Applications of Lieb-Robinson Bounds to Quantum Dynamics with and without Disorder

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Contents

Abstract

This dissertation presents new results on two problems concerning the dynamics of certain classes of interacting many-body systems in quantum mechanics. Chapter 1 is an introduction, which includes mathematical preliminaries and a summary of the results.

In Chapter 2 we introduce the notion of *transmission time* to study the dynamics of disordered quantum spin chains and prove results relating its behavior to many-body localization properties. We also study two versions of the so-called Local Integrals of Motion (LIOM) representation of spin chain Hamiltonians and their relation to dynamical many-body localization. We prove that uniform-in-time dynamical localization expressed by a zero-velocity Lieb-Robinson bound implies the existence of a LIOM representation of the dynamics as well as a weak converse of this statement. We also prove that for a class of spin chains satisfying a form of exponential dynamical localization, sparse perturbations result in a dynamics in which transmission times diverge at least as a power law of distance, with a power for which we provide lower bound that diverges with increasing sparseness of the perturbation.

In Chapter 3 we introduce a class of UV-regularized two-body interactions for fermions in $\mathbb{R}^d$ and prove a Lieb-Robinson estimate for the dynamics of this class of many-body systems. As a step toward this result, we also prove a propagation bound of Lieb-Robinson type for Schrödinger operators. We apply the propagation bound to prove the existence of infinite-volume dynamics as a strongly continuous group of automorphisms on the CAR algebra.

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CHAPTER 1

Introduction

Quantum mechanics is the mathematical theory that describes the microscopic world at low energy, i.e. it applies to particles moving at speeds which are small relative to the speed of light. Quantum mechanics, and its relativistic extensions described by quantum field theories, are to date arguably the most successful physical theories ever devised. Some of our most indispensable technologies are possible because of our understanding of quantum mechanics; examples include the transistor, the microchip, the laser and magnetic resonant imaging (MRI).

In this dissertation we are interested in many-body quantum systems. Roughly speaking, by a many-body system we mean a model, or a collection of models, with a large number of degrees of freedom which are interacting. One is inevitable led to such models in the study of bulk matter. We study two classes of many-body quantum models in this thesis: quantum spin systems and many-body fermion models in the continuum. Quantum spin systems model systems of particles, called spins, which are constrained to lie at fixed points in space. The other class of models describe particles, e.g. electrons, which can move throughout d-dimensional space. The mathematical preliminaries of quantum mechanics, quantum spin systems and Fermion systems will be presented in this chapter.

In this thesis we present two main groups of results: The existence of a strongly continuous dynamics for a class of many-body fermion systems in $\mathbb{R}^d$, and some results concerning slow propagation in some disordered quantum spin chains. A common theme in our results is a tool known as a Lieb-Robinson bound. Lieb-Robinson bounds provide a bound on the speed of propagation of information under the dynamics of a quantum mechanical system. Such bounds have been extensively developed for quantum spin systems, where they have been instrumental in proving a number of important results. The Lieb-Robinson bounds for fermions in the continuum presented in this thesis are the first of their kind.

1.1. Mathematical Formulations of Quantum Mechanics

There are several mathematically rigorous formulations of quantum mechanics, all of which are equivalent in a certain sense. We will not attempt to give a survey of them all here, rather we will only present the two formulations that are directly used in this thesis. First we present the Hilbert space approach to quantum mechanics, then the approach based on C^*-algebras. Afterwards we discuss dynamics.

1.1.1. The Hilbert Space Formulation. A quantum system is described by a complex Hilbert space $\mathcal{H}$. The bounded linear operators on $\mathcal{H}$, denoted by $\mathcal{B}(\mathcal{H})$ describe the physical observables of the system, i.e. the quantities that in principle can be measured in the lab (In physics observables refer to *self-adjoint* linear operators. In many important applications it is also necessary to consider observables which are unbounded operators). Typical examples of observables are position, energy or angular momentum. The state of the system is then encoded as a positive, normalized linear functional on $\mathcal{B}(\mathcal{H})$. A linear functional $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is

1. **positive** if $\omega(AA^*) \geq 0$ for every $A \in \mathcal{B}(\mathcal{H})$,
2. **normalized** if $\omega(\mathbb{1}) = 1$, where $\mathbb{1}$ denotes the identity operator.

A positive, normalized linear functional ω is called a **state** (on $\mathcal{B}(\mathcal{H})$). It is a fact that a state is necessarily bounded with (operator) norm equal to 1. Given a state ω on $\mathcal{B}(\mathcal{H})$, the expectation value of an observable $A \in \mathcal{B}(\mathcal{H})$ is given by $\omega(A)$. Some authors require states to be weak* continuous, in which case there is a unique positive semi-definite trace class operator $\rho \in \mathcal{B}(\mathcal{H})$ satisfying $\mathrm{Tr}(\rho) = 1$ such that $\omega(A) = \mathrm{Tr}(\rho A)$. Such a ρ is called a **density operator**, and density operators are in 1-1 correspondence with the weak* continuous states through the mapping $\rho \mapsto \omega_\rho$, where $\omega_\rho(A) = \mathrm{Tr}(A\rho)$. Sometimes a density operator is also referred to as a state due to this correspondence. For quantum systems described by finite dimensional Hilbert spaces, all states have an associated density operator, which in this case is called a **density matrix**. The simplest example of a state is a **vector state**, which is defined by a density operator of the form $\rho = |\psi\rangle\langle\psi|$ for a unit vector $\psi \in \mathcal{H}$.

EXAMPLE 1.1.1. All elementary particles carry an intrinsic angular momentum, which is called spin. The spin state of an elementary particle is described by a finite dimensional Hilbert space

$\mathcal{H} = \mathbb{C}^n$. The simplest non-trivial quantum system is therefore described by the Hilbert space $\mathbb{C}^2$. We will refer to a quantum system described by a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$ as a spin.

1.1.2. The C^*-algebra formulation. There is another approach to formulating quantum mechanics, called the algebraic formulation. In this approach one forgoes the Hilbert space and begins with a C^*-algebra $\mathcal{A}$ which represents the observables of the system. States are defined to be positive, normalized linear functionals on $\mathcal{A}$. Given a state ω on $\mathcal{A}$ and an observable $A \in \mathcal{A}$, one still interprets $\omega(A)$ as the expectation value of the observable A when the system is in the state ω. The following theorem provides a connection between the two formulations of quantum mechanics presented.

THEOREM 1.1.1 (Gelfand-Naimark). *Every C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a norm closed, $*$-subalgebra of bounded operators on some Hilbert space $\mathcal{H}$.*

1.1.3. Dynamics. Given a Hilbert space $\mathcal{H}$ describing a quantum system the observable representing energy holds special significance. This observable is called the **Hamiltonian**, and it is typically denoted by H, and it may be time-dependent. For systems in which energy is conserved the Hamiltonian is time-independent. In many models of interest it is necessary to consider an unbounded Hamiltonian, in which case we require H to be a densely-defined self-adjoint operator. The significance of the Hamiltonian is that it determines the dynamics of the system. Given a time-independent Hamiltonian H, the functional calculus for densely-defined self-adjoint operators permits us to define, for each $t \in \mathbb{R}$, the unitary operator $U(t) : \mathcal{H} \to \mathcal{H}$ by $U(t) = e^{-itH}$. The maps $U(t)$ have the following properties:

1. **Strong continuity:** $\lim_{t\to 0} U(t)\psi = \psi$ for each $\psi \in \mathcal{H}$.
2. **Group property:** $U(t)U(s) = U(t+s)$ for every $t, s \in \mathbb{R}$.
3. **Strong differentiability** $\lim_{t\to 0} \frac{1}{t}(U(t) - \mathbb{1})\psi = -iH\psi$ for every $\psi \in \mathcal{D}(H)$

The collection $\{U(t) : t \in \mathbb{R}\}$ is called a strongly continuous one-parameter group of unitaries on $\mathcal{H}$. The **Heisenberg dynamics** on $\mathcal{B}(\mathcal{H})$ is defined to be the family of maps $\tau_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, where for each $t \in \mathbb{R}$, $\tau_t(A) = U(t)^* A U(t)$. The family $\{\tau_t : t \in \mathbb{R}\}$ also possesses a group property: $\tau_t \tau_s = \tau_{t+s}$ for every $t, s \in \mathbb{R}$. Given a state ω on $\mathcal{B}(\mathcal{H})$, the time evolution of ω is given by the family of states ω_t determined by $\omega_t(A) = \omega(\tau_t(A))$. In the case that ω is a vector state corresponding to a unit vector $\psi \in \mathcal{H}$ we have that for every $A \in \mathcal{B}(\mathcal{H})$,

$$\omega_t(A) = \langle \psi, U(t)^* A U(t)\psi \rangle = \langle U(t)\psi, AU(t)\psi \rangle. \tag{1.1}$$

Therefore we may define the time evolution of the unit vector ψ by $\psi(t) \equiv U(t)\psi$. Provided $\psi \in \mathcal{D}(H)$, $\psi(t)$ satisfies the **Schrödinger equation**:

$$\frac{d\psi}{dt} = -iH\psi \tag{1.2}$$

EXAMPLE 1.1.2. **The free particle in $\mathbb{R}^d$:** A free quantum particle in d-dimensional space is modeled by the Hilbert space $L^2(\mathbb{R}^d)$ with the Hamiltonian

$$H_0 = -\Delta, \tag{1.3}$$

where Δ is the Laplacian with domain $H^2(\mathbb{R}^d)$. Physically H_0 represents the observable of kinetic energy. Given a measurable set $B \subseteq \mathbb{R}^d$, the operator χ_B on $L^2(\mathbb{R}^d)$ given by, $(\chi_B\psi)(x) = \chi_B(x)\psi(x)$, where χ_B is the indicator function of B (we abuse notation by allowing χ_B to represent both the function and the operator), is the observable which asks whether the particle is located in the set B. Given a pure state ψ, the expected value of χ_B is

$$\langle \psi, \chi_B\psi \rangle = \int_{x \in B} |\psi(x)|^2 \mathrm{d}x. \tag{1.4}$$

The expected value of χ_B represents the probability that the particle is observed inside the set B. Therefore for this model we interpret $|\psi|^2$ as the probability density for the position of the particle.

EXAMPLE 1.1.3. **Schrödinger operator on $\mathbb{R}^d$:** A particle moving in d-dimensional space under the influence of a conservative force with potential function V is described by the Hilbert space $L^2(\mathbb{R}^d)$ and the Hamiltonian

$$H = -\Delta + V, \tag{1.5}$$

where we use V to denote the multiplication operator corresponding to the function V. The energy is composed of two parts, the kinetic energy described by the free particle Hamiltonian $-\Delta$, and the potential energy V. Operators of the form (1.5) are called **Schrödinger operators**. Under various assumptions on V, the operator in (1.5) can be shown to be essentially self adjoint on some domain. For example, if $\sup_{x\in\mathbb{R}^d} |V(x)| < \infty$, then H is self-adjoint on the Sobolev space $H^2(\mathbb{R}^d)$.

EXAMPLE 1.1.4. **A tight binding model:** Tight binding models are reductions of models of a particle in $\mathbb{R}^d$. In these models there is assumed to be an underlying lattice through which the particle moves. This lattice could model the atoms in a crystalline solid, for example. Typically this lattice is modeled by a metric graph. For this example we consider it to be modeled by $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. In a tight binding model the wave function is taken to be a function on $\mathbb{Z}^d$, instead of a function of a continuous variable in $\mathbb{R}^d$. This reduction is reasonable when the particle spends most of its time in close proximity of one of the lattice sites and the time spent jumping between lattice sites is small compared to the typical time the particle is bound to a site. The Hilbert space is then given by $\ell^2(\mathbb{Z}^d)$. Given a normalized $\psi \in \ell^2(\mathbb{Z}^d)$, $|\psi(x)|^2$ gives the probability of observing the particle at $x \in \mathbb{Z}^d$. The free particle in the tight binding model is described by the Hamiltonian

$$H_0 = -\Delta, \tag{1.6}$$

where Δ is the discrete Laplacian:

$$(\Delta\psi)(x) = \sum_{y\sim x}(\psi(y) - \psi(x)) \tag{1.7}$$

where $x \sim y$ if and only if $\|x - y\|_1 = 1$. Interactions between the particle and the lattice can be modeled by adding a multiplication operator V to the Hamiltonian: $(V\psi)(x) = V(x)\psi(x)$, where V physically represents the potential energy at the site x.

When the Hamiltonian is time-dependent the dynamics becomes slightly more complicated. Suppose that $H(t)$ is a time-dependent Hamiltonian, which we assume is bounded and norm continuous in t. The Schrödinger equation reads

$$i\frac{d\psi}{dt} = H(t)\psi(t) \tag{1.8}$$

$$\psi(0) = \psi_0 \tag{1.9}$$

If one can solve the operator valued equation:

$$U'(t) = -iH(t)U(t) \tag{1.10}$$

$$U(0) = \mathbb{1} \tag{1.11}$$

then it is easily verified that $U(t)$ is a unitary for every $t \in \mathbb{R}$, and $\psi(t) = U(t)\psi_0$ solves the Schrödinger equation. The solution to Eq. (1.10) exists and can be expressed using the absolutely convergent **Dyson series**

$$U(t) = \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} H(s_1)H(s_2)\cdots H(s_n)\mathrm{d}s_n\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1. \tag{1.12}$$

There is one last tool relating to dynamics that we will cover here. Often one has a Hamiltonian H_0 for which the dynamics is relatively well understood, and perturbations of it are considered. We call H_0 the free Hamiltonian, and H_I the perturbation. For simplicity we will assume that both H_0 and H_I are bounded operators. Then there is a decomposition of the dynamics generated by $H = H_0 + H_I$ given by,

$$e^{-itH} = e^{-itH_0}U(t), \tag{1.13}$$

where $U(t)$ is a unitary operator satisfying the following operator differential equation

$$U'(t) = -i\tau_t^{H_0}(H_I)U(t). \tag{1.14}$$

We see that $U(t)$ is the unitary operator generated by the time-dependent Hamiltonian $\tau_t^{H_0}(H_I)$. This decomposition of the dynamics is called the **interaction picture**.

1.2. Quantum Many-Body Models

In this section we develop the mathematics of the many-body models studied in this thesis. Before doing so, we must address how to form composite systems. Suppose we are given two quantum mechanical systems described by $\mathcal{H}_1$ and $\mathcal{H}_2$. We take it as an axiom that the system which is composed of $\mathcal{H}_1$ and $\mathcal{H}_2$ is described by the Hilbert space tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$: $\mathcal{H}_1 \otimes \mathcal{H}_2$. The bounded operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ also have a tensor product structure, and there is a special physical interpretation of operators tensored with the identity map. If $A \in \mathcal{B}(\mathcal{H}_1)$ is an observable, then the observable $A \otimes \mathbb{1}_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\mathbb{1}_2$ is the identity on $\mathcal{H}_2$, is interpreted as an observation of A on $\mathcal{H}_1$ as a subsystem of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Similarly, for $B \in \mathcal{B}(\mathcal{H}_2)$, $\mathbb{1}_1 \otimes B$, where $\mathbb{1}_1$ is the identity on $\mathcal{H}_1$, represents an observation of B on $\mathcal{H}_2$ as a subsystem. We will abuse notation and simply write $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes B$, where the position of $\mathbb{1}$ in the tensor product indicates which identity map it is.

1.2.1. Quantum Spin Systems. A **quantum spin** is a quantum mechanical system described by a finite dimensional Hilbert space. We will usually refer to a quantum spin simply as a **spin**. The **spin** of the system, s, is a half-integer: $s \in \{0, 1/2, 1, 3/2, ...\}$. The dimension of the corresponding Hilbert space is given by $2s + 1$. So, for example a spin 1/2 particle is described by a 2-dimensional Hilbert space.

To define a quantum spin system we start with a countable metric space (Γ, d), which we refer to as the lattice, and an assignment of each $x \in \Gamma$ to a finite-dimensional Hilbert space $\mathcal{H}_x$, called an **on-site Hilbert space**. By $\mathcal{P}_0(\Gamma)$ we denote the finite subsets of Γ. For every $\Lambda \in \mathcal{P}_0(\Gamma)$, the Hilbert space for subsystem on Λ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$. The corresponding algebra of observables is $\mathcal{B}(\mathcal{H}_\Lambda) = \mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x)$. If $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\Gamma)$ with $\Lambda_1 \subseteq \Lambda_2$, then there is a natural embedding of $\mathcal{A}_{\Lambda_1}$ into $\mathcal{A}_{\Lambda_2}$ given by

$$A \mapsto A \otimes \mathbb{1}_{\Lambda_2 \setminus \Lambda_1}, \tag{1.15}$$

where $\mathbb{1}_{\Lambda_2\setminus\Lambda_1} \in \mathcal{A}_{\Lambda_2\setminus\Lambda_1}$ is the identity operator. The map defined by Eq. (1.15) is in fact a $*$-isometry. The natural inclusions permit us to define the **algebra of local observables**, $\mathcal{A}_\Gamma^{\mathrm{loc}}$ as the inductive limit

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\in\mathcal{P}_0(\Gamma)} \mathcal{A}_\Lambda \tag{1.16}$$

A model is specified by assigning a Hamiltonian H_Λ to each finite volume $\Lambda \in \mathcal{P}_0(\Gamma)$. This provides a Heisenberg dynamics $\tau_t^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ given by $\tau_t^{H_\Lambda}(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$. In fact, since we can take $H_\Lambda \in \mathcal{A}_\Gamma^{\mathrm{loc}}$, the map $\tau_t^{H_\Lambda}$ is in fact a *-algebra automorphism on $\mathcal{A}_\Gamma^{\mathrm{loc}}$. The **quantum spin system** in finite volume Λ is then given by the triple $(\mathcal{H}_\Lambda, \mathcal{A}_\Lambda, H_\Lambda)$. In the case that $\Gamma = \mathbb{Z}$ (or a subinterval of $\mathbb{Z}$) we call the system a **quantum spin chain**. Generally speaking we are interested in models where the finite volume Hamiltonians are related in some way. A natural and convenient way to specify a model is through an **interaction**, which is a map $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_\Gamma^{\mathrm{loc}}$ with the property that for every $X \in \mathcal{P}_0(\Gamma)$,

$$\Phi(X) = \Phi(X)^* \in \mathcal{A}_X. \tag{1.17}$$

We can then define, for each $\Lambda \in \mathcal{P}_0(\Gamma)$, a **local Hamiltonian** by

$$H_\Lambda = \sum_{X \subseteq \Lambda} \Phi(X). \tag{1.18}$$

Physically, the term $\Phi(X)$ represents the contribution of $|X|$-body interactions between the spins situated on the sites in X to the total energy of the system. For example, if only 2-body interactions are present, then we will have $\Phi(X) \neq 0$ only if $|X| = 2$.

An important class of interactions are the finite range interactions. For simplicity consider $\Gamma = \mathbb{Z}$. An interaction is said to be **finite range** if there is an integer $R > 0$ such that the interaction can be expressed in the form

$$\Phi(X) = \begin{cases} h_x & \text{if } X = [x, x+R] \text{ for some } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{1.19}$$

In the case that $R = 1$ we call Φ a nearest neighbor interaction.

Example 1.2.1. **The XY Chain:** Let $\Gamma = \mathbb{Z}$. Consider a quantum spin system with $\mathcal{H}_x = \mathbb{C}^2$ for each $x \in \mathbb{Z}$. The Pauli spin matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.20}$$

For a matrix (or operator) $A \in \mathcal{B}(\mathbb{C}^2)$, let A_x denote the same operator acting on $\mathcal{H}_x$, so $A_x \in \mathcal{A}_{\{x\}} \subseteq \mathcal{A}_\Gamma^{\mathrm{loc}}$. So, for any $\Lambda \ni x$, A_x can be thought of as $A \otimes \mathbb{1}_{\Lambda \setminus \{x\}}$. The XY chain is the quantum spin model obtained from the interaction Φ defined by

$$\Phi(X) = \begin{cases} \mu_x((1+\gamma_x)\sigma_x^1 \otimes \sigma_{x+1}^1 + (1-\gamma_x)\sigma_x^2 \otimes \sigma_{x+1}^2) & \text{if } X = \{x, x+1\} \text{ for some } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{1.21}$$

where $(\mu_x)_{x\in\mathbb{Z}}$ and $(\gamma_x)_{x\in\mathbb{Z}}$ are real sequences. The local Hamiltonians on intervals $[a, b]$ are then given by

$$H_{[a,b]} = \sum_{x=a}^{b-1} \mu_x((1+\gamma_x)\sigma_x^1 \otimes \sigma_{x+1}^1 + (1-\gamma_x)\sigma_x^2 \otimes \sigma_{x+1}^2) \tag{1.22}$$

It is natural to ask whether we can define an infinite volume quantum spin system, i.e. can we discuss the limit at $\Lambda \uparrow \Gamma$ of the finite volume systems $(\mathcal{H}_\Lambda, \mathcal{A}_\Lambda, H_\Lambda)$ in some mathematically reasonable way. As a first step in this direction, we define the **quasi-local observables**, $\mathcal{A}_\Gamma$, to be the norm completion of $\mathcal{A}_\Gamma^{\mathrm{loc}}$:

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}^{\|\cdot\|} \tag{1.23}$$

A_Γ is then a C^*-algebra, which we take to be the algebra of observables of the infinite volume quantum spin system. We can similarly define $\mathcal{A}_\Lambda$ for any subset Λ of Γ, where Λ is not necessarily finite. The **support** of an observable $A \in \mathcal{A}_\Gamma$, denoted by $\mathrm{supp}(A)$, is defined to be the smallest subset X of Γ such that $A \in \mathcal{A}_X$ (if $A = \mathbb{1}$, we define $\mathrm{supp}(A) = \varnothing$). The physical interpretation of the support of an observable A is that A represents some observation of the spins at the lattice sites in $\mathrm{supp}(A)$. We now see the utility of the algebraic formulation of quantum mechanics. The algebra of observables for the infinite system has a natural definition in terms of the local algebras $\mathcal{A}_\Lambda$. On the other hand, there is no natural way to define the limit of the Hilbert spaces $\mathcal{H}_\Lambda$, as the tensor product of infinitely many Hilbert spaces is not well-defined. Now we look to dynamics and ask what we can say about the limit of $\tau_t^{H_\Lambda}$ as $\Lambda \uparrow \Gamma$. In general we cannot establish the existence of this limit, but if the limit exists strongly then we can show the limiting object has some nice properties:

Proposition 1.2.1. *Suppose $(\mathcal{H}_\Lambda, \mathcal{A}_\Lambda, H_\Lambda)$ is a family of finite quantum spin systems indexed by $\Lambda \in \mathcal{P}_0(\Gamma)$. Suppose there exists an increasing, exhaustive sequence $\Lambda_n \in \mathcal{P}_0(\Gamma)$ such that for* *each $A \in \mathcal{A}_\Gamma^{loc}$ and each $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \tau_t^{H_{\Lambda_n}}(A) \equiv \tau_t(A) \tag{1.24}$$

exists in the operator norm topology. Then

$$\lim_{n\to\infty} \tau_t^{H_{\Lambda_n}}(A) \equiv \tau_t(A) \tag{1.25}$$

exists for every $A \in \mathcal{A}_\Gamma$, and $\{\tau_t : t \in \mathbb{R}\}$ is a one-parameter group of automorphisms of $\mathcal{A}_\Gamma$. If the convergence in Eq. (1.24) is uniform for t in compact subsets of $\mathbb{R}$, then $\{\tau_t : t \in \mathbb{R}\}$ is also strongly continuous.

Proof. Since each $\tau_t^{H_\Lambda}$ is an isometry of $\mathcal{A}_\Gamma^{\text{loc}}$, a straightforward application of the bounded linear transformation theorem (see [**50**]) shows that $\tau_t^{H_\Lambda}$ extends to an automorphism of $\mathcal{A}_\Gamma$. For any $n, m \in \mathbb{N}$, and for any $A' \in \mathcal{A}_\Gamma^{\text{loc}}$ we have that

$$\begin{aligned} \|\tau_t^{H_{\Lambda_n}}(A) - \tau_t^{H_{\Lambda_m}}(A)\| &\le \|\tau_t^{H_{\Lambda_n}}(A - A')\| + \|\tau_t^{H_{\Lambda_m}}(A - A')\| + \|\tau_t^{H_{\Lambda_n}}(A') - \tau_t^{H_{\Lambda_m}}(A')\| \\ &\le 2\|A - A'\| + \|\tau_t^{H_{\Lambda_n}}(A') - \tau_t^{H_{\Lambda_m}}(A')\| \end{aligned} \tag{1.26}$$

Since $\mathcal{A}_\Gamma^{\text{loc}}$ is dense in $\mathcal{A}_\Gamma$ and $\tau_t^{H_{\Lambda_n}}(A')$ is a Cauchy sequence this shows that $\tau_t^{H_{\Lambda_n}}(A)$ is Cauchy, hence convergent. Since each $\tau_t^{H_\Lambda}$ is an automorphism of $\mathcal{A}_\Gamma$, it immediately follows that τ_t is an automorphism of $\mathcal{A}_\Gamma$. We must establish the group property and the strong continuity of $\{\tau_t : t \in \mathbb{R}\}$. Given $A \in \mathcal{A}_\Gamma$ and $t, s \in \mathbb{R}$, the inequality

$$\begin{aligned} \|\tau_t^{H_\Lambda}(\tau_s^{H_\Lambda}(A)) - \tau_t(\tau_s(A))\| &\le \|\tau_t^{H_\Lambda}(\tau_s^{H_\Lambda}(A) - \tau_s(A))\| + \|(\tau_t^{H_\Lambda} - \tau_t)(\tau_s(A))\| \\ &= \|\tau_s^{H_\Lambda}(A) - \tau_s(A)\| + \|(\tau_t^{H_\Lambda} - \tau_t)(\tau_s(A))\| \end{aligned} \tag{1.27}$$

allows us to conclude that $\tau_{t+s}(A) = \tau_t(\tau_s(A))$. Now assume the convergence is uniform for t in compact subsets of $\mathbb{R}$. Given the group property, it suffices to prove that τ_t is strongly continuous at $t = 0$. Given $A \in \mathcal{A}_\Gamma$, we have the inequality

$$\|\tau_t(A) - A\| \le \|(\tau_t - \tau_t^{H_\Lambda})(A)\| + \|\tau_t^{H_\Lambda}(A) - A\| \tag{1.28}$$

for each $\Lambda \in \mathcal{P}_0(\Gamma)$. Given $\epsilon > 0$, we may choose Λ such that $\|(\tau_t - \tau_t^{H_\Lambda})(A)\| \leq \epsilon$ for every $t \in [-1, 1]$ and then choose $\delta \in (0, 1)$ such that $|t| < \delta$ implies $\|\tau_t^{H_\Lambda}(A) - A\| < \epsilon$. This proves the strong continuity of τ_t. □

When the limit $\lim_{\Lambda \uparrow \Gamma} \tau_t^{H_\Lambda}$ exists, we refer to it as the **thermodynamic limit** of the dynamics. It is of great interest to know when this limit exists and when the resulting family of automorphisms forms a strongly continuous one-parameter family of automorphisms. A sufficient condition for the existence of this limit is that the family of local dynamics $\tau_t^{H_\Lambda}$ satisfy a locality condition called a Lieb-Robinson bound. Loosely speaking, a Lieb-Robinson bounds the speed at which the dynamics $\tau_t^{H_\Lambda}$ can spread out a local observable. To see what we mean by this, suppose $A \in \mathcal{A}_X$ for some $X \subseteq \Lambda \in \mathcal{P}_0(\Gamma)$. Regardless of how small X is, in general we will have $\mathrm{supp}(\tau_t^{H_\Lambda}(A)) = \Lambda$ for $t \neq 0$. Despite this, in many important situations the observable $\tau_t^{H_\Lambda}(A)$ can be well approximated by an observable supported on some set $X(t)$ which typically becomes larger as t does, but may be much smaller than Λ for small $|t|$. Before we can make this notion mathematically precise, we introduce the commutator and discuss how it can be used to measure locality. Given $A, B \in \mathcal{A}_\Gamma$, the **commutator**, $[A, B]$, of A and B is given by $[A, B] = AB - BA$. Suppose $A, B \in \mathcal{A}_\Gamma$ and $\mathrm{supp}(A) \cap \mathrm{supp}(B) = \varnothing$. Then it is easy to see that $[A, B] = 0$. There is an important converse to this fact: If $A \in \mathcal{A}_\Gamma$, and $[A, B] = 0$ for every $B \in \mathcal{A}_X$, then $\mathrm{supp}(A) \cap X = \varnothing$. In fact, we can take this result a step further with the following proposition:

PROPOSITION 1.2.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complex Hilbert spaces. Suppose that, for $\epsilon > 0$, $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ satisfies,*

$$\|[A, \mathbb{1} \otimes B]\| \leq \epsilon \|B\|, \tag{1.29}$$

for every $B \in \mathcal{B}(\mathcal{H}_2)$. Then there exists $A' \in \mathcal{B}(\mathcal{H}_1)$ with $\|A'\| \leq \|A\|$ such that

$$\|A' \otimes \mathbb{1} - A\| \leq \epsilon \tag{1.30}$$

A proof of proposition 1.2.2 can be found in [**68**]. The proposition enables us to study the spread of an observables support under the dynamics by studying commutator bounds. When the local Hamiltonians are generated by an interaction, Lieb-Robinson bounds follow from certain decay conditions on the interaction. These decay conditions are conveniently expressed in terms of a decay function called an **F-function**. An F-function for Γ is a function $F : [0,\infty) \to (0,\infty)$ satisfying the following properties

1. F is non-increasing
2. (uniformly integrable) $\|F\| \equiv \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) < \infty$
3. (convolution inequality) $C_F = \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{z\in\Gamma} F(d(x,z))F(d(z,y)) < \infty$

It is easy to verify that if F is an F-function for Γ and $g : [0,\infty) \to (0,\infty)$ is a concave function, then $F_g : [0,\infty) \to [0,\infty)$ given by $F_g(x) = e^{-g(x)}F(x)$ is also an F-function for Γ with $\|F_g\| \leq \|F\|$ and $C_{gF} \leq C_F$. We also note that if F is an F-function for Γ, then F is also an F-function for any subset of Γ.

Example 1.2.2. Suppose $\Gamma = \mathbb{Z}^\nu$ for some $\nu \in \mathbb{N}$. Then for any $\epsilon > 0$, the function

$$F(x) = \frac{1}{1 + x^{\nu+\epsilon}} \tag{1.31}$$

is an F-function for Γ.

Given an interaction Φ and an F-function F for Γ, we define the **F-norm** of Φ to be the quantity

$$\|\Phi\|_F \equiv \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{X\in\mathcal{P}_0(\Gamma):\\ x,y\in X}} \|\Phi(X)\|. \tag{1.32}$$

When $\|\Phi\|_F < \infty$ we say that Φ is **F-norm bounded**. When an interaction Φ satisfies an F-norm bound with an F-function of the form $F_\mu(x) = e^{-\mu x}F(x)$ for some $\mu > 0$, where F is some F-function, we say that Φ is **short range**.

The Φ-boundary of a set $X \in \mathcal{P}_0(\Gamma)$ is given by $\partial_\Phi X = \{x \in X : \exists Z \in \mathcal{P}_0(\Gamma) \text{ with } x \in Z, Z \cap X^c \neq \varnothing, \text{ and } \Phi(Z) \neq 0\}$. We are now ready to state a result on Lieb-Robinson bounds.

Theorem 1.2.1. *Suppose that Φ is an interaction which is F-norm bounded for an F-function F. Suppose that $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, where $X, Y \in \mathcal{P}_0(\Gamma)$ with $X \cap Y = \varnothing$. Then for any*

$\Lambda \in \mathcal{P}_0(\Gamma)$ *with* $X \cup Y \subseteq \Lambda$,

$$\|[\tau_t^{H_\Lambda}(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F}(e^{2\|\Phi\|_F C_F |t|} - 1)D(X,Y), \tag{1.33}$$

where

$$D(X,Y) = \min\left\{\sum_{x\in X}\sum_{y\in\partial_\Phi Y} F(d(x,y)), \sum_{y\in Y}\sum_{x\in\partial_\Phi X} F(d(x,y))\right\} \tag{1.34}$$

The following example investigates the consequences of this theorem for the case of short range interactions.

EXAMPLE 1.2.3. Suppose Φ is a short range interaction, with corresponding F-function $F(x) = e^{-\mu x}F(x)$ for some F-function F. Then theorem 1.2.1 holds, with

$$D(X,Y) \leq \|F\| \min\{|\partial_\Phi X|, |\partial_\Phi Y|\} e^{-\mu d(X,Y)},$$

It follows that for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ and $\Lambda \supseteq X \cup Y$,

$$\|[\tau_t^{H_\Lambda}(A), B]\| \leq \frac{2\|A\|\|B\|\|F\|}{C_F} \min\{|\partial_\Phi X|, |\partial_\Phi Y|\} e^{\mu(v|t| - d(X,Y))}, \tag{1.35}$$

(1.36)

where

$$v = \frac{2\|\Phi\|_F C_F}{\mu} \tag{1.37}$$

is called the **Lieb-Robinson velocity**. Suppose now that $\Lambda \in \mathcal{P}_0(\Gamma)$ is a large set, and $X \subseteq \Lambda$. We can imagine that X is a small subset of Λ. Fix $\epsilon > 0$, and for all $t \in \mathbb{R}$, define $X(t) = \{x \in \Lambda : d(x,X) \leq v|t| + \epsilon\}$. Let $A \in \mathcal{A}_X$. For any $B \in \mathcal{A}_{\Lambda\setminus X(t)}$ Eq. (1.35) implies that

$$\|[\tau_t^{H_\Lambda}(A), B]\| \leq \frac{2\|F\|\|A\|\|B\|}{C_F} \min\{|\partial_\Phi X|, |\partial_\Phi Y|\} e^{-\mu\epsilon} \tag{1.38}$$

An application of proposition 1.2.2 shows that there is an observable $A(t) \in A_{X(t)}$ such that

$$\|A(t) - A\| \leq \frac{2\|F\|\|A\|}{C_F} \min\{|\partial_\Phi X|, |\partial_\Phi Y|\} e^{-\mu\epsilon}. \tag{1.39}$$

This illustrates that for short range interactions, any disturbance which propagates faster than the Lieb-Robinson velocity is exponentially suppressed in the distance it is ahead of $v|t|$. It also shows that the dynamics can at most, up to a small error, grow the support of local observables linearly in time. When the supports of local observables grow under the dynamics in this way, we say the dynamics is **ballistic**.

A particular application of the Lieb-Robinson bound for quantum spin systems is the existence of a strongly continuous thermodynamic limit, which follows from the next theorem and Proposition 1.2.1

THEOREM 1.2.2. *Suppose that Φ is an interaction satisfying an F-norm bound. Then for any increasing, absorbing sequence (Λ_n) of finite subsets of Γ, any $t \in \mathbb{R}$, and any $A \in \mathcal{A}_\Gamma^{loc}$, the limit*

$$\lim_{n\to\infty} \tau_t^{H_{\Lambda_n}}(A) \tag{1.40}$$

exists, and the convergence is uniform for t in compact subsets of $\mathbb{R}$.

