}^8(3)$. *The pairing process results in* $(\ )(2)(\ )(21)(1)(1)(2)(21)$, *where the paired letters are indicated with braces. Hence, we have the following values of* $p([1,k])$ *for* $1 \leqslant k \leqslant 8$*: 0, 1, 1, 2, 2, 3, 3, 3. Note that the letters in the fourth and seventh factors are left unpaired.*

Similarly, if we take $\mathbf{h} = (\ )(2)(2)(21)(2)(1)(21)(21) \in \mathcal{H}^8(3)$, *we obtain* $(\ )(2)(2)(21)(2)(1)(21)(21)$.

Thus, we have the following values of $p([1,k])$ *for* $1 \leqslant k \leqslant 8$*: 0, 1, 2, 2, 2, 3 ,4, 5. In this case all the letters in* $\mathbf{h}$ *are paired.*

Definition 111. *Let* $\mathbf{h} = h^m \dots h^2 h^1 \in \mathcal{H}^m(3)$. *The crystal operator* f_i *for* $1 \leqslant i < m$ *on* $\mathbf{h}$ *is defined as follows. The operator* f_i *only depends on* $h^{i+1}h^i$ *and the parity of* $p([1, i-1])$ *of Definition 109. In the following cases, we indicate only the changes in* $h^{i+1}h^i$ *under* f_i *as the remainder of* $\mathbf{h}$ *remains invariant:*

(1) $(21)(x) \xrightarrow{i} 0$, *where* $(x) \in \{(\ ), (1), (2), (21)\}$,

(2) $(x)(\ ) \xrightarrow{i} 0$, *where* $(x) \in \{(\ ), (1), (2), (21)\}$,

(3) $(x)(x) \xrightarrow{i} 0$, *where* $(x) \in \{(\ ), (1), (2)\}$,

(4) $(1)(21) \xrightarrow{i} (21)(2)$,

(5) $(2)(21) \xrightarrow{i} (21)(1)$,

(6) $(\ )(x) \xrightarrow{i} (x)(\ )$, *where* $(x) \in \{(1), (2)\}$,

(7) $(\ )(21) \xrightarrow{i} (2)(1) \xrightarrow{i} (21)(\ )$, *if* $p([1, i-1])$ *is even,*

(8) $(\ )(21) \xrightarrow{i} (1)(2) \xrightarrow{i} (21)(\ )$, *if* $p([1, i-1])$ *is odd.*

The operator e_i *is defined similarly. One reverses the changes introduced in cases (4) to (8) and annihilates* $\mathbf{h}$ *when the following occurs at* $h^{i+1}h^i$*:*

(1)' $(x)(21) \xrightarrow{i} 0$, *where* $(x) \in \{(\ ), (1), (2), (21)\}$,

(2)' $(\ )(x) \xrightarrow{i} 0$, *where* $(x) \in \{(\ ), (1), (2), (21)\}$,

(3)' $(x)(x) \xrightarrow{i} 0$, *where* $(x) \in \{(\ ), (1), (2)\}$.

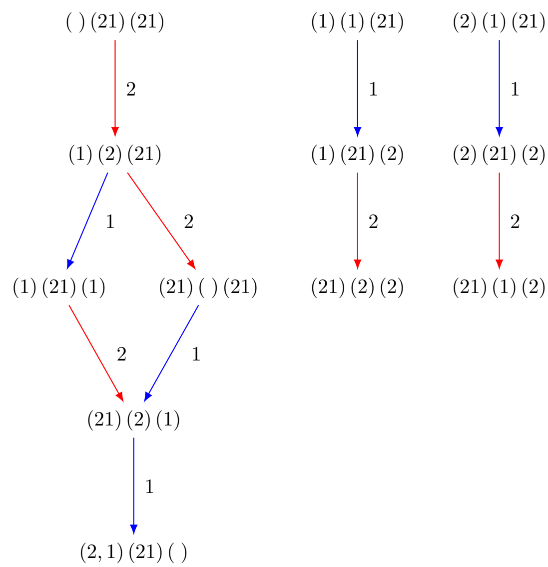


FIGURE 3.3. The crystal graph for $\mathcal{H}^3(3)$ restricted to decreasing factorizations with four letters.

Similar to Definition 48, the weight map is defined as $\mathrm{wt}(\mathbf{h}) = (\mathsf{len}(h^1), \mathsf{len}(h^2), \ldots, \mathsf{len}(h^m))$*. Meanwhile,* $\varphi_i(\mathbf{h})$ *(resp.* $\varepsilon_i(\mathbf{h})$*) is defined to be the largest nonnegative integer* k *such that* $f_i^k(\mathbf{h}) \neq 0$ *(resp.* $e_i^k(\mathbf{h}) \neq 0$*).*

It is not difficult to check that the operators f_i and e_i defined above preserve the relation $\equiv_{\mathcal{H}_0}$ on $\mathcal{H}^m(3)$ whenever they do not annihilate the decreasing factorizations. Furthermore, the structure above defines an abstract, seminormal A_{m-1} crystal on $\mathcal{H}^m(3)$.

We note that one may also verify that the crystal is a Stembridge crystal by checking that the axioms formulated in [**Ste03**] are satisfied. Figure 3.3 displays the crystal graph on $\mathcal{H}^3(3)$ restricted to decreasing factorizations that use exactly 4 letters.

3.4.2. Nonlocality. In this subsection, we show that it is impossible to construct a crystal on $\mathcal{H}^m$ with the following properties for f_i:

(1) f_i only changes the i-th and $(i+1)$-th decreasing factors;
(2) f_i is determined by the first $(i+1)$ factors;
(3) $f_i(\mathbf{h}) \equiv_{\mathcal{H}_0} \mathbf{h}$ and $\mathsf{ex}[f_i(\mathbf{h})] = \mathsf{ex}(\mathbf{h})$, for all $\mathbf{h} \in \mathcal{H}^m$ with $f_i(\mathbf{h}) \neq 0$.

Let $\mathbf{h}_1 = h_1^m \dots h_1^2 h_1^1 \in \mathcal{H}^m$ and suppose that $f_i(\mathbf{h}_1) \neq 0$. If we write $f_i(\mathbf{h}_1) = h_2^m \dots h_2^2 h_2^1$, then the above assumptions imply that $h_1^{i+1} h_1^i \dots h_1^1 \equiv_{\mathcal{H}_0} h_2^{i+1} h_2^i \dots h_2^1$. Obviously the crystal on $\mathcal{H}^m(3)$ defined in Section 3.4.1 satisfies these assumptions.

Suppose that a crystal structure with the above assumptions exists on $\mathcal{H}^4(4)$. Consider the Schur expansion of the stable Grothendieck polynomial in 4 variables for $w = 12132$:

$$\mathfrak{G}_{12132}(x_1, x_2, x_3, x_4; \beta) = s_{221} + \beta(2s_{222} + 3s_{2211}) + \beta^2(6s_{2221} + 6s_{22111}) + \cdots .$$

(Note that s_{22111} is zero in four variables and hence could be omitted). The linear term in β implies that there are two connected components with highest weight $(2, 2, 2, 0)$ (lowest weight $(0, 2, 2, 2)$) for the crystal $\mathcal{H}^4(4)$ with excess 1. All decreasing factorizations mentioned below are those of $w = 12132$ with 4 factors and excess 1.

There are two decreasing factorizations of weight $(2, 2, 2, 0)$: ( )(21)(21)(32) and ( )(21)(32)(32). Focus on the connected component with highest weight ( )(21)(32)(32) and try to complete the crystal graph from top to bottom. Since the only decreasing factorization of weight $(2, 2, 1, 1)$ with the first and second factors both being (32) is (2)(1)(32)(32), we can compute the action of f_3 on this highest weight element. By some similar arguments we can fill in part of the crystal graph as indicated in Figure 3.4 with the above assumptions. The dashed spaces are undetermined.

Yet note that the red f_2 highlighted in the graph changed the first factor from (3) to (2). Hence, Condition (1) is violated, providing a counterexample that crystals with the above conditions always exist on $\mathcal{H}^m(n)$ for $n > 3$.

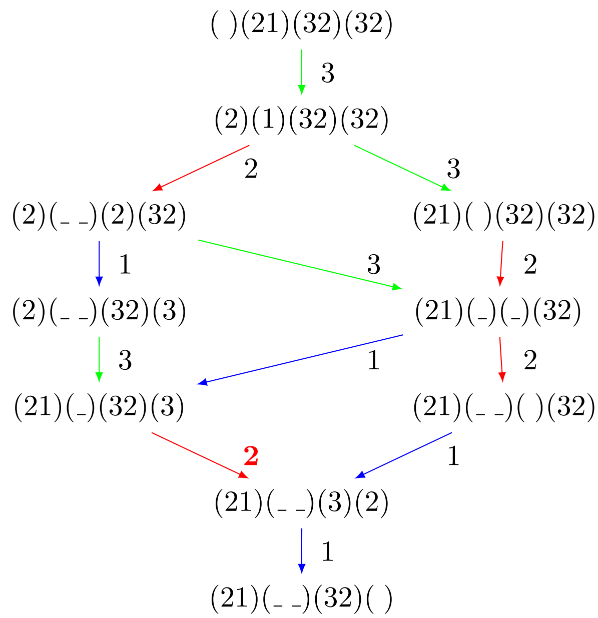


FIGURE 3.4. Partial filling of the connected component of $\mathcal{H}^4(3)$ containing highest weight element ( )(21)(32)(32).

CHAPTER 4

Uncrowding map on hook-valued tableaux

This chapter is based on joint work with Jianping Pan, Joseph Pappe and Anne Schilling published in [**PPPS20**].

4.1. Hook-valued tableaux

In Section 4.1.1, we define hook-valued tableaux [**Yel17**] and in Section 4.1.2 we review the crystal structure on hook-valued tableaux as introduced in [**HS20**].

4.1.1. Hook-valued tableaux. A semistandard Young tableau U of hook shape is a tableau of the form

$$U = \begin{array}{|c|c|c|c|} \cline{1-1} \ell_p \\ \cline{1-1} \vdots \\ \cline{1-1} \ell_1 \\ \hline x & a_1 & \dots & a_q \\ \hline \end{array}\,,$$

where the integer entries weakly increase from left to right and strictly increase from bottom to top. In this case, $\mathsf{H}(U) = x$ is called the *hook entry* of U, $\mathsf{L}(U) = (\ell_1, \ell_2, \ldots, \ell_p)$ is the *leg* of U, and $\mathsf{A}(U) = (a_1, a_2, \ldots, a_q)$ is the *arm* of U. Both the arm and the leg of U are allowed to be empty. Additionally, the *extended leg* of U is defined as $\mathsf{L}^+(U) = (x, \ell_1, \ell_2, \ldots, \ell_p)$. We denote by $\max(U)$ (resp. $\min(U)$) the maximal (resp. minimal) entry in U.

Definition 112. *Fix a partition λ. A semistandard hook-valued tableau (or hook-valued tableau for short) T of shape λ is a filling of the Young diagram for λ with (nonempty) semistandard Young tableaux of hook shape such that:*

(i) $\max(A) \leqslant \min(B)$ *whenever the cell containing A is in the same row, but left of the cell containing B;*

(ii) $\max(A) < \min(C)$ *whenever the cell containing A is in the same column, but below the cell containing C.*

The set of all hook-valued tableaux of shape λ (respectively, with entries at most m) is denoted by $\mathsf{HVT}(\lambda)$ *(respectively,* $\mathsf{HVT}^m(\lambda)$*).*

Given a hook-valued tableau T, its arm excess *is the total number of integers in the arms of all cells of T, while its* leg excess *is the total number of integers in the legs of all cells of T.*

Remark 113. *In the special case when a hook-valued tableau has arm excess 0, it is also called a* set-valued tableau*. Similarly, a* multiset-valued tableau *is a hook-valued tableau with leg excess 0. We use the notation* $\mathsf{SVT}(\lambda)$ *(resp.* $\mathsf{SVT}^m(\lambda)$*) and* $\mathsf{MVT}(\lambda)$ *(resp.* $\mathsf{MVT}^m(\lambda)$*) for the set of all set-valued tableaux of shape λ (resp. with entries at most m) and the set of all multiset-valued tableaux of shape λ (resp. with entries at most m), respectively.*

4.1.2. Crystal structure on hook-valued tableaux. Hawkes and Scrimshaw [**HS20**] defined a crystal structure on hook-valued tableaux. We review their definition here.

Definition 114 ([**HS20**], Definition 4.1). *Let C be a hook-valued tableau of column shape. The column reading word $R(C)$ is obtained by reading the extended leg in each cell from top to bottom, followed by reading all of the remaining entries, arranged in a weakly increasing order.*

For a hook-valued tableau T, its column reading word is formed by concatenating the column reading words of all of its columns, read from left to right, that is,

$$R(T) = R(C_1)R(C_2)\ldots R(C_\ell),$$

where ℓ is the number of columns of T and C_i is the ith column of T.

Example 115. *Let T be the hook-valued tableau*

$$T = \begin{array}{|l|l|l} \cline{1-2} \begin{matrix} 4 \\ 33 \end{matrix} & \begin{matrix} \\ 5 \end{matrix} & \\ \hline \begin{matrix} 2 \\ 11 \end{matrix} & \begin{matrix} 4 \\ 334 \end{matrix} & \multicolumn{1}{l|}{\begin{matrix} \\ 4445 \end{matrix}} \\ \hline \end{array}.$$

The column reading words for the columns of T are respectively 432113, 54334 and 4445, so that

$$R(C) = 432113543344445.$$

Definition 116. *[**HS20**, Definition 4.3] Let $T \in \mathsf{HVT}^m(\lambda)$. For any $1 \leqslant i < m$, we employ the following pairing rules. Assign $-$ to every i in $R(T)$ and assign $+$ to every $i+1$ in $R(T)$. Then, successively pair each $+$ that is adjacent and to the left of a $-$, removing all paired signs until nothing can be paired.*

The operator f_i acts on T according to the following rules in the given order. If there is no unpaired $-$, then f_i annihilates T. Otherwise, locate the cell c with entry the hook-valued tableau $B = T(c)$ containing the i corresponding to the rightmost unpaired $-$.

(M) *If there is an $i+1$ in the cell above c with entry $B^{\uparrow}$, then f_i removes an i from $\mathsf{A}(B)$ and adds $i+1$ to $\mathsf{A}(B^{\uparrow})$.*

(S) *Otherwise, if there is a cell to the right of c with entry $B^{\rightarrow}$, such that it contains an i in $\mathsf{L}^+(B^{\rightarrow})$, then f_i removes the i from $\mathsf{L}^+(B^{\rightarrow})$ and adds $i+1$ to $\mathsf{L}(B)$.*

(N) *Else, f_i changes the i in B into an $i+1$.*

Similarly, the operator e_i acts on T according to the following rules in the given order. If there is no unpaired $+$, then e_i annihilates T. Otherwise, locate the cell c with entry the hook-valued tableau $B = T(c)$ containing the entry $i+1$ corresponding to the leftmost unpaired $+$.

(M) *If there is an i in the cell below c with entry $B^{\downarrow}$, then e_i removes the $i+1$ from $\mathsf{A}(B)$ and adds i to $\mathsf{A}(B^{\downarrow})$.*

(S) *Otherwise, if there is a cell to the left of c with entry $B^{\leftarrow}$, such that it contains an $i+1$ in $\mathsf{L}(B^{\leftarrow})$, then e_i removes the $i+1$ from $\mathsf{L}(B^{\leftarrow})$ and adds i to $\mathsf{L}^+(B)$.*

(N) *Else, e_i changes the $i+1$ in B into an i.*

Based on the pairing procedure above, $\varphi_i(T)$ is the number of unpaired $-$, whereas $\varepsilon_i(T)$ is the number of unpaired $+$.

We remark that the definition of crystal operators on HVT specializes to the definition on SVT in [**MPS20**] or the one on MVT in [**HS20**] when the arm excess or leg excess of the tableaux is set to 0, respectively.

Example 117. *Consider the following hook-valued tableau T:*

$$T = \begin{array}{|l|l|}\hline \begin{array}{l}4\\34\end{array} & \begin{array}{l}5\\4\end{array} \\ \hline \begin{array}{l}2\\11\end{array} & \begin{array}{l}3\\233\end{array} \\ \hline \end{array}.$$

Then, e_3 annihilates T, whereas

$$e_1(T) = \begin{array}{|l|l|}\hline \begin{array}{l}4\\34\end{array} & \begin{array}{l}5\\4\end{array} \\ \hline \begin{array}{l}\\ \\11\end{array} & \begin{array}{l}3\\2\\133\end{array} \\ \hline \end{array}, \quad f_1(T) = \begin{array}{|l|l|}\hline \begin{array}{l}4\\34\end{array} & \begin{array}{l}5\\4\end{array} \\ \hline \begin{array}{l}2\\12\end{array} & \begin{array}{l}3\\233\end{array} \\ \hline \end{array}, \quad f_3(T) = \begin{array}{|l|l|}\hline \begin{array}{l}4\\34\end{array} & \begin{array}{l}5\\44\end{array} \\ \hline \begin{array}{l}2\\11\end{array} & \begin{array}{l}3\\23\end{array} \\ \hline \end{array}.$$

For a given cell (r, c) in row r and column c in a hook-valued tableau T, let $L_T(r, c)$ be the leg of $T(r, c)$, let $\mathsf{A}_T(r, c)$ be arm of $T(r, c)$, let $H_T(r, c)$ be the hook entry of $T(r, c)$, and let $L_T^+(r, c)$ be the extended leg of $T(r, c)$.

4.2. Uncrowding map on hook-valued tableaux

In Section 4.2.1, we first review the uncrowding map on set-valued tableaux. In Section 4.2.2, we give a new uncrowding map on hook-valued tableaux and prove some of its properties in Section 4.2.3. The relation to the uncrowding map on multiset-valued tableaux is given in Section 4.2.4. In Section 4.2.5, we give the inverse of the uncrowding map on hook-valued tableaux, called the crowding map. In Section 4.2.6, an alternative definition of the uncrowding map on hook-valued tableaux is provided.

4.2.1. Uncrowding map on set-valued tableaux. For set-valued tableaux, there exists an uncrowding operator, which maps a set-valued tableau to a pair of tableaux, one being a semistandard Young tableau and the other a flagged increasing tableau (see for example [**Len00**, **Buc02**, **BM12**, **RTY18**]). In this setting, the uncrowding operator intertwines with the crystal operators on set-valued tableaux and semistandard Young tableaux, respectively [**MPS20**].

Consider partitions λ, μ with $\lambda \subseteq \mu$ and $\lambda_1 = \mu_1$. A *flagged increasing tableau* (introduced in [**Len00**] and called (strict) *elegant fillings* by various authors [**LP07, BM12, Pat16**]) is a row and column strict filling of the skew shape μ/λ such that the positive integer entries in the i-th row of the tableau are at most $i-1$ for all $1 \leqslant i \leqslant \ell(\mu)$, where $\ell(\mu)$ is the length of partition μ. In particular, the bottom row is empty. The set of all flagged increasing tableaux is denoted by $\mathcal{F}$. The set of all flagged increasing tableaux of shape μ/λ with $\lambda_1 = \mu_1$ is denoted by $\mathcal{F}(\mu/\lambda)$.

We now review the uncrowding operation on set-valued tableaux. We call a cell in a set-valued tableau a *multicell* if it contains more than one letter.

Definition 118. *Define the* uncrowding operation *on $T \in \mathsf{SVT}(\lambda)$ as follows. First identify the topmost row r in T with a multicell. Let x be the largest letter in row r that lies in a multicell; remove x from the cell and perform RSK row bumping with x into the rows above. The resulting tableau, whose shape differs from λ by the addition of one cell, is the output of this operation.*

The uncrowding map *on set-valued tableaux*

$$\mathcal{U}_{\mathsf{SVT}} : \mathsf{SVT}(\lambda) \longrightarrow \bigsqcup_{\mu \supseteq \lambda} \mathsf{SSYT}(\mu) \times \mathcal{F}(\mu/\lambda) \tag{4.2.1}$$

is defined as follows. Let $T \in \mathsf{SVT}(\lambda)$ with leg excess ℓ.

(1) *Initialize $P_0 = T$ and $Q_0 = F_0$, where F_0 is the unique flagged increasing tableau of shape λ/λ.*
(2) *For each $1 \leqslant i \leqslant \ell$, P_i is obtained from P_{i-1} by applying the uncrowding operation. Let C be the cell in $\mathsf{shape}(P_i)/\mathsf{shape}(P_{i-1})$. If C is in row r', then F_i is obtained from F_{i-1} by adding cell C with entry $r' - r$.*
(3) *Set $\mathcal{U}_{\mathsf{SVT}}(T) = (P, F) := (P_\ell, F_\ell)$.*

It was proved in [**Buc02**, Section 6] that $\mathcal{U}_{\mathsf{SVT}}$ in (4.2.1) is a bijection. Monical, Pechenik and Scrimshaw [**MPS20**] proved that $\mathcal{U}_{\mathsf{SVT}}$ intertwines with the crystal operators on set-valued tableaux (see also [**MPPS20**]). A similar uncrowding algorithm for multiset-valued tableaux was given in [**HS20**, Section 3.2].

