Algebraic and Boolean Methods for
Computation and Certification of Ramsey-type Numbers

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Abstract

Ramsey numbers and their variants are among the most interesting and well-studied numbers in combinatorics. This dissertation explores them through the lenses of algebraic and Boolean methods.

Many combinatorial problems have natural encodings as systems of polynomial equations where the system is feasible if and only if the original problem has a solution. When a system $f_1 = \cdots = f_m = 0$ has no solution over an algebraically closed field, Hilbert's Nullstellensatz guarantees the existence of a certain polynomial identity $\sum_{i=1}^{m} \beta_i f_i = 1$ called a *certificate*. The *degree* of a certificate is the maximal degree of the β_i and is related to the complexity of the underlying problem. For example, if a suitable encoding of a combinatorial problem gives constant degree bounds, then the problem is in P.

In Chapter 2, we give a general method to encode a broad class of Ramsey-type problems, including the problems of computing Ramsey, Schur, and van der Waerden numbers, as systems of polynomial equations and construct Nullstellensatz certificates when they have no solution. The degrees of these certificates are given in terms of winning strategies of *Builder-Painter games* which generalize the notion of the *(restricted) online Ramsey numbers*. Additionally, the degrees are strictly smaller than those given by the best known general bounds for these ideals.

Later in Chapter 2 we study the classical Ramsey numbers through Alon's Combinatorial Nullstellensatz and construct "Ramsey polynomials" that give lower bounds for Ramsey numbers when they are not identically zero. We call the coefficients of these polynomials *ensemble numbers* and investigate their combinatorial meaning.

Our main contributions in Chapter 3 deal with computing *Rado numbers.* Given an equation $\mathcal{E}$, the k-color Rado number $R_k(\mathcal{E})$ is the smallest number n such that every k-coloring of $\{1, 2, \ldots, n\}$ contains a monochromatic solution to $\mathcal{E}$. We encode the problem of computing Rado numbers as an instance of the *Boolean satisfiability problem (SAT)* by constructing Boolean formulas $F_n^k(\mathcal{E})$

that are satisfiable if and only if $R_k(\mathcal{E}) > n$. Using SAT solvers we determine hundreds of new Rado number values.

We observe that many equations $\mathcal{E}$ have the property that not all the integers in $[n]$ are used in proving upper bounds $R_k(\mathcal{E}) \leq n$. Moreover, for certain families of equations we can describe the integers that *are* used using a single set of polynomials. We exploit this property and use a modified encoding to compute new values for *infinite* families of three-color Rado numbers, namely formulas for $R_3(x - y = bz)$, $R_3(a(x - y) = (a - 1)z)$ for $a \geq 3$, and $R_3(a(x - y) = bz)$ for $b \geq 1, a \geq b + 2, gcd(a, b) = 1$.

The *degree of regularity* of an equation $\mathcal{E}$ is the largest number of colors k for which $R_k(\mathcal{E})$ is finite. We prove several new bounds on the degree of regularity for classes of linear equations, and using this we compute the degree of regularity of $ax + by = cz$ for small values of a, b, and c. Moreover, we classify the degree of regularity for some equations of the form $a(x + y) = bz$; this improves on Rado's original result for the degree of regularity of these equations. Finally, we answer a conjecture of Golowich and show that for all $m, k \geq 3$ there are m-variable linear equations with degree of regularity at most k.

In Chapter 4, we study the Ramsey properties of integer sequences. A *D-diffsequence* is a sequence whose consecutive differences lie in a prescribed set D. We focus on the case where D is the set F of Fibonacci numbers. In particular, using combinatorial words and word morphisms, we construct a 4-coloring of $\mathbb{Z}^+$ that avoids 4-term F-diffsequences and a 2-coloring of $\mathbb{Z}^+$ that avoids 5-term arithmetic progressions with common difference in F. These colorings improve on the best known Ramsey results for the Fibonacci numbers. We also give some related results and experimental data on diffsequences involving Lucas and Perrin numbers.

Lastly, in Chapter 5 we show how our SAT methods can be applied to other combinatorial problems. Here we give new bounds and exact values for Ramsey numbers involving book and wheel graphs. Furthermore, we compute several exact values of *Turán numbers* for complete bipartite graphs and two-dimensional analogues of the *Sidon-Ramsey numbers*. We also apply SAT solving to geometric problems involving angles in finite fields and compute maximal sizes of sets that avoid certain angles in $\mathbb{F}_q^n$.

The Appendix contains large tables of data, including a 3500-entry table of two-color Rado numbers and colorings of large sets of integers that avoid certain monochromatic sequences.

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First and foremost, I am thankful to my advisor Jesús A. De Loera for introducing me to the subject of Ramsey numbers and many other beautiful areas of mathematics. This dissertation arose from a conversation in early 2020 involving working out $R(3,3) \leq 6$ on a blackboard and a suggestion to try studying Ramsey numbers with the Nullstellensatz. I never looked back. Jesús supported me with countless ideas and encouragement along the way, and I was inspired by his legendary enthusiasm and work ethic. His mentorship guided me through this long journey, and I will forever be grateful for his support.

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My work on Rado numbers would not have been nearly as strong without the help of Yuan Chang, who as an undergraduate contributed immensely to the code used in our computations. His work probably tripled (at least!) the number of Rado numbers we could compute, and I am thankful for all his work and helpful discussions in Summer 2021 and beyond.

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CHAPTER 1

Introduction

To state the contributions of this thesis, we begin with an introduction to *Ramsey theory.* The discussion in Sections 1.1 and 1.2 is meant to be self-contained and accessible to non-experts; those familiar with the subject may safely skip the proofs in these sections. Sections 1.3 and 1.4 give background on the methods used in this thesis. Our main contributions are stated in Section 1.5.

1.1. Ramsey Theory

Ramsey theory is the study of patterns that emerge in sufficiently large mathematical objects. In particular, it often deals with structures that are preserved under set partitioning or coloring. A classical example is the so-called *friendship theorem*, which states that at a party with six guests, there is necessarily either a group of three guests who all know each other, or a group of three guests who are all mutual strangers. However, the same is not true of every party of five guests: if each guest is arranged in a circle and knows only the two adjacent guests, then there is no group of three mutual acquaintances or three mutual strangers.

This phenomenon is a special case of *Ramsey's theorem*, first proven by the logician Frank Plumpton Ramsey in 1926 [**115**]. The simplest version of the theorem is stated below.

THEOREM 1.1.1 (Ramsey). *Fix positive integers r and s. Then there exists an n such that every red-blue edge coloring of K_n contains either a red K_r or a blue K_s.*

A major question in combinatorics asks how small n can be. The smallest such n is called the *Ramsey number* $R(r, s)$.

DEFINITION 1.1.1. *For given positive integers r and s, the* Ramsey number *$R(r, s)$ is the smallest n such that every red-blue edge coloring of K_n contains either a red K_r or a blue K_s. Equivalently, $R(r, s)$ is the smallest n such that every graph on n vertices contains either a clique of size r or an independent set of size s.*

We give some simple properties of Ramsey numbers below.

Proposition 1.1.1. *Let r and s be positive integers. The Ramsey numbers $R(r,s)$ satisfy the following properties.*

(i) $R(r,s) = R(s,r)$.

(ii) $R(1,s) = 1$.

(iii) $R(2,s) = s$.

Despite their apparent simplicity, the Ramsey numbers are notoriously hard to compute when $r, s \geq 3$. The following quote by Paul Erdős illustrates this difficulty:

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

In fact, only nine such values are known, and we display them in Table 1.1 below. We omit the entries below the main diagonal because $R(r,s) = R(s,r)$.

Table 1.1. Ramsey numbers $R(r,s)$

$r \backslash s$	1	2	3	4	5	6	7	8	9	
1	1	1	1	1	1	1	1	1	1	...
2		2	3	4	5	6	7	8	9	...
3			6	9	14	18	23	28	36	
4				18	25					

Several of these numbers can be calculated by hand using elementary methods. The calculation of $R(3,3) = 6$ is well-known, and it was featured as a problem on the Kürschák and Putnam competitions in 1947 and 1953, respectively [**114**]. The exact values of $R(3,4), R(3,5)$, and $R(4,4)$ were established by Greenwood and Gleason in 1955 [**63**]. The following lemma is the key ingredient in proving upper bounds for these small nontrivial Ramsey numbers, and it can also be used to prove Ramsey's theorem.

Lemma 1.1.1. *The Ramsey numbers $R(r,s)$ satisfy $R(r,s) \le R(r,s-1)+R(r-1,s)$. Moreover, if $R(r,s-1)$ and $R(r-1,s)$ are both even, then $R(r,s) \le R(r,s-1)+R(r-1,s)-1$.*

Proof. Let $n = R(r,s-1)+R(r-1,s)$, and consider an arbitrary red-blue edge coloring of K_n. We wish to show that there is either a red K_r or blue K_s. Consider the $n-1$ edges incident to some vertex v. Then either at least $R(r,s-1)$ of them are blue or $R(r-1,s)$ of them are red since if these both do not hold, then $\deg(v) < n-1$. Without loss of generality, suppose at least $R(r,s-1)$ of these edges are blue. Then by the definition of $R(r,s-1)$, there is either a red K_r or a blue K_{s-1} contained in the blue neighbors of v. In the former case we are done, and in the latter case, the graph induced by the vertices of the blue K_{s-1} together with v is a blue K_s. The case where at least $R(r-1,s)$ edges incident to v are red follows by a similar argument.

Now suppose that $R(r,s-1)$ and $R(r-1,s)$ are both even, and consider an arbitrary red-blue edge coloring of K_{n-1}. Let r_v denote the red-degree of a vertex v, and let E_r be the number of red edges. Then we have

$$\sum_{v\in V} r_v = 2E_r.$$

Notice that $|V| = n-1$ is odd, so it follows that there is some v such that r_v is even.

Now suppose for this vertex v that $r_v < R(r-1,s)$. Since r_v and $R(r-1,s)$ are both even, then this inequality can be strengthened to $r_v < R(r-1,s)-1$. Then the number of blue edges b_v incident to v is at least $R(r,s-1)$, because otherwise $b_v \le R(r,s-1)-1$ and we would have

$$n-2 = r_v + b_v < R(r-1,s)-1+R(r,s-1)-1 = n-2,$$

a contradiction. Therefore $r_v \ge R(r-1,s)$ or $b_v \ge R(r,s-1)$, and the result follows from the previous argument. $\square$

Ramsey's theorem is a consequence of Proposition 1.1.1 and Lemma 1.1.1, and we can use it to obtain a crude upper bound on Ramsey numbers.

Proof of Theorem 1.1.1. Using the facts that $R(r,s) = R(s,r)$ and $R(1,s)$ is finite for all s, it follows by induction and Lemma 1.1.1 that $R(r,s)$ is finite for all s.

We claim that $R(r,s) \le \binom{r+s-2}{r-1}$. Observe that this inequality is true for $r = s = 1$ since $R(1,s) = 0$ for all s. Now suppose it is true whenever $r+s \le k$. If $r+s = k+1$, then by Lemma

1.1.1, it follows that

$$R(r,s) \leq R(r-1,s) + R(r,s-1) \leq \binom{r+s-3}{r-2} + \binom{r+s-3}{r-1} = \binom{r+s-2}{r-1}$$

as desired. □

Using Stirling's approximation, the bound shown in the above proof gives the bound

$$R(s,s) \leq (1+o(1))\frac{4^{s-1}}{\sqrt{\pi(s-1)}}. \tag{1.1}$$

We can obtain a lower bound using the probabilistic method.

THEOREM 1.1.2. *For all $s \geq 3$, the Ramsey number $R(s,s)$ satisfies $R(s,s) > 2^{s/2}$.*

PROOF. For $s = 3$, we have $R(3,3) = 6$, and the desired inequality holds. Now suppose $s \geq 4$, and let $n = 2^{s/2}$. Let p denote the probability that a randomly chosen red-blue edge coloring of K_n does not contain a monochromatic K_s. If this probability is positive, then we are done.

Observe that for each K_s in K_n, the probability that it is monochromatic is $2^{1-\binom{s}{2}}$. Since there are $\binom{n}{s}$ copies of K_s, the probability that at least one is monochromatic is at most $\binom{n}{s}2^{1-\binom{s}{2}}$. Therefore since $n = 2^{s/2}$ and $s! > 2^s$ for $s > 4$, it follows that

$$1 - p \leq \binom{n}{s}2^{1-\binom{s}{2}} \leq \frac{n!}{s!(n-s)!}2^{1-\binom{s}{2}} \leq \frac{2^{s^2/2+1-\binom{s}{2}}}{s!} = \frac{2^{s/2+1}}{s!} < 2^{1-s/2} < 1.$$

Therefore p is positive, and we are done. □

Theorem 1.1.2 and (1.1) show (roughly) that the diagonal Ramsey numbers $R(s,s)$ satisfy

$$\sqrt{2}^s \leq R(s,s) \leq 4^s. \tag{1.2}$$

The precise asymptotics of Ramsey numbers are a major open question in Ramsey theory, but there have been no improvements to the bases $\sqrt{2}$ and 4 in (1.2), which have stood since 1947 [**48**, **49**], until a recent preprint improved the upper bound to $(4-c)^s$ for a small constant c [**27**]. The best lower bound $R(s,s) > \frac{\sqrt{2}}{e}s2^{s/2}(1+o(1))$ was shown by Spencer in 1975 by applying the Lovász Local Lemma [**127**].

We turn back to the computation of precise values of Ramsey numbers. Lemma 1.1.1 also enables us to calculate four of the values in Table 1.1.

PROPOSITION 1.1.2. *The Ramsey numbers* $R(3,3), R(3,4), R(3,5)$*, and* $R(4,4)$ *satisfy*

$$R(3,3) = 6,\ \ R(3,4) = 9,\ \ R(3,5) = 14,\ \ R(4,4) = 18.$$

PROOF. First, we will show $R(3,3) = 6$. To show the lower bound $R(3,3) \geq 6$, observe in Figure 1.1 that coloring the edges of a 5-cycle in K_5 red and all other edges blue avoids a red K_3 and a blue K_3. For the upper bound $R(3,3) \leq 6$, suppose towards contradiction that there is a red-blue edge coloring of K_6 that contains no red K_3 and no blue K_3. Consider a vertex v in K_6. By the pigeonhole principle, at least 3 of the edges incident to v must be the same color, say red. Label these three red edges $\{v, w_1\}, \{v, w_2\}, \{v, w_3\}$. Then if the edge $\{w_1, w_2\}$ is red, then $\{v, w_1, w_2\}$ is a red K_3, so $\{w_1, w_2\}$ must be blue. By similar reasoning, it follows that $\{w_1, w_3\}$ and $\{w_2, w_3\}$ must be blue as well. But then the vertices w_1, w_2, and w_3 form a blue K_3, which is a contradiction.

Next we will show that $R(3,4) = 9$. Since $R(2,4) = 4$ and $R(3,3) = 6$ are even, by Lemma 1.1.1, we have $R(3,4) \leq R(2,4) + R(3,3) - 1 = 9$. For the lower bound, the edge coloring of K_8 in Figure 1.1 does not contain a red K_3 or a blue K_4. Here, if the vertices are labeled 1 to 8 in order around the circle, the edge ij (with $i < j$) is colored red if and only if $j - i \equiv 1, 4 \pmod 8$.

Moving to $R(3,5)$, applying Lemma 1.1.1 again we have $R(3,5) \leq R(2,5)+R(3,4) = 5+9 = 14$. For the lower bound, the edge coloring of K_{13} in Figure 1.1 does not contain a red K_3 or a blue K_5. If the vertices are labeled 1 to 13 in order around the circle, the edge ij (with $i < j$) is colored red if and only if $j - i \equiv 1, 5 \pmod{13}$.

For $R(4,4)$, we have $R(4,4) \leq R(3,4) + R(4,3) = 2R(3,4) = 18$. For the lower bound, the edge coloring of K_{17} in Figure 1.1 does not contain a red K_4 or a blue K_4. If the vertices are labeled 1 to 17 in order around the circle, the edge ij (with $i < j$) is colored red if and only if $j - i \equiv 1, 2, 4, 8$ (mod 17).

$\square$

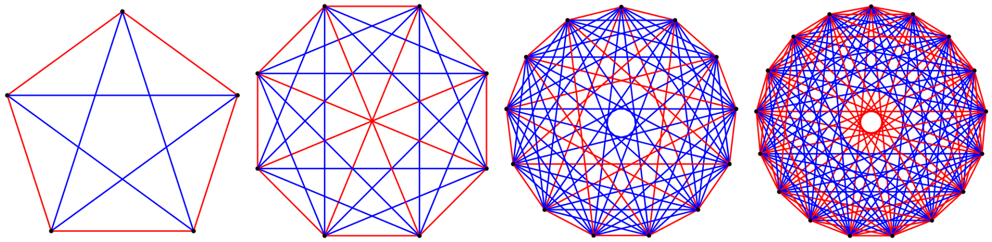


FIGURE 1.1. From left to right, colorings that give tight lower bounds for $R(3,3), R(3,4), R(3,5)$, and $R(4,4)$

We remark that this proof method does not give tight upper bounds for any of the other known nontrivial Ramsey numbers. The value of $R(3,6) = 18$ was first proven in 1964 by Kéry [**83**], though a short, elementary proof in English was given by Cariolaro in 2007 [**28**]. Kalbfleisch proved the lower bound $R(4,5) \geq 25$ in 1965 [**80**], and one year later proved $R(3,7) \geq 23$ and $R(3,9) \geq 36$ in his thesis [**81**]. The next exact value was confirmed in 1968 when Graver and Yackel proved $R(3,7) \leq 23$ [**61**].

The calculations of the remaining known values all required extensive use of computers. Grinstead and Roberts established the lower bound $R(3,8) \geq 28$ and the upper bound $R(3,9) \leq 36$ in 1982 [**64**]. McKay and Zhang established the upper bound $R(3,8) \leq 28$ in 1992 [**104**]. The value $R(4,5) = 25$ was established by McKay and Radziszowski in 1995 [**102**].

No other exact values have been computed in the past twenty-seven years, though various bounds have been improved. The smallest unknown diagonal Ramsey number, $R(5,5)$ is known to satisfy $43 \leq R(5,5) \leq 48$ [**5, 51**]. The other "next" unknown values, $R(3,10)$ and $R(4,6)$ satisfy $40 \leq R(3,10) \leq 42$ and $36 \leq R(4,6) \leq 41$ [**52, 53, 55, 103**].

There are many generalizations of Ramsey numbers that extend the definition to more colors and graphs other than K_r and $\overline{K_s}$. We give one version of generalized Ramsey numbers below.

DEFINITION 1.1.2. *Let $G_1, \ldots, G_k$ be graphs. The* Ramsey number $R(G_1, \ldots, G_k)$ *is the smallest number n such that every edge k-coloring of K_n contains a copy of G_i in color i.*

We remark that the classical Ramsey number $R(r,s)$ is the same as the generalized Ramsey number $R(K_r, K_s)$. We will use the term "Ramsey number" to both the classical Ramsey numbers $R(r,s)$ and the generalized Ramsey numbers in Definition 1.1.2. In the case that $G_i = K_{j_i}$ for all i, we will simply write $R(j_1, \ldots, j_k)$ instead of $R(K_{j_1}, \ldots, K_{j_k})$. These numbers are sometimes referred to as *multicolor Ramsey numbers.* Greenwood and Gleason showed that $R(3,3,3) = 17$ [**63**] in 1955, but it took over sixty years to prove the next known value, $R(3,3,4) = 30$ [**34**]. The numbers $R(G_1, G_2)$ have known formulas for some families of G_1 and G_2, for instance Chvátal showed that $R(K_{1,n}, K_m) = n(m-1)+1$ [**32**]. However, in general computing values of $R(G_1, G_2)$ is still a difficult problem. A "dynamic survey" of the best known bounds for hundreds of Ramsey numbers is maintained at [**114**].

The existence of generalized Ramsey numbers is guaranteed by the following theorem.

THEOREM 1.1.3. *The Ramsey numbers $R(G_1, \ldots, G_k)$ are finite for all $G_1, \ldots, G_k$.*

PROOF. We will prove that $R(r_1, \ldots, r_k)$ is finite for all choices of $r_1, \ldots, r_k$. This is sufficient because every graph G_i is a subgraph of the complete graph $K_{|V(G_i)|}$, so we have $R(G_1, \ldots, G_k) \leq R(|V(G_1)|, \ldots, |V(G_k)|$.

We will proceed by induction on k. For $k = 2$, this is simply Theorem 1.1.1. For $k > 2$, assume the numbers $R(r_1, \ldots, r_m)$ exist for $m < k$ and let $n = R(r_1, \ldots, r_{k-2}, R(r_{k-1}, r_k))$. We claim $R(r_1, \ldots, r_k) \leq n$. Consider a k-coloring χ of the edges of K_n. Let χ' be the $(k-1)$-coloring of K_n that agrees with χ except every edge assigned color k is instead assigned color $k-1$. By the definition of n, there is either a clique of size r_i in color i for some i, $1 \leq i \leq k-2$, or else there is a clique of size $R(r_{k-1}, r_k)$ in color $k-1$. In the former case, we are done, so suppose there is a clique of size $R(r_{k-1}, r_k)$ in color $k-1$. Now de-identify colors $k-1$ and k, so that the edges in this clique are colored in these two colors. By the definition of $R(r_{k-1}, r_k)$, there is either a clique of size r_{k-1} in color $k-1$ or a clique of size r_k in color k contained within this copy of $K_{R(r_{k-1}, r_k)}$, which is contained in K_n. Therefore every coloring χ produces a clique of size r_i in color i for some i, and this completes the proof. $\square$

1.2. Arithmetic Ramsey Theory

A large part of this thesis deals not with graphs, but with numbers in *arithmetic* Ramsey theory. Often we are interested in solutions to an equation over the set $\{1, 2, \ldots, n\}$, which we henceforth denote by $[n]$. One of the most important results in this area is *Schur's theorem*, and the proof follows quickly from the existence of the numbers $R(r_1, \ldots, r_k)$.

THEOREM 1.2.1 (Schur, 1916). *For all integers k, there exists a number n such that every k-coloring of $[n]$ contains a monochromatic solution to the equation $x + y = z$.*

PROOF. Let $n = R(3, \ldots, 3)$, where there are k threes in the argument of R. Consider an arbitary k-coloring χ of $[n-1]$. Assign the vertices of K_n the labels 1 through n. Let ψ be the k-coloring of K_n where the edge ij is colored $\chi(|i-j|)$.

By the definition of n, there exists a monochromatic triangle abc with $a < b < c$. Let $x = b - a$, $y = c - b$, and $z = c - a$. Then $x + y = z$ and $\chi(x) = \chi(y) = \chi(z)$, so we are done. □

Schur's motivation for proving Theorem 1.2.1 was to attack Fermat's Last Theorem, and in doing so he was able to prove that the equation $x^n + y^n \equiv z^n \pmod{p}$ has nontrivial solutions for sufficiently large primes p.

Like Ramsey numbers, we are interested in the *smallest* number n where the pattern, in this case a monochromatic solution to $x + y = z$, appears.

DEFINITION 1.2.1. *The* Schur number $S(k)$ *is the smallest number n such that every k-coloring of $[n]$ contains a monochromatic solution to the equation $x + y = z$.*

As a simple example, consider the Schur number $S(2)$. It is straightforward to check that the coloring of $\{1, 2, 3, 4\}$ where 1 and 4 are colored red and 2 and 3 are colored blue does not contain a monochromatic solution to $x+y = z$. But every coloring of $\{1, 2, 3, 4, 5\}$ does contain a solution. Without loss of generality, suppose 1 is red. Then 2 is blue since $1 + 1 = 2$. Moreover, 4 is red since $2 + 2 = 4$. Then 3 is blue since $1 + 2 = 3$, and then 5 cannot be either color since $1 + 4 = 5$ and $2 + 3 = 5$. Hence $S(2) = 4$.