PROOF. Let $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$ and denote $\mathrm{supp}(A) = X$. Take $m \geq 1$ large enough so $X \subseteq \Lambda_m$. For any $n \geq m$ we have

$$\tau_t^{H_{\Lambda_n}}(A) - \tau_t^{H_{\Lambda_m}}(A) = \int_0^t \frac{d}{ds}\left(\tau_s^{\Lambda_n}(\tau_{t-s}^{H_{\Lambda_m}}(A))\right) \mathrm{d}s. \tag{1.41}$$

A simple computation shows that

$$\frac{d}{ds}\tau_s^{\Lambda_n}(\tau_{t-s}^{H_{\Lambda_m}}(A)) = i\tau_s^{H_{\Lambda_n}}\left([H_{\Lambda_n} - H_{\Lambda_m}, \tau_{t-s}^{H_{\Lambda_m}}(A)\right). \tag{1.42}$$

Therefore for $t > 0$,

$$\|\tau_t^{H_{\Lambda_n}}(A) - \tau_t^{H_{\Lambda_m}}(A)\| \leq \sum_{Z \in S_{\Lambda_n}(\Lambda_m)} \int_0^t \|[\tau_s^{\Lambda_m}(A), \Phi(Z)\|\mathrm{d}s \tag{1.43}$$

where

$$S_{\Lambda_n}(\Lambda_m) = \{Z \subseteq \Lambda_n : Z \cap \Lambda_m \neq \varnothing \text{ and } Z \cap \Lambda_m^c \neq \varnothing\}. \tag{1.44}$$

By dividing the sum on Z and applying Theorem 1.2.1,

$$\|\tau_t^{H_{\Lambda_n}}(A) - \tau_t^{H_{\Lambda_m}}(A)\| \leq 2\|A\|t \sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X\neq\varnothing}} \|\Phi(Z)\| \tag{1.45}$$

$$+ \frac{2\|A\|}{C_F}\int_0^t (e^{2C_F\|\Phi\|_F s}-1)\mathrm{d}s \sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X=\varnothing}} \|\Phi(Z)\sum_{\substack{x\in X\\ z\in Z}} F(d(x,z)) \tag{1.46}$$

Note that

$$\sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X\neq\varnothing}} \|\Phi(Z)\| \leq \sum_{x\in X}\sum_{z\in\Lambda_n\setminus\Lambda_m}\sum_{\substack{Z\subseteq\Lambda_n\\ x,z\in Z}} \|\Phi(Z)\| \tag{1.47}$$

and

$$\sum_{\substack{Z\in S_{\Lambda_n}(\Lambda_m):\\ Z\cap X=\varnothing}} \|\Phi(Z)\|\sum_{z\in Z}\sum_{\substack{x\in X\\ z\in Z}} F(d(x,z)) \leq \sum_{x\in X}\sum_{z'\in\Lambda_n\setminus\Lambda_m}\sum_{z\in\Lambda_n} F(d(x,z))\sum_{\substack{Z\subseteq\Lambda_n:\\ z,z'\in Z}} \|\Phi(Z)\| \tag{1.48}$$

$$\leq \|\Phi\|_F \sum_{x\in X}\sum_{z'\in\Lambda_n\setminus\Lambda_m}\sum_{z\in\Lambda_n} F(d(x,z))F(d(z,z')) \tag{1.49}$$

$$\leq C_F \sum_{x\in X}\sum_{z'\in\Lambda_n\setminus\Lambda_m} F(d(x,z')). \tag{1.50}$$

It follows that

$$\|\tau_t^{H_{\Lambda_n}}(A) - \tau_t^{H_{\Lambda_m}}(A)\| \leq 2\|A\|\|\Phi\|_F\left(\int_0^t e^{2\|\Phi\|_F C_F s}\mathrm{d}s\right)\sum_{x\in X}\sum_{z\in\Lambda_n\setminus\Lambda_m} F(d(x,z)). \tag{1.51}$$

Given the F-function conditions, this shows that $\tau_t^{H_{\Lambda_n}}(A)$ is a Cauchy sequence in $\mathcal{A}_\Gamma$, uniformly for t in compact subsets of $\mathbb{R}$. □

1.2.2. Many-Body Fermion Systems. We begin by discussing the model of a single fermion, and then define the Fock space which is a model in which the number of fermion particles can be arbitrary.

A quantum particle moving in d-dimensional space is described by the Hilbert space $L^2(\mathbb{R}^d)$. If the particle moves through a force field generated by a potential energy function $V:\mathbb{R}^d\to\mathbb{R}$, then the Hamiltonian H is given by

$$H = -\Delta + V, \tag{1.52}$$

where Δ is the Laplacian and we abuse notation by using the symbol V to refer to the multiplication operator on $L^2(\mathbb{R}^d)$ given by $(Vf)(x) = V(x)f(x)$. Some assumptions will need to be made about V that ensure the operator (1.52) is self-adjoint (or at least essentially self-adjoint) on some dense subspace of $L^2(\mathbb{R}^d)$. For example, if $V \in L^\infty(\mathbb{R}^d)$ then H is self-adjoint on the domain $H^2(\mathbb{R}^d)$. Later we will be precise regarding our assumptions on V.

In macroscopic systems it is impossible to know how many microscopic particles make up the system. It therefore seems reasonable to consider a quantum system in which the number of particles is itself an observable. The quantum system of n particles with Hilbert space $L^2(\mathbb{R}^d)$ is given by the n-fold Hilbert space tensor product

$$L^2(\mathbb{R}^d)^{\otimes n}, \tag{1.53}$$

which is naturally isomorphic to the space $L^2(\mathbb{R}^{dn})$. We will often represent an element $\psi \in L^2(\mathbb{R}^d)^{\otimes n}$ as a function $\psi(x_1, x_2, ..., x_n)$ in n d-dimensional coordinates $x_i \in \mathbb{R}^d$. The coordinate x_i can be thought of as representing the location of the ith particle. Subatomic particles of a given species are, as far as we can tell, indistinguishable. Therefore physically relevant wave functions for n identical particles should have a symmetry under particle permutations which preserves the physical information of the wave function. For identical fermions, this symmetry is anti-symmetry of the wave function: an n-particle fermion wave function ψ must satisfy

$$\psi(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = \operatorname{sgn}(\sigma)\psi(x_1, x_2, ..., x_n) \tag{1.54}$$

for every $(x_1, x_2, ..., x_n) \in (\mathbb{R}^d)^n$. Therefore the Hilbert space for n-identical fermions is actually $(L^2(\mathbb{R}^d)^{\otimes n})^-$, the antisymmetric n-fold tensor product. A system of identical fermions where the particle number is itself an observable is given by the **(antisymmetic) Fock space $\mathfrak{F}^-$**,

$$\mathfrak{F}^- = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^d)^{\otimes n})^-, \tag{1.55}$$

where $L^2(\mathbb{R}^d)^{\otimes 0} \equiv \mathbb{C}$ is the **vacuum** Hilbert space, representing a state of the system with no particles. $(L^2(\mathbb{R}^d)^{\otimes n})^-$ naturally embeds onto a subspace of $\mathfrak{F}^-$, which we call the n-particle subspace. It is sometimes useful to consider $\mathfrak{F}^-$ as a closed subspace of the full Fock space $\mathfrak{F}$, given by

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^d)^{\otimes n}. \tag{1.56}$$

For each $n \in \mathbb{N}$, consider the operator $A_n : L^2(\mathbb{R}^d)^{\otimes n} \to (L^2(\mathbb{R}^d)^{\otimes n})^-$ given by,

$$(A_n f)(x_1, \cdots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(x_{\sigma(1)}, ..., x_{\sigma(n)}) \tag{1.57}$$

It is easy to show that A_n is an orthogonal projection. If we take $A_0 = \mathbb{1}$, then the antisymmetrization operator $A : \mathfrak{F} \to \mathfrak{F}^-$ is the orthogonal projection given by

$$A = \bigoplus_{n=0}^{\infty} A_n. \tag{1.58}$$

In Fock space the particle number is itself an observable, and its corresponding operator N is the self-adjoint operator with domain

$$\mathcal{D}(N) = \{(\psi_n) \in \mathfrak{F} : \sum_{n=0}^{\infty} n^2 \|\psi_n\|^2 < \infty\} \tag{1.59}$$

given by,

$$N(\psi_n) = (n\psi_n). \tag{1.60}$$

We call N the **number operator**. It is clear that N restricts to a self-adjoint operator on $\mathfrak{F}^-$.

We now discuss ways to construct models on Fock space. Suppose we are given a self-adjoint operator H_1 with domain $\mathcal{D}$ on $L^2(\mathbb{R}^d)$, representing the energy of a single particle. Consider the operator

$$H_n' = H_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes H_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \mathbb{1} \otimes H_1 \tag{1.61}$$

acting on the domain $\widehat{\bigotimes}_{k=1}^{n}\mathcal{D}$, where $\widehat{\bigotimes}$ indicates the *algebraic* tensor product of the vector spaces $\mathcal{D}$. It is possible to show that H_n' is closable, and its closure $H_n = \overline{H_n'}$ is a self-adjoint operator on $(L^2(\mathbb{R}^d))^{\otimes n}$ (e.g. see section VIII.10 on tensor products of unbounded operators in [**76**]). H_n represents the Hamiltonian of n non-interacting particles where the energy of the single particles is modeled by H_1. We can then define the **second quantization** of H_1, $d\Gamma(H_1)$ on $\mathfrak{F}$ by

$$d\Gamma(H_1) = \bigoplus_{n=0}^{\infty} H_n, \tag{1.62}$$

taking $H_0 = 0$. $d\Gamma(H_1)$ is a self-adjoint operator on $\mathfrak{F}$, physically representing a non-interacting system of particles where each particle has energy described by H_1. It is not hard to see that $d\Gamma(H_1)$ is reduced by the subspace $\mathfrak{F}^-$. We will abuse notation slightly and use $d\Gamma(H_1)$ to denote the restriction to $\mathfrak{F}^-$. Later we will discuss some ways to add interactions between particles to the model.

There are a few distinguished operators on $\mathfrak{F}$, called **creation and annihilation operators**, which we now discuss. Given $\varphi \in L^2(\mathbb{R}^d)$, denote by $b(\varphi)$ and $b^*(\varphi)$ the operators on $\mathfrak{F}$ which act on $\psi = (\psi_n) \in \mathfrak{F}$ by,

$$(b(\varphi)\psi)_n(x_1, ..., x_n) = \sqrt{n+1}\int_{\mathbb{R}^d} \overline{\varphi(x)}\psi_{n+1}(x, x_1, ..., x_{n+1})\mathrm{d}x \tag{1.63}$$

and

$$(b^*(\varphi)\psi)_n(x_1, ..., x_n) = \sqrt{n}\varphi(x_1)\psi_{n-1}(x_2, ..., x_n). \tag{1.64}$$

The operator $b^*(\varphi)$ creates a particle in the state φ and its adjoint $b(\varphi)$ annihilates a particle. These are densely defined, unbounded closed operators on the full Fock space $\mathfrak{F}$. We define the creation and annihilation operators restricted to $\mathfrak{F}^-$ by

$$a(\varphi) = Ab(\varphi)A \text{ and } a^*(\varphi) = Ab^*(\varphi)A. \tag{1.65}$$

It is easy to check that $\mathfrak{F}^-$ is invariant under $a(\varphi)$, so in fact

$$a(\varphi) = b(\varphi)A \text{ and } a^*(\varphi) = Ab^*(\varphi). \tag{1.66}$$

Given $\psi \in \mathfrak{F}^-$,

$$(a(\varphi)\psi)_n(x_1, ..., x_n) = \sqrt{n+1} \int_{\mathbb{R}^d} \overline{\varphi(x)} \psi_{n+1}(x, x_1, ..., x_n) \mathrm{d}x \tag{1.67}$$

and

$$(a^*(\varphi)\psi)_n(x_1, ..., x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i-1} \varphi(x_i) \psi_{n-1}(x_1, ..., \hat{x_i}, ..., x_n) \tag{1.68}$$

where $\hat{x_i}$ indicates that the variable x_i is absent. The creation and annihilation operators on $\mathfrak{F}^-$ satisfy the **canonical anticommutation relations (CAR)**:

$$\{a^*(\varphi), a^*(\psi)\} = \{a(\varphi), a(\psi)\} = 0 \tag{1.69}$$

$$\{a^*(\varphi), a(\psi)\} = \langle \varphi, \psi \rangle \mathbb{1} \tag{1.70}$$

where $\{A, B\} = AB + BA$ is the anticommutator of the operators A and B. The CAR relations imply that $(a^*(\varphi))^2 = 0$ for every $\varphi \in L^2(\mathbb{R}^d)$, thus making it impossible to create two fermions in the same state. In physics this is referred to as the Pauli exclusion principle. It follows from the CAR relations that the creation and annihilation operators are bounded on $\mathfrak{F}^-$, in fact $\|a(\varphi)\| = \|a^*(\varphi)\| = \|\varphi\|$.

Suppose that we have a Hamiltonian on $\mathfrak{F}^-$ given by second quantization: $H = d\Gamma(H_1)$ for some one particle Hamiltonian H_1. The Heisenberg dynamics generated by $d\Gamma(H_1)$ has a particularly simple action on elements of the CAR algebra. Let τ_t denote the Heisenberg dynamics generated by $d\Gamma(H_1)$ on $\mathcal{B}(\mathfrak{F}^-)$. It is not difficult to show that for any $\varphi \in L^2(\mathbb{R}^d)$,

$$\tau_t(a(\varphi)) = a(e^{-itH_1}\varphi). \tag{1.71}$$

Therefore the Heisenberg dynamics on the CAR algebra is completely determined by the one-particle evolution. Since the unitary e^{-itH_1} is strongly continuous, the map $t \mapsto \tau_t(a(\varphi))$ is operator norm continuous.

We will see that many interesting operators can be expressed in terms of the creation and annihilation operators. The norm closure of the algebra generated by $\{a(\varphi), a^*(\varphi) : \varphi \in L^2(\mathbb{R}^d)\}$ is called the **CAR algebra**. We may also introduce the *fields* a_x and a_x^* whose actions are given by

$$(a_x\psi)_n(x_1,...,x_n) = \sqrt{n+1}\psi_{n+1}(x,x_1,...,x_n) \tag{1.72}$$

and

$$(a_x^*\psi)_n(x_1,....,x_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(-1)^{i-1}\delta(x-x_i)\psi_{n-1}(x_1,...,\hat{x}_i,...,x_n) \tag{1.73}$$

a_x and a_x^* can be realized rigorously as operator valued distributions, which produce an operator after integration against a function:

$$a(\varphi) = \int_{\mathbb{R}^d}\overline{\varphi(x)}a_x\mathrm{d}x \tag{1.74}$$

$$a^*(\varphi) = \int_{\mathbb{R}^d}\varphi(x)a_x^*\mathrm{d}x \tag{1.75}$$

Formally, using Eqs. (1.67) and (1.68) $a_x = a(\delta_x)$ and $a_x^* = a^*(\delta_x)$. The number operator N can formally be expressed in terms of the fields by,

$$N = \int_{\mathbb{R}^d} a_x^*a_x\mathrm{d}x, \tag{1.76}$$

as a simple calculation shows. Given a measurable set $\Lambda \subseteq \mathbb{R}^d$ the observable of the number of particles in Λ, N_Λ, can be expressed as

$$N_\Lambda = \int_\Lambda a_x^*a_x\mathrm{d}x \tag{1.77}$$

More generally, given a multiplication operator V acting on $L^2(\mathbb{R}^d)$, we can formally write

$$\int_{\mathbb{R}^d} V(x)a_x^*a_x\mathrm{d}x \tag{1.78}$$

It is not difficult to see that this is the same as the second quantization, $d\Gamma(V)$, of V. Therefore the number operator N_Λ is the second quantization of the multiplication operator χ_Λ, which is the one particle observable of the particle being in the region Λ.

Given a one-particle Hamiltonian H_1 on $L^2(\mathbb{R}^d)$, the second quantization $d\Gamma(H_1)$ represents the energy of the system of non-interacting particles each of which has energy described by H_1. A more realistic model will incorporate interactions between the particles into the Hamiltonian. A common type of interaction is a two-body interaction, where the potential energy only depends on the relative separation of pairs of particles. Given an n-particle wave function $\psi \in (L^2(\mathbb{R}^d))^{\otimes n}$, such an interaction may have the form

$$(W\psi)(x_1, ..., x_n) = \sum_{i<j} W(x_i - x_j)\psi(x_1, ..., x_n), \tag{1.79}$$

where W is a real-valued function on $\mathbb{R}^d$ satisfying $W(x) = W(-x)$. Formally, the operator W acting on the Fock space $\mathfrak{F}^-$ may be written as a double integral:

$$W = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)a_x^* a_y^* a_y a_x \, \mathrm{d}x\mathrm{d}y \tag{1.80}$$

We obtain an interacting model H by adding the interactions W to the original second quantization $d\Gamma(H_1)$: $H = d\Gamma(H_1) + W$. In general certain assumptions will need to be made on W in order to guarantee that the operator H is essentially self-adjoint on some dense subspace of $\mathfrak{F}^-$.

1.3. Summary of Results

Here we provide a summary of the results presented in the subsequent chapters of this thesis. Chapter 2 of this dissertation is the paper *slow propagation in some disordered quantum spin chains* [**66**], on which I was coauthor with my advisor Bruno Nachtergaele. Chapter 3 is the paper *Lieb-Robinson bounds and strongly continuous dynamics for a class of many-body fermion models in* $\mathbb{R}^d$ [**41**], on which I was coauthor with Martin Gebert, Bruno Nachtergaele and Robert Sims. Additional background is discussed in the introductory sections of the respective chapters.

1.3.1. Summary of Chapter 2 Results. The results of chapter 2 concern a phenomena called ‘localization’ which we have not yet discussed. Localization in quantum mechanics is a phenomena typically associated with *disordered* systems. Roughly, localization refers to an absence of transport. Unfortunately it is impossible to give a precise definition of localization in quantum mechanics. The issue is that what constitutes ‘transport’ in quantum mechanics is model dependent. Even in a particular model, there may be several distinct ways of quantifying transport. The study of localization in quantum mechanics began in 1958 when the physicist Philip W. Anderson introduced what is now called the Anderson model [**7**]. The Anderson model is a tight binding model given by the Hamiltonian

$$H_\lambda = -\Delta + \lambda V_\omega \tag{1.81}$$

acting on $\ell^2(\mathbb{Z}^d)$. Here $\lambda \in \mathbb{R}$ and V_ω is a random multiplication operator. For $\psi \in \ell^2(\mathbb{Z}^d)$, it acts as $(V_\omega \psi)(x) = \omega_x \psi(x)$, where $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ is a sequence of real valued random variables. If we consider ω to be an element of $\prod_{x\in\mathbb{Z}^d}\mathbb{R}$, then an Anderson model is precisely determined by specifying a probability measure on $\prod_{x\in\mathbb{Z}^d}\mathbb{R}$. A common choice is the product measure determined by a fixed distribution, so that $(\omega_x)_{x\in\mathbb{Z}^d}$ is an i.i.d. sequence. The following theorem is well known [**5**]:

THEOREM 1.3.1. *For the Anderson model 1.81, suppose that the sequence $(\omega_x)_{x\in\mathbb{Z}^d}$ is i.i.d., and that ω_0 has absolutely continuous distribution with compact support. Then there is a number $\lambda_d \geq 0$ such that if $|\lambda| > \lambda_d$ there are positive constants C and η such that*

$$\sum_{\substack{y\in\mathbb{Z}^d:\\ \|y-x\|_2\geq R}} \mathbb{E}\big[\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\lambda}\delta_x\rangle|\big] \leq Ce^{-\eta R} \tag{1.82}$$

holds for all $R > 0$. Furthermore, if $d = 1$, then $\lambda_d = 0$.

The property expressed by Eq. (1.82) is called (exponential) dynamical localization. Given a particle initially located at $x \in \mathbb{Z}$, the probability of observing it at site $y \in \mathbb{Z}$ at time t is given by $|\langle \delta_y, e^{-itH_\lambda}\delta_x\rangle|^2$. Thus property Eq. (1.82) says that the probability of observing a particle which started at $x \in \mathbb{Z}$ a distance R away decays exponentially in R. In other words, the particle is localized under the dynamics. To understand the significance of this effect we investigate the $\lambda = 0$ case, when H_λ is the Hamiltonian for the free particle. For $\lambda = 0$, $H_\lambda = -\Delta$, and by use of Fourier series it is possible to explicitly show that for any normalized $\psi \in \ell^2(\mathbb{Z}^d)$ with finite second moments,

$$\langle e^{-itH_0}\psi, X^2 e^{-itH_0}\psi\rangle = O(t^2) \tag{1.83}$$

as $t \to \infty$, where X^2 is the squared position operator $X^2\delta_x = \|x\|_2^2\delta_x$. Therefore for the free particle expected distance squared of the particle from the origin increases like t^2, regardless of the initial spatial probability distribution. This dynamical effect is called ballistic propagation. Going further, using the fact that H_0 has absolutely continuous spectrum one can show that $\lim_{t\to\infty} |\langle \delta_y, e^{-itH_0}\psi\rangle| = 0$ (see Theorem 2.6 in [**5**] and the discussion thereafter). It follows that the probability of observing the particle in any bounded region of $\mathbb{Z}^d$ tends to 0 as $t \to \infty$.

Anderson's insight was that a disordered potential landscape can lead to a complete absence of propagation under the dynamics. Besides dynamical localization, there are several other notions of localization for the tight binding model, which are discussed in depth in [**5**]. Several of these involve the structure of the Hamiltonian spectrum or eigenvalues. In this dissertation we focus entirely on dynamics, and so do not go into these other notions of localization.

The Anderson model is a single particle model, and it is natural to ask whether analogous results hold in many-body systems. To date, the effects of disorder on quantum many body systems are not well understood, physically or mathematically. One of the challenges is to first determine what the appropriate localization properties are in the many-body setting. In the Anderson model the absence of particle transport indicated localization. In a quantum spin system, for example, the particles are fixed in space, so a different notion of localization is needed. One idea is that it more natural to look at localization under the Heisenberg dynamics for quantum spin systems. Here, localization would be indicated by the support of time evolved local observables spreading extremely slowly, or not at all. The work presented here is concerned only with dynamical localization, so we will not discuss other ways in which quantum many-body systems may be localized. The phenomena of localization in quantum many-body systems is referred to as many-body localization (MBL). For a review article including a discussion of MBL indicators for quantum spin systems, see [**1**].

We adopt the following definition of dynamical localization for a quantum spin chain:

Definition 1.3.1. Let $F : \mathbb{Z}_+ \to (0, \infty)$ be a non-increasing function with the property $\lim_{x\to\infty} F(x) = 0$. We say that a family $\{H_\Lambda : \Lambda \subset \mathbb{Z} \text{ finite intervals}\}$ of random local Hamiltonians $H_\Lambda \in \mathcal{A}_\Lambda$ exhibits dynamical localization with decay function F if there exists a constant

$\beta \geq 0$ and a function $\chi : \mathbb{N} \to (0, \infty)$ such that for any sets $X, Y \subseteq \Lambda$ with $Y \subset [\min X, \max X]^c$,

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \sup_{\substack{A\in\mathcal{A}_X^1 \\ B\in\mathcal{A}_Y^1}} \frac{\|[\tau_t^{H_\Lambda}(A), B]\|}{\chi(|X|)(1+|t|^\beta)}\right] \leq F(d(X,Y)). \tag{1.84}$$

Here $d(X,Y) = \min\{|x-y| : x \in X, y \in Y\}$ is the usual set distance. We call dynamical localization with decay function F, where $F(x) \propto e^{-\eta x}$ for some $\eta > 0$, exponential dynamical localization.

Example 1.3.1. **The Disordered XY Chain:** The isotropic XY in an external, disordered field in the $\hat{z}$ direction is a spin 1/2 chain with the following family of Hamiltonians

$$H_{[a,b]} = \sum_{x=a}^{b-1} (\sigma_x^1 \otimes \sigma_{x+1}^1 + \sigma_x^2 \otimes \sigma_{x+1}^x) + \lambda \sum_{x=a}^{b} \omega_x \sigma_x^3, \tag{1.85}$$

where $\lambda \in \mathbb{R}$, $a < b$, and $(\omega_x)_{x\in\mathbb{Z}}$ is a sequence of random variables. If the sequence $(\omega_x)_{x\in\mathbb{Z}}$ is i.i.d. with an absolutely continuous, compactly supported distribution then the family $\{H_{[a,b]}\}$ is known to be exponentially dynamically localized according to definition 1.3.1 [**46**].

Consider a quantum spin chain with algebra of (quasilocal) observables $\mathcal{A}_\mathbb{Z}$. We wish to quantify the speed of propagation of a family of Hamiltonians on finite subsystems. The following definition provides a tool for this:

Definition 1.3.2. Given a sequence of Hamiltonians $H_n \in \mathcal{A}_{[0,n]}$ and a sequence of positive numbers (ϵ_n), define the transmission time, $t_n(\epsilon_n)$ of H_n as,

$$t_n(\epsilon_n) = \inf\{|t| : \sup_{\substack{A\in\mathcal{A}_0^1 \\ B\in\mathcal{A}_n^1}} \|[\tau_t^{H_n}(A), B]\| > \epsilon_n\}. \tag{1.86}$$

In essence, the transmission time $t_n(\epsilon_n)$ measures the time required for a disturbance in the chain to propagate and reach strength ϵ_n a distance n away. One can imagine that an instrument is used to measure when the signal from site 0 arrives at site n, but this instrument has a sensitivity and can only detect signals with strength ϵ_n. It is trivial to see that $t_n(\epsilon)$ is monotone increasing in ϵ.

We investigate the behavior of the transmission time for a system satisfying a typical Lieb-Robinson bound and for a system exhibiting dynamical localization. Suppose that the sequence

H_n satisfies,

$$\sup_{\substack{A\in\mathcal{A}_x^1\\ B\in\mathcal{A}_y^1}} \|[\tau_t^{H_n}(A), B]\| \le C(e^{\mu v|t|}-1)e^{-\mu|x-y|} \tag{1.87}$$

for $x \neq y$, uniformly in n. The bound implies that,

$$t_n(\epsilon_n) \ge \frac{1}{\mu v}\log(1+\frac{\epsilon_n e^{\mu n}}{C}), \tag{1.88}$$

in which case

$$\limsup_{n\to\infty} \frac{n}{t_n(\epsilon_n)} \le v \tag{1.89}$$

provided ϵ_n decays subexponentially in n. This shows that for large n, $t_n(\epsilon_n) \gtrsim \frac{n}{v}$, so the bound on the speed of propagation puts a lower bound on the transmission time, and we see that the transmission time grows at least linearly in n.

The following proposition considers what happens when the system is dynamically localized.

PROPOSITION 1.3.1. *Suppose that a sequence $H_n \in \mathcal{A}_{[0,n]}$ of random Hamiltonians exhibits dynamical localization with decay function F given by $F(x) = e^{-\eta x^\rho}$ for some $\rho \in (0,1]$. Then for any positive γ and α such that $\beta\gamma + \alpha < 1$,*

$$\frac{e^{\gamma\eta n^\rho}}{t_n(e^{-\alpha\eta n^\rho})} \to 0 \tag{1.90}$$

almost surely.

In the previous proposition, we see that for dynamically localized systems where the decay function F is a stretched exponential, the transmission time grows at least as fast as a stretched exponential in n, which demonstrates that transport is extremely slow.

We now present one of the main results of Chapter 2. In the case of exponential dynamical localization there exists a family of perturbations such that the perturbed model still has long transmission times. These perturbations consist of additional nearest neighbor interactions that occur with low density at random positions. For this class of perturbations we can prove that the transmission time grows super linearly provided the perturbations are sufficiently sparse.

Theorem 1.3.2. *Let $H_n^0 \in \mathcal{A}_{[0,n]}$ be a sequence of random Hamiltonians defined over the probability space $(\Omega_0, \mathrm{Pr}_0)$ which are exponentially dynamically localized in the sense of Definition 2.2.1 ($\rho = 1$). Let $(\delta_x)_{x=0}^\infty$ be an i.i.d. sequence of Bernoulli random variables over the probability space $(\Omega_1, \mathrm{Pr}_1)$, with $\mathrm{Pr}_1(\delta_0 = 0) = p \in (0, 1]$. Let $(\psi_x)_{x=0}^\infty$ denote a uniformly bounded sequence with $\psi_x \in \mathcal{A}_{[x,x+1]}$ for all x. Consider the sequence of random Hamiltonians*

$$H_n(\omega) = H_n^0(\omega_0) + \sum_{x=0}^{n-1} \delta_x(\omega_1)\psi_x; \tag{1.91}$$

over the probability space $\Omega_0 \times \Omega_1$ equipped with the product measure. If t_n is the transmission time of H_n, then for any $\gamma > 0$ and $\alpha \in (0, 1/3)$ satisfying

$$\eta \left(\frac{1 - 3\alpha}{1 - \alpha} \right) > 2[(\beta + 1)\gamma - 1] \log \left(\frac{1}{p} \right), \tag{1.92}$$

$$\frac{n^\gamma}{t_n(e^{-\alpha\eta n})} \to 0 \tag{1.93}$$

in probability.

This theorem can also be generalized to apply directly in the thermodynamic limit under certain reasonable assumptions (see Theorem 2.2.4).

There is another way of thinking about many-body localization that has gained some traction in the physics literature. Here, many-body localization is thought to emerge from the existence of of an extensive set of local conserved quantities (observables) called local integrals of motion (LIOMs). Heuristic definitions of LIOMs have been given in the physics literature [**51**], [**82**] and LIOMs are thought to account for most (if not all) of the phenomena of MBL [**53**].

In this thesis we provide two different rigorous definitions of LIOMs and explore the relationship between LIOMs and dynamical localization. Specifically, we prove that both of our definitions of LIOMs lead to dynamical localization, and prove a partial converse.

Our first definition of LIOMs is based on the discussion in [**51**].

Definition 1.3.3 (**LIOMs of the first kind**). Let $H_n \in \mathcal{A}_{[0,n]}$ be a sequence of random Hamiltonians. We say that the sequence H_n has LIOMs of the first kind if the following conditions are satisfied:

(1) There is a sequence of random unitary maps $U_n \in \mathcal{A}_{[0,n]}$ such that

$$U_n^* H_n U_n = \sum_{X \subseteq [0,n]} \sum_{\mathbf{m} \in \prod_{x \in X}\{2,\dots,d_x\}} \phi_n(\mathbf{m}, X) \prod_{x \in X} S_{\mathbf{m}_x;x}, \tag{1.94}$$

where $S_{m;x}$ is the operator supported at the site x given by the matrix,

$$(S_{m;x})_{jk} = \delta_{j,1}\delta_{k,1} - \delta_{j,m}\delta_{k,m} \tag{1.95}$$

and the $\phi_n(\mathbf{m}, X)$ are random variables satisfying

$$\sup_n \mathbb{E}\left[\sup_{x,y \in [0,n]} \frac{1}{F(|x-y|)} \sum_{\substack{X \subseteq [0,n]: \\ x,y \in X}} \left\| \sum_{\mathbf{m} \in \prod_{x \in X}\{2,\dots,d_x\}} \phi_n(\mathbf{m}, X) \prod_{x \in X} S_{\mathbf{m}_x;x} \right\| \right] < \infty. \tag{1.96}$$

for some non-increasing function $F : \mathbb{Z}_+ \to (0,\infty)$ satisfying $\lim_{x\to\infty} F(x) = 0$.

(2) The sequence of unitary maps U_n is quasi-local, in the sense that for all disjoint finite subsets $X, Y \subset \Gamma$,

$$\sup_n \mathbb{E} \sup_{\substack{A \in \mathcal{A}_X^1 \\ B \in \mathcal{A}_Y}} \|[U_n^* A U_n, B]\| \leq \sum_{\substack{x \in X \\ y \in Y}} G(|x-y|), \tag{1.97}$$

for some non-increasing function $G : \mathbb{Z}_+ \to (0,\infty)$ satisfying $\lim_{x\to\infty} G(x) = 0$.

The LIOMs in Definition 1.3.3 are the quasi-local operators $\{U_n^* S_{m;x} U_n\}$. We prove that LIOMs of the first kind lead to a propagation bound. Under certain assumptions on the decay functions F and G in Definition 1.3.3 (for example, if F and G were decaying exponentials) this propagation bound readily implies dynamical localization.

Theorem 1.3.3. *Suppose that the sequence of Hamiltonians H_n has LIOMs of the first kind. Let X and Y be finite disjoint subsets of $\mathbb{Z}_+$. For a set $Z \subset \mathbb{Z}_+$, let $Z_{n,\lambda} = \{x \in [0,n] : d(x,Z) \leq \lambda d(X,Y)\}$. Then for $\lambda \in (0, 1/2)$,*

$$\sup_{\substack{A \in \mathcal{A}_X^1 \\ B \in \mathcal{A}_Y^1}} \|[\tau_t^{H_n}(A), B]\| \leq 2\left[D_{n,X,\lambda} + D_{n,Y,\lambda} + |t| C_n \sum_{\substack{x \in X_{n,\lambda} \\ y \in Y_{n,\lambda}}} F(|x-y|)\right], \tag{1.98}$$

where $D_{n,X,\lambda}$ and $D_{n,Y,\lambda}$ are nonnegative random variables satisfying,

$$\mathbb{E}D_{n,X,\lambda} \leq \sum_{\substack{x\in X \\ y\in X^c_{n,\lambda}}} G(|x-y|) \quad \textit{and} \quad \mathbb{E}D_{n,Y,\lambda} \leq \sum_{\substack{x\in Y \\ y\in Y^c_{n,\lambda}}} G(|x-y|), \tag{1.99}$$

and

$$C_n(\omega) = \sup_{x,y\in[0,n]} \frac{1}{F(|x-y|)} \sum_{\substack{X\subseteq[0,n]: \\ x,y\in X}} \left\| \sum_{\mathbf{m}\in\prod_{x\in X}\{2,\ldots,d_x\}} \phi_n(\mathbf{m}, X) \prod_{x\in X} S_{\mathbf{m}_x;x} \right\|, \tag{1.100}$$

where by the assumptions in Definition 2.2.2 we have $\sup_n \mathbb{E}C_n < \infty$.

Our second definition of LIOMs was motivated by the discussion in [**21**].

Definition 1.3.4 (**LIOMs of the second kind**). Suppose that Φ is a (random) finite range interaction with a thermodynamic limit τ_t generated by the derivation δ. We say the interaction has LIOMs of the second kind if there exists a family $\{I_x\}_{x\in\mathbb{Z}}$ of self-adjoint, uniformly bounded quasi-local observables I_x satisfying the following:

(1) There is a non-increasing function $F : \mathbb{Z}_+ \to (0,\infty)$, with $\lim_{n\to\infty} F(n) = 0$, such that for all $x \in \mathbb{Z}$,

$$\mathbb{E} \sup_{A\in\mathcal{A}^1_Y} \|[I_x, A]\| \leq F(d(x,Y)). \tag{1.101}$$

(2) For each $x \in \mathbb{Z}$,

$$\delta(I_x) = 0. \tag{1.102}$$

(3) For each $A \in \mathcal{A}^{\mathrm{loc}}$,

$$\delta(A) = \lim_{n\to\infty} \sum_{x=-n}^{n} [I_x, A], \tag{1.103}$$

almost surely, i.e. the family $\sum_{x=-n}^{n} I_x$ of quasi-local Hamiltonians almost surely generate the same dynamics in the thermodynamic limit as Φ.