4.2.2. Uncrowding map on hook-valued tableaux. In [**HS20**], the authors ask for an uncrowding map for hook-valued tableaux which intertwines with the crystal operators. Here we provide such an uncrowding map by uncrowding the arm excess in a hook-valued tableaux to obtain a set-valued tableaux. An alternative obtained by uncrowding the leg excess first is given in Section 4.2.4.

Definition 119. *The* uncrowding bumping $\mathcal{V}_b \colon \mathsf{HVT} \to \mathsf{HVT}$ *is defined by the following algorithm:*

(1) *Initialize T as the input.*
(2) *If the arm excess of T equals zero, return T.*
(3) *Else, find the rightmost column that contains a cell with nonzero arm excess. Within this column, find the cell with the largest value in its arm. (In French notation this is the topmost cell with nonzero arm excess in the specified column.) Denote the row index and column index of this cell by r and c, respectively. Denote the cell as (r, c), its rightmost arm entry by a, and its largest leg entry by ℓ.*
(4) *Look at the column to the right of (r, c) (i.e. column $c + 1$) and find the smallest number that is greater than or equal to a.*
   - *If no such number exists, attach an empty cell to the top of column $c + 1$ and label the cell as $(\tilde{r}, c + 1)$, where $\tilde{r}$ is its row index. Let k be the empty character.*
   - *If such a number exists, label the value as k and the cell containing k as $(\tilde{r}, c + 1)$ where $\tilde{r}$ is the cell's row index.*

   We now break into cases:

   (a) *If $\tilde{r} \neq r$, then remove a from $\mathsf{A}_T(r, c)$, replace k with a, and attach k to the arm of $\mathsf{A}_T(\tilde{r}, c + 1)$.*
   (b) *If $\tilde{r} = r$ then remove $(a, \ell] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_T(r, c)$ where $(a, \ell] = \{a + 1, a + 2, \ldots, \ell\}$, remove a from $\mathsf{A}_T(r, c)$, insert $(a, \ell] \cap \mathsf{L}_T(r, c)$ into $\mathsf{L}_T(\tilde{r}, c+1)$, replace the hook entry of $(\tilde{r}, c + 1)$ with a, and attach k to $\mathsf{A}_T(\tilde{r}, c + 1)$.*
(5) *Output the resulting tableau.*

See Figures 4.1 and 4.2 for illustration.

Lemma 120. *The map $\mathcal{V}_b$ is well-defined. More precisely, for $T \in \mathsf{HVT}$ we have $\mathcal{V}_b(T) \in \mathsf{HVT}$.*

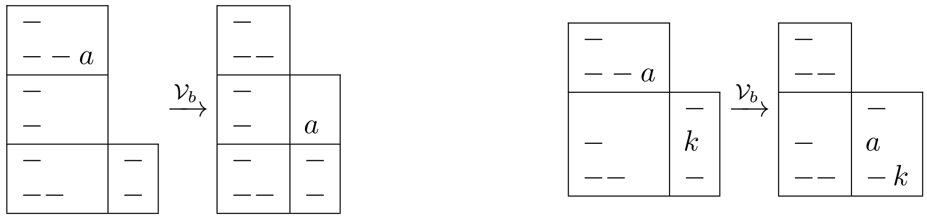


FIGURE 4.1. When $\tilde{r} \neq r$. Left: $(\tilde{r}, c+1)$ is a new cell; Right: $(\tilde{r}, c+1)$ is an existing cell.

FIGURE 4.2. When $\tilde{r} = r$. Left: $(r, c+1)$ is a new cell; Right: $(r, c+1)$ is an existing cell.

PROOF. It suffices to check that $\mathcal{V}_b$ preserves the semistandardness condition of both the entire hook-valued tableau and the filling within each cell. We break into two cases depending on whether Step (4)a or (4)b in Definition 119 is applied.

Case 1: Assume Step (4)a is applied. To verify semistandardness within each cell, it suffices to check cells (r, c) and $(\tilde{r}, c+1)$. The semistandardness within cell (r, c) is clearly preserved as the only change to the hook-shaped tableau in cell (r, c) is that an entry was removed from $\mathsf{A}_T(r, c)$. We now check the semistandardness condition within cell $(\tilde{r}, c+1)$. We have that $\mathcal{V}_b$ either created the cell $(\tilde{r}, c+1)$ and inserted the number a in it or $\mathcal{V}_b$ replaced k with a and appended k to the arm of cell $(\tilde{r}, c+1)$. In both cases, the tableau in cell $(\tilde{r}, c+1)$ is a semistandard hook-shaped tableau. In the second case this is true since k is weakly greater than $\mathsf{H}_T(\tilde{r}, c+1)$ and k is the smallest number weakly greater than a in column $c+1$.

We now check the semistandardness of the entire tableau. Note that it suffices to check the semistandardness in row $\tilde{r}$ and column $c+1$. Since $\tilde{r} < r$, the semistandardness in row $\tilde{r}$ is preserved as a is larger than every number in $(\tilde{r}, c)$ and k remains in the same cell. Also, the semistandardness in column $c+1$ is preserved as k is chosen to be the smallest number in column $c+1$ that is weakly greater than a.

Case 2: Assume Step (4)b is applied. The semistandardness within cell (r, c) is clearly preserved as the only change to (r, c) is that entries from $\mathsf{L}_T(r, c)$ and $\mathsf{A}_T(r, c)$ are removed. We now check the semistandardness condition within cell $(r, c+1)$. If $(a, \ell] \cap \mathsf{L}_T(r, c) = \emptyset$, then a is weakly larger than all elements of (r, c). In this case, the semistandardness within cell $(r, c+1)$ follows from the argument in Case 1. If $(a, \ell] \cap \mathsf{L}_T(r, c) \neq \emptyset$, then a is not weakly larger than all elements of (r, c). After applying $\mathcal{V}_b$ the semistandardness condition in the leg of $(r, c+1)$ will still hold as $a < x < z$ for all $x \in (a, \ell] \cap \mathsf{L}_T(r, c)$, where z is the smallest value in $\mathsf{L}_T(r, c+1)$. Similarly, the semistandardness condition in the arm of $(r, c+1)$ holds as $a < k$ or k is the empty character. Thus, the semistandardness condition in each cell is preserved. The semistandardness of row r is preserved as all numbers strictly greater than a in (r, c) are moved to $(r, c+1)$ along with a. The semistandardness condition within column $c+1$ is preserved as every number in $(r+1, c+1)$ is strictly greater than ℓ and every number in $(r-1, c+1)$ is strictly less than a.

$\square$

Definition 121. *The uncrowding insertion* $\mathcal{V}\colon \mathsf{HVT} \to \mathsf{HVT}$ *is defined as* $\mathcal{V}(T) = \mathcal{V}_b^d(T)$, *where the integer* $d \geqslant 1$ *is minimal such that* $\mathsf{shape}(\mathcal{V}_b^d(T))/\mathsf{shape}(\mathcal{V}_b^{d-1}(T)) \neq \emptyset$ *or* $\mathcal{V}_b^d(T) = \mathcal{V}_b^{d-1}(T)$.

A *column-flagged increasing tableau* is a tableau whose transpose is a flagged increasing tableau. Let $\hat{\mathcal{F}}$ denote the set of all column-flagged increasing tableaux. Let $\hat{\mathcal{F}}(\mu/\lambda)$ denote the set of all column-flagged increasing tableaux of shape μ/λ.

Definition 122. *Let* $T \in \mathsf{HVT}(\lambda)$ *with arm excess* α. *The uncrowding map*

$$\mathcal{U}\colon \mathsf{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$$

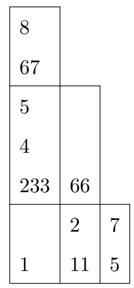
is defined by the following algorithm:

(1) *Let* $P_0 = T$ *and let* Q_0 *be the column-flagged increasing tableau of shape* λ/λ.

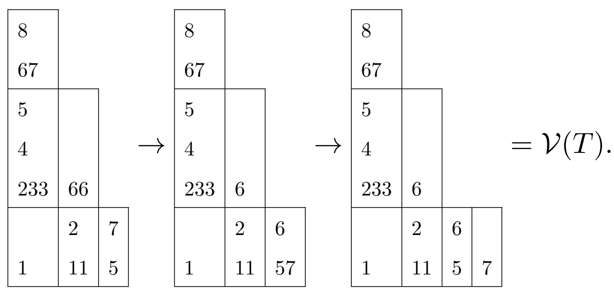
(2) *For* $1 \leqslant i \leqslant \alpha$, *let* $P_{i+1} = \mathcal{V}(P_i)$. *Let* c *be the index of the rightmost column of* P_i *containing a cell with nonzero arm excess and let* $\tilde{c}$ *be the column index of the cell* $\mathsf{shape}(P_{i+1})/\mathsf{shape}(P_i)$. *Then* Q_{i+1} *is obtained from* Q_i *by appending the cell* $\mathsf{shape}(P_{i+1})/\mathsf{shape}(P_i)$ *to* Q_i *and filling this cell with* $\tilde{c} - c$.

Define $\mathcal{U}(T) = (P(T), Q(T)) := (P_\alpha, Q_\alpha)$.

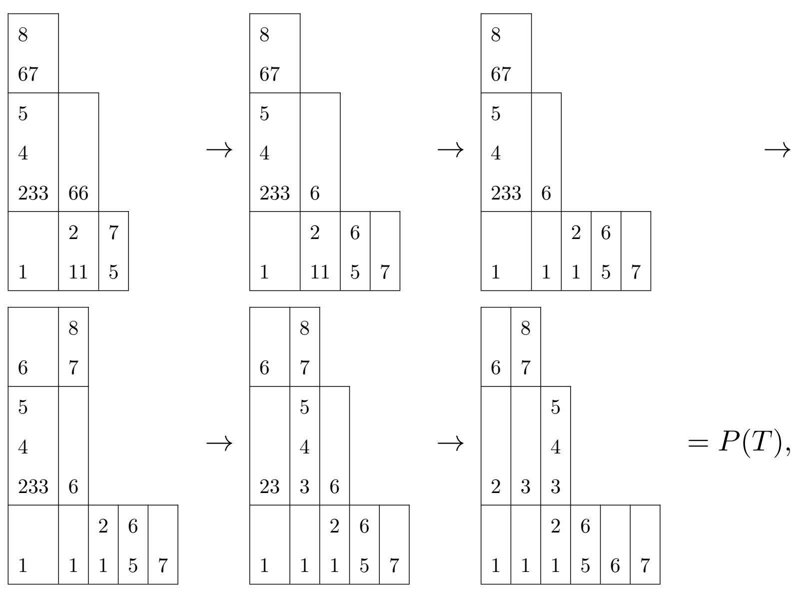
EXAMPLE 123. *Let* T *be the hook-valued tableau*

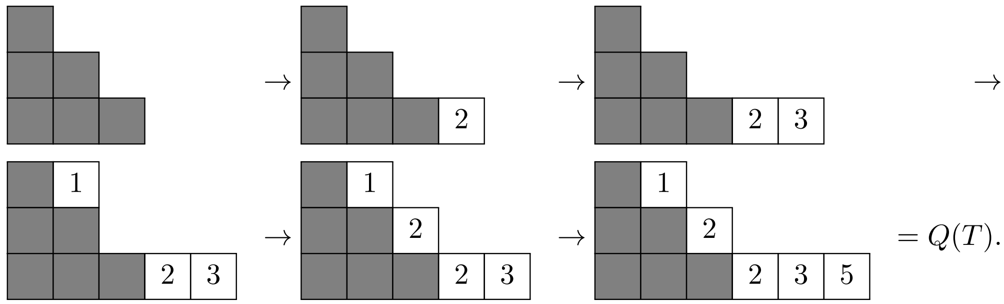


Then, we obtain the following sequence of tableaux $\mathcal{V}_b^i(T)$ *for* $0 \leqslant i \leqslant 2 = d$ *when computing the first uncrowding insertion:*



Continuing with the remaining uncrowding insertions, we obtain the following sequences of tableaux for the uncrowding map:





Corollary 124. *Let $T \in \mathsf{HVT}$. Then $P(T)$ is a set-valued tableau.*

Proof. By Lemma 120 and Definition 121, we have that $\mathcal{V}(T)$ is a hook-valued tableau. Note that if the arm excess of T is nonzero, then the arm excess of $\mathcal{V}(T)$ is one less than that of T. Since $P(T) = \mathcal{V}^{\alpha}(T)$, where α is the arm excess of T, we have that the arm excess of $P(T)$ is zero. Thus, $P(T)$ is a set-valued tableau. $\square$

Definition 125. *Let $T \in \mathsf{HVT}$ and let d be minimal such that $\mathcal{V}(T) = \mathcal{V}_b^d(T)$. The insertion path p of $T \to \mathcal{V}(T)$ is defined as follows:*

- *If $d = 0$, set $p = \emptyset$.*
- *Otherwise, let (r_0, c_0) be the rightmost and topmost cell of T containing a cell with nonzero arm excess. For all $1 \leqslant j \leqslant d$, let $c_j = c_0 + j$ and let $r_j = \tilde{r}$ be $\tilde{r}$ in Definition 119 when $\mathcal{V}_b$ is applied to $\mathcal{V}_b^{j-1}(T)$. Set $p = ((r_0, c_0), (r_1, c_1), \ldots, (r_d, c_d))$.*

Lemma 126. *Let $T \in \mathsf{HVT}$. Then $Q(T)$ is a column-flagged increasing tableau.*

Proof. By construction, the positive integer entries in column i of $Q(T)$ are at most $i-1$. Let m be the smallest nonnegative integer such that $\mathcal{V}^m(T) = P(T)$. Let $p^i = ((r_0^i, c_0^i), (r_1^i, c_1^i), \ldots, (r_{d_i}^i, c_{d_i}^i))$ for $0 \leqslant i < m$ be the insertion path of $\mathcal{V}^i(T) \to \mathcal{V}^{i+1}(T)$. Since $c_0^{i+1} \leqslant c_0^i$ for all $0 \leqslant i < m$, the entries in each row of $Q(T)$ are strictly increasing. To check that the entries in each column of $Q(T)$ are strictly increasing, it suffices to show that if $c_0^{i+1} = c_0^i$ then p^{i+1} lies weakly below p^i. In other words, it suffices to check that $c_0^{i+1} = c_0^i$ implies that $r_j^{i+1} \leqslant r_j^i$ for all $0 \leqslant j \leqslant d_i$. We prove this by induction on j. Note that $r_0^{i+1} \leqslant r_0^i$ by the definition of $\mathcal{U}$. Assume by induction that $r_j^{i+1} \leqslant r_j^i$. This implies that the a when applying $\mathcal{V}_b$ to $\mathcal{V}_b^j(\mathcal{V}^i(T))$ is weakly smaller than the a when applying $\mathcal{V}_b$ to $\mathcal{V}_b^j(\mathcal{V}^{i-1}(T))$. Thus, we must have $r_{j+1}^{i+1} \leqslant r_{j+1}^i$. $\square$

4.2.3. Properties of the uncrowding map. Let T be a hook-valued tableau. Define $R_i(T)$ as the induced subword of $R(T)$ consisting only of the letters i and $i+1$. In the next lemma, we use the same notation as in Definition 119. Furthermore, two words are *Knuth equivalent* if one can be transformed to the other by a sequence of Knuth equivalences on three consecutive letters

$$xzy \equiv zxy \quad \text{for } x \leqslant y < z, \qquad yxz \equiv yzx \quad \text{for } x < y \leqslant z.$$

LEMMA 127. *For $T \in \mathsf{HVT}$, $R_i(T) = R_i(\mathcal{V}_b(T))$ unless T satisfies one of the following three conditions:*

(a) *$a = i$ or $a = i+1$ and column $c+1$ contains both an i and an $i+1$,*

(b) *$\tilde{r} = r$, $i \in (a, \ell] \cap \mathsf{L}_T(r, c)$, $k = i$, and column $c+1$ contains an $i+1$,*

(c) *$\tilde{r} = r$, $a = i$, $i+1 \in (a, \ell] \cap \mathsf{L}_T(r, c)$, and (r, c) contains another i besides a.*

Moreover, $R_i(T)$ is Knuth equivalent to $R_i(\mathcal{V}_b(T))$.

PROOF. Let $R_i(T) = r_1 r_2 \ldots r_m$. We break into cases based on the value of a.

Case 1: Assume $a \neq i, i+1$.

Assume Step (4)a is applied by $\mathcal{V}_b$. If $k \neq i, i+1$, then $R_i(T) = R_i(\mathcal{V}_b(T))$ as the position of all letters i and $i+1$ remains the same. Let $k = i$. We have that k is the only i in column $c+1$. Hence, when k gets bumped from $\mathsf{L}_T(\tilde{r}, c+1)$ and appended to $\mathsf{A}_T(\tilde{r}, c+1)$, the relative position of k to the other letters i and $i+1$ in $R_i(T)$ does not change. Thus, $R_i(T) = R_i(\mathcal{V}_b(T))$. Let $k = i+1$. Note that column $c+1$ cannot have a cell containing an i as k is the smallest number weakly greater than a. Hence, moving k from $\mathsf{L}_T(\tilde{r}, c+1)$ to $\mathsf{A}_T(\tilde{r}, c+1)$ will not change $R_i(T)$. Therefore, we once again have that $R_i(T) = R_i(\mathcal{V}_b(T))$.

Assume Step (4)b is applied by $\mathcal{V}_b$. Consider the subcase when $(a, \ell] \cap \mathsf{L}_T(r, c) = \emptyset$. By a similar argument to the previous paragraph, we have that $R_i(T) = R_i(\mathcal{V}_b(T))$. Next, consider the subcase when $i+1 \in (a, \ell] \cap \mathsf{L}_T(r, c)$. This implies that $a < i$ and the only time $i+1$ occurs in column c is in $\mathsf{L}_T(r, c)$. Note that if an i exists in column c, it must be contained in $\mathsf{L}_T(r, c)$. We also have that $k \geqslant i+1$ or k is the empty character and no cell in column $c+1$ contains an i. Thus, removing $(a, \ell] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_T(r, c)$, replacing k with $(a, \ell] \cap \mathsf{L}_T(r, c)$ in $\mathsf{L}_T(r, c+1)$, and appending k to $\mathsf{A}_T(r, c+1)$ does not change $R_i(T)$. Therefore $R_i(T) = R_i(\mathcal{V}_b(T))$. Let $i \in (a, \ell] \cap \mathsf{L}_T(r, c)$ and

$i+1 \notin (a,\ell] \cap \mathsf{L}_T(r,c)$. Note that the only place $i+1$ can occur in column c is as $\mathsf{H}_T(r+1,c)$ and the only place i can occur in column c is in $\mathsf{L}_T(r,c)$. This implies that removing $(a,\ell] \cap \mathsf{L}_T(r,c)$ from $\mathsf{L}_T(r,c)$, replacing k with $(a,\ell] \cap \mathsf{L}_T(r,c)$ in $\mathsf{L}_T(r,c+1)$ and appending k to $\mathsf{A}_T(r,c+1)$ will not change $R_i(T)$ unless both $i+1$ and i show up in column $c+1$. This can only occur when $k=i$ which implies that $R_i(T) = r_1 \dots i\ i{+}1\ k \dots r_m$ and $R_i(\mathcal{V}_b(T)) = r_1 \dots i{+}1\ i\ k \dots r_m$. We see that $R_i(T)$ and $R_i(\mathcal{V}_b(T))$ only differ by a Knuth relation implying they are Knuth equivalent. Assume that $i, i+1 \notin (a,\ell] \cap \mathsf{L}_T(r,c) \neq \emptyset$. If $a > i+1$ the positions of all letters i and $i+1$ remain the same after $\mathcal{V}_b$ is applied. If $a < i$, then the positions of all letters i and $i+1$ also remain the same unless $k=i$ or $k=i+1$. In both of these special subcases, it can be checked that still $R_i(T) = R_i(\mathcal{V}_b(T))$.

Case 2: Assume $a = i$.

Assume Step (4)a is applied by $\mathcal{V}_b$. If column $c+1$ does not contain both an i and an $i+1$, then we have $R_i(T) = R_i(\mathcal{V}_b(T))$. However, if both an i and an $i+1$ are in column $c+1$, then $R_i(T) = r_1 \dots i\ i+1\ i \dots r_m$ and $R_i(\mathcal{V}_b(T)) = r_1 \dots i+1\ i\ i \dots r_m$ which are Knuth equivalent.