The calculation of the third Schur number $S(3) = 14$ is a more difficult exercise, but still doable by hand. Computing $S(k)$ for $k \geq 4$ is still yet more difficult, and almost surely requires machine assistance. The number $S(4) = 45$ was first computed in 1966 by Golomb and Baumert using a backtracking algorithm [**56**]. The largest known Schur number is $S(5) = 161$, which was computed by Heule in 2017-2018 [**71**]. This calculation required a staggering amount of computational power using *SAT solvers* and parallelized computation. The key technique used is called the *cube and conquer* method, which partitions the problem into many tractable subproblems and solves them in parallel. The computation required 14 years of CPU time, and another 36 years of CPU time for the verification of the calculation. However, the cube and conquer procedure allowed this to be done in mere days of real time on a supercomputer. The verification of this result is especially notable because it produced a proof over 2 petabytes in size, which is arguably the longest proof in history. We will discuss SAT solvers in detail in Section 1.3.

Another celebrated theorem in arithmetic Ramsey theory is *van der Waerden's theorem*, which concerns *arithmetic progressions*, sequences of the form $x, x+d, \ldots, x+(\ell-1)d$. Van der Waerden attributed the statement of the theorem to Baudet, but in fact it was originally conjectured by Schur while he studied the distribution of quadratic residues over $\mathbb{Z}_p$ [**60**].

THEOREM 1.2.2 (Van der Waerden, 1927 [**130**]). *For all positive integers k and ℓ, there exists a smallest number $n = w(\ell; k)$ such that every k-coloring of $[n]$ contains a monochromatic ℓ-term arithmetic progression, i.e. a sequence of the form $x, x+d, x+2d, \ldots, x+(\ell-1)d$ for some positive integer d.*

The numbers $w(\ell; k)$ are called *van der Waerden numbers.* There exist elementary proofs of van der Waerden's theorem using a double induction (see [**60**] or [**91**]), but we omit them here.

As with Ramsey numbers, finding precise bounds on van der Waerden numbers is difficult, and perhaps even more so. The best known lower bounds are due to a finite field construction due to Berlekamp [**13**], who showed $w(p+1; 2) \geq p2^p$. The upper bounds given for $w(\ell; k)$ in the original proof of van der Waerden's theorem grow like the notorious Ackermann function. Shelah later made a striking improvement in [**124**], giving upper bounds that were primitive recursive, but are nonetheless described as "wowzer" functions in [**60**] due to their fast growth. The best known bounds today are due to Gowers [**59**], who showed $w(\ell; k) \leq 2^{2^{k^{2^{2^{\ell+9}}}}}$. While a massive breakthrough, this bound is still huge compared to the values of known van der Waerden numbers (see Table 1.2 below). It is a major open question what the "true" bounds for van der Waerden numbers are; Graham asked whether $w(\ell; 2) \leq 2^{\ell^2}$ and offered a \$1000 prize for an answer [**91**].

Schur's theorem has a number of generalizations, the most notable of which is *Rado's theorem*, proven by Schur's Ph.D. student Richard Rado. The full version of theorem extends Schur's theorem to arbitrary systems of equations with integer coefficients. To state the theorem, we need to use the following definition.

TABLE 1.2. Table of van der Waerden numbers $w(\ell; k)$

$\ell \backslash k$	2	3	4
3	9	27	76
4	35	293	
5	178		
6	1132		

DEFINITION 1.2.2. *Let A be an $\ell \times m$ matrix, and denote the columns of A by $a_1, \ldots, a_m$. We say that A satisfies the* columns condition *if there exists a partition $\{C_1, \ldots, C_k\}$ of $\{1, \ldots, m\}$ such that the following conditions hold. Define $s_i = \sum_{j \in C_i} a_j$ to be the vector sum of the columns whose indices are in C_i.*

(i) $s_1 = 0$

(ii) s_k *can be written as a (rational) linear combination of the elements of* $\bigcup_{1 \le j < k} \{a_i\}_{i \in C_j}$ *for all* $k \ge 2$.

THEOREM 1.2.3 (Rado, 1933 [**113**]). *Let $Ax = 0$ be a system of linear equations with integer coefficients. Then for every positive integer k, there exists an n such that every k-coloring of $[n]$ contains a monochromatic solution to $Ax = 0$ if and only if A satisfies the columns condition.*

We will be most interested in the case where there is only a single equation, in which case Theorem 1.2.3 becomes the following.

THEOREM 1.2.4. *Let $\mathcal{E}$ be the equation $\sum_{i=1}^{m} a_i x_i = 0$ in the variables x_i with nonzero integer coefficients a_i. Then for every positive integer k, there exists an n such that every k-coloring of $[n]$ contains a monochromatic solution to $\mathcal{E}$ if and only if there exists a nonempty subset of the a_i that sums to 0.*

We will prove only Theorem 1.2.4, but the proof of Theorem 1.2.3 is similar. We present this proof because for one direction the coloring construction involved is similar to those in Chapter 3, and the other direction serves as a nice application of van der Waerden's theorem. Before the proof, we need the following two corollaries which extend van der Waerden's theorem.

COROLLARY 1.2.1. *Let $a, b, k, \ell \in \mathbb{Z}^+$ with $a \le b$. Then every k-coloring of $[bw(\ell; k)]$ contains a monochromatic ℓ-term arithmetic progression whose gap is a multiple of a.*

Proof. Let χ be an arbitrary k-coloring of $[bw(\ell;k)]$. Define a coloring χ' of $[w(\ell;k)]$ by $\chi'(x) = \chi(ax)$. By van der Waerden's theorem, there is an ℓ-term arithmetic progression $y, y+d, \ldots, y+(\ell-1)d$ with $\chi'(y) = \chi'(y+d) = \cdots = \chi'(y+(\ell-1)d)$. But then $\chi(ay) = \chi(ay+ad) = \cdots = \chi(ay+a(\ell-1)d)$, so $ay, ay+ad, ay+2ad, \ldots, ay+a(\ell-1)d$ is a monochromatic arithmetic progression with gap ad as desired. $\square$

Corollary 1.2.2. *Let $n = w(\ell;k)$, and suppose the set $S := a[1,n] = \{a, 2a, \ldots, na\}$ is k-colored. Then S contains a monochromatic ℓ-term arithmetic progression.*

Proof. Let χ be an arbitrary k-coloring of S. Then define the k-coloring χ' of $[n]$ by $\chi'(x) = \chi(ax)$. By van der Waerden's theorem, there is an arithmetic progression $x, x+d, \ldots, x+(\ell-1)d$ with $\chi'(x) = \cdots = \chi'(x+(\ell-1)d)$. Then $\chi(ax) = \chi(ax+ad) = \cdots = \chi(ax+ad(\ell-1))$, so $ax, ax+ad, \ldots, ax+ad(\ell-1)$ is a monochromatic ℓ-term arithmetic progression contained in S. $\square$

We are now able to prove Theorem 1.2.4.

Proof of Theorem 1.2.4. We follow the proof given in [**91**]. Fix an equation $\mathcal{E}$: $\sum_{i=1}^m a_i x_i = 0$ with $a_i \neq 0$ for all i. Suppose first that there is no subset of the a_i that sums to zero. Let p be a prime that does not divide any sum of a nonempty subset of the a_i. We will construct a $(p-1)$-coloring χ of the positive integers that contains no monochromatic solutions to $\mathcal{E}$.

Each positive integer x has a unique p-ary expansion $x = \sum_{j=0}^t c_j p^j$ with $0 \leq c_j \leq p-1$. Let j^* be the smallest j such that $c_j \neq 0$, and let $\chi(x) = c_{j^*}$.

Suppose for the sake of contradiction that $x_1, \ldots, x_m$ is a monochromatic solution to $\mathcal{E}$ with $\chi(x_1) = \cdots = \chi(x_m) = \alpha$. Let e be the smallest positive integer such that there is some index i with $p^e \nmid x_i$. Now let $I = \{i : x_i \equiv \alpha \pmod{p^e}\}$ and let $I' = [m] \setminus I = \{i : x_i \equiv 0 \pmod{p^e}\}$, and note that I is nonempty. Now consider the equation $\mathcal{E}$ modulo p: we are left with

$$0 \equiv \sum_{i \in I} c_i x_i \equiv \alpha \sum_{i \in I} x_i \pmod{p^e}.$$

Since $1 \leq \alpha \leq p-1$, we have $p^e \mid \sum_{i \in I} x_i$, and so $p \mid \sum_{i \in I} x_i$. Recall that I is nonempty, so this is a contradiction.

For the other direction, we proceed by induction on the number of colors k. Let S be the largest subset of the a_i that sums to 0. Without loss of generality, suppose $S = \{a_1, \dots, a_r\}$ and $a_1 > 0$. If $r = m$, then the solution $x_1 = \cdots = x_r = 1$ is monochromatic, so suppose $r < m$. Let $s = \sum_{j=r+1}^{m} a_j$, and observe that $s \neq 0$. Set x_1 to be a positive integer large enough so that $x_1 + s > 0$, and set $x_2 = \cdots = x_r = x_1 + s$, and set $x_{r+1} = \cdots = x_m = a_1$. Then we have $\sum_{i=1}^{m} a_i x_i = a_1 x_1 + \sum_{i=2}^{r} a_i(x_1 + s) + \sum_{i=r+1}^{m} a_i a_1 = a_1 x_1 - a_1(x_1 + s) + a_1 s = 0$, so $(x_1, \dots, x_m)$ is a monochromatic solution.

Now suppose the theorem holds for k colors, and we will show that it holds for $k+1$ colors. Let $R_k(\mathcal{E})$ denote the smallest number n such that every k-coloring of $[n]$ contains a monochromatic solution to $\mathcal{E}$. We once again suppose that $\sum_{i=1}^{r} a_i = 0$ with $r < m$ maximal and let $s = \sum_{i=r+1}^{m} a_i \neq 0$. Moreover, let $A := \sum_{i=1}^{m} |a_i|$.

We will show that $R_{k+1}(\mathcal{E}) \leq Aw(R_k(\mathcal{E}) + 1; k) =: n$. Let χ be an arbitrary $(k+1)$-coloring of $[n]$. As with the $k = 1$ case, we will set $x_2 = x_3 = \cdots = x_r$ and $x_{r+1} = \cdots = x_m$ so that $\mathcal{E}$ becomes the three-variable equation $a_1(x_1 - x_2) + s x_m = 0$.

Since $1 \leq |s| < A$, by Corollary 1.2.1, there is a monochromatic $(R_k(\mathcal{E}) + 1)$- term arithmetic progression whose gap is a multiple of $|s|$. Let $y, y + d|s|, \dots, y + R_k(\mathcal{E})d|s|$ be such an arithmetic progression.

There are now two cases to consider. First, assume there exists some element $z \in [R_k(\mathcal{E})]$ such that $\chi(zda_1) = \chi(y)$. If $s > 0$, then setting $x_1 = y$, $x_2 = y + zda_1$, $x_m = zda_1$ is a monochromatic solution to $\mathcal{E}$, and if $s < 0$, then $x_1 = y + zda_1$, $x_2 = y$, $x_m = zda_1$ is a monochromatic solution. Otherwise, we have $\chi(zda_1) \neq \chi(y)$ for $1 \leq d \leq R_k(\mathcal{E})$, so the set $\{da_1, 2da_1, \dots, R_k(\mathcal{E})da_1\}$ has only k colors, so by the induction hypothesis and Corollary 1.2.2, the proof is complete. $\square$

We will refer to both Theorem 1.2.3 and Theorem 1.2.4 as *Rado's theorem.* We also introduce the notion of *regularity*, which will simplify the statements of many of our results.

Definition 1.2.3. *An equation $\mathcal{E}$ is* k-regular *if there exists an n such that every k-coloring of $[n]$ contains a monochromatic solution to $\mathcal{E}$. If $\mathcal{E}$ is k-regular for all $k \geq 1$, then we say that $\mathcal{E}$ is regular.*

For example, Schur's theorem states that the equation $x + y = z$ is regular. Rado's theorem states that the equation $2x_1 + 3x_2 - 5x_3 - 7x_4 = 0$ is regular since $2 + 3 - 5 = 0$, but the equation $2x_1 + 2x_2 - x_3 = 0$ is not since no subset of the coefficients sums to 0.

Some of the central numbers in this thesis are the Ramsey-type numbers associated to Rado's theorem, which are appropriately called *Rado numbers.* We study the Rado numbers in great detail in Chapter 3.

Definition 1.2.4. *Given an equation $\mathcal{E}$ and integer $k \geq 1$, the* Rado number *$R_k(\mathcal{E})$ is the smallest number n such that every k-coloring of $[n]$ contains a monochromatic solution to $\mathcal{E}$. If no such number exists, then we say $R_k(\mathcal{E}) = \infty$.*

Note that the Schur numbers $S(k)$ are simply the Rado numbers $R_k(x+y = z)$. An interesting–and difficult–question is when Rado numbers for a given equation are finite (exist) and when they are not. For example, $R_2(2x + 2y = z) = 34$, but $R_3(2x + 2y = z) = \infty$. There is a known characterization of 2-regular linear equations, which was also given by Rado.

Theorem 1.2.5. *For $m \geq 3$, the equation $\sum_{i=1}^{m} a_i x_i = 0$ is 2-regular if and only if there exist coefficients a_i and a_j with opposite sign.*

However, there is no known general characterization of k-regular equations for arbitrary k. Another interesting statistic is the *degree of regularity* of an equation.

Definition 1.2.5. *The* degree of regularity *of an equation $\mathcal{E}$, denoted $dor(\mathcal{E})$, is the largest integer k for which $\mathcal{E}$ is k-regular. If $\mathcal{E}$ is regular, then we say $dor(\mathcal{E}) = \infty$.*

From the example above, we see that $dor(2x + 2y = z) = 2$. Rado made several observations on the degree of regularity in his thesis, and proved the following results for three variable linear homogeneous equations (see also [**18**]).

Theorem 1.2.6 (Rado). *The following results on degree of regularity hold.*

(i) If $\mathcal{E}$ is the equation $a(x + y) = z$ with $a \in \mathbb{Q}$ and $a \neq 2^k$ for all $k \in \mathbb{Z}$, then $dor(\mathcal{E}) \leq 3$.

(ii) For all $k \in \mathbb{Z}$, $dor(2^k(x + y) = z) = \infty$ or $dor(2^k(x + y) = z) \leq 5$.

(iii) Let $a, b, c, \alpha \in \mathbb{Z}$. If $\alpha \neq 0$ and p is a prime such that $p \nmid abc(a+b)$, then $dor(ax + by + p^{\alpha}cz) = \infty$ or $dor(ax + by + p^{\alpha}cz) \leq 5$

(iv) Let $a, b, c \in \mathbb{Z}$. If $\alpha, \beta, \gamma \in \mathbb{Z}$ are pairwise distinct and $p \nmid abc$, then $dor(p^{\alpha}ax + p^{\beta}by + p^{\gamma}cz) \leq 7$.

In general it is difficult to determine the degree of regularity of a given equation. The question of whether there is an equation of degree of regularity exactly k was raised by Rado and answered in 2010 by Alexeev and Tsimerman [**2**], who showed that the equation $\mathcal{L}_k$ given by

$$\sum_{i=1}^{k} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{k} \frac{2^i}{2^i - 1} \right) x_0$$

has degree of regularity precisely k. Notice, however, that the equation $\mathcal{L}_k$ has $k+1$ variables. If the number of variables is fixed, then the answer is not so clear. That is, given a fixed m, for every $k \geq 1$, can we find a linear homogeneous equation $\mathcal{E}$ in m variables with $dor(\mathcal{E}) = k$? Rado himself conjectured that the answer is no, and this is known as *Rado's boundedness conjecture.*

Conjecture 1.2.1 (Rado's boundedness conjecture). *Given a positive integer m, there is a universal constant $\Delta = \Delta(m)$ such that every linear homogeneous equation $\mathcal{E}$ in m variables is regular or satisfies $dor(\mathcal{E}) < \Delta(m)$.*

In other words, Rado's boundedness conjecture says that nonregular equations in a fixed number of variables have a bound on their degree of regularity. The only nontrivial case where this is known is for $m = 3$; Fox and Kleitman proved that $\Delta = 24$ is sufficient [**54**].

We end our discussion of Rado numbers and regularity here, but we will return to and build upon these results in Chapter 3. The next two sections give background on the methods used in this dissertation.

1.3. Boolean Satisfiability

Many of the problems in this thesis can be encoded in terms of the satisfiability of Boolean formulas. The *Boolean satisfiability problem* (SAT) is of fundamental importance in computer science and complexity theory (see [**126**] for an introduction). SAT was shown by Cook and Levin to be NP-complete, and there is no known polynomial-time algorithm to decide whether a given formula is satisfiable. However, recent computational advances have produced powerful tools known as *SAT solvers* that can determine the satisfiability of large Boolean formulas and are often efficient in practice.

We recall the following standard terminology, all of which can be found in [**21**]. A *literal* is a Boolean variable x or its negation, which we denote $\bar{x}$. A *clause* is a logical disjunction of literals, e.g. $x \vee \bar{y} \vee z$. A formula ϕ is in *conjuctive normal form* (CNF) if it is a logical conjunction of clauses, e.g. $\phi = (x \vee \bar{y}) \wedge (\bar{x} \vee z) \wedge (x \vee y \vee z)$. We will sometimes write clauses as sets of literals and formulas as sets of clauses, e.g. $\phi = \{\{x, \bar{y}\}, \{\bar{x}, z\}, \{x, y, z\}\}$. All formulas we consider will be written in conjunctive normal form, and this is the standard input for most SAT solvers. A formula is *satisfiable* if there is a truth assignment to the variables such that at least one literal in each clause is true, and *unsatisfiable* otherwise.

1.3.1. Algorithms for satisfiability. The core algorithm behind many modern SAT solvers is *Conflict-Driven Clause Learning* (CDCL), though they are equipped with numerous preprocessing methods, heuristics, and other procedures to solve instances efficiently. We will discuss only a barebones version of the CDCL algorithm, but a thorough treatment can be found in [**21**]. A key procedure for simplifying clauses in the formula is *unit propagation*. Consider the formula $\phi_1 = \{\{x_1, \bar{x}_2, x_3\}, \{x_2\}, \{\bar{x_2}, x_4\}, \{\bar{x}_1, x_4\}.\{x_1, x_2, \bar{x}_4\}\}$. Notice that if ϕ_1 is satisfiable, then x_2 must be assigned true. Then we can remove the literal $\bar{x}_2$ from each clause where it appears and delete each clause where x_2 appears, so ϕ_1 is equivalent to the formula $\phi_2 = \{\{x_1, x_3\}, \{x_4\}, \{\bar{x}_1, x_4\}\}$. Now we see that x_4 must be assigned true and obtain $\phi_3 = \{\{x_1, x_3\}, \{\bar{x}_1\}\}$. Applying unit propagation two more times, we set x_1 to false and x_3 to true, and obtain the satisfying assignment $x_1 =$ false, $x_2 = x_3 = x_4 =$ true.

CDCL begins by selecting a variable, assigning it true or false, and then applying unit propagation to build an *implication graph*. The vertices of the implication graph are literals, and there is a directed edge (x, y) if x being assigned true is the "reason" y is assigned true; more precisely, an edge (x, y) is constructed when there is a clause C containing $\bar{x}$ and y, and all other literals in C besides y are false. In the example above, the implication graph is $x_2 \to x_4 \to \bar{x}_1 \to x_3$. A *conflict* in the implication graph occurs when a literal and its negation are both vertices (assigned to true). When a conflict occurs, we find a *cut* in the implication graph that led to the conflict

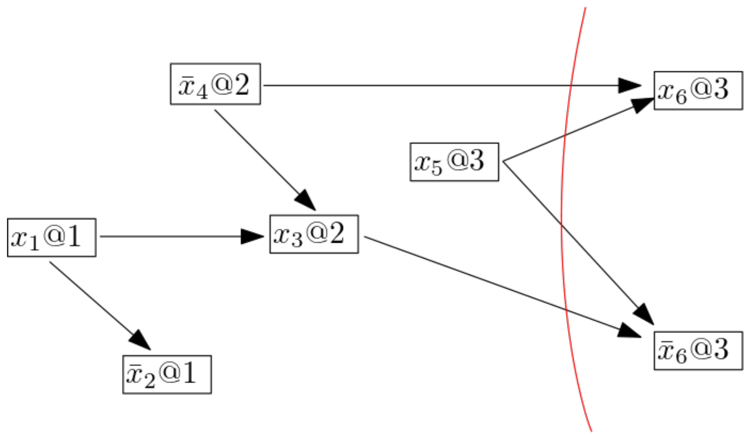


FIGURE 1.2. Illustration of the implication graph for Example 1.3.1. Each node is of the form $\ell@d$, which corresponds to the literal ℓ being assigned true at decision level d. The cut is displayed in red.

and learn a new clause that contains the negations of all the literals on one side of the cut edges. The solver then backtracks to an appropriate decision level and begins the search anew. We give an illustration of CDCL and the implication graph with a simple example.

EXAMPLE 1.3.1. Consider the formula

$$\phi = (\bar{x}_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (x_4 \vee \bar{x}_5 \vee x_6) \wedge (\bar{x}_3 \vee \bar{x}_5 \vee \bar{x}_6) \wedge (x_5 \vee x_7 \vee x_8) \wedge (\bar{x}_4 \vee \bar{x}_7 \vee x_9) \wedge (x_1 \vee \bar{x}_8 \vee x_9).$$

Suppose x_1 is arbitrarily assigned to true. Then x_2 is assigned false by unit propagation. At the next decision level, x_4 is assigned false, and then x_3 is assigned true. Continuing, we assign x_5 true, then a conflict arises: x_6 must be assigned both true and false. Since the assigning $x_3, \bar{x}_4$, and x_5 to true led to this conflict, we *learn* the clause $(\bar{x}_3 \vee x_4 \vee \bar{x}_5)$.

1.3.2. SAT solvers and combinatorics. In recent years, SAT solvers have proven to be extremely useful in computing precise bounds for Ramsey-type numbers. Among the first such results are the calculations of the van der Waerden numbers $w(6;2)$ and $w(4;3)$ [**86, 87**]. Another significant result is the solution to the *Pythagorean triples problem*, which asks for the Rado number $R_2(x^2 + y^2 = z^2)$; a monetary prize for this result was also offered by Graham. Heule, Kullmann, and Marek proved that $R_2(x^2 + y^2 = z^2) = 7825$ in [**74**]. This was notable on a theoretical level because it was not known at the time whether $x^2 + y^2 = z^2$ was 2-regular, and it is still not known whether $R_k(x^2 + y^2 = z^2) < \infty$ when $k = 3$. On a computational level this was also remarkable because of the sheer size of the proof, which, like the proof of $S(5) = 161$, required terabytes of memory to produce. These computational successes are founded in part on the *cube and conquer* paradigm. Cube and conquer pairs a CDCL solver with a complementary *look-ahead solver* to partition a hard CNF formula ϕ into *cubes*. A cube is simply a partial assignment to the variables in ϕ. The look-ahead solver selects variables according to a heuristic and splits the formula into (potenitally millions of) subproblems that consist of ϕ together with unit clauses that make up a cube. The heuristics involved in this partitioning attempt to make each subproblem much easier to solve, and then subproblems can be passed to the CDCL solver independently and in parallel. Empirical data suggests that cube and conquer often performs better than running a single CDCL or look-ahead solver. We will not discuss the finer details of cube and conquer or the heuristics involved here, and we refer the reader to [**71**, **73**, **74**].