The arguments in [**21**] can be adapted to prove that the existence of LIOMs satisfying Definition 1.3.4 leads to dynamical localization with $\beta = 1$. The following proposition provides a partial converse, by proving that dynamical localization with $\beta = 0$ implies the existence of LIOMs satisfying Definition 1.3.4.

THEOREM 1.3.4. *Suppose a model with finite-range interactions is dynamically localized with decay function F uniformly in time ($\beta = 0$), and that F has a finite first moment: $\sum_{x=1}^{\infty} xF(x) < \infty$. Then the model has LIOMs of the second kind. Moreover, a LIOM representation (canonical in the sense of [**21**]) can be given explicitly by the following expression:*

$$\tilde{h}_x = \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \tau_t(h_x)dt. \tag{1.104}$$

where T_n is a suitably chosen (random) strictly increasing sequence in $\mathbb{N}$. The terms $\tilde{h}_x$ are time-invariant, and there is a constant $C > 0$ such that

$$\mathbb{E}(\sup_{B\in\mathcal{A}_Y^1} \|[\tilde{h}_x, B]\|) \leq CF(d(x,Y)) \tag{1.105}$$

for every $x \in \mathbb{Z}$.

1.3.2. Summary of Chapter 3 Results. In this chapter we study the dynamics of a class of interacting many-body fermions in the continuum. Our goal is to prove that the Heisenberg dynamics of the interacting model in the thermodynamic limit gives strongly continuous (i.e. operator norm continuous) observable evolution in time when we restrict the class of observables to the CAR algebra. In quantum spin systems, the proof of the existence of the strongly continuous thermodynamic limit given in Theorem 1.2.2 is facilitated by Lieb-Robinson bounds on the finite volume Heisenberg dynamics which are uniform in the system volume. Our strategy for interacting fermion systems is the same. We first prove propagation bounds on the Heisenberg dynamics when the interaction terms of the model are restricted to finite volumes. This Lieb-Robinson bound for the many-body fermion system allows us to prove strong continuity of the thermodynamic limit.

We will first introduce the class of models we study. The exact assumptions we make on the various parameters of the model will be stated in the theorems of this section. Further discussion can be found in Chapter 3. To construct our model, we start with a one-particle Hamiltonian H_1, which we take to be a Schödinger operator:

$$H_1 = -\Delta + V, \tag{1.106}$$

where Δ is the d-dimensional Laplacian and V is a multiplication operator. The corresponding non-interacting, or free, system is given by the second quantization $d\Gamma(H_1)$ of H_1. A typical two-body interaction in finite volume would have the form,

$$W_\lambda^0 = \frac{1}{2}\int_\Lambda \int_\Lambda W(x-y) a_x^* a_y^* a_y a_x \, \mathrm{d}x\mathrm{d}y \tag{1.107}$$

where W is a real-valued, even function, and $\Lambda \subseteq \mathbb{R}^d$ is a bounded, measurable set. We then have a family of models,

$$H_\Lambda^0 = d\Gamma(H_1) + W_\Lambda^0 \tag{1.108}$$

parameterized by finite volume subsets of $\mathbb{R}^d$. We would like to study the thermodynamic limit of these models as $\Lambda \uparrow \mathbb{R}^d$. A fundamental difficulty is that the operators W_Λ^0 are unbounded, even when the interaction function W is essentially bounded. This is due to the fact that an arbitrary number of fermions can occupy a volume Λ (the number operator N_Λ is unbounded). The unboundedness of the interaction terms makes it mathematically difficult (if not impossible) to obtain a propagation bound on the dynamics. To address this problem in a physically reasonable way, we smear the fermions in space in a way that effectively gives them a finite size. We introduce the smeared interaction given by,

$$W_\Lambda^\sigma = \frac{1}{2}\int_\Lambda \int_\Lambda W(x-y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma) \, \mathrm{d}x\mathrm{d}y \tag{1.109}$$

where $\varphi_x^\sigma : \mathbb{R}^d \to \mathbb{R}$ denotes the L^1 normalized gaussian centered at x with standard deviation $\sigma > 0$:

$$\varphi_x^\sigma(z) = \frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}} e^{-\frac{\|z-x\|_2^2}{2\sigma^2}}. \tag{1.110}$$

The physical effect of smearing the fermions can be seen by investigating the smeared number operator

$$N_\Lambda^\sigma = \int_\Lambda a^*(\varphi_x^\sigma) a(\varphi_x^\sigma) \, \mathrm{d}x, \tag{1.111}$$

which is the observable representing the number of smeared fermions that are observed in Λ. We have that,

$$\|N_\Lambda\| \leq \int_\Lambda \|a^*(\varphi_x^\sigma)\|\|a(\varphi_x^\sigma)\|\mathrm{d}x = \|\varphi_x^\sigma\|_2^2|\Lambda| = \frac{|\Lambda|}{(4\pi\sigma^2)^{d/2}}. \tag{1.112}$$

This shows that the smeared fermions have a size effectively that of a d-dimensional sphere of radius $\sigma > 0$. In particular, only a finite number of smeared fermions can occupy any finite volume Λ. The interaction term W_Λ^σ can similarly be shown to satisfy

$$\|W_\Lambda^\sigma\| \leq \frac{1}{2}\frac{|\Lambda|^2}{(4\pi\sigma^2)^d}\|W\|_\infty \tag{1.113}$$

Smearing the fermions effectively cuts out the high energy portion of the interaction W_Λ^0. This removal of the high energy region is called a UV cutoff. Note that W_Λ^σ is still generally unbounded for infinite volume Λ. We now have a two parameter family of models $H_\Lambda^\sigma = d\Gamma(H_1) + W_\Lambda^\sigma$. All of our results will apply to this family of models when $\sigma > 0$.

If $\sigma > 0$ approaches 0, the effective size of the smeared fermions approaches 0. It seems reasonable to expect that for small $\sigma > 0$, the models H_Λ^σ provide a decent approximation to the point particle model H_Λ^0. In fact, if this were not the case then the models H_Λ^σ should really be discarded. The following proposition shows that the smeared fermion model converges to the point particle model in a certain sense.

Proposition 1.3.2. *Let Λ be a measurable subset of $\mathbb{R}^d$. For $t \in \mathbb{R}$ we denote by $U_\Lambda^\sigma(t) = e^{-itH_\Lambda^\sigma}$ and $U_\Lambda^0(t) = e^{-itH_\Lambda^0}$ the unitary groups generated by H_Λ^σ and H_Λ^0, respectively. Then*

$$\lim_{\sigma\downarrow 0} U_\Lambda^\sigma(t)\psi = U_\Lambda^0(t)\psi$$

for each $\psi \in \mathfrak{F}^-$, uniformly for t in compact subsets of $\mathbb{R}$.

Now let τ_t^Λ denote the Heisenberg dynamics on $\mathcal{B}(\mathfrak{F}^-)$ generated by H_Λ^σ, so $\tau_t^\Lambda(A) = e^{itH_\Lambda^\sigma}Ae^{-itH_\Lambda^\sigma}$. In addition we denote the dynamics non-interacting system by $\tau_t^\emptyset$, where $\tau_t^\emptyset(A) = e^{itd\Gamma(H_1)}Ae^{-itd\Gamma(H_1}$. In order to prove the strong continuity of the thermodynamic limit we need a Lieb-Robinson type bound on expressions of the form

$$\{\tau_t^\Lambda(a(f)), a^*(g)\} \text{ and } \{\tau_t^\Lambda(a(f))a(g)\} \tag{1.114}$$

for $f, g \in L^2(\mathbb{R}^d)$ with disjoint support. Using the interaction picture we can relate τ_t^Λ, the dynamics with interactions, to $\tau_t^\emptyset$ in the following way.

$$\tau_t^\Lambda(a(f)) = \tau_t^\emptyset(a(f)) + i\int_0^t \mathrm{d}s\, \tau_s^\Lambda\left(\left[W_\Lambda^\sigma, \tau_{t-s}^\emptyset(a(f))\right]\right) \tag{1.115}$$

Since $\tau_t^\emptyset(a(f)) = a(e^{-itH_1}f)$, we prove a propagation bound on the one-particle dynamics in order use Eq. (1.115). The following theorem gives such a bound.

THEOREM 1.3.5. *Assume that $V : \mathbb{R}^d \to \mathbb{C}$ has the form*

$$V(x) = \int_{\mathbb{R}^d} \mathrm{d}\mu(k)\, e^{-ik\cdot x} \tag{1.116}$$

where μ : Borel$(\mathbb{R}^d) \to \mathbb{R}$ is a real-valued finite Borel measure on $\mathbb{R}^d$ with compact support, which is also even in the sense that $\mu(A) = \mu(-A)$ for every Borel subset A of $\mathbb{R}^d$. Consider the Schrödinger operator $H_1 = -\Delta + V$. Then there exist constants $C_1, C_2, C_3 > 0$ depending on d, μ, and σ such that the estimate

$$\left|\langle e^{-itH_1}f, \varphi_x^\sigma\rangle\right| \le C_1 e^{C_2|t|\ln|t|}\int_{\mathbb{R}^d} \mathrm{d}y\, e^{-\frac{C_3}{t^2+1}|x-y|}|f(y)| \tag{1.117}$$

holds for all $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$.

The idea is now to combine Eq. (1.115) with Theorem 1.3.5 iteratively to prove a Lieb-Robinson type bound on the expressions in Eq. (1.114). The following theorem shows the result of carrying out this procedure.

THEOREM 1.3.6. *Let V satisfying the conditions specified in Theorem 1.3.5, and let $W \in L^\infty(\mathbb{R}^d)$ be real-valued and satisfy $W(-x) = W(x)$ and $|W(x)| \le Ce^{-a|x|}$ for some constants $C, a > 0$. Then, there exist continuous functions $C(t), a(t) > 0$ such that for all bounded and* *measurable* $\Lambda \subset \mathbb{R}^d$*, and* $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$*, one has the following bounds:*

$$\|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \langle e^{-it(-\Delta+V)} f, g\rangle \mathbb{1}\| \leq \|f\|_1 \|g\|_1 e^{C(t)} e^{-a(t)d(supp(f),supp(g))} \tag{1.118}$$

$$\|\{\tau_t^\Lambda(a(f)), a(g)\}\| \leq \|f\|_1 \|g\|_1 e^{C(t)} e^{-a(t)d(supp(f),supp(g))} \tag{1.119}$$

where $d(supp(f), supp(g))$ *denotes the distance between the essential supports of* f *and* g*.*

The main theorem of Chapter 3 is presented next.

THEOREM 1.3.7. *There exists a strongly continuous one-parameter group of automorphism of the CAR algebra over* $L^2(\mathbb{R}^d)$*,* $\{\tau_t\}_{t\in\mathbb{R}}$*, such that*

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \tau_t^\Lambda(a(f)) = \tau_t(a(f)), \text{ for all } f \in L^2(\mathbb{R}^d). \tag{1.120}$$

CHAPTER 2

Slow propagation in some disordered quantum spin chains

2.1. Introduction

Anderson localization in random Schrödinger operators is quite well understood. Mathematical proofs of this phenomenon have been given under a variety of conditions. See the recent book by Aizenman and Warzel for an overview of the state-of-the-art [**5**]. The physical phenomenon is a drastic slowdown of transport in the system's dynamics, which is seen as the consequence of a change in the nature of the spectrum from continuous spectrum (extended states) to pure point spectrum (localized states).

The problem of Many-Body Localization (MBL) is the question of what happens to localization properties in the presence of interactions. Although Anderson in his work that started the subject of localization [**7**] envisioned the phenomenon for interacting systems, research on MBL picked up only relatively recently stimulated by papers by Basko, Aleiner, and Altshuler [**10**], Oganesyan and Huse [**73**], and Pal and Huse [**74**].

Quantum spin system with, for example, nearest neighbor interactions, are among the simplest interacting quantum many-body systems and much of the recent work on MBL dealt with one of just three one-dimensional quantum spin models: the XY chain, the quantum Ising chain, and the XXZ chain. The small number of rigorous results that have been obtained so far are also mostly restricted to these three models. Exponential dynamical localization, uniformly in time, was proved for a class of disordered XY chains by exploiting their connection to Anderson models [**1**, **46**, **83**]. Imbrie studied the quantum Ising chain with random couplings and fields [**52**]. Localization properties in the low-energy region, called the droplet-regime, of the ferromagnetic XXZ chain were proved in [**11**, **12**, **34**, **35**, **36**].

For a single quantum particle, the study of localization for a long time focused on spectral properties. i.e., proving the occurrence of point spectrum with associated eigenvectors that satisfy exponential decay. Later, multi-scale analysis [**42**] and the fractional-moment method [**2**] emerged as two powerful tools to study dynamical localization. Systems of N interacting particles can be analyzed by extending these methods, as along as N is fixed [**3**, **24**].

The first main result of this work is the proof of a relation between uniform dynamical localization and the existence of Local Integrals of Motion (LIOM). The LIOM picture [**21**, **81**] has been proposed as the mechanism by which systems exhibiting MBL do not thermalize under their own (closed system) dynamics and, in particular, that violate the Eigenfunction Thermalization Hypothesis (ETH). We give two definitions of LIOMs, consistent with the different ways this concept has been considered in the literature. For lack of a better name, we call them LIOMs of the first kind Definition 2.2.2 and LIOMS of the second kind (Definition 2.2.3). The first kind implies dynamical localization of the form generically expected for strongly disordered quantum spin chains. The second kind, as we show, exist when we have uniform-in-time dynamical localization, such as has been proved to occur in the random XY chain [**46**].

In interacting many-body systems it is most natural to express localization in terms of dynamical properties directly. A good (but not typcial) example is the zero-velocity Lieb-Robinson bound proved for the disordered XY chain in [**46**]. In this work, we introduce the notion of *transmission time*, as the smallest time a signal or disturbance can reach a prescribed strength a given distance away from the source. See Definition 2.2.4. For exponentially localized systems, we expect transmission times grow exponentially with the distance. We then prove that exponentially localized systems perturbed by sparse disorder, have transmission times that grow at least as a power law and we we give a lower bound for the power that diverges with increasing sparseness of the perturbation. A large power indicates sub-diffusive behavior. We model the sparse disorder by adding a uniformly bounded but otherwise arbitrary nearest-neighbor term to the Hamiltoian at locations determined by a Bernoulli process with small probability of success.

De Roeck and coworkers have argued that MBL, interpreted as the complete absence of transport, is only possible in one-dimensional systems. They argue that diffusion of energy is inevitable in higher dimensions [**26**, **27**, **29**, **31**, **59**, **88**]. We only study one dimensional systems in this work, and therefore we do not have results that either support or contradict these arguments. Rather, for one-dimensional systems our results implies a degree of robustness of localization phenomena in the sense of slow propagation. Others have investigated stability of MBL in spin chains under the influence of regions of low disorder or coupling to a heat bath [**43**], in a kicked quantum spin chain model [**14**] and by extensive numerical calculation for the Heisenberg chain [**86**]. The latter studies consider properties of the spectral form factor (i.e., the Fourier transform of a two-point function) to look for an indicator of an MBL-type transition. It would be interesting to supplement these studies with information about transmission times in these models.

In Section 2.2 we introduce several definitions related to MBL and describe our main results. The proofs are in Section 2.3. Two applications are discussed in Section 2.4. Some auxiliary facts are collected in an appendix.

2.2. Many-body localization properties and main results

In this section we define several properties associated with localized many-body systems. We focus on characteristics of the dynamics in terms of which our main results are formulated and restrict ourselves to the one-dimensional setting. All notions make sense for multi-dimensional systems but, as discussed in the introduction, the phenomenon of many-body localization as it is commonly understood may well be restricted to one dimension.

We will consider subsystems of a chain of quantum systems labeled by $x \in \mathbb{Z}$, with a finite-dimensional Hilbert space $\mathcal{H}_x$ for each $x \in \mathbb{Z}$. The Hilbert space of the subsystem associated with a finite set $X \subset \mathbb{Z}$, is given by $\mathcal{H}_X = \bigotimes_{x\in X} \mathcal{H}_x$, and the observables measurable in this subsystem are given by $\mathcal{A}_X := \mathcal{B}(\mathcal{H}_X)$. The elements of $\mathcal{A}^{\mathrm{loc}} := \bigcup_{X\subset\mathbb{Z}} \mathcal{A}_X$, where the union is over finite subsets, are called the *local observables*, whereas the norm completion of $\mathcal{A}^{\mathrm{loc}}$, denoted by $\mathcal{A}_{\mathbb{Z}}$, is the algebra of quasi-local observables. We denote the closed unit ball of $\mathcal{A}_X$ by $\mathcal{A}_X^1$.

A convenient way to specify a model is with an interaction, which is a map Φ assigning to each finite set $X \subset \mathbb{Z}$ an element $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$. Associated to the interaction Φ is the family of local Hamiltonians $H_\Lambda = \sum_{X\subset\Lambda} \Phi(X) \in \mathcal{A}_\Lambda$, defined for each finite subset $\Lambda \subset \mathbb{Z}$. The Heisenberg dynamics generated by a family of local Hamiltonians determined by an interaction Φ is defined in the usual way:

$$\tau_t^{H_\Lambda}(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \tag{2.1}$$

The interactions Φ may be random, meaning the following: There is a probability space $(\Omega, \mathcal{F}, \Pr)$, and to each $\omega \in \Omega$ there is assigned an interaction $\Phi(\omega)$. We assume weak measurability of the random operators $\omega \mapsto \Phi(\omega)(X)$ for each finite $X \subset \mathbb{Z}$.

A finite range interaction is one for which there exists $R \geq 0$ such that $\Phi(X) = 0$ unless $\operatorname{diam} X \leq R$. R is then the *range* of the interaction. A common way to introduce a model with a finite-range interaction is to specify self-adjoint $h_x \in \mathcal{A}_{[x,x+R]}$, for each $x \in \mathbb{Z}$.

2.2.1. Dynamical Localization. In the single-particle setting, dynamical localization refers to the absence of ballistic or diffusive propagation in the system's Schrödinger evolution. Initially localized wave functions remain localized for all time under the dynamics. A natural analogue of this property in the setting of quantum spin chains is localization of the Heisenberg dynamics. We consider a general notion of dynamical localization expressed by the following definition.

Definition 2.2.1. Let $F : \mathbb{Z}_+ \to (0, \infty)$ be a non-increasing function with the property $\lim_{x\to\infty} F(x) = 0$.
(i) We say that a family $\{H_\Lambda : \Lambda \subset \mathbb{Z} \text{ finite intervals}\}$ of random local Hamiltonians $H_\Lambda \in \mathcal{A}_\Lambda$ exhibits dynamical localization with decay function F if there exists a constant $\beta \geq 0$ and a function $\chi : \mathbb{N} \to (0, \infty)$ such that for any sets $X, Y \subseteq \Lambda$ with $Y \subset [\min X, \max X]^c$, the random variable

$$C_{\Lambda;X,Y} \equiv \sup_{t\in\mathbb{R}} \sup_{\substack{A\in\mathcal{A}_X^1 \\ B\in\mathcal{A}_Y^1}} \frac{\|[\tau_t^{H_\Lambda}(A), B]\|}{\chi(|X|)(1+|t|^\beta)} \tag{2.2}$$

satisfies

$$\mathbb{E}C_{\Lambda;X,Y} \leq F(d(X,Y)) \tag{2.3}$$

Here $d(X,Y) = \min\{|x-y| : x \in X, y \in Y\}$ is the usual set distance.

(ii) If F is of the form $F(x) = e^{-\eta x}$ we say the family $\{H_\Lambda\}$ exhibits exponential dynamical localization. In this case η^{-1} is called (a bound for the) localization length.
(iii) If F is of the form $F(x) = e^{-\eta x^\rho}$ for some $\rho \in (0,1)$, we say the family $\{H_\Lambda\}$ exhibits stretched exponential dynamical localization.

(iv) We say the family $\{H_\Lambda\}$ exhibits dynamical localization with decay function F uniformly in time if it satisfies (i) with $\beta = 0$.

The following lemma shows that if a family of local Hamiltonians is dynamically localized and the corresponding family of local dynamics has a thermodynamic limit, then the infinite volume dynamics is also dynamically localized with the same decay function.

LEMMA 2.2.1. *Suppose that $\{H_\Lambda\}$ is a family of dynamically localized Hamiltonians with decay function F, and that the corresponding family of dynamics $\{\tau_t^{H_\Lambda}\}$ has a thermodynamic limit. In other words, there is an exhaustive sequence $\Lambda_n \uparrow \mathbb{Z}$ such that almost surely,*

$$\lim_{n\to\infty} \tau_t^{H_{\Lambda_n}} \equiv \tau_t \tag{2.4}$$

strongly for all $t \in \mathbb{R}$, where τ_t is a $*$-automorphism of $\mathcal{A}_{\mathbb{Z}}^{loc}$. Then for any finite set $X \subset \mathbb{Z}$ and any set $Y \subseteq [\min X, \max X]^c$, the random variable

$$C_{X,Y} \equiv \sup_{t\in\mathbb{R}} \sup_{\substack{A\in\mathcal{A}_X^1 \\ B\in\mathcal{A}_Y^1}} \frac{\|[\tau_t(A), B]\|}{\chi(|X|)(1+|t|^\beta)} \tag{2.5}$$

satisfies

$$\mathbb{E}C_{X,Y} \leq F(d(X,Y)) \tag{2.6}$$

PROOF. First let $X, Y \subset \mathbb{Z}$ be finite, with $Y \subset [\min X, \max X]^c$. It follows immediately that,

$$C_{X,Y} = \sup_{t\in\mathbb{R}} \sup_{\substack{A\in\mathcal{A}_X^1 \\ B\in\mathcal{A}_Y^1}} \frac{\|[\tau_t(A), B]\|}{\chi(|X|)(1+|t|^\beta)} \leq \liminf_{n\to\infty} C_{\Lambda_n;X,Y}. \tag{2.7}$$

By Fatou's lemma, $\mathbb{E}C_{X,Y} \leq F(d(X,Y))$. Now suppose $Y \subseteq [\min X, \max X]^c$ is infinite. For any sequence of finite sets $Y_n \uparrow Y$, by using local approximations and the fact that C_{X,Y_n} is monotone in n we obtain

$$C_{X,Y} \leq \lim_{n\to\infty} C_{X,Y_n}, \tag{2.8}$$

which proves the lemma. $\square$

2.2.2. Local Integrals of Motion. The lack of ergodicity seen in MBL systems can be 'explained' as a consequence the emergence of an extensive set of local conserved quantities, called local integrals of motion (LIOMs). In this section we propose precise definitions of LIOMs. Heuristic definitions of LIOMs have been given in the physics literature, [**51**], [**82**]. LIOMs are thought to account for most of the phenomena of MBL. See, for example, the review paper [**53**]. To address the variety seen in the physics literature we formulate two distinct definitions. Specifically, Definition 2.2.2 given below is modeled after the discussion in [**51**], while Definition 2.2.3 was motivated by [**21**]. We refer to them as LIOMs of the first kind and LIOMs of the second kind, respectively. We briefly discuss the relation between the two at the end of this section.

In the following definition we restrict our attention to quantum spin chains, for simplicity. The definition can also be formulated in higher-dimensions. Let $d_x \geq 2$ denote the dimension of the Hilbert space at $x \in \mathbb{Z}$.

Definition 2.2.2 (**LIOMs of the first kind**). Let $H_n \in \mathcal{A}_{[0,n]}$ be a sequence of random Hamiltonians. We say that the sequence H_n has LIOMs of the first kind if the following conditions are satisfied:

(1) There is a sequence of random unitary maps $U_n \in \mathcal{A}_{[0,n]}$ such that

$$U_n^* H_n U_n = \sum_{X \subseteq [0,n]} \sum_{\mathbf{m} \in \prod_{x \in X}\{2,\dots,d_x\}} \phi_n(\mathbf{m}, X) \prod_{x \in X} S_{\mathbf{m}_x;x}, \tag{2.9}$$

where $S_{m;x}$ is the operator supported at the site x given by the matrix,

$$(S_{m;x})_{jk} = \delta_{j,1}\delta_{k,1} - \delta_{j,m}\delta_{k,m} \tag{2.10}$$

and the $\phi_n(\mathbf{m}, X)$ are random variables satisfying

$$\sup_n \mathbb{E}\left[\sup_{x,y \in [0,n]} \frac{1}{F(|x-y|)} \sum_{\substack{X \subseteq [0,n]: \\ x,y \in X}} \left\| \sum_{\mathbf{m} \in \prod_{x \in X}\{2,\dots,d_x\}} \phi_n(\mathbf{m}, X) \prod_{x \in X} S_{\mathbf{m}_x;x} \right\| \right] < \infty. \tag{2.11}$$

for some non-increasing function $F : \mathbb{Z}_+ \to (0, \infty)$ satisfying $\lim_{x \to \infty} F(x) = 0$.

(2) The sequence of unitary maps U_n is quasi-local, in the sense that for all disjoint finite subsets $X, Y \subset \Gamma$,

$$\sup_n \mathbb{E} \sup_{\substack{A \in \mathcal{A}_X^1 \\ B \in \mathcal{A}_Y}} \|[U_n^* A U_n, B]\| \leq \sum_{\substack{x \in X \\ y \in Y}} G(|x-y|), \tag{2.12}$$

for some non-increasing function $G : \mathbb{Z}_+ \to (0, \infty)$ satisfying $\lim_{x\to\infty} G(x) = 0$.

Remark 2.2.1. *The LIOMs in definition 2.2.2 are the quasi-local operators $U_n S_{m;x} U_n^*$. The key feature of the family $\{S_{m;x}\}_{m=2}^{d_x}$ is that the operators are uniformly bounded, are mutually commuting, and generate a maximal abelian subalgebra of observables. Any other set of observables with these properties could be used in the definition instead.*

The following theorem shows that the Heisenberg dynamics generated by a Hamiltonian with LIOMs of the first kind satisfies the type of propagation bound expressing dynamical localization.

Theorem 2.2.1. *Suppose that the sequence of Hamiltonians H_n has LIOMs of the first kind. Let X and Y be finite disjoint subsets of $\mathbb{Z}_+$. For a set $Z \subset \mathbb{Z}_+$, let $Z_{n,\lambda} = \{x \in [0, n] : d(x, Z) \leq \lambda d(X, Y)\}$. Then for $\lambda \in (0, 1/2)$,*

$$\sup_{\substack{A \in \mathcal{A}_X^1 \\ B \in \mathcal{A}_Y^1}} \|[\tau_t^{H_n}(A), B]\| \leq 2\left[D_{n,X,\lambda} + D_{n,Y,\lambda} + |t| C_n \sum_{\substack{x \in X_{n,\lambda} \\ y \in Y_{n,\lambda}}} F(|x-y|)\right], \tag{2.13}$$

where $D_{n,X,\lambda}$ and $D_{n,Y,\lambda}$ are nonnegative random variables satisfying,

$$\mathbb{E} D_{n,X,\lambda} \leq \sum_{\substack{x \in X \\ y \in X_{n,\lambda}^c}} G(|x-y|) \quad \textit{and} \quad \mathbb{E} D_{n,Y,\lambda} \leq \sum_{\substack{x \in Y \\ y \in Y_{n,\lambda}^c}} G(|x-y|), \tag{2.14}$$

and

$$C_n(\omega) = \sup_{x,y \in [0,n]} \frac{1}{F(|x-y|)} \sum_{\substack{X \subseteq [0,n]: \\ x,y \in X}} \left\| \sum_{\mathbf{m} \in \prod_{x \in X} \{2,\ldots,d_x\}} \phi_n(\mathbf{m}, X) \prod_{x \in X} S_{\mathbf{m}_x; x} \right\|, \tag{2.15}$$

where by the assumptions in Definition 2.2.2 we have $\sup_n \mathbb{E} C_n < \infty$.

The proof of this theorem is given in Section 2.3.1.

It is natural to ask whether the existence of LIOMs also follows from dynamical localization. Indeed, the existence of LIOMs and dynamical localization are regarded as equivalent properties in the physics literature. It turns out to be convenient to use a slightly different notion of LIOMs to prove a result in this direction.

DEFINITION 2.2.3 (**LIOMs of the second kind**). Suppose that Φ is a (random) finite range interaction with a thermodynamic limit τ_t generated by the derivation δ. We say the interaction has LIOMs of the second kind if there exists a family $\{I_x\}_{x\in\mathbb{Z}}$ of self-adjoint, uniformly bounded quasi-local observables I_x satisfying the following:

(1) There is a non-increasing function $F : \mathbb{Z}_+ \to (0, \infty)$, with $\lim_{n\to\infty} F(n) = 0$, such that for all $x \in \mathbb{Z}$,

$$\mathbb{E} \sup_{A\in\mathcal{A}_Y^1} \|[I_x, A]\| \leq F(d(x, Y)). \tag{2.16}$$

(2) For each $x \in \mathbb{Z}$,

$$\delta(I_x) = 0. \tag{2.17}$$

(3) For each $A \in \mathcal{A}^{\mathrm{loc}}$,

$$\delta(A) = \lim_{n\to\infty} \sum_{x=-n}^{n} [I_x, A], \tag{2.18}$$

almost surely, i.e. the family $\sum_{x=-n}^{n} I_x$ of quasi-local Hamiltonians almost surely generate the same dynamics in the thermodynamic limit as Φ.

REMARK 2.2.2. *In Definition 2.2.3 we do not assume that the LIOMs I_x commute. From the time invariance it is necessary that $I_x \in \ker\delta$, thus if $\ker\delta$ is abelian the LIOMs will commute. We expect $\ker\delta$ to be abelian almost surely, generically for continuous randomness. Note that in finite volumes, $\delta(\cdot) = [H, \cdot]$ for a local Hamiltonian H, and simplicity of the spectrum of H is equivalent to $\ker\delta$ being an abelian algebra.*

The following proposition connects dynamical localization uniform in time with the ‘canonical LIOMs’ introduced in [**21**].

THEOREM 2.2.2. *Suppose a model with finite-range interactions is dynamically localized with decay function F uniformly in time ($\beta = 0$), and that F has a finite first moment: $\sum_{x=1}^{\infty} xF(x) < \infty$. Then the model has LIOMs of the second kind. Moreover, a LIOM representation (canonical in the sense of [**21**]) can be given explicitly by the following expression:*

$$\tilde{h}_x = \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \tau_t(h_x)dt. \tag{2.19}$$

where T_n is a suitably chosen (random) strictly increasing sequence in $\mathbb{N}$. The terms $\tilde{h}_x$ are time-invariant, and there is a constant $C > 0$ such that

$$\mathbb{E}(\sup_{B\in\mathcal{A}_Y^1} \|[\tilde{h}_x, B]\|) \leq CF(d(x,Y)) \tag{2.20}$$

for every $x \in \mathbb{Z}$.

The proof of this result can be found in Section 2.3.1.

In the definition of LIOMs of the first kind, Definition 2.2.2, nothing is said on the dependence of the unitaries and the interaction coefficients on the length, n, of the chain. One could expect however, that a random interaction Φ can be defined by

$$\Phi(X) = \lim_{n\to\infty} \sum_{\mathbf{m}\in\{1,\ldots,d-1\}^{|X|}} \phi_n(\mathbf{m}, X) \prod_{x\in X} S_{\mathbf{m}_x;x}, \tag{2.21}$$

where it should be understood that n here refers to a finite spin chain labeled by $[-n, n]$. Using the notion of *local convergence in F-norm* (see [**71**, Definition 3.7]), it is then straightforward to define conditions that ensure the existence of a commuting family of LIOMs of the second kind.

2.2.3. Transmission Times.

DEFINITION 2.2.4. Given a Hamiltonian $H \in \mathcal{A}_{[0,n]}$ and an $\epsilon > 0$ define the transmission time, $t(\epsilon)$ of H as,

$$t(\epsilon) = \inf\{|t| : \sup_{\substack{A\in\mathcal{A}_0^1 \\ B\in\mathcal{A}_n^1}} \|[\tau_t^H(A), B]\| > \epsilon\}. \tag{2.22}$$

Suppose we have a sequence $H_n \in \mathcal{A}_{[0,n]}$ of Hamiltonians with associated transmission times $t_n(\epsilon)$. It is reasonable to expect that dispersive effects may cause the commutator defining the transmission time to never exceed some fixed $\epsilon > 0$ for large values of n. If this occurs then $t_n(\epsilon)$ will cease to be a meaningful quantity. For this reason we should consider a sequence ϵ_n, suitably decaying in n, and instead consider the sequence of transmission times $t_n(\epsilon_n)$. We note that some authors prefer the term 'scrambling time' instead of transmission time [**23**].

A natural question to ask is whether the transmission time is consistent with the propagation bounds imposed by a Lieb-Robinson bound. Suppose that the sequence H_n satisfies,

$$\sup_{\substack{A \in \mathcal{A}_x^1 \\ B \in \mathcal{A}_y^1}} \|[\tau_t^{H_n}(A), B]\| \leq C(e^{\mu v|t|} - 1)e^{-\mu|x-y|} \tag{2.23}$$

for $x \neq y$, uniformly in n. Such bounds are known to hold for a broad class of quantum spin models on general lattices [**64**]. The bound implies that,

$$t_n(\epsilon_n) \geq \frac{1}{\mu v} \log(1 + \frac{\epsilon_n e^{\mu n}}{C}), \tag{2.24}$$

in which case

$$\limsup_{n \to \infty} \frac{n}{t_n(\epsilon_n)} \leq v \tag{2.25}$$

provided ϵ_n decays subexponentially in n.

We consider slow transport in a quantum spin chain to be characterized by super-linear growth of the transmission time. For stretched exponential dynamically localized spin chains the transmission time grows as a stretched exponential, as the next proposition shows.

Proposition 2.2.1. *Suppose that a sequence $H_n \in \mathcal{A}_{[0,n]}$ of random Hamiltonians exhibits dynamical localization with decay function F given by $F(x) = e^{-\eta x^\rho}$ for some $\rho \in (0, 1]$. Then for any positive γ and α such that $\beta\gamma + \alpha < 1$,*

$$\frac{e^{\gamma \eta n^\rho}}{t_n(e^{-\alpha \eta n^\rho})} \to 0 \tag{2.26}$$

almost surely.