Assume Step (4)b is applied by $\mathcal{V}_b$. Consider the subcase when $(a,\ell] \cap \mathsf{L}_T(r,c) = \emptyset$. By a similar argument to the previous paragraph, we have that $R_i(T) = R_i(\mathcal{V}_b(T))$ unless both an i and an $i+1$ are in column $c+1$ in which case $R_i(T)$ and $R_i(\mathcal{V}_b(T))$ are only Knuth equivalent. Consider the subcase given by $i+1 \in (a,\ell] \cap \mathsf{L}_T(r,c)$. Note that no cell in column $c+1$ can contain an i, the only cell that could contain an $i+1$ in column $c+1$ is $(r,c+1)$, and the only cell containing letters i or $i+1$ in column c is (r,c). This implies that it suffices to look at the changes to (r,c) and $(r,c+1)$. We see that $R_i(T) = r_1 \dots i+1\ \underbrace{i \dots i\, a}_{\gamma} \dots r_m$ and $R_i(\mathcal{V}_b(T)) = r_1 \dots \underbrace{i \dots i}_{\gamma-1}\ i+1\ a$ where $\gamma \geqslant 1$ is the number of letters i in cell (r,c) including a. We see that $R_i(T)$ and $R_i(\mathcal{V}_b(T))$ are Knuth equivalent. Consider the subcase when $i+1 \notin (a,\ell] \cap \mathsf{L}_T(r,c) \neq \emptyset$. We have that both i and $i+1$ cannot be in a cell in column $c+1$ and an $i+1$ cannot be in column c. Thus applying $\mathcal{V}_b$ does not change $R_i(T)$ giving us that $R_i(T) = R_i(\mathcal{V}_b(T))$.

Case 3: Assume $a = i+1$.

Assume Step (4)a is applied by $\mathcal{V}_b$. If column $c+1$ does not contain both i and $i+1$, then we have that $R_i(T) = R_i(\mathcal{V}_b(T))$. However, if both i and $i+1$ occur in column $c+1$, then $R_i(T) = r_1 \dots i{+}1\ i{+}1\ i \dots r_m$ and $R_i(\mathcal{V}_b(T)) = r_1 \dots i{+}1\ i\ i{+}1 \dots r_m$ which are Knuth equivalent.

Assume Step (4)b is applied by $\mathcal{V}_b$. If $(a, \ell] \cap \mathsf{L}_T(r, c) = \emptyset$, then $R_i(T) = R_i(\mathcal{V}_b(T))$ unless both i and $i+1$ occur in column $c+1$. In this exceptional case, we have that $R_i(T)$ and $R_i(\mathcal{V}_b(T))$ are only Knuth equivalent by a similar argument to the previous paragraph. If $(a, \ell] \cap \mathsf{L}_T(r, c) \neq \emptyset$, then $k > i+1$ or k is the empty character and no cell in column $c+1$ contains an $i+1$. Thus applying $\mathcal{V}_b$ does not change $R_i(T)$ giving us that $R_i(T) = R_i(\mathcal{V}_b(T))$. $\square$

Remark 128. *In general, the full reading words are not Knuth equivalent under the uncrowding map. For example, take the following hook-valued tableau T, which uncrowds to a set-valued tableau S:*

$$T = \begin{array}{|c|c|} \hline 4 & \\ 3 & \\ 2 & 5 \\ 12 & 4 \\ \hline \end{array} \to \begin{array}{|c|c|c|} \hline & 4 & \\ 2 & 3 & 5 \\ 1 & 2 & 4 \\ \hline \end{array} = S.$$

The reading word changed from 4321254 *to* 2143254, *which are not Knuth equivalent.*

Proposition 129. *Let $T \in \mathsf{HVT}$.*

(1) *If $f_i(T) = 0$, then $f_i(P(T)) = 0$.*
(2) *If $e_i(T) = 0$, then $e_i(P(T)) = 0$.*

Proof. Since $P(T) = \mathcal{V}_b^s(T)$ for some $s \in \mathbb{N}$ and Knuth equivalence is transitive, we have that $R_i(T)$ is Knuth equivalent to $R_i(P(T))$ by the previous lemma. As $f_i(T) = 0$, we have that every i in $R_i(T)$ is i-paired with an $i+1$ to its left. This property is preserved under Knuth equivalence giving us that $f_i(P(T)) = 0$. The same reasoning implies (2). $\square$

Lemma 130. *Let $T \in \mathsf{HVT}$.*

(1) *If $f_i(T) \neq 0$, then $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T)) \neq 0$.*
(2) *If $e_i(T) \neq 0$, then $e_i(\mathcal{V}_b(T)) = \mathcal{V}_b(e_i(T)) \neq 0$.*

Proof. We are going to prove (1). Part (2) follows since e_i and f_i are partial inverses.

Let a, ℓ, k, r, c, and $\tilde{r}$ be defined as in Definition 119 when $\mathcal{V}_b$ is applied to T. Similarly, define a', ℓ', k', r', c', and $\tilde{r}'$ for when $\mathcal{V}_b$ is applied to $f_i(T)$. Let $R_i(T) = r_1 r_2 \ldots r_m$ and $R_i(\mathcal{V}_b(T)) = r'_1 r'_2 \ldots r'_m$ be the corresponding reading words. Let $(\hat{r}, \hat{c})$ denote the cell containing the rightmost unpaired i in T, where $\hat{r}$ and $\hat{c}$ are its row and column index respectively. We break into cases based on the position of $(\hat{r}, \hat{c})$ to (r, c).

Case 1: Assume $(\hat{r}, \hat{c}) = (r, c)$. We break into subcases based on how f_i acts on T.

- Assume that $(r+1, c)$ contains an $i+1$.

  As every entry in (r, c) must be strictly smaller than the values in $(r+1, c)$ and (r, c) must contain an i, we have that $\ell = i$ or $a = i$. If $\ell = i$, then ℓ is i-paired with the $i+1$ in $(r+1, c)$. Hence a is always equal to i and a must correspond to the rightmost unpaired i of T. Thus, f_i acts on T by removing a from (r, c) and appending an $i+1$ to $\mathsf{A}_T(r+1, c)$. Note that $(a, \ell] \cap \mathsf{L}_T(r, c) = \emptyset$ implying $\mathcal{V}_b$ acts on T by removing a from $\mathsf{A}_T(r, c)$, replacing k in $(\tilde{r}, c+1)$ with a, and appending k to $\mathsf{A}_T(\tilde{r}, c+1)$ where $\tilde{r} \leqslant r$. We break into subcases based upon where the values of i and $i+1$ are in column $c+1$ utilizing the fact that column $c+1$ cannot contain an i without an $i+1$ (since the arm excess of cell $(r+1, c)$ is zero and cell (r, c) contains the rightmost unpaired i).

  Assume that column $c+1$ does not contain an i. Since a corresponds to the rightmost unpaired i in T and column $c+1$ does not contain an i, we have that the rightmost unpaired i in $\mathcal{V}_b(T)$ is precisely a in the cell $(\tilde{r}, c+1)$. Note that $(\tilde{r}+1, c+1)$ does not contain an $i+1$ in $\mathcal{V}_b(T)$ as $k \geqslant i+1$ or k is the empty character. Similarly, we have that $(\tilde{r}, c+2)$ does not contain an i. Thus, f_i acts on $\mathcal{V}_b(T)$ by changing a to an $i+1$ in $(\tilde{r}, c+1)$. We now consider $\mathcal{V}_b(f_i(T))$. When applying $\mathcal{V}_b$ to $f_i(T)$, a' is precisely the $i+1$ appended to $\mathsf{A}_T(r+1, c)$ and k' is the same as k. Since $\tilde{r}' = \tilde{r} < r+1$, we have that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $i+1$ from $\mathsf{A}_{f_i(T)}(r+1, c)$, replacing k with an $i+1$ in $(\tilde{r}, c+1)$, and appending k to $\mathsf{A}_{f_i(T)}(\tilde{r}, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

  Assume that column $c+1$ contains both an i and an $i+1$ in the same cell. Note that this implies that $k = i$. Since a is the rightmost unpaired i in T and the only cell in column $c+1$ that contained an $i+1$ or an i is $(\tilde{r}, c+1)$, we have that the rightmost unpaired i in $\mathcal{V}_b(T)$ is the i appended to $\mathsf{A}_T(\tilde{r}, c+1)$. Since $(\tilde{r}, c+1)$ contains an $i+1$, we have that $(\tilde{r}+1, c+1)$ cannot contain an $i+1$ and $(\tilde{r}, c+2)$ cannot contain an i. Thus, f_i acts on $\mathcal{V}_b(T)$ by changing the i in $\mathsf{A}_{\mathcal{V}_b(T)}(\tilde{r}, c+1)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. When applying $\mathcal{V}_b$ to $f_i(T)$, a' is precisely the $i+1$ appended to $\mathsf{A}_T(r+1, c)$ and k' is the $i+1$ in $(\tilde{r}, c+1)$. Since $\tilde{r}' = \tilde{r} < r+1$, we have that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $i+1$ from $\mathsf{A}_{f_i(T)}(r+1, c)$, replacing $i+1$ in $(\tilde{r}, c+1)$ with the $i+1$ from $\mathsf{A}_{f_i(T)}(r+1, c)$, and appending an $i+1$ to $\mathsf{A}_{f_i(T)}(\tilde{r}, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Assume that column $c+1$ contains both an i and an $i+1$ in different cells. Note that this implies that $k = i$. Since a corresponds to the rightmost unpaired i in $R_i(T)$ and the only $i+1$ and i in column $c+1$ are in cells $(\tilde{r}+1, c+1)$ and $(\tilde{r}, c+1)$ respectively, we have that the rightmost unpaired i in $R_i(\mathcal{V}_b(T))$ corresponds to the i appended to $\mathsf{A}_T(\tilde{r}, c+1)$. By assumption, we have that $(\tilde{r}+1, c+1)$ contains an $i+1$. Thus, f_i acts on $\mathcal{V}_b(T)$ by removing the i from $\mathsf{A}_{\mathcal{V}_b(T)}(\tilde{r}, c+1)$ and appending an $i+1$ to $\mathsf{A}_{\mathcal{V}_b(T)}(\tilde{r}+1, c+1)$. We now consider $\mathcal{V}_b(f_i(T))$. When applying $\mathcal{V}_b$ to $f_i(T)$, a' is precisely the $i+1$ appended to $\mathsf{A}_T(r+1, c)$ and k' is the $i+1$ in cell $(\tilde{r}+1, c+1)$. If $\tilde{r}' = r+1$, then $i+1$ is weakly larger than every value in $(r+1, c)$. Thus, either $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r+1, c) = \emptyset$ or $\tilde{r}' < r+1$. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $i+1$ from $\mathsf{A}_{f_i(T)}(r+1, c)$, replacing the $i+1$ in $\mathsf{H}_{f_i(T)}(\tilde{r}+1, c+1)$ with the $i+1$ removed from $\mathsf{A}_{f_i(T)}(r+1, c)$, and appending an $i+1$ to $\mathsf{A}_{f_i(T)}(\tilde{r}+1, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

- Assume that $(r+1, c)$ does not contain an $i+1$ and $(r, c+1)$ contains an i.

  Under these assumptions, we have that no cell in column c can contain an $i+1$. This implies that column $c+1$ must contain an $i+1$. The cell $(r+1, c+1)$ cannot have an $i+1$ as this would force $(r+1, c)$ to also have an $i+1$. Thus, $(r, c+1)$ must contain an $i+1$ in its leg. By our assumption we have that f_i acts on T by removing the i from $(r, c+1)$ and appending an $i+1$ to $\mathsf{L}_T(r, c)$. We break into subcases according to where the rightmost unpaired i sits inside the cell (r, c). If the rightmost unpaired i is in $\mathsf{H}_T(r, c)$, then $a \geqslant i$ which would either contradict the hook entry being the rightmost unpaired i or cell $(r, c+1)$ containing an i. Thus, we only need to consider the subcases where the rightmost unpaired i is either in the leg or arm of (r, c).

Assume that the rightmost unpaired i is in $\mathsf{L}_T(r, c)$ for this entire paragraph. This implies that $\ell = i$. Since $(r, c+1)$ contains an i, we have that $a < i$. If $\tilde{r} < r$, then $\mathcal{V}_b$ acts on T by removing a from (r, c), replacing k with a in $(\tilde{r}, c+1)$, and appending k to $\mathsf{A}_T(\tilde{r}, c+1)$. Since $a, k < i$, we have that $\mathcal{V}_b$ does not change position of the rightmost unpaired i. Note that $(r+1, c)$ still does not contain an $i+1$ while $(r, c+1)$ still contains an i. Thus, f_i acts on $\mathcal{V}_b(T)$ by removing the i from $(r, c+1)$ and appending an $i+1$ to $\mathsf{L}_{\mathcal{V}_b(T)}(r, c)$. We now consider $\mathcal{V}_b(f_i(T))$. Note that (r', c'), a', and k' are the same as (r, c), a, and k respectively. Thus, $\mathcal{V}_b$ acts in the same way as before. This gives us that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. If $\tilde{r} = r$, then k is precisely the i in cell $(r, c+1)$. We see that $\mathcal{V}_b$ acts on T by removing $(a, i] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_T(r, c)$ and a from $\mathsf{A}_T(r, c)$, replacing k with $((a, i] \cap \mathsf{L}_T(r, c)) \cup \{a\}$, and appending k to $\mathsf{A}_T(r+1, c)$. Since there is an $i+1$ in $\mathsf{L}_{\mathcal{V}_b(T)}(r, c+1)$, we see that the rightmost unpaired i in $\mathcal{V}_b(T)$ is precisely k in $\mathsf{A}_{\mathcal{V}_b(T)}(r, c+1)$. Note that $(r+1, c+1)$ does not contain an $i+1$ and $(r, c+2)$ does not contain an i because $(r, c+1)$ contains an $i+1$. Thus, f_i acts on $\mathcal{V}_b(T)$ by changing the i in $\mathsf{A}_{\mathcal{V}_b(T)}(r, c+1)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We have that a' is the same as a and k' is the $i+1$ in $(r, c+1)$. We have $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c') = \{i+1\} \cup ((a, i] \cap \mathsf{L}_T(r, c))$. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $\{i+1\} \cup ((a, i] \cap \mathsf{L}_T(r, c))$ from $\mathsf{L}_{f_i(T)}(r, c)$ and a from $\mathsf{A}_{f_i(T)}(r, c)$, replacing $i+1$ with $\{i+1\} \cup ((a, i] \cap \mathsf{L}_T(r, c)) \cup \{a\}$ in $(r, c+1)$, and appending an $i+1$ to $\mathsf{A}_{f_i(T)}(r, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Assume that the rightmost unpaired i is in $\mathsf{A}_T(r, c)$. This implies that $a = i$ and forces a to correspond to the rightmost unpaired i. We also have that k is the i in $(r, c+1)$. Since i is weakly greater than all values in (r, c), we have that $(a, \ell] \cap \mathsf{L}_T(r, c) = \emptyset$. Thus, $\mathcal{V}_b$ acts on T by removing a from (r, c), replacing k with a in $(r, c+1)$, and appending k to $\mathsf{A}_T(r, c+1)$. Since a was the rightmost unpaired i in T and cell

$(r, c+1)$ contains an $i+1$ in its leg, we have that the rightmost unpaired i in $\mathcal{V}_b(T)$ is k in $\mathsf{A}_{\mathcal{V}_b(T)}(r, c+1)$. As $i+1$ is in $(r, c+1)$, we have that $(r+1, c+1)$ cannot contain an $i+1$ and $(r, c+2)$ cannot contain an i. This implies that f_i acts on $\mathcal{V}_b(T)$ by changing the i in $\mathsf{A}_{\mathcal{V}_b(T)}(r, c+1)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We have that a' is the same as a and k' is equal to the $i+1$ in $(r, c+1)$. Note that $(a', \ell'] \cap \mathsf{L}_T(r, c) = \{i+1\}$. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $i+1$ from $\mathsf{L}_{f_i(T)}(r, c)$ and a from $\mathsf{A}_{f_i(T)}(r, c)$, replacing the $i+1$ in $(r, c+1)$ with $\{i+1, a\}$, and appending an $i+1$ to $\mathsf{A}_{f_i(T)}(r, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

- Assume that $(r+1, c)$ does not contain an $i+1$ and $(r, c+1)$ does not contain an i. We break into subcases based on where the rightmost unpaired i sits inside (r, c).

  Assume that rightmost unpaired i is in the hook entry of (r, c) for the remainder of this paragraph. Note that this implies that $a > i$ and the rightmost unpaired i in $\mathcal{V}_b(T)$ is still the hook entry of (r, c). We see that $\mathcal{V}_b$ does not insert an $i+1$ into $(r+1, c)$ nor an i into $(r, c+1)$. This implies that f_i acts on T and $\mathcal{V}_b(T)$ in the same way by changing the hook entry of (r, c) into an $i+1$. Next, we note that (r', c'), a', k', and $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ are the same as (r, c), a, k, and $(a, \ell] \cap \mathsf{L}_T(r, c)$ respectively. Thus, $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same manner without affecting the hook entry of (r, c). Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

  Assume that the rightmost unpaired i is in the leg of (r, c) for the remainder of this paragraph. This implies that $a \neq i$. First, we assume that $a > i$ or $\tilde{r} < r$. Under this extra assumption, we observe that the action of $\mathcal{V}_b$ does not change the position of the rightmost unpaired i. Also, $\mathcal{V}_b$ does not insert an $i+1$ into $(r+1, c)$ nor an i into $(r, c+1)$. We see that f_i acts on T and $\mathcal{V}_b(T)$ in the same way by changing the i in the leg of (r, c) into an $i+1$. Next, we note that (r', c'), a', and k' are the same as (r, c), a, and k respectively. If $a > i$, we have that $a \geqslant i+1$ implying that $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c') = (a, \ell] \cap \mathsf{L}_T(r, c)$. Thus, either $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c') = (a, \ell] \cap \mathsf{L}_T(r, c)$ or $\tilde{r} < r$. This implies that $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same manner and does not affect the i or $i+1$ in the leg of (r, c). Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. Next, assume that $\tilde{r} = r$ and $a < i$. This implies that $(a, \ell] \cap \mathsf{L}_T(r, c) \neq \emptyset$ as $i \in (a, \ell] \cap \mathsf{L}_T(r, c)$. We have that $\mathcal{V}_b$ acts on T by removing $(a, \ell] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_T(r, c)$ and a from $\mathsf{A}_T(r, c)$, replacing k with $((a, l] \cap \mathsf{L}_T(r, c)) \cup \{a\}$ in $(r, c+1)$, and appending k to $\mathsf{A}_T(r, c+1)$. By assumption, there was no i in $(r, c+1)$ to begin with. Thus, we have that the rightmost unpaired i of $\mathcal{V}_b(T)$ is the i in $(r, c+1)$ that replaced k. Since $k \geqslant i+1$ or k is the empty character, we have that the cell $(r+1, c+1)$ does not contain an $i+1$ and the cell $(r, c+2)$ does not contain an i. Hence, f_i acts on $\mathcal{V}_b(T)$ by replacing the i in $\mathsf{L}_{\mathcal{V}_b(T)}(r, c+1)$ with an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We have that f_i acts on T by changing the i in $\mathsf{L}_T(r, c)$ to an $i+1$. We see that a' and k' are the same as a and k respectively. Since $i > a$, we have that $i+1 > a$ or in other words $i+1 \in (a', \ell'] \cap \mathsf{L}_T(r, c)$. This implies that $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c') = (((a', \ell'] \cap \mathsf{L}_T(r, c)) \cup \{i+1\}) - \{i\}$. We have $\mathcal{V}_b$ acts on $f_i(T)$ by removing $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r, c)$ from $\mathsf{L}_{f_i(T)}(r, c)$ and a from $\mathsf{A}_{f_i(T)}(r, c)$, replacing k with $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r, c)$ in $(r, c+1)$, and appending k to $\mathsf{A}_{f_i(T)}(r, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Assume that the rightmost unpaired i is in $\mathsf{A}_T(r, c)$ and $\tilde{r} < r$ or $(a, \ell] \cap \mathsf{L}_T(r, c) = \emptyset$ for this entire paragraph. Under this assumption, f_i acts on T by changing the rightmost i in the arm of (r, c) to an $i+1$. Also, $\mathcal{V}_b$ acts on T by removing a from $\mathsf{A}_T(r, c)$, replacing k in $(\tilde{r}, c+1)$ with a, and appending k to $\mathsf{A}_T(\tilde{r}, c+1)$. First, we make the additional assumption that $i < a$. Since we assume the rightmost unpaired i is in the arm of (r, c) and $i < a$, we have the rightmost unpaired i in $\mathcal{V}_b(T)$ is in the same position as in T. Note that the cell $(r+1, c)$ still does not contain an $i+1$ and the cell $(r, c+1)$ still does not contain an i. Thus, we have that f_i acts on $\mathcal{V}_b(T)$ by changing the rightmost i in $\mathsf{A}_{\mathcal{V}_b}(r, c)$ into an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We see that a'