1.3.3. Proofs of unsatisfiability. Computer-generated results are often met with skepticism. Perhaps the most famous proof by computer is the proof of the Four Color Theorem by Appel and Haken [**6**], part of which included checking nearly two-thousand cases with computer assistance. This raises a fundamental question: does “the computer says this result is true” really constitute a proof? Computers are prone to bugs, crashes, and other inexplicable errors, so one might argue that computer results are more like experiments than rigorous proofs. SAT solver proofs are no different; the solvers themselves are complicated programs that involve thousands of lines of code. How can we say that they work correctly one-hundred percent of the time, especially on large, hard problems with long solve times? This is a valid concern: subtle errors have in fact been found in otherwise well-trusted, state of the art solvers [**79**]. Some of this is a matter of philosophy: Are humans really more reliable? Contradictory results have been published in top mathematics journals (see [**85**, **123**]). It is not unusual to find small errors or typos in peer-reviewed literature. For these reasons there is a growing movement to formalize and catalog results in Lean in libraries such as `mathlib` (see `https://github.com/leanprover-community`). Another common complaint about computer-generated proofs is that they are inelegant, or not understandable by humans. While this is often the case (the 2000-terabyte proof of $S(5) = 161$ is hardly bedtime reading, for instance), again, we ask: Are human proofs really better? One of the famous long human proofs is the classification of finite simple groups, a result that spans thousands of pages and many authors. Is the proof fully understood? This is not meant as an insult to the contributors to this result, but rather to say that there are few, if any, mathematicians who could recreate the proof alone.

So why, then, should one trust a SAT solver proof? If a formula ϕ is satisfiable, then the solver outputs a truth assignment to the variables (a certificate) which can be verified to satisfy each clause in ϕ efficiently. But when ϕ is unsatisfiable, the answer is less clear. For unsatisfiable instances, solvers can output a proof which can then be checked independently by a simpler, more trustworthy algorithm. A commonly used proof format is *DRAT (deletion resolution asymmetric tautology).* A DRAT proof consists of a sequence of clauses C_i that preserve satisfiability of the original formula ϕ, i.e. $\phi \cup \bigcup_{i=1}^{k} \{C_i\}$ is equisatisfiable to ϕ for all k. At any time some of these clauses can be deleted from the proof; this is done in practice to save space when the proof is verified. A proof of unsatisfiability culminates in the empty clause, which is by definition unsatisfiable. The clauses C_i are called *RAT clauses*, which are defined as follows.

Definition 1.3.1. *A clause C has the property AT (asymmetric tautology) with respect to a formula ϕ if $\phi \cup \bar{C} := \phi \cup \bigcup_{\ell \in C} \{\{\bar{\ell}\}\}$ results in a conflict. In other words, assigning all the literals in C to false and applying unit propagation results in a conflict.*

A clause C has the property RAT (resolution asymmetric tautology) with respect to ϕ if there exists a literal $\ell \in C$ such that for all clauses $D \in \phi$ with $\bar{\ell} \in D$, the clause $C \cup (D \setminus \{\bar{\ell}\})$ has property AT with respect to ϕ.

The RAT property is useful for two reasons: it can be checked in polynomial time, and the processing and solving steps done by SAT solvers can be translated into RAT clauses. We will not discuss the implementation of the verification algorithm or the generation of DRAT proofs by solvers, but more details can be found in [**72**], for example.

1.4. Polynomial Systems of Equations and Solvability

Another way we encode combinatorial problems is through *systems of polynomial equations.* There is a rich body of literature surrounding the relationships between combinatorics and polynomials. We could not hope to detail the full extent of this work in this thesis. Here we give some highlights that are relevant to our work, and we refer the reader to the forthcoming survey [**39**].

Among the first instances of this paradigm is the graph coloring ideal given by Bayer [**10**]. Recall that a graph $G = (V, E)$ is k-colorable if there is a function $\chi : V \to \{1, \ldots, k\}$ such that $\chi(u) \neq \chi(v)$ for all $\{u, v\} \in E$.

ENCODING A. *Given a graph $G = (V, E)$, the following system of equations has a solution over $\mathbb{C}$ if and only if G is k-colorable.*

$$x_v^k - 1 = 0 \quad \forall v \in V,$$
$$\sum_{i=0}^{k-1} x_u^i x_v^{k-1-i} = 0 \quad \forall \{u, v\} \in E.$$

Each vertex has a corresponding variable x_v, and the first set of equations forces each x_v to be assigned a k-th root of unity, which in turn correspond to the k colors. The second set of equations forces adjacent vertices to take on different roots of unity (colors); this can be seen by rewriting the equations as $(x_u^k - x_v^k)/(x_u - x_v) = 0$. Moreover, there is a one-to-one correspondence between proper k-colorings of G and solutions to this system of equations.

When a system of equations over an algebraically closed field has *no* solution, Hilbert's Nullstellensatz says that there exists a polynomial identity that certifies this fact.

THEOREM 1.4.1 (Hilbert, 1893 [**75**]). *Let $f_1 = \cdots = f_m = 0$ be a system of equations over an algebraically closed field K. Then this system has no solution if and only if there exist polynomials α_i such that*

$$\sum_{i=1}^{m} \alpha_i f_i = 1.$$

The identity $\sum_{i=1}^{m} \alpha_i f_i = 1$ is called a *Nullstellensatz certificate*. The *degree* of a certificate is the maximal degree of the polynomials α_i, though we note that some authors define the degree to be the maximal degree among all $\alpha_i f_i$. The certificate degree is a rough measure of how difficult it is to prove that a system has no solution; systems with higher minimum Nullstellensatz degree are in some sense "harder" to refute. There is a more precise link between computing Nullstellensatz certificates and complexity theory. We say that an problem of size n has an $O(g(n))$-*encoding* as a system of polynomial equations if the number of variables, number of equations, and the bit-sizes of the number of monomials, all coefficients, and all exponents in the equations are all at most $g(n)$.

For example, Encoding A is an $O(n^2)$-encoding for graphs of order n. Given an NP-complete problem L with an $O(g(n))$-encoding for some polynomial $g(n)$, under the assumption $P \neq NP$, for every $n \geq 0$ there must be instances I of L, where the minimum degree of a Nullstellensatz certificate is at least n (for a detailed proof, see [**99**]).

The fundamental idea behind this fact is that computing Nullstellensatz certificates is equivalent to solving a *linear* system of equations. For a fixed degree d, one can write out the certificate polynomials α_i as arbitrary polynomials of degree d, expand the identity $\sum_{i=1}^{m} \alpha_i f_i = 1$, and equate the coefficients of the monomials on both sides. The resulting system is a linear system in the coefficients of the α_i. If this new linear system has no solution, then one can increment d and try again with a larger system. This is the *Nullstellensatz Linear Algebra algorithm* (NulLA) developed in [**42**].

There are bounds on the degrees of Nullstellensatz certificates for general systems of linear equations, so NulLA is indeed an algorithm. However, these bounds are exponential and sharp in the most general case. Kollár [**84**] showed for all algebraically closed fields K, there exists a a polynomial system $f_1 = \cdots = f_m = 0$ of degree $d > 2$ in n variables whose minimal degree Nullstellensatz certificate has degree d^m. One such system is

$$x_1^d = x_1 x_n^{d-1} - x_2^d = \cdots = x_{n-2} x_n^{d-1} - x_{n-1}^d = x_{n-1} x_n^{d-1} - 1 = 0.$$

Fortunately, there is a significant improvement on this bound when the systems of equations have additional structure. Brownawell [**23**] observed that result of Lazard [**93**] gives a *linear* bound on the Nullstellensatz certificate degree in these cases.

LEMMA 1.4.1. *Let K be an algebraically closed field, and let $f_1 = \cdots = f_m = 0$ be an infeasible system of equations of degree at most d. If the f_i have no common zeros at infinity, then the system has a Nullstellensatz certificate of degree at most $n(d-1)$.*

While Lemma 1.4.1 gives a stark improvement over the exponential degree bounds, even a linear degree bound is far from practical. A non 3-colorable graph on a modest 20 vertices whose minimal certificate degree is the maximum possible in Lemma 1.4.1 would require solving for the coefficients of $\binom{60}{20} > 10^{15}$ monomials for each equation in Encoding A. The practical success of NulLA is in part due to many optimizations, for instance modifying Encoding A to be an encoding over $\bar{\mathbb{F}}_2$ rather than $\mathbb{C}$, and adding redundant "degree cutting" equations that do not affect the feasibility of the system but lower the certificate degree. But more remarkably, the extensive computational experiments done in [**42**] and [**99**] did not find a certificate degree of degree more than four, even on difficult benchmarks.

Since the hypothesis $P \neq NP$ is widely regarded as true, we should expect to find a family of instances whose minimal degree Nullstellensatz certificates *grow*. However, it seemed difficult to find a graph with high minimal Nullstellensatz degree "in the wild," and it was not obvious how to construct one. A few years later, Lauria and Nordström [**92**] constructed a family with linear degree growth using a reduction from the so-called functional pigeonhole principle, which had previously known lower bounds in the *polynomial calculus* proof system. While this quashed any hopes of NulLA as a polynomial-time algorithm for 3-coloring, the point remains that in practice, Nullstellensatz certificate computations can be effective.

A related problem is to find combinatorial interpretations for Nullstellensatz certificates. For Encoding A (using $\bar{\mathbb{F}}_2$ instead of $\mathbb{C}$), De Loera, Hillar, Malkin, and Omar characterized graphs that have a degree one minimal Nullstellensatz certificate in terms of oriented cycles [**41**]. Moreover, this class of graphs can be recognized in polynomial time. Another combinatorial characterization of these graphs that does not impose any directed graph structures is given in [**94**]. No such characterization is known for graphs whose minimal degree Nullstellensatz certificate is equal to 4 or higher (note that degrees 2 and 3 are not possible; see [**99**]). In [**100**], Margulies, Onn, and Pasechnik study the partition problem, which takes as input a set of positive integers and asks whether it can be partitioned into two disjoint subsets whose sums are equal. The following encoding describes this problem using polynomial equations over $\mathbb{C}$.

Encoding B. *Given a set $S = \{s_1, \ldots, s_n\}$ of integers, there is no partition of S into two equal-sum subsets if and only if the following system of equations has no solution over $\mathbb{C}$.*

$$x_i^2 - 1 = 0, \forall i \in S, \qquad \sum_{i \in S} s_i x_i = 0.$$

They characterize the form of minimal degree certificates using Encoding B in terms of carefully chosen subsets of $[n]$, and moreover give combinatorial interpretations for the monomials involved in the certificates as well as their coefficients.

1.5. Contributions

Broadly stated, the contributions of this dissertation deal with both the theoretical and practical complexity of computing Ramsey-type numbers. On the theoretical side, we study encodings of "Ramsey-type" problems using polynomial equations and ideals. For each problem, we give a sequence of polynomial systems that become feasible precisely when the Ramsey-type number is reached. The solutions to these system correspond to "Ramsey colorings" of the respective objects (integers, graph edges, points, etc.) for the problem, meaning those colorings that successfully avoid the given monochromatic substructure.

Using these encodings, we give a general framework to construct Nullstellensatz certificates for these infeasible systems. These certificates have combinatorial meaning in terms of *Builder-Painter games*. The Builder-Painter game was first described in [**36**] for the classical Ramsey numbers. We fix graphs $G_1, \dots, G_k$ and an integer n. The game is played on the graph K_n, and each turn Builder selects an edge, and Painter assigns it a color in $[k]$. Builder's objective is to construct a monochromatic copy of G_i in color i for some i; Painter's goal is to delay this as long as possible. Notice that if $n \geq R(G_1, \dots, G_k)$, then Builder is guaranteed to win eventually. The *restricted online Ramsey number* $\tilde{R}(G_1, \dots, G_k; n)$ is the smallest number of turns for which Builder is guaranteed a victory no matter what Painter does. In the case $G_i = K_{r_i}$ for all i, we simply write $\tilde{R}(r_1, \dots, r_k; n)$ for $\tilde{R}(G_1, \dots, G_k; n)$. These certificates have degree strictly smaller than the maximum given in Lemma 1.4.1, and the minimal degree is bounded above by generalizations of the restricted online Ramsey numbers. Our results for the graphical Ramsey numbers are given below.

Theorem 2.3.1. *The Ramsey number $R(G_1, \dots, G_k)$ is at most n if and only if there is no solution to the following system over $\overline{\mathbb{F}_2}$, where $K_n = (V, E)$ is the complete graph on n vertices. Moreover, when the system has solutions, the number of solutions to this system is equal to the* *number of graphs of order n that avoid copies of G_i in color i. In particular, when $k = 2$, $G_1 = K_r$, and $G_2 = K_s$, this is the number of Ramsey graphs $RG(n, r, s)$.*

$$p_{H,i} := \prod_{e \in E(H)} x_{i,e} = 0 \qquad \forall i, 1 \le i \le k, \quad \forall H \subseteq K_n,\ H \cong G_i, \tag{2.3}$$

$$q_e := 1 + \sum_{i=1}^{k} x_{i,e} = 0 \qquad \forall e \in E, \tag{2.4}$$

$$u_{i,j,e} := x_{i,e} x_{j,e} = 0 \qquad \forall e \in E,\ \forall i, j,\ i \neq j. \tag{2.5}$$

When $k = 2$, $G_1 = K_r$, and $G_2 = K_s$, the Ramsey ideal *$RI(n, r, s)$ is ideal of the polynomial ring $\overline{\mathbb{F}_2}[x_{1,e}, x_{2,e}]_{e \in E(K_n)}$ generated by the polynomials p_H, q_e and $u_{i,j,e}$. Then we have*

$$RI(n, r, s) \supseteq RI(n, r+1, s) \supseteq \cdots \supseteq RI(n, n, s) \supseteq RI(n, n+1, s) = RI(n, n+2, s) = \ldots$$

and

$$RI(n, r, s) \supseteq RI(n, r, s+1) \supseteq \cdots \supseteq RI(n, r, n) \supseteq RI(n, r, n+1) = RI(n, r, n+2) = \ldots$$

THEOREM 2.3.2. *If $n \ge R(G_1, \ldots, G_k)$, then there is an explicit Nullstellensatz certificate of degree $\tilde{R}(G_1, \ldots, G_k; n) - 1$ that the statement $R(G_1, \ldots, G_k) > n$ is false using the encoding in Theorem 2.3.1. In particular, in the case of 2-color classical Ramsey numbers, this implies that if $n \ge R(r, s)$, then there exists a Nullstellensatz certificate of degree $\tilde{R}(r, s; n) - 1$ that the statement $R(r, s) > n$ is false.*

We emphasize that graphical Ramsey numbers do not have any special properties that are used to prove Theorems 2.3.1 and 2.3.2. In fact, a similar encoding and certificate construction work for *Ramsey-type problems*, which we define precisely in Chapter 2. Ramsey-type problems include some of the most famous in Ramsey theory, including the problems of computing Schur, Rado, van der Waerden, and Hales-Jewett numbers. We will also define a Builder-Painter game for other Ramsey-type problems in Chapter 2. These results are consolidated in our metatheorem, Theorem 2.3.3 which is a generalization of Theorems 2.3.1 and 2.3.2 to other Ramsey-type problems; the bounds of the certificate degrees are given in terms of analogues of the restricted online Ramsey numbers.

Our second contribution in Chapter 2 deals with Alon's *Combinatorial Nullstellensatz*. This is a popular technique where combinatorial problems are encoded by a single polynomial $f(x_1, \ldots, x_n)$, and the combinatorial property of interest is true depending on whether f vanishes or not at certain testing points. This approach has been used with great success in many situations (see, for example, [**4**, **46**, **66**, **77**, **82**, **118**, **132**] and the references therein).

We show that lower bounds for Ramsey numbers can be obtained by showing that a certain *Ramsey polynomial* $f_{r,s,n}$ is not identically zero. Its coefficients are new combinatorial numbers $E_{n,k,r,H}$ which we call *ensemble numbers*. We show that $E_{n,k,r,H}$ equals the number of ways to choose two distinct edges from k-tuples of r-cliques inside K_n such that, every edge in a subgraph H is chosen an odd number of times and every edge in its complement $\bar{H}$ is chosen an even number of times. We give a detailed example computing a value of $E_{n,k,r,H}$ in Section 2.3.

Theorem 2.4.2 shows that the numbers $E_{n,k,r,H}$ can be used to find lower bounds for the diagonal Ramsey number $R(r, r)$, and it is an analogue of Theorem 7.2 in [**4**].

THEOREM 2.4.2. *If*

$$\sum_{k\ odd} 2^k \left(\binom{r}{2}^2 - \binom{r}{2}\right)^{\binom{r}{2}-k} E_{n,k,r,H} \neq \sum_{k\ even} 2^k \left(\binom{r}{2} - \binom{r}{2}^2\right)^{\binom{r}{2}-k} E_{n,k,r,H}$$

for some H, then $R(r, r) > n$.

The use of NulLA or the Combinatorial Nullstellensatz as a practical means to compute or certify upper bounds for Ramsey numbers is limited. We encountered difficulty attempting to certify bounds for numbers as low as $R(3, 4)$, and it is unlikely that NulLA can compute new Ramsey number upper bounds due to the size of the polynomial systems involved. Our computational focus shifted to the *Rado numbers*, and inspired by the successes in computing $S(5)$ and $R_2(x^2+y^2 = z^2)$, we turned to SAT solvers.

Our first computational contribution is the calculation of many new Rado numbers for three variable linear homogeneous equations $ax + by = cz$. These computations were carried out by SAT solvers and use an encoding that, given an equation $\mathcal{E}$ and positive integers k and n, produces a formula $F_n^k(\mathcal{E})$ that is satisfiable if and only if $R_k(\mathcal{E}) > n$. This encoding is based on the one given in [**71**] and is described in Chapter 3.

Theorem 3.6.1. *The values of the following Rado numbers are known.*

(i) $R_2(ax + by = cz)$ *for* $1 \le a, b, c \le 20$.

(ii) $R_3(a(x - y) = bz)$ *for* $1 \le a, b \le 15$.

(iii) $R_3(a(x + y) = bz)$ *for* $1 \le a, b \le 10$.

(iv) $R_3(ax + by = cz)$ *for* $1 \le a, b, c \le 6$.

(v) $R_4(x - y = az)$ *for* $1 \le a \le 4$.

(vi) $R_4(a(x - y) = z)$ *for* $1 \le a \le 5$.

Most of the values given in Theorem 3.6.1 are new, though some of the lower values were previously known. The results of Theorem 3.6.1 suggest that several patterns hold for the Rado numbers $R_3(a(x - y) = bz)$, $1 \le a, b \le 15$, and we show that these patterns continue for higher values of a and b.

Theorem 3.4.1. *The values of the following Rado numbers are known:*

(i) $R_3(x - y = (m - 2)z) = m^3 - m^2 - m - 1$ *for* $m \ge 3$.

(ii) $R_3(a(x - y) = (a - 1)z) = a^3 + (a - 1)^2$ *for* $a \ge 3$.

(iii) $R_3(a(x - y) = bz) = a^3$ *for* $b \ge 1$, $a \ge b + 2$, $gcd(a, b) = 1$.

These are among the first infinite families of three color Rado numbers that are known. In particular, a corollary of Theorem 3.4.1 is an exact formula for the *generalized Schur numbers* given by $S(m, k) = R_k(x_1 + \cdots + x_{m-1} = x_m)$ for $k = 3$, though we note this result was obtained independently in [**22**]. In Chapter 3 we also prove several results on degree of regularity. These results, combined with additional computations, yield the following.

Theorem 3.6.2. *The degree of regularity of the equation* $ax + by = cz$ *is known for all* $1 \le a, b, c \le 5$.

We are also able to give the degree of regularity for every equation of the form $a(x + y) = bz$ with $a, b \in \mathbb{Z}^+$ satisfying $a \le 5$ or $b \le 2$.

THEOREM 3.6.3. *Let $a, b \in \mathbb{Z}$ with $gcd(a,b) = 1$ and either $a \le 5$ or $b \le 2$. Then*

$$dor(a(x+y) = bz) = \begin{cases} \infty & a = b = 1, \ \text{or } a = 1, b = 2, \\ 2 & a \ge 2b \text{ or } b \ge 4a, \\ 3 & \text{otherwise.} \end{cases}$$

A simple corollary of Theorem 3.6.3 is an improvement on Theorem 1.2.6 (*ii*), a complete characterization of the degree of regularity for the equations $a(x+y) = bz$ with $\frac{a}{b} = 2^\ell$.

COROLLARY 3.6.1. *Let $a, b \in \mathbb{Z}^+$ with $\frac{a}{b} = 2^\ell$, $\ell \in \mathbb{Z}$. Then*

$$dor(a(x+y) = bz) = \begin{cases} \infty & \ell = -1, 0, \\ 2 & \text{otherwise.} \end{cases}$$

We also answer a related conjecture of Golowich [**57**].

CONJECTURE 3.6.1. *For each positive integer k there is an integer $m(k)$ such that for any $m \ge m(k)$, any linear homogeneous equation in m variables with nonzero integer coefficients not all of the same sign is $k-$regular.*

This conjecture suggests that for a given k and sufficiently many variables, 2-regularity implies k-regularity. However, we give counterexamples showing that this is not enough, and for all $m, k \ge 3$ there is a linear homogeneous equation in m variables that is not k-regular.

THEOREM 3.6.4. *For all $m, k \ge 3$, there is a linear homogeneous equation $\mathcal{E}$ in m variables that is not k-regular. In particular,*

$$x_1 + \cdots + x_{m-1} = \lceil (m-1)^{\frac{k-1}{k-2}} \rceil x_m$$

is not k-regular. Thus Conjecture 3.6.1 is false.

As a corollary, for $m \ge 3$ there exists a linear homogeneous equation in m variables with degree of regularity exactly 2.

COROLLARY 3.6.2. *For all $m \geq 3$, there is a linear homogeneous equation $\mathcal{E}$ in m variables with $dor(\mathcal{E}) = 2$. In particular,*

$$dor(x_1 + \cdots + x_{m-1} = (m-1)^2 x_m) = 2.$$

In Chapter 4, we study Ramsey-type results involving two types of sequences, *diffsequences* and *arithmetic progressions with prescribed gap set.* We recall some terminology from [**91**]. Given a set D, a D*-diffsequence* is a sequence where the differences between consecutive terms are all elements of D. The Ramsey-type number associated to diffsequences is denoted $n = \Delta(D, \ell; k)$, the smallest number n such that every k-coloring of $[n]$ contains a D-diffsequence of length ℓ. For arithmetic progressions, we define the number $n = n(AP_D, \ell; k)$ to be the smallest number n such that every k-coloring of $[n]$ contains an arithmetic progression of length ℓ whose common difference lies in D. If the numbers $\Delta(D, \ell; k)$ exist for all ℓ, then we say that D is k*-accessible.* The largest k for which D is k-accessible is called the *degree of accessibility* of D, denoted $doa(D)$. Similarly, if the numbers $n(AP_D, \ell; k)$ exist for all ℓ, then we say the set AP_D is k-regular, and the largest k for which AP_D is k-regular is called the *degree of regularity* of AP_D, denoted $dor(AP_D)$ (note the similarity to the definition of degree of regularity for *equations*, defined in terms of the Rado numbers).

Landman and Robertson studied the existence of and bounds for the numbers $\Delta(D, \ell; k)$ for various choices of D, with particular emphasis on translates of the set of primes [**90**]. More recent work by Clifton [**33**] and Chokshi, Clifton, Landman, and Sawin [**31**] has examined diffsequences involving sets such as $D = \{2^i : i \geq 0\}$ and given bounds on $\Delta(D, \ell; 2)$.

Another interesting choice for D is the set of Fibonacci numbers $F = \{1, 2, 3, 5, 8, 13, \ldots\}$. Ramsey results involving sequences that satisfy the Fibonacci recurrence, among other linear recurrences, have been studied in [**16, 69, 89, 108, 109**]. Landman and Robertson showed that F is 2-accessible and left the matter of determining $doa(F)$ as an open question [**90, 91**]. In [**7**], Ardal, Gunderson, Jungić, Landman, and Williamson showed that $dor(AP_F) \leq 5$ by constructing an explicit 6-coloring of $\mathbb{Z}^+$ that does not contain any monochromatic 2-term F-diffsequences. Moreover, they proved that $1 \leq dor(AP_F) \leq 3$ and gave several values of $\Delta(F, \ell; k)$. Our results build upon the work in [**7**]: we give improvements on the bounds for $dor(AP_F)$ and $doa(F)$. In particular, this shows $doa(F) = 1$.