Proof. For $\beta = 0$ it is easy to see that $\Pr(t_n(e^{-\alpha\eta n^\rho}) = \infty \text{ eventually}) = 1$. Assume $\beta > 0$. By assumption,

$$\sup_{\substack{A\in\mathcal{A}_0^1 \\ B\in\mathcal{A}_n}} \|\tau_t^{H_n}(A), B]\| \le \chi(1)C_n(1+|t|^\beta), \tag{2.27}$$

where $\mathbb{E}C_n \le e^{-\eta n^\rho}$. Choose any δ such that $\beta\gamma + \alpha < \delta < 1$. Let

$$A_n = \left\{\chi(1)C_n \le e^{-\delta\eta n^\rho}\right\}.$$

By Markov's inequality,

$$\Pr(A_n^c) \le \chi(1)\frac{\mathbb{E}C_n}{e^{-\delta\eta n^\rho}} \le \chi(1)e^{-(1-\delta)\eta n^\rho}.$$

It follows from the Borel-Cantelli lemma that $\Pr(\mathbb{1}_{A_n} = 1 \text{ eventually}) = 1$. (2.27) implies that,

$$\mathbb{1}_{A_n} t_n(e^{-\alpha\eta n^\rho})^\beta \ge \mathbb{1}_{A_n}\left(\frac{e^{-\alpha\eta n^\rho}}{\chi(1)C_n} - 1\right) \ge (e^{(\delta-\alpha)\eta n^\rho} - 1)\mathbb{1}_{A_n}$$

Therefore

$$\mathbb{1}_{A_n}\frac{e^{\gamma\eta n^\rho}}{t_n(e^{-\alpha\eta n^\rho})} \le \frac{e^{\gamma\eta n^\rho}}{(e^{(\delta-\alpha)\eta n^\rho} - 1)^{1/\beta}}$$

Since $\gamma < (\delta - \alpha)/\beta$ and $\mathbb{1}_{A_n} = 1$ eventually with probability 1, it follows that $\frac{e^{\gamma\eta n^\rho}}{t_n(e^{-\alpha\eta n^\rho})} \to 0$ almost surely.

□

In the case of exponential dynamical localization there exists a family of perturbations such that the perturbed model still has long transmission times. These perturbations consist of additional nearest neighbor interactions that occur with low density at random positions. For this class of perturbations we can prove that the transmission time grows super linearly provided the perturbations are sufficiently sparse[1].

Theorem 2.2.3. *Let $H_n^0 \in \mathcal{A}_{[0,n]}$ be a sequence of random Hamiltonians defined over the probability space $(\Omega_0, \Pr_0)$ which are exponentially dynamically localized in the sense of Definition 2.2.1 ($\rho = 1$). Let $(\delta_x)_{x=0}^\infty$ be an i.i.d. sequence of Bernoulli random variables over the probability space*

[1] After this work appeared on the arXiv, similar perturbations were considered by De Roeck, Huveneers, and Olla, who proved subdiffusive dynamics in classical Hamiltonian chains [**28**].

$(\Omega_1, \mathrm{Pr}_1)$, *with* $\mathrm{Pr}_1(\delta_0 = 0) = p \in (0, 1]$. *Let* $(\psi_x)_{x=0}^{\infty}$ *denote a uniformly bounded sequence with* $\psi_x \in \mathcal{A}_{[x,x+1]}$ *for all* x. *Consider the sequence of random Hamiltonians*

$$H_n(\omega) = H_n^0(\omega_0) + \sum_{x=0}^{n-1} \delta_x(\omega_1)\psi_x; \tag{2.28}$$

over the probability space $\Omega_0 \times \Omega_1$ *equipped with the product measure. If* t_n *is the transmission time of* H_n, *then for any* $\gamma > 0$ *and* $\alpha \in (0, 1/3)$ *satisfying*

$$\eta\left(\frac{1-3\alpha}{1-\alpha}\right) > 2[(\beta+1)\gamma - 1]\log\left(\frac{1}{p}\right), \tag{2.29}$$

$$\frac{n^{\gamma}}{t_n(e^{-\alpha\eta n})} \to 0 \tag{2.30}$$

in probability.

Unfortunately we do not know how to prove a similar robustness result for models with a decay function F that decays slower than exponentially. For example, certain anisotropic XY chains are only known to exhibit stretched exponential dynamical localization, as we note in Section 2.4.1

Theorem 2.2.3 concerns finite volume Hamiltonians. The following theorem shows that in certain cases one can work directly with the thermodynamic limit.

THEOREM 2.2.4. *Suppose that* Φ_0 *is a random interaction over the probability space* $(\Omega_0, \mathrm{Pr}_0)$ *whose finite volume Hamiltonians are exponentially dynamically localized. Suppose that* $(\delta_x)_{x\in\mathbb{Z}}$ *is a sequence of i.i.d. Bernoulli random variables over the probability space* $(\Omega_1, \mathrm{Pr}_1)$, *with* $\mathrm{Pr}_1(\delta_0 = 0) = p \in (0, 1]$. *Let* $(\psi_x)_{x\in\mathbb{Z}}$ *denote a uniformly bounded sequence with* $\psi_x \in \mathcal{A}_{[x,x+1]}$ *for all* x. *Let* Φ_2 *be the random nearest neighbor interaction given by,*

$$\Phi_2(\{x, x+1\}) = \delta_x\psi_x \tag{2.31}$$

for all $x \in \mathbb{Z}$. *Define the random interaction* $\Phi(\omega) = \Phi_0(\omega_0) + \Phi_1(\omega_1)$ *over the probability space* $\Omega_0 \times \Omega_1$ *equipped with the product measure. If, for almost every* $\omega_0 \in \Omega_0$, *there is a (possibly random)* F*-function* F *such that* Φ_0 *is* F*-normed, then the thermodynamic limit,* τ_t, *of* Φ *exists* *almost surely. For any fixed $r \in \mathbb{N}$, define*

$$t_n(\epsilon) = \inf\{|t| : \sup_{\substack{A\in\mathcal{A}^1_{[-r,0]} \\ B\in\mathcal{A}^1_{[n,\infty)}}} \|[\tau_t(A), B]\| > \epsilon\}. \tag{2.32}$$

Then for any $\gamma > 0$ and $\alpha \in (0, 1/3)$ satisfying

$$\eta\left(\frac{1-3\alpha}{1-\alpha}\right) > 2[(\beta+1)\gamma - 1]\log\left(\frac{1}{p}\right), \tag{2.33}$$

$$\frac{n^\gamma}{t_n(e^{-\alpha\eta n})} \to 0 \tag{2.34}$$

in probability.

2.3. Proofs of Main Results

2.3.1. Proofs of results about LIOMs. Showing that LIOMs of the fist kind imply dynamical localization is a straightforward application of the quasi-locality properties of the LIOMs.

Proof of Theorem 2.2.1. For any $A \in \mathcal{A}^1_X$, $B \in \mathcal{A}^1_Y$,

$$\|[\tau_t^{H_n}(A), B]\| = \|[\tau_t^{\tilde{H}_n}(\tilde{A}), \tilde{B}]\|, \tag{2.35}$$

where $\tilde{O} = U_n^* O U_n$ for an observable O. Using the quasi-locality of the unitary U_n specified in Eq. (2.12), by a standard application of conditional expectations (see, for example, [**71**, Section IV.A]), we can find (random) local observables $A_\lambda \in \mathcal{A}_{X_{n,\lambda}}$ and $B_\lambda \in \mathcal{A}_{Y_{n,\lambda}}$, with $\|A_{n,\lambda}\|, \|B_{n,\lambda}\| \le 1$ such that,

$$\|\tilde{A} - A_\lambda\| \le D_{n,X,\lambda} \tag{2.36}$$

$$\|\tilde{B} - B_\lambda\| \le D_{n,Y,\lambda}, \tag{2.37}$$

where $D_{n,X,\lambda}$ and $D_{n,Y,\lambda}$ have the desired expectation bound. Therefore,

$$\|[\tau_t^{\tilde{H}_n}(\tilde{A}), \tilde{B}]\| \le 2\left(D_{X,\lambda,n} + D_{Y,\lambda,n}\right) + \|[\tau_t^{\tilde{H}_n}(A_\lambda), B_\lambda]\|. \tag{2.38}$$

Now,

$$\|[\tau_t^{\tilde{H}_n}(A_\lambda), B_\lambda]\| = \|[\tau_t^{\tilde{H}_{X,Y}}(A_\lambda), B_\lambda]\| \tag{2.39}$$

where

$$\tilde{H}_{X,Y}(\omega) = \sum_{\substack{Z\subset[0,n]:\\ Z\cap X_{n,\lambda}, Z\cap Y_{n,\lambda}\neq\varnothing}} \sum_{\mathbf{m}\in\prod_{x\in Z}\{2,\ldots,d_x\}} \phi_n(\mathbf{m}, Z) \prod_{z\in Z} S_{\mathbf{m}_z;z} \tag{2.40}$$

Note that $\tilde{H}_{X,Y}$ consist of the terms of $\tilde{H}_n$ which do not in general commute with either A_λ or B_λ. If $f(t) = [\tau_t^{\tilde{H}_{X,Y}}(A_\lambda), B_\lambda]$, then

$$f'(t) = i[[\tilde{H}_{X,Y}, \tau_t^{\tilde{H}_{X,Y}}(A_\lambda)], B_\lambda] = -i[f(t), \tilde{H}_{X,Y}] - i[[B_\lambda, \tilde{H}_{X,Y}], \tau_t^{\tilde{H}_{X,Y}}(A_\lambda)] \tag{2.41}$$

Since the first term on the right is norm preserving, we have that,

$$\|[\tau_t^{\tilde{H}_{X,Y}}(A_\lambda), B_\lambda]\| \leq 4|t|\|\tilde{H}_{X,Y}\|. \tag{2.42}$$

The estimate,

$$\begin{aligned}
\|\tilde{H}_{X,Y}\| &\leq \sum_{\substack{Z\subset[0,n]:\\ Z\cap X_{n,\lambda}, Z\cap Y_{n,\lambda}\neq\varnothing}} \left\| \sum_{\mathbf{m}\in\prod_{x\in Z}\{2,\ldots,d_x\}} \phi_n(\mathbf{m}, Z) \prod_{z\in Z} S_{\mathbf{m}_z;z} \right\| \\
&\leq \sum_{\substack{x\in X_{n,\lambda}\\ y\in Y_{n,\lambda}}} \sum_{\substack{Z:\\ x,y\in Z}} \left\| \sum_{\mathbf{m}\in\prod_{x\in Z}\{2,\ldots,d_x\}} \phi_n(\mathbf{m}, Z) \prod_{z\in Z} S_{\mathbf{m}_z;z} \right\| \leq C_n(\omega) \sum_{\substack{x\in X_{n,\lambda}\\ y\in Y_{n,\lambda}}} F(|x-y|),
\end{aligned}$$

together with (2.38) completes the proof. □

The existence of LIOMs of the second kind for uniform-in-time dynamically localized systems follows from a combination of quasi-locality arguments and compactness.

Proof of Theorem 2.2.2. We first show how to construct a sequence T_n for which the limit in (2.19) exists almost surely for any dynamics that is sufficiently localized uniformly in time. For $A \in \mathcal{A}_X^1$ and $T > 0$, define

$$A_T = \frac{1}{T}\int_0^T \tau_t(A)dt.$$

A_T is random since τ_t is.

For each $N \in \mathbb{N}$, let Π_N denote the conditional expectation $\mathcal{A}^{\mathrm{loc}} \to \mathcal{A}_{X(N)}$ defined as the limit of the normalized partial trace over the complement of $X(N) = \{y \in \mathbb{Z} : d(y, X) < N\}$ (see [**71**, Section 4.2]). Since the dynamics τ_t is assumed to satisfy (2.3), we have

$$\mathbb{E}(\sup_T \|\Pi_N(A_T) - A_T\|) \leq CF(N) \tag{2.43}$$

where $C = 2\chi(|X|)$. In particular, $\sum_{N=1}^{\infty} F(N) < \infty$ implies that

$$\lim_N \sup_T \|\Pi_N(A_T) - A_T\| = 0 \text{ almost surely} \tag{2.44}$$

Since $\mathcal{A}^1_{X(N)}$ is compact, there exists a sequence $(T_n^{(N)})_{n\geq 1}$, and $A(N) \in \mathcal{A}^1_{X(N)}$ such that

$$\lim_n \Pi_N(A_{T_n^{(N)}}) = A(N).$$

We can pick the sequences $(T_n^{(N)})_{n\geq 1}$ such that $(T_n^{(N+1)})_{n\geq 1}$ is a subsequence of $(T_n^{(N)})_{n\geq 1}$, for all N. Fix $\epsilon > 0$, and let $N \leq M$. Choose $K(N, M)$ such that for all $n \geq K(N, M)$, we have

$$\|\Pi_N(A_{T_n^{(N)}}) - A(N)\| \leq \epsilon, \ \|\Pi_M(A_{T_n^{(M)}}) - A(M)\| \leq \epsilon.$$

Since $N \leq M$, $(T_n^{(M)})_{n\geq 1}$ is a subsequence of $(T_n^{(N)})_{n\geq 1}$. Therefore, we also have

$$\|\Pi_N(A_{T_n^{(M)}}) - A(N)\| \leq \epsilon, \text{ for all } n \geq K(N, M).$$

Using these bounds we have

$$\begin{aligned}
\|A(N) - A(M)\| &\leq 2\epsilon + \|\Pi_N(A_{T_n^{(M)}}) - \Pi_M(A_{T_n^{(M)}})\| \\
&\leq 2\epsilon + \|\Pi_N(A_{T_n^{(M)}}) - A_{T_n^{(M)}}\| + \|\Pi_M(A_{T_n^{(M)}}) - A_{T_n^{(M)}}\| \\
&\leq 2\epsilon + \sup_T \|\Pi_N(A_T) - A_T\| + \sup_T \|\Pi_M(A_T) - A_T\|.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this estimate along with (2.44) shows that $(A(N))_N$ is almost surely a Cauchy sequence in $\mathcal{A}_{\mathbb{Z}}$. Denote its limit by $\tilde{A}$.

We can now pick an increasing sequence K_N such that for all $n \geq K_N$ we have

$$\|\Pi_N(A_{T_n^{(N)}}) - A(N)\| \leq \frac{1}{N}.$$

Then

$$\lim_N \Pi_N(A_{T_{K_N}^{(N)}}) = \lim_N A(N) = \tilde{A}.$$

Since we also have

$$\|\Pi_N(A_{T_{K_N}^{(N)}}) - A_{T_{K_N}^{(N)}}\| \leq \sup_T \|\Pi_N(A_T) - A_T\|,$$

we can conclude the convergence of the sequence of time averages:

$$\lim_N A_{T_{K_N}^{(N)}} = \tilde{A}. \tag{2.45}$$

The time-invariance of $\tilde{A}$ is obvious from the fact that it is the limit of time averages as in (2.45). By taking the $\limsup$ of (2.43) we also obtain a quasi-locality estimate for $\tilde{A}$:

$$\mathbb{E}(\|[\tilde{A}, B]\|) \leq CF(d(X, \mathrm{supp} B)) \tag{2.46}$$

We can now apply this to $A = h_x$ and, possibly after taking another subsequence, obtain a sequence of times T_n such that for all $x \in \mathbb{Z}$,

$$\tilde{h}_x = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} \tau_t(h_x) dx. \tag{2.47}$$

are well-defined, time-invariant, and quasi-local. The model is assumed to be finite range, so the constant C can be chosen to be uniform in x.

Finally, the quasi-local Hamiltonians $\tilde{H}_\Lambda$ defined by

$$\tilde{H}_\Lambda = \sum_{x \in \Lambda} \tilde{h}_x,$$

generate the same dynamics τ_t in the thermodynamic limit. To see the last point we once more have to argue we can interchange two limits, which we do next.

Let X be finite, $A \in \mathcal{A}_X^1$, and $\epsilon > 0$. Fix a sufficiently large positive integer M such that for all Λ containing $X(M)$ we have

$$\sum_{x\in\Lambda}[h_x, A] = \delta(A). \tag{2.48}$$

Then, we have

$$\|\delta(A) - \tilde{\delta}(A)\| \leq \left\|\sum_{x\in\Lambda}[h_x, A] - \sum_{x\in\Lambda}[\tilde{h}_x, A]\right\| + \sum_{x\notin\Lambda}\|[\tilde{h}_x, A]\| \tag{2.49}$$

Then, for any $L, n \in \mathbb{N}$, starting from (2.49), we obtain the following estimate:

$$\begin{aligned}
\|\delta(A) - \tilde{\delta}(A)\| &\leq \left\|\sum_{x\in X(M+L)}[h_x, A] - \sum_{x\in X(M+L)}[\tilde{h}_x, A]\right\| + \sum_{x\notin X(M+L)}\|[\tilde{h}_x, A]\| \\
&= \left\|\left[\left(\sum_{x\in X(M+L)}\frac{1}{T_n}\int_0^{T_n}\tau_t^{(X(M+L))}(h_x)\right), A\right] - \sum_{x\in X(M+L)}[\tilde{h}_x, A]\right\| \\
&\quad + \sum_{x\notin X(M+L)}\|[\tilde{h}_x, A]\| \\
&\leq \sum_{x\in X(M+L)\setminus X(M)}\left(\sup_{t\in\mathbb{R}}\|[\tau_t^{(X(M+L))}(h_x), A]\| + \|[\tilde{h}_x, A]\|\right) \\
&\quad + \left\|\sum_{x\in X(M)}\left[\frac{1}{T_n}\int_0^{T_n}\tau_t^{(X(M+L))}(h_x)\,dt - \tilde{h}_x, A\right]\right\| + \sum_{x\notin X(M+L)}\|[\tilde{h}_x, A]\|
\end{aligned}$$

Therefore, almost surely

$$\begin{aligned}
\|\delta(A) - \tilde{\delta}(A)\| &\leq \liminf_{L\to\infty}\sum_{x\notin X(M)}\left(\sup_{t\in\mathbb{R}}\|[\tau_t^{(X(M+L))}(h_x), A]\| + \|[\tilde{h}_x, A]\|\right) \\
&\quad + \left\|\sum_{x\in X(M)}\left[\frac{1}{T_n}\int_0^{T_n}\tau_t(h_x)\,dt - \tilde{h}_x, A\right]\right\|
\end{aligned}$$

Letting $n \to \infty$ in this inequality gives,

$$\|\delta(A) - \tilde{\delta}(A)\| \leq \liminf_{L\to\infty}\sum_{x\notin X(M)}\left(\sup_{t\in\mathbb{R}}\|[\tau_t^{(X(M+L))}(h_x), A]\| + \|[\tilde{h}_x, A]\|\right)$$

almost surely. By Fatou's lemma,

$$\mathbb{E}\liminf_{L\to\infty}\sum_{x\notin X(M)}\left(\sup_{t\in\mathbb{R}}\|[\tau_t^{(X(M+L))}(h_x),A]\|+\|[\tilde{h}_x,A]\|\right)\le 4C\sum_{d=M}^{\infty}F(d)$$

This upper bound is summable in M, therefore,

$$\lim_{M\to\infty}\liminf_{L\to\infty}\sum_{x\notin X(M)}\left(\sup_{t\in\mathbb{R}}\|[\tau_t^{(X(M+L))}(h_x),A]\|+\|[\tilde{h}_x,A]\|\right)=0$$

almost surely, which proves that $\delta(A)=\tilde{\delta}(A)$ with probability 1. □

2.3.2. Proofs of results about transmission time. We will prove Theorem 2.2.3 by utilizing the interaction picture decomposition of the Heisenberg dynamics $\tau_t^{H_n}=\tau_t^{H_n^I}\circ\tau_t^{H_n^0}$, where H_n^I is the time dependent random Hamiltonian given by,

$$H_n^I(\omega,t)=\sum_{x=0}^{n-1}\delta_x(\omega_1)\tau_t^{H_n^0(\omega_0)}(\psi_x) \tag{2.50}$$

We make use of this decomposition of the dynamics in the following way: for an integer $d_n\in[0,n]$, for any $A\in\mathcal{A}_0^1$ by quasilocality of the the dynamics $\tau_t^{H_n^0}$ we can write

$$\tau_t^{H_n^0(\omega_0)}(A)=\tilde{A}(\omega_0,t)+E(\omega_0,t), \tag{2.51}$$

where $\text{supp}(\tilde{A})\subset[0,d_n]$, $\|\tilde{A}\|\le 1$ and

$$\|E(\omega_0,t)\|\le\chi(1)C_{d_n}(\omega_0)(1+|t|^{\beta}) \tag{2.52}$$

where $\mathbb{E}C_{d_n}\le e^{-\eta(d_n+1)}$. Eq. (2.51) gives the following bound,

$$\sup_{\substack{A\in\mathcal{A}_0^1\\ B\in\mathcal{A}_n^1}}\|[\tau_t^{H_n(\omega)}(A),B]\|\le 2\chi(1)C_{d_n}(\omega_0)(1+|t|^{\beta})+\sup_{\substack{A\in\mathcal{A}_{[0,d_n]}^1\\ B\in\mathcal{A}_n^1}}\|\tau_t^{H_n^I(\omega)}(A),B]\|. \tag{2.53}$$

To proceed we will need to derive a suitable Lieb-Robinson bound for the dynamics $\tau_t^{H_n^I}$. The first step in deriving such a bound is to write H_n^I in terms of a suitable time dependent random interaction.

First we introduce some notation. Let $\Lambda_n = [0,n]$ and $\Lambda_{n;x}(m) = \{y \in \Lambda_n : d(y, \{x, x+1\}) \leq m\}$. We write

$$\tau_t^{H_n^0(\omega_0)}(\psi_x) = \sum_{m \geq 0} \psi_{n;x}^{(m)}(\omega_0, t), \tag{2.54}$$

where

$$\psi_{n;x}^{(m)}(t) = \begin{cases} \mathrm{Tr}_{\mathcal{H}_{\Lambda_n \setminus \Lambda_{n;x}(0)}} \left(\tau_t^{H_n^0}(\psi_x) \right) & \text{if } m = 0 \\ [\mathrm{Tr}_{\mathcal{H}_{\Lambda_n \setminus \Lambda_{n;x}(m)}} - \mathrm{Tr}_{\mathcal{H}_{\Lambda_n \setminus \Lambda_{n;x}(m-1)}}] \left(\tau_t^{H_n^0}(\psi_x) \right) & \text{if } m \geq 1 \end{cases} \tag{2.55}$$

Here Tr denotes the normalized partial trace operator. Note that the sum in Eq. (2.54) is actually a finite sum, since $\psi_{n;x}^{(m)} = 0$ for any m such that $\Lambda_{n;x}(m-1) = \Lambda_n$.

PROPOSITION 2.3.1. *supp*$(\psi_{n;x}^{(m)}(t)) \subseteq \Lambda_{n;x}(m)$ *for all* $m \geq 0$ *and*

$$\|\psi_{n;x}^{(m)}(t)\| \leq \begin{cases} \|\psi_x\| & \textit{if } m = 0 \\ \|\psi_x\| C_{n;x}^{(m)} (1 + |t|^\beta) & \textit{if } m \geq 1 \end{cases} \tag{2.56}$$

where $C_{n;x}^{(m)}$ *is a non-negative random variable satisfying*

$$\mathbb{E} C_{n;x}^{(m)} \leq 2\chi(2) e^{-\eta m} \tag{2.57}$$

PROOF. $\mathrm{supp}(\psi_{n;x}^{(m)}(t)) \subseteq \Lambda_{n;x}(m)$ follows from properties of the partial trace. The bound $\|\psi_{n;x}^{(0)}(t)\| \leq \|\psi_x\|$ is immediate. For $m \geq 1$,

$$\begin{aligned} \|\psi_{n;x}^{(m)}(t)\| &\leq \|\tau_t^{H_n^0}(\psi_x) - \mathrm{Tr}_{\mathcal{H}_{\Lambda_n \setminus \Lambda_{n;x}(m)}} \left(\tau_t^{H_n^0}(\psi_x) \right) \| \\ &\quad + \|\tau_t^{H_n^0}(\psi_x) - \mathrm{Tr}_{\mathcal{H}_{\Lambda_n \setminus \Lambda_{n;x}(m-1)}} \left(\tau_t^{H_n^0}(\psi_x) \right) \| \end{aligned} \tag{2.58}$$

$$\leq \|\psi_x\| \chi(2) \left(C_{\Lambda_n; \Lambda_x, \Lambda_n \setminus \Lambda_{n;x}(m)} + C_{\Lambda_n; \Lambda_x, \Lambda_n \setminus \Lambda_{n;x}(m-1)} \right) |t|^\beta \tag{2.59}$$

$$\equiv \|\psi_x\| C_{n;x}^{(m)} (1 + |t|^\beta). \tag{2.60}$$

The expectation bound on $C_{n;x}^{(m)}$ follows from the assumptions. $\square$

The decomposition given in Eq. (2.54) provides a way to write $H_n^I(t)$ in terms of a random interaction. Define $\Phi_n(\omega,t) : \mathcal{P}(\Lambda_n) \to \mathcal{A}_{\Lambda_n}$ by,

$$\Phi_n(\omega,t)(X) = \sum_{\substack{(x,m):\\ \Lambda_{n;x}(m)=X}} \delta_x(\omega_1)\psi_{n;x}^{(m)}(\omega_0,t). \tag{2.61}$$

Then $H_n^I = \sum_{X\subseteq[0,n]} \Phi_n(X)$ follows from Eq. (2.54).

We will use Theorem 3.1 of [**71**] in order to obtain a Lieb-Robinson bound for the dynamics generated by H_n^I. If we apply that theorem directly to Φ_n, with a suitable decaying function F, we obtain a Lieb-Robinson bound with a time growth factor of

$$\exp\left(\int_0^t \sup_{x,y\in[0,n]} \frac{1}{F(|x-y|)} \sum_{\substack{X\subseteq[0,n]\\ x,y\in X}} \|\Phi_n(\omega,s)(X)\| ds\right) \tag{2.62}$$

This will not be of any use to us, as

$$\sup_{x,y\in[0,n]} \frac{1}{F(|x-y|)} \sum_{\substack{X\subseteq[0,n]\\ x,y\in X}} \|\Phi_n(\omega,s)(X)\| \tag{2.63}$$

will be of order 1 due to the presence of non-zero δ_x. To remedy this we observe that the methods used in [**71**] produce Lieb-Robinson bounds which are independent of on-site terms in the interaction and also do not depend on the dimension of the Hilbert spaces at each site. This allows us to define a new lattice for the model, which is effectively a subset of $[0,n]$, by identifying certain spins which forces certain interaction terms to become on-site terms. As we explain below, we will be able to obtain a better Lieb-Robinson bound using this method. Specifically, given $\Gamma \subset [0,n]$, we can define the lattice to obtain a Lieb-Robinson bound for the dynamics generated by H_n^I with a time growth factor of

$$\exp\left(\int_0^t \sup_{x,y\in\Gamma} \frac{1}{F(|x-y|)} \sum_{\substack{X\subseteq[0,n]\\ x,y\in X}} \|\Phi_n(\omega,s)(X)\| ds\right). \tag{2.64}$$

Note than in Eq. (2.64) the supremum in the exponent is taken over pairs of points $x, y \in \Gamma$, as opposed to in Eq. (2.62) where all possible pairs of points in $[0, n]$ enter. The sum in Eq. (2.64) therefore excludes any interaction term whose support does not contain a point of Γ. The arguments for obtaining such a Lieb-Robinson bound given the subset Γ are given in detail in the appendix.

It remains to specify how Γ should be chosen. We know that intervals I of length $L \sim \log_{1/p}(n)$ with the property that $\delta_x = 0$ for all $x \in I$ exist with high probability. The interaction terms $\Phi_n(X)$ decay exponentially in the diameter of X, so the sum of all interaction terms linking sites $x, y \in I$ will decay exponentially in the distance $d(\{x,y\}, I^c)$. This suggests that we take Γ to consist of the intervals I with a collar of length ℓ removed from both sides. The interaction terms linking sites $x, y \in \Gamma$ will then decay at least as fast as $e^{-\eta\ell}$. Taking ℓ to be a fraction of L leads to power law decay in n of the interaction strength. The following Lemma makes this precise.

LEMMA 2.3.1. *Fix $n \in \mathbb{N}$ and consider the time dependent random interaction Φ_n given by Eq. (2.61). Let $\theta \in (0,1)$ be arbitrary. Consider an event $E \subset \Omega_1$ with the following two properties:*

(i) $(\delta_1, ..., \delta_{n-1})$ is fixed on E

(ii) There are two disjoint intervals $I_j = [a_j, b_j]$, $j = 1, 2$, with $|I_j| \geq \theta \log_{1/p}(n)$ such that $\delta_x\big|_E = 0$ for each $x \in I_1 \cup I_2$.

For $\sigma \in [0, 1/2)$, let $\ell = \lfloor \sigma\theta \log_{1/p}(n) \rfloor$ and define the collared intervals $\tilde{I}_j = [a_j + \ell, b_j - \ell]$. Then for any $x, y \in \tilde{I}_1 \cup \tilde{I}_2$,

$$\mathbb{1}_E(\omega_1) \sum_{\substack{X \subseteq [0,n]: \\ x,y \in X}} \|\Phi_n(\omega, t)(X)\| \leq B_{E;x,y}(\omega_0)(1 + |t|^\beta), \tag{2.65}$$

where there is a constant $\tilde{C}$, depending only on η, such that $B_{E;x,y}$ satisfies,

$$\mathbb{E} B_{E;x,y} \leq \tilde{C} n^{-\frac{\lambda\eta\sigma\theta}{\log(1/p)}} e^{-(1-\lambda)\eta\frac{|x-y|}{2}} \tag{2.66}$$

for any $\lambda \in (0,1)$.

PROOF. First note that for any points $x < y$ in $[0, n]$ the following inequality holds,

$$\sum_{\substack{X \subseteq [0,n]: \\ x,y \in X}} \|\Phi_n(\omega, t)(X)\| \leq \sum_{z=0}^{n-1} \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} \delta_z(\omega_1) \|\psi_{n;z}^{(m)}(\omega_0, t)\|. \tag{2.67}$$

This follows from the fact that $\max\{|z-x|, |z-y+1|\}$ is the smallest integer m such that $x, y \in \Lambda_{n;z}(m)$. Without loss of generality assume $a_1 < a_2$, and take $x \leq y \in \tilde{I}_2 \cup \tilde{I}_2$. Suppose $x \in \tilde{I}_s$, $y \in \tilde{I}_r$ with $s \leq r$. On the event E, $\delta_z(\omega_1) = 0$ if $z \in I_1 \cup I_2$, so we have the bound

$$\begin{aligned}
&\mathbb{1}_E(\omega_1) \sum_{z=0}^{n-1} \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} \delta_z(\omega_1) \|\psi_{n;z}^{(m)}(\omega_0, t)\| \\
&\leq \sum_{z \notin I_1 \cup I_2} \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} \|\psi_{n;z}^{(m)}(\omega_0, t)\| \\
&\leq (\sup_x \|\psi_{x,x+1}\|) \sum_{z \notin I_r \cup I_s} \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} C_{n;z}^{(m)}(\omega_0)(1 + |t|^\beta) \equiv B_{E;x,y}(\omega_0)(1 + |t|^\beta).
\end{aligned} \tag{2.68}$$

By Proposition 2.3.1,

$$\mathbb{E}B_{E;x,y} \leq 2(\sup_x \|\psi_x\|)\chi(2) \sum_{z \notin I_r \cup I_s} \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} e^{-\eta m} \tag{2.69}$$

We have,

$$\sum_{z \notin I_r \cup I_s} \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} e^{-\eta m} = \left(\sum_{z=0}^{a_r - 1} + \sum_{z=b_r+1}^{a_s - 1} + \sum_{z=b_s+1}^{n-1} \right) \sum_{\substack{m \geq \\ \max\{|z-x|,|z-y+1|\}}} e^{-\eta m} \tag{2.70}$$

We first estimate,

$$\left(\sum_{z=0}^{a_r-1}+\sum_{z=b_s+1}^{n-1}\right)\sum_{\substack{m\geq\\ \max\{|z-x|,|z-y+1|\}}} e^{-\eta m} \leq \left[\sum_{z=0}^{a_r-1}\sum_{m=y-z-1}^{\infty} e^{-\eta m}+\sum_{z=b_s+1}^{n-1}\sum_{m=z-x}^{\infty} e^{-\eta m}\right]$$

$$\leq \sum_{k=y-a_r}^{\infty}\sum_{m=k}^{\infty} e^{-\eta m}+\sum_{k=b_s-x}^{\infty}\sum_{m=k}^{\infty} e^{-\eta m}$$

$$=\frac{1}{(1-e^{-\eta})^2}\left[e^{-\eta(y-a_r)}+e^{-\eta(b_s-x)}\right]$$

$$\leq \frac{1}{(1-e^{-\eta})^2}e^{-\eta[(y-x)+\ell]}, \tag{2.71}$$

where we used that $b_s-y, x-a_r\geq \ell$ in the last line. The remaining sum in Eq. (2.70) vanishes when $r=s$. If $r<s$ then,

$$\sum_{z=b_r+1}^{a_s-1}\sum_{\substack{m\geq\\ \max\{z-x,y-z-1\}}} e^{-\eta m} \leq \sum_{z=b_r+1}^{\lceil\frac{y+x-1}{2}\rceil-1}\sum_{m=y-z-1}^{\infty} e^{-\eta m}+\sum_{z=\lceil\frac{y+x-1}{2}\rceil}^{a_s-1}\sum_{m=z-x}^{\infty} e^{-\eta m}$$

$$\leq \sum_{k=y-\lceil\frac{y+x-1}{2}\rceil}^{\infty}\sum_{m=k}^{\infty} e^{-\eta m}+\sum_{k=\lceil\frac{y+x-1}{2}\rceil-x}^{\infty}\sum_{m=k}^{\infty} e^{-\eta m}$$

$$\leq \frac{1}{(1-e^{-\eta})^2}\left[e^{-\eta(y-\lceil\frac{y+x-1}{2}\rceil)}+e^{-\eta(\lceil\frac{y+x-1}{2}\rceil-x)}\right]$$

$$\leq \frac{1}{(1-e^{-\eta})^2}e^{\frac{\eta}{2}}e^{-\eta(\frac{y-x}{2})} \tag{2.72}$$

If $r<s$, then $|x-y|\geq 2\ell$ and

$$e^{-\eta(\frac{y-x}{2})}=e^{-\eta\lambda(\frac{y-x}{2})}e^{-\eta(1-\lambda)(\frac{y-x}{2})}\leq e^{-\lambda\eta\ell}e^{-(1-\lambda)\eta(\frac{y-x}{2})} \tag{2.73}$$

Therefore,

$$\sum_{z\notin I_1\cup I_2}\sum_{\substack{m\geq\\ \max\{|z-x|,|z-y+1|\}}} e^{-\eta m}\leq \frac{1}{(1-e^{-\eta})^2}\left(e^{-\eta(|x-y|+\ell)}+(1-\delta_{s,r})e^{\frac{\eta}{2}}e^{-\lambda\eta\ell}e^{-(1-\lambda)\eta\frac{|x-y|}{2}}\right), \tag{2.74}$$

which together with Eq. (2.69) proves the lemma. □

We now use Lemma 2.3.1 to prove that a Lieb-Robinson bound holds for the dynamics $\tau_t^{H_n^I}$ on an event contained in E which has probability nearly that of E for large n.