and k' are the same as a and k respectively. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing a from (r,c), replacing k with a in $(\tilde{r},c)$, and appending k to $\mathsf{A}_{f_i(T)}(\tilde{r},c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. Next, we make the assumption that $a = i$ and column $c+1$ does not contain both an i and an $i+1$. We have that the rightmost unpaired i in $\mathcal{V}_b(T)$ is precisely the i that replaced k in $(\tilde{r},c+1)$. We also have that $k \geqslant i+1$ or k is the empty character implying that the cell $(\tilde{r}+1,c+1)$ does not contain an $i+1$ and the cell $(\tilde{r},c+2)$ does not contain an i. This implies that f_i acts on $\mathcal{V}_b(T)$ by changing the i in $\mathsf{L}^+_{\mathcal{V}_b(T)}(\tilde{r},c+1)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We see that a' is the $i+1$ in (r,c) created by appying f_i and k' is the same as k. Thus, $\mathcal{V}_b$ acts on $f_i(T)$ by removing the $i+1$ from (r,c), replacing k with an $i+1$ in $(\tilde{r},c)$, and appending k to $\mathsf{A}_{f_i(T)}(\tilde{r},c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. Next, we assume that $a = i$ and column $c+1$ contains both an i and an $i+1$ in the same cell. Note that this implies that $k = i$. Since a corresponded to the rightmost unpaired i in T and the only cell in column $c+1$ that contains an $i+1$ or an i is $(\tilde{r},c+1)$, we have that the rightmost unpaired i in $\mathcal{V}_b(T)$ corresponds to the i appended to $\mathsf{A}_T(\tilde{r},c+1)$. Since $(\tilde{r},c+1)$ contains an $i+1$ in $\mathcal{V}_b(T)$, we have that $(\tilde{r}+1,c+1)$ cannot contain an $i+1$ and $(\tilde{r},c+2)$ cannot contain an i. Thus, f_i acts on $\mathcal{V}_b(T)$ by changing the i in $\mathsf{A}_{\mathcal{V}_b(T)}(\tilde{r},c+1)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We see that a' is the $i+1$ in (r,c) obtained after applying f_i and k' is the $i+1$ in cell $(\tilde{r},c+1)$. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing the $i+1$ from (r,c), replacing k' with an $i+1$ in $(\tilde{r},c+1)$, and appending k' to $\mathsf{A}_{f_i(T)}(\tilde{r},c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. Finally, we make the assumption that $a = i$ and column $c+1$ contains both an i and an $i+1$ but in different cells. We once again have that $k = i$, but now we have that $(\tilde{r}+1,c+1)$ contains an $i+1$. We have that the rightmost unpaired i in $\mathcal{V}_b(T)$ is the i that was appended to $\mathsf{A}_T(\tilde{r},c+1)$. Since $(\tilde{r}+1,c+1)$ contains an $i+1$, we have that f_i acts on $\mathcal{V}_b(T)$ by removing the i from $\mathsf{A}_{\mathcal{V}_b(T)}(\tilde{r},c+1)$ and appending an $i+1$ to $\mathsf{A}_{\mathcal{V}_b(T)}(\tilde{r}+1,c+1)$. We now consider $\mathcal{V}_b(f_i(T))$. We see that a' is the $i+1$ in (r,c) obtained after applying f_i and k' the $i+1$ in cell $(\tilde{r}+1,c+1)$. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing the $i+1$ from (r,c), replacing k' with an $i+1$ in

$(\tilde{r}+1, c+1)$, and appending k' to $\mathsf{A}_{f_i(T)}(\tilde{r}+1, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Assume that the rightmost unpaired i is in the arm of (r, c), $\tilde{r} = r$, and $(a, \ell] \cap \mathsf{L}_T(r, c) \neq \emptyset$ for this entire paragraph. First, we make the additional assumption that $i < a$. This gives us that $\mathcal{V}_b(T)$ is attained from T by removing $(a, \ell] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_T(r, c)$ and a from $\mathsf{A}_T(r, c)$, replacing k in cell $(r, c+1)$ with $((a, \ell] \cap \mathsf{L}_T(r, c)) \cup \{a\}$, and appending k to $\mathsf{A}_T(r, c+1)$. Since $k, a > i$, we have that the rightmost unpaired i in $\mathcal{V}_b(T)$ remains the same as in T. We also have that the cell $(r+1, c)$ does not contain an $i+1$ and the cell $(r, c+1)$ does not contain an i. Thus, f_i acts on $\mathcal{V}_b(T)$ by changing the rightmost i in $\mathsf{A}_{\mathcal{V}_b(T)}(r, c)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We have that f_i acts on T by changing the rightmost i in $\mathsf{A}_T(r, c)$ to an $i+1$. We see that a', k', and $(a', l'] \cap \mathsf{L}_{f_i(T)}(r', c')$ are the same as a, k, and $(a, \ell] \cap \mathsf{L}_T(r, c)$ respectively. This implies that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $(a, \ell] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_{f_i(T)}(r, c)$ and a from $\mathsf{A}_{f_i(T)}(r, c)$, replacing k in cell $(r, c+1)$ with $((a, l] \cap \mathsf{L}_T(r, c)) \cup \{a\}$, and appending k to $\mathsf{A}_{f_i(T)}(r, c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. Next, we assume that $a = i$ and (r, c) contains an $i+1$. Since $a = i$, the $i+1$ in (r, c) must be in its leg. Also as a is the rightmost unpaired i of T, we must have that (r, c) contains another i besides a. This gives us that $\mathcal{V}_b(T)$ is attained from T by removing $(a, \ell] \cap \mathsf{L}_T(r, c)$ from $\mathsf{L}_T(r, c)$ and a from $\mathsf{A}_T(r, c)$, replacing k in cell $(r, c+1)$ with $((a, \ell] \cap \mathsf{L}_T(r, c)) \cup \{a\}$, and appending k to $\mathsf{A}_T(r, c+1)$. Note that the i inserted into $(r, c+1)$ becomes i-paired while an i in (r, c) becomes unpaired. This implies that the rightmost unpaired i in $\mathcal{V}_b(T)$ still sits in the cell (r, c). We see that the cell $(r+1, c)$ still does not contain an $i+1$; however, the cell $(r, c+1)$ now contains an i. This implies that f_i acts on $\mathcal{V}_b(T)$ by removing the i from the cell $(r, c+1)$ and appending an $i+1$ to $\mathsf{L}_{\mathcal{V}_b(T)}(r, c)$. We now consider $\mathcal{V}_b(f_i(T))$. We have that f_i acts on T by changing a into an $i+1$. We have that a' is the $i+1$ obtained from applying f_i and k' is same as k. We see that $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ is the same as $(a, \ell] \cap \mathsf{L}_T(r, c)$ excluding the $i+1$. We have that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ from $\mathsf{L}_{f_i(T)}(r, c)$ and $i+1$ from $\mathsf{A}_{f_i(T)}(r, c)$, leaving the $i+1$ in $\mathsf{L}_{f_i(T)}(r, c)$, replacing k in $(r, c+1)$

with $((a',\ell']\cap \mathsf{L}_{f_i(T)}(r',c'))\cup\{a'\}$, and appending k to $\mathsf{A}_{f_i(T)}(r,c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. Finally, we assume that $a = i$ and $i+1$ is not in the cell (r,c). This gives us that $\mathcal{V}_b(T)$ is attained from T by removing $(a,\ell]\cap \mathsf{L}_T(r,c)$ from $\mathsf{L}_T(r,c)$ and a from $\mathsf{A}_T(r,c)$, replacing k in cell $(r,c+1)$ with $((a,\ell]\cap \mathsf{L}_T(r,c))\cup\{a\}$, and appending k to $\mathsf{A}_T(r,c+1)$. Since $k \geqslant j > i+1$ for all $j \in (a,\ell]\cap \mathsf{L}_T(r,c)$, we have that the i inserted into the cell $(r,c+1)$ is the rightmost unpaired i in $\mathcal{V}_b(T)$. Note that the cell $(r+1,c+1)$ does not contain an $i+1$ and the cell $(r,c+2)$ does not contain an i. Thus, f_i acts on $\mathcal{V}_b(T)$ by changing the i in $(r,c+1)$ to an $i+1$. We now consider $\mathcal{V}_b(f_i(T))$. We have that f_i acts on T by changing a into an $i+1$. We have that a' is the $i+1$ obtained from applying f_i and k' is the same as k. We see that $(a',\ell']\cap \mathsf{L}_{f_i(T)}(r',c') = (a,\ell]\cap \mathsf{L}_T(r,c)$. We have that $\mathcal{V}_b$ acts on $f_i(T)$ by removing $(a,\ell]\cap \mathsf{L}_T(r,c)$ from $\mathsf{L}_{f_i(T)}(r,c)$ and $i+1$ from $\mathsf{A}_{f_i(T)}(r,c)$, replacing k in $(r,c+1)$ with $((a,\ell]\cap \mathsf{L}_T(r,c))\cup\{a'\}$, and appending k to $\mathsf{A}_{f_i(T)}(r,c+1)$. We see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Case 2: Assume that $\hat{r} < r$ and $\hat{c} = c$.

Note that $a > i$. By Lemma 127 we have that $R_i(T) = R_i(\mathcal{V}_b(T))$ unless $a = i+1$ and column $c+1$ contains both an i and an $i+1$. However, even in this special case, we see that the rightmost unpaired i of $\mathcal{V}_b(T)$ is in the same position as the rightmost unpaired i of T. We also see that $\mathcal{V}_b(T)$ does not change whether or not cell $(\hat{r}+1,c)$ contains an $i+1$ and whether or not cell $(\hat{r},c+1)$ contains an i. Thus, f_i acts on the same i and in the same way for both T and $\mathcal{V}_b(T)$. Since $a > i$, we have that k' is the same as k. Note that the only way for f_i to affect the cell (r,c) in T is if $\hat{r} = r-1$ and (r,c) contains an $i+1$. However, even in this special case, we see that (r',c'), a', l', and $(a',\ell']\cap \mathsf{L}_{f_i(T)}(r',c')$ are the same as (r,c), a, ℓ, and $(a,\ell]\cap \mathsf{L}_T(r,c)$. Thus, $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same way. Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Case 3: Assume that $\hat{c} < c$.

Let $\tilde{i}$ denote the rightmost unpaired i of T. From the proof of Lemma 127, we have that $\mathcal{V}_b$ does not change whether or not the i's to the right of $\tilde{i}$ in $R_i(T)$ are i-paired. Thus, the rightmost unpaired i in $R_i(T)$ and $R_i(\mathcal{V}_b(T))$ are in the same position. As $\mathcal{V}_b$ does not affect any column to the left of column c, we have that the rightmost unpaired i for $\mathcal{V}_b(T)$ is in the same position as the rightmost unpaired i for T. Note that $\mathcal{V}_b$ also does not affect whether or not cell $(\hat{r}+1, \hat{c})$ contains an $i+1$ and whether or not cell $(\hat{r}, \hat{c}+1)$ contains an i. Thus, f_i acts on the rightmost unpaired i in T and $\mathcal{V}_b(T)$ in exactly the same way. Next, we note that (r', c'), a', k', and $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ are the same as (r, c), a, k, and $(a, \ell] \cap \mathsf{L}_T(r, c)$ respectively. Thus, $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same way. Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Case 4: Assume that $\hat{r} \leqslant r$ and $\hat{c} = c+1$.

Under this assumption, we have that column $c+1$ does not contain an $i+1$ and $a \neq i+1$ since the cells in column $c+1$ do not contain any arms. We break into subcases.

- Assume that $k \neq i$. This implies that the rightmost unpaired i in $\mathcal{V}_b(T)$ is in the same position as the rightmost unpaired i in T. We see that $\mathcal{V}_b$ does not change whether or not cell $(\hat{r}+1, c+1)$ contains an $i+1$ and whether or not cell $(\hat{r}, c+2)$ contains an i. Thus, f_i acts on the rightmost unpaired i in T and $\mathcal{V}_b(T)$ in exactly the same way. We also observe that (r', c'), a', ℓ', k', and $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ are the same as a, ℓ, k, and $(a, \ell] \cap \mathsf{L}_{f_i(T)}(r, c)$ respectively. Thus, $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same way. Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.
- Assume that $k = i$. We see that the rightmost unpaired i in $\mathcal{V}_b(T)$ is the i that was appended to $\mathsf{A}_T(\hat{r}, c+1)$. Note that $\mathcal{V}_b$ does not change whether or not cell $(\hat{r}+1, c+1)$ contains an $i+1$ and whether or not cell $(\hat{r}, c+2)$ contains an i. We first make the extra assumption that $(\hat{r}, c+2)$ in T contains an i. This implies that f_i acts on $\mathcal{V}_b(T)$ and T in the same way by removing the i from the hook entry of $(\hat{r}, c+2)$ and appending an $i+1$ to the leg of $(\hat{r}, c+1)$. We also have that (r', c'), a', ℓ', k', and $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ are equal to (r, c), a, ℓ, k, and $(a, \ell] \cap \mathsf{L}_{f_i(T)}(r, c)$ respectively. Thus, $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same way. Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$. We now assume that $(\hat{r}, c+2)$ does not contain an i. This implies that f_i acts on $\mathcal{V}_b(T)$ by

changing the i in $\mathsf{A}_{\mathcal{V}_b(T)}(\hat{r}, c+1)$ to an $i+1$ and acts on T similarly by changing the i in $\mathsf{L}_{\mathcal{V}_b(T)}(\hat{r}, c+1)$ to an $i+1$. Note that (r', c'), a', ℓ', and $(a', \ell'] \cap \mathsf{L}_{f_i(T)}(r', c')$ are equal to (r, c), a, ℓ, and $(a, \ell] \cap \mathsf{L}_{f_i(T)}(r, c)$ respectively while k' is the $i+1$ in $\mathsf{L}_{f_i(T)}(\hat{r}, c+1)$. Thus, besides the value of the number that is bumped from the leg of $(\hat{r}, c+1)$ to its arm, we have $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same way. Looking at $f_i(\mathcal{V}_b(T))$ and $\mathcal{V}_b(f_i(T))$, we see that $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Case 5: Assume that $\hat{r} > r$ and $\hat{c} = c$ or $c+1$.

Under this assumption, we have that $\mathcal{V}_b$ does not change the cells $(\hat{r}, \hat{c})$, $(\hat{r}+1, \hat{c})$, and $(\hat{r}, \hat{c}+1)$. We also have that $R_i(T) = R_i(\mathcal{V}_b(T))$ implying that the rightmost unpaired i in $\mathcal{V}_b(T)$ is in the same position as the rightmost unpaired i in T. Thus, f_i acts on the rightmost unpaired i in $\mathcal{V}_b(T)$ and T in the same way. Note that $i+1$ cannot be in column $\hat{c}$ implying that f_i can only make changes to the legs and hook entries of $(\hat{r}, \hat{c})$ and $(\hat{r}, \hat{c}+1)$. Since these changes only affect the legs and hook entries of cells outside of the possible cells that $\mathcal{V}_b$ can change, we have that $\mathcal{V}_b$ acts on T and $f_i(T)$ in the same way. Therefore, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

Case 6: Assume that $\hat{c} \geqslant c+2$.

Let $\tilde{i}$ denote the rightmost unpaired i of T. From the proof of Lemma 127, we have that $\mathcal{V}_b$ does not change whether or not the $i+1$'s to the left of $\tilde{i}$ are i-paired. Thus, the rightmost unpaired i in $R_i(T)$ and $R_i(\mathcal{V}_b(T))$ are in the same position. As $\mathcal{V}_b$ does not affect any column to the right of column $c+1$, we have that the rightmost unpaired i for $\mathcal{V}_b(T)$ is in the same position as the rightmost unpaired i for T. Note that $\mathcal{V}_b$ also does not affect whether or not cell $(\hat{r}+1, \hat{c})$ contains an $i+1$ and whether or not cell $(\hat{r}, \hat{c}+1)$ contains an i. Since the cells that f_i and $\mathcal{V}_b$ could change are different and the rightmost unpaired i does not change, we have that the actions of f_i and $\mathcal{V}_b$ on T are independent and $f_i(\mathcal{V}_b(T)) = \mathcal{V}_b(f_i(T))$.

□

THEOREM 131. *Let $T \in \mathsf{HVT}$.*

(1) *If $f_i(T) \neq 0$, we have $f_i(P(T)) = P(f_i(T))$ and $Q(T) = Q(f_i(T))$.*
(2) *If $e_i(T) \neq 0$, we have $e_i(P(T)) = P(e_i(T))$ and $Q(T) = Q(e_i(T))$.*

PROOF. Part (2) follows from part (1) since e_i and f_i are partial inverse. We prove part (1) here.

Let $T \in \mathsf{HVT}$ with arm excess α such that $f_i(T) \neq 0$ for some i. Then $f_i(P(T)) = P(f_i(T))$ follows from Lemma 130, as $P(T)$ is obtained by successive applications of $\mathcal{V}$ on T and each application of $\mathcal{V}$ is a string of applications of $\mathcal{V}_b$.

Since crystal operators do not change arm excess, we may employ the notation in Definition 122 and denote the pair of insertion and recording tableaux produced at the j-th step for $0 \leqslant j \leqslant \alpha$ of the uncrowding map $\mathcal{U}$ for T and $f_i(T)$ as $(P_j(T), Q_j(T))$ and $(P_j(f_i(T)), Q_j(f_i(T)))$, respectively. As crystal operators do not change the shape of T, we have $\mathsf{shape}(P_j(f_iT)) = \mathsf{shape}(f_i(P_j(T))) = \mathsf{shape}(P_j(T))$ for all $0 \leqslant j \leqslant \alpha$. Hence

(4.2.2)
$$\mathsf{shape}(P_{j+1}(T))/\mathsf{shape}(P_j(T)) = \mathsf{shape}(P_{j+1}(f_i(T)))/\mathsf{shape}(P_j(f_i(T))) \qquad \text{for all } 0 \leqslant j \leqslant \alpha - 1.$$

Next we show $Q_j(T) = Q_j(f_i(T))$ for all $0 \leqslant j \leqslant \alpha$ by induction. When $j = 0$, $Q_0(T) = Q_0(f_i(T))$ since $\mathsf{shape}(P_0(T)) = \mathsf{shape}(P_0(f_i(T))) = \mathsf{shape}(T)$.

Suppose $Q_j(T) = Q_j(f_i(T))$ for a given $j \geqslant 0$. It suffices to show that the cells

$$\mathsf{shape}(Q_{j+1}(T))/\mathsf{shape}(Q_j(T)) = \mathsf{shape}(P_{j+1}(T))/\mathsf{shape}(P_j(T)) \qquad \text{and}$$
$$\mathsf{shape}(Q_{j+1}(f_i(T)))/\mathsf{shape}(Q_j(f_i(T))) = \mathsf{shape}(P_{j+1}(f_i(T)))/\mathsf{shape}(P_j(f_i(T)))$$

in $Q_{j+1}(T)$ and $Q_{j+1}(f_i(T))$ are at the same position with the same entry. By (4.2.2), the cells are in the same position, say in column $\tilde{c}$. By Definition 116, f_i does not move elements in the arm to a different column, so the columns in which we start the uncrowding insertion $\mathcal{V}$ on $P_j(T)$ and $P_j(f_i(T))$ are the same, say c, by Definition 122. Hence the cells $\mathsf{shape}(Q_{j+1}(T))/\mathsf{shape}(Q_j(T))$ and $\mathsf{shape}(Q_{j+1}(f_i(T)))/\mathsf{shape}(Q_j(f_i(T)))$ are at the same position with entry $\tilde{c} - c$. The theorem follows. $\square$

Hawkes and Scrimshaw [**HS20**, Theorem 4.6] proved that $\mathsf{HVT}^m(\lambda)$ is a Stembridge crystal by checking the Stembridge axioms. This also follows directly from our analysis above.

COROLLARY 132. *The crystal* $\mathsf{HVT}^m(\lambda)$ *of Definition 116 is a Stembridge crystal of type* A_{m-1}*.*

Proof. According to [**MPS20**], $\mathsf{SVT}^m(\mu)$ is a Stembridge crystal of type A_{m-1}. By Theorem 131, the map

$$\mathcal{U}\colon \mathsf{HVT}^m(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} \mathsf{SVT}^m(\mu) \times \hat{\mathcal{F}}(\mu/\lambda),$$

is a strict crystal morphism (see for example [**BS17**, Chapter 2]). The statement follows. $\square$

4.2.4. Uncrowding map on multiset-valued tableaux. The uncrowding map on hook-valued tableaux described above turns out to be a generalization of the uncrowding map on multiset-valued tableaux by Hawkes and Scrimshaw [**HS20**, Section 3.2]. We will prove that this is indeed the case in this section. Let us recall the definition of the uncrowding map in [**HS20**, Section 3.2].