Theorem 4.2.1. *The degree of accessibility of the Fibonacci numbers F is at most three.*

Theorem 4.2.2. *The set AP_F of arithmetic progressions whose gaps are Fibonacci numbers is not 2-regular. Moreover, $dor(AP_F) = 1$.*

We conclude Chapter 4 by studying diffsequences when D is the set L of *Lucas numbers* or set P of (nonzero) *Perrin numbers*. We show in Proposition 4.3.1 that $dor(L) \leq 3$, though this proof is far simpler than that of Theorem 4.2.1. In addition, it is simple to modify the SAT encoding for Rado numbers to suit our sequences. We give experimental results and compute values of $\Delta(L, \ell; k)$ and $\Delta(P, \ell; k)$ via SAT solving.

Chapter 5 gives some miscellaneous computational results. In Section 5.1 we use SAT solvers to compute bounds for Ramsey numbers involving *book* and *wheel* graphs B_n and W_n. The precise definitions of these graphs are given in Section 5.1.

Theorem 5.1.1. $R(B_4, B_5) = R(B_3, B_6) = 19$*,* $R(W_5, W_7) \geq 15$.

Section 5.2 showcases applications of our SAT methods to other combinatorial problems. We give several tables of experimental data involving *Turán numbers*, *Sidon-Ramsey numbers*, and sets avoiding angles in vector spaces over finite fields. In Section 5.3, we mention some open questions raised in this thesis and suggest methods and avenues for future research.

CHAPTER 2

Ramsey Theory and Hilbert's Nullstellensatz

2.1. Ramsey Numbers and Complexity Theory

While we know in practice computing Ramsey numbers is extremely difficult (and considered harder than fighting a war with an alien civilization), it is not clear what is the appropriate computational complexity class to show hardness of computing Ramsey numbers $R(r,s)$. For example, the closely related *arrowing decision problem* asks whether given three graphs F, G, H is there is a red-blue edge-coloring of F that contains neither a red G or a blue H? This decision problem was shown to be in co-NP for fixed choices of G, H [**24**]. Later Schaefer [**122**] showed that in general it is in the polynomial hierarchy to answer this queries, but it is not clear what to do with this complexity question when F, G, H are complete graphs K_N, K_r, K_s because there is only one value $R(r,s)$ for each input N, r, s, hence it is not clear how it can be hard for any of the usual classes like NP. See details in [**24, 67, 122**].

In recent years, Pak and collaborators [**78, 111, 112**] have proposed another way to measure complexity is by looking at counting sequences. We propose that their point of view could be another way to assert hardness of $R(r,s)$ by counting of Ramsey graphs: *Ramsey* (r,s)*-graphs*, are graphs with no red clique of size r, and no independent set of size s. Clearly, the number of vertices of a Ramsey (r,s)-graph is less than the Ramsey number $R(r,s)$. We are interested in the number of Ramsey graphs on n vertices denoted by $RG(n,r,s)$. What is the complexity of counting the sequence of numbers $\{RG(n,r,s)\}_{n=1}^{\infty}$? From Ramsey's theorem this sequence consists of $R(r,s)-1$ positive numbers and then an infinite tail of zeroes.

The #SAT problem asks how many satisfying assignments there are to a given Boolean formula ϕ. For any given r and s, it is possible to construct formulas whose satisfying assignments are in one-to-one correspondence with Ramsey (r,s)-graphs. We defer the details to Encoding C in Chapter 5. We give some examples of $RG(n,r,s)$ in Table 2.1, which are computed using the #SAT solver

RELSAT [**9**]. The hardness of $R(r,s)$ can then be rephrased as the question of whether the counting function $RG(n,r,s)$ is in $\#P$.

TABLE 2.1. Tables of $RG(n,r,s)$ for small n,r,s.

n	$RG(n,3,3)$	$RG(n,3,4)$	$RG(n,3,5)$	$RG(n,3,6)$	$RG(n,3,7)$	$RG(n,4,4)$
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	6	7	7	7	7	8
4	18	40	41	41	41	62
5	12	322	387	388	388	892
6	0	2812	5617	5788	5789	22484
7	0	13842	113949	133080	133500	923012
8	0	17640	2728617	4569085	4681281	55881692
9	0	0	55650276	220280031	245743539	4319387624

2.2. Polynomial Ideals, Ramsey-type Problems, and Analogues of Restricted Online Ramsey Numbers

Our first contribution, Theorem 2.3.1, reintroduces the sequence $\{RG(n,r,s)\}_{n=1}^{\infty}$ as the number of solutions of certain zero-dimensional ideals over the polynomial ring $\overline{\mathbb{F}_2}[x_1,\ldots,x_n]$. The solutions are indicator vectors that yield all Ramsey graphs (note, here they are not counted up to symmetry or automorphism classes). Some simple properties of $RG(n,r,s)$, such as the fact that $RG(n,r,s) \leq RG(n,r+1,s)$, follow immediately from Theorem 2.3.1, and the first value of n for which the system of equations in Theorem 2.3.1 has no solution is equal to the Ramsey number.

The proof of Theorems 2.3.1 and 2.3.2 do not rely on the graph-theoretic properties of Ramsey numbers specifically, and in fact they apply to a much larger class of problems in Ramsey theory. In particular, we can modify the encoding in Theorem 2.3.1 to suit several well-known problems in Ramsey theory, such as computing Schur, Rado, and van der Waerden numbers. We express these problems using the general framework below.

Definition 2.2.1. *Let k be a positive integer, and let $\{S_n\}$ be a sequence of sets. For c in $[k]$, the set of colors, let $\mathcal{P}_n^c$ be a subset of S_n. A triple $A := (\{S_n\}, \{\mathcal{P}_n^c\}; k)$ is a* Ramsey-type problem *if the following hold:*

(i) $S_i \subseteq S_{i+1}$ *for* $i \geq 1$,

(ii) $\mathcal{P}_i^c \subseteq \mathcal{P}_{i+1}^c$ *for* $i \geq 1$, $1 \leq c \leq k$,

(iii) There exists an integer N such that for all $i \geq N$ and every k-coloring of S_i there is a color c and some element $X \in \mathcal{P}_i^c$ where each element of X is assigned color c.

The smallest such N is called the Ramsey-type number *for A, and is denoted $R(A)$.*

We see that in the problem of computing classical Ramsey numbers $R(r, s)$, we have $S_n = E(K_n) = \{(i, j) : 1 \leq i < j \leq n\}$. The families $\mathcal{P}_n^1$ and $\mathcal{P}_n^2$ consist of all the sets of edges of induced subgraphs of K_n containing r and s vertices, respectively. As another example, the problem of computing Schur numbers asks for the smallest n such that every k-coloring $[n]$ contains a monochromatic solution to the equation $x + y = z$. In this case we have $S_n = [n]$, and for all c we have $\mathcal{P}_n^c = \{\{x, y, z\} : \{x, y, z\} \subseteq [n], x + y = z\}$.

We will see in Section 2.3 that the encoding in Theorem 2.3.1 can be modified to give bounds for many other Ramsey-type numbers, including Schur, Rado, van der Waerden, and Hales-Jewett numbers [**60**, **91**]. As mentioned, these bounds are given in terms of a certain two-player game. One of the first connections between Ramsey theory and games is the *Hales-Jewett theorem*, which roughly says that certain generalizations of tic-tac-toe cannot end in draws [**68**]. More general "Ramsey games" were introduced by Beck [**11**]. A particular game he studied was the *van der Waerden game* in which two players select alternately select integers from $[n]$ and a player wins when they have selected a length ℓ arithmetic progression. Beck went on to introduce the (unrestricted) *online Ramsey numbers* $\tilde{R}(r, s)$ in [**12**], and Kurek and Ruciński independently studied them in [**88**]. The online Ramsey numbers are defined similarly as their restricted counterparts, except Builder is not restricted to K_n, and instead may choose any edge from an infinite set of vertices. Work by Conlon [**35**] has shown that for infinitely many r,

$$\tilde{R}(r, r) \leq 1.001^{-r} \binom{R(r, r)}{2},$$

for infinitely many r, giving an exponential improvement for the online Ramsey numbers versus the number of edges in $K_{R(r,r)}$. Another recent result [**36**] gives the lower bound

$$\tilde{R}(r,r) \geq 2^{(2-\sqrt{(2)})r-O(1)}.$$

We are most interested in the restricted online Ramsey numbers $\tilde{R}(r,r;n)$ and their analogues for other Ramsey-type problems. For diagonal Ramsey numbers, the best known upper bound is $\tilde{R}(r,r;n) \leq \binom{n}{2} - \Omega(n \log n)$ when $n = R(r,r)$ [**58**]. We can define numbers analogous to the restricted online Ramsey numbers for Ramsey-type problems in terms of another Builder-Painter game. For a fixed n, we define this game as follows.

For each turn, Builder selects one object from S_n and Painter assigns it a color in $[k]$. Builder wins once there is a color c and an element $X \in \mathcal{P}_n^c$ where every element of X is assigned color c. Define the number $\tilde{R}_k(\mathcal{P}_n^1, \ldots, \mathcal{P}_n^k; S_n)$ to be the smallest number of turns for which Builder is guaranteed a victory. We will call these numbers *restricted online Schur, Rado, van der Waerden, etc.* numbers as appropriate. In this notation, the restricted online Ramsey number $\tilde{R}(r,s;n)$ is equal to $\tilde{R}_2(\mathcal{P}_n^1, \mathcal{P}_n^2; S_n)$ with $\mathcal{P}_n$ and S_n defined as above for the Ramsey number $R(r,s)$. Theorem 2.3.3 generalizes Theorems 2.3.1 and 2.3.2.

THEOREM 2.3.3. *Let $A = (\{S_n\}, \{\mathcal{P}_n^c\}; k)$ be a Ramsey-type problem. Then for each n, the Ramsey-type number for A is strictly greater than n if and only if the following system of equations has no solution over $\overline{\mathbb{F}_2}$.*

$$\begin{aligned}
p_{X,c} &:= \prod_{s \in X} x_{c,s} = 0 && \forall X \in \mathcal{P}_n^c,\ 1 \leq c \leq k \\
q_s &:= 1 + \sum_{i=1}^{k} x_{i,s} = 0 && \forall s \in S_n, \\
u_{i,j,s} &:= x_{i,s} x_{j,s} = 0 && \forall s \in S_n,\ \forall i,j,\ 1 \leq i < j \leq k.
\end{aligned}$$

If $n \geq R(A)$, then the minimal degree of a Nullstellensatz certificate for this system is at most $\tilde{R}_k(\mathcal{P}_n^1, \ldots, \mathcal{P}_n^k; S_n) - 1$.

Moreover, the number of solutions to this system is equal to the number of k-colorings of S_n such that for every color c, each set $X \in \mathcal{P}_n^c$ contains an object that is not assigned color c.

For example, in the case of Schur numbers, the number of solutions to this system is exactly the number of k-colorings of $[n]$ that do not contain any monochromatic solutions to $x + y = z$. In Section 2.3 we give some examples of values of $\tilde{R}(r, s; n)$ and $\tilde{R}(\mathcal{P}_n^1, \ldots, \mathcal{P}_n^k; S_n)$ and discuss the Nullstellensatz certificates for the associated polynomial systems.

2.3. Ramsey and Hilbert's Nullstellensatz

We have seen in Section 1.4 that combinatorial problems, including coloring, finding independent sets, partitions, etc. can be encoded as a system of polynomial equations (see, e.g., [**10**, **25**, **40**, **44**, **45**, **76**, **98**, **101**]). A Nullstellensatz certificate for such a combinatorial polynomial system is therefore a proof that a combinatorial theorem is true. We are interested on bounding the Nullstellensatz degree for our Ramsey systems.

Recall from the discussion in Section 1.4 that while the most general bounds for Nullstellensatz certificates are exponential, Lemma 1.4.1 shows that for "combinatorial ideals," the bounds are much better, linear in the number of variables. Over finite fields there are degree bounds that are independent of the number of variables [**62**], and a recent paper [**105**] gives substantial improvements to these bounds. The bounds we give in Theorems 2.3.2 and 2.3.3 for our systems of equations are better than the above bounds. Moreover, it has been documented that in practice the degrees of Nullstellensatz certificates of NP-hard problems (e.g., non-3-colorability), tend to be small "in practice" (see, for example, [**43**, **94**, **99**] and the references therein), especially when the polynomial encodings are over finite fields. Note also that when we know the degree of the Nullstellensatz certificate, one can compute explicit coefficients of the Nullstellensatz certificate using a linear algebra system derived by equating the monomials of the identity. This has been exploited in practical computation with great success, see [**42**, **43**, **94**].

We now prove Theorem 2.3.1 of our encoding for Ramsey numbers over $\overline{\mathbb{F}_2}$ below.

Proof of Theorem 2.3.1. Suppose there is a solution $\mathbf{x}$ to the system over $\overline{\mathbb{F}_2}$. For each edge e of K_n and each color i, the system has a variable $x_{i,e}$. The polynomials $u_{i,j,e}$ guarantee that for a given e, at most one variable $x_{i,e}$ is nonzero. From the polynomials q_e, we then see that exactly one index i such that $x_{i,e} = 1$, and let $\phi(\mathbf{x})$ be the coloring χ where $\chi(e)$ is this index. Color each edge e of K_n with the color $\chi(e)$. In the equations involving the polynomials p_H, for each subgraph H of K_n with $H \cong G_i$, there is at least one edge e in H with $x_{i,e} = 0$. Therefore $\chi(e) \neq i$, so there is no monochromatic copy of G_i in color i.

Conversely, if we have a coloring χ of the edges of K_n with no monochromatic G_i in color i, then let $\psi(\chi)$ be the solution $\mathbf{x}$ where

$$x_{i,e} = \begin{cases} 1 & \text{if } \chi(e) = i, \\ 0 & \text{otherwise.} \end{cases}$$

One can check easily that $\mathbf{x}$ satisfies the system of equations. The maps ϕ and ψ are inverses of each other, and so the number of solutions to the system is equal to the number of colorings of K_n with no monochromatic G_i in color i.

For the first chain of ideals, observe that for a fixed i, the polynomial $\prod_{e \in E(H)} x_{i,e}$ divides $\prod_{e \in E(H')} x_{i,e}$ if and only if H is a subgraph of H'. Since every copy of K_{r+1} in K_n contains a copy of K_r as a subgraph, in the ideal $RI(n, r+1, s)$, every polynomial of the form $\prod_{e \in E(H')} x_{i,e}$ with $H' \cong K_{r+1}$ is divisible by a generator $\prod_{e \in E(H)} x_{i,e}$ of $RI(n, r, s)$ with $H \cong K_r$. The ideals in the chain are equal for $r > n$ since in this case K_r is not a subgraph of K_n. The proof for the second chain of ideals is similar. $\square$

Before we prove Theorem 2.3.2, we show a special case as a warm-up example. There is a simple certificate of the fact that $R(r, 2) \leq r$.

Example 2.3.1. For all r, there exists a Nullstellensatz certificate of degree $\binom{r}{2} - 1$ of the statement $R(r, 2) \leq r$.

Proof. Label the edges of K_r from 1 to $n = \binom{r}{2}$. The following identity is a certificate that $R(r, 2) \leq r$. Polynomials in parentheses are part of the system of equations in Theorem 2.3.1.

$$
\begin{aligned}
1 = & (1 + x_{1,1} + x_{2,1}) + x_{1,1}(1 + x_{1,2} + x_{2,2}) + x_{1,1}x_{1,2}(1 + x_{1,3} + x_{2,3}) + \dots \\
& + x_{1,1}x_{1,2} \cdots x_{1,n-1}(1 + x_{1,n} + x_{2,n}) \\
& + x_{2,1} + x_{1,1}(x_{2,2}) + x_{1,1}x_{1,2}(x_{2,3}) + \cdots + x_{1,1}x_{1,2} \cdots x_{1,n-1}(x_{2,n}) \\
& + (x_{1,1} \cdots x_{1,n})
\end{aligned}
$$

□

In the proof of Theorem 2.3.2, we show how to translate a strategy for Builder into a Nullstellensatz certificate. This method can be used to construct a certificate for all (known) upper bounds for $R(G_1, \dots, G_s)$. Notably, better strategies for Builder yield lower degree certificates. In Example 2.3.1, this is not a concern since the order in which Builder selects edges does not matter, and in fact $\tilde{R}(r, 2) = \binom{r}{2}$. Painter can simply use the first color for every edge, and Builder wins only when all $\binom{r}{2}$ edges are selected.

The proofs of Theorems 2.3.2 and 2.3.3 are similar, and in fact Theorem 2.3.2 follows from Theorem 2.3.3, but for the sake of concreteness we begin with Theorem 2.3.2.

Proof of Theorem 2.3.2. Number the edges of K_n from 1 to $\binom{n}{2}$. A t-turn game state g is a set $\{(e_{i_1}, c_1), (e_{i_2}, c_2), \dots, (e_{i_t}, c_t)\}$ of pairs of edges $e_{i_j} \in E$ chosen by Builder and colors $c_j \in [k]$ chosen by Painter. A game is complete if there is some color $c \in [k]$ where Painter has colored a monochromatic G_c in color c. Let $d := \tilde{R}(G_1, \dots, G_k; n)$. If Builder follows an optimal strategy for choosing edges, then the game lasts at most d turns, that is $t \leq d$.

For a t-turn game state g, define the monomial $\pi(g)$ to be

$$\pi(g) := \prod_{j=1}^{t} x_{c_j, e_{i_j}}.$$

Similarly, for any monomial $f = \prod_{j=1}^{t} x_{c_j, e_{i_j}}$ with distinct e_{i_j}, let $\sigma(f)$ denote the game state $\{(e_{i_1}, c_1), (e_{i_2}, c_2), \dots, (e_{i_t}, c_t)\}$.

We will describe an algorithm to construct a Nullstellensatz certificate of the form

$$\sum_{i=1}^{k} \sum_{H \cong G_i} \beta_{H,i} p_{H,i} + \sum_{e \in E} \gamma_e q_e = 1. \tag{2.1}$$

Denote the left-hand side of Equation 2.1 by L. For each $i \in [k]$, initialize $\beta_{H,i}$ to 0 for all $H \cong G_i$. Initialize γ_e to 0 for all edges e except e_1, and set $\gamma_{e_1} := 1$. Then repeat the following:

(1) Expand and simplify L so that L is a sum of monomials. If $L = 1$, then we are done.

(2) Otherwise, at least one term in L is a nonconstant monomial f.

(3) If $\sigma(f)$ is a completed game state, then $p_{H,i}$ divides f for some color i and $H \cong G_i$. Then set

$$\beta_{H,i} \leftarrow \beta_{H,i} + \frac{f}{p_{H,i}}.$$

This results in $L \leftarrow L + f$, which cancels the original f in the certificate since it is an expression over $\overline{\mathbb{F}_2}$, which has characteristic 2.

(4) If $\sigma(f)$ is not a completed game state, then let e be an edge that Builder should choose in an optimal strategy from the game state $\sigma(f)$. Set

$$\gamma_e \leftarrow \gamma_e + f.$$

Since $fq_e = f + \sum_{i=1}^{k} fx_{i,e}$, we obtain $L \leftarrow L + f + \sum_{i=1}^{k} fx_{i,e}$. This results in the cancellation of f in L, but adds k additional terms (one for each of Painter's k choices for coloring e) to L. Note that if $\sigma(f)$ is a t-turn game state, then $\sigma(fx_{i,e})$ is a $(t+1)$-turn game state for all i.

By the symmetry of K_n, it does not matter which edge Builder selects first. Therefore each nonconstant term that appears in L corresponds to a game state where Builder (but not necessarily Painter) has followed an optimal strategy. Since terms that correspond to completed games are cancelled out in step 3, this procedure terminates, resulting in a Nullstellensatz certificate. Because Builder follows an optimal strategy, the maximal degree of any term in any γ_e is $d-1$, so the degree of the certificate is $d-1$.

□

To illustrate the importance of Builder's strategy in this method, observe that one can construct a degree 7 certificate for the statement $R(3,3) \leq 6$ using the following strategy that mimics the proof in Proposition 1.1.2: For the first five turns, Builder selects each edge incident to some vertex v. No matter how Painter colors these edges, three must be colored the same color. Call these edges vw_1, vw_2, vw_3. Then for the next three turns, Builder selects the edges w_1w_2, w_1w_3, and w_2w_3, and Painter must construct a monochromatic triangle. However, if Builder plays poorly and selects, for example, the edges $(1,2), (2,3), (3,4), (4,5), (5,6), (1,6), (1,4), (2,5)$, and $(3,6)$, then no matter what Painter does there are *no* monochromatic triangles, and this leads to a higher degree certificate.

The proof of Theorem 2.3.2 shows that the polynomials can "simulate" a tree of Builder-Painter games. However, in general the degrees of certificates can be strictly smaller than the bounds given in Theorems 2.3.2. For example, a result from [**88**] implies $\tilde{R}(3,3;6) = 8$. However, there exists a Nullstellensatz certificate of degree 5 using the encoding in Theorem 2.3.1, which is better than the bound given in Theorem 2.3.2. We now give the proof of Theorem 2.3.3.

Proof of Theorem 2.3.3. Let $A = (\{S_n\}, \{\mathcal{P}_n^c\}; k)$ be a Ramsey-type problem. A t-turn game state g after t is a set $\{(s_{i_1}, c_1), (s_{i_2}, c_2), \ldots, (s_{i_t}, c_t)\}$ of objects $s \in S_n$ chosen by Builder and colors $c_j \in [k]$ chosen by Painter. A game is complete if there is a color $c \in [k]$ and some element $X \in \mathcal{P}_n^c$ where Painter has colored all the elements of X color c. Let $d := \tilde{R}_k(\mathcal{P}_n; S_n)$. If Builder follows an optimal strategy for choosing edges, then the game lasts at most d turns, that is $t \leq d$.

For a game state g_t, define the monomial $\pi(g_t)$ to be

$$\pi(g_t) := \prod_{j=1}^{t} x_{c_j, s_{i_j}}.$$

Similarly, for any monomial $f = \prod_{j=1}^{t} x_{c_j, s_{i_j}}$ with distinct s_{i_j}, let $\sigma(f)$ denote the game state $\{(s_{i_1}, c_1), (s_{i_2}, c_2), \ldots, (s_{i_t}, c_t)\}$.

We will describe an algorithm to construct a Nullstellensatz certificate of the form

$$\sum_{(X,c) \in \mathcal{P}_n} \beta_{X,c} p_{X,c} + \sum_{s \in S_n} \gamma_s q_s = 1. \tag{2.2}$$

Denote the left-hand side of Equation 2.2 by L. Initialize $\beta_{X,c}$ to 0 for all $(X, c) \in \mathcal{P}_n$. Let s^* be an object that Builder selects first in an optimal strategy. Initialize γ_{s^*} to 1 and γ_s to 0 for all other $s \in S_n$. Then repeat the following:

(1) Expand and simplify L so that L is a sum of monomials. If $L = 1$, then we are done.

(2) Otherwise, at least one term in L is a nonconstant monomial f.

(3) If $\sigma(f)$ is a completed game state, then $p_{X,c}$ divides f for some $(X, c) \in \mathcal{P}_n$. Then set

$$\beta_{X,c} \leftarrow \beta_{X,c} + \frac{f}{p_{X,c}}.$$

This results in $L \leftarrow L + f$, which cancels the original f in the certificate since it is an expression over $\overline{\mathbb{F}_2}$, which has characteristic 2.

(4) If $\sigma(f)$ is not a completed game state, then let s be an object that Builder should choose in an optimal strategy from the game state $\sigma(f)$. Set

$$\gamma_s \leftarrow \gamma_s + f.$$

Since $fq_s = f + \sum_{i=1}^{k} fx_{i,s}$, we obtain $L \leftarrow L + f + \sum_{i=1}^{k} fx_{i,e}$. This results in the cancellation of f in L, but adds k additional terms (one for each of Painter's k choices for coloring s) to L. Note that if $\sigma(f)$ is a t-turn game state, then $\sigma(fx_{i,s})$ is a $(t+1)$-turn game state for all i.