LEMMA 2.3.2. *Assume the hypotheses and notation of Lemma 2.3.1, with the additional assumption that* $|I_j| \leq \frac{3}{2}\log_{1/p}(n)$ *for* $j = 1, 2$. *Then for any* $\nu \in (0,1)$ *there is an event* $W_E \subset \Omega_0$ *such that for any* $\xi \in (0,1)$ *there are positive constants* c_0 *and* c_1*, which depend only on* ν, ξ, λ *and* η*, such that*

$$\mathbb{1}_{W_E}(\omega_0)\mathbb{1}_E(\omega_1) \sup_{\substack{A \in \mathcal{A}^1_{[0,a_1]} \\ B \in \mathcal{A}^1_n}} \|[\tau_t^{H_n^I(\omega)}(A), B]\| \leq c_0(e^{c_1 n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}}(|t|+|t|^{\beta+1})} - 1)e^{-\xi\frac{\nu(1-\lambda)\eta}{2}d(I_1,I_2)}. \tag{2.75}$$

Furthermore, the event W_E *satisfies,*

$$\Pr(W_E) \geq 1 - \tilde{C}' n^{-\frac{(1-\nu)\lambda\eta\sigma\theta}{\log(1/p)}} \log_{1/p}(n) \tag{2.76}$$

where

$$\tilde{C}' = \frac{3\tilde{C}}{1 - e^{-\frac{(1-\nu)(1-\lambda)\eta}{2}}} \tag{2.77}$$

PROOF. For a fixed pair x, y in $\tilde{I}_1 \cup \tilde{I}_2$, by Markov's inequality and Lemma 2.3.1,

$$\Pr(B_{E;x,y} \leq n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}} e^{-\nu(1-\lambda)\eta\frac{|y-x|}{2}}) \geq 1 - \tilde{C} n^{-\frac{(1-\nu)\lambda\eta\sigma\theta}{\log(1/p)}} e^{-(1-\nu)(1-\lambda)\eta\frac{|x-y|}{2}} \tag{2.78}$$

Let

$$W_E = \{B_{E;x,y} \leq n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}} e^{-\nu(1-\lambda)\eta\frac{|y-x|}{2}} \text{ for all } x, y \in \tilde{I}_1 \cup \tilde{I}_2\} \tag{2.79}$$

It follows that,

$$
\begin{aligned}
\Pr(W_E) &\geq 1 - \tilde{C} n^{-\frac{(1-\nu)\lambda\eta\sigma\theta}{\log(1/p)}} \sum_{\substack{x \leq y: \\ x,y \in \tilde{I}_1 \cup \tilde{I}_2}} e^{-(1-\nu)(1-\lambda)\eta\frac{|x-y|}{2}} \\
&\geq 1 - \tilde{C} n^{-\frac{(1-\nu)\lambda\eta\sigma\theta}{\log(1/p)}} \sum_{x \in \tilde{I}_1 \cup \tilde{I}_2} \sum_{y=x}^{\infty} e^{-(1-\nu)(1-\lambda)\eta\frac{|x-y|}{2}} \\
&\geq 1 - \frac{3\tilde{C}}{1 - e^{-\frac{(1-\nu)(1-\lambda)\eta}{2}}} n^{-\frac{(1-\nu)\lambda\eta\sigma\theta}{\log(1/p)}} \log_{1/p}(n) \qquad (2.80)
\end{aligned}
$$

Let F be any F-function on $\mathbb{Z}$ such that for any $c > 0$,

$$
\sup_{x \in \mathbb{Z}} \frac{e^{-c|x|}}{F(|x|)} < \infty. \qquad (2.81)
$$

Then by Lemma 2.3.1 and the definition of W_E we have that,

$$
\begin{aligned}
&\mathbb{1}_{E \cap W_E}(\omega) \sup_{x,y \in \tilde{I}_1 \cup \tilde{I}_2} \frac{1}{e^{-\xi\frac{\nu(1-\lambda)\eta}{2}|x-y|} F(|x-y|)} \sum_{\substack{X \subseteq [0,n]: \\ x,y \in X}} \|\Phi_n(\omega, t)(X)\| \\
&\leq n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}} \sup_{x,y \in \tilde{I}_1 \cup \tilde{I}_2} \frac{e^{-\nu(1-\xi)(1-\lambda)\eta\frac{|y-x|}{2}}}{F(|x-y|)} \\
&\leq n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}} \sup_{x \in \mathbb{Z}} \frac{e^{-\frac{\nu(1-\xi)(1-\lambda)\eta}{2}|x|}}{F(|x|)} \qquad (2.82)
\end{aligned}
$$

The result now follows from Proposition 2.5.2 in the appendix, using the collection $\mathcal{I} = \{\tilde{I}_1, \tilde{I}_2\}$. $\square$

From Lemma 2.3.2, we see that the best Lieb-Robinson bound will be obtained on events E where the intervals I_1 and I_2 are as far apart as possible. This in fact occurs with high probability: Let $\theta \in (0, 1)$ and suppose F_n is the event that there are two intervals of consecutive 0's of length at least $\theta \log_{1/p}(n)$ in n i.i.d. Bernoulli trials, such that the distance r_n between the intervals satisfies $\lim r_n/n = 1$. Then the probability of F_n tends to 1 as n tends to infinity. This can be seen by noting that if $\theta' \in (\theta, 1)$, then the longest run R_n of zeros in $\lfloor n^{\theta'} \rfloor$ i.i.d. Bernoulli trials has the property that

$$
\frac{R_n}{\theta' \log_{1/p}(n)} \to 1 \qquad (2.83)
$$

in probability. Therefore, with a probability tending to 1, there is an interval of length at least $\theta\log_{1/p}(n)$ in both the first and last $\lfloor n^{\theta'}\rfloor$ trials in n Bernoulli trials. The distance between these two intervals is at least $n-2n^{\theta'}$.

Proof of Theorem 2.2.3. We will prove the result for $\beta>0$. The case $\beta=0$ requires only minor modifications. We will show that under the hypotheses of the theorem there is a sequence of events Q_n with $\lim_{n\to\infty}\Pr(Q_n)=1$, and a deterministic sequence x_n satisfying $\lim_{n\to\infty} n^\gamma/x_n=0$ such that

$$\mathbb{1}_{Q_n} t_n(e^{-\alpha\eta n})\geq x_n. \tag{2.84}$$

From this it easily follows that $n^\gamma/t_n(e^{-\alpha\eta n})\to 0$ in probability.

Let $\kappa\in(\alpha,1)$. Our starting point is Eq. (2.53), with $d_n=\lfloor\kappa n\rfloor$. Consider the event $F_n=\{2\chi(1)C_{d_n}\leq n^{-(\gamma\beta+1)}e^{-\alpha\eta n}\}$. By Markov's inequality,

$$\Pr(F_n)\geq 1-2\chi(1)n^{\gamma\beta+1}e^{-(\kappa-\alpha)\eta n}. \tag{2.85}$$

It follows from Eq. (2.53) that,

$$\mathbb{1}_{F_n}(\omega_0)\sup_{\substack{A\in\mathcal{A}_0^1\\ B\in\mathcal{A}_n^1}}\|[\tau_t^{H_n(\omega)}(A),B]\|\leq(1+|t|^\beta)n^{-\gamma\beta+1}e^{-\alpha\eta n}+\sup_{\substack{A\in\mathcal{A}_{[0,d_n]}^1\\ B\in\mathcal{A}_n^1}}\|\tau_t^{H_n^I(\omega)}(A),B]\|. \tag{2.86}$$

Choose $\theta\in(0,1)$, and let $G_n\subset\Omega_1$ denote a sequence of events in which there are two runs of zeros in the list $(\delta_{d_n},...,\delta_{n-1})$ of length at least $\theta\log_{1/p}(n)$ and no more than $\frac{3}{2}\log_{1/p}(n)$, and such that if r_n denotes the distance between the two runs, $\lim_{n\to\infty} r_n/n\to(1-\kappa)$. We have observed that such a sequence can be chosen with $\lim_{n\to\infty}\Pr(G_n)=1$. Write,

$$G_n=\bigsqcup_{E\in\mathcal{F}_n} E, \tag{2.87}$$

where $\mathcal{F}_n$ is the set of events $E \subset \Omega_1$ on which $(\delta_{d_n}, \delta_{d_n+1}, ...\delta_{n-1})$ is fixed. Consider an event $E \in \mathcal{F}_n$. By Lemma 2.3.2 we have that,

$$\mathbb{1}_{W_E}(\omega_0)\mathbb{1}_E(\omega_1) \sup_{\substack{A\in\mathcal{A}^1_{[0,d_n]}\\ B\in\mathcal{A}^1_n}} \|[\tau_t^{H^I_n(\omega)}(A), B]\| \leq c_0(e^{c_1 n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}}(|t|+|t|^{\beta+1})} - 1)e^{-\xi\frac{\nu(1-\lambda)\eta}{2}r_n}. \tag{2.88}$$

Note that Eq. (2.85) and Lemma 2.3.2 imply that for each $E \in \mathcal{F}_n$,

$$\Pr(W_E \cap F_n) \geq 1 - 2\chi(1)n^{\gamma\beta+1}e^{-(\kappa-\alpha)\eta n)} - \tilde{C}' n^{-\frac{(1-\nu)\lambda\eta\sigma\theta}{\log(1/p)}} \log_{1/p}(n) \equiv X_n. \tag{2.89}$$

Clearly $X_n \to 1$ as $n \to \infty$. Now define $Q_n = \sqcup_{E\in\mathcal{F}_n} E \cap W_E \cap F_n$. By independence and Eq. (2.89),

$$\Pr(Q_n) = \sum_{E\in\mathcal{F}_n} \Pr(E)\Pr(W_E \cap F_n) \geq X_n \sum_{E\in\mathcal{F}_n} \Pr(E) = X_n \Pr(G_n), \tag{2.90}$$

which shows that $\Pr(Q_n) \to 1$ as $n \to \infty$.

We now show that the transmission time has a deterministic lower bound on the event Q_n. Eqs. (2.86) and (2.88) give the bound,

$$\mathbb{1}_{Q_n}(\omega) \sup_{\substack{A\in\mathcal{A}^1_0\\ B\in\mathcal{A}^1_n}} \|[\tau_t^{H_n(\omega)}(A), B]\| \leq (1+|t|^\beta)n^{-(\gamma\beta+1)}e^{-\alpha\eta n} + c_0(e^{c_1 n^{-\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}}(|t|+|t|^{\beta+1})} - 1)e^{-\xi\frac{\nu(1-\lambda)\eta}{2}r_n} \tag{2.91}$$

It follows that

$$\mathbb{1}_{Q_n} t_n(e^{-\alpha\eta n}) \geq \min\{(\tfrac{1}{2}n^{\gamma\beta+1} - 1)^{\frac{1}{\beta}}, Y_n\} \equiv x_n, \tag{2.92}$$

where

$$Y_n = \left[\frac{n^{\frac{\lambda\eta\sigma\theta}{\log(1/p)}}}{2c_1}\log\left(1 + \frac{1}{2c_0}e^{(\xi\frac{\nu(1-\lambda)}{2}\frac{r_n}{n} - \alpha)\eta n}\right)\right]^{\frac{1}{\beta+1}}. \tag{2.93}$$

Since $\lim_{n\to\infty} r_n/n = (1-\kappa)$, we have that

$$\frac{n^{(\frac{\nu\lambda\eta\sigma\theta}{\log(1/p)}+1)/(\beta+1)}}{Y_n} \tag{2.94}$$

converges to a positive constant, provided

$$\xi\frac{\nu(1-\lambda)(1-\kappa)}{2} > \alpha \tag{2.95}$$

One can check that Eq. (2.95) can be satisfied only if $\alpha < 1/3$. In this case, ν and ξ close to 1 can be chosen so Eq. (2.95) is satisfied only if

$$\kappa \in (\alpha, 1-2\alpha) \text{ and } \lambda \in (0, 1-\frac{2\alpha}{1-\kappa}). \tag{2.96}$$

If Eq. (2.95) is satisfied, then Eq. (2.94) implies that

$$\lim_{n\to\infty}\frac{n^\gamma}{Y_n} = 0 \tag{2.97}$$

provided

$$\eta > \frac{[\gamma(\beta+1)-1]}{\nu\lambda\sigma\theta}\log(1/p). \tag{2.98}$$

We conclude that if Eq. (2.98) is satisfied, then $n^\gamma/x_n \to 0$.

We can choose parameters so Eq. (2.98) is satisfied if η is larger than

$$\inf\frac{[\gamma(\beta+1)-1]}{\nu\lambda\sigma\theta}\log(1/p) = \frac{2[\gamma(\beta+1)-1]}{1-\frac{2\alpha}{1-\alpha}}\log(1/p), \tag{2.99}$$

where the infimum is taken over parameter values satisfying Eqs. (2.96) and (2.95). $\square$

The following general proposition is needed to adapt the proof of Theorem 2.2.3 to the thermodynamic limit.

PROPOSITION 2.3.2. *Suppose $\Phi_1, \Phi_2 : \mathcal{P}_0(\mathbb{Z}) \to \mathcal{A}_{\mathbb{Z}}^{loc}$ are two F-norm bounded interactions with respect to some F-function. Let $H_\Lambda^j = \sum_{X\subseteq\Lambda}\Phi_j(X)$ denote the corresponding local Hamiltonians for each finite volume $\Lambda \subset \mathbb{Z}$. Let τ_t denote the thermodynamic limit of the model $\Phi_1 + \Phi_2$. Then* *the following limit holds,*

$$\tau_t = \lim_{\Lambda_2 \uparrow \mathbb{Z}} \lim_{\Lambda_1 \uparrow \mathbb{Z}} \tau_t^{H^1_{\Lambda_1} + H^2_{\Lambda_2}} \tag{2.100}$$

where the limits are taken along any increasing, exhaustive sequences of finite subsets of $\mathbb{Z}$. For each finite $\Lambda \subset \mathbb{Z}$, $\lim_{\Lambda_1 \uparrow \mathbb{Z}} \tau_t^{H^1_{\Lambda_1} + H^2_{\Lambda}}$ can be expressed in terms of the interaction picture:

$$\lim_{\Lambda_1 \uparrow \mathbb{Z}} \tau_t^{H^1_{\Lambda_1} + H^2_{\Lambda}} = \tau_t^{\Lambda, I} \circ \tau_t^0, \tag{2.101}$$

where τ_t^0 is the thermodynamic limit of the model Φ_1, and $\tau_t^{\Lambda,I}$ is the dynamics generated by the time-dependent, quasi-local Hamiltonian $\tau_t^0(H^2_\Lambda)$.

Armed with Proposition 2.3.2, the proof of Theorem 2.2.4 is nearly identical to the proof of Theorem 2.2.3. Using the decomposition (2.101), one can show that the bound (2.75) in Lemma 2.3.2 holds with $\tau_t^{H_n^I}$ replaced by $\tau_t^{\Lambda,I}$, uniformly for intervals $\Lambda \supseteq [0, n]$. One can then obtain the bound (2.91) with $\tau_t^{H_n}$ replaced by $\tau_t^{I;\Lambda} \circ \tau_t^0$. Taking the limit $\Lambda \uparrow \mathbb{Z}$ gives this bound for the thermodynamic limit, and the proof proceeds exactly as before.

2.4. Applications

As mentioned before, MBL in the sense of dynamical localization without an energy restriction, has been rigorously established only for the random XY chain and partial results exists for the quantum Ising chain. Naturally, applications of the results in this paper, at the moment, are also restricted to these two models. An extension we will not discuss in detail here is to fermion chains. Our arguments go through without change as long the same obvious analogous conditions are satisfied. Generalizing in another direction, one could consider non-random quasi-periodic chains with localization properties such as the Fibonacci chain [**60**] or the fermion models studied by Mastropietro [**61**, **62**].

2.4.1. The Disordered XY Chain. Consider three real-valued sequences μ_j, γ_j and ω_j. These sequences may be random. The finite volume anisotropic XY Hamiltonian in an external field in the z-direction is given by the Hamiltonian

$$H_n^{XY} = \sum_{j=0}^{n-1} \mu_j[(1+\gamma_j)\sigma_j^x\sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y\sigma_{j+1}^y] + \lambda\sum_{j=0}^{n}\omega_j\sigma_j^z, \tag{2.102}$$

acting on $\bigotimes_{x=0}^n \mathbb{C}^2$. Here $\sigma_j^x, \sigma_j^y, \sigma_j^z \in \mathcal{A}_j$ denote the Pauli spin matrices acting on the jth spin. It is well known that the many-body XY Hamiltonian can be written in terms of an effective one-body Hamiltonian via the Jordan-Wigner transformation [**58**]:

$$H_n^{XY} = \mathcal{C}^* M_n \mathcal{C}, \tag{2.103}$$

where $\mathcal{C}^t = (c_0, ..., c_n, c_0^*, ..., c_n^*)$ is a column vector of operators c_j given by

$$c_j = \frac{1}{2}(\sigma_j^x - i\sigma_j^y)\prod_{k=0}^{j-1}\sigma_k^z,$$

and M_n is a 2×2 block matrix,

$$M_n = \begin{pmatrix} A_n & B_n \\ -B_n & -A_n \end{pmatrix}$$

with

$$A_n = \begin{pmatrix} \omega_0 & -\mu_0 & 0 & 0 & 0 \\ -\mu_0 & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -\mu_n \\ 0 & 0 & 0 & -\mu_n & \omega_n \end{pmatrix},$$

and

$$B_n = \begin{pmatrix} 0 & -\mu_0\gamma_0 & 0 & 0 & 0 \\ \mu_0\gamma_0 & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -\mu_n\gamma_n \\ 0 & 0 & 0 & \mu_n\gamma_n & 0 \end{pmatrix}.$$

The following result was proved in [**46**]:

THEOREM 2.4.1. *Suppose that the matrices M_n are exponentially dynamically localized in the following sense: there exist positive constants C and η such that for any integers $n \geq 0$ and $j, k \in [0, n+1]$,*

$$\mathbb{E}[\sup_{t\in\mathbb{R}} |(e^{-itM_n})_{j,k}| + |(e^{-itM_n})_{j,n+k+1}|] \leq Ce^{-\eta|j-k|}. \tag{2.104}$$

Then the Heisenberg dynamics $\tau_t^{H_n^{XY}}$ of the XY-chain is exponentially dynamically localized, uniformly in time, with $\chi(x) = 4^x$.

Theorem 2.4.1 shows that if the sequences μ_j, γ_j and ω_j are such that dynamical localization for the M_n holds, then Theorem 2.2.3 applies to the XY chain. If, in addition $\sup_j \mu_j$ and $\sup_j \gamma_j$ are almost surely finite, then the XY chain satisfies the hypotheses of Theorem 2.2.4.

There are several instances in which the matrices M_n are known to satisfy (2.104). For example, if $\gamma_j = 0$ and $\mu_j = 1$ for all j, and the ω_j are i.i.d. with compactly supported density, then $B_n = 0$ and A_n is the finite volume Anderson model. In this case it is well known that (2.104) holds [**55**]. In [**37**] a large class of random block operators were shown to exhibit exponential dynamical localization at high disorder. Under the assumption that μ_j and γ_j are deterministic and bounded, and that the ω_j are i.i.d. with sufficiently smooth distribution, this class of random block operators includes M_n and (2.104) holds for sufficiently large $|\lambda|$. Therefore in these models the conditions of theorems 2.2.2, 2.2.3 and 2.2.4 are satisfied.

The anisotropic case was also investigated in [**22**]. The methods there prove localization of the M_n for ω_j with compactly supported distribution contained in $(-\infty, -2)$ or $(2, \infty)$. For these results smoothness of the distribution is not needed, however the method produces a bound with a stretched exponential, not an exponential as in (2.104). This localization bound is shown to imply a uniform in time localization bound for the XY chain where the decay is given by a stretched exponential. Therefore disordered anisotropic XY models have LIOMs, as shown by Theorem 2.2.2, but our results do not imply robustness of long transmission times under perturbation.

2.4.2. The Quantum Ising Chain. Another model that has been widely discussed in the literature is the quantum Ising with random coefficients. For concreteness, consider the following family of Hamiltonians for a spin-1/2 systems on a chain $[a,b]\subset\mathbb{Z}$:

$$H_{[a,b]}=\sum_{x=a}^{b-1}J_x\sigma_x^3\sigma_{x+1}^3+\sum_{x=a}^{b}\gamma\Gamma_x\sigma_x^1+h_x\sigma_x^3, \tag{2.105}$$

where $(J_x),(\Gamma_x)$, and (h_x) are three independent sequences of i.i.d. random variables, each with bounded density of compact support.

Mathematical work by John Imbrie and a variety of numerical results point towards the existence of a description of this model in terms of LIOMs of the first kind (Definition 2.2.2). To state the various claims we need to introduce the assumptions made by Imbrie [**52**]. Let $\lambda_\alpha^{[a,b]}$ denote an enumeration of the eigenvalues, which are almost surely simple.

Imbrie's Assumption: There exist γ_0, such that for all $\gamma\in(-\gamma_0,\gamma_0)$, there exists constants $\nu,C>0$, such that for all $\delta>0, a<b\in\mathbb{Z}$ we have

$$\Pr(\min_{\alpha\neq\beta}|\lambda_\alpha^{[a,b]}-\lambda_\beta^{[a,b]}|<\delta)\leq\delta^\nu C^{b-a+1}. \tag{2.106}$$

In [**52**] Imbrie uses a systematic perturbation theory which, under his assumptions, he argues combines with a multi-scale analysis to prove detailed properties about the eigenvectors of the Hamiltonians $H_{[a,b]}$ for sufficiently small γ, uniformly in the length of the chain. We should note, however, that among experts in the multiscale analysis approach to proving localization there is no agreement that such an argument can indeed be carried out along the lines described in [**52**].

In the review paper [**53**, Section 4.3] the following implications of the perturbation analysis of [**52**] are stated: $H_{[a,b]}$ is diagonalized by a quasi-local unitary transformation and the resulting energy eigenvalues when labeled by Ising configurations take the form of a random Ising model with multi-spin interactions of strong decay, i.e., something very similar to the LIOM picture we define in Definition 2.2.2. The LIOM representation is explained by starting from Imbrie's localization property for the eigenvectors $\psi^{[a,b]}$ which reads as follows: there exists $\kappa>0$ such that for all sufficiently long finite intervals $[a,b]$ containing the origin one has

$$\left|1-\mathbb{E}\left[\sum_\alpha\rho_\alpha|\langle\psi_\alpha,\sigma_0^3\psi_\alpha\rangle|\right]\right|\leq\gamma^\kappa,$$

where ρ_α is a probability distribution such as

$$\rho_\alpha = \frac{e^{-\beta\lambda_\alpha^{[a,b]}}}{\sum_\gamma e^{-\beta\lambda_\gamma^{[a,b]}}}.$$

In the spirit of these results it appears that the disordered quantum Ising chain may indeed be a model where the exponential dynamical localization of Definition 2.2.1 and the LIOM picture of Definition 2.2.2 indeed both hold.

2.5. Appendix: Lieb-Robinson Bounds

In this appendix we develop a bound on the velocity of propagation under the Heisenberg dynamics which ignores interaction terms supported in a given subset of the lattice. We use the results of [**71**], in which Lieb-Robinson bounds which do not depend on on-site interactions are developed for Hamiltonians expressed in terms of time-dependent interactions.

Let (Γ, d) denote a countable metric space, and let $\mathcal{P}_0(\Gamma)$ denote the collection of finite subsets of Γ. Assign a spin Hilbert space $\mathcal{H}_x$ to each $x \in \Gamma$. The algebra of local observables is given by $\mathcal{A}^{\text{loc}} = \cup_{X \in \mathcal{P}_0(\Gamma)} \mathcal{A}_X$, where $\mathcal{A}_X = \bigotimes_{x \in X} \mathcal{B}(\mathcal{H})$. A time-dependent interaction $\Phi : \mathbb{R} \times \mathcal{P}_0(\Gamma)$ is called continuous if $t \mapsto \Phi(t, X)$ is norm continuous for every $X \in \mathcal{P}_0(\Gamma)$.

To measure the spatial decay of the interaction we introduce the notion of an F-function. Let (Γ, d) denote a countable metric space. Then an F-function on (Γ, d) is a function $F : [0, \infty) \to (0, \infty)$ such that

(1) F is non-increasing.

(2) F is integrable, i.e.,

$$\|F\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty. \tag{2.107}$$

(3) F satisfies the convolution identity,

$$C_F = \sup_{x,y \in \Gamma} \frac{1}{F(d(x,y))} \sum_{z \in \Gamma} F(d(x,z))F(d(z,y)) < \infty. \tag{2.108}$$

If $\mu > 0$, it is easy to show that $F_\mu(x) = e^{-\mu x}F(x)$ also defines an F function on (Γ, d) with $\|F_\mu\| \leq \|F\|$ and $C_{F_\mu} \leq C_F$.

Given an F-function F, we denote by $\mathcal{B}_F$ the set of continuous interactions $\Phi : \mathbb{R}\times\mathcal{P}_0(\Gamma) \to \mathcal{A}^{\mathrm{loc}}$ such that the function on $\mathbb{R}$

$$t \mapsto \sup_{x,y\in\Gamma} \frac{1}{F(d(x,y))} \sum_{\substack{x,y\in X \\ |X|>1}} \|\Phi(t,X)\| \tag{2.109}$$

is locally bounded.

THEOREM 2.5.1 (Theorem 3.1 in [**71**]). *Let $\Phi \in \mathcal{B}_{F_\mu}$ for some F-function F and $\mu > 0$, and let $X, Y \in \mathcal{P}_0(\Gamma)$ with $X \cap Y = \varnothing$. Then for any $\Lambda \in \mathcal{P}_0(\Gamma)$ with $X \cup Y \subseteq \Lambda$, we have*

$$\sup_{\substack{A\in\mathcal{A}_X^1 \\ B\in\mathcal{A}_Y^1}} \|[\tau_t^{H_\Lambda}(A), B]\| \leq \frac{2\|F\|}{C_{F_\mu}} \min\{|X|, |Y|\}(e^{2C_{F_\mu}I(t)} - 1)e^{-\mu d(X,Y)} \tag{2.110}$$

for every $t \in \mathbb{R}$, where

$$I(t) = \int_{\min\{0,t\}}^{\max\{0,t\}} \sup_{x,y\in\Gamma} \frac{e^{\mu d(x,y)}}{F(d(x,y))} \sum_{\substack{x,y\in X \\ |X|>1}} \|\Phi(s,X)\| ds. \tag{2.111}$$

We will now apply the previous theorem to obtain a Lieb-Robinson bound which ignores interaction terms in certain parts of the lattice. For simplicity we restrict ourselves to one-dimensional finite volume systems. Neither of these restrictions is essential.

Suppose that we have a quantum spin chain $\mathcal{H} = \bigotimes_{x=0}^n \mathcal{H}_x$ on the interval $\Lambda_n = [0, n] \subset \mathbb{Z}_+$ together with a time-dependent Hamiltonian $H(t)$ generated by an interaction $\Phi(t) : \mathcal{P}(\Lambda_n) \to \mathcal{B}(\mathcal{H})$. Let $\mathcal{I} = \{I_j\}_{j=1}^m$ be a collection of disjoint subintervals $I_j = [a_j, b_j] \subset \Lambda_n$, satisfying $b_j < a_{j+1}$. For purposes of notation let $b_0 = 0$ and $a_{m+1} = n$. We seek to define an equivalent spin chain in which the spins located on the sites $[b_j, a_{j+1}]$ are identified. Define the contracted lattice $\Gamma_\mathcal{I}$ by,

$$\Gamma_\mathcal{I} = \cup_{j=1}^m [a_j, b_j) \cup \{n\}$$

Define a map $\mathcal{C} : \Lambda_n \to \Gamma_{\mathcal{I}}$ by,

$$\mathcal{C}(x) = \begin{cases} a_j & \text{if } x \in [b_{j-1}, a_j] \text{ for some } j = 1, 2, ..., m+1 \\ x & \text{Otherwise} \end{cases} \tag{2.112}$$

Note that $\mathcal{C}$ maps a site in Λ_n to its corresponding site in $\Gamma_{\mathcal{I}}$. For each $x \in \Gamma_{\mathcal{I}}$, define

$$\mathcal{H}'_x = \bigotimes_{z \in \mathcal{C}^{-1}(\{x\})} \mathcal{H}_z \tag{2.113}$$

Then $\bigotimes_{x=0}^n \mathcal{H}_x = \bigotimes_{x \in \Gamma_{\mathcal{I}}} \mathcal{H}'_x$, and an observable which has support X in $\mathcal{A}_{\Lambda_n}$ has support $\mathcal{C}(X)$ in $\mathcal{A}_{\Gamma_{\mathcal{I}}}$. Define an interaction $\tilde{\Phi}(t)$ on $\Gamma_{\mathcal{I}}$ by,

$$\tilde{\Phi}(t)(X) = \sum_{\substack{Z \subseteq \Lambda_n \\ \mathcal{C}(Z) = X}} \Phi(t)(Z) \tag{2.114}$$

Then $\tilde{\Phi}$ and Φ generate the same Hamiltonian. With this setup we have the following proposition.

THEOREM 2.5.2. *Suppose d is a metric on $\Gamma_{\mathcal{I}}$. Let $\mu > 0$ and let F denote any F-function on $(\Gamma_{\mathcal{I}}, d)$. Then for any $X, Y \subseteq \Lambda_n$ with $\mathcal{C}(X) \cap \mathcal{C}(Y) = \varnothing$ we have,*

$$\sup_{\substack{A \in \mathcal{A}^1_X \\ B \in \mathcal{A}^1_Y}} \|[\tau^H_t(A), B]\| \leq \frac{2\|F\|}{C_{F_\mu}} \min\{|\mathcal{C}(X)|, |\mathcal{C}(Y)|\}(e^{2C_{F_\mu} I(t)} - 1)e^{-\mu d(\mathcal{C}(X), \mathcal{C}(Y))} \tag{2.115}$$

holds for all $t \in \mathbb{R}$, where

$$I(t) = \int_{\min\{0,t\}}^{\max\{0,t\}} \sup_{x,y \in \Gamma_{\mathcal{I}}} \frac{e^{\mu d(x,y)}}{F(d(x,y))} \sum_{\substack{X \subseteq \Gamma_{\mathcal{I}}: \\ x,y \in X, \\ |X|>1}} \|\tilde{\Phi}(s)(X)\| ds. \tag{2.116}$$

PROOF. Apply Theorem 2.5.1 to the spin model $\tilde{\Phi}$. $\square$

A few remarks about this theorem need to be made. Note that

$$\sum_{\substack{X \subseteq \Gamma_{\mathcal{I}}: \\ x,y \in X, \\ |X|>1}} \|\tilde{\Phi}(t)(X)\| = \sum_{\substack{X \subseteq \Gamma_{\mathcal{I}}: \\ x,y \in X, \\ |X|>1}} \| \sum_{\substack{Z \subseteq \Lambda_n \\ \mathcal{C}(Z) = X}} \Phi(t)(Z)\| \leq \sum_{\substack{Z \subseteq \Lambda_n: \\ x,y \in Z, \\ |\mathcal{C}(Z)|>1}} \|\Phi(t)(Z)\| \tag{2.117}$$

for any pair $x, y \in \Gamma_{\mathcal{I}}$. If $Z \subset [b_{j-1}, a_j]$ for some j, then $\mathcal{C}(Z)$ will contain at most one point of $\Gamma_{\mathcal{I}}$. Therefore Theorem 2.5.2 provides an upper bound on the speed of propagation which excludes elements from the original interaction with support Z.

While Theorem 2.5.2 was stated for an arbitrary metric d on $\Gamma_{\mathcal{I}}$, there are two natural metrics which both allow $(\Gamma_{\mathcal{I}}, d)$ to be isometrically embedded into $\mathbb{Z}_+$. One choice to simply restrict the usual metric on $\mathbb{Z}_+$ to $\Gamma_{\mathcal{I}}$. Another choice is to define d so that $(\Gamma_{\mathcal{I}}, d)$ isometrically embeds into $[0, L]$, where $L = \sum_{j=1}^{m}(b_j - a_j)$. With either of these metrics, given an F-function F on $\mathbb{Z}_+$ with the usual metric, the constants in Theorem 2.5.2 can be chosen to be $c_0 = 2\|F\|/C_{F_\mu}$ and $c_1 = 2C_{F_\mu}$. In particular, these constants do not depend on n or the collection of intervals $\mathcal{I}$. This follows from the fact that $\Gamma_{\mathcal{I}}$ isometrically embeds into $(\mathbb{Z}_+, |\cdot|)$ when equipped with either of these metrics.

CHAPTER 3

Lieb-Robinson bounds and strongly continuous dynamics for a class of many-body fermion systems in $\mathbb{R}^d$

3.1. Introduction

The goal of this paper is to study propagation estimates for interacting fermion systems in $\mathbb{R}^d$, $d \geq 1$, and to apply them to construct the infinite-volume dynamics for a class of such systems as a strongly continuous one-parameter group of automorphisms of the standard canonical anti-commutation relations (CAR) algebra. We introduce a class of short-range, ultra-violet-regularized two-body interactions for which this is possible. Without the use of an ultra-violet (UV) cut-off of some kind, such a result cannot be expected to hold. See, for example, the discussion in [**13**, Introduction to Section 6.3]. Nevertheless, as Sakai notes in the last paragraph of his book [**79**], constructing the dynamics for interacting systems is one of the most important problems. To address this problem, a common approach is to consider the dynamics in representations of the algebra of observables associated with a class of sufficiently regular states. This is not our approach here. Instead we introduce a UV regularization of the interactions. This allow us to construct the infinite system dynamics as automorphisms of the CAR algebra of observables that depend continuously on time. A typical situation where it is advantageous to consider the dynamics on the observables algebra of the infinite system is in non-equilibrium statistical mechanics, where until now one would either use a quasi-free dynamics (as, e.g., in [**39**]) or work in a lattice setting where UV regularization is provided by the lattice (see, e.g., [**8**,**9**,**45**,**49**], and [**77**,**78**] for the original and fundamental existence result for quantum spin systems.).

One broad class of models in which UV degrees of freedom are naturally absent are mean field models and related limiting regimes and the dynamics of such models have been studied including in infinite volume. For example, well-posedness for the Hartree equation in infinite volume, which describes the mean field limit [**33**], has been proved by Lewin and Sabin in [**56**].

The regularization we adopt in this paper is smearing the interactions by Gaussians parameterized by $\sigma > 0$ in such a way that the pair interaction between point particles is recovered in the limit $\sigma \to 0$ (See Appendix 3.6 for a proof). Formally, in second quantization, this leads to a Hamiltonian of the form

$$H_\Lambda^\sigma = \int_{\mathbb{R}^d} (\nabla a_x^* \nabla a_x + V(x) a_x^* a_x) dx + \frac{1}{2} \int_\Lambda \int_\Lambda W(x-y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma) dx dy, \tag{3.1}$$

where V is an external potential such as a smooth periodic function, and W is a short-range two-body interaction. We defer stating precise conditions on V and W until Section 3.2. The smearing is only needed in the interaction and one can take for φ_x^σ an L^1-normalized Gaussian of width σ and centered at $x \in \mathbb{R}^d$. The parameter σ can be interpreted as the size of the particles and, as discussed in Appendix 3.6, restricted to the N-particle Hilbert space, for any finite number of particles N, in either a finite or infinite volume, the dynamics converges to the standard Schrödinger dynamics generated by the self-adjoint Hamiltonian H_N given by

$$H_N = \sum_{k=1}^N (-\Delta_k + V(x_k)) + \sum_{1 \le k < l \le N} W(x_k - x_l). \tag{3.2}$$

Having a state-independent definition of the dynamics has both conceptual and practical advantages. From early on it was realized however that the subtle, non-robust, property of (thermodynamic) stability may be an obstacle to using perturbation series to define Heisenberg dynamics for infinite systems in the continuum [**32**]. Therefore, it is not surprising that attempts were made to construct toy models of interacting theories for which stability could be proved. An early example is [**84**]. In [**85**] an infinite-volume dynamics for interacting fields is obtained using relativistic locality (Minkowski space). The only previous Euclidean construction of infinite-system dynamics on the CAR algebra over $L^2(\mathbb{R}^d)$ that explicitly considers a regularized pair interaction, as far as we are aware, is by Narnhofer and Thirring [**72**]. In that work the authors were motivated by the desire to preserve the Galilean invariance of the dynamics, which led them to employ a somewhat contrived UV regularization. The smearing of the form (3.1) used here is, we believe, more natural and likely to faithfully reproduce the low-energy physics.