Definition 133. *Let $T \in \mathsf{MVT}(\lambda)$. The uncrowding map*

$$\Upsilon : \mathsf{MVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} \mathsf{SSYT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$$

sends T to a pair of tableaux using the following algorithm:

(1) *Set $U_{\lambda_1+1} = \emptyset$ and F_{λ_1+1} be the unique column-flagged increasing tableau of shape $\emptyset/\emptyset$.*
(2) *Let $1 \leqslant k \leqslant \lambda_1$ and assume that the pair (U_{k+1}, F_{k+1}) is defined. The pair (U_k, F_k) is defined recursively from (U_{k+1}, F_{k+1}) using the following two steps:*
   (a) *Define U_k as the RSK row insertion tableau from the word*

   $$R(C_k)R(C_{k+1}) \cdots R(C_{\lambda_1}),$$

   where C_j is the j-th column of T for every $1 \leqslant j \leqslant \lambda_1$. In other words, if we denote by $T_{\geqslant k}$ the tableau formed by the columns weakly to the right of the k-th column of T, U_k is obtained by performing RSK row insertion using the column reading word of $T_{\geqslant k}$.
   (b) *Form the tableau F_k of shape $\mathsf{shape}(U_k)/\mathsf{shape}(T_{\geqslant k})$ as follows. Shift F_{k+1} by one column to the right and fill the boxes in the same positions into F_k; for every unfilled box in the shape $\mathsf{shape}(U_k)/\mathsf{shape}(U_{k+1})$, label each box in column i with entry $i-1$.*

Define $\Upsilon(T) = (U, F) := (U_1, F_1)$.

Example 134. *Let T be the multiset-valued tableau*

$$T = \begin{ytableau} 45 \\ 233 & 345 \\ 1 & 11 & 4 \end{ytableau}.$$

Then, we obtain the following pairs of tableaux for the uncrowding map Υ:

$$(U_4, F_4) = (\emptyset, \emptyset)$$

$$(U_3, F_3) = \left(\begin{ytableau} 4 \end{ytableau}, \begin{ytableau} *(gray) \end{ytableau} \right)$$

$$(U_2, F_2) = \left(\begin{ytableau} 3 & 5 \\ 1 & 1 & 4 & 4 \end{ytableau}, \begin{ytableau} *(gray) & 1 \\ *(gray) & *(gray) & 2 & 3 \end{ytableau} \right)$$

$$(U_1, F_1) = \left(\begin{ytableau} 4 & 5 \\ 2 & 3 & 3 & 5 \\ 1 & 1 & 1 & 3 & 4 & 4 \end{ytableau}, \begin{ytableau} *(gray) & 1 \\ *(gray) & *(gray) & 1 & 3 \\ *(gray) & *(gray) & *(gray) & 2 & 3 & 5 \end{ytableau} \right) = (U, F) = \Upsilon(T).$$

Proposition 135. *Let $T \in \mathsf{MVT}(\lambda)$. Then $\mathcal{U}(T) = \Upsilon(T)$. In other words, the uncrowding map as defined in Definition 122 is equivalent to the uncrowding map of Definition 133 in [**HS20**, Section 3.2].*

Proof. Recall from Definition 122, that the pair of uncrowding and recording tableaux for $\mathcal{U}(T)$ is denoted by $(P(T), Q(T)) = \mathcal{U}(T)$. Similarly, let us denote $(U(T), F(T)) := \Upsilon(T)$.

Assume that $S \in \mathsf{MVT}(\lambda)$ is highest weight, that is, $e_i(S) = 0$ for $i \geqslant 1$. By [**HS20**, Proposition 3.10], row i of S only contains the letter i. Thus its weight is some partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. By Proposition 129 and Theorem 131, $P(S) \in \mathsf{SSYT}$ is highest weight. As weights of tableaux are preserved under uncrowding, the weight of $P(S)$ is equal to μ. By a similar argument using [**HS20**, Theorem 3.17], $U(S) \in \mathsf{SSYT}$ is also highest weight with weight μ. Since highest weight semistandard Young tableaux are uniquely determined by their weights, we have $P(S) = U(S)$.

Recall that as long as $f_i T \neq 0$ for $T \in \mathsf{MVT}(\lambda)$, we have $U(f_i T) = f_i U(T)$ by [**HS20**, Theorem 3.17] and $P(f_i T) = f_i P(T)$ by Theorem 131. Now let $T \in \mathsf{MVT}(\lambda)$ be arbitrary. Then $T =$

$f_{i_1} \cdots f_{i_k}(S)$ for some sequence of $i_1, \ldots, i_k$ and S highest weight. Hence,

$$P(T) = P(f_{i_1} \cdots f_{i_k} S) = f_{i_1} \cdots f_{i_k} P(S) = f_{i_1} \cdots f_{i_k} U(S) = U(f_{i_1} \cdots f_{i_k} S) = U(T).$$

It remains to show that $Q(T) = F(T)$ for all $T \in \mathsf{MVT}(\lambda)$. To do this, we show that the newly created boxes of the uncrowding map up to a specified column in Definition 133 are in the same positions as those for the uncrowding insertion in Definition 122. For every $Y \in \mathsf{MVT}(\mu)$ and for every $1 \leqslant j \leqslant \mu_1$, denote by $Y_{\geqslant j}$ the tableau formed by the rightmost j columns of Y; here $Y_{\geqslant \mu_1+1}$ is the empty tableau.

Let $T \in \mathsf{MVT}(\lambda)$ be arbitrary. For $1 \leqslant k \leqslant \lambda_1+1$, let $P^{(k)}$ be the tableau obtained by performing the uncrowding map $\mathcal{U}$ on T on the columns from right to left up to and including the k-th column of T; here $P^{(\lambda_1+1)} = T$. In other words, $P^{(k)} = \mathcal{V}^{\alpha_k}(T)$ as in Definition 121, where α_k is the arm excess of $T_{\geqslant k}$. As the entries to the left of column k of T are untouched by the uncrowding insertion in Definition 121, for every $1 \leqslant k \leqslant \lambda_1+1$, we have $(P^{(k)})_{\geqslant k} = P(T_{\geqslant k}) = U(T_{\geqslant k})$. It follows that for every $1 \leqslant k \leqslant \lambda_1$, up to horizontal shifts, the newly formed boxes in $\mathsf{shape}(P^{(k)})/\mathsf{shape}(P^{(k+1)}) = \mathsf{shape}[(P^{(k)})_{\geqslant k+1}]/\mathsf{shape}[(P^{(k+1)})_{\geqslant k+1}]$ and $\mathsf{shape}([U(T_{\geqslant k})]_{\geqslant k+1})/\mathsf{shape}([U(T_{\geqslant k+1})]_{\geqslant k+1})$ are in the same positions. Since the entries in these boxes both record the difference in column indices relative to the k-th column for each $1 \leqslant k \leqslant \lambda_1$ and since the recording tableaux for both maps are formed from the union of these boxes, we conclude that $Q(T) = F(T)$, completing the proof. $\square$

4.2.5. Crowding map. In this section, we give a description of the "inverse" of the uncrowding map.

We begin by introducing some notation. Let $F \in \hat{\mathcal{F}}$ with e entries. For each cell (r, c) in F with entry $F(r, c)$, define the corresponding *destination column* to be $d(r, c) = c - F(r, c)$. Define the *crowding order* on F by ordering all the cells in F with a filling, first determined by their destination column (smallest to largest) and then by column index (largest to smallest). Denote the order by $(r_1, c_1), (r_2, c_2), \ldots, (r_e, c_e)$. Set $\alpha(F) = (\alpha_1, \alpha_2, \ldots, \alpha_e)$, where $\alpha_i = F(r_i, c_i)$. Let the arm excess for a column of a hook-valued tableau be the sum of arm excesses of all its cells.

Definition 136. *Let $h \in \mathsf{HVT}$ and let (r, c) be a cell in h with $c > 1$ and with at most one element in $\mathsf{A}_h(r, c)$. If $\mathsf{A}_h(r, c)$ is empty, we also require that the cell (r, c) is a corner cell in h. Then we define the* crowding bumping *$\mathcal{C}_b$ on the pair $[h, (r, c)]$ by the following algorithm:*

(1) *If $\mathsf{A}_h(r, c)$ is nonempty, set m to be the only element in $\mathsf{A}_h(r, c)$ and $b = \max\{x \in \mathsf{L}_h^+(r, c) \mid x \leqslant m\}$. Otherwise, set $m = \mathsf{H}_h(r, c)$ and $b = \max(\mathsf{L}_h^+(r, c))$.*

(2) *Find the largest r' such that $\mathsf{H}_h(r', c-1) \leqslant b$. If $r' = r$, set $q = \mathsf{H}_h(r, c)$. Otherwise, set $q = b$. In either case, append q to $\mathsf{A}_h(r', c-1)$.*

(3) (a) *If r' from Step 2 equals r, perform either of the following:*
   - (i) *If $\mathsf{A}_h(r, c)$ is nonempty, move the set $\{x \in \mathsf{L}_h(r, c) \mid q < x \leqslant m\}$ from $\mathsf{L}_h(r, c)$ to $\mathsf{L}_h(r', c-1)$ and keep it strictly increasing. Remove m from $\mathsf{A}_h(r, c)$ and set $H_h(r, c) = m$.*
   - (ii) *Otherwise, $\mathsf{A}_h(r, c)$ is empty, so move $\mathsf{L}_h(r, c)$ into $\mathsf{L}_h(r', c-1)$ and keep it to be strictly increasing. Remove cell (r, c) from h.*

   (b) *Otherwise, $r' \neq r$ and perform either of the following:*
   - (i) *Suppose that $\mathsf{A}_h(r, c)$ is nonempty. Replace q in $\mathsf{L}_h^+(r, c)$ with m. Remove m from $\mathsf{A}_h(r, c)$.*
   - (ii) *If instead $\mathsf{A}_h(r, c)$ is empty, then remove cell (r, c) from h.*

Denote the resulting (not necessarily semistandard) hook-valued tableau by h'. We write $\mathcal{C}_b([h, (r, c)]) = [h', (r', c-1)]$. We also define the projections *p_1 and p_2 by $p_1 \circ \mathcal{C}_b([h, (r, c)]) = h'$ and $p_2 \circ \mathcal{C}_b([h, (r, c)]) = (r', c-1)$. See Figures 4.3 and 4.4 for illustration.*

Figure 4.3. When $r' = r$. Left: (i) $\mathsf{A}_h(r, c) \neq \emptyset$. Right: (ii) $\mathsf{A}_h(r, c) = \emptyset$.

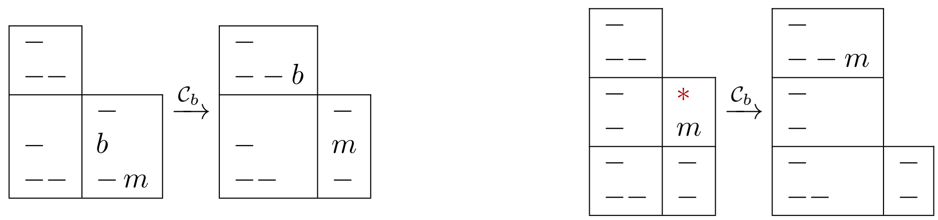


FIGURE 4.4. When $r' \neq r$. Left: $\mathsf{A}_h(r,c) \neq \emptyset$. Right: $\mathsf{A}_h(r,c) = \emptyset$.

EXAMPLE 137. *We compute $\mathcal{C}_b$ in two examples:*

$$T = \begin{array}{|c|c|} \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline \begin{array}{c} \\ \\ \\ 1\,1 \end{array} & \begin{array}{c} 5 \\ 4 \\ 3 \\ 2\,4 \end{array} \\ \hline \end{array}\,, \quad \mathcal{C}_b([T,(1,2)]) = \left[\, \begin{array}{|c|c|} \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline \begin{array}{c} 4 \\ 3 \\ 1\,1\,2 \end{array} & \begin{array}{c} \\ 5 \\ 4 \end{array} \\ \hline \end{array}\,, (1,1) \,\right] = [\, T' \,, (1,1) \,].$$

$$S = \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline \begin{array}{c} 2 \\ 1 \end{array} & \begin{array}{c} 3 \\ 2 \end{array} \\ \hline \end{array}\,, \quad \mathcal{C}_b([S,(1,2)]) = \left[\, \begin{array}{|c|} \hline 33 \\ \hline \begin{array}{c} 2 \\ 1 \end{array} \\ \hline \end{array}\,, (2,1) \,\right] = [\, S' \,, (2,1) \,].$$

REMARK 138. *In Definition 136,*

- *if $r' = r$, then h' is always semistandard and has the same weight as h;*
- *if $r' \neq r$ and $\mathsf{A}_h(r,c)$ is empty, then h' might have fewer letters than h. In Example 137, S contains 5 letters while S' only contains 4. This happens precisely when $\mathsf{L}_h(r,c)$ is nonempty.*

In principle, the arm in cell $(r', c-1)$ could be greater than the q that is to be inserted. However, we only consider the cases as defined in the order described by the next paragraph. We refer to Proposition 144 which states that all tableaux we deal with in this section are indeed semistandard hook-valued tableaux.

Let $(S,F) \in \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$ with crowding order $(r_1,c_1),(r_2,c_2),\ldots,(r_e,c_e)$ and $\alpha(F) = (\alpha_1,\alpha_2,\ldots,\alpha_e)$. For all $0 \leqslant j \leqslant e-1$ and for all $0 \leqslant s \leqslant \alpha_{j+1}$, define $T_j^{(s)}$ recursively by setting

$T_0^{(0)} := S$ and

$$T_j^{(s)} := \begin{cases} p_1 \circ \mathcal{C}_b([T_j^{(s-1)}, (r_{j+1}, c_{j+1})]) & \text{when } s > 0, \\ T_{j-1}^{(\alpha_j)} & \text{when } s = 0 \text{ and } j > 0. \end{cases}$$

Additionally, define $T_e^{(0)} := T_{e-1}^{(\alpha_e)}$.

Thus we obtain the following sequence

$$S = T_0^{(0)} \xrightarrow[(r_1,c_1)]{p_1 \circ \mathcal{C}_b^{\alpha_1}} T_1^{(0)} \xrightarrow[(r_2,c_2)]{p_1 \circ \mathcal{C}_b^{\alpha_2}} T_2^{(0)} \xrightarrow[(r_3,c_3)]{p_1 \circ \mathcal{C}_b^{\alpha_3}} \cdots \xrightarrow[(r_e,c_e)]{p_1 \circ \mathcal{C}_b^{\alpha_e}} T_e^{(0)}.$$

Remark 139. *The tableaux $T_j^{(s)}$ are well-defined. We check the conditions in Definition 136. Let $h = T_j^{(s)}$ for some $0 \leqslant j \leqslant e-1$ and for some $0 \leqslant s < \alpha_{j+1}$, with cell (r, c).*

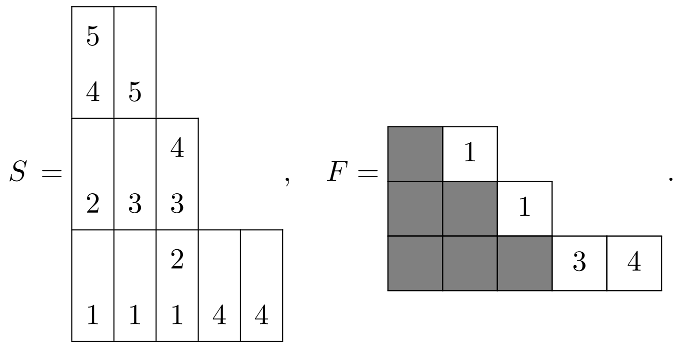
- *Since $F \in \hat{\mathcal{F}}$, we always have $c > 1$.*
- *The case that $\mathsf{A}_h(r, c)$ is empty can only occur in $T_{j-1}^{(0)}$ for some $j > 0$. In this case, $(r, c) = (r_j, c_j)$, which is a corner cell.*
- *Consider the α_j steps in $T_{j-1}^{(0)} \xrightarrow[(r_j,c_j)]{p_1 \circ \mathcal{C}_b^{\alpha_j}} T_j^{(0)}$. We first delete cell (r_j, c_j), which has no arm. Then at every step after that, we move leftward one column at a time. Before we reach column $d(r_j, c_j)$, there is exactly one column with arm excess being 1 and the rest has zero arm excess among columns to the right of $d(r_j, c_j)$ since recall that the cells (r_j, c_j) are ordered from smallest to largest destination column. Once we reach column $d(r_j, c_j)$, the cell there may contain more than one arm element, but we then go to (r_{j+1}, c_{j+1}), which is a corner cell instead. Thus there is at most one element in $\mathsf{A}_h(r, c)$.*

Definition 140. *With the same notation as above, define the* insertion path *of $T_{j-1}^{(0)} \to T_j^{(0)}$ for $1 \leqslant j \leqslant e$ to be*

$$\mathsf{path}_j := \left((r_j^{(0)}, c_j^{(0)}), (r_j^{(1)}, c_j^{(1)}), \ldots, (r_j^{(\alpha_j)}, c_j^{(\alpha_j)}) \right),$$

where $(r_j^{(s)}, c_j^{(s)}) := p_2 \circ \mathcal{C}_b^s([T_{j-1}^{(0)}, (r_j, c_j)])$ for $0 \leqslant s \leqslant \alpha_j$.

EXAMPLE 141. *Consider the following pair of tableaux* $(S, F) \in \mathsf{HVT}((5,3,2)) \times \hat{\mathcal{F}}((5,3,2)/((3,2,1)))$,



The crowding order is $(1,5), (1,4), (3,2), (2,3)$. *The insertion path and destination column for each of them are:*

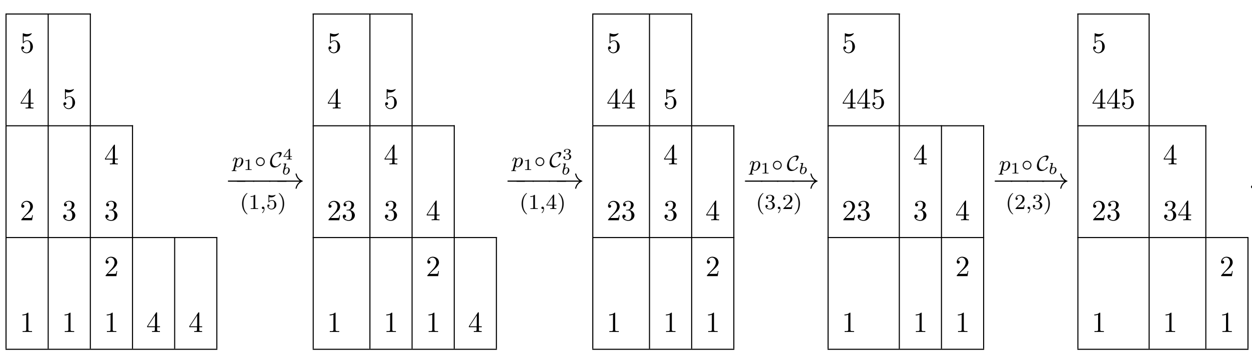
$$\mathsf{path}_1 = ((1,5),(1,4),(2,3),(2,2),(2,1)),\ d(1,5) = 1,$$

$$\mathsf{path}_2 = ((1,4),(2,3),(2,2),(3,1)),\ d(1,4) = 1,$$

$$\mathsf{path}_3 = ((3,2),(3,1)),\ d(3,2) = 1,$$

$$\mathsf{path}_4 = ((2,3),(2,2)),\ d(2,3) = 2.$$

We obtain the sequence from the algorithm:



LEMMA 142. *If* $d(r_j, c_j) = d(r_{j+1}, c_{j+1})$, *then* path_{j+1} *is weakly above* path_j.

PROOF. By the definition of crowding order, $d(r_j, c_j) = d(r_{j+1}, c_{j+1})$ implies $c_j > c_{j+1}$. Set $z_j := c_j - c_{j+1}$. Then we have $c_j^{(s+z_j)} = c_j - z_j - s = c_{j+1} - s = c_{j+1}^{(s)}$ for $0 \leqslant s \leqslant \alpha_{j+1}$. We need to show that $r_{j+1}^{(s)} \geqslant r_j^{(s+z_j)}$ for $0 \leqslant s \leqslant \alpha_{j+1}$. Computing $T_{j-1}^{(s)}$ from $T_{j-1}^{(s-1)}$ for $1 \leqslant s \leqslant \alpha_j$, we denote b and q in Step 1 and Step 2 of Definition 136 by $b_j^{(s)}$ and $q_j^{(s)}$.

Since (r_{j+1}, c_{j+1}) is a corner cell in $T_{j-1}^{(z_j)}$, we have $r_{j+1}^{(0)} \geqslant r_j^{(z_j)}$. We prove that, for $1 \leqslant s \leqslant \alpha_{j+1}$, we have that $q_{j+1}^{(s)} \geqslant q_j^{(s+z_j)}$, which implies $b_{j+1}^{(s)} \geqslant b_j^{(s+z_j)}$ and thus $r_{j+1}^{(s)} \geqslant r_j^{(s+z_j)}$.