For each nonconstant monomial f that appears in L, its corresponding game state $\sigma(L)$ is one where Builder (but not necessarily Painter) has followed an optimal strategy. Since terms that correspond to completed games are cancelled out in step 3, this procedure terminates and results in a Nullstellensatz certificate. Because Builder follows an optimal strategy, the maximal degree of any term in any γ_s is at most $d - 1$, so the degree of the certificate is at most $d - 1$. $\square$

As an application of Theorem 2.3.3, let $\mathcal{E}$ be a linear equation, and let $R_k(\mathcal{E})$ denote the k-color Rado number for $\mathcal{E}$, the smallest n such that every k-coloring of $[n]$ contains a monochromatic solution to $\mathcal{E}$. Let $X_{n,\mathcal{E}}$ be the set of all solutions over $[n]$ to $\mathcal{E}$. Let $\mathcal{P}_n^c := X_{n,\mathcal{E}}$ for all c. If $R_k(\mathcal{E})$ exists, then $(\{[n]\}, \{\mathcal{P}_n^c\}; k)$ is a Ramsey-type problem, and we have the following corollary.

COROLLARY 2.3.1. *Let $\mathcal{E}$ be the linear equation $\sum_{j=1}^{t} a_i y_i = a_0$ with a finite Rado number $R_k(\mathcal{E})$. Let $X_{n,\mathcal{E}} = \{(m_1, \ldots, m_t) : \sum_{j=1}^{t} a_j m_j = a_0,\ 1 \le m_j \le n\}$ be the set of solutions over $[n]$ to $\mathcal{E}$. Then for every n, the following system has no solution over $\overline{\mathbb{F}_2}$ if and only if $n \ge R_k(\mathcal{E})$.*

$$\prod_{j=1}^{t} x_{i,m_j} = 0 \qquad \forall (m_1, \ldots, m_t) \in X_{n,\mathcal{E}},\ 1 \le i \le k,$$

$$1 + \sum_{i=1}^{k} x_{i,m} = 0 \qquad 1 \le m \le n,$$

$$x_{i,m} x_{j,m} = 0 \qquad 1 \le m \le n, 1 \le i < j \le k.$$

The degree of a minimal Nullstellensatz certificate for this system has degree at most $\tilde{R}_k(X_{n,\mathcal{E}}, \ldots, X_{n,\mathcal{E}}; [n]) - 1$.

EXAMPLE 2.3.2. Let $\mathcal{E}$ denote the equation $x+3y = 3z$, and let $X_{9,\mathcal{E}}$ be the solutions to $\mathcal{E}$ over $[9]$ as above. It is known that $R_2(\mathcal{E}) = 9$ [**91**]. However, Builder can select, in order, the integers 4,6,9,3, and 7 to win the Builder-Painter game in at most 5 turns: since $(6, 4, 6)$ is a solution, 4 and 6 must be different colors, and then since $(9, 6, 9)$ and $(3, 3, 4)$ are solutions, 4 and 9 must be one color and 3 and 6 are the other color. But then $(3, 6, 7)$ and $(9, 4, 7)$ are solutions, so there is a monochromatic solution no matter which color Painter selects for 7. Therefore $\tilde{R}_2(X_{9,\mathcal{E}}, X_{9,\mathcal{E}}; [9]) \le 5$, and the minimal degree of a Nullstellensatz certificate for the system of equations in Corollary 2.3.1 is at most 4. In fact, some computations show the minimal degree is 2.

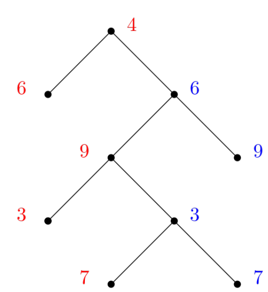


FIGURE 2.1. Depiction of the game in Example 2.3.2. Leaves of the tree denote completed games where Builder has won.

Similarly, the encoding in Theorem 2.3.1 for the Schur number $S(2) = R_2(x + y = z)$ also gives an example of Nullstellensatz certificates that are smaller than the ones given by games. It is well-known that $S(2) = 5$, and from the encoding in Theorem 2.3.3, we have $S(2) \leq 5$ if and only if the following system of equations has no solutions over $\overline{\mathbb{F}_2}$.

$$1 + x_{1,i} + x_{2,i} = 0, \qquad 1 \leq i \leq 5,$$

$$\left.\begin{array}{l} x_{i,1}x_{i,2} = 0, \quad x_{i,2}x_{i,4} = 0, \\ x_{i,1}x_{i,3}x_{i,4} = 0, \quad x_{i,1}x_{i,4}x_{i,5} = 0, \quad x_{i,2}x_{i,3}x_{i,5} = 0. \end{array}\right\} i = 1, 2$$

A computer search shows that the number $\tilde{R}_2(X_{5,x+y=z}, X_{5,x+y=z}; [5]) = 5$, where $X_{5,x+y=z}$ is the set of positive integer solutions to $x + y = z$ in $[1, 5]$. The following identity (over $\overline{\mathbb{F}_2}$) is a degree 3 Nullstellensatz certificate for the above system of equations, which is an improvement on the bound in Theorem 2.3.3.

$$\begin{aligned}
1 = {} & (x_{2,5} + x_{1,4}x_{1,5} + x_{1,5}x_{2,3}x_{2,4})(1 + x_{1,1} + x_{2,1}) + \\
& (x_{1,1}x_{1,3} + x_{2,1}x_{2,5} + x_{1,1}x_{1,5}x_{2,4} + x_{1,1}x_{2,3}x_{2,5} + x_{1,3}x_{1,5}x_{2,4} + x_{1,4}x_{1,5}x_{2,1})(1 + x_{1,2} + x_{2,2}) + \\
& (x_{1,1}x_{2,5} + x_{2,4}x_{1,5} + x_{1,1}x_{1,5}x_{2,4})(1 + x_{1,3} + x_{2,3}) + \\
& (x_{1,5} + x_{1,1}x_{1,3}x_{1,5} + x_{1,1}x_{1,3}x_{2,2} + x_{1,2}x_{2,1}x_{2,5})(1 + x_{1,4} + x_{2,4}) + \\
& (1 + x_{1,1}x_{1,3})(1 + x_{1,5} + x_{2,5}) + \\
& (x_{1,3} + x_{2,3}x_{2,5} + x_{2,4}x_{1,5})x_{1,1}x_{1,2} + (x_{2,1}x_{1,5} + x_{2,1}x_{2,5})x_{1,2}x_{2,4} \\
& (x_{2,5} + x_{1,4}x_{1,5})x_{2,1}x_{2,2} + (x_{1,1}x_{1,3} + x_{1,1}x_{1,5} + x_{1,3}x_{3,5})x_{2,2}x_{2,4} + (x_{1,5} + x_{2,2})x_{1,1}x_{1,3}x_{1,4} + \\
& (x_{1,1}x_{1,4}x_{1,5}) + x_{2,4}(x_{1,2}x_{1,3}x_{1,5}) + x_{1,5}(x_{2,1}x_{2,3}x_{2,4}) + x_{1,2}(x_{2,1}x_{2,4}x_{2,5}) + x_{1,1}(x_{2,2}x_{2,3}x_{2,5}).
\end{aligned}$$

Recall the van der Waerden number $w(\ell; k)$ is the smallest n such that every k-coloring of $[n]$ contains a monochromatic ℓ-term arithmetic progression [**60**]. Let $AP_{n,\ell}$ denote the set of all

ℓ-term arithmetic progressions in $[n]$. Then setting $\mathcal{P}_n^c := AP_{n,\ell}$ for all c, then $(\{[n]\}, \{\mathcal{P}_n^c\}; k)$ is a Ramsey-type problem as well.

COROLLARY 2.3.2. *For every n, the following system has no solution over $\overline{\mathbb{F}_2}$ if and only if $n \geq w(\ell; k)$.*

$$\prod_{j=1}^{t} x_{i,m_j} = 0 \qquad \forall (m_1, \dots, m_t) \in AP_{n,t},\ 1 \leq i \leq k,$$

$$1 + \sum_{i=1}^{k} x_{i,m} = 0 \qquad 1 \leq m \leq n,$$

$$x_{i,m} x_{j,m} = 0 \qquad 1 \leq m \leq n, 1 \leq i < j \leq k.$$

The minimal degree of a Nullstellensatz certificate for this system is at most

$$\tilde{R}_r(AP_{n,k}, \dots, AP_{n,k}; [n]) - 1.$$

The number of solutions to this system is the number of k-colorings of $[n]$ that contain no t-term monochromatic arithmetic progressions.

We give one last consequence of Theorem 2.3.3. For fixed parameters t and n, a *combinatorial line* is a nonconstant sequence of points $v^1, \dots, v^t$, where $v^i \in [t]^n$ such that for every coordinate j, the sequence $(v_j^i)_{i=1}^t$ is either constant or $v_j^i = i$ for all i. The *Hales-Jewett number* $HJ(t,k)$ is the smallest number n such that every k-coloring of $[t]^n$ contains a monochromatic combinatorial line [**60**]. Let $L_{t,n}$ denote the set of all combinatorial lines on t^n. If we set $\mathcal{P}_n^c = L_{t,n}$ for all c, then $(\{[t]^n, \{\mathcal{P}_n^c\}; k)$ is a Ramsey-type problem.

COROLLARY 2.3.3. *For every n, the following system has no solution over $\overline{\mathbb{F}_2}$ if and only if $n \geq HJ(t,k)$.*

$$\prod_{j=1}^{t} x_{i,v^j} = 0 \qquad \forall (v^1, \dots, v^t) \in L_{t,n},\ 1 \leq i \leq k,$$

$$1 + \sum_{i=1}^{k} x_{i,v} = 0 \qquad v \in [t]^n,$$

$$x_{i,v} x_{j,v} = 0 \qquad v \in [t]^n, 1 \leq i < j \leq k.$$

The number of solutions to this system is the number of k-colorings of $[t]^n$ that do not contain any monochromatic combinatorial lines.

2.4. Ramsey and Alon's Combinatorial Nullstellensatz

In this section we introduce a way to encode the problem of finding a lower bound for $R(r,s)$ in terms of properties of a single polynomial. We also define a family of numbers $E_{n,k,r,H}$ whose values can provide bounds for $R(r,r)$. The following theorem of Alon, the "Combinatorial Nullstellensatz," has been used to solve many problems in combinatorics and graph theory (see, for example, [**4**, **46**, **66**, **77**, **82**, **132**] and the references therein).

THEOREM 2.4.1 (Alon, [**4**]). *Let F be a field, and let $f \in F[x_1, \dots, x_n]$. Let $\deg(f) = \sum_{i=1}^n t_i$ with each t_i a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ is nonzero. Then if $S_1, \dots, S_n$ are subsets of F with $|S_i| > t_i$, then there exist $s_1 \in S_1, \dots, s_n \in S_n$ such that $f(s_1, \dots, s_n) \neq 0$.*

Here we apply the Combinatorial Nullstellensatz to show that lower bounds for Ramsey numbers can be obtained by showing that a certain polynomial is not identically zero. Consider the following polynomial, where $K_n = (V,E)$ is the complete graph on n vertices:

$$f(x) = f_{r,s,n}(x) = \left(\prod_{S \subset V,\ |S|=r} \left(\sum_{e \in E(S)} x_e - \binom{r}{2} \right) \right) \left(\prod_{S \subset V,\ |S|=s} \left(\sum_{e \in E(S)} x_e + \binom{s}{2} \right) \right). \tag{2.3}$$

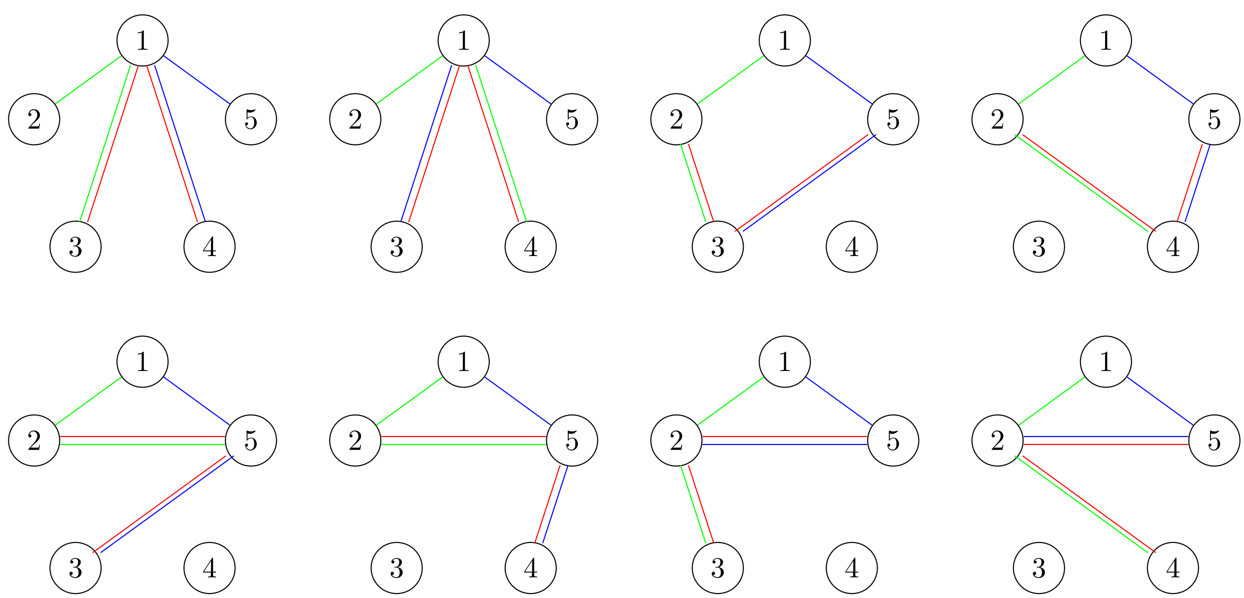
Every 2-coloring of the edges of G corresponds to an assignment $c : \{x_e\}_{e \in E} \to \{-1, 1\}$. If an edge e is colored with the first color, then we set $c(x_e) = 1$, and if e is colored with the second color, then $c(x_e) = -1$. Then $f(c(x)) = 0$ if and only if G contains an r-clique in the first color or an s-clique in the second color. Therefore if $f(c(x_1), \dots, c(x_{|E|})) = 0$ for all colorings c, then $R(r,s) \leq n$.

Since we only consider the values of f on $\{-1,1\}^{|E|}$, we may instead consider the multilinear representative of f in the ideal $\langle x_e^2 - 1 \rangle_{e \in E}$. This representative can be obtained by deleting each variable with an even exponent from each term in f. By Theorem 2.4.1, this representative is the zero polynomial if and only if $R(r,s) \leq |V|$.

In the proof of Theorem 2.4.2, we focus on the case when $r = s$ and study the polynomial $f_{r,r,n}$. Before proving Theorem 2.4.2, we give an example of a value of an ensemble number $E_{n,k,r,H}$, which are coefficients of the multilinear representative of $f_{r,r,n}$. We recall the definition of $E_{n,k,r,H}$ below.

We call a collection of k r-cliques a (k,r)-*ensemble*. For each clique in the ensemble, we select exactly two edges, and if each edge in H is selected an odd number of times and each edge of $\bar{H}$ is selected an odd number of times, then we call this a *valid covering* of a subgraph H. The ensemble number $E_{n,k,r,H}$ is the total number of valid coverings of H counted from all (k,r)-ensembles of K_n.

If H is the graph on five vertices with edge set $\{(1,2),(1,5)\}$, then $E_{5,3,3,H} = 8$. The figure below depicts all eight ways. Each graph has an associated $(3,3)$-ensemble $\mathcal{E}$, and each 3-clique in $\mathcal{E}$ is assigned a distinct color $c \in \{\text{red,green,blue}\}$. The two edges from that 3-clique are colored c. Edges colored with more than one color are drawn as multiple edges. In the upper left figure, for example, the associated ensemble is $\{\{1,2,3\},\{1,3,4\},\{1,4,5\}$. The edges $(1,2)$ and $(1,3)$ were chosen from the clique $\{1,2,3\}$, the edges $(1,3)$ and $(1,4)$ were chosen from the clique $\{1,3,4\}$, and the edges $(1,4)$ and $(1,5)$ were chosen from the clique $\{1,4,5\}$. The edges of H are chosen exactly once, and all other edges are chosen zero or two times. Note that for some $(3,3)$-ensembles, such as $\{\{1,2,3\},\{1,2,4\},\{3,4,5\}\}$, it is impossible to choose edges in H an odd number of times.



We now prove Theorem 2.4.2.

Proof of Theorem 2.4.2. We will use the symbol $\equiv$ to denote equivalent representatives in the ideal $I := \langle x_e^2 - 1 \rangle_{e \in E}$. Consider the product of the two terms in f that arise from a fixed r-subset S of V. Expanding this product and using the relations $x_e^2 \equiv 1$ gives

$$\left(\sum_{e \in E(S)} x_e - \binom{r}{2} \right) \left(\sum_{e \in E(S)} x_e + \binom{r}{2} \right) \equiv \left(\sum_{\{e,e'\} \in \binom{E(S)}{2}} 2x_e x_{e'} + \binom{r}{2} - \binom{r}{2}^2 \right).$$

Taking the product of these terms over all r-subsets of V gives

$$f(x) \equiv \prod_{S \subset V,\ |S|=r} \left(\sum_{\{e,e'\} \in \binom{E(S)}{2}} 2x_e x_{e'} + \binom{r}{2} - \binom{r}{2}^2 \right). \tag{2.4}$$

After expanding the product, we may write f as a sum of monomials of the form $\prod_{e \in E} x_e^{b_e}$. Two monomials of this form are equivalent modulo I if and only if the parity of b_e is the same for all edges e. If H is a subgraph of G, it follows that every monomial that satisfies the condition that b_e is odd if and only if $e \in E(H)$ is equivalent to the squarefree monomial $m_H := \prod_{e \in E(H)} x_e$. Therefore, f is equivalent to a sum of the form

$$f \equiv \sum_{H \subseteq G} a_H m_H. \tag{2.5}$$

We now calculate the coefficients a_H in terms of $E_{n,k,r,H}$.

In the expansion of the right-hand side of (2.4), we have the following combinatorial interpretation of the terms. Each term in the product corresponds to an r-subset S of V. For each S, the term represents a choice of picking either a pair of edges (one of the $2x_e x_{e'}$ terms in the sum), or zero edges (the term $\binom{r}{2} - \binom{r}{2}^2$) from $E(S)$. Therefore the coefficient a_H is

$$a_H = \sum_{k=0}^{\binom{n}{r}} 2^k \left(\binom{r}{2}^2 - \binom{r}{2} \right)^{\binom{r}{2}-k} E_{n,k,r,H}. \tag{2.6}$$

If a_H is nonzero for some graph H, then the multilinear representative of f is nonzero, and by Theorem 2.4.1 there exists a coloring of K_n that makes f nonzero, and in this case it follows that

$R(r,r) > n$. Setting the expression (2.6) to be not equal to zero and rearranging terms by the parity of k concludes the proof. $\square$

As an example, we give the values of $E_{n,k,r,H}$ for $n = 5, r = 3$, and H a graph of order five with an even number of edges. We denote by G_1 the graph with edge set $\{\{1,2\},\{1,3\},\{2,3\},\{1,4\}\}$ and G_2 the graph with edge set $\{\{1,2\},\{2,3\},\{3,4\},\{3,5\}\}$.

H \ k	0	1	2	3	4	5	6	7	8	9	10
$\overline{K_5}$	1	0	0	20	30	132	220	540	585	460	60
$P_3 \cup \overline{K_2}$	0	1	2	8	44	106	280	496	612	413	86
$K_2 \cup K_2 \cup K_1$	0	0	4	12	28	124	276	484	628	404	88
$K_{1,4}$	0	0	3	4	36	132	242	588	516	428	99
G_1	0	0	2	8	32	120	292	504	592	392	106
G_2	0	0	1	10	34	114	292	510	590	390	107
P_5	0	0	1	8	40	112	282	520	592	384	109
$C_4 \cup K_1$	0	0	2	8	28	136	272	504	612	376	110
$K_2 \cup K_3$	0	0	0	12	36	108	292	516	588	388	108
$\overline{K_{1,4}}$	0	0	0	8	24	120	328	504	552	392	120
$\overline{G_1}$	0	0	0	6	30	118	318	514	554	386	122
$\overline{G_2}$	0	0	0	4	36	116	308	524	556	380	124
$\overline{P_5}$	0	0	0	4	32	132	288	524	576	364	128
$\overline{C_4 \cup K_1}$	0	0	0	4	36	116	308	524	556	380	124
$\overline{K_2 \cup K_3}$	0	0	0	4	36	108	348	444	636	340	132
$\overline{P_3 \cup \overline{K_2}}$	0	0	0	0	24	128	344	512	520	384	136
$\overline{K_2 \cup K_2 \cup K_1}$	0	0	0	0	20	144	324	512	540	368	140
K_5	0	0	0	0	0	144	400	480	480	400	144

Unfortunately, we were unable to find any nontrivial patterns in the above data, and it appears to be difficult to compute the numbers $E_{n,k,r,H}$ in general.

CHAPTER 3

Rado Numbers, SAT Solvers, and Degree of Regularity

3.1. Rado Numbers

Recall that for a given equation $\mathcal{E}$, the *k-color Rado number* $R_k(\mathcal{E})$ is the smallest number n such that every k-coloring of $[n]$ induces a monochromatic solution to $\mathcal{E}$, or infinity if no such n exists. In the latter case, this is equivalent to the existence of a k-coloring of $\mathbb{Z}^+$ with no monochromatic solution to $\mathcal{E}$ (see [**91**]).

The 2-color Rado numbers for various types of equations (linear, nonlinear, nonhomogeneous, etc.) have been studied the most widely, and many computations can be found in, for example, [**74**, **106**, **117**, **121**]. Here we focus on homogeneous three-variable linear equations. These are an interesting case for two reasons. First, Rado numbers for two variable linear homogeneous equations are completely known.

Proposition 3.1.1. *For all $k \in \mathbb{Z}^+$, the Rado number $R_k(ax = by)$ satisfies*

$$R_k(ax = by) = \begin{cases} 1 & \text{if } a = b, \\ \infty & \text{otherwise.} \end{cases}$$

A proof of Proposition 3.1.1 can be found in [**106**]. The second reason is that Rado numbers for (linear homogeneous) equations in $m = 3$ variables bound those for equations with $m > 3$ variables. For $k = 2$ colors, this is useful because we know exactly when $R_2(\mathcal{E})$ is finite by Theorem 1.2.5. The following proposition is also given in [**106**].

Proposition 3.1.2. *Suppose an equation $\mathcal{E}'$ can be obtained from setting equal some of the variables in the equation $\mathcal{E}$. Then $R_k(\mathcal{E}) \leq R_k(\mathcal{E}')$ for all k.*

For instance, we see that $R_2(w+2x+y = z) \leq R_2(2x+2y = z)$ by setting $w = y$. Unfortunately, there is no known formula for the general family of Rado numbers $R_2(ax + by = cz)$, $a, b, c > 0$.

However, formulas for the Rado numbers $R_2(a(x-y)=bz)$ and $R_2(a(x+y)=bz)$ are known. The first proof for the former family is attributed to an unpublished work of Burr and Loo, but a proof can be found in [**91**]. The formula for the latter family is due to Harborth and Maasberg, and it includes fourteen cases. Both formulas are quasipolynomial in a and b subject to algebraic conditions on a and b. Other special cases of 2-color Rado numbers are known as well, for instance a formula for $R_2(\sum_{i=1}^{m-1} a_i x_i = x_m - c)$ was proven for $c=0$ by Guo and Sun [**65**] and then extended for constants $c \geq 1 - \sum_{i=1}^{m-1} a_i$ by Schaal and Zinter [**120**].

There are some computations of 3-color Rado numbers scattered throughout the literature [**106**, **116**, **119**], but Rado numbers with more than two colors have not been studied as often. We present a systematic study of these numbers.

Recall that the *generalized Schur numbers* are given by $S(m,k) = R_k(x_1 + \cdots + x_{m-1} = x_m)$. In [**17**] it was shown that $S(m,3) \geq m^3 - m^2 - m - 1$, and it was conjectured in [**1**] and later proved in [**22**] that $S(m,3) = m^3 - m^2 - m - 1$. Myers showed in [**106**] that the numbers $R_k(x-y=(m-2)z)$ give an upper bound for $S(m,k)$, and several more values of $R_k(x-y=(m-2)z)$ were shown to be equal to $S(m,k)$, thus giving exact values for more generalized Schur numbers. Myers went on to make the following conjecture in [**106**].