Before summarizing our results, we point out that *defining* a dynamics on the CAR algebra over $L^2(\mathbb{R}^d)$ is by itself not the issue. Including pair interactions in a densely defined self-adjoint Hamiltonian on Fock space has been accomplished a long time ago. The corresponding one-parameter group of unitaries can be used to define a dynamics as a group of automorphisms on the bounded operators on Fock space, which includes the CAR algebra. This dynamics, however, is in general *not strongly continuous*. This is because the commutator of the unregularized interaction term with a creation or annihilation operator is unbounded.

Our proof of convergence of the thermodynamic limit of the infinite-volume dynamics hinges on a propagation estimate of Lieb-Robinson type [**57**] for systems in which the interaction is only active in a bounded volume Λ, with estimates that are uniform in Λ. Let $\tau_t^\Lambda(\cdot)$ denote the Heisenberg evolution with the interactions restricted to Λ (see (3.26) for the precise definition) and define the one-particle Schrödinger evolution in the usual way:

$$f_t = e^{-it(-\Delta+V)}f, \quad f \in L^2(\mathbb{R}^d), \quad t \in \mathbb{R}. \tag{3.3}$$

Lieb-Robinson Bound for Schrödinger operators. Let V be given as the Fourier transform of a finite Borel measure of compact support on $\mathbb{R}^d$. For $\sigma > 0$ and $x \in \mathbb{R}^d$, denote by φ_x^σ the L^1 normalized Gaussian on $\mathbb{R}^d$ with mean x and variance σ. Then, there exist constants $C_1, C_2, C_3 > 0$, such that for all $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}$ one has

$$\left|\langle e^{-it(-\Delta+V)}f, \varphi_x^\sigma\rangle\right| \le C_1 e^{C_2|t|\ln|t|} \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-\frac{C_3}{t^2+1}|x-y|}|f(y)|. \tag{3.4}$$

A more detailed estimate and explicit constants are given in Proposition 3.3.1 and Corollary 3.3.1. For discrete Schrödinger operators on graphs a Lieb-Robinson type propagation estimate holds for any real-valued diagonal potential [**4**].

Let τ_t^Λ, be the Heisenberg dynamics generated by H_Λ in (3.1) for bounded $\Lambda \subset \mathbb{R}^d$, and $t \in \mathbb{R}$. We will prove the following result as Theorem 3.2.5.

Propagation Bound for many-body fermion dynamics. *Let $W \in L^\infty(\mathbb{R}^d)$ be real-valued and satisfying $W(-x) = W(x)$ and $|W(x)| \le Ce^{-a|x|}$, for some $C, a > 0$. Then, there exist continuous functions $C(t), a(t) > 0$ such that for all bounded and measurable $\Lambda \subset \mathbb{R}^d$, and $f, g \in$*

$L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$*, one has the following bounds:*

$$\|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \langle e^{-it(-\Delta+V)}f, g\rangle \mathbb{1}\| \leq \|f\|_1 \|g\|_1 e^{C(t)} e^{-a(t) d(supp(f), supp(g))} \tag{3.5}$$

$$\|\{\tau_t^\Lambda(a(f)), a(g)\}\| \leq \|f\|_1 \|g\|_1 e^{C(t)} e^{-a(t) d(supp(f), supp(g))} \tag{3.6}$$

where $d(supp(f), supp(g))$ *denotes the distance between the essential supports of* f *and* g*.*

Explicit forms of $C(t)$ and $a(t)$ are given in Theorem 3.2.5. This Lieb-Robinson type bound provides localization estimates for general elements in the CAR algebra by the usual algebraic relations in the same way as for lattice fermion systems as in [**18**, **48**, **69**].

As an application of this propagation bound above, which is of independent interest, we then prove the existence and continuity of the infinite systems dynamics. See Theorem 3.2.6 for the precise statement. There are other approaches to proving the convergence of the dynamics in the thermodynamic limit. Using propagation bounds, however, yields a short and intuitive proof.

Strongly continuous infinite-volume dynamics. *There exists a strongly continuous one-parameter group of automorphism of the CAR algebra over* $L^2(\mathbb{R}^d)$*,* $\{\tau_t\}_{t\in\mathbb{R}}$*, such that*

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \tau_t^\Lambda(a(f)) = \tau_t(a(f)), \text{ for all } f \in L^2(\mathbb{R}^d). \tag{3.7}$$

The strategy for proving existence of the thermodynamic limit of the Heisenberg dynamics using propagation bounds appears to be quite general and has been employed successfully for lattice systems [**13**, **67**, **71**]. This method works whenever the interactions restricted to a bounded region are described by a bounded self-adjoint operator. It is worth noting that the free part of the dynamics does not require a cut-off for this result to hold. Due to its uniformity in Λ, the propagation bound (3.5) extends to the infinite-system dynamics.

Several generalizations of the propagation bounds could be considered. For Schrödinger operators, we expect that the restrictions on V can be relaxed. The many-body bounds are derived here for regularized pair interactions only. Our approach can handle k-body terms with virtually no changes. A different type of extension of obvious interest would be to consider fermions in an external magnetic field. In contrast, constructing the many-body dynamics for boson systems one has to face an additional element of unboundedness that has long been understood to force one to consider a weaker topology to express the continuity in time [**89**]. Already for boson lattice systems, such as oscillator lattices, Lieb-Robinson bounds can be derived but one finds bounds that are no longer in terms of the operator norm of the observables [**6**, **65**]. Such bounds can nevertheless still be used to prove the existence of infinite-systems dynamics [**67**]. Another approach to define the dynamics of infinite oscillator lattices was developed by Buchholz [**19**], who constructs a strongly continuous dynamics on the Resolvent Algebra [**20**].

The existence of propagation bounds of Lieb-Robinson type and the strongly continuous infinite-volume dynamics for many-body systems with Hamiltonians of the form (3.1) provide a new avenue for applications. For example, if we choose for V a periodic potential, such that $-\Delta + V$ has a band structure with a gap, the non-interacting many-body ground state at suitable fermion density is gapped. We expect this gap to persist in the presence of interactions as in (3.1) with W sufficiently small. Stability of the ground state gap has been proved for broad classes of lattice systems [**15**,**16**,**25**,**30**,**40**,**47**,**54**,**63**,**70**,**80**]. We believe that an analogous result for the continuum systems studied in this paper is now within reach.

3.2. Model and statement of main results

Let $d \geq 1$ and take Δ to be the Laplace operator on $\mathbb{R}^d$. For any real-valued $V \in L^\infty(\mathbb{R}^d)$, we will denote by

$$H_1 = -\Delta + V \tag{3.8}$$

the corresponding (self-adjoint) Schrödinger operator with domain $H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, see [**75**] for more details. As required, we will impose further conditions on V, e.g. see (3.18).

Our goal is to analyze a class of operators on the fermionic Fock space. We will follow closely the notation in [**13**], see specifically Section 5.2.1, and refer the reader there for more details. Let us denote by

$$\mathfrak{F}^- = \bigoplus_{n=0}^{\infty} \left(L^2(\mathbb{R}^d)^{\otimes n}\right)^- \tag{3.9}$$

the anti-symmetric Fock space (Hilbert space) generated by $L^2(\mathbb{R}^d)$. In the above, $L^2(\mathbb{R}^d)^{\otimes n}$ is short for $\bigotimes_{k=1}^n L^2(\mathbb{R}^d)$ and $(\,\cdot\,)^-$ denotes anti-symmetrization. For each $f \in L^2(\mathbb{R}^d)$, take $a(f) \in \mathcal{B}(\mathfrak{F}^-)$, the bounded linear operators over $\mathfrak{F}^-$, to be the annihilation operator corresponding to f, and denote by $a^*(f)$, its adjoint, the corresponding creation operator. It is well-known that these creation and annihilation operators satisfy the canonical anti-commutation relations (CAR)

$$\big\{a(f), a(g)\big\} = 0 \quad \text{and} \quad \big\{a(f), a^*(g)\big\} = \langle f, g\rangle \mathbb{1} \quad \text{for all } f, g \in L^2(\mathbb{R}^d) \tag{3.10}$$

where $\{A, B\} = AB + BA$ denotes the anti-commutator, $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{R}^d)$, and $\mathbb{1}$ is the identity acting on $\mathfrak{F}^-$. In addition, one has that

$$\|a^*(f)\| = \|a(f)\| = \|f\|_2 \quad \text{for all } f \in L^2(\mathbb{R}^d) \tag{3.11}$$

where here, and in the following, $\|\cdot\|_p$ will refer to the L^p-norm for $p \in [1, \infty]$ and $\|\cdot\|$ will denote the operator norm.

The models we will consider are defined in terms of a particular class of annihilation and creation operators. Let $\sigma > 0$, take $x \in \mathbb{R}^d$, and consider the Gaussian $\varphi_x^\sigma : \mathbb{R}^d \to \mathbb{R}$ with

$$\varphi_x^\sigma(y) = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-\frac{|y-x|^2}{2\sigma^2}} \quad \text{for all } y \in \mathbb{R}^d\,. \tag{3.12}$$

We say that φ_x^σ is centered at $x \in \mathbb{R}^d$ with variance σ^2. We have chosen an L^1-normalization, i.e. $\|\varphi_x^\sigma\|_1 = 1$ for all $x \in \mathbb{R}^d$. Given (3.11), it is clear that for any $x \in \mathbb{R}^d$,

$$\|a^*(\varphi_x^\sigma)\|^2 = \|a(\varphi_x^\sigma)\|^2 = \|\varphi_x^\sigma\|_2^2 = (4\pi\sigma^2)^{-d/2} =: C_\sigma \tag{3.13}$$

where we have introduced the notation $C_\sigma > 0$ as this quantity will enter our estimates frequently.

For any bounded and measurable set $\Lambda \subset \mathbb{R}^d$, we will analyze the operator

$$H_\Lambda^\sigma = \mathrm{d}\Gamma(H_1) + W_\Lambda^\sigma \tag{3.14}$$

acting on $\mathfrak{F}^-$, where $\mathrm{d}\Gamma(H_1)$ denotes the second quantization of H_1, again see [**13**] for the definition, and the interaction W_Λ^σ is given by

$$W_\Lambda^\sigma = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\,\mathrm{d}y\, W_\Lambda(x,y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma) \tag{3.15}$$

where $W_\Lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ has the form $W_\Lambda(x,y) = \chi_{\Lambda\times\Lambda}(x,y) W(x-y)$ for some real-valued $W \in L^\infty(\mathbb{R}^d)$ and $\chi_{\Lambda\times\Lambda}(x,y)$ denotes the indicator function of the set $\Lambda\times\Lambda$. In this case, for each fixed $\sigma > 0$ and any $\Lambda \subset \mathbb{R}^d$ that is bounded and measurable, we have that

$$\|W^\sigma_\Lambda\| \leq \frac{1}{2} C_\sigma^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}x \, \mathrm{d}y \, |W_\Lambda(x,y)| \leq \frac{1}{2} \left(\frac{1}{4\pi\sigma^2} \right)^d \|W\|_\infty |\Lambda|^2 \tag{3.16}$$

where $|\Lambda|$ denotes the Lebesgue measure of Λ. Thus for bounded W, $W^\sigma_\Lambda \in \mathcal{B}(\mathfrak{F}^-)$ for each choice of $\sigma > 0$ and $\Lambda \subset \mathbb{R}^d$ as above. We conclude that, in these cases, H^σ_Λ is a well-defined self-adjoint operator on the anti-symmetric Fock space $\mathfrak{F}^-$.

As we progress, more assumptions will need to be made on W. For ease of later reference, we state them here.

Assumption 3.2.1 (On W). *Let $W : \mathbb{R}^d \to \mathbb{R}$ satisfy*

(1) $W \in L^\infty(\mathbb{R}^d)$ is real-valued;

(2) W is symmetric, i.e. $W(-x) = W(x)$ for almost every $x \in \mathbb{R}^d$;

(3) W is short-range, i.e. there are positive numbers a and c_W for which

$$|W(x)| \leq c_W e^{-a|x|} \quad \textit{for almost every } x \in \mathbb{R}^d. \tag{3.17}$$

3.2.1. Bounds on the propagator of one-particle Schrödinger operators. In this section, we derive propagation bounds for one-particle Schrödinger operators with the form H_1 as in (3.8). To make a precise statement, we require the following from the potential V.

Assumption 3.2.2 (On V). *Let $V : \mathbb{R}^d \to \mathbb{C}$ have the form*

$$V(x) = \int_{\mathbb{R}^d} \mathrm{d}\mu(k) \, e^{-ik\cdot x} \tag{3.18}$$

where μ : Borel($\mathbb{R}^d$) $\to \mathbb{R}$ is a Borel measure on $\mathbb{R}^d$ satisfying:

(1) μ has support contained in a ball, i.e. there is some $M > 0$ and $\operatorname{supp} \mu \subset B_M(0)$;

(2) $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are non-negative finite measures on Borel($\mathbb{R}^d$), i.e. $\mu^\pm(\mathbb{R}^d) < \infty$. We set $|\mu| = \mu^+ + \mu^-$;

(3) μ is even, i.e. $\mu(A) = \mu(-A)$ for all $A \in$ Borel($\mathbb{R}^d$).

Under these assumptions, V is the Fourier transform of a signed, compactly supported, finite measure μ, which is real-valued and bounded. Two parameters that will appear in estimates are

C_μ and M, to characterize V. They need not be chosen optimally but should satisfy

$$\int_{\mathbb{R}^d} \mathrm{d}|\mu|(x) \le C_\mu, \quad \sup\{|k| \mid k \in \operatorname{supp}(|\mu|)\} \le M. \tag{3.19}$$

Two classes of examples of potentials V satisfying Assumption 3.2.2 are the following.

(i) Let V satisfy Assumption 3.2.2 and suppose the corresponding measure μ has a density $f \in L^1(\mathbb{R}^d)$. Then, these assumptions imply that f has compact support, that $f(x) = f(-x)$, and that

$$V(x) = \int_{\mathbb{R}^d} \mathrm{d}k\, f(k) e^{-ik\cdot x}. \tag{3.20}$$

For example, our class of potentials V includes $V(x) = \operatorname{sinc}^k(x)$ for all $k \in \mathbb{N}$ for which the density is the k-fold convolution of the indicator function $f(y) = 1_{[-1,1]}(y)$.

(ii) Let V satisfy Assumption 3.2.2 and suppose the corresponding measure μ satisfies: There is some $N \in \mathbb{N}$, points $\{a_n\}_{n=1}^N$ in $\mathbb{R}^d$, and numbers $\{b_n\}_{n=1}^N$ in $\mathbb{R}$ for which

$$\mu(A) = \frac{1}{2}\sum_{n=1}^{N} b_n(\delta_{a_n}(A) + \delta_{-a_n}(A)) \quad \text{for any } A \in \operatorname{Borel}(\mathbb{R}^d). \tag{3.21}$$

Here $\delta_{(\cdot)}$ denotes the Dirac measure. This form gives rise to potentials V with

$$V(x) = \sum_{n=1}^{N} b_n \cos(a_n \cdot x). \tag{3.22}$$

The main result of this section is:

Theorem 3.2.3 (Lieb-Robinson bound for Schrödinger operators). *Let V satisfy Assumption 3.2.2 and consider the Schrödinger operator $H_1 = -\Delta + V$ as defined in (3.8). Then there exist constants $C_1, C_2, C_3 > 0$ depending on $d, \mu,$ and σ such that the estimate*

$$\left|\langle e^{-itH_1} f, \varphi_x^\sigma\rangle\right| \le C_1 e^{C_2|t|\ln|t|} \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-\frac{C_3}{t^2+1}|x-y|} |f(y)| \tag{3.23}$$

holds for all $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$.

The constants $C_1, C_2,$ and C_3 are derived in the proof of Corollary 3.3.1.

Remarks 3.2.4. (i) Theorem 3.2.3 relies, in general, on the smoothness of the class of test functions φ_y^σ that we used to probe the locality properties of the dynamics. For example, we can see from an explicit computation in the case of $V = 0$, that the exponential decay does generally not hold when the Gaussians is replaced by a non-smooth functions such as, for example a characteristic function. Using the formula

$$(e^{-it(-\Delta)}\psi)(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}y \, e^{\frac{i|x-y|^2}{4t}} \psi(y), \tag{3.24}$$

valid for $t \neq 0$ and general $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, a straightforward calculation, in the case $d = 1$, shows that the leading behavior of $|(e^{-it(-\Delta)}\chi_{[-1,1]})(x)|$, for $t > 0$ and $|x|$ large is given by

$$|(e^{-it(-\Delta)}\chi_{[-1,1]})(x)| \sim 2\sqrt{\frac{t}{\pi}} \frac{x}{x^2-1} \sin\frac{x}{2t}.$$

(ii) In the case $V = 0$ and for the Gaussians φ_y^σ another explicit computation (see e.g. [**87**, Sec. 7.3]), shows that

$$\left|\left(e^{-it(-\Delta)}\varphi_y^\sigma\right)(x)\right| = \frac{1}{(2\pi)^{d/2}} \frac{e^{-\frac{\sigma^2|x-y|^2}{8t^2+2\sigma^4}}}{(4t^2+\sigma^4)^{d/4}}. \tag{3.25}$$

Using this form one immediately sees that an estimate analogous to (3.4) holds with a Gaussian distance dependence of the kernel.

(iii) In view of the first two remarks it is clear that Theorem 3.2.3 is far from optimal. It is also likely that similar bounds hold for a broader class of potentials V.

3.2.2. Lieb-Robinson bounds and the thermodynamic limit. Our next results concern Lieb-Robinson bounds for the Heisenberg dynamics associated to the operator H_Λ^σ defined in (3.14). We begin by recalling the notion of dynamics on Fock space.

As before, let $\mathcal{B}(\mathfrak{F}^-)$ denote the bounded linear operators on $\mathfrak{F}^-$. For each $\sigma > 0$ and any bounded, measurable $\Lambda \subset \mathbb{R}^d$, we define the Heisenberg dynamics associated to H_Λ^σ for each $t \in \mathbb{R}$ as the map $\tau_t^\Lambda : \mathcal{B}(\mathfrak{F}^-) \to \mathcal{B}(\mathfrak{F}^-)$ defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda^\sigma} A e^{-itH_\Lambda^\sigma} \quad \text{for all } A \in \mathcal{B}(\mathfrak{F}^-)\,. \tag{3.26}$$

We note that although $\tau_t^\Lambda(A)$ depends on σ, we have suppressed this in our notation.

We will analyze this dynamics on the CAR algebra generated by the set $\{a(f), a^*(f) : f \in L^2(\mathbb{R}^d)\}$. Again, we refer to [**13**, Sec 5.2] for more details. In particular, we will focus our attention to operators $A = a(f) \in \mathcal{B}(\mathfrak{F}^-)$ for some $f \in L^2(\mathbb{R}^d)$.

Let us also recall the free dynamics, i.e. the case where $W = 0$ and there is no interaction. We will denote this well-studied, free dynamics of the CAR algebra by

$$\tau_t^{\emptyset}(a(f)) = e^{it\mathrm{d}\Gamma(H_1)}a(f)e^{-it\mathrm{d}\Gamma(H_1)}, \qquad f \in L^2(\mathbb{R}^d), \quad t \in \mathbb{R}\,. \tag{3.27}$$

A straight forward calculation shows

$$\tau_t^{\emptyset}(a(f)) = a(f_t) \qquad \text{where} \qquad f_t = e^{-itH_1}f. \tag{3.28}$$

Our goal is to examine the behavior of τ_t^{Λ} as Λ tends to $\mathbb{R}^d$, and in particular, we wish to establish the existence of a dynamics in this thermodynamic limit. To do so, we regard τ_t^{Λ} as a perturbation of the infinite volume free dynamics $\tau_t^{\emptyset}$ on the finite volume Λ. In this case, the key to constructing the thermodynamic limit is an appropriate form of the Lieb-Robinson bound. To express the Lieb-Robinson bound for this model, we find it convenient to introduce the non-negative function

$$F_t^{\Lambda}(f,g) = \|\{\tau_t^{\Lambda}(a(f)), a^*(g)\} - \{\tau_t^{\emptyset}(a(f)), a^*(g)\}\| + \|\{\tau_t^{\Lambda}(a(f)), a(g)\}\| \tag{3.29}$$

Iteration is at the heart of most Lieb-Robinson bounds, and in the present context, our proof will show that the function F_t^{Λ} above iterates more simply than either term on the right-hand-side of (3.29). In any case, we find the following Lieb-Robinson bound.

Theorem 3.2.5 (Lieb-Robinson bound). *Fix $\sigma > 0$. Let V satisfy Assumption 3.2.2, W satisfy Assumption 3.2.1, and for each $t \in \mathbb{R}$ and any bounded, measurable set $\Lambda \subset \mathbb{R}^d$, denote by τ_t^{Λ} the dynamics associated to H_{Λ}^{σ} as defined in (3.26). For any $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, the bound*

$$F_t^{\Lambda}(f,g) \leq D(t)(e^{P_3(t)} - 1)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, e^{-\frac{c_t|x-y|}{4}}|f(x)||g(y)| \tag{3.30}$$

holds for functions $D(t) \sim e^{c|t||\ln|t||}$, P_3 a polynomial of degree $6d+1$ in $|t|$, and $c_t \sim \frac{1}{1+t^2}$. Explicit values for these functions are given in Section 3.4, see specifically Lemma 3.4.3 and Lemma 3.4.4.

Remark 3.2.1. *Here it is crucial to subtract the free time evolution*

$$\{\tau_t^{\emptyset}(a^*(f)), a(g)\} = \langle f_t, g\rangle \mathbb{1} \tag{3.31}$$

since $|\langle f_t, g\rangle|$ does not, in general, decay exponentially; see Remark 3.2.4 (i).

Our main application concerns the existence of a dynamics in the thermodynamic limit. In Section 3.5 we show how the following theorem is a consequence of the Λ-independent bounds proven in Theorem 3.2.5.

Theorem 3.2.6. *Under the assumptions of Theorem 3.2.5 there exists a strongly continuous one-parameter group of automorphisms of the CAR algebra over $L^2(\mathbb{R}^d)$, $\{\tau_t\}_{t\in\mathbb{R}}$, such that for all $f \in L^2(\mathbb{R}^d)$ and any increasing sequence (Λ_n) of bounded subsets of $\mathbb{R}^d$ such that $\cup_n \Lambda_n = \mathbb{R}^d$,*

$$\lim_{n\to\infty} \tau_t^{\Lambda_n}(a(f)) = \tau_t(a(f)) \tag{3.32}$$

in the operator norm topology, with convergence uniform in t in compact subsets of $\mathbb{R}$.

3.3. Lieb-Robinson Bound for Schrödinger Operators. Proof of Theorem 3.2.3

In this section we use the notation $H_0 = -\Delta$ and $H_1 = -\Delta + V$. To prove Theorem 3.2.3 we will use a Dyson series expansion for e^{itH_1}:

$$e^{-itH_1} = e^{-itH_0} + \sum_{n=1}^{\infty}(-i)^n \int_0^t \mathrm{d}t_n \cdots \int_0^{t_2} \mathrm{d}t_1\, e^{-i(t-t_n)H_0} V e^{-i(t_n-t_{n-1})H_0} V \cdots V e^{it_1H_0}. \tag{3.33}$$

Since V is bounded (by Assumption 3.2.2), this series is absolutely convergent in norm. We are interested in estimating $\left|\left(e^{-itH_1}\varphi_y^\sigma\right)(x)\right|$, where φ_y^σ is the Gaussian function given in (3.12). Using the Fourier representation of V (3.18), the integrand of the n-th term in the expansion (3.33) applied to φ_y^σ can be expressed as follows:

$$\begin{aligned}&\left(e^{-i(t-t_n)H_0} V e^{-i(t_n-t_{n-1})H_0} V \cdots V e^{it_1H_0}\varphi_y^\sigma\right)(x)\\ &= \int_{\mathbb{R}^d} \mathrm{d}\mu(k_n) \cdots \int_{\mathbb{R}^d} \mathrm{d}\mu(k_1) A(t_1, ..., t_n, t, k_1, ..., k_n, y, x)\end{aligned} \tag{3.34}$$

where

$$A(t_1,...,t_n,t,k_1,...,k_n,y,x)=\big(e^{-i(t-t_n)H_0}V_{k_n}e^{-i(t_n-t_{n-1})H_0}V_{k_{n-1}}\cdots V_{k_1}e^{it_1H_0}\varphi_y^\sigma\big)(x). \tag{3.35}$$

Here V_k is the multiplication operator by $V_k(x)=e^{-ik\cdot x}$.

LEMMA 3.3.1. *Let $n\in\mathbb{N}$. For all $k_1,...,k_n,y,x\in\mathbb{R}^d$, $t\geq t_n\geq ...\geq t_1\geq 0$ one has*

$$\big|A(t_1,...,t_n,t,k_1,...,k_n,y,x)\big|=\frac{1}{(2\pi)^{d/2}}\frac{e^{-\frac{\sigma^2}{8t^2+2\sigma^4}\left|(x-y)-2\sum_{l=0}^n(t_{l+1}-t_l)\sum_{j=1}^l k_j\right|^2}}{(4t^2+\sigma^4)^{d/4}}, \tag{3.36}$$

where we use the conventions $t_{n+1}=t$, $t_0=0$ and $\sum_{j=1}^0=0$. Furthermore,

$$\big|\big(e^{-itH_0}\varphi_y^\sigma\big)(x)\big|=\frac{1}{(2\pi)^{d/2}}\frac{e^{-\frac{\sigma^2|x-y|^2}{8t^2+2\sigma^4}}}{(4t^2+\sigma^4)^{d/4}}. \tag{3.37}$$

PROOF. Let $\mathcal{F}$ be the unitary Fourier transform on $\mathbb{R}^d$ and $\mathcal{F}^*$ be its inverse then we obtain

$$A(t_1,...t_n,t,k_1,...,k_n,y,x)=\big(\mathcal{F}^*\mathcal{F}e^{-i(t-t_n)H_0}\mathcal{F}^*\mathcal{F}V_{k_n}\mathcal{F}^*\cdots\mathcal{F}V_{k_1}\mathcal{F}^*\mathcal{F}e^{it_1H_0}\mathcal{F}^*\mathcal{F}\varphi_y^\sigma\big)(x). \tag{3.38}$$

Now, for all $t\in\mathbb{R}$, $k\in\mathbb{R}^d$ and $\psi\in L^2(\mathbb{R}^d)$ we have $\mathcal{F}e^{-itH_0}\mathcal{F}^*\psi=e^{-i(\cdot)^2t}\psi$ and $\mathcal{F}V_k\mathcal{F}^*\psi=\psi(\cdot-k)$. Therefore,

$$A(t_1,...t_n,t,k_1,...,k_n,y,x)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}k\,e^{ikx}\prod_{l=0}^n e^{-i(t_{l+1}-t_l)\left(k-\sum_{j=l+1}^n k_j\right)^2}\big(\mathcal{F}\varphi_y^\sigma\big)(k-\sum_{j=1}^n k_j),$$

where we use the convention that $\sum_{j=n+1}^n=0$. Next we use that

$$(\mathcal{F}\varphi_y^\sigma)(k)=\frac{1}{(2\pi)^{d/2}}e^{-ik\cdot y}e^{-\frac{\sigma^2|k|^2}{2}}$$

and perform a change of variables to obtain

$$A(t_1,...t_n,t,k_1,...,k_n,y,x)=\frac{e^{i\sum_{j=1}^n k_j\cdot(x-y)}}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}k\,e^{ik\cdot(x-y)}\prod_{l=0}^n e^{-i(t_{l+1}-t_l)\left|k+\sum_{j=1}^l k_j\right|^2}e^{-\frac{\sigma^2|k|^2}{2}}, \tag{3.39}$$

where we use the convention $\sum_{j=1}^{0} = 0$. Multiplying out $\big|k + \sum_{j=1}^{l} k_j\big|^2 = |k|^2 + 2k \cdot \sum_{j=1}^{l} k_j + \big|\sum_{j=1}^{l} k_j\big|^2$, using $\sum_{l=0}^{n}(t_{l+1} - t_l) = t_{n+1} - t_0 = t$ and taking the absolute value give

$$\big|A(t_1, ...t_n, t, k_1, ..., k_n, y, x)\big| = \frac{1}{(2\pi)^d}\Big|\int_{\mathbb{R}^d} \mathrm{d}k\, e^{ik\cdot(x-y)} e^{-i|k|^2 t} e^{-2i\sum_{l=0}^{n}(t_{l+1}-t_l)\sum_{j=1}^{l} k\cdot k_j} e^{-\frac{\sigma^2|k|^2}{2}}\Big|.$$

Now, a calculation shows that for any $c > 0$ and $a, b \in \mathbb{R}$

$$\frac{1}{(2\pi)^{1/2}}\Big|\int_{\mathbb{R}} \mathrm{d}k\, e^{ikb} e^{-k^2(c+ia)}\Big| = \frac{e^{-\frac{cb^2}{4(a^2+c^2)}}}{(4(a^2+c^2))^{1/4}}. \tag{3.40}$$

Hence,

$$\big|A(t_1, ...t_n, t, k_1, ..., k_n, y, x)\big| = \frac{1}{(2\pi)^{d/2}} \frac{1}{(4t^2+\sigma^4)^{d/4}} e^{-\frac{\sigma^2}{8t^2+2\sigma^4}\big|(x-y)-2\sum_{l=0}^{n}(t_{l+1}-t_l)\sum_{j=1}^{l} k_j\big|^2}, \tag{3.41}$$

which proves (3.36). Identity (3.37) follows from an explicit calculation using the integral kernel of e^{-itH_0}, see e.g. [**87**, Sec. 7.3]. $\square$

With this lemma we have arrived at the following estimate. For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned}\big|\big(e^{-itH_1} - e^{-itH_0}\big)\varphi_y^\sigma(x)\big| \le \frac{1}{(2\pi)^{d/2}} \frac{1}{(4t^2+\sigma^4)^{d/4}} \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \mathrm{d}|\mu|(k_1) \cdots \int_{\mathbb{R}^d} \mathrm{d}|\mu|(k_n) \\ \times \int_0^{|t|} \mathrm{d}t_n \cdots \int_0^{t_2} \mathrm{d}t_1\, e^{-\frac{\sigma^2}{8t^2+2\sigma^4}\big|(x-y)-2\sum_{l=0}^{n}(t_{l+1}-t_l)\sum_{j=1}^{l} k_j\big|^2}\Big),\end{aligned} \tag{3.42}$$

where, as before, we use the convention $\sum_{j=1}^{0} = 0$ and recall that $|\mu| = \mu^+ + \mu^-$. By estimating the RHS of this estimate, we will obtain the following proposition. Recall the definitions of C_μ and M in (3.19).

Proposition 3.3.1. *For all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, we have*

$$\big|\big(e^{-itH_1} - e^{-itH_0}\big)\varphi_y^\sigma(x)\big| \le \frac{1}{(2\pi)^{d/2}} \frac{1}{(4t^2+\sigma^4)^{d/4}} \Big(e^{-\frac{\sigma^2|x-y|^2}{32t^2+8\sigma^4}}\big(e^{C_\mu|t|}-1\big) + \frac{1}{\sqrt{2\pi}} e^{-\frac{|x-y|}{4|t|M}\left(\ln\frac{|x-y|}{4MC_\mu t^2}-1\right)} e^{C_\mu|t|}\Big).$$

Using (3.37) to bound $|e^{-itH_0}\varphi_y^\sigma(x)|$, it is then straightforward to obtain an estimate for $|e^{-itH_1}\varphi_y^\sigma(x)|$. In our application, however, the following simplified estimate is easier to use.

Corollary 3.3.1. *There exist constants C_1, C_2, and C_3, such that for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, we have*

$$\left|e^{-itH_1}\varphi_y^\sigma(x)\right| \leq C_1 e^{C_2|t|\ln|t| - C_3\frac{|x-y|}{1+t^2}}. \tag{3.43}$$

Proof of Proposition 3.3.1. Note that, as expected from translation invariance, the RHS of (3.42) depends only on $x-y$. Therefore, w.l.o.g., we can assume $y=0$. Let I_n, $n \geq 1$, denote the n-th term of the sum in the RHS of (3.42):

$$I_n = \int_{\mathbb{R}^d} \mathrm{d}|\mu|(k_1)\cdots\int_{\mathbb{R}^d}\mathrm{d}|\mu|(k_n)\int_0^{|t|}\mathrm{d}t_n\cdots\int_0^{t_2}\mathrm{d}t_1\, e^{-\frac{\sigma^2}{8t^2+2\sigma^4}\left|(x-y)-2\sum_{l=0}^n(t_{l+1}-t_l)\sum_{j=1}^l k_j\right|^2}. \tag{3.44}$$

Given $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, we split the sum over n in (3.42) as follows:

$$B_0 = \sum_{n=1}^{N_0} I_n, \quad B_1 = \sum_{n=N_0+1}^{\infty} I_n, \text{ with } N_0 = \left[\frac{|x|}{(4|t|M)}\right], \tag{3.45}$$

where $[a]$ denotes the integer part of a and M is as in (3.19).

First, if $\frac{|x|}{(4|t|M)} \geq 1$, the first sum is non-empty and we estimate its terms as follows. Let $n \leq |x|/(4|t|M)$, and note that for $k_j \in B_M(0)$ with $1 \leq j \leq l \leq n$ and $|t| = t_{n+1} \geq t_n \geq ... \geq t_0 = 0$

$$\left|\sum_{l=0}^n (t_{l+1}-t_l)\sum_{j=1}^l k_j\right| \leq Mn\sum_{l=0}^n(t_{l+1}-t_l) = Mn|t|. \tag{3.46}$$

Therefore, we obtain

$$\left|x - 2\sum_{l=0}^n(t_{l+1}-t_l)\sum_{j=1}^l k_j\right|^2 \geq \frac{1}{4}|x|^2 \tag{3.47}$$

and

$$B_0 \leq e^{-\frac{\sigma^2|x|^2}{32t^2+8\sigma^4}}\sum_{n=1}^{N_0}\frac{C_\mu^n|t|^n}{n!} \leq e^{-\frac{\sigma^2|x|^2}{32t^2+8\sigma^4}}(e^{C_\mu|t|}-1). \tag{3.48}$$

To estimate the terms in B_1, note that the integrand in (3.44) is bounded by 1. Hence, using C_μ defined in (3.19),

$$B_1 \leq \sum_{n=N_0+1}^{\infty}\frac{(C_\mu|t|)^n}{n!} \leq \frac{(C_\mu|t|)^{N_0+1}}{(N_0+1)!}e^{C_\mu|t|}. \tag{3.49}$$

Stirling's formula yields for all $m \geq 1$ the bound

$$\frac{1}{m!} \leq \frac{1}{\sqrt{2\pi}} e^{m - m\ln m}. \tag{3.50}$$

Using this and $N_0 + 1 \geq |x|/(4|t|M)$, we obtain

$$B_1 \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|}{4|t|M}(\ln \frac{|x|}{4MC_\mu t^2} - 1)} e^{C_\mu |t|}. \tag{3.51}$$

If $\frac{|x|}{(4|t|M)} < 1$, $B_0 = 0$ and we estimate B_1 as in (3.51) with $N_0 = 0$.