We prove $q_{j+1}^{(s)} \geqslant q_j^{(s+z_j)}$ by induction on s. First we check the case $k = 1$. If $r_{j+1}^{(0)} > r_j^{(z_j)}$, then it is obvious that $q_{j+1}^{(1)} > q_j^{(z_j+1)}$. Otherwise if $r_{j+1}^{(0)} = r_j^{(z_j)}$, we consider the following cases. $q_j^{(z_j)}$ is the only element in $\mathsf{A}_{T_{j-1}^{(z_j)}}(r_{j+1}, c_{j+1})$. Let $x = \mathsf{H}_{T_{j-1}^{(z_j)}}(r_{j+1}, c_{j+1})$, $y = \max(\mathsf{L}_{T_{j-1}^{(z_j)}}(r_{j+1}, c_{j+1}))$ and $y' = \max\{z \in \mathsf{L}^+_{T_{j-1}^{(z_j)}}(r_{j+1}, c_{j+1}) \mid z \leqslant q_j^{(z_j)}\}$. See Figure 4.5 for illustration.

Case (1): If $r_j^{(z_j+1)} = r_j^{(z_j)}$, then $q_j^{(z_j+1)} = x$. If $r_{j+1}^{(1)} = r_{j+1}^{(0)}$, then $q_{j+1}^{(1)} = q_j^{(z_j)}$. If $r_{j+1}^{(1)} \neq r_{j+1}^{(0)}$, then $q_{j+1}^{(1)}$ equals y when $y > y'$ and $q_j^{(z_j)}$ when $y = y'$. In both cases $q_{j+1}^{(1)} \geqslant x = q_j^{(z_j+1)}$.

$$\boxed{\begin{array}{ll} y & \\ - & \\ y' & \\ * & \\ x & q_j^{(z_j)} \end{array}} \qquad \boxed{\begin{array}{l} y \\ - \\ q_j^{(z_j)} \end{array}} \qquad \boxed{\begin{array}{l} y \\ - \\ q_j^{(z_j)} \\ * \\ x \end{array}}$$

FIGURE 4.5. Cell $(r_{j+1}^{(0)}, c_{j+1}^{(0)}) = (r_j^{(z_j)}, c_j^{(z_j)})$ in $T_{j-1}^{(z_j)}$ (left); in $T_j^{(0)}$, case(1) (middle), case(2) (right).

Case (2): If $r_j^{(z_j+1)} \neq r_j^{(z_j)}$, then $q_j^{(z_j+1)} = y'$. In this case we have $\mathsf{H}_{T_{j-1}^{(z_j)}}(r_{j+1}+1, c_{j+1}-1) \leqslant y' \leqslant y$. Since $\mathsf{H}_{T_j^{(0)}}(r_{j+1}+1, c_{j+1}-1)$ is smaller or equal to y', we have that $r_{j+1}^{(1)} \neq r_{j+1}^{(0)}$. Therefore $q_{j+1}^{(1)}$ equals y when $y > y'$ and $q_j^{(z_j)}$ when $y = y'$. In this case $q_{j+1}^{(1)} \geqslant y' = q_j^{(z_j+1)}$.

Now we have proved the base case $s = 1$. Next, suppose it holds for some $s \geqslant 1$ that $q_{j+1}^{(s)} \geqslant q_j^{(s+z_j)}$ and $r_{j+1}^{(s)} \geqslant r_j^{(s+z_j)}$. The statement is similar to the argument of the base case. If $r_{j+1}^{(s)} > r_j^{(z_j+s)}$, it is obvious that $q_{j+1}^{(s+1)} > q_j^{(s+1+z_j)}$ and thus $r_{j+1}^{(s+1)} \geqslant r_j^{(s+1+z_j)}$. If $r_{j+1}^{(s)} = r_j^{(z_j+s)}$, we discuss the following cases. $q_j^{(s+z_j)}$ is the only element in $\mathsf{A}_{T_{j-1}^{(s+z_j)}}(r_j^{(s+z_j)}, c_j^{(s+z_j)})$. Let $x = \mathsf{H}_{T_{j-1}^{(s+z_j)}}(r_j^{(s+z_j)}, c_j^{(s+z_j)})$, $y = \max(\mathsf{L}_{T_{j-1}^{(s+z_j)}}(r_j^{(s+z_j)}, c_j^{(s+z_j)}))$ and $y' = \max\{z \in \mathsf{L}^+_{T_{j-1}^{(s+z_j)}}(r_j^{(s+z_j)}, c_j^{(s+z_j)}) \mid z \leqslant q_j^{(s+z_j)}\}$. See Figure 4.6 for illustration.

Case (1): If $r_j^{(s+1+z_j)} = r_j^{(s+z_j)}$, then $q_j^{(s+1+z_j)} = x$. If $r_{j+1}^{(s+1)} = r_{j+1}^{(s)}$, then $q_{j+1}^{(s+1)} = q_j^{(s+z_j)} \geqslant x$. If $r_{j+1}^{(s+1)} \neq r_{j+1}^{(s)}$, then $q_{j+1}^{(s+1)} = \max\{z \in \mathsf{L}^+_{T_j^{(s)}}(r_{j+1}^{(s)}, c_{j+1}^{(s)}) \mid z \leqslant q_{j+1}^{(s)}\} \geqslant q_j^{(s+z_j)} \geqslant x$. So in either case we have $q_{j+1}^{(s+1)} \geqslant q_j^{(s+1+z_j)}$.

$$\boxed{\begin{array}{ll} y & \\ - & \\ y' & \\ * & \\ x & q_j^{(s+z_j)} \end{array}} \qquad \boxed{\begin{array}{ll} y & \\ - & \\ q_j^{(s+z_j)} & q_{j+1}^{(s)} \end{array}} \qquad \boxed{\begin{array}{ll} y & \\ - & \\ q_j^{(s+z_j)} & \\ * & \\ x & q_{j+1}^{(s)} \end{array}}$$

FIGURE 4.6. Cell $(r_{j+1}^{(s)}, c_{j+1}^{(s)}) = (r_j^{(s+z_j)}, c_j^{(s+z_j)})$ in $T_{j-1}^{(s+z_j)}$ (left); in $T_j^{(s)}$, case(1) (middle), case(2) (right).

Case (2): If $r_j^{(s+1+z_j)} \neq r_j^{(s+z_j)}$, then $q_j^{(s+1+z_j)} = y'$. In this case we have $\mathsf{H}_{T_{j-1}^{(s+z_j)}}(r_j^{(s+z_j)} + 1, c_j^{(s+z_j)} - 1) \leqslant y' \leqslant q_j^{(s+z_j)}$. Since $\mathsf{H}_{T_j^{(s)}}(r_{j+1}^{(s)} + 1, c_{j+1}^{(s)} - 1)$ is smaller or equal to $q_j^{(s+z_j)}$, we have that $r_{j+1}^{(s+1)} \neq r_{j+1}^{(s)}$. Therefore $q_{j+1}^{(s+1)} = \max\{z \in \mathsf{L}^+_{T_j^{(s)}}(r_{j+1}^{(s)}, c_{j+1}^{(s)}) \mid z \leqslant q_{j+1}^{(s)}\}$. By induction we have $q_j^{(s+z_j)} \leqslant q_{j+1}^{(s)}$, thus $q_{j+1}^{(s+1)} \geqslant q_j^{(s+z_j)} \geqslant y' = q_j^{(s+1+z_j)}$. This completes the proof. $\square$

LEMMA 143. *With the notations as above, let* $0 \leqslant j \leqslant e-1$, $0 \leqslant s < \alpha_{j+1}$ *and* $\mathcal{C}_b([T_j^{(s)}, (r, c)]) = [T_j^{(s+1)}, (r', c-1)]$ *for some* r, c, r'. *Then in* $T_j^{(s+1)}$, *column* $c-1$ *is the rightmost column with nonzero arm excess and* $(r', c-1)$ *is the topmost cell in column* $c-1$ *with nonzero arm excess.*

PROOF. In any path_j, consider the arm excess of its columns. Those with column index c such that $d(r_j, c_j) < c < c_j$ started with arm excess 0, then changed to arm excess 1 when the insertion path passed through that column, and immediately decreased to 0.

Thus the $q_j^{(s)}$ that is being moved to cell $(r', c-1)$ is always at the rightmost column containing nonzero arm excess. When $c-1 > d(r_j, c_j)$, the arm excess of the column $c-1$ is exactly 1, $(r', c-1)$ is also the topmost cell containing an arm. For $c - 1 = d(r_j, c_j)$, the path path_j has reached its destination. At that point, any column to the right of $d(r_j, c_j)$ has 0 arm excess. It follows from Lemma 142 that the cell $(r_j^{(\alpha_j)}, c_j^{(\alpha_j)})$ is also the topmost cell containing an arm. $\square$

PROPOSITION 144. *The tableau* $T_j^{(s+1)}$ *is a semistandard hook-valued tableau for all* $0 \leqslant j \leqslant e-1$ *and for all* $0 \leqslant s < \alpha_{j+1}$.

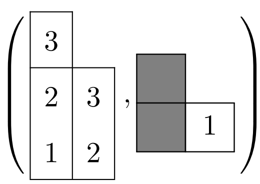
PROOF. We only need to check that the q in Step 2 of Definition 136 is greater or equal to the hook entry and arm of the cell q is to be inserted into. When q is the only arm element, it is obvious that q is greater or equal to the hook entry.

The case when q is not the only arm element can only happen when we reach the destination column of the path. By the proof of Lemma 142, we have that for $q_{j+1}^{(s)} \geqslant q_j^{(s+z_j)}$ for $s \geqslant 1$ and for j such that $d(r_j, c_j) = d(r_{j+1}, c_{j+1})$. Hence the statement follows by setting $k = \alpha_{j+1}$. $\square$

Before we define the "inverse" of the uncrowding map $\mathcal{U} : \mathsf{HVT}(\lambda) \to \sqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$, we need to restrict our domain to a subset K_λ of $\sqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$, as the image of $\mathcal{U}$ is not all of $\sqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$. We define:

$$\begin{aligned} \mathsf{K}_\lambda(\mu) :=& \{(S, F) \in \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda) \mid \mathsf{weight}(T_j^{(s)}) = \mathsf{weight}(S), \forall\, 0 \leqslant j \leqslant e-1, \forall\, 0 \leqslant s \leqslant \alpha_{j+1}\}, \\ \mathsf{K}_\lambda :=& \bigsqcup_{\mu \supseteq \lambda} \mathsf{K}_\lambda(\mu). \end{aligned}$$

Remark 145. *From the perspective of the uncrowding map, the set-valued tableau S in Example 137 cannot be obtained from a shape $(1,1)$ hook-valued tableau via the uncrowding map as explained in Remark 138. We say the cell $(1,2)$ in S practices social distancing. In this case,*

 $\notin \mathsf{K}_{(1,1)}$.

The (S, F) in Example 141 is in $\mathsf{K}_{(3,2,1)}(5,3,2)$.

Definition 146. *We can now define the crowding map $\mathcal{C}$ for any partition λ as follows,*

$$\begin{aligned} \mathcal{C} \colon \mathsf{K}_\lambda &\longrightarrow \mathsf{HVT}(\lambda) \\ (S, F) &\mapsto T_e^{(0)}. \end{aligned}$$

Proposition 147. *The image of the uncrowding map $\mathcal{U} : \mathsf{HVT}(\lambda) \to \sqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$ is a subset of K_λ. Moreover, we have $\mathcal{C} \circ \mathcal{U} = \mathbf{1}_{\mathsf{HVT}(\lambda)}$.*

Proof. We show that if $\tilde{h} = \mathcal{V}_b(h)$, where $h \in \mathsf{HVT}$, $\mathcal{V}_b$ is as defined in Definition 119 and $\tilde{h}$ is obtained by moving some letter(s) from the cell (r, c) to $(\tilde{r}, c+1)$ (potentially adding a box), then $\mathcal{C}_b([\tilde{h}, (\tilde{r}, c+1)]) = [h', (r', c)]$ satisfies $[h', (r', c)] = [h, (r, c)]$.

We follow the notation used in Definitions 119 and 136. Thus $a = \max(\mathsf{A}_h(r,c))$. We have that $\mathsf{H}_h(\tilde{r},c) \leqslant a$. If cell $(r+1,c)$ is in h, then $\mathsf{H}_h(r+1,c) > a$.

Case (1): $\tilde{r} \neq r$.

Case (1A): If cell $(\tilde{r},c+1)$ is not in h, then h' is obtained by adding cell $(\tilde{r},c+1)$ and moving a from $\mathsf{A}_h(r,c)$ to $\mathsf{H}_h(\tilde{r},c+1)$. Under the action of $\mathcal{C}_b$, by Step 1, $b = a$ and $r' = r$. $\mathcal{C}_b$ appends a to $\mathsf{A}_{\tilde{h}}(r,c)$ and removes cell $(\tilde{r},c+1)$, which recovers h.

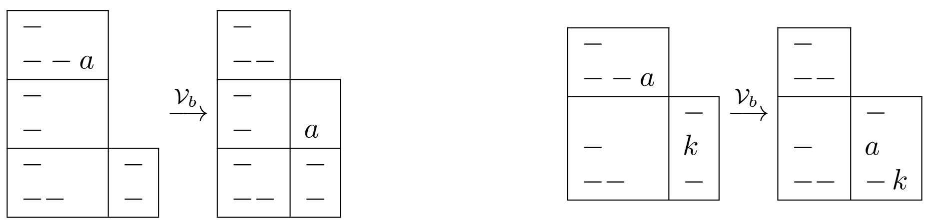


FIGURE 4.7. Left: case (1A): $(\tilde{r},c+1)$ is not in h. Right: case (1B): $(\tilde{r},c+1)$ is in h.

Case (1B): If cell $(\tilde{r},c+1)$ is in h, then $k \in \mathsf{L}_h^+(\tilde{r},c+1)$ is the smallest number that is greater than or equal to a in column $c+1$. h' is obtained by removing a from $\mathsf{A}_h(r,c)$, replacing k with a, and attaching k to $\mathsf{A}_h(\tilde{r},c+1)$. Under the action of $\mathcal{C}_b$, by Step 1, we can see that $m = k$, $b = a$ and $r' = r$. By Step 3(b)i, $q = b = a$, and a is appended to $\mathsf{A}_{\tilde{h}}(r,c)$ and $q = a$ in $\mathsf{L}_{\tilde{h}}(\tilde{r},c+1)$ is replaced with $m = k$. In the end, m is removed from $\mathsf{A}_{\tilde{h}}(\tilde{r},c+1)$. We recover h.

Case (2): $\tilde{r} = r$. Let $\ell = \max(\mathsf{L}_h^+(r,c))$.

Case (2A): If cell $(r,c+1)$ is not in h, $\mathcal{V}_b$ adds cell $(r,c+1)$, removes the part of $\mathsf{L}_h(r,c)$ that is greater than a to $\mathsf{L}_h(r,c+1)$ and moves a from $\mathsf{A}_h(r,c)$ to $\mathsf{H}_h(r,c+1)$. Under the action of $\mathcal{C}_b$, by Step 1, $m = a$ and $b = \ell$. Thus $r' = r$. By Step 3(a)ii, we move $\mathsf{L}_{\tilde{h}}(r,c+1)$ into $\mathsf{L}_{\tilde{h}}(r,c)$ and we recover h.

FIGURE 4.8. Left: Case (1A): $(r,c+1)$ is not in h. Right: Case (1B): $(r,c+1)$ is in h.

Case (2B): If cell $(r, c+1)$ is in h, $\tilde{h}$ is obtained by moving the part of $\mathsf{L}_h(r, c)$ that is greater than a to $\mathsf{L}_h(r, c+1)$, moving a from $\mathsf{A}_h(r, c)$ to $\mathsf{H}_h(r, c+1)$, and appending k to $\mathsf{A}_h(r, c+1)$. Under the action of $\mathcal{C}_b$, by Step 1, $m = k$ and $b = \ell$. Then $r' = r$ and $q = a$. By Step 3(a)i, we move the set $\{x \in \mathsf{L}_{\tilde{h}}(r, c) \mid a < x \leqslant k\}$ from $\mathsf{L}_{\tilde{h}}(r, c+1)$ into $\mathsf{L}_{\tilde{h}}(r, c)$, which is the set that was moved from cell (r, c) by $\mathcal{V}_b$. Removing k from $\mathsf{A}_{\tilde{h}}(r, c+1)$ and setting $\mathsf{H}_{\tilde{h}}(r, c+1) = k$, we recover h.

Now we have proven $\mathcal{C}_b([\tilde{h}, (\tilde{r}, c+1)]) = [h', (r', c)] = [h, (r, c)]$. It follows that for any $(S, F) = \mathcal{U}(h)$, we have that $T_j^{(s)}$ is semistandard and has the same weight as S for all $0 \leqslant j \leqslant e-1$, for all $0 \leqslant s \leqslant \alpha_{j+1}$. Thus $\mathsf{image}(\mathcal{U}) \subset \mathsf{K}_\lambda$ and $\mathcal{C} \circ \mathcal{U} = \mathbf{1}_{\mathsf{HVT}(\lambda)}$. □

Proposition 148. *K_λ is a subset of the image of $\mathcal{U} : \mathsf{HVT}(\lambda) \to \sqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$. Moreover, $\mathcal{U} \circ \mathcal{C} = \mathbf{1}_{\mathsf{K}_\lambda}$.*

Proof. Let $(S, F) \in \mathsf{K}_\lambda$, then for all $0 \leqslant j < e$ and for all $0 \leqslant s < \alpha_{j+1}$, $\mathcal{C}_b([T_j^{(s)}, (r, c)]) = [T_j^{(s+1)}, (r', c-1)]$ for some r, c, r'. We show that $\mathcal{V}_b(T_j^{(s+1)}) = T_j^{(s)}$ for all $0 \leqslant j < e$ and for all $0 \leqslant s < \alpha_{j+1}$. Following the notation in Definition 119, we first locate the rightmost column that contains nonzero arm excess, then determine the topmost cell in row $\tilde{r}$ in that column with nonzero arm excess. We denote by a the largest arm element in that cell.

By Lemma 143, in $T_j^{(s+1)}$, column $c-1$ is the rightmost column with nonzero arm excess and $(r', c-1)$ is the topmost cell in column $c-1$ with nonzero arm excess.

Case (1): $r' = r$. In this case either cell $(r+1, c-1)$ does not exist in $T_j^{(s)}$, or $\mathsf{H}_{T_j^{(s)}}(r+1, c-1) > b$.

Case (1A): $\mathsf{A}_{T_j^{(s)}}(r, c) = \emptyset$. $m = \mathsf{H}_{T_j^{(s)}}(r, c)$ and $b = \max(\mathsf{L}^+_{T_j^{(s)}}(r, c))$. Since $r' = r$, $q = m$, $T_j^{(s+1)}$ is obtained by appending m to $\mathsf{A}_{T_j^{(s)}}(r, c-1)$, moving $\mathsf{L}_{T_j^{(s)}}(r, c)$ into $\mathsf{L}_{T_j^{(s)}}(r, c-1)$, and removing cell (r, c) from $T_j^{(s)}$. Note that everything in $\mathsf{L}_{T_j^{(s)}}(r, c)$ is greater than m and everything in $\mathsf{L}_{T_j^{(s)}}(r, c-1)$ is smaller or equal to m.

For the $\mathcal{V}_b$ action, we have $a = m$ and b is the greatest letter in $\mathsf{L}_{T_j^{(s+1)}}(r, c-1)$. Since every letter in $T_j^{(s+1)}(r'', c)$ is smaller than m for $r'' < r$, we have $\tilde{r} = r$. $\mathcal{V}_b$ acts on $T_j^{(s+1)}$ by adding the cell (r, c), setting the hook entry to be m, and moving $(m, b] \cap \mathsf{L}_{T_j^{(s+1)}}(r, c-1)$ to $\mathsf{L}_{T_j^{(s+1)}}(r, c)$. Then we recover $T_j^{(s)}$.

FIGURE 4.9. Left: Case (1A): $\mathsf{A}_{T_j^{(s)}}(r,c)=\emptyset$. Right: Case (1B): $\mathsf{A}_{T_j^{(s)}}(r,c)\neq\emptyset$.

Case (1B): $\mathsf{A}_{T_j^{(s)}}(r,c)\neq\emptyset$. m is the only element in $\mathsf{A}_{T_j^{(s)}}(r,c)$, $q=\mathsf{H}_{T_j^{(s)}}(r,c)$ and $b=\max\{x\in \mathsf{L}^+_{T_j^{(s)}} \mid x\leqslant m\}$. $T_j^{(s+1)}$ is obtained by appending q to $\mathsf{A}_{T_j^{(s)}}(r,c-1)$, setting $\mathsf{H}_{T_j^{(s)}}(r,c)$ to be m, deleting $\mathsf{A}_{T_j^{(s)}}$, and moving $\{x\in \mathsf{L}_{T_j^{(s)}(r,c)} \mid q<x\leqslant m\}$ to $\mathsf{L}_{T_j^{(s)}}(r,c-1)$.