Conjecture 3.1.1 (Myers). *$R_3(x-y=(m-2)z) = m^3 - m^2 - m - 1$ for $m \geq 3$.*

In this chapter we focus on computing Rado numbers for three variable linear homogeneous equations using SAT-based methods described in Section 3.2. In particular, we show Conjecture 3.1.1 is true. Our main result is Theorem 3.4.1, which was stated in Section 1.5, and as a corollary we obtain the following.

Corollary 3.1.1. *$S(m,3) = m^3 - m^2 - m - 1$ for $m \geq 3$.*

Theorem 3.6.1 gives calculations for many 3-color and 4-color Rado numbers. We collect the 3-color Rado number values we computed in Theorem 3.6.1 in Tables 3.1 to 3.8. We also give the additional values $R_3(ax+ay=bz)$ for $3 \leq a \leq 6$, $11 \leq b \leq 20$ as well as our values for $R_4(a(x-y)=bz)$ in Section 3.8 (Tables 3.9 and 3.10). Underlined entries in these tables correspond to equations whose coefficients are not coprime.

Table 3.1. 3-color Rado numbers $R_3(a(x-y)=bz)$

$b \backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	14	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375
2	43	$\underline{14}$	31	$\underline{14}$	125	$\underline{27}$	343	$\underline{64}$	729	$\underline{125}$	1331	$\underline{216}$	2197	$\underline{343}$	3375
3	94	61	$\underline{14}$	73	125	$\underline{14}$	343	512	$\underline{27}$	1000	1331	$\underline{64}$	2197	2744	$\underline{125}$
4	173	$\underline{43}$	109	$\underline{14}$	141	$\underline{31}$	343	$\underline{14}$	729	$\underline{125}$	1331	$\underline{27}$	2197	$\underline{343}$	3375
5	286	181	186	180	$\underline{14}$	241	343	512	729	$\underline{14}$	1331	1728	2197	2744	$\underline{27}$
6	439	$\underline{94}$	$\underline{43}$	$\underline{61}$	300	$\underline{14}$	379	$\underline{73}$	$\underline{31}$	$\underline{125}$	1331	$\underline{14}$	2197	$\underline{343}$	$\underline{125}$
7	638	428	442	456	470	462	$\underline{14}$	561	729	1000	1331	1728	2197	$\underline{14}$	3375
8	889	$\underline{173}$	633	$\underline{43}$	665	$\underline{109}$	644	$\underline{14}$	793	$\underline{141}$	1331	$\underline{31}$	2197	$\underline{343}$	3375
9	1198	856	$\underline{94}$	892	910	$\underline{61}$	896	896	$\underline{14}$	1081	1331	$\underline{73}$	2197	2744	$\underline{125}$
10	1571	$\underline{286}$	1171	$\underline{181}$	$\underline{43}$	$\underline{186}$	1190	$\underline{180}$	1206	$\underline{14}$	1431	$\underline{241}$	2197	$\underline{343}$	$\underline{31}$
11	2014	1508	1530	1552	1574	1596	1618	1584	1575	1580	$\underline{14}$	1849	2197	2744	3375
12	2533	$\underline{439}$	$\underline{173}$	$\underline{94}$	2005	$\underline{43}$	2053	$\underline{61}$	$\underline{109}$	$\underline{300}$	2024	$\underline{14}$	2341	$\underline{379}$	$\underline{141}$
13	3134	2432	2458	2484	2510	2536	2562	2588	2574	2530	2541	2544	$\underline{14}$	2913	3375
14	3823	$\underline{638}$	3039	$\underline{428}$	3095	$\underline{442}$	$\underline{43}$	$\underline{456}$	3207	$\underline{470}$	3113	$\underline{462}$	3146	$\underline{14}$	3571
15	4606	3676	$\underline{286}$	3736	$\underline{94}$	$\underline{181}$	3826	3856	$\underline{186}$	$\underline{61}$	3795	$\underline{180}$	3835	3836	$\underline{14}$

Table 3.2. 3-color Rado numbers $R_3(a(x+y)=bz)$

$b \backslash a$	1	2	3	4	5	6	7	8	9	10
1	14	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	1	$\underline{14}$	243	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$	∞	$\underline{\infty}$
3	54	54	$\underline{14}$	384	2000	$\underline{\infty}$	∞	∞	$\underline{\infty}$	∞
4	∞	$\underline{1}$	108	$\underline{14}$	875	$\underline{243}$	4459	$\underline{\infty}$	∞	$\underline{\infty}$
5	∞	105	135	180	$\underline{14}$	864	3430	3072	12393	$\underline{\infty}$
6	∞	$\underline{54}$	$\underline{1}$	$\underline{54}$	750	$\underline{14}$	3087	$\underline{384}$	$\underline{243}$	$\underline{2000}$
7	∞	455	336	308	875	756	$\underline{14}$	1536	8748	7500
8	∞	$\underline{\infty}$	432	$\underline{1}$	1000	$\underline{108}$	2744	$\underline{14}$	8019	$\underline{875}$
9	∞	∞	$\underline{54}$	585	1125	$\underline{54}$	3087	1224	$\underline{14}$	6000
10	∞	$\underline{\infty}$	1125	$\underline{105}$	$\underline{1}$	$\underline{135}$	3430	$\underline{180}$	7290	$\underline{14}$

Table 3.3. $R_3(ax + by = z)$

$b \backslash a$	1	2	3	4	5	6
1	14	43	94	173	286	439
2		∞	1093	∞	2975	4422
3			∞	∞	∞	∞
4				∞	∞	∞
5					∞	∞
6						∞

Table 3.4. $R_3(ax+by = 2z)$

$b \backslash a$	1	2	3	4	5	6
1	1	14	54	∞	70	126
2		$\underline{14}$	61	$\underline{43}$	181	$\underline{94}$
3			243	∞	395	648
4				$\underline{\infty}$	∞	$\underline{1093}$
5					∞	∞
6						$\underline{\infty}$

Table 3.5. $R_3(ax+by = 3z)$

$b \backslash a$	1	2	3	4	5	6
1	54	1	27	54	89	195
2		54	31	∞	140	108
3			$\underline{14}$	109	186	$\underline{43}$
4				384	220	∞
5					2000	1074
6						$\underline{\infty}$

Table 3.6. $R_3(ax+by = 4z)$

$b \backslash a$	1	2	3	4	5	6
1	∞	∞	1	64	100	∞
2		$\underline{1}$	∞	$\underline{14}$	∞	$\underline{54}$
3			108	73	105	∞
4				$\underline{14}$	180	$\underline{61}$
5					141	∞
6						$\underline{31}$

Table 3.7. $R_3(ax+by = 5z)$

$b \backslash a$	1	2	3	4	5	6
1	∞	45	60	1	125	150
2		105	1	∞	125	70
3			135	100	125	108
4				180	141	∞
5					$\underline{14}$	300
6						864

Table 3.8. $R_3(ax+by = 6z)$

$b \backslash a$	1	2	3	4	5	6
1	∞	40	81	∞	1	216
2		$\underline{54}$	81	$\underline{1}$	90	$\underline{27}$
3			$\underline{1}$	∞	135	$\underline{14}$
4				$\underline{54}$	∞	$\underline{31}$
5					750	241
6						$\underline{14}$

Recall that if the number $R_k(\mathcal{E})$ is finite for a fixed k, we say that the equation $\mathcal{E}$ is *k-regular*. If $\mathcal{E}$ is k-regular for all $k \geq 1$, we say $\mathcal{E}$ is *regular*. The *degree of regularity* of an equation $\mathcal{E}$, denoted $dor(\mathcal{E})$ is the largest k for which $\mathcal{E}$ is k-regular. For example, $R_2(2x + 2y = z) = 34$, but we will see that $R_3(2x + 2y = z) = \infty$, so $dor(2x + 2y = z) = 2$. When $\mathcal{E}$ is regular, we say $dor(\mathcal{E}) = \infty$, so $dor(x + y = z) = \infty$ by Theorem 1.2.1. It is possible to define degree of regularity for coloring other sets besides $\mathbb{Z}^+$, for example $\mathbb{Z}$, $\mathbb{Q}$, or other rings R as in [**18**, **54**]. We are concerned only with the positive integers here, though some of our results can be extended to larger sets.

One of the most interesting open questions concerning regularity is Conjecture 1.2.1, Rado's boundedness conjecture. In [**54**] Fox and Kleitman showed that Rado's boundedness conjecture is true if it is true for the case of homogeneous equations. They also proved the first nontrivial case of the conjecture by showing that if a linear homogeneous equation in three variables is 24-regular, then it is regular. However, it is not known if 24 is the best possible constant. Moreover, in [**54**], several coloring lemmas give more precise bounds on the degree of regularity of 3-variable linear homogeneous equations. We contribute further improvements on their results and compute the degree of regularity of all sufficiently small equations $ax + by = cz$. In particular, we were unable to find any nonregular, 4-regular equations of the form $ax + by = cz$.

This chapter is organized as follows: In Section 3.2 we describe our SAT encoding and the computational methods we used to compute Rado numbers. Sections 3.3 and 3.4 show how to obtain the lower and upper bounds, respectively, for the numbers in Theorem 3.4.1. Section 3.5 gives several lemmas used to obtain improved bounds on the degree of regularity of certain families of equations. We accumulate all of these results to prove Theorem 3.6.1 and our other theorems on degree of regularity in Section 3.6. The remainder of the chapter contains more computational data and details on our methods. Section 3.7 gives additional details on the computational aspects of this project. Sections 3.8 and 3.9 contain additional tables of Rado numbers and degrees of regularity for various equations. Finally, Section 3.10 gives additional details on part of the method used to prove Theorem 3.4.1.

In this section we explain how to encode the problem of finding Rado numbers as an instance of SAT and describe additional techniques used to increase performance. The code used for our computations can be found at [**30**].

Given an equation $\mathcal{E}$ and positive integers n, k, we construct a formula $F_n^k(\mathcal{E})$ that is satisfiable if and only if there exists a k-coloring of $[n]$ that does not contain a monochromatic solution to $\mathcal{E}$. Therefore if $F_n^k(\mathcal{E})$ is satisfiable, then $R_k(\mathcal{E}) > n$, and otherwise $R_k(\mathcal{E}) \leq n$. We use the variables v_j^i that are assigned the value true if and only if the integer j is colored with color i. Following the language used in [**71**], a formula $F_n^k(\mathcal{E})$ consists of three different types of clauses: *positive, negative,* and *optional*.

- Positive clauses encode that every number j is assigned at least one color, and are of the form $v_j^1 \vee v_j^2 \vee \cdots \vee v_j^k$ for $1 \leq j \leq n$.
- Negative clauses encode that there are no monochromatic solutions to $\mathcal{E}$. If $(x_1, x_2, \ldots, x_m)$ is a solution to $\mathcal{E}$, then its corresponding negative clauses are $\bar{v}_{x_1}^i \vee \cdots \vee \bar{v}_{x_m}^i$ for $1 \leq i \leq k$. Every solution $x \in [n]^m$ contributes these k negative clauses to $F_n^k(\mathcal{E})$.
- Optional clauses encode that every number j is assigned at most one color, and are of the form $\bar{v}_j^{i_1} \vee \bar{v}_j^{i_2}$ for $1 \leq j \leq n$ and $1 \leq i_1 < i_2 \leq k$. These clauses are not strictly necessary since they do not affect the satisfiability of $F_n^k(\mathcal{E})$, but they ensure that satisfying assignments are in one-to-one correspondence with k-colorings of $[n]$ that avoid monochromatic solutions to $\mathcal{E}$.

Example 3.2.1. The clauses in the formula $F_4^3(x + y = z)$ are:

Positive clauses:

$$(v_1^1 \vee v_1^2 \vee v_1^3) \wedge (v_2^1 \vee v_2^2 \vee v_2^3) \wedge (v_3^1 \vee v_3^2 \vee v_3^3) \wedge (v_4^1 \vee v_4^2 \vee v_4^3)$$

Negative clauses:

$$(\overline{v}_1^1 \vee \overline{v}_1^1 \vee \overline{v}_2^1) \wedge (\overline{v}_2^1 \vee \overline{v}_1^1 \vee \overline{v}_3^1) \wedge (\overline{v}_3^1 \vee \overline{v}_1^1 \vee \overline{v}_4^1) \wedge$$
$$(\overline{v}_1^1 \vee \overline{v}_2^1 \vee \overline{v}_3^1) \wedge (\overline{v}_2^1 \vee \overline{v}_2^1 \vee \overline{v}_4^1) \wedge (\overline{v}_1^1 \vee \overline{v}_3^1 \vee \overline{v}_4^1) \wedge$$
$$(\overline{v}_1^2 \vee \overline{v}_1^2 \vee \overline{v}_2^2) \wedge (\overline{v}_2^2 \vee \overline{v}_1^2 \vee \overline{v}_3^2) \wedge (\overline{v}_3^2 \vee \overline{v}_1^2 \vee \overline{v}_4^2) \wedge$$
$$(\overline{v}_1^2 \vee \overline{v}_2^2 \vee \overline{v}_3^2) \wedge (\overline{v}_2^2 \vee \overline{v}_2^2 \vee \overline{v}_4^2) \wedge (\overline{v}_1^2 \vee \overline{v}_3^2 \vee \overline{v}_4^2) \wedge$$
$$(\overline{v}_1^3 \vee \overline{v}_1^3 \vee \overline{v}_2^3) \wedge (\overline{v}_2^3 \vee \overline{v}_1^3 \vee \overline{v}_3^3) \wedge (\overline{v}_3^3 \vee \overline{v}_1^3 \vee \overline{v}_4^3) \wedge$$
$$(\overline{v}_1^3 \vee \overline{v}_2^3 \vee \overline{v}_3^3) \wedge (\overline{v}_2^3 \vee \overline{v}_2^3 \vee \overline{v}_4^3) \wedge (\overline{v}_1^3 \vee \overline{v}_3^3 \vee \overline{v}_4^3)$$

Optional clauses:

$$(\overline{v}_1^1 \vee \overline{v}_1^2) \wedge (\overline{v}_1^1 \vee \overline{v}_1^3) \wedge (\overline{v}_1^2 \vee \overline{v}_1^3) \wedge (\overline{v}_2^1 \vee \overline{v}_2^2) \wedge (\overline{v}_2^1 \vee \overline{v}_2^3) \wedge (\overline{v}_2^2 \vee \overline{v}_2^3) \wedge$$
$$(\overline{v}_3^1 \vee \overline{v}_3^2) \wedge (\overline{v}_3^1 \vee \overline{v}_3^3) \wedge (\overline{v}_3^2 \vee \overline{v}_3^3) \wedge (\overline{v}_4^1 \vee \overline{v}_4^2) \wedge (\overline{v}_4^1 \vee \overline{v}_4^3) \wedge (\overline{v}_4^2 \vee \overline{v}_4^3)$$

If we input $F_4^3(x+y=z)$ into a SAT solver, it will output satisfiable. The 3−coloring $1, 2, 3, 4$, for example, avoids monochromatic solutions. We remark that even though some clauses, such as $\overline{v}_1^1 \vee \overline{v}_1^1 \vee \overline{v}_2^1$, contain redundant literals, these literals are removed in a preprocessing step.

We will use this SAT encoding to prove Theorem 3.6.1. In Section 3.7 we give practical details on this encoding and how to generate formulas efficiently.

3.3. Lower bounds for Rado Number Families

In order to prove Theorem 3.4.1, we require lower bounds for each equation. The main results of this section are explicit colorings that give general lower bounds for the three families of equations in Theorem 3.4.1. The following lemma gives a general lower bound on the Rado numbers $R_k(a(x-y)=bz)$ for any number of colors k.

Lemma 3.3.1. *Suppose $a, b \geq 1$ and $\gcd(a,b)=1$. Then $R_k(a(x-y)=bz) \geq a^k$.*

Proof. Let $v_a(n)$ denote the highest power of a that divides n. Then $v_a : [1, a^k-1] \to \{0, 1, \ldots, k-1\}$ defines a $k-$coloring of $[1, a^k-1]$ that has no monochromatic solutions of $a(x-y) = bz$. To see this, suppose (x, y, z) is a monochromatic solution in color c. If $x \leq y$, then there is no $z \in [1, a^k - 1]$ that satisfies $a(x-y) = bz$, so suppose $x > y$. Then $x = a^c x'$ and $y = a^c y'$, where $a \nmid x', y'$. Since $\gcd(a,b) = 1$, $v_a(z) = v_a(bz) = v_a(a(x-y)) = v_a(a^{c+1}(x'-y')) \geq c+1$, a contradiction. $\square$

For the case $b = a-1$, we have an improved lower bound on $R_3(a(x-y) = (a-1)z)$.

LEMMA 3.3.2. $R_3(a(x-y) = (a-1)z) \geq a^3 + (a-1)^2$.

PROOF. We will construct a coloring of $[1, a^3 + (a-1)^2 - 1]$ that induces no monochromatic solutions to $a(x-y) = (a-1)z$. Define

$$\chi(i) := \begin{cases} 0 & v_a(i) = 2 \text{ or } (v_a(i) = 0 \text{ and } (i < a^2 - a \text{ or } i > a^3 - a)), \\ 1 & v_a(i) = 1, \\ 2 & \text{otherwise.} \end{cases}$$

Let (x,y,z) be a positive integer solution to $a(x-y) = (a-1)z$. Suppose $\chi(x) = \chi(y) = 0$. If $v_a(x) = v_a(y) \geq 2$, then since a and $a-1$ are relatively prime, $v_a(z) = v_a((a-1)z) = v_a(a(x-y)) \geq 3$, and $\chi(z) \neq 0$. If $v_a(x) = 2$ and $v_a(y) = 0$, then $v_a(z) = v_a(a(x-y)) = 1$, so $\chi(z) = 1$. The case $v_a(x) = 0$ and $v_a(y) = 2$ is similar. Note that $x > y$ since z must be positive. If $v_a(x) = v_a(y) = 0$, $x \geq a^3 - a$ and $y \leq a^2 - a$, then $a(x-y) \geq a(a^3 - a^2) > (a-1)(a^3 - a)$. Then $z \in [a^3 - a, a^3 + (a-1)^2 - 1]$. Since $(a-1)z = a(x-y)$, it follows that $v_a(z) \geq 1$, so $v_a(z) \neq 0$. But the only value $z \in [a^3 - a, a^3 + (a-1)^2 - 1]$ with $v_a(z) \geq 2$ is a^3, so $\chi(z) \neq 0$. Now if $v_a(x) = v_a(y) = 0$ and $x, y \in [1, a^2 - a)$ or $x, y \in (a^3 - a, a^3 + (a-1)^2 - 1]$, then $x - y < a^2 - a$, so $(a-1)z = a(x-y) < a^3 - a^2$. Then $v_a(z) \in \{1, 2\}$. If $v_a(z) = 1$, then $\chi(z) = 1 \neq 0$. If $v_a(z) = 2$, then $z \geq a^2$ and $(a-1)z \geq a^3 - a^2$, a contradiction.

Now suppose $\chi(x) = \chi(y) = 1$. Then $v_a(z) = v_a(a(x-y)) \geq 2$, so $\chi(z) \neq 1$.

If $\chi(x) = \chi(y) = 2$, then either $v_a(x) = v_a(y) = 0$, or $x = a^3$. In the former case we have $v_a(z) = v_a(a(x-y)) \geq 1$. We have $z \neq a^3$ since this implies $x - y = a^3 - a^2$, but this is impossible since $x, y \in [1, a^3 - (a-1)^2 - 1]$, and it follows that $\chi(z) \neq 2$. If $x = a^3$, then $y \neq a^3$ and $v_a(y) = 0$. Therefore $v_a(z) = v_a(a(x-y)) = 1$, so $\chi(z) \neq 2$.

Therefore there are no monochromatic solutions to $a(x-y) = (a-1)z$. $\square$

We illustrate the colorings given by Lemmas 3.3.1 and 3.3.2 in Figures 3.1 and 3.2.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63

FIGURE 3.1. 3-colorings of [26] and [63] that avoid monochromatic solutions to $3(x-y)=z$ and $4(x-y)=z$, respectively.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72

FIGURE 3.2. 3-colorings of [30] and [72] that avoid monochromatic solutions to $3(x-y)=2z$ and $4(x-y)=3z$, respectively.

We also require a related result from [**17**] that constructs explicit colorings that give lower bounds for the generalized Schur numbers. We will see that this result, along with Proposition 3.1.2, gives the tight lower bound for the Rado numbers $R_3(x+(m-2)y=z)$.

LEMMA 3.3.3 (Beutelspacher and Brestovansky). *The generalized Schur numbers $S(k,m)$ satisfy*

$$S(k,m) \geq m^k - m^{k-1} - m^{k-2} - \cdots - m - 1.$$

3.4. Modified SAT Encoding and Proof of Theorem 3.4.1

In this section we prove Theorem 3.4.1 using an encoding of the Rado number problem similar to that in Section 3.2. The key difference is that in this new encoding, indices of variables are indexed by *polynomials* rather than fixed integers.

Let $\mathcal{E}$ be a linear equation in m variables, and let S be a set of polynomials. Let $C \subseteq S^m$ be a set of solutions to $\mathcal{E}$. The variable v_s^i is assigned the value true if and only if the expression $s \in S$ is assigned color i. Positive and optional clauses are constructed similarly to the method in Section 3.2. The negative clauses are constructed from the solutions in C. For example, if $S = \{ia : 1 \leq i \leq 7\}$, and $\mathcal{E}$ is the equation $x - y = 5z$, then $(x,y,z) = (7a, 2a, a)$ is a solution. If $(7a, 2a, a) \in C$, then we add the negative clause $\bar{v}_{7a}^1 \vee \bar{v}_{2a}^1 \vee \bar{v}_a^1$ to our formula. The following lemma formalizes this procedure and describes how to use these formulas to compute Rado numbers for families of equations.

Lemma 3.4.1. *Let* $\Sigma = \{\sigma_1, \ldots, \sigma_\ell\}$ *be a finite alphabet of parameters. Let* $\mathcal{E}$ *be a linear equation in the variables* $x_1, \ldots, x_m$ *with coefficients in* Σ. *Let* S *be a set of expressions over* Σ, *and let* $C \subset S^m$ *be a set of solutions to* $\mathcal{E}$. *We define* $F_{k,S,C}(\mathcal{E})$ *to be the corresponding formula for the* k*-color Rado number generated by the clauses from* C *as follows.*

$$F_{k,S,C}(\mathcal{E}) := Pos_{k,S} \wedge Neg_{k,C} \wedge Opt_{k,S}, \textit{ where}$$

$$Pos_{k,S} := \bigwedge_{s \in S} \left(\bigvee_{i=1}^{k} v_s^i \right),$$

$$Neg_{k,C} := \bigwedge_{(s_1,\ldots,s_m) \in C} \bigwedge_{i=1}^{k} \bigvee_{j=1}^{m} \bar{v}_{s_j}^i,$$

$$Opt_{k,S} := \bigwedge_{s \in S} \bigwedge_{1 \le i_1 < i_2 \le k} (\bar{v}_s^{i_1} \vee \bar{v}_s^{i_2}).$$

Let $A \subset \mathbb{Z}^\ell$. *If* $1 \le s(a) = s(a_1, \ldots, a_\ell) \le f(a)$ *for all* $s \in S$, $a \in A$ *and* $F_{k,S}(\mathcal{E})$ *is unsatisfiable, then* $R_k(\mathcal{E}) \le f(a)$ *for all* $a \in A$.

In other words, if substituting values for parameters in a formula F always gives a valid formula, i.e., one whose variables with indices bounded between 1 and n, then the unsatisfiability of F gives an upper bound on the Rado numbers for a family of equations. For each equation $\mathcal{E}$ in the family, F is an unsatisfiable subformula (possibly with some variables identified, for instance v_{2a} and v_{a^2} are the same when $a = 2$) in the corresponding Rado number formula for $\mathcal{E}$. Such formulas are also called *unsatisfiable cores*. We give a simple example to illustrate the idea behind Lemma 3.4.1.