The proposition is proved by combining the estimates (3.48) and (3.51). □

Proof of Corollary 3.3.1. We may again restrict ourselves to the case $y = 0$. Proposition 3.3.1 and (3.37) together immediately give the estimate

$$\begin{aligned} \left| e^{-itH_1} \varphi_0^\sigma(x) \right| \leq & \frac{1}{(2\pi)^{d/2}} \frac{1}{(4t^2 + \sigma^4)^{d/4}} \\ & \times \left[e^{-\frac{\sigma^2 |x|^2}{8t^2 + 2\sigma^4}} + e^{-\frac{\sigma^2|x|^2}{32t^2 + 8\sigma^4}} (e^{C_\mu|t|} - 1) + \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{4M|t|}(\ln \frac{|x|}{4MC_\mu t^2} - 1)} e^{C_\mu |t|} \right]. \end{aligned}$$

The last term in the square bracket is the estimate (3.51) for B_1, which we can simplify by considering two cases for $(x,t) \in \mathbb{R}^{d+1}$, namely $\frac{|x|}{4MC_\mu t^2} \geq e^2$, and $\frac{|x|}{4MC_\mu t^2} < e^2$. In the first case, we have

$$B_1 \leq \frac{e^{-\frac{|x|}{4|t|M} + C_\mu |t|}}{\sqrt{2\pi}}. \tag{3.52}$$

On the other hand, if $\frac{|x|}{4MC_\mu t^2} < e^2$, we use the inequality $e^{-u\ln u} \leq e^{\frac{1}{e}}$ for all $u > 0$, to obtain

$$B_1 \leq \frac{1}{\sqrt{2\pi}} e^{\frac{1}{e}} e^{e^2 C_\mu |t| (\ln C_\mu |t| + 1) + C_\mu |t|} e^{-\frac{|x|}{4MC_\mu t^2} + e^2}. \tag{3.53}$$

By bounding B_1 by the sum of the RHSs of (3.52) and (3.53) and making a few more easy simplifications we arrive at the following bound:

$$\begin{aligned} \left| e^{-itH_1} \varphi_0^\sigma(x) \right| \leq & \frac{1}{(2\pi\sigma^2)^{d/2}} \\ & \times \left[e^{-\frac{\sigma^2|x|^2}{32t^2 + 8\sigma^4}} e^{C_\mu |t|} + \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|}{4|t|M} + C_\mu |t|} + \frac{1}{\sqrt{2\pi}} e^{\frac{1}{e}} e^{e^2 C_\mu |t| (\ln C_\mu |t| + 1) + C_\mu |t|} e^{-\frac{|x|}{4MC_\mu t^2} + e^2} \right]. \end{aligned}$$

To estimate the Gaussian decay of the first term between the square brackets by a simple exponential, we use that for all $u \in \mathbb{R}$, $u^2 \geq u - 1/4$. Furthermore, since $u \ln u \geq u - 1$, we also have

$$e^{C_\mu |t|} \leq e^{1 + C_\mu |t| \ln C_\mu |t|}.$$

Using this and replacing the constant prefactors by their maximum, we find

$$\left| e^{-itH_1} \varphi_0^\sigma(x) \right| \leq \frac{1}{(2\pi\sigma^2)^{d/2}} e^{\frac{1}{8\sigma^2} + \frac{1}{e} + e^2} e^{e^2 C_\mu |t| (\ln C_\mu |t| + 2)} \left(e^{-\frac{\sigma^2 |x|}{32t^2 + 8\sigma^4}} + e^{-\frac{|x|}{4tM}} + e^{-\frac{|x|}{4MC_\mu t^2}} \right).$$

Finally, we use the estimate

$$\begin{aligned} \min\left\{ \frac{\sigma^2}{32t^2 + 8\sigma^4}, \frac{1}{4tM}, \frac{1}{4MC_\mu t^2} \right\} &\geq \frac{1}{t^2(\frac{32}{\sigma^2} + 4MC_\mu) + 4tM + 8\sigma^2} \\ &\geq \frac{1}{t^2(\frac{32}{\sigma^2} + 4MC_\mu + 2) + 2M^2 + 8\sigma^2} \\ &\geq \frac{1}{\frac{32}{\sigma^2} + 4MC_\mu + 2(M^2+1) + 8\sigma^2} \frac{1}{t^2+1} \end{aligned} \tag{3.54}$$

to bound the sum of three exponentials by

$$3e^{-C_3 \frac{|x|}{t^2+1}}, \text{ with } C_3 = \frac{1}{\frac{32}{\sigma^2} + 4MC_\mu + 2(M^2+1) + 8\sigma^2}.$$

It is now straightforward to find suitable values for C_1 and C_2 for which the bound given in the corollary holds. □

3.4. Many-body Lieb-Robinson bound. Proof of Theorem 3.2.5

The main goal of this section is to prove Theorem 3.2.5. We do so in three steps. First, in Section 3.4.1 we establish a basic estimate which facilitates an iteration scheme; this is the content of Lemma 3.4.1. Next, in Section 3.4.2, we estimate a kernel function which, among other things, ultimately justifies the convergence of our iteration, see Lemma 3.4.2 and Lemma 3.4.3. Finally, in Section 3.4.3 we perform the iteration and verify the bound claimed in Theorem 3.2.5.

3.4.1. A Preliminary Bound. In this section, we provide an estimate on the basic quantity of interest in Theorem 3.2.5. Let us briefly recall the set-up. We have fixed $\sigma > 0$, taken V and W satisfying Assumption 3.2.2 and Assumption 3.2.1 respectively, and introduced, see (3.29), the non-negative function

$$F_t^\Lambda(f,g) = \left\|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \{\tau_t^\emptyset(a(f)), a^*(g)\}\right\| + \left\|\{\tau_t^\Lambda(a^*(f)), a^*(g)\}\right\| \tag{3.55}$$

for any bounded, measurable set $\Lambda \subset \mathbb{R}^d$, $t \in \mathbb{R}$, and functions $f, g \in L^2(\mathbb{R}^d)$. Our first estimate is as follows.

LEMMA 3.4.1. *Under the assumptions described above, for any $t \geq 0$, we find that*

$$\begin{aligned} F_t^\Lambda(f,g) \ \leq\ & C_\sigma \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x\, K_{t-s}(f,x) |\langle e^{-isH_1}\varphi_x^\sigma, g\rangle| \\ & + C_\sigma \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x\, K_{t-s}(f,x) F_s^\Lambda(\varphi_x^\sigma, g) \end{aligned} \tag{3.56}$$

where $C_\sigma > 0$ is as in (3.13) and with kernel function $K_t(f,x)$ given by

$$K_t(f,x) = \|W\|_1 |\langle e^{-itH_1} f, \varphi_x^\sigma\rangle| + 2\left(|W| * |\langle e^{-itH_1} f, \varphi_{(\cdot)}^\sigma\rangle|\right)(x)\,. \tag{3.57}$$

PROOF. We begin by recalling a useful perturbation formula. Fix a bounded, measurable set $\Lambda \subset \mathbb{R}^d$ and take $\sigma > 0$, $t \geq 0$, and $f \in L^2(\mathbb{R}^d)$. In this case,

$$\tau_t^\Lambda(a(f)) = \tau_t^\emptyset(a(f)) + i \int_0^t \mathrm{d}s\, \tau_s^\Lambda\left(\left[W_\Lambda^\sigma, \tau_{t-s}^\emptyset(a(f))\right]\right) \tag{3.58}$$

a proof of which can be found in [**13**, Prop. 5.4.1]. Note that

$$\begin{aligned} \left[W_\Lambda^\sigma, \tau_{t-s}^\emptyset(a(f))\right] =& \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}x \mathrm{d}y\, W_\Lambda(x,y) \left[a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma), a(f_{t-s})\right] \\ =& \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}x \mathrm{d}y\, W_\Lambda(x,y) \left[a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma), a(f_{t-s})\right] a(\varphi_y^\sigma) a(\varphi_x^\sigma) \end{aligned} \tag{3.59}$$

where we have set $f_t = e^{-itH_1} f$ and used (3.10). Further calculating, we find

$$\begin{aligned} \left[a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma), a(f_{t-s})\right] =& a^*(\varphi_x^\sigma)\left\{a^*(\varphi_y^\sigma), a(f_{t-s})\right\} - \left\{a^*(\varphi_x^\sigma), a(f_{t-s})\right\} a^*(\varphi_y^\sigma) \\ =& \langle f_{t-s}, \varphi_y^\sigma\rangle a^*(\varphi_x^\sigma) - \langle f_{t-s}, \varphi_x^\sigma\rangle a^*(\varphi_y^\sigma). \end{aligned} \tag{3.60}$$

Using now the symmetry of W, we obtain

$$
\begin{aligned}
&\{\tau_t^\Lambda(a(f)), a^*(g)\} - \{\tau_t^\emptyset(a(f)), a^*(g)\} \\
=&i\int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, W_\Lambda(x,y)\langle f_{t-s}, \varphi_y^\sigma\rangle \left\{\tau_s^\Lambda\left(a^*(\varphi_x^\sigma)\right)\tau_s^\Lambda\left(a(\varphi_y^\sigma)\right)\tau_s^\Lambda\left(a(\varphi_x^\sigma)\right), a^*(g)\right\}.
\end{aligned}
$$

With the anti-commutator relation

$$
\{ABC, D\} = \{A,D\}BC - A\{B,D\}C + AB\{C,D\}\,, \tag{3.61}
$$

the norm bound

$$
\begin{aligned}
&\left\|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \{\tau_t^\emptyset(a(f)), a^*(g)\}\right\| \\
\leq& C_\sigma \int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y |W(x-y)||\langle f_{t-s}, \varphi_y^\sigma\rangle| \\
&\times \left(\left\|\left\{\tau_s^\Lambda(a^*(\varphi_x^\sigma)), a^*(g)\right\}\right\| + \left\|\left\{\tau_s^\Lambda(a(\varphi_y^\sigma)), a^*(g)\right\}\right\| + \left\|\left\{\tau_s^\Lambda(a(\varphi_x^\sigma)), a^*(g)\right\}\right\|\right)
\end{aligned} \tag{3.62}
$$

readily follows. Here we have used the bound $|W_\Lambda(x,y)| \leq |W(x-y)|$ for all $x, y \in \mathbb{R}^d$. Similar arguments yield the bound

$$
\begin{aligned}
&\left\|\{\tau_t^\Lambda(a^*(f)), a^*(g)\}\right\| \\
\leq& C_\sigma \int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, |W(x-y)||\langle f_{t-s}, \varphi_y^\sigma\rangle| \\
&\times \left(\left\|\left\{\tau_s^\Lambda(a^*(\varphi_x^\sigma)), a^*(g)\right\}\right\| + \left\|\left\{\tau_s^\Lambda(a^*(\varphi_y^\sigma)), a^*(g)\right\}\right\| + \left\|\left\{\tau_s^\Lambda(a(\varphi_x^\sigma)), a^*(g)\right\}\right\|\right).
\end{aligned} \tag{3.63}
$$

Our goal, as in most Lieb-Robinson bounds, is to derive bounds which can be iterated. Since neither (3.62) nor (3.63) iterate separately, we bound their sum, i.e. the function $F_t^\Lambda(f,g)$ introduced in (3.55) above. Recalling that $\{\tau_t^\emptyset(a(f)), a^*(g)\} = \langle f_t, g\rangle \mathbb{1}$, we find

$$
\begin{aligned}
F_t^\Lambda(f,g) \leq& C_\sigma \int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, |W(x-y)||\langle f_{t-s}, \varphi_y^\sigma\rangle| \\
&\times \left(2F_s^\Lambda(\varphi_x^\sigma, g) + 2|\langle e^{-isH_1}\varphi_x^\sigma, g\rangle| + F_s^\Lambda(\varphi_y^\sigma, g) + |\langle e^{-isH_1}\varphi_y^\sigma, g\rangle|\right) \\
=& C_\sigma \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x\, K_{t-s}(f,x)\left(F_s^\Lambda(\varphi_x^\sigma, g) + |\langle e^{-isH_1}\varphi_x^\sigma, g\rangle|\right),
\end{aligned} \tag{3.64}
$$

where we have introduced $K_t(f,x)$ as in (3.57). This is the claim in (3.56). □

Looking at the bound proven in Lemma 3.4.1, in particular (3.56), it is natural to begin an iteration. To ensure convergence, we first provide an estimate on the kernel function $K_t(f,x)$.

3.4.2. Estimating the Kernel. In this section, we provide two useful apriori estimates on the kernel function $K_t(f,x)$ defined in (3.57) of Lemma 3.4.1. First, we estimate this function for general $f \in L^2(\mathbb{R}^d)$. This result is stated in Lemma 3.4.2 below. In Lemma 3.4.3, we provide a similar estimate in the special case that f is a Gaussian.

Before we prove our first estimate, it will be convenient to introduce the following notation. For any $r > 0$, set $G_r : \mathbb{R}^d \to \mathbb{R}$ to be

$$G_r(x) = e^{-r|x|} \quad \text{for all } x \in \mathbb{R}^d \,. \tag{3.65}$$

LEMMA 3.4.2. *Let $t \geq 0$, $f \in L^2(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$. We have*

$$K_t(f,x) \leq P_1(t)e^{C_2 t|\ln t|}\big(G_{b_t} * |f|\big)(x) \tag{3.66}$$

where G_{b_t} is as in (3.65) with

$$4b_t = \frac{aC_3}{a(t^2+1)+C_3} \tag{3.67}$$

and $P_1 : \mathbb{R} \to (0,\infty)$ is the polynomial of degree 2d given by

$$P_1(t) = C_1(2c_W D_3^2 (4b_t)^{-d} + \|W\|_1) \,. \tag{3.68}$$

Here $C_1, C_2, C_3 > 0$ are as in the proof of Theorem 3.2.3, a and c_W are as in (3.17), and $D_3 > 0$ is as in Lemma 3.7.1.

PROOF. We first note that

$$K_0(f,x) \leq \|W\|_1(\varphi_0^\sigma * |f|)(x) + 2\big((|W| * \varphi_0^\sigma) * |f|\big)(x) \tag{3.69}$$

which may be further estimated as in (3.66).

Now, for $t > 0$, we recall the bound in (3.4):

$$\big|\langle e^{-itH_1} f, \varphi_x^\sigma\rangle\big| \leq C_1 e^{C_2 t|\ln t|} \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-\frac{C_3}{t^2+1}|x-y|}|f(y)| = C_1 e^{C_2 t|\ln t|}(G_{a_t} * |f|)(x) \tag{3.70}$$

where we use the notation in (3.65) and set

$$a_t = \frac{C_3}{t^2+1} \tag{3.71}$$

In this case, the bound

$$\begin{aligned} K_t(f,x) &= \|W\|_1 |\langle e^{-itH_1} f, \varphi_x^\sigma\rangle| + 2\big(|W| * |\langle e^{-itH_1} f, \varphi_{(\cdot)}^\sigma\rangle|\big)(x) \\ &\le C_1 e^{C_2 t|\ln t|} \|W\|_1 \big(G_{a_t} * |f|\big)(x) + 2C_1 e^{C_2 t|\ln t|}\big(|W| * \big(G_{a_t} * |f|\big)\big)(x) \end{aligned} \tag{3.72}$$

is clear. Moreover, the exponential decay of W, see (3.17) in Assumption 3.2.1, implies

$$\begin{aligned} \big(|W| * \big(G_{a_t} * |f|\big)\big)(x) &\le c_W \int_{\mathbb{R}^d} \mathrm{d}z\, e^{-a|x-z|}\big(G_{a_t} * |f|\big)(z) \\ &\le c_W \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-4b_t|x-z|} e^{-4b_t|z-y|} |f(y)| \end{aligned} \tag{3.73}$$

where we have used (3.67), and in particular, that $4b_t \le \min\{a, a_t\}$. By Lemma 3.7.1 there is $D_3 > 0$, depending only on d, such that

$$\int_{\mathbb{R}^d} dz e^{-4b_t|x-z|} e^{-4b_t|z-y|} \le \frac{D_3^2}{(4b_t)^d} e^{-b_t|x-y|}\,. \tag{3.74}$$

We conclude that

$$\big(|W| * \big(G_{a_t} * |f|\big)\big)(x) \le c_W \frac{D_3^2}{(4b_t)^d} (G_{b_t} * |f|)(x) \tag{3.75}$$

and note that (3.66) follows from the point-wise bound $G_{a_t}(x) \le G_{b_t}(x)$. □

We now turn to the special case of a Gaussian.

Lemma 3.4.3. *Let $t \ge 0$, $\sigma > 0$, and $x, y \in \mathbb{R}^d$. We have*

$$K_t(\varphi_x^\sigma, y) \le P_2(t) e^{C_2 t|\ln t|} G_{c_t}(x-y) \tag{3.76}$$

where G_{c_t} is as in (3.65) with

$$c_t = \frac{aC_3}{16(a(t^2+1) + C_3(1+a\sigma^2))} \tag{3.77}$$

and $P_2 : \mathbb{R} \to \mathbb{R}$ is the polynomial of degree $4d$ given by

$$P_2(t) = \frac{e^{\frac{1}{8\sigma^2}}}{(2\pi\sigma^2)^{d/2}} P_1(t) \frac{D_3^2}{(4c_t)^d} \,. \tag{3.78}$$

Here we use freely the notation established in Lemma 3.4.2.

Proof. Applying Lemma 3.4.2 and the simple bound $u^2 \geq u - 1/4$, for all $u \in \mathbb{R}$, we find that

$$\begin{aligned} K_t(\varphi_\sigma^x, y) &\leq P_1(t) e^{C_2 t |\ln t|} \big(G_{b_t} * \varphi_x^\sigma\big)(y) \\ &= \frac{P_1(t) e^{\frac{1}{8\sigma^2}}}{(2\pi\sigma^2)^{d/2}} e^{C_2 t |\ln t|} \int_{\mathbb{R}^d} \mathrm{d}z\, e^{-b_t |y-z|} e^{-\frac{|x-z|}{2\sigma^2}} . \end{aligned} \tag{3.79}$$

Recalling also Lemma 3.7.1, the bound

$$\int_{\mathbb{R}^d} \mathrm{d}z\, e^{-b_t |y-z|} e^{-\frac{|x-z|}{2\sigma^2}} \leq D_3^2 \big(\frac{1}{4c_t}\big)^d e^{-c_t |x-y|}, \tag{3.80}$$

follows from the estimate

$$4c_t \leq \min\left\{ b_t, \frac{1}{2\sigma^2} \right\} . \tag{3.81}$$

This proves the assertion. $\square$

3.4.3. Iterating the Bound. In this section, we will iterate the bound proven in Lemma 3.4.1, i.e. (3.56), and complete the proof of Theorem 3.2.5.

We first note that iteration of (3.56) produces, for any $N \in \mathbb{N}$, a bound of the form

$$F_t^\Lambda(f, g) \leq \sum_{n=1}^{N} a_n(t, f, g) + R_N(t, f, g) \tag{3.82}$$

where for each $1 \leq n \leq N$ the terms

$$\begin{aligned} a_n(t, f, g) = C_\sigma^n \int_0^t \mathrm{d}t_1 \int_{\mathbb{R}^d} \mathrm{d}x_1\, K_{t-t_1}(f, x_1) \int_0^{t_1} \mathrm{d}t_2 \int_{\mathbb{R}^d} \mathrm{d}x_2\, K_{t_1 - t_2}(\varphi_{x_1}^\sigma, x_2) \cdots \\ \times \int_0^{t_{n-1}} \mathrm{d}t_n \int_{\mathbb{R}^d} \mathrm{d}x_n\, K_{t_{n-1} - t_n}(\varphi_{x_{n-1}}^\sigma, x_n) |\langle e^{-i t_n H_1} \varphi_{x_n}^\sigma, g\rangle| \end{aligned} \tag{3.83}$$

and similarly, the remainder is given by

$$
\begin{aligned}
R_N(t,f,g) = C_\sigma^N \int_0^t \mathrm{d}t_1 \int_{\mathbb{R}^d} \mathrm{d}x_1\, K_{t-t_1}(f,x_1) \int_0^{t_1} \mathrm{d}t_2 \int_{\mathbb{R}^d} \mathrm{d}x_2\, K_{t_1-t_2}(\varphi_{x_1}^\sigma, x_2) \cdots \\
\times \int_0^{t_{N-1}} \mathrm{d}t_N \int_{\mathbb{R}^d} \mathrm{d}x_N\, K_{t_{N-1}-t_N}(\varphi_{x_{N-1}}^\sigma, x_N) F_{t_N}^\Lambda(\varphi_{x_N}^\sigma, g)
\end{aligned} \tag{3.84}
$$

Next, we estimate the terms $a_n(t,f,g)$.

Lemma 3.4.4. *Let $t>0$ and $n \in \mathbb{N}$. We find the following bound*

$$
a_n(t,f,g) \le D(t) \frac{P_3(t)^n}{n!} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-\frac{c_t|x-y|}{4}} |f(x)||g(y)| \tag{3.85}
$$

where $P_3 : \mathbb{R} \to \mathbb{R}$ is a polynomial in t of degree $6d+1$ with

$$
P_3(t) = \frac{C_\sigma e^{C_2} D_3 t P_2(t)}{c_t^d} \quad \textit{and} \quad D(t) = C_1 e^{C_2} D_3 \frac{P_1(t)}{P_2(t)} e^{C_2 t |\ln t|}. \tag{3.86}
$$

All quantities appearing above are as in Lemma 3.4.2 and Lemma 3.4.3.

Proof. Fix $t>0$. For our estimate, it will be convenient to recall some bounds from Section 3.4.2. First, let $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Lemma 3.4.2 shows that for $0 \le t_1 \le t$,

$$
\begin{aligned}
K_{t-t_1}(f,x) &\le P_1(t-t_1) e^{C_2(t-t_1)|\ln(t-t_1)|} (G_{b_{t-t_1}} * |f|)(x) \\
&\le P_1(t) e^{C_2(t-t_1)|\ln(t-t_1)|} (G_{c_t} * |f|)(x)
\end{aligned} \tag{3.87}
$$

where we have used that P_1 is increasing in t and that $b_{t-t_1} \ge b_t \ge c_t$. Next, an application of Lemma 3.4.3 shows that for any $0 \le t_j \le t_{j-1} \le t$ and all $x, y \in \mathbb{R}^d$,

$$
\begin{aligned}
K_{t_{j-1}-t_j}(\varphi_x^\sigma, y) &\le P_2(t_{j-1}-t_j) e^{C_2(t_{j-1}-t_j)|\ln(t_{j-1}-t_j)|} G_{c_{t_{j-1}-t_j}}(x-y) \\
&\le P_2(t) e^{C_2(t_{j-1}-t_j)|\ln(t_{j-1}-t_j)|} G_{c_t}(x-y)
\end{aligned} \tag{3.88}
$$

Here we used that the polynomial $P_2(t)$ is increasing in t and the function c_t is decreasing in t. Similarly, arguing as in (3.70), we see that

$$
|\langle e^{-it_n H_1} \varphi_{x_n}^\sigma, g\rangle| \le C_1 e^{C_2 t_n |\ln t_n|} (G_{a_{t_n}} * |g|)(x_n) \le C_1 e^{C_2 t_n |\ln t_n|} (G_{c_t} * |g|)(x_n) \tag{3.89}
$$

for any $0 \leq t_n \leq t$ and $x_n \in \mathbb{R}^d$. As a final observation, note that for parameters $0 = t_{n+1} \leq t_n \leq \cdots \leq t_j \leq t_{j-1} \leq \cdots \leq t_1 \leq t_0 = t$, we have the bound

$$\begin{aligned}
\sum_{j=1}^{n+1}(t_{j-1}-t_j)|\ln(t_{j-1}-t_j)| &\leq \sum_{j=1}^{n+1}(t_{j-1}-t_j)\ln(t_{j-1}-t_j)1\{t_{j-1}-t_j \geq 1\} + n + 1 \\
&\leq \ln t\, 1\{t \geq 1\} t + n + 1 \\
&\leq t|\ln t| + n + 1, \qquad (3.90)
\end{aligned}$$

where $1\{\cdot\}$ stands for the indicator function, we used that $x|\ln x| \leq 1$ for all $x \in (0, 1]$, and in the argument of the ln, we use the bound $t_{j-1} - t_j \leq t$, for $j = 1, \ldots, n+1$, to produce a telescopic sum.

Putting all this together we find that

$$\begin{aligned}
a_n(t, f, g) \quad &\leq \quad C_1 C_\sigma^n P_1(t) P_2(t)^{n-1} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n e^{C_2 \sum_{j=1}^{n+1}(t_{j-1}-t_j)|\ln(t_{j-1}-t_j)|} \\
&\quad \times \int_{\mathbb{R}^d} dx_1 \cdots \int_{\mathbb{R}^d} dx_n (G_{c_t} * |f|)(x_1) G_{c_t}(x_1 - x_2) \cdots G_{c_t}(x_{n-1} - x_n)(G_{c_t} * |g|)(x_n) \\
&\leq \quad C_1 C_\sigma^n P_1(t) P_2(t)^{n-1} e^{C_2 t|\ln t|} e^{(n+1)C_2} \frac{t^n}{n!} \\
&\quad \times \int_{\mathbb{R}^d} dx_1 (G_{c_t} * |f|)(x_1)(G_{c_t} * \cdots * G_{c_t} * |g|)(x_1). \qquad (3.91)
\end{aligned}$$

The latter integral can be further estimated as

$$\begin{aligned}
\int_{\mathbb{R}^d} dx (G_{c_t} * |f|)(x)(G_{c_t} * \cdots * G_{c_t} * |g|)(x) \quad &= \quad \int_{\mathbb{R}^d} dx |f(x)| (G_{c_t} * \cdots * G_{c_t} * |g|)(x) \\
&\leq \quad c_t^d \left(\frac{D_3}{c_t^d}\right)^{n+1} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy e^{-\frac{c_t|x-y|}{4}} |f(x)||g(y)| \qquad (3.92)
\end{aligned}$$

by using the point-wise estimate in Lemma 3.7.1 on the $n+1$-fold convolution of G_{c_t} with itself. This is the bound claimed in (3.85). $\square$

We can now complete the proof of Theorem 3.2.5.

Proof of Theorem 3.2.5. Given (3.82) and the estimate in Lemma 3.4.4, i.e. (3.85), it is clear that we need only show that the remainder term $R_N(t)$, see (3.84), goes to zero as $N \to \infty$. We will see that this is the case uniformly for t in compact sets.

Fix $T > 0$ and let $t \in [-T, T]$. We argue as in the proof of Lemma 3.4.4. The only difference between the term $a_N(t, f, g)$ and $R_N(t)$ is the final factor in the integrand: more precisely, $|\langle e^{-it_N H_1}\varphi^\sigma_{x_n}, g\rangle|$ is replaced with $F^\Lambda_{t_N}(\varphi^\sigma_{x_N}, g)$. In this case, the naive bound

$$F^\Lambda_{t_N}(\varphi^\sigma_{x_N}, g) \leq 6\|\varphi^\sigma_{x_N}\|_2\|g\|_2 = 6\sqrt{C_\sigma}\|g\|_2 \tag{3.93}$$

will suffice. In fact,

$$\begin{aligned} R_N(t) &\leq 6\sqrt{C_\sigma}\|g\|_2 C_\sigma^N P_1(t)P_2(t)^{N-1}\int_0^t dt_1 \cdots \int_0^{t_{N-1}} dt_n e^{C_2\sum_{j=1}^N (t_{j-1}-t_j)|\ln(t_{j-1}-t_j)|} \\ &\quad \times \int_{\mathbb{R}^d} dx_1 \cdots \int_{\mathbb{R}^d} dx_N (G_{c_t} * |f|)(x_1)G_{c_t}(x_1 - x_2)\cdots G_{c_t}(x_{N-1} - x_N) \\ &\leq 6\sqrt{C_\sigma}\|g\|_2 C_\sigma^N P_1(t)P_2(t)^{N-1} e^{C_2 t|\ln t|} e^{NC_2}\frac{t^N}{N!} \\ &\quad \times \int_{\mathbb{R}^d} dx_N (G_{c_t} * \cdots * G_{c_t} * |f|)(x_N). \end{aligned} \tag{3.94}$$

Now, another application of Lemma 3.7.1 demonstrates that

$$\begin{aligned} \int_{\mathbb{R}^d} dx (G_{c_t} * \cdots * G_{c_t} * |f|)(x) &\leq \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy (G_{c_t} * \cdots G_{c_t})(x - y)|f(y)| \\ &\leq \left(\frac{D_3}{c_t^d}\right)^N c_t^d \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy e^{-\frac{c_t|x-y|}{4}}|f(y)| \\ &= \left(\frac{D_3}{c_t^d}\right)^N c_t^d \|G_{c_t/4}\|_1 \|f\|_1 \end{aligned} \tag{3.95}$$

where here we have used that $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Since all the quantities in these estimates are explicit, it is clear that

$$\lim_{N\to 0} \sup_{t\in[-T,T]} R_N(t) = 0 \tag{3.96}$$

which completes the proof of Theorem 3.2.5. $\square$

3.5. The infinite-system dynamics. Proof of Theorem 3.2.6

Our proof of Theorem 3.2.6 will make essential use of the following direct consequences of the propagation bounds of Theorem 3.2.5.

Let V and W satisfy Assumptions 3.2.2 and 3.2.1, respectively. Then, there exist continuous functions $\widetilde{C}(\cdot), \widetilde{a}(\cdot) > 0$, such that with φ_x^σ the Gaussians introduced in (3.12), and $f \in L^2(\mathbb{R}^d)$ of compact support, denoted by $\mathrm{supp}(f)$, we have

$$\|\{\tau_t^\Lambda(a(f)), a^\#(\varphi_z^\sigma)\}\| \leq \|f\|_1 e^{\widetilde{C}(|t|)} e^{-\widetilde{a}(|t|)d(\mathrm{supp}(f),z)} \tag{3.97}$$

where $a^\#(\cdot)$ refers to either an annihilation or creation operator, compare with (3.5). This estimate follows from Theorem 3.2.5 and Theorem 3.2.3 where we again use arguments as in (3.79) and (3.80).

Apart from this key estimate, the proof below uses a combination of several ideas introduced in [**13**, **17**, **64**, **67**].

Proof of Theorem 3.2.6. We first prove (3.32) for $f \in L^2(\mathbb{R}^d)$ of compact support, say $\mathrm{supp} f \subset X \subset \mathbb{R}^d$, for some compact X. Let $(\Lambda_n)_{n\geq 1}$ be an increasing sequence of bounded, measurable sets such that $\bigcup_n \Lambda_n = \mathbb{R}^d$. To show that $(\tau_t^{\Lambda_n}(a(f)))_{n\geq 1}$ is Cauchy, uniformly in $t \in [-T, T]$ for $T > 0$, note that, for any $\Lambda_m \subseteq \Lambda_n$, the operator

$$W_{\Lambda_n}^\sigma - W_{\Lambda_m}^\sigma = \int_{\Lambda_n\times\Lambda_n\setminus\Lambda_m\times\Lambda_m} \mathrm{d}x\mathrm{d}y\, W(x-y) a^*(\varphi_x^\sigma)a^*(\varphi_y^\sigma)a(\varphi_y^\sigma)a(\varphi_x^\sigma)$$

is bounded by our assumptions. Hence, the generator of the strongly continuous dynamics $\tau_t^{\Lambda_n}$ is a bounded perturbation of the generator of $\tau_t^{\Lambda_m}$. In this case, we can apply the same perturbation formula as in (3.58) to compare the two dynamics. The following identity then holds

$$\begin{aligned} &\tau_t^{\Lambda_n}(a(f)) - \tau_t^{\Lambda_m}(a(f)) \\ &= \frac{i}{2}\int_0^t \mathrm{d}s \int_{\Lambda_n\times\Lambda_n\setminus\Lambda_m\times\Lambda_m} \mathrm{d}x\mathrm{d}y\, W(x-y)\tau_t^{\Lambda_n}\left(\left[a^*(\varphi_x^\sigma)a^*(\varphi_y^\sigma)a(\varphi_y^\sigma)a(\varphi_x^\sigma), \tau_{t-s}^{\Lambda_m}(a(f))\right]\right). \end{aligned} \tag{3.98}$$

Note the identity

$$
\begin{aligned}
\left[a^*(\varphi_x^\sigma)a^*(\varphi_y^\sigma)a(\varphi_y^\sigma)a(\varphi_x^\sigma), \tau_{t-s}^{\Lambda_m}(a(f))\right] =& a^*(\varphi_x^\sigma)a^*(\varphi_y^\sigma)a(\varphi_y^\sigma)\left\{a(\varphi_x^\sigma), \tau_{t-s}^{\Lambda_m}(a(f))\right\} \\
&- a^*(\varphi_x^\sigma)a^*(\varphi_y^\sigma)\left\{a(\varphi_y^\sigma), \tau_{t-s}^{\Lambda_m}(a(f))\right\}a(\varphi_x^\sigma) \\
&+ a^*(\varphi_x^\sigma)\left\{a^*(\varphi_y^\sigma), \tau_{t-s}^{\Lambda_m}(a(f))\right\}a(\varphi_y^\sigma)a(\varphi_x^\sigma) \\
&- \left\{a^*(\varphi_x^\sigma), \tau_{t-s}^{\Lambda_m}(a(f))\right\}a^*(\varphi_y^\sigma)a(\varphi_y^\sigma)a(\varphi_x^\sigma).
\end{aligned}
\tag{3.99}
$$

Bounding (3.98) in norm, applying (3.97), and using the symmetry of W, we then find for any $T > 0$ constants C and $b > 0$, such that for every $t \in [-T, T]$ and every $n > m$,

$$
\|\tau_t^{\Lambda_n}(a(f)) - \tau_t^{\Lambda_m}(a(f))\| \leq C\|\varphi_0^\sigma\|_2^3\|f\|_1 \int_{\Lambda_n\times\Lambda_n\setminus\Lambda_m\times\Lambda_m} \mathrm{d}x\mathrm{d}y\, |W(x-y)|e^{-bd(X,x)},
\tag{3.100}
$$

which converges to 0 as $n > m \to \infty$ since $|W(x-y)|e^{-bd(X,x)} \in L^1(\mathbb{R}^{2d})$. This shows that for compactly supported f, the sequence $(\tau_t^{\Lambda_n}(a(f)))_{n\geq 1}$ is Cauchy (in norm) uniformly for $t \in [-T, T]$. Thus, the limit exists and gives rise to an isometry from $\mathcal{P}_c$, the set algebraically generated by $\{a(f), a^*(f) : f \in L^2(\mathbb{R}^d)$ of compact support$\}$, into $\mathcal{A}(L^2(\mathbb{R}^d))$. Equation (3.98) can be applied to see this limit is independent of the sequence (Λ_n). As $\mathcal{P}_c$ is dense in $\mathcal{A}(L^2(\mathbb{R}^d))$, this isometry extends uniquely to a homomorphism, τ_t, of $\mathcal{A}(L^2(\mathbb{R}^d))$. It is straightforward to verify that $\tau_t \circ \tau_s = \tau_{t+s}$ and, in particular, that τ_{-t} is the inverse of τ_t. Hence, $\{\tau_t \mid t \in \mathbb{R}\}$ is a one-parameter group of automorphisms of the CAR algebra.