For the $\mathcal{V}_b$ action, $a=q$ and b is the greatest letter in $\mathsf{L}_{T_j^{(s+1)}}(r,c-1)$. Since every letter in $T_j^{(s+1)}(r'',c)$ is smaller than q for $r''<r$ and $m\geqslant q$, $\tilde{r}=r$. $\mathcal{V}_b$ acts on $T_j^{(s+1)}$ by setting $\mathsf{H}_{T_j^{(s+1)}}(r,c)=q$, $\mathsf{A}_{T_j^{(s+1)}}(r,c)=m$, and moving $(q,b]\cap \mathsf{L}_{T_j^{(s+1)}}(r,c-1)$ to $\mathsf{L}_{T_j^{(s+1)}}(r,c)$. We recover $T_j^{(s)}$.

Case (2): $r'\neq r$.

Case (2A): $\mathsf{A}_{T_j^{(s)}}(r,c)=\emptyset$. Note that in this case, $\mathcal{C}_b$ will move m somewhere else and remove the cell (r,c). Since $\mathsf{weight}(T_j^{(s+1)})=\mathsf{weight}(T_j^{(s)})$, we must have that $\mathsf{L}_{T_j^{(s)}}(r,c)=\emptyset$. So $b=q=m$. $T_j^{(s+1)}$ is obtained from $T_j^{(s)}$ by appending m to $\mathsf{A}_{T_j^{(s)}}(r',c-1)$ and removing the cell (r,c).

For the $\mathcal{V}_b$ action, $a=m$. Since every letter in $T_j^{(s+1)}(r'',c)$ is smaller than m for $r''<r$, a new cell (r,c) is added, $\tilde{r}=r$. $\mathcal{V}_b$ acts on $T_j^{(s+1)}$ by moving m to $\mathsf{H}_{T_j^{(s+1)}}(r,c)$. We recover $T_j^{(s)}$.

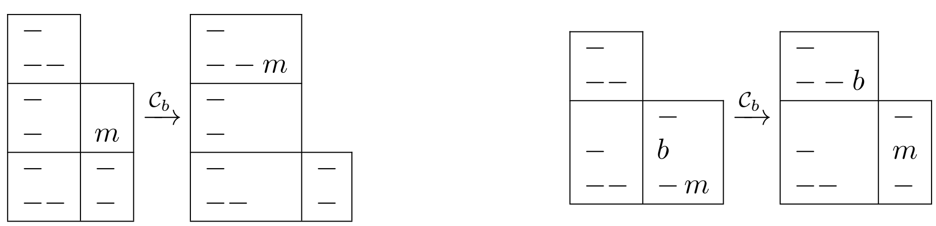


FIGURE 4.10. Left: case (2A): $\mathsf{A}_{T_j^{(s)}}(r,c)=\emptyset$. Right: case (2B): $\mathsf{A}_{T_j^{(s)}}(r,c)\neq\emptyset$.

Case (2B): $\mathsf{A}_{T_j^{(s)}}(r,c)\neq\emptyset$. m is the only element in $\mathsf{A}_{T_j^{(s)}}(r,c)$, $q=b=\max\{x\in\mathsf{L}^+_{T_j^{(s)}}(r,c)\mid x\leqslant m\}$. $T_j^{(s+1)}$ is obtained by appending b to $\mathsf{A}_{T_j^{(s)}}(r',c-1)$, replacing b in $\mathsf{L}_{T_j^{(s)}}(r,c)$ with m, and removing m from $\mathsf{A}_{T_j^{(s)}}(r,c)$.

For the $\mathcal{V}_b$ action, $a=b$. Since every letter in $T_j^{(s+1)}(r'',c)$ is smaller than b for $r''<r$, m is the smallest letter that is greater or equal to b in column c. Hence $\tilde{r}=r$. $\mathcal{V}_b$ acts on $T_j^{(s+1)}$ by removing

b from $\mathsf{A}_{T_j^{(s+1)}}(r', c-1)$, replacing m in $\mathsf{L}_{T_j^{(s+1)}}(r,c)$ with b, and attaching m to $\mathsf{A}_{T_j^{(s+1)}}(r,c)$. We recover $T_j^{(s)}$.

Therefore we have $\mathcal{V}_b(T_j^{(s+1)}) = T_j^{(s)}$ for all $0 \leqslant j \leqslant e-1$, for all $0 \leqslant s < \alpha_j$, and $\mathcal{V}(T_{j+1}^{(0)}) = T_j^{(0)}$. It follows that we also recover the recording tableau F. Thus $\mathcal{U}(T_e^{(0)}) = (S, F)$. $\square$

COROLLARY 149. *The uncrowding map $\mathcal{U}$ is a bijection between $\mathsf{HVT}(\lambda)$ and K_λ with inverse $\mathcal{C}$.*

4.2.6. Alternative uncrowding on hook-valued tableaux. In Section 4.2.2, we defined an uncrowding map sending hook-valued tableaux to pairs of tableaux with one being set-valued and the other being column-flagged increasing. As hook-valued tableaux were introduced as a generalization of both set-valued tableaux and multiset-valued tableaux, it is natural to ask if there is an uncrowding map taking hook-valued tableaux to pairs of tableaux with one being multiset-valued. In this section we provide such a map.

DEFINITION 150. *The* multiset uncrowding bumping $\tilde{\mathcal{V}}_b \colon \mathsf{HVT} \to \mathsf{HVT}$ *is defined by the following algorithm:*

(1) *Initialize T as the input.*
(2) *If the leg excess of T equals zero, return T.*
(3) *Find the topmost row that contains a cell with nonzero leg excess. Within this column, find the cell with the largest value in its leg. (This is the rightmost cell with nonzero leg excess in the specified row.) Denote the row index and column index of this cell by r and c, respectively. Denote the cell as (r,c), its largest leg entry by ℓ, and its rightmost arm entry by a.*
(4) *Look at the row above (r,c) (i.e. row $r+1$) and find the leftmost number that is strictly greater than ℓ.*
    - *If no such number exists, attach an empty cell to the end of row $r+1$ and label the cell as $(r+1,\tilde{c})$, where $\tilde{c}$ is its column index. Let k be the empty character.*
    - *If such a number exists, label the value as k and the cell containing k as $(r+1,\tilde{c})$ where $\tilde{c}$ is the cell's column index.*

    We now break into cases:

(a) *If $\tilde{c} \neq c$, then remove ℓ from $\mathsf{L}_T(r,c)$, replace k with ℓ, and attach k to the leg of $\mathsf{L}_T(r+1,\tilde{c})$.*

(b) *If $\tilde{c} = c$ then remove $[\ell, a] \cap \mathsf{A}_T(r,c)$ from $\mathsf{A}_T(r,c)$ where $[\ell, a] \cap \mathsf{A}_T(r,c)$ is the multiset $\{z \in \mathsf{A}_T(r,c) \mid \ell \leqslant z \leqslant a\}$. Remove ℓ from $\mathsf{L}_T(r,c)$, insert $[\ell,a] \cap \mathsf{A}_T(r,c)$ into $\mathsf{A}_T(r+1,\tilde{c})$, replace the hook entry of $(r+1,\tilde{c})$ with ℓ, and attach k to $\mathsf{L}_T(r+1,\tilde{c})$.*

(5) *Output the resulting tableau.*

Definition 151. *The multiset uncrowding insertion $\tilde{\mathcal{V}} \colon \mathsf{HVT} \to \mathsf{HVT}$ is defined as $\tilde{\mathcal{V}}(T) = \tilde{\mathcal{V}}_b^d(T)$, where the integer $d \geqslant 1$ is minimal such that $\mathsf{shape}(\tilde{\mathcal{V}}_b^d(T))/\mathsf{shape}(\tilde{\mathcal{V}}_b^{d-1}(T)) \neq \emptyset$ or $\tilde{\mathcal{V}}_b^d(T) = \tilde{\mathcal{V}}_b^{d-1}(T)$.*

Definition 152. *Let $T \in \mathsf{HVT}(\lambda)$ with leg excess α. The multiset uncrowding map*

$$\tilde{\mathcal{U}} \colon \mathsf{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} \mathsf{MVT}(\mu) \times \mathcal{F}(\mu/\lambda)$$

is defined by the following algorithm:

(1) *Let $\tilde{P}_0 = T$ and let $\tilde{Q}_0$ be the flagged increasing tableau of shape λ/λ.*

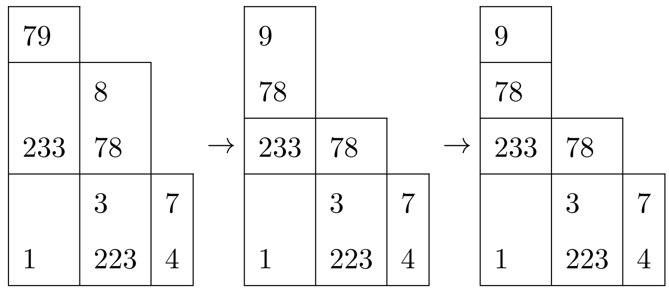
(2) *For $1 \leqslant i \leqslant \alpha$, let $\tilde{P}_{i+1} = \tilde{\mathcal{V}}(\tilde{P}_i)$. Let r be the index of the topmost row of $\tilde{P}_i$ containing a cell with nonzero leg excess and let $\tilde{r}$ be the row index of the cell $\mathsf{shape}(\tilde{P}_{i+1})/\mathsf{shape}(\tilde{P}_i)$. Then $\tilde{Q}_{i+1}$ is obtained from $\tilde{Q}_i$ by appending the cell $\mathsf{shape}(\tilde{P}_{i+1})/\mathsf{shape}(\tilde{P}_i)$ to $\tilde{Q}_i$ and filling this cell with $\tilde{r} - r$.*

Define $\tilde{\mathcal{U}}(T) = (\tilde{P}(T), \tilde{Q}(T)) := (\tilde{P}_\alpha, \tilde{Q}_\alpha)$.

Example 153. *Let T be the hook-valued tableau*

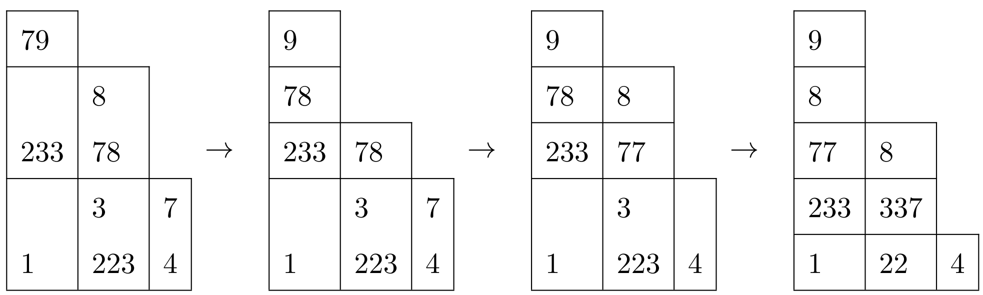
$$T = \begin{array}{|c|c|c|} \cline{1-1} 79 & \multicolumn{2}{c}{} \\ \cline{1-2} \begin{matrix} \\ 233 \end{matrix} & \begin{matrix} 8 \\ 78 \end{matrix} & \multicolumn{1}{c}{} \\ \hline \begin{matrix} \\ 1 \end{matrix} & \begin{matrix} 3 \\ 223 \end{matrix} & \begin{matrix} 7 \\ 4 \end{matrix} \\ \hline \end{array} .$$

Then, we obtain the following sequence of tableaux $\tilde{\mathcal{V}}_b^i(T)$ *for* $0 \leqslant i \leqslant 2 = d$ *when computing the first multiset uncrowding insertion:*

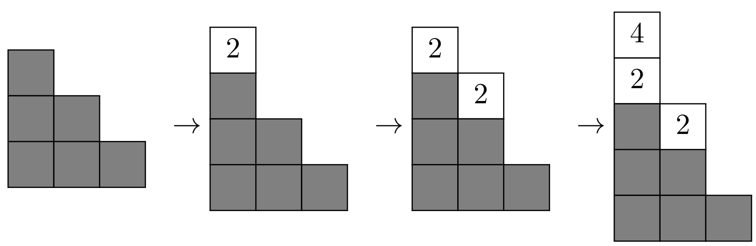


$= \tilde{\mathcal{V}}(T)$.

Continuing with the remaining multiset uncrowding insertions, we obtain the following sequences of tableaux for the multiset uncrowding map:



$= \tilde{P}(T)$,



$= \tilde{Q}(T)$.

Proposition 154. *Let* $T \in \mathsf{HVT}$. *Then* $\tilde{\mathcal{U}}(T)$ *is well-defined.*

Proof. The statement follows from a similar argument to the proofs found in Corollary 124 and Lemma 126. $\square$

Similar to the uncrowding map $\mathcal{U}$, the multiset uncrowding map $\tilde{\mathcal{U}}$ interwines with the corresponding crystal operators.

Theorem 155. *Let* $T \in \mathsf{HVT}$.

(1) *If* $f_i(T) = 0$, *then* $f_i(\tilde{P}(T)) = 0$.
(2) *If* $e_i(T) = 0$, *then* $e_i(\tilde{P}(T)) = 0$.

(3) *If $f_i(T) \neq 0$, we have $f_i(\tilde{P}(T)) = \tilde{P}(f_i(T))$ and $\tilde{Q}(T) = \tilde{Q}(f_i(T))$.*

(4) *If $e_i(T) \neq 0$, we have $e_i(\tilde{P}(T)) = \tilde{P}(e_i(T))$ and $\tilde{Q}(T) = \tilde{Q}(e_i(T))$.*

PROOF. The proof follows similarly to those found in Proposition 129, Lemma 130, and Theorem 131. □

4.3. Applications

In this section, we provide the expansion of the canonical Grothendieck polynomials $G_\lambda(x; \alpha, \beta)$ in terms of the stable symmetric Grothendieck polynomials $G_\mu(x; \beta = -1)$ and in terms of the dual stable symmetric Grothendieck polynomials $g_\mu(x; \beta = 1)$ using techniques developed in [**BM12**]. We first review the basic definitions and Schur expansions of the two polynomials.

Recall that the stable symmetric Grothendieck polynomial is the generating function of set-valued tableaux

$$G_\mu(x; -1) = \sum_{S \in \mathsf{SVT}(\mu)} (-1)^{|S|-|\mu|} x^{\mathsf{weight}(S)}.$$

Its Schur expansion can be obtained from the crystal structure on set-valued tableaux [**MPS20**]

$$G_\mu(x; -1) = \sum_{\substack{S \in \mathsf{SVT}(\mu) \\ e_i(S)=0 \;\; \forall i}} (-1)^{|S|-|\mu|} \; s_{\mathsf{weight}(S)}.$$

DEFINITION 156. *The reading word $\mathsf{word}(S) = w_1 w_2 \cdots w_n$ of a set-valued tableau $S \in \mathsf{SVT}(\mu)$ is obtained by reading the elements in the rows of S from the top row to the bottom row in the following way. In each row, first ignore the smallest element of each cell and read all remaining elements in descending order. Then read the smallest elements of each cell in ascending order.*

EXAMPLE 157. *The reading word of $P(T)$ in Example 123 is $\mathsf{word}(P(T)) = 8675423362111567$.*

EXAMPLE 158. *The highest weight set-valued tableaux of shape (2) are*

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline & 2 \\ 1 & 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline & 3 \\ & 2 \\ 1 & 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline & 4 \\ & 3 \\ & 2 \\ 1 & 1 \\ \hline \end{array}\,, \quad \ldots,$$

which gives the Schur expansion

$$G_{(2)}(x;-1) = s_2 - s_{21} + s_{211} - s_{2111} \pm \cdots .$$

The *dual stable symmetric Grothendieck polynomials* $g_\mu(x;1)$ are dual to $G_\mu(x;-1)$ under the Hall inner product on the ring of symmetric functions.

DEFINITION 159. *A* reverse plane partition *of shape μ is a filling of the cells in the Ferrers diagram of μ with positive integers, such that the entries are weakly increasing in rows and columns. We denote the collection of all reverse plane partitions of shape μ by $\mathsf{RPP}(\mu)$ and the set of all reverse plane partitions by RPP.*

The evaluation $\mathsf{ev}(R)$ *of a reverse plane partition $R \in \mathsf{RPP}$ is a composition $\alpha = (\alpha_i)_{i \geqslant 1}$, where α_i is the total number of columns in which i appears. The* reading word $\mathsf{word}(R)$ *is obtained by first circling the bottommost occurrence of each letter in each column, and then reading the circled letters row-by-row from top to bottom and left to right within each row.*

EXAMPLE 160. *Consider the reverse plane partition*

$$R = \begin{array}{|c|c|c} \hline 1 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \in \mathsf{RPP}((3,2)).$$

By circling the bottommost occurrence of each letter in each column, we obtain

$$R = \begin{array}{|c|c|c} \hline 1 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \ \mathsf{ev}(R) = (2,1,1), \ \mathsf{word}(R) = 2113.$$

Lam and Pylyavskyy [**LP07**] showed that the dual stable symmetric Grothendieck polynomials $g_\mu(x;1)$ are generating functions of reverse plane partitions of shape μ

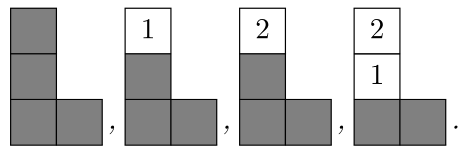
$$g_\mu(x;1) = \sum_{R \in \mathsf{RPP}(\mu)} x^{\mathsf{ev}(R)}.$$

They also provided the Schur expansion of the dual stable symmetric Grothendieck polynomials [**LP07**, Theorem 9.8]

$$g_\mu(x;1) = \sum_F s_{\mathsf{innershape}(F)},$$

where the sum is over all flagged increasing tableaux whose outer shape is μ.

Example 161. *When $\mu = (\mu_1)$ is a partition with only one row, we have $g_{(\mu_1)}(x;1) = s_{(\mu_1)}$. The flagged increasing tableaux of outer shape $(2,1,1)$ are*



Hence $g_{211}(x;1) = s_{211} + 2s_{21} + s_2$.

According to [**BM12**], a symmetric function f_α over the ring R is said to have a *tableaux Schur expansion* if there is a set of tableaux $\mathbb{T}(\alpha)$ and a weight function $\mathsf{wt}_\alpha \colon \mathbb{T}(\alpha) \to R$ so that

$$f_\alpha = \sum_{T \in \mathbb{T}(\alpha)} \mathsf{wt}_\alpha(T) s_{\mathsf{shape}(T)}.$$

Furthermore, any symmetric function with such a property has the following expansion in terms of $G_\mu(x;-1)$ and $g_\mu(x;1)$.

Theorem 162. *[**BM12**, Theorem 3.5] Let f_α be a symmetric function with a tableaux Schur expansion $f_\alpha = \sum_{T \in \mathbb{T}(\alpha)} \mathsf{wt}_\alpha(T) s_{\mathsf{shape}(T)}$ for some $\mathbb{T}(\alpha)$. Let $\mathbb{S}(\alpha)$ and $\mathbb{R}(\alpha)$ be defined as sets of set-valued tableaux and reverse plane partitions, respectively, by*

$$S \in \mathbb{S}(\alpha) \text{ if and only if } P(\mathsf{word}(S)) \in \mathbb{T}(\alpha), \text{ and}$$

$$R \in \mathbb{R}(\alpha) \text{ if and only if } P(\mathsf{word}(R)) \in \mathbb{T}(\alpha),$$

where $P(w)$ is the RSK insertion tableau of the word w. We also extend wt_α to $\mathbb{S}(\alpha)$ and $\mathbb{R}(\alpha)$ by setting $\mathsf{wt}_\alpha(X) := \mathsf{wt}_\alpha(P(\mathsf{word}(X)))$ for any $X \in \mathbb{S}(\alpha)$ or $\mathbb{R}(\alpha)$. Then we have

$$f_\alpha = \sum_{R \in \mathbb{R}(\alpha)} \mathsf{wt}_\alpha(R) G_{\mathsf{shape}(R)}(x;-1), \text{ and}$$

$$f_\alpha = \sum_{S \in \mathbb{S}(\alpha)} \mathsf{wt}_\alpha(S)(-1)^{|S|-|\mathsf{shape}(S)|} g_{\mathsf{shape}(S)}(x;1).$$

Proposition 163. *The canonical Grothendieck polynomials have a tableaux Schur expansion.*

PROOF. Recall the uncrowding map on set-valued tableaux of Definition 118

$$\mathcal{U}_{\mathsf{SVT}} : \mathsf{SVT}(\mu) \longrightarrow \bigsqcup_{\nu \supseteq \mu} \mathsf{SSYT}(\nu) \times \mathcal{F}(\nu/\mu).$$

By Corollary 149, we have a bijection

$$\mathcal{U} : \mathsf{HVT}(\lambda) \to \mathsf{K}_\lambda = \bigsqcup_{\mu \supseteq \lambda} \mathsf{K}_\lambda(\mu).$$

Note that $\mathsf{K}_\lambda \subseteq \bigsqcup_{\mu \supseteq \lambda} \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$. Denote

$$\phi_\lambda(S) = |\{F \in \hat{\mathcal{F}} \mid (S, F) \in \mathsf{K}_\lambda\}|.$$

Note that sometimes $\phi_\lambda(S) = 0$.