Example 3.4.1. Let $\mathcal{E}$ be the equation $x + (a-2)y = z$ for a parameter a. Consider the Rado numbers $R_2(\mathcal{E}) = a^2 - a - 1$ for $a \ge 3$. Let $S = \{1, a-1, a, a^2-2a+1, a^2-a-1\}$. Then the formula $F_{2,S}(\mathcal{E})$ is

$$(v_1^1 \vee v_1^2) \wedge (v_a^1 \vee v_a^2) \wedge (v_{a-1}^1 \vee v_{a-1}^2) \wedge (v_{a^2-2a+1}^1 \vee v_{a^2-2a+1}^2) \wedge (v_{a^2-a-1}^1 \vee v_{a^2-a-1}^2) \wedge$$

$$(\overline{v}_1^1 \vee \overline{v}_{a-1}^1) \wedge (\overline{v}_1^2 \vee \overline{v}_{a-1}^2) \wedge (\overline{v}_{a^2-2a+1}^1 \vee \overline{v}_{a-1}^1) \wedge (\overline{v}_{a^2-2a+1}^2 \vee \overline{v}_{a-1}^2) \wedge$$

$$(\overline{v}_1^1 \vee \overline{v}_a^1 \vee \overline{v}_{a^2-2a+1}^1) \wedge (\overline{v}_1^2 \vee \overline{v}_{a^2-a-1}^2 \vee \overline{v}_{a^2-2a+1}^2) \wedge$$

$$(\overline{v}_1^1 \vee \overline{v}_a^1 \vee \overline{v}_{a^2-2a+1}^1) \wedge (\overline{v}_1^2 \vee \overline{v}_{a^2-a-1}^2 \vee \overline{v}_{a^2-2a+1}^2) \wedge$$

$$(\overline{v}_{a-1}^1 \vee \overline{v}_a^1 \vee \overline{v}_{a^2-a-1}^1) \wedge (\overline{v}_{a-1}^2 \vee \overline{v}_a^2 \vee \overline{v}_{a^2-a-1}^2) \wedge$$

$$(\overline{v}_1^1 \vee \overline{v}_1^2) \wedge (\overline{v}_a^1 \vee \overline{v}_a^2) \wedge (\overline{v}_{a-1}^1 \vee \overline{v}_{a-1}^2) \wedge (\overline{v}_{a^2-2a+1}^1 \vee \overline{v}_{a^2-2a+1}^2) \wedge (\overline{v}_{a^2-a-1}^1 \vee \overline{v}_{a^2-a-1}^2).$$

It a routine check that for all $a \geq 3$ and $p(a) \in S$ we have $1 \leq p(a) \leq a^2 - a - 1$. The formula $F_{2,S}(\mathcal{E})$ is unsatisfiable, so Lemma 3.4.1 implies $R_2(\mathcal{E}) \leq a^2 - a - 1$ for $a \geq 3$.

The formula in Example 3.4.1 is simple enough that one could show it is unsatisfiable by hand. But for more colors and formulas with many clauses, this becomes far more difficult, and SAT solvers are necessary. We are able to prove Conjecture 3.1.1 using Lemma 3.4.1 and a solver's assistance.

Proof of Theorem 3.4.1. For the Rado numbers $R_3(x - y = (m-2)z)$, in Lemma 3.4.1 let $\Sigma = \{m\}$, and let $\mathcal{E}$ be the equation $x - y = (m-2)z$. The set S is a family of 685 polynomials and the set C contains 9468 solutions to $\mathcal{E}$. It is straightforward to show that $1 \leq s(m) \leq m^3 - m^2 - m - 1$ for all $m \geq 3$ and all $s \in S$. The formula $F_{3,S}(\mathcal{E})$ was shown to be unsatisfiable in 0.03 seconds by Satch, proving $R_3(x - y = (m-2)z) \leq m^3 - m^2 - m - 1$ for $m \geq 3$. By Proposition 3.1.2 and Lemma 3.3.3, we have $R_3(\mathcal{E}) \geq S(m, 3) \geq m^3 - m^2 - m - 1$ for $m \geq 3$, and so $R_3(x - y = (m-2)z) = m^3 - m^2 - m - 1$ for $m \geq 3$.

For the Rado numbers $R_3(a(x - y) = (a-1)z)$, let $\Sigma = \{a\}$, and let $\mathcal{E}$ denote the equation $a(x - y) = (a-1)z$. We constructed a set S of 1365 polynomials and a set C of 20811 solutions to $\mathcal{E}$. It is again straightforward to show that $1 \leq s(a) \leq a^3 + (a-1)^2$ for all $a \geq 16$ and all $s \in S$. The formula $F_{3,S,C}(\mathcal{E})$ was shown to be unsatisfiable in 0.05 seconds by Satch, proving $R_3(\mathcal{E}) \leq a^3 + (a-1)^2$ for $a \geq 16$. By Lemma 3.3.2 and Theorem 3.6.1, it follows that $R_3(\mathcal{E}) = a^3 + (a-1)^2$ for $a \geq 3$.

For the Rado numbers $R_3(a(x-y)=bz)$, let $\Sigma=\{a,b\}$, and let $\mathcal{E}$ denote the equation $a(x-y)=bz$. We constructed a set S of 40645 polynomials and a set C of 490897 solutions to $\mathcal{E}$. All polynomials $p(a,b)$ were verified to satisfy $1\leq p(a,b)\leq a^3$ for all integers a,b satisfying $a\geq 16, b\geq 1$, and $a\geq b+2$ using the software GloptiPoly 3 [**70**]. Some additional valid inequalities were added to the region specified in GloptiPoly 3 using elementary calculus techniques. The formula $F_{3,S,C}$ was shown to be unsatisfiable in 1.72 seconds by Satch, proving $R_3(a(x-y)=bz)\leq a^3$ for all (a,b) satisfying $a\geq 16$, $b\geq 1$, and $a\geq b+2$. By Theorem 3.6.1 and Lemma 3.3.1, $R_3(a(x-y)=bz)=a^3$ for $a\geq 3, b\geq 1, a\geq b+2$ with $\gcd(a,b)=1$.

For each of these formulas, the sets S and C were constructed using a heuristic search procedure. We give details of this procedure in Section 3.10.

□

We remark also that the sets S are not necessarily of minimum, or even minimal size. The problem of extracting general a minimal unsatisfiable core of an unsatisfiable formula is in general difficult [**107**]. We have made no serious effort to optimize the sizes of S and C, and this would be an interesting direction for future research.

An interesting consequence of Theorem 3.4.1 is that it gives an upper bound for the "restricted online" Rado numbers from Chapter 2.

Corollary 3.4.1. *The following bounds for restricted online Rado numbers hold.*

(i) $\tilde{R}_3(x-y=(m-2)z; m^3-m^2-m-1)\leq 685$ *for* $m\geq 3$.

(ii) $\tilde{R}_3(a(x-y)=(a-1)z; a^3+(a-1)^2)\leq 1365$ *for* $a\geq 16$.

(iii) $\tilde{R}_3(a(x-y)=bz; a^3)\leq 40645$ *for* $a\geq 16, b\geq 1$, $a\geq b+2$, $gcd(a,b)=1$.

Proof. Given an equation E with coefficients a and possibly b that belongs to one of the three families, let S be the corresponding set of polynomials from the proof of Theorem 3.4.1. Builder can evaluate all of the polynomials in S at a and b and choose only integers from the resulting set. The SAT solving computations of the proof of Theorem 3.4.1 then guarantee Builder's victory in at most $|S|$ turns. □

In particular, Corollary 3.4.1 shows that the restricted online Rado numbers for these families are bounded by a constant while the ordinary Rado numbers themselves grow.

3.5. Coloring Lemmas for Degree of Regularity

Working towards the goal of computing the degree of regularity and Rado number for as many equations as possible, in this section we collect several results on colorings that avoid monochromatic solutions. These colorings give upper bounds on the degree of regularity of certain equations, and this allows us to avoid doing unnecessary computations. We are especially interested in cases where we can show that the degree of regularity of an equation is at most three. In these cases a computation of a (finite) 3-color Rado number is a proof that the degree of regularity equals three.

The following result gives two algebraic conditions that guarantee an upper bound on the degree of regularity for a class of equations. Roughly speaking, if certain coefficients in an equation are too large or small compared to the sum of all the coefficients, then this prevents k-regularity. A version of the first condition was proved in [**54**].

Lemma 3.5.1. *Let $\mathcal{E}$ be the equation $a_1x_1 + \cdots + a_{m-1}x_{m-1} = a_m x_m$ with $a_1 \leq a_2 \leq \cdots \leq a_{m-1}$ and $a_i > 0$ for all i. Let $S := \sum_{i=1}^{m-1} a_i$. Then $\mathcal{E}$ is not k-regular if one of the following conditions holds:*

(i) $S \leq \frac{a_1^{k-1}}{a_m^{k-2}}$, *(ii)* $S \leq a_1^{\frac{1}{k-1}} a_m^{1-\frac{1}{k-1}}$.

Proof. For the first condition, let $d := \left(\frac{S}{a_m}\right)^{\frac{1}{k-1}}$. Define the coloring $\chi(n) := \lceil \log_d n \rceil \pmod{k}$. Suppose $(x_1, \ldots, x_m)$ is a solution to $\mathcal{E}$, and let $M := \max\{x_1, \ldots, x_{m-1}\}$. Let i be the unique integer such that $M \in (d^{i-1}, d^i]$. Then $x_m = \frac{\sum_{i=1}^{m-1} a_i x_i}{a_m} \leq d^{k-1}M \leq d^{i+k-1}$. By hypothesis we have $d^{k-1} \leq \left(\frac{a_1}{a_m}\right)^{k-1}$. Therefore $x_m \geq \frac{a_1 M}{a_m} \geq dM > d^i$. We have shown that $\chi(x_m) \in [i+1, i+k-1]$, so $\chi(x_m) \neq \chi(M)$ and there are no χ-monochromatic solutions to $\mathcal{E}$.

For the second condition, suppose $S \leq a_1^{\frac{1}{k-1}} a_m^{1-\frac{1}{k-1}}$ and that $(x_1, \ldots, x_m)$ is a solution to $\mathcal{E}$. Again let $M = \max\{x_1, \ldots, x_{m-1}\}$, and let $d = \left(\frac{a_1}{a_m}\right)^{\frac{1}{k-1}}$. Define a k-coloring $\chi(n) = \lceil \log_d(n) \rceil \pmod{k}$. Suppose $M \in (d^{i+1}, d^i]$, so $\chi(M) = i \pmod{k}$. Note $d < 1$ since $a_1 < S <= a_1^{\frac{1}{k-1}} a_m^{1-\frac{1}{k-1}}$ implies $a_1 < a_m$. Then $x_m = \frac{\sum_{i=1}^{m-1} a_i x_i}{a_m} \leq \frac{SM}{a_m} \leq dM \leq d^{i+1}$. Moreover, $x_m \geq \frac{a_1 M}{a_m} = d^{k-1}M > d^{i+k}$. Therefore $\lceil \log_d x_m \rceil \in [i+k-1, i+1]$, so $\chi(x_m) \neq \chi(M)$ and there are no χ-monochromatic solutions to $\mathcal{E}$. $\square$

Recall that for any prime p, the *p-adic valuation* $v_p(x)$ is the largest integer n such that p^n divides x. Many useful colorings come from p-adic valuations and studying the divisibility properties of an equation's coefficients. Indeed, this idea is used directly in the proof of Rado's theorem (Theorem 1.2.4). We will freely use the fact that $v_p(xy) = v_p(x) + v_p(y)$ for all integers x and y. In [**54**] the following result was shown:

LEMMA 3.5.2. *Suppose $\mathcal{E}$ is an equation of the form $ax + by + cz = 0$ with $v_p(a), v_p(b), v_p(c)$ all distinct for some prime p. Then $\mathcal{E}$ has degree of regularity at most 3.*

If the condition in Lemma 3.5.2 is strengthened to distinct p-adic valuations modulo 3, then we obtain an improved bound on the degree of regularity.

EXAMPLE 3.5.1. Let $\mathcal{E}$ denote the equation $x + 2y = 4z$. Consider the 3-coloring $\chi(n) = v_2(n) \pmod 3$, so that, for example, $\chi(2) = \chi(2^4) = \chi(2^7) = \cdots = 1$. If (x, y, z) is a solution to $\mathcal{E}$ and $\chi(x) = \chi(y) = \chi(z)$, then $v_2(x), v_2(2y)$, and $v_2(4z)$ are all distinct since these values are all different modulo 3. Let $\alpha = \min\{v_2(x), v_2(2y), v_2(4z)\}$. Then reducing each side of $\mathcal{E}$ modulo $p^{\alpha+1}$ gives a contradiction. Therefore χ induces no monochromatic solutions to $\mathcal{E}$.

The following lemma generalizes the proof in the example above.

LEMMA 3.5.3. *Let $\mathcal{E}$ be the equation $\sum_{i=1}^{m} a_i x_i = 0$. If there is a prime p for which $v_p(a_i) \not\equiv v_p(a_j) \pmod k$ for all $i \neq j$, then $\mathcal{E}$ is not k-regular.*

PROOF. Define a k-coloring $\chi(n) := v_p(n) \pmod k$. Suppose $(x_1, \ldots, x_m)$ is a monochromatic solution to $\mathcal{E}$. Then $v_p(a_i x_i) \neq v_p(a_j x_j)$ for $i \neq j$ since these values are distinct modulo k. Let $\alpha = \min_{i=1}^{m}\{v_p(a_i x_i)\}$. Then $\sum_{i=1}^{m} a_i x_i \not\equiv 0 \pmod{p^{\alpha+1}}$, so $\sum_{i=1}^{m} a_i x_i \neq 0$, a contradiction. $\square$

The following two results are similar to Lemma 5 and Lemma 6 in [**54**]. Here we show that under additional assumptions on $v_p(a + b)$ and $v_p(b + c)$, respectively, it follows that the degrees of regularity of certain equations are at most six, which is stronger than the corresponding best bounds in [**54**]. We also show that another hypothesis on the order of a particular group element further improves this upper bound to four.

Lemma 3.5.4. *Let $\mathcal{E}$ denote the equation $ax + by + cz = 0$. If $\mathcal{E}$ is not regular and $0 = v_p(a) = v_p(b) = v_p(a+b) < v_p(c) =: r$, then $dor(\mathcal{E}) < 6$. If additionally the element $-ab^{-1}$ in the multiplicative group $G = \mathbb{Z}_{p^r}^{\times}$ has even order, then $dor(\mathcal{E}) < 4$.*

Proof. Since $v_p(a) = v_p(b) = 0$, let $g := -ab^{-1} \in G$. Since $a + b \not\equiv 0 \pmod{p^r}$, it follows that g is not the identity element of G. Let Γ denote the graph with vertex set $\{1, \dots, p^r - 1\}$ and edges (x, y) if $x \equiv gy \pmod{p^r}$ (see Figure 3.3 for an example). Since $v_p(a+b) = 0$, it follows that $-ab^{-1} \not\equiv 1 \pmod{p}$, so Γ is loopless. Then Γ is a union of disjoint cycles, and each cycle has size $\frac{ord(g)}{p^i}$ for some i. If $ord(g)$ is even, then all of the cycles in Γ have even length, and so Γ is 2-colorable (note the conditions $0 = v_p(a) = v_p(b) = v_p(a+b)$ imply $p \neq 2$). Otherwise, Γ is 3-colorable since each vertex has degree at most 2. Let C_1 be a proper vertex coloring of Γ that uses the fewest number of colors. We will construct a (4- or 6-) coloring C to show that $\mathcal{E}$ is not 4- or 6- regular and conclude $dor(\mathcal{E}) < 4$ or $dor(\mathcal{E}) < 6$, respectively.

Let $q := p^{2r}$. For all $n \in \mathbb{N}$, write $n = q^{\alpha} n'$ with $n' \not\equiv 0 \pmod{q}$. Define the coloring C_2 to be

$$C_2(n) = \begin{cases} 1 & \text{if } n' \not\equiv 0 \pmod{p^r}, \\ 2 & \text{if } n' \equiv 0 \pmod{p^r}. \end{cases}$$

Let C be the product coloring

$$C(n) = \begin{cases} (C_1(n'), 1) & \text{if } C_2(n) = 1, \\ (C_1(n'/p^r), 2) & \text{if } C_2(n) = 2. \end{cases}$$

We claim that C is a coloring with no monochromatic solutions to $\mathcal{E}$.

Suppose (x, y, z) is a monochromatic solution to $\mathcal{E}$. Write $x = q^{\alpha} x', y = q^{\beta} y', z = q^{\gamma} z'$ with $x', y', z' \nmid q$ and $ax + by + cz = 0$. Without loss of generality, we may assume that at least one of α, β, and γ is zero, and in each case we will show a contradiction.

Suppose first that $C_2(x) = C_2(y) = C_2(z) = 1$.

Case 1: Suppose $\alpha = 0$. Then if $\beta > 0$, then we may reduce $\mathcal{E}$ modulo p^r to obtain a contradiction since $b, c \equiv 0 \pmod{p^r}$, but $ax \not\equiv 0 \pmod{p^r}$. So we may assume $\beta = 0$. Now suppose $ax + by \equiv 0 \pmod{p^r}$. Then $y \equiv gx \pmod{p^r}$. But this is impossible since x and y would have different colors by the coloring C_1 (recall that Γ is loopless). Therefore $ax+by \not\equiv 0 \pmod{p^r}$, and so $ax+by+cz \not\equiv 0 \pmod{p^r}$, a contradiction.

Case 2: The case $\beta = 0$ is similar to the case $\alpha = 0$.

Case 3: Suppose $\gamma = 0$ and $\alpha, \beta > 0$. Then $ax + by \equiv 0 \pmod{q}$, but $cz \not\equiv 0 \pmod{q}$, and so $ax + by + cz \not\equiv 0 \pmod{q}$, a contradiction. Therefore $\alpha = 0$ or $\beta = 0$, and the proof follows from one of the previous cases.

If $C_2(x) = C_2(y) = C_2(z) = 2$, then in all cases we may divide x', y', and z' by p^r, and the proof follows similarly.

$\square$

LEMMA 3.5.5. *Let $\mathcal{E}$ denote the equation $ax + by + cz = 0$. If $\mathcal{E}$ is not regular and $0 = v_p(a) < v_p(b) = v_p(c) = v_p(b + c) =: r$ for some prime p, then $dor(\mathcal{E}) < 6$. Write $b = p^r b'$ and $c = p^r c'$. If additionally the element $g := -b'c'^{-1}$ in the multiplicative group $G = \mathbb{Z}_{p^s}^{\times}$ has even order, then $dor(\mathcal{E}) < 4$.*

PROOF. Since $v_p(b + c) = r$ and $v_p(b') = v_p(c') = 0$, $g \in G$ and g is not the identity element of G. Let Γ denote the graph with vertex set $\{1, \ldots, p^r - 1\}$ and edges (x, y) if $x \equiv gy \pmod{p^r}$. Then Γ is a union of disjoint cycles, and each cycle has size $\frac{ord(g)}{p^i}$ for some i. Note that Γ is not loopless since $v_p(b+c) = r$. If g has even order, then Γ is 2-colorable, and otherwise Γ is 3-colorable. Let C_1 be a proper vertex coloring of Γ that uses the fewest number of colors. We will construct a (4- or 6-) coloring C to show that $\mathcal{E}$ is not 4- or 6- regular and conclude $dor(\mathcal{E}) < 4$ or $dor(\mathcal{E}) < 6$, respectively.

Let $q := p^{2r}$, and for all $n \in \mathbb{N}$, write $n = q^{\alpha} n'$ with $n' \not\equiv 0 \pmod{p^{2r}}$. Define the coloring C_2 to be

$$C_2(n) = \begin{cases} 1 & \text{if } n' \not\equiv 0 \pmod{p^r}, \\ 2 & \text{if } n' \equiv 0 \pmod{p^r}. \end{cases}$$

Let C be the product coloring

$$C(n) = \begin{cases} (C_1(n'), 1) & \text{if } C_2(n) = 1, \\ (C_1(n'/p^r), 2) & \text{if } C_2(n) = 2. \end{cases}$$

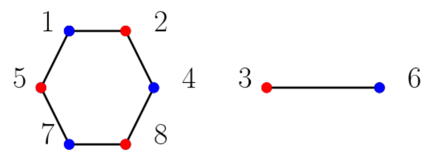


FIGURE 3.3. 2-coloring of graph with vertex set [8] and edges (x, y) if $x \equiv 2y \pmod 9$.

Suppose (x, y, z) is a monochromatic solution to $\mathcal{E}$ with respect to C. Write $x = q^\alpha x', y = q^\beta y', z = q^\gamma z'$ with $x', y', z' \nmid q$ and $ax + by + cz = 0$. Without loss of generality, we may assume that at least one of α, β, and γ is zero, and in each case we will show a contradiction.

Suppose $C_2(x') = C_2(y') = C_2(z') = 1$. Case 1: Suppose $\alpha = 0$. Then $ax + by + cz \not\equiv 0 \pmod{p^r}$. Case 2: Suppose $\alpha > 0$ and $\beta = 0$. Then if $\gamma > 0$, then $ax + by + cz \not\equiv 0 \pmod q$, so assume $\gamma = 0$. If $by + cz \equiv 0 \pmod q$, then $z = z' \equiv -b'c'^{-1}y' \pmod{p^r}$. But by coloring C_1 it follows that z and y have different colors, a contradiction. The case $\gamma = 0$ is similar.

Suppose $C_2(x') = C_2(y') = C_2(z') = 2$. If $\alpha = 0$, then $ax + by + cz \not\equiv 0 \pmod q$. The cases $\beta = 0, \gamma = 0$ are similar to above. □

3.6. Proofs of Theorem 3.6.1 and Results on Degree of Regularity

We begin this section with a proof of Theorem 3.6.1. Most of the work was carried out by computer, though we use Lemma 3.5.1 when the Rado numbers are infinite.

PROOF OF THEOREM 3.6.1. For each finite number $R_k(ax+by = cz)$, we produced a k-coloring of $[R_k(ax + by = cz) - 1]$ that contained no monochromatic solutions to $ax + by = cz$ and verified using a SAT solver that the formula $F^k_{R_k(ax+by=cz)}(ax+by = cz)$ from the encoding in Section 3.2 is unsatisfiable. For the remaining cases, we concluded $R_3(ax+by = cz) = \infty$ using Lemma 3.5.1. □

We are now able to prove Theorem 3.6.2.

PROOF OF THEOREM 3.6.2. Let $\mathcal{E}$ denote the equation $ax + by = cz$. For each triple (a, b, c) that satisfies $1 \le a, b, c \le 5$ and $a \le b$, we performed the following calculations. If a nonempty subset of $\{a, b, -c\}$ sums to zero, then $dor(\mathcal{E}) = \infty$ by Theorem 1.2.4. If $v_p(a), v_p(b)$, and $v_p(c)$ are all distinct modulo 3 for some prime p, or if one of the inequalities $a+b \le \frac{a^2}{c}$ or $a+b \le \sqrt{ac}$ holds, then $dor(\mathcal{E}) = 2$ by Lemma 3.5.3, Lemma 3.5.1, and Theorem 1.2.5.

In all other cases we see that $dor(\mathcal{E}) \leq 3$ by either Theorem 3.6.1, Lemma 3.5.1, Lemma 3.5.4, or Lemma 3.5.5. The computation of a finite Rado number in Theorem 3.6.1 gives $dor(\mathcal{E}) = 3$. $\square$

Recall from Theorem 1.2.6 (ii) that Rado proved the following result on the family of equations $a(x + y) = bz$.

THEOREM 3.6.1 (Rado). *If $a/b \neq 2^k$ for all $k \in \mathbb{Z}$, then $dor(a(x + y) = bz) \leq 3$.*

The following lemma strengthens Rado's result to include the case when $a/b = 2^k$ for some integer k.