To prove the strong continuity in t, it suffices to note that, for $f \in L^2(\mathbb{R}^d)$ and of compact support, the continuity of $t \mapsto \tau_t^{\Lambda_n}(a(f)) - \tau_t^{\emptyset}(a(f))$ carries over to the limiting function $t \mapsto \tau_t(a(f)) - \tau_t^{\emptyset}(a(f))$ due to the uniform convergence on compact intervals. Then, since $\tau_t^{\emptyset}$ is already known to be strongly continuous, $\tau_t(a(f))$ must be too. Finally, an $\epsilon/3$ argument shows that the strong continuity extends to the full CAR algebra. □

3.6. Appendix A: Convergence of the $\sigma \to 0$ limit

We prove that, for any fixed finite number of fermions, the UV-regularized dynamics converges to the standard one as σ tends to 0. For this we consider arbitrary, not necessarily bounded, measurable $\Lambda \subset \mathbb{R}_d$. When Λ is not bounded, the interaction operator W_Λ^σ is generally unbounded.

Therefore, we start by providing a more careful definition of H^σ_Λ. Note that (3.15) defines a bounded operator $W^\sigma_{\Lambda;n}$ on the n-particle subspace $\big(L^2(\mathbb{R}^d)^{\otimes n}\big)^-$ for each n. Define the operator $W^0_{\Lambda;n}$ on $\big(L^2(\mathbb{R}^d)^{\otimes n}\big)^-$ to be multiplication by the function $\sum_{1\le k<l\le n} W_\Lambda(x_k - x_l)$. For $\sigma \ge 0$, define

$$H_{1;n} + W^\sigma_{\Lambda;n} = \sum_{k=1}^{n}(-\Delta_k + V(x_k)) + W^\sigma_{\Lambda;n} \tag{3.101}$$

acting on $\big(L^2(\mathbb{R}^d)^{\otimes n}\big)^-$. For real-valued $V, W \in L^\infty(\mathbb{R}^d)$ this operator is self-adjoint on the domain $\mathcal{D}(H_{1;n}) = (H^2(\mathbb{R}^d)^{\otimes n})^-$. With $\sigma \ge 0$, let H^σ_Λ be the operator acting on Fock space by

$$H^\sigma_\Lambda = \bigoplus_{n=0}^{\infty}(H_{1;n} + W^\sigma_{\Lambda;n})$$

with domain

$$\mathcal{D}(H^\sigma_\Lambda) = \{(\psi_n) \in \mathfrak{F}^- : \psi_n \in \mathcal{D}(H_{1;n}) \text{ and } \sum_{n=0}^{\infty} \|(H_{1;n} + W^\sigma_{\Lambda;n})\psi_n\|_2^2 < \infty\}.$$

This operator is well-known to be self-adjoint, see e.g. [**90**, Exercise 5.43].

Theorem 3.6.1. *For real-valued $V, W \in L^\infty(\mathbb{R}^d)$, take H_1 as in* (3.8)*. Then, for any measurable set $\Lambda \subset \mathbb{R}^d$*

$$H^\sigma_\Lambda \to H^0_\Lambda$$

in the strong resolvent sense as $\sigma \downarrow 0$.

Using a slight modification of [**76**, Thm VIII.20(a)], the above theorem readily implies:

Corollary 3.6.1. *For $t \in \mathbb{R}$ we denote by $U^\sigma_\Lambda(t) = e^{-itH^\sigma_\Lambda}$ and $U^0_\Lambda(t) = e^{-itH^0_\Lambda}$ the unitary groups generated by H^σ_Λ and H^0_Λ, respectively. Then*

$$\lim_{\sigma\downarrow 0} U^\sigma_\Lambda(t)\psi = U^0_\Lambda(t)\psi$$

for each $\psi \in \mathfrak{F}^-$, uniformly for t in compact subsets of $\mathbb{R}$.

Remark 3.6.1. *Theorem 3.6.1 and Corollary 3.6.1 apply more generally to any self-adjoint operator H_1.*

Proof of Theorem 3.6.1. We start by reducing the proof Theorem 3.6.1 to what is essentially the 2-particle situation. Recall that for $\sigma \geq 0$,

$$H_\Lambda^\sigma = \bigoplus_{n=0}^{\infty} (H_{1;n} + W_{\Lambda;n}^\sigma).$$

For each n, let $\mathcal{D}_n^- = (\mathcal{S}(\mathbb{R}^d)^{\otimes n})^-$ be the antisymmetrized n-fold tensor product of the Schwarz space $\mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)^{\otimes n} = \bigotimes_{j=1}^n \mathcal{S}(\mathbb{R}^d)$ is the set of finite linear combinations of functions ψ of the form $\psi(x_1, ..., x_n) = \psi_1(x_1) \cdots \psi_n(x_n)$, with each $\psi_j \in \mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{S}(\mathbb{R}^d)$ is a core for H_1, D_n^- is a core for $H_{1;n}$ by the Corollary to [**76**, Thm VIII.33]. It follows that $\mathcal{D}_n^-$ is a core for $H_{1;n} + W_{\Lambda;n}^\sigma$ for every $\sigma \geq 0$. If

$$\mathcal{D}^- = \{(\psi_n) \in \mathfrak{F}^- : \psi_n \in \mathcal{D}_n^- \text{ and } \exists N \text{ with } \psi_n = 0 \, \forall n \geq N\}, \tag{3.102}$$

then it is not difficult to see that $\mathcal{D}^-$ is a core for H_Λ^σ for every $\sigma \geq 0$ (this is essentially Example 2 in [**76**, Sec VIII]).

As is well-known, strong resolvent convergence follows from strong convergence on a common core, see e.g. [**76**, Thm VIII.25(a)], and thus to prove Theorem 3.6.1, it suffices to establish that

$$\lim_{\sigma \downarrow 0} H_\Lambda^\sigma \psi = H_\Lambda^0 \psi \tag{3.103}$$

for every $\psi \in \mathcal{D}^-$. Given the form of $\mathcal{D}^-$, see (3.102), it is clear that (3.103) will follow if

$$\lim_{\sigma \downarrow 0} W_{\Lambda;n}^\sigma \psi = W_{\Lambda;n}^0 \psi \text{ for every } \psi \in \mathcal{D}_n^-. \tag{3.104}$$

We need only prove (3.104). To this end, let $b^*(\cdot)$ and $b(\cdot)$ denote the creation and annihilation operators on the full Fock space $\mathfrak{F} = \bigoplus_{n=0}^\infty L^2(\mathbb{R}^d)^{\otimes n}$; for any $f \in L^2(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d)^{\otimes n}$,

$$(b(f)\psi)(x_1, ..., x_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} \mathrm{d}x f(x) \psi(x, x_1, ..., x_{n-1})$$

and

$$(b^*(f)\psi)(x_1, ..., x_{n+1}) = \sqrt{n+1} f(x_1) \psi(x_2, ..., x_{n+1}).$$

For any $f, g \in L^2(\mathbb{R}^d)$, the operator $b^*(f)b^*(g)b(g)b(f)$ is reduced by the n-particle subspace, therefore for $\sigma > 0$ we may define

$$\tilde{W}^\sigma_{\Lambda;n} = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, W_\Lambda(x,y) b^*(\varphi^\sigma_x)b^*(\varphi^\sigma_y)b(\varphi^\sigma_y)b(\varphi^\sigma_x)\Big|_{\bigotimes_{k=1}^n L^2(\mathbb{R}^d)},$$

which is a bounded operator on $L^2(\mathbb{R}^d)^{\otimes n}$. Also define $\tilde{W}^0_{\Lambda;n}$ by

$$(\tilde{W}^0_{\Lambda;n}h)(x_1,...,x_n) = \frac{n(n-1)}{2} W_\Lambda(x_1,x_2)h(x_1,...,x_n)$$

for $h \in L^2(\mathbb{R}^d)^{\otimes n}$. With these definitions, for every $\sigma \geq 0$,

$$W^\sigma_{\Lambda;n} = A_n \tilde{W}^\sigma_{\Lambda;n}\Big|_{(\bigotimes_{k=1}^n L^2(\mathbb{R}^d))^-}$$

where A_n is the antisymmetrization projection $L^2(\mathbb{R}^d)^{\otimes n} \to (L^2(\mathbb{R}^d)^{\otimes n})^-$. We conclude that: if we prove that $\lim_{\sigma\downarrow 0} \tilde{W}^\sigma_{\Lambda;n}\psi = \tilde{W}^0_{\Lambda;n}\psi$ for every ψ of the form $\psi(x_1,...,x_n) = \psi_1(x_1)\cdots\psi_n(x_n)$, where $\psi_1,...,\psi_n \in \mathcal{S}(\mathbb{R}^d)$, then (3.104) follows. Moreover, since for every $\sigma \geq 0$ $\tilde{W}^\sigma_{\Lambda;n}$ acts nontrivially only on the first two particles, it will suffice to prove that $\tilde{W}^\sigma_{\Lambda;2} \to \tilde{W}^0_{\Lambda;2}$ strongly as $\sigma \downarrow 0$.

Let $\sigma > 0$ and introduce the function $\Phi^\sigma : \mathbb{R}^{2d} \to \mathbb{R}$ by setting

$$\Phi^\sigma(x,y) = \varphi^\sigma_0(x)\varphi^\sigma_0(y) = \frac{1}{(2\pi\sigma^2)^d} e^{-\frac{1}{2\sigma^2}(|x|^2+|y|^2)} \quad \text{for any } x,y \in \mathbb{R}^d. \tag{3.105}$$

It is clear that Φ^σ is L^1-normalized, and moreover, a simple calculation shows that for $\psi \in L^2(\mathbb{R}^{2d})$,

$$\begin{aligned}
&(\tilde{W}^\sigma_{\Lambda;2}\psi)(x_1,x_2) \\
&\quad = \int_{\mathbb{R}^{4d}} \mathrm{d}x\mathrm{d}y\mathrm{d}z_1\mathrm{d}z_2\, W_\Lambda(x,y)\psi(z_1,z_2)\varphi^\sigma_{z_1}(x)\varphi^\sigma_{x_1}(x)\varphi^\sigma_{z_2}(y)\varphi^\sigma_{x_2}(y) \\
&\quad = (\Phi^\sigma * (W_\Lambda(\Phi^\sigma * \psi)))(x_1,x_2).
\end{aligned}$$

We are now ready to conclude the proof of the theorem.

Let $\psi \in \mathcal{S}(\mathbb{R}^d)^{\otimes 2}$. Then

$$\tilde{W}^\sigma_{\Lambda;2}\psi = \Phi^\sigma * (W_\Lambda(\Phi^\sigma * \psi)) = \Phi^\sigma * (W_\Lambda\psi) + \Phi^\sigma * (W_\Lambda(\Phi^\sigma * \psi - \psi))$$

The first term above converges to the desired limit. In fact, convolutions with appropriately scaled L^1-functions converge in L^p-norm, see e.g. [**38**, Thm 8.14 a)], and thus since $\psi \in L^2(\mathbb{R}^{2d})$,

$$\Phi^\sigma * (W_\Lambda \psi) \to W_\Lambda \psi \quad \text{in } L^2(\mathbb{R}^{2d}) \text{ as } \sigma \to 0\,.$$

We handle the remainder with Young's inequality, i.e. the bound

$$\|\Phi^\sigma * (W_\Lambda(\Phi^\sigma * \psi - \psi))\|_2 \le \|\Phi^\sigma\|_1 \|W_\Lambda(\Phi^\sigma * \psi - \psi)\|_2 \le \|W\|_\infty \|\Phi^\sigma * \psi - \psi\|_2.$$

A further application of [**38**, Thm 8.14 a)] shows that

$$\lim_{\sigma \downarrow 0} \|\Phi^\sigma * \psi - \psi\|_2 = 0$$

which proves the result. □

3.7. Appendix B: Several Fourier transforms

In this section we aim at proving the following Lemma:

LEMMA 3.7.1. *Let $a > 0$, $n \in \mathbb{N}$ with $n \ge 2$, and $x \in \mathbb{R}^d$. If $G_a(x) = e^{-a|x|}$, then there exists a constant $D_3 > 0$ such that*

$$(\underbrace{G_a * G_a * \cdots * G_a}_{n-1 \text{ convolutions}})(x) \le \left(\frac{D_3}{a^d}\right)^n a^d e^{-\frac{a|x|}{4}}. \tag{3.106}$$

To prove this, we first compute several Fourier transforms. We let $\mathcal{F}$ denote the unitary Fourier transform and $\mathcal{F}^*$ its inverse.

LEMMA 3.7.2. *Let $a > 0$ and $G_a(x) = e^{-a|x|}$. Then for $\xi \in \mathbb{R}^d$*

$$\big(\mathcal{F}G_a\big)(\xi) = \frac{2^{d/2}\Gamma(\frac{d+1}{2})}{\sqrt{\pi}} \frac{a}{(a^2 + \xi^2)^{\frac{d+1}{2}}}, \tag{3.107}$$

where Γ denotes the Gamma function.

PROOF. Let $\xi \in \mathbb{R}^d$. We compute using the spherical symmetry of G_a

$$\begin{aligned}\big(\mathcal{F}G_a\big)(\xi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x e^{-a|x|} e^{-ix\xi} \\ &= |\xi|^{\frac{2-d}{2}} \int_0^\infty \mathrm{d}r\, r^{\frac{d}{2}}\, J_{\frac{d-2}{2}}(|\xi| r)\, e^{-ar} \end{aligned} \tag{3.108}$$

where $J_\nu(y)$ denotes the Bessel function of first kind. Computing this integral using [**44**, Sec. 6.621 eq. 1] gives

$$\big(\mathcal{F}G_a\big)(\xi) = \frac{2^{d/2}\Gamma(\frac{d+1}{2})}{\sqrt{\pi}} \frac{a}{(a^2+|\xi|^2)^{\frac{d+1}{2}}}, \tag{3.109}$$

which is the assertion. □

LEMMA 3.7.3. *Let $a > 0$, $\xi \in \mathbb{R}^d$ and*

$$H_a(\xi) = \frac{a}{(a^2+|\xi|^2)^{\frac{d+1}{2}}}. \tag{3.110}$$

Then for $n \in \mathbb{N}$ we obtain

$$\big(\mathcal{F}^* H_a^n\big)(x) = \frac{2^{\frac{2-(d+1)n}{2}} a^{\frac{d-(d-1)n}{2}} |x|^{\frac{d(n-1)+n}{2}} K_{\frac{d(n-1)+n}{2}}(a|x|)}{\Gamma\big(\frac{(1+d)n}{2}\big)}, \tag{3.111}$$

where Γ denotes the Gamma function and $K_\nu(y)$ the modified Bessel function.

PROOF. We compute, using again the Fourier transform for spherically symmetric functions,

$$\begin{aligned}\big(\mathcal{F}^* H_a^n\big)(x) &= \frac{a^n}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}\xi \Big(\frac{1}{a^2+|\xi|^2}\Big)^{\frac{n(d+1)}{2}} e^{ix\xi} \\ &= a^n |x|^{\frac{2-d}{2}} \int_0^\infty \mathrm{d}r\, r^{\frac{d}{2}}\, J_{\frac{d-2}{2}}(|x| r) \Big(\frac{1}{a^2+r^2}\Big)^{\frac{n(d+1)}{2}}, \end{aligned} \tag{3.112}$$

where $J_\nu(y)$ is the Bessel function of first kind. Integrating the latter with the help of [**44**, Sec. 6.565 eq. 4], gives

$$\big(\mathcal{F}^* H_a^n\big)(x) = \frac{2^{1-\frac{(d+1)n}{2}} a^{\frac{d-(d-1)n}{2}} |x|^{\frac{d(n-1)+n}{2}} K_{\frac{d(n-1)+n}{2}}(a|x|)}{\Gamma\big(\frac{(1+d)n}{2}\big)}. \tag{3.113}$$

□

Hence, from Lemma 3.7.2 and Lemma 3.7.3 we obtain

$$\mathcal{F}^*\big((\mathcal{F}G_a)^n\big)(x) = \frac{\big(\Gamma(\frac{d+1}{2})\big)^n 2^{\frac{2-n}{2}} a^{\frac{d-(d-1)n}{2}} |x|^{\frac{d(n-1)+n}{2}} K_{\frac{d(n-1)+n}{2}}(a|x|)}{\pi^{n/2}\Gamma\big(\frac{(d+1)n}{2}\big)}. \tag{3.114}$$

LEMMA 3.7.4. *Let $\eta > 0$. The modified Bessel function K_η satisfies for $y > 0$ the bound*

$$0 \leq K_\eta(y) \leq \frac{4^\eta}{y^\eta} e^{-\frac{y}{4}} \Gamma(\eta). \tag{3.115}$$

PROOF. We write the modified Bessel function K_η as

$$K_\eta(y) = \int_0^\infty \mathrm{d}t\, e^{-y\cosh(t)} \cosh(\eta t), \tag{3.116}$$

see [**44**, Sec. 8.432 eq. 1]. Using $\frac{1}{2}e^x \leq \cosh(x) \leq e^x$ valid for all $x \geq 0$, we obtain

$$\int_0^\infty \mathrm{d}t\, e^{-y\cosh(t)} \cosh(\eta t) \leq \int_0^\infty \mathrm{d}t\, e^{-\frac{ye^t}{2}} e^{\eta t} = \int_1^\infty \mathrm{d}u\, e^{-\frac{uy}{2}} u^{\eta-1}, \tag{3.117}$$

where we performed the change of variables $u = e^t$ in the last line. Now, for $u \geq 1$ and $y > 0$ we have $e^{-\frac{uy}{2}} \leq e^{-\frac{y}{4}} e^{-\frac{uy}{4}}$ and therefore

$$\begin{aligned} \int_1^\infty \mathrm{d}u\, e^{-\frac{uy}{2}} u^{\eta-1} &\leq e^{-\frac{y}{4}} \int_0^\infty \mathrm{d}u\, e^{-\frac{uy}{4}} u^{\eta-1} \\ &= \frac{4^\eta}{y^\eta} e^{-\frac{y}{4}} \int_0^\infty \mathrm{d}u\, e^{-u} u^{\eta-1} = \frac{4^\eta}{y^\eta} e^{-\frac{y}{4}} \Gamma(\eta). \end{aligned} \tag{3.118}$$

$\square$

PROOF OF LEMMA 3.7.1. Starting with (3.114) and using Lemma 3.7.4 we obtain

$$\begin{aligned} (G_a * G_a * \cdots * G_a)(x) &= (2\pi)^{d(n-1)/2} \mathcal{F}^*\big((\mathcal{F}G_a)^n\big)(x) \\ &\leq \frac{\big(\Gamma(\frac{d+1}{2})\pi^{\frac{d-1}{2}} 2^{\frac{3d+1}{2}}\big)^n \pi^{-\frac{d}{2}} 2^{\frac{1}{2}} a^{-d(n-1)} \Gamma\big(\frac{d(n-1)+n}{2}\big) e^{-\frac{a|x|}{4}}}{\Gamma\big(\frac{(d+1)n}{2}\big)} \\ &\leq D_3^n a^{-d(n-1)} e^{-\frac{a|x|}{4}} \end{aligned} \tag{3.119}$$

for some explicit constant $D_3 > 0$ depending on d, where we used that $(2/\pi^d)^{\frac{1}{2}} \leq 1$ and

$$\frac{\Gamma(\frac{d(n-1)+n}{2})}{\Gamma\big(\frac{(d+1)n}{2}\big)} \leq 1. \tag{3.120}$$

This gives the assertion. ∎

Bibliography

[1] H. Abdul-Rahman, B. Nachtergaele, R. Sims, and G. Stolz, *Localization properties of the disordered xy spin chain. a review of mathematical results with an eye toward many-body localization*, Ann. Phys. (Berlin), 529 (2017), p. 1600280. arXiv:1610.01939.

[2] M. Aizenman and S. Molchanov, *Localization at large disorder and at extreme energies: An elementary derivations*, Commun. Math. Phys., 157 (1993), pp. 245–278.

[3] M. Aizenman and S. Warzel, *Localization bounds for multiparticle systems*, Commun. Math. Phys., 290 (2009), pp. 903–934.

[4] ———, *Absolutely continuous spectrum implies ballistic transport for quantum particles in a random potential on tree graphs*, J. Math. Phys., 53 (2012), p. 095205.

[5] ———, *Random operators. Disorder effects on quantum spectra and dynamics.*, vol. 168 of Graduate Studies in Mathematics, Amer. Math. Soc., 2015.

[6] L. Amour, P. Levy-Bruhl, and J. Nourrigat, *Dynamics and Lieb-Robinson estimates for lattices of interacting anharmonic oscillators*, Colloq. Math., 118 (2010), pp. 609–648.

[7] P. W. Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev., 109 (1958), pp. 1492–1505.

[8] S. Bachmann, A. Bols, W. De Roeck, and M. Fraas, *Quantization of conductance in gapped interacting systems*. arXiv:1707.06491, 2017.

[9] ———, *A many-body index for quantum charge transport*, Commun. Math. Phys., (2019). arXiv:1810.07351. To appear in Commun. Math. Phys.

[10] D. M. Basko, I. L. Aleiner, , and B. L. Altshuler, *Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states*, Annals of Physics, 321 (2006), pp. 1126–1205.

[11] V. Beaud and S. Warzel, *Low-energy Fock-space localization for attractive hard-core particles in disorder*, Ann. H. Poincaré, 18 (2017), pp. 3143–3166.

[12] ———, *Bounds on the entanglement entropy of droplet states in the XXZ spin chain*, J. Math. Phys., 59 (2018), p. 012109. arXiv:1709.10428.

[13] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. 2, Springer Verlag, 2 ed., 1997.

[14] P. Braun, D. Waltner, M. Akila, B. Gutkin, and T. Guhr, *Transition from quantum chaos to localization in spin chains*. .arXiv:1902.06265, 2019.

[15] S. Bravyi, M. Hastings, and S. Michalakis, *Topological quantum order: stability under local perturbations*, J. Math. Phys., 51 (2010), p. 093512.

[16] S. Bravyi and M. B. Hastings, *A short proof of stability of topological order under local perturbations*, Commun. Math. Phys., 307 (2011), p. 609.

[17] S. Bravyi, M. B. Hastings, and F. Verstraete, *Lieb-Robinson bounds and the generation of correlations and topological quantum order*, Phys. Rev. Lett., 97 (2006), p. 050401.

[18] J.-B. Bru and W. de Siquiera Pedra, *Lieb-Robinson Bounds for Multi-Commutators and Applications to Response Theory*, vol. 13 of Springer Briefs in Mathematical Physics, Springer Nature, 2017.

[19] D. Buchholz, *The resolvent algebra for oscillating lattice systems: Dynamics, ground and equilibrium states*, Commun. Math. Phys., 353 (2017), pp. 691–716. arXiv:1605.05259.

[20] D. Buchholz and H. Grundling, *The resolvent algebra: A new approach to canonical quantum systems*, J. Funct. Analysis, 254 (2008), pp. 2725–2779.

[21] A. Chandran, I. Kim, G. Vidal, and D. Abanin, *Constructing local integrals of motion in the many-body localized phase*, Phys. Rev. B, 91 (2015), p. 085425.

[22] J. Chapman and G. Stolz, *Localization for random block operators related to the XY spin chain*, Ann. H. Poincaré, 16 (2015), pp. 405–435.

[23] C.-F. Chen and A. Lucas, *Finite speed of quantum scrambling with long range interactions*, Phys. Rev. Lett., 123 (2019).

[24] V. Chulaevsky and Y. Suhov, *Multi-particle Anderson localisation: induction on the number of particles*, Math. Phys. Anal. Geom., 12 (2009), pp. 117–139.

[25] J. Cirac, S. Michalakis, D. Perez-Garcia, and N. Schuch, *Robustness in projected entangled pair states*, Phys. Rev. B, 88 (2013), p. 115108.

[26] W. De Roeck and F. Huveneers, *Stability and instability towards delocalization in many-body localization systems*, Phys. Rev. B, 95 (2017), p. 155129.

[27] W. De Roeck, F. Huveneers, M. Müller, and M. Schiulaz, *Absence of many-body mobility edges*, Phys. Rev. B, 93 (2016), p. 014203.

[28] W. De Roeck, F. Huveneers, and S. Olla, *Subdiffusion in one-dimensional Hamiltonian chains with sparse interactions*, J. Stat. Phys., (2020).

[29] W. De Roeck and J. Imbrie, *Many-body localization: stability and instability*, Phil. Trans. R. Soc. A, 375 (2017), p. 20160422.

[30] W. De Roeck and M. Salmhofer, *Persistence of exponential decay and spectral gaps for interacting fermions*, Commun. Math. Phys., 365 (2019), pp. 773–796.

[31] W. De Roeck and M. Schütz, *Local perturbations perturb exponentially–locally*, J. Math. Phys., 56 (2015), p. 061901.

[32] F. Dyson, *Divergence of perturbation theory in quantum electrodynamics*, Phys. Rev., 85 (1952), pp. 631–632.

[33] A. Elgart, L. Erdös, B. Schlein, and H.-T. Yau, *Nonlinear Hartree equation as the mean field limit of weakly coupled fermions*, J. Math. Pures Appl., 83 (2004), pp. 1241–1273.

[34] A. Elgart, A. Klein, and G. Stolz, *Droplet localization in the random XXZ model and its manifestations*, J. Phys. A: Math. Theor., 51 (2018), p. 01LT02.

[35] ———, *Manifestations of dynamical localization in the disordered xxz spin chain*, Commun. Math. Phys., 361 (2018), pp. 1083–1113 —. arXiv:1708.00474.

[36] ———, *Many-body localization in the droplet spectrum of the random XXZ quantum spin chain*, J. Funct. Anal., 275 (2018), pp. 211–258. arXiv:1703.07483.

[37] A. Elgart, M. Shamis, and S. Sodin, *Localisation for non-monotone Schrödiner operators*, J. Eur. Math. Soc., 16 (2014), pp. 909–924.

[38] G. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, 1999.

[39] J. Fröhlich, M. Merkli, and D. Ueltschi, *Dissipative transport: Thermal contacts and tunnelling junctions*, Ann. Henri Poincaré, 4 (2003), pp. 897–945.

[40] J. Fröhlich and A. Pizzo, *Lie-Schwinger block-diagonalization and gapped quantum chains*. arXiv:1812.02457, 2018.

[41] M. Gebert, B. Nachtergaele, J. Reschke, and R. Sims, *Lieb-Robinson bounds and stronlgy continuous dynamics for a class of many-body fermion systems in* $\mathbb{R}^d$, Ann. H. Poincaré, 21 (2020), pp. 3609–3637.

[42] F. Germinet and A. Klein, *Bootstrap multiscale analysis and localization in random media*, Commun. Math. Phys., 222 (2001), pp. 415–448.

[43] M. Goihl, J. Eisert, and C. Krumnow, *Are many-body localized systems stable in the presence of a small bath?*, Phys. Rev. B, 99 (2019), p. 195145. arXiv:1902.04371.

[44] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Elsevier/Academic Press, Amsterdam, seventh ed., 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.

[45] H. Grundling and G. Rudolph, *Dynamics for QCD on an infinite lattice*, Commun. Math. Phys., 349 (2017), pp. 1163–1202.

[46] E. Hamza, R. Sims, and G. Stolz, *Dynamical localization in disordered quantum spin systems*, Commun. Math. Phys., 315 (2012), pp. 215–239.

[47] M. B. Hastings, *The stability of free Fermi Hamiltonians*, J. Math. Phys., 60 (2019), p. 042201. arXiv:1706.02270.

[48] M. B. Hastings and T. Koma, *Spectral gap and exponential decay of correlations*, Commun. Math. Phys., 265 (2006), pp. 781–804.

[49] M. B. Hastings and S. Michalakis, *Quantization of hall conductance for interacting electrons on a torus*, Commun. Math. Phys., 334 (2015), pp. 433–471.

[50] J. Hunter and B. Nachtergaele, *Applied Analysis*, World Scientific, 2001.

[51] D. Huse, R. Nandkishore, and V. Oganesyan, *Phenomenology of fully many-body-localized systems*, Phys. Rev. B, 90 (2014), p. 174202.

[52] J. Imbrie, *On many-body localization for quantum spin chains*, J. Stat. Phys., 163 (2016), pp. 998–1048.

[53] J. Imbrie, V. Ros, and A. Scardicchio, *Review: Local integrals of motion in many-body localized systems*, Ann. Phys. (Berlin), 529 (2017), p. 1600278.

[54] I. Klich, *On the stability of topological phases on a lattice*, Ann. Phys., 325 (2010), pp. 2120–2131.

[55] H. Kunz and B. Souillard, *Sur le spectre des opérateurs aux différences finies aléatoires*, Commun. Math. Phys., 78 (1980), pp. 201–246.

[56] M. Lewin and J. Sabin, *The Hartree equation for infinitely many particles. I. Well-posedness theory*, Commun. Math. Phys., 334 (2015), pp. 117–170.

[57] E. Lieb and D. Robinson, *The finite group velocity of quantum spin systems*, Commun. Math. Phys., 28 (1972), pp. 251–257.

[58] E. Lieb, T. Schultz, and D. Mattis, *Two soluble models of an antiferromagnetic chain*, Ann. Phys. (N.Y.), 16 (1961), pp. 407–466.

[59] D. Luitz, F. Huveneers, and W. De Roeck, *How a small quantum bath can thermalize long localized chains*, Phys. Rev. Lett., 119 (2017), p. 150602.

[60] N. Macé, N. Laflorencie, and F. Alet, *Many-body localization in a quasiperiodic fibonacci chain*, SciPost Phys., 6 (2019), p. 050.

[61] V. Mastropietro, *Coupled identical localized fermionic chains with quasi-random disorder*, Phys. Rev. B, 95 (2017), p. 075155.

[62] ———, *Localization in interacting fermionic chains with quasi-random disorder*, Commun. Math. Phys., 351 (2017), pp. 283–309.

[63] S. Michalakis and J. Zwolak, *Stability of frustration-free Hamiltonians*, Commun. Math. Phys., 322 (2013), pp. 277–302.

[64] B. Nachtergaele, Y. Ogata, and R. Sims, *Propagation of correlations in quantum lattice systems*, J. Stat. Phys., 124 (2006), pp. 1–13.

[65] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, *Lieb-Robinson bounds for harmonic and anharmonic lattice systems*, Commun. Math. Phys., 286 (2009), pp. 1073–1098.

[66] B. Nachtergaele and J. Reschke, *Slow propagation in some disordered quantum spin chains*, J. Stat. Phys., 182 (2021).

[67] B. Nachtergaele, B. Schlein, R. Sims, S. Starr, and V. Zagrebnov, *On the existence of the dynamics for anharmonic quantum oscillator systems*, Rev. Math. Phys., 22 (2010), pp. 207–231.

[68] B. Nachtergaele, V. Scholz, and R. F. Werner, *Local approximation of observables and commutator bounds*, in Operator Methods in Mathematical Physics, J. Janas, P. Kurasov, A. Laptev, and S. Naboko, eds., vol. 227 of Operator Theory: Advances and Applications, Birkäuser Verlag, 2013, pp. 143–149.

[69] B. Nachtergaele, R. Sims, and A. Young, *Lieb-Robinson bounds, the spectral flow, and stability for lattice fermion systems*, in Mathematical Results in Quantum Physics, F. Bonetto, D. Borthwick, E. Harrell, and M. Loss, eds., vol. 717 of Contemporary Mathematics, Amer. Math. Soc., 2018, pp. 93–115.

[70] ———, *Quasi-locality bounds for quantum lattice systems and perturbations of gapped ground states.* in preparation, 2018.

[71] ———, *Quasi-locality bounds for quantum lattice systems. I. Lieb-Robinson bounds, quasi-local maps, and spectral flow automorphisms*, J. Math. Phys., 60 (2019), p. 061101.

[72] H. Narnhofer and W. Thirring, *Quantum field theories with Galilei-invariant interactions*, Phys. Rev. Lett., 64 (1990), pp. 1863–1866.

[73] V. Oganesyan and D. Huse, *Localization of interacting fermions at high temperature*, Phys. Rev. B, 75 (2007), p. 155111.

[74] A. Pal and D. Huse, *The many-body localization phase transition*, Phys. Rev. B, 82 (2010), p. 174411.

[75] M. Reed and B. Simon, *Fourier Analysis, Self-Adjointness. Methods of Modern Mathematical Physics*, vol. 2, Academic Press, 1975.

[76] M. Reed and B. Simon, *Functional Analsyis. Methods of Modern Mathematical Physics*, vol. 1, Academic Press, revised and enlarged ed., 1980.

[77] D. W. Robinson, *Statistical mechanics of quantum spin systems II*, Commun. Math. Phys., 7 (1968), pp. 337–348.

[78] D. Ruelle, *Statistical Mechanics: Rigorous Results*, Benjamin, New York, 1969.

[79] S. Sakai, *Operator algebras in dynamical systems*, vol. 41 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1991. The theory of unbounded derivations in C^*-algebras.

[80] N. Schuch, D. Perez-Garcia, and I. Cirac, *Classifying quantum phases using matrix product states and peps*, Phys. Rev. B, 84 (2011), p. 165139.

[81] M. Serbyn, Z. Papić, and D. A. Abanin, *Local conservation laws and the structure of the many-body localized states*, Phys. Rev. Lett., 111 (2013), p. 127201.

[82] M. Serbyn, Z. Papic, and D. A. Abanin, *Universal slow growth of entanglement in interacting strongly disordered systems*, Phys. Rev. Lett., 110 (2013), p. 260601.

[83] R. Sims and S. Warzel, *Decay of determinantal and Pfaffian correlation functionals in one-dimensional lattices*, Commun. Math. Phys., 347 (2016), pp. 903–931.

[84] R. F. Streater, *On certain non-relativistic quantized fields*, Commun. Math. Phys., 7 (1968), pp. 93–98.

[85] R. F. Streater and I. F. Wilde, *The time evolution of quantized fields with bounded quasi-local interaction density*, Commun. Math. Phys., 17 (1970), pp. 21–32.

[86] J. Šũntajs, J. Bonča, T. Prosen, and L. Vidmar, *Quantum chaos challenges many-body localization.* arXiv:1905.06345, 2019.

[87] G. Teschl, *Mathematical methods in quantum mechanics*, vol. 99 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2009. With applications to Schrödinger operators.

[88] T. Thiery, F. Huveneers, M. Müller, and W. De Roeck, *Many-body delocalization as a quantum avalanche*, Phys. Rev. Lett., 121 (2018), p. 140601.

[89] A. F. Verbeure, *Many-Body Boson Systems*, Theoretical and Mathematical Physics, Springer-Verlag, 2011.

[90] J. Weidmann, *Linear Operators in Hilbert Spaces*, vol. 68 of Graduate Texts in Mathematics, Springer, 1980.