Given $H \in \mathsf{HVT}(\lambda)$, we have $\mathcal{U}(H) = (S, F) \in \mathsf{SVT}(\mu) \times \hat{\mathcal{F}}(\mu/\lambda)$ for some $\mu \supseteq \lambda$ and $|\mu| = |\lambda| + a(H)$. We can also obtain $\mathcal{U}_{\mathsf{SVT}}(S) = (T, Q) \in \mathsf{SSYT}(\nu) \times \mathcal{F}(\nu/\mu)$ for some $\nu \supseteq \mu$ and $|\nu| = |H|$. The weights of H, S and T are the same. When H is highest weight, that is $e_i(H) = 0$ for all i, then S and T are also of highest weight and $\mathsf{weight}(H) = \mathsf{shape}(T)$. Denote by $\mathsf{HVT}_h(\lambda), \mathsf{SVT}_h(\lambda), \mathsf{SSYT}_h(\lambda)$ the subset of highest weight elements in $\mathsf{HVT}(\lambda), \mathsf{SVT}(\lambda), \mathsf{SSYT}(\lambda)$, respectively.

Applying [**HS20**, Theorem 4.6] and the above correspondence, we obtain

$$\begin{aligned}
G_\lambda(x; \alpha, \beta) &= \sum_{H \in \mathsf{HVT}_h(\lambda)} \alpha^{a(H)} \beta^{\ell(H)} s_{\mathsf{weight}(H)} = \sum_{\mu \supseteq \lambda} \sum_{(S,F) \in \mathsf{K}_\lambda(\mu)} \alpha^{|\mu| - |\lambda|} \beta^{|S| - |\mu|} s_{\mathsf{weight}(S)} \\
&= \sum_{\mu \supseteq \lambda} \sum_{S \in \mathsf{SVT}_h(\mu)} \phi_\lambda(S) \alpha^{|\mu| - |\lambda|} \beta^{|S| - |\mu|} s_{\mathsf{weight}(S)} \\
&= \sum_{\mu \supseteq \lambda} \sum_{\nu \supseteq \mu} \sum_{T \in \mathsf{SSYT}_h(\nu)} \sum_{Q \in \mathcal{F}(\nu/\mu)} \phi_\lambda(\mathcal{U}_{\mathsf{SVT}}^{-1}(T, Q)) \alpha^{|\mu| - |\lambda|} \beta^{|\nu| - |\mu|} s_{\mathsf{weight}(T)} \\
&= \sum_{\mu \supseteq \lambda} \sum_{\nu \supseteq \mu} \sum_{T \in \mathsf{SSYT}_h(\nu)} \alpha^{|\mu| - |\lambda|} \beta^{|\nu| - |\mu|} \sum_{Q \in \mathcal{F}(\nu/\mu)} \phi_\lambda(\mathcal{U}_{\mathsf{SVT}}^{-1}(T, Q)) s_{\mathsf{shape}(T)} \\
&= \sum_{T \in \mathbb{T}(\lambda)} \mathsf{wt}_\lambda(T) s_{\mathsf{shape}(T)},
\end{aligned}$$

where $\mathbb{T}(\lambda) = \{T \in \mathsf{SSYT}_h(\nu) \mid \nu \supseteq \lambda\}$ and

$$\mathsf{wt}_\lambda(T) = \sum_{\mu : \lambda \subseteq \mu \subseteq \mathsf{shape}(T)} \alpha^{|\mu| - |\lambda|} \beta^{|\mathsf{shape}(T)| - |\mu|} \sum_{Q \in \mathcal{F}(\mathsf{shape}(T)/\mu)} \phi_\lambda(\mathcal{U}_{\mathsf{SVT}}^{-1}(T, Q)).$$

$\square$

Corollary 164. *The canonical Grothendieck polynomials have $G_\mu(x;-1)$ and $g_\mu(x;1)$ expansions:*

$$G_\lambda(x;\alpha,\beta) = \sum_{R\in\mathbb{R}(\lambda)} \mathsf{wt}_\lambda(R)G_{\mathsf{shape}(R)}(x;-1),$$

$$G_\lambda(x;\alpha,\beta) = \sum_{S\in\mathbb{S}(\lambda)} \mathsf{wt}_\lambda(S)(-1)^{|S|-|\mathsf{shape}(S)|}g_{\mathsf{shape}(S)}(x;1).$$

Example 165. *We compute the first two terms in $G_{(2)}(x;\alpha,\beta) = s_2+\beta s_{21}+2\alpha s_3+2\alpha\beta s_{31}+\cdots$. The semistandard Young tableaux involved are*

$$\mathbb{T}((2)) = \left\{ \begin{ytableau} 1 & 1 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 1 & 1 & 1 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 1 & 1 \end{ytableau}, \dots \right\}.$$

Labelling the tableaux $T_1, T_2, T_3, T_4, \dots,$ we have $\mathsf{wt}_{(2)}(T_1) = 1, \mathsf{wt}_{(2)}(T_2) = \beta, \mathsf{wt}_{(2)}(T_3) = 2\alpha,$ $\mathsf{wt}_{(2)}(T_4) = 2\alpha\beta$. Next we compute the elements in $\mathbb{R}((2))$ and $\mathbb{S}((2))$ that correspond to T_1 and T_2:

$$\{R\in\mathbb{R}((2)) \mid P(\mathsf{word}(R)) = T_1\} = \{ \begin{ytableau} 1 & 1 \end{ytableau}, \begin{ytableau} 1 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 1 & 1 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 1 \\ 1 \\ 1 & 1 \end{ytableau}, \dots \}$$

$$\{R\in\mathbb{R}((2)) \mid P(\mathsf{word}(R)) = T_2\} = \{ \begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 2 \\ 1 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 2 \\ 2 \\ 1 & 1 \end{ytableau}, \dots \}$$

$$\{S\in\mathbb{S}((2)) \mid P(\mathsf{word}(S)) = T_1\} = \{ \begin{ytableau} 1 & 1 \end{ytableau} \}$$

$$\{S\in\mathbb{S}((2)) \mid P(\mathsf{word}(S)) = T_2\} = \{ \begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{array}{|c|c|} \hline & 2 \\ \cline{2-2} 1 & 1 \\ \hline \end{array} \}.$$

Applying the expansion formulas, we obtain

$$\begin{aligned} G_{(2)}(x;\alpha,\beta) =& (G_{(2)}(x;-1) + G_{(21)}(x;-1) + G_{(22)}(x;-1) + G_{(211)}(x;-1) + \cdots) \\ &+ \beta(G_{(21)}(x;-1) + G_{(22)}(x;-1) + 2G_{(211)}(x;-1) + \cdots) + \cdots \\ G_{(2)}(x;\alpha,\beta) =& g_{(2)}(x;1) + \beta(g_{(21)}(x;1) - g_{(2)}(x;1)) + \cdots. \end{aligned}$$

APPENDIX A

Conjectures for weakly decreasing factorizations

The appendix discusses conjectures concerning the relation between crystal on multiset-valued tableaux and a $\star$-crystal on weakly decreasing factorizations. All of these conjectures were developed by the author through SageMath computer explorations [**Sag20**] and many of them are analogous to the results in [**MPPS20**].

Definition 166. *Let n be a positive integer. A* weakly decreasing factorization *of $w \in \mathcal{H}_0(n)$ into m* factors *is a product of the form*

$$\mathbf{h} = h^m \dots h^2 h^1,$$

where the sequence in each factor

$$h^i = h^i_1 h^i_2 \dots h^i_{\ell_i}$$

is either empty or weakly decreasing for each $1 \leqslant i \leqslant m$ and $\mathbf{h} \equiv_{\mathcal{H}_0} w$ in the 0-Hecke monoid $\mathcal{H}_0(n)$. The set of all possible weakly decreasing factorizations into m factors shall be denoted as $\mathcal{H}^m_{weak}$.

We call $\mathsf{ex}(\mathbf{h}) = \mathsf{len}(\mathbf{h}) - \ell$ the excess *of $\mathbf{h}$, where $\mathsf{len}(\mathbf{h})$ is the length of $\mathbf{h}$ as a word and ℓ is the length of a reduced expression for w.*

We say that $\mathbf{h}$ is fully-commutative *if w is fully-commutative.*

A.0.1. Weak $\star$-crystal. This subsection introduces a crystal on fully-commutative weakly decreasing factorizations.

Definition 167. *We define the* weak $\star$-crystal *$\mathcal{H}^{m,\star}_{weak}$ as follows. As a set, $\mathcal{H}^{m,\star}_{weak}$ consists of all fully-commutative weakly decreasing factorizations in $\mathcal{H}^m_{weak}$.*

Let $\mathbf{h} = h^m \dots h^2 h^1 \in \mathcal{H}^{m,\star}_{weak}$. The weight *function is defined as*

$$\mathrm{wt}(\mathbf{h}) = (\mathsf{len}(h^1), \mathsf{len}(h^2), \dots, \mathsf{len}(h^m)).$$

For every $1 \leqslant i < m$, we define the i-pairing of $\mathbf{h}$ as follows.

- *Perform the following step in weakly decreasing order on elements of h^{i+1}, removing previously paired letters in h^i from consideration at each iteration.*
- *Suppose that the letter in h^{i+1} under consideration is x.*

Case (a): *If there was another copy of x in h^{i+1} that was previously paired to an $x+1$ in h^i, identify the smallest letter y that is yet to be paired in h^i such that $y > x+1$, if possible. Pair this x with this y, if y exists.*

Case (b): *Otherwise, identify the smallest letter y that is yet to be paired in h^i such that $y > x$, if possible. Pair this x with this y, if y exists.*

Case (c): *Else, if there is no such y from the cases above, x is unpaired.*

After the above i-pairing, if all letters in h^i have been paired, then $f_i^\star$ annihilates $\mathbf{h}$. Otherwise, let y be the largest unpaired letter in h^i. The crystal operator $f_i^\star$ acts on $\mathbf{h}$ in either of the following ways:

(1) *If $y-1 \in h^{i+1}$ and $y \in h^i$, then remove y from h^i, add $y-1$ to h^{i+1}.*

(2) *Otherwise, remove y from h^i and add y to h^{i+1}.*

Similarly, if all letters in h^{i+1} have been i-paired, then $e_i^\star$ annihilates $\mathbf{h}$. Otherwise, let x be the smallest unpaired letter in h^{i+1}. The crystal operator $e_i^\star$ acts on $\mathbf{h}$ in either of the following ways:

(1) *If $x \in h^{i+1}$ and $x+1 \in h^i$, then remove x from h^{i+1}, add $x+1$ to h^i.*

(2) *Otherwise, remove x from h^{i+1} and add x to h^i.*

EXAMPLE 168. *Let $\mathbf{h} = (22111)(333)(55444) \in \mathcal{H}^{3,\star}_{weak}$.*
We have

$$f_1^\star(\mathbf{h}) = (22111)(3333)(5544), \qquad e_1^\star(\mathbf{h}) = 0,$$

$$f_2^\star(\mathbf{h}) = 0, \qquad e_2^\star(\mathbf{h}) = (2211)(3331)(55444).$$

A.0.2. Residue map.

DEFINITION 169. *Given $T \in \mathsf{MVT}^m(\lambda)$, we define the residue map $\mathsf{res} : \mathsf{MVT}^m(\lambda) \to \mathcal{H}^m_{weak}$ as follows. Associate each cell (i,j), where $i,j \geqslant 1$, with its content $\ell(\lambda)+j-i$. Produce a weakly* *decreasing factorization* $\mathbf{h} = h^m h^{m-1} \ldots h^2 h^1$ *by declaring* h^k *to be the (possibly empty) sequence formed by taking the contents of all cells in* T *containing* k *and then arranging the contents in weakly decreasing order. We define* $\mathsf{res}(T) := \mathbf{h}$.

EXAMPLE 170. *Consider the following tableau* T *given by*

$$T = \begin{array}{|c|c|c|}\cline{1-1} 566 & \multicolumn{2}{c}{} \\ \cline{1-1} 34 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 33 & \multicolumn{1}{c}{} \\ \hline 11 & 112 & 2333 \\ \hline \end{array}\quad ,$$

whose contents of each cell are labeled in red:

$$T = \begin{array}{|c|c|c|}\cline{1-1} 566_{\color{red}1} & \multicolumn{2}{c}{} \\ \cline{1-1} 34_{\color{red}2} & \multicolumn{2}{c}{} \\ \cline{1-2} 2_{\color{red}3} & 33_{\color{red}4} & \multicolumn{1}{c}{} \\ \hline 11_{\color{red}4} & 112_{\color{red}5} & 2333_{\color{red}6} \\ \hline \end{array}\quad ,$$

The third factor is formed as follows: searching through all cells with entry 3 (counting multiplicity), three of them has content 6, two has content 4 and one has content 2; hence the third factor reads 666442. We have $\mathsf{res}(T) = (11)(1)(2)(666442)(63)(5544)$.

As the definition of residue map is not too different as provided in Definition 56, we expect that its image is contained in $\mathcal{H}^{m,\star}_{\text{weak}}$.

CONJECTURE 171. *The image of the residue map lies within* $\mathcal{H}^{m,\star}_{weak}$.

It turns out that the residue map serves as a crystal morphism between the crystal on $\mathcal{H}^{m,\star}_{\text{weak}}$ with the crystal on multiset-valued tableaux given by [**HS20**]. The following conjecture has been verified for partitions up to size 6, excess 5 and maximum entry 4.

CONJECTURE 172. *The residue map intertwines the crystal operators on multiset-valued tableaux with those of* $\mathcal{H}^{m,\star}_{weak}$.

In other words, the following diagram commutes:

$$\begin{tikzcd}
\mathsf{MVT}^m(\lambda) \arrow[r, "\mathsf{res}"] \arrow[d, "f_i"] & \mathcal{H}^{m,\star}_{weak} \arrow[d, "f_i^\star"] \\
\mathsf{MVT}^m(\lambda) \arrow[r, "\mathsf{res}"] & \mathcal{H}^{m,\star}_{weak}.
\end{tikzcd}$$

EXAMPLE 173. *Consider the following tableau T:*

$$T = \begin{array}{|c|c} \hline 223 & \\ \hline 111 & \multicolumn{1}{c|}{123} \\ \hline \end{array},$$

Then, one can verify that we have the following commutative diagram:

$$\begin{tikzcd}
T = \begin{array}{|c|c} \hline 223_{\color{red}1} & \\ \hline 111_{\color{red}2} & \multicolumn{1}{c|}{123_{\color{red}3}} \\ \hline \end{array} \arrow[r, "\mathsf{res}"] \arrow[d, "f_1"] & (31)(311)(3222) \arrow[d, "f_1^\star"] \\
f_1(T) = \begin{array}{|c|c} \hline 2223_{\color{red}1} & \\ \hline 11_{\color{red}2} & \multicolumn{1}{c|}{123_{\color{red}3}} \\ \hline \end{array} \arrow[r, "\mathsf{res}"] & (31)(3111)(322).
\end{tikzcd}$$

A.0.3. $\star$-insertion. Similar to decreasing factorizations, one may represent a weakly decreasing factorization $\mathbf{h} = h^m h^{m-1} \cdots h^1$, where $h^i = h^i_1 h^i_2 \ldots h^i_{\ell_i}$, by a *weakly decreasing Hecke biword*

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} m & \ldots & m & \ldots & 1 & \ldots & 1 \\ h^m_1 & \ldots & h^m_{\ell_m} & \ldots & h^1_1 & \ldots & h^1_{\ell_1} \end{bmatrix}.$$

In addition, we say that $[\mathbf{k}, \mathbf{h}]^t$ is *fully commutative* if $\mathbf{h}$ is fully-commutative.

Here, we define the weak $\star$-insertion on fully-commutative weakly decreasing Hecke biwords.

DEFINITION 174. *Fix a fully-commutative decreasing Hecke biword $[\mathbf{k}, \mathbf{h}]^t$. The insertion is done by reading the columns of this biword from right to left.*

Begin with (P_0, Q_0) being a pair of empty tableaux. For every integer $i \geqslant 0$, we recursively construct (P_{i+1}, Q_{i+1}) from (P_i, Q_i) as follows. Let $[q, x]^t$ be the i-th column (from the right) of $[\mathbf{k}, \mathbf{h}]^t$. Suppose that we are inserting x into column C of P_i.

- **Case 1:** *If C is empty or $x < \min(C)$, then form P_{i+1} by appending x to column C and form Q_{i+1} by adding q in the corresponding position to Q_i. Terminate and return (P_{i+1}, Q_{i+1}).*
- **Case 2:** *Otherwise, if $x \notin C$, locate the largest y in C with $y < x$. Bump y with x and insert y into the next column of P_i.*
- **Case 3:** *Otherwise, $x \in C$, then locate the largest y in C with $x \leqslant y$ and interval $[x, y]$ contained in C. Column C remains unchanged and y is to be inserted into the next column of P_i.*

Denote $(P, Q) = (P_\ell, Q_\ell)$ if $[\mathbf{k}, \mathbf{h}]^t$ has length ℓ. We define the weak $\star$-insertion *by $\star([\mathbf{k}, \mathbf{h}]^t) = (P, Q)$.*

In addition, for a Young diagram of shape λ, a *reverse semistandard Young tableau* of shape λ is a filling of the Young diagram with positive integers such that the entries weakly decrease along rows from left to right and strictly decrease along columns from bottom to top.

Denote $\mathsf{col}(T)$ as the column reading word of a tableau T.

Example 175. *Let*

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} 3 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 & 1 & 3 & 2 & 2 \end{bmatrix}.$$

By performing weak $\star$-insertion with columns inserted from right to left, we obtain

$$\emptyset \to \begin{array}{|c|}\hline 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\cline{1-1} 1 & \multicolumn{2}{c}{} \\ \hline 3 & 2 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-2} 1 & 1 & \multicolumn{1}{c}{} \\ \hline 3 & 2 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 & 2 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|}\cline{1-3} 1 & 1 & 1 & \multicolumn{1}{c}{} \\ \hline 3 & 3 & 2 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|c|}\cline{1-3} 1 & 1 & 1 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & 2 & 2 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|c|c|}\cline{1-3} 1 & 1 & 1 & \multicolumn{3}{c}{} \\ \hline 3 & 3 & 3 & 2 & 2 & 2 \\ \hline\end{array} = P$$

$$\emptyset \to \begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|}\cline{1-3} 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 1 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|c|}\cline{1-3} 2 & 2 & 2 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 & 2 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|c|c|c|c|}\cline{1-3} 2 & 2 & 2 & \multicolumn{3}{c}{} \\ \hline 1 & 1 & 1 & 2 & 3 & 3 \\ \hline\end{array} = Q.$$

We conjecture that the weak $\star$-insertion is a bijection.

Conjecture 176. *The weak $\star$-insertion is a bijection from the set of all fully-commutative weakly decreasing Hecke biwords to the set of all pairs of tableaux (P, Q) of the same shape, where both P is reverse semistandard, Q is semistandard and the word $\mathsf{col}(P)$ is fully-commutative.*

Moreover, we have the following conjecture, which has been verified for all elements in $\mathcal{H}^{m,\star}_{\text{weak}}$ up to excess 5, for words in $\mathcal{H}_0(5)$.

Conjecture 177. *Let $\mathbf{h} \in \mathcal{H}^{m,\star}_{weak}$ be a fully-commutative weakly decreasing Hecke factorization. Let $(P^\star(\mathbf{h}), Q^\star(\mathbf{h})) = \star(\mathbf{h})$ be the insertion and recording tableaux under the weak $\star$-insertion of Definition 174. Then $f_i^\star(\mathbf{h}) \neq 0$ if and only if $f_i(Q^\star(\mathbf{h})) \neq 0$. Furthermore, if $f_i^\star(\mathbf{h}) \neq 0$, then $Q^\star(f_i^\star(\mathbf{h})) = f_i Q^\star(\mathbf{h})$.*

In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}^{m,\star}_{weak} \arrow[r, "Q^\star"] \arrow[d, "f_i^\star"] & \mathsf{SSYT}^m \arrow[d, "f_i"] \\
\mathcal{H}^{m,\star}_{weak} \arrow[r, "Q^\star"] & \mathsf{SSYT}^m.
\end{tikzcd}$$

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