LEMMA 3.6.1. *$R_3(x + y = bz) = \infty$ for $b \geq 4$, and $R_3(a(x + y) = z) = \infty$ for $a \geq 2$. Moreover, $dor(a(x + y) = bz) \leq 3$ unless $a = b = 1$ or $a = 1, b = 2$.*

PROOF OF THEOREM 3.6.3. Let $a, b \in \mathbb{Z}^+$, and assume $gcd(a, b) = 1$. Denote by $\mathcal{E}_{a,b}$ the equation $a(x + y) = bz$. Theorem 1.2.4 says $dor(\mathcal{E}_{a,b}) = \infty$ if and only if $a = b = 1$ or $a = 1, b = 2$. For all other a and b, Theorem 1.2.5 and Theorem 1.2.6 give $2 \leq \mathcal{E}_{a,b} \leq 3$.

From Lemma 3.5.1 (i), we have $\mathcal{E}_{a,b}$ is not 3-regular if $2a \leq \frac{a^2}{2b}$, and from Lemma 3.5.1 (ii), it follows that $\mathcal{E}_{a,b}$ is not 3-regular if $2a \leq \sqrt{ab}$. After rearranging, we see that $a(x + y) = bz$ is not 3-regular if $a \geq 2b$ or $b \geq 4a$. Then in these cases we immediately have $dor(\mathcal{E}_{a,b}) = 2$. For the remaining cases, the Rado number calculations from Theorem 3.6.1 (see Tables 3.2 and 3.9) give $dor(\mathcal{E}_{a,b}) = 2$. $\square$

We now prove Theorem 3.6.4 and Corollary 3.6.2

PROOF OF THEOREM 3.6.4 AND COROLLARY 3.6.2. The proof of Theorem 3.6.4 is immediate from Lemma 3.5.1 condition (ii) since $S = m - 1 \leq \lceil (m - 1)^{\frac{k-1}{k-2}} \rceil^{\frac{k-2}{k-1}}$. Corollary 3.6.2 follows from Theorem 1.2.5 and setting $k = 3$ in Theorem 3.6.4. $\square$

3.7. Rado CNF File Generation

For the remainder of this chapter, we discuss some of the computational details from our calculations. The steps for computing a Rado number for a given linear equation $\mathcal{E}$ are the following:

- Generate the CNF file encoding $F_n^k(\mathcal{E})$, including symmetry-breaking clauses if desired.
- Determine the satisfiability of $F_n^k(\mathcal{E})$ with a SAT solver.

- Adjust n to find the smallest n for which $F_n^k(\mathcal{E})$ is unsatisfiable.

In the following subsections, we explain how to achieve each step of the computation procedure.

3.7.1. Generating CNF Files. The first step in computing a Rado number is writing a formula $F_n^k(\mathcal{E})$ to a file in DIMACS `.cnf` format (see [**21**], Chapter 2). For many Rado numbers, especially those with 3 or fewer colors, this step took far longer than the actual SAT solving. This is in contrast to the paper [**71**], where the solutions to the single equation $x+y=z$ in [161] can be generated easily, but the solving process is extremely difficult. Our work involved computing with thousands of easier instances and larger values of n. The main bottleneck is negative clause generation since it requires enumerating all solutions to $\mathcal{E}$ in $[n]$, and there are $O(n^2)$ solutions for the three-variable case.

For efficient solution generation to homogeneous linear equations, we used the built-in function `isolve` in Maple to parameterize the solutions. We then used SymPy to parse the output of Maple and generate the solutions with values in $[1,n]$.

Example 3.7.1. If we want to generate all integer solutions in the interval $[1,1000]$ for the equation $43x-5y=13z$, we can feed $43x-5y=13z$ into Maple's `isolve` function, which gives the output

$$\{x=i, y=6i-13j, z=i+5j\}.$$

Since we want all integer solutions within $[1,1000]$, we can loop i from 1 to 1000 and manipulate the inequality

$$1 \leq y = 6i-13j \leq 1000$$

into an inner loop where j is looped from $\lceil \frac{1000-6i}{-13} \rceil$ to $\lfloor \frac{1-6i}{-13} \rfloor$. For z, we can simply check whether $1 \leq i+5j \leq 1000$ is satisfied or not inside the loops to determine if (x,y,z) is a solution.

Some of the formulas in our computations contain millions of clauses, and writing these clauses to a file is a time-consuming part of CNF file generation. For example, the CNF file which encodes $F_{16397}^3(5x+5y=19z)$ contains more than 20 million clauses, of which only 65588 are positive or optional.

The algorithm to generate all the positive and optional clauses is done though PYTHON. Since the parameterization of the equation $\mathcal{E}$ is also passed to PYTHON, we also used PYTHON to generate the CNF files.

3.7.2. Symmetry Breaking. Symmetry breaking is a SAT solving technique that can lead to significant speedups by preventing the solver from looking in isomorphic areas of the search space. In our case, permuting colors in a coloring has no effect on whether it avoids monochromatic solutions to a given equation. Therefore, it can be helpful to add additional clauses to our formula to forbid color permutations.

We can break this symmetry in the formula $F_4^3(x + y = z)$, for example, by adding the clauses (v_1^1) and (v_2^2). These clauses force number 1 to be red and number 2 to be blue. In general, if $\mathcal{E}$ is a linear homogeneous equation in three variables and we have a solution (x, y, z) where two of x, y, and z are equal to each other, then we can add clauses that force the two equal variables to be the first color and the remaining variable to be the second color. For Rado numbers $R_k(\mathcal{E})$ with $k > 3$, we can also add clauses that break the symmetries on the other colors (see [**71**]).

Generating a larger set of symmetry breaking clauses with more sophisticated preprocessing is more difficult and requires far more time than normal file generation. We included only a simple preprocessing step in our solving process. The benefit of this preprocessing becomes more apparent when n and k grow. Without symmetry breaking, SATCH takes nearly 15 minutes to determine that $F_{45}^4(x+y = z)$ is unsatisfiable, but only a few seconds after adding symmetry breaking clauses.

3.7.3. SAT Solvers. Most of the SAT solving computations were done with the solver SATCH v0.4.17, developed by Biere [**19**]. We initially used SATCH because it is remarkably fast at proving upper bounds for many 3-color Rado numbers and relatively easy to use. For example, SATCH is able to prove the upper bounds for the values in the first column and last row of Table 3.1 in under 10 seconds for each equation. Later, as we moved towards larger CNF files and more colors, SATCH started to struggle. We also experimented with CADICAL [**20**] and the multithreaded SAT solver GLUCOSE [**8**]. In general, SATCH performed better on smaller instances, but GLUCOSE solved larger instances up to two times as quickly.

3.7.4. Binary Search. In order to compute the exact value of a Rado number $R_k(\mathcal{E})$, we often must determine the satisfiability of $F_n^k(\mathcal{E})$ for many values of n. A convenient property of the formulas $F_n^k(\mathcal{E})$ is that if $m < n$, then we can obtain the formula $F_m^k(\mathcal{E})$ simply by deleting all the clauses that contain variables v_j^i with $j > m$. Therefore, once we have a formula $F_u^k(\mathcal{E})$ that is unsatisfiable, we have an upper bound $R_k(\mathcal{E}) \leq u$, and we no longer need to do any solution (negative clause) generation. After obtaining a lower bound $R_k(\mathcal{E}) > \ell$ with a satisfiable formula $F_\ell^k(\mathcal{E})$, we can compute the exact value of the Rado number $R_k(\mathcal{E})$ using binary search to jump between ℓ and u. Our initial guesses for suitable bounds on $R_k(\mathcal{E})$ were made largely through trial and error. However, even with the poor estimates $10 \leq R_3(x - y = bz) \leq 5000$ for $1 \leq b \leq 15$, it is possible to compute the exact values for all of these numbers in under two hours.

3.8. Additional Rado Number Calculations

In this section we give additional bounds and exact values for various Rado numbers.

3.8.1. 3-color Rado Numbers. Table 3.9 gives $R_3(ax+ay = bz)$ for $3 \leq a \leq 6$, $11 \leq b \leq 20$.

TABLE 3.9. $R_3(ax + ay = bz)$

b \ a	3	4	5	6
11	2019	847	1958	1188
12	$\underline{\infty}$	$\underline{54}$	2400	$\underline{1}$
13	∞	1710	3445	1963
14	∞	$\underline{455}$	3675	$\underline{336}$
15	$\underline{\infty}$	5408	$\underline{54}$	$\underline{105}$
16	∞	$\underline{\infty}$	5725	$\underline{432}$
17	∞	∞	8330	4743
18	$\underline{\infty}$	$\underline{\infty}$	12069	$\underline{54}$
19	∞	∞	16397	6726
20	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{1025}$

3.8.2. 4-color Rado Numbers. Table 3.10 gives some values for the 4-color Rado numbers $R_4(a(x-y)=bz)$. These numbers are considerably more difficult to compute than $R_3(a(x-y)=bz)$, and it took the solver CaDiCaL [**20**] up to 20 hours to prove some of the upper bounds. Notably, $R_4(x-y=(m-2)z)=m^4-m^3-m^2-m-1$ for $4 \le m \le 6$, which implies $S(4,4)=171$, $S(5,4)=469$, and $S(6,4)=1037$ by Lemma 3.3.3 and Proposition 3.1.2.

Table 3.10. $R_4(a(x-y)=bz)$

$b \backslash a$	1	2	3	4	5
1	45	56	81	256	625
2	171	$\underline{45}$	103	$\underline{56}$	
3	469	>225	$\underline{45}$		
4	1037				

3.9. Degree of Regularity Values

Tables 3.11 to 3.15 give the values of $dor(ax+by=cz)$ for all a,b,c with $1 \le a,b,c \le 5$.

Table 3.11. $dor(ax+by=z)$

$b \backslash a$	1	2	3	4	5
1	∞	∞	∞	∞	∞
2	∞	2	3	2	3
3	∞	3	2	2	2
4	∞	2	2	2	2
5	∞	3	2	2	2

Table 3.12. $dor(ax+by=2z)$

$b \backslash a$	1	2	3	4	5
1	∞	∞	3	2	3
2	∞	∞	∞	∞	∞
3	3	∞	3	2	3
4	2	∞	2	2	2
5	3	∞	3	2	2

TABLE 3.13. $dor(ax + by = 3z)$

$b \backslash a$	1	2	3	4	5
1	3	∞	∞	3	3
2	∞	3	∞	2	3
3	∞	∞	∞	∞	∞
4	3	2	∞	3	3
5	3	3	∞	3	3

TABLE 3.14. $dor(ax + by = 4z)$

$b \backslash a$	1	2	3	4	5
1	2	2	∞	∞	3
2	2	∞	2	∞	2
3	∞	2	3	∞	3
4	∞	∞	∞	∞	∞
5	3	2	3	∞	3

TABLE 3.15. $dor(ax + by = 5z)$

$b \backslash a$	1	2	3	4	5
1	2	3	3	∞	∞
2	3	3	∞	2	∞
3	3	∞	3	3	∞
4	∞	2	3	3	∞
5	∞	∞	∞	∞	∞

3.10. Heuristic Search Procedure in Proof of Theorem 3.4.1

Here we detail the method FINDPOLYNOMIALS which we used to find the sets S of polynomials in the proof of Theorem 3.4.1. We give the version of FINDPOLYNOMIALS used for the equation $a(x - y) = (a - 1)z$. The procedures for the other two equations were similar, but had minor modifications.

In brief, we initialize S to a set of polynomials S_0, and we use an auxiliary set of "gaps" G to add more polynomials to S. The procedure BOUNDEDINTEGERPOLYNOMIAL returns true if and only if all of its arguments are polynomials $p(a) \in \mathbb{Z}[a]$ that satisfy $1 \le p(a) \le a^3 + (a-1)^2$ for all $a \ge 16$. The FINDPOLYNOMIALS procedure is not guaranteed to produce a set of clauses that yields an unsatisfiable formula, and it took several attempts to come up with suitable choices for the initial sets S_0 and G_0. As an example, for the equation $a(x-y) = (a-1)z$, we set $S_0 = \{1, a-1, a, a+1, a^2-1, a^2, a^2+1, a^3, a^3+(a-1)^2\}$, $G_0 = \{1, a-1, a, (a-1)^2, a^2\}$, and $\text{maxIterations} = 3$. Files containing the polynomials and clauses used in our formulas can be found in [**30**].

Algorithm 1 FINDPOLYNOMIALS(S_0, G_0,maxIterations)

```
S ← S_0
G ← G_0
for i = 1 to maxIterations do
    for p, q in S do
        r ← (p−q)/(a−1)
        if BOUNDEDINTEGERPOLYNOMIAL(r) then
            G ← G ∪ {r}
        end if
    end for
    for p ∈ S, q ∈ G do
        r_+ ← p + (a − 1)q
        if BOUNDEDINTEGERPOLYNOMIAL(r_+) then
            S ← S ∪ {r_+}
        end if
        r_− ← p − (a − 1)q
        if BOUNDEDINTEGERPOLYNOMIAL(r_−) then
            S ← S ∪ {r_−}
        end if
    end for
end for
C ← ∅
for p, q ∈ S do
    x ← p
    y ← p − (a − 1)q
    z ← aq                         ▷ Here (x, y, z) is a solution to a(x − y) = (a − 1)z.
    if BOUNDEDINTEGERPOLYNOMIAL(x, y, z) then
        S ← S ∪ {x, y, z}
        C ← C ∪ {{x, y, z}}
    end if
end for
return S, C
```

CHAPTER 4

Diffsequences and Arithmetic Progressions Involving Fibonacci Numbers

The main goal of this chapter is to prove Theorems 4.2.1 and 4.2.2, which are stated in Section 1.5. Recall that a D-*diffsequence* is a sequence $x_1, \dots, x_\ell$ satisfying $x_{i+1} - x_i \in D$ for $1 \le i \le \ell - 1$ and that $\Delta(D, \ell; k)$ is the smallest n such that every k-coloring of $[n]$ contains a monochromatic ℓ-term D-diffsequence. Our proofs of these results involve combinatorial words that produce colorings that avoid either F-diffsequences or arithmetic progressions with gap in F of a certain length. The search for these words was aided by the Online Encyclopedia of Integer Sequences (OEIS) and the computational power of SAT solvers. We also use SAT solvers to compute other exact values of $\Delta(D, \ell; k)$ for other sets D.

This chapter is organized as follows. In Section 4.1, we formally define some of the objects studied in this paper and recall some well-known properties of Fibonacci numbers and combinatorial words. Section 4.2 contains the proofs of Theorems 4.2.1 and 4.2.2. We conclude in Section 4.3 with some experimental data.

4.1. Combinatorial Words and Fibonacci Numbers

In this section we collect several results that are used in the proofs of Theorems 4.2.1 and 4.2.2. We also fix the following notation for numbers and objects used throughout this chapter.

We let $F := \{1, 2, 3, 5, 8, \dots\}$ denote the set of Fibonacci numbers, and let $G := \{1, 4, 17, 72, \dots\} = \{\frac{f}{2} : f \in F\} \cap \mathbb{Z}$. We let f_n be the n-th term of the Fibonacci sequence, where $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n > 2$. Similarly, we set $g_n := \frac{f_{3n}}{2}$. We denote the Lucas numbers ℓ_n by $\ell_0 = 2, \ell_1 = 1$, $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \ge 2$. For any real number r, we denote the fractional part of r by $\{r\} := r - \lfloor r \rfloor$. We let ϕ denote the golden ratio $\phi := \frac{1+\sqrt{5}}{2}$. When a number $\Delta(D, \ell; k)$ does not exist, we write $\Delta(D, \ell; k) = \infty$, and similarly for $n(AP_D, \ell; k)$.

The following lemma consists of two well-known results that give exact formulas for the Fibonacci numbers f_n and Lucas numbers ℓ_n in terms of ϕ.

LEMMA 4.1.1. *The following identities for Fibonacci numbers f_n and Lucas numbers ℓ_n hold.*

(i) $f_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$ *for* $n \geq 1$.

(ii) $\ell_n = \phi^n + (-\phi)^{-n}$ *for* $n \geq 0$.

There are straightforward proofs of Lemma 4.1.1 using induction or generating functions (see [**131**], for example). An immediate consequence of Lemma 4.1.1 is the following identity.

COROLLARY 4.1.1. *For* $n \geq 2$, $\frac{f_n}{\phi} - f_{n-1} = (-1)^{n+1}\phi^{-n}$.

We recall several standard definitions on combinatorial words (see [**15**] or [**3**] for an introduction). A *word* is a (possibly infinite) sequence of symbols of a finite, nonempty alphabet Σ. Given two words w_1 and w_2, we write $w_1 w_2$ for the concatenation of w_1 and w_2. A *word morphism* is a map α from the set of words over an alphabet Σ to the set of words over an alphabet Σ' satisfying $\alpha(xy) = \alpha(x)\alpha(y)$ for all words x and y. Practically speaking, we need only specify a morphism on Σ, and the word $\alpha(x)$ is obtained by replacing each instance of σ in x by $\alpha(\sigma)$, for each $\sigma \in \Sigma$.

In this thesis we consider only words over the alphabet $\{0, 1\}$. Of particular interest is the n-th *Fibonacci word* F_n, which is given by

$$F_0 = 0, \quad F_1 = 01, \quad F_n = F_{n-1}F_{n-2} \text{ if } n \geq 2.$$

The *infinite Fibonacci word* is the limit $F_\infty = 010010100100\ldots$, the unique word that contains F_n as a prefix for all n. We use the infinite Fibonacci word to define two new words, S and T, which provide us colorings used in the proof of Theorems 4.2.1 and 4.2.2, respectively.

DEFINITION 4.1.1. *Let μ be the word morphism given by* $0 \mapsto 10$, $1 \mapsto 01$, *and let ν be the word morphism given by* $0 \mapsto 1$, $1 \mapsto 00$. *The words S and T are given by*

$$S := \mu(F_\infty) = 1001101001\ldots, \quad T := \nu(F_\infty) = 1001100100\ldots.$$

The morphism μ is known as the *Thue-Morse morphism*. The fixed point of μ (which is unique up to binary complement) is the famous Thue-Morse infinite word $0110100110010\ldots$, which has many interesting properties. One motivation for Thue's study of his eponymous word was *pattern avoidance*. Thue showed that it is *overlap-free*, meaning it does not contain any consecutive string of the form $0x0x0$ or $1x1x1$ (see [**15**] for a proof in English). In this light it is not surprising that μ should appear in the construction of words avoiding the patterns of diffsequences and arithmetic progressions.

The following lemma lists some key properties of the words F_∞, S, and T. In particular, it shows that F_∞ is *Sturmian*. Sturmian words are well-studied and have several useful properties (see [**3**], Chapter 9). In particular, the positions of the ones in a Sturmian word are given by terms in a *Beatty sequence*, a sequence of the from $a_n = \lfloor n\alpha \rfloor$ for some positive irrational α. This property allows us to determine the positions of ones in S and T as well. We refer the interested reader to [**14**], [**3**], and the references therein for additional results on Sturmian words.

LEMMA 4.1.2. *The following results on the words F_∞, S, and T hold.*

(i) The infinite Fibonacci word F_∞ satisfies

$$F_\infty(n) = \begin{cases} 0 & \textit{if } n = \lfloor m\phi \rfloor \textit{ for some integer } m, \\ 1 & \textit{otherwise.} \end{cases}$$

(ii) If $n \neq \lfloor m\phi \rfloor$ for all integers m, then there exist integers m' and m'' such that $n+1 = \lfloor m'\phi \rfloor$ and $n-1 = \lfloor m''\phi \rfloor$.

(iii) The word S satisfies

$$S(n) = \begin{cases} 0 & \textit{if } n \textit{ is even and } \frac{n}{2} = \lfloor m\phi \rfloor \textit{ for some integer } m, \\ 1 & \textit{if } n \textit{ is even and } \frac{n}{2} \neq \lfloor m\phi \rfloor \textit{ for all integers } m, \\ 0 & \textit{if } n \textit{ is odd and } \frac{n+1}{2} \neq \lfloor m\phi \rfloor \textit{ for all integers } m, \\ 1 & \textit{if } n \textit{ is odd and } \frac{n+1}{2} = \lfloor m\phi \rfloor \textit{ for some integer } m. \end{cases}$$

(iv) The word T satisfies

$$T(n) = \begin{cases} 1 & \textit{if } n = 2\lfloor m\phi \rfloor - m \textit{ for some integer } m, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. The statement (i) was proven in [**128**]. The statement (ii) follows easily from the fact that $1 < \phi < 2$, and (iii) follows from (i) and the definition of S. The proof of (iv) was originally given by Michel Dekking on the OEIS entry A287772 for νF_∞. For completeness, we give a slightly different proof here.

By (i), the positions of the zeros in F_∞ are given by the sequence $\lfloor m\phi \rfloor$, and by the definition of ν, every 1 in T is obtained from a 0 in F_∞. It therefore suffices to show that ν maps the 0 at position $\lfloor m\phi \rfloor$ to the 1 at position $2\lfloor m\phi \rfloor - m$. This is easy to verify for $m = 1$, and suppose it holds for $m = k$.

We consider two cases, noting that $\lfloor (k+1)\phi \rfloor - \lfloor k\phi \rfloor \in \{1, 2\}$ for all integers k. First, if $\lfloor (k+1)\phi \rfloor - \lfloor k\phi \rfloor = 1$, then the k-th and $(k+1)$-th zeros in F_∞ are adjacent, and so in T, the k-th and $(k+1)$-th ones are adjacent. By the induction hypothesis, the k-th one is in position $2\lfloor k\phi \rfloor - k$ and the $(k+1)$-th one is in position $2\lfloor k\phi \rfloor - k + 1 = 2\lfloor (k+1)\phi \rfloor - (k+1)$ as desired.

Now suppose $\lfloor (k+1)\phi \rfloor - \lfloor k\phi \rfloor = 2$. Then there is a one between the k-th and $(k+1)$-th zeros in F_∞. Therefore in T, by the definition of ν and the induction hypothesis, the $(k+1)$-th one is in position $2\lfloor k\phi \rfloor - k + 3 = 2\lfloor (k+1)\phi \rfloor - (k+1)$, which completes the proof. $\square$

4.2. Proofs of Theorems 4.2.1 and 4.2.2

In this section we prove our main results, Theorems 4.2.1 and 4.2.2. In each case we construct a coloring of $\mathbb{Z}^+$ and show by contradiction that it does not contain a suitable monochromatic diffsequence or arithmetic progression. Our main technique in the proofs of Theorems 4.2.1 and 4.2.2 is constructing a sequence of numbers $\{m_i\phi\}$ whose fractional parts are either strictly increasing or strictly decreasing. If there exist i and j such that $|\{m_i\phi\} - \{m_j\phi\}| \geq 1$, then this is a contradiction.

4.2.1. Proof of Theorem 4.2.1. To prove F is not 4-accessible, we must find a 4-coloring of $\mathbb{Z}^+$ with no k-term F-diffsequences for some positive integer k. Instead of working directly with such a 4-coloring, we will use a 2-coloring that avoids k-term G-diffsequences. The following lemma shows that the existence of this 2-coloring is enough to prove that F is not 4-accessible.

LEMMA 4.2.1. *Suppose G is not 2-accessible, i.e., $\Delta(G, k; 2) = \infty$ for some k. Then F is not 4-accessible.*

PROOF. Let $\chi : \mathbb{Z}^+ \to \{1, 2\}$ be a 2-coloring of $\mathbb{Z}^+$ that does not contain a monochromatic k-term G-diffsequence. Then define a 4-coloring $\chi' : \mathbb{Z}^+ \to \{c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}\}$ by

$$\chi'(n) = \begin{cases} c_{1,\chi(\frac{n+1}{2})} & n \text{ odd}, \\ c_{2,\chi(\frac{n}{2})} & n \text{ even}. \end{cases}$$

Now suppose towards contradiction that χ' contains a k-term monochromatic F-diffsequence $n_1, \ldots, n_k$. By the construction of χ', each term in the diffsequence has the same parity. Suppose first that $n_1, \ldots, n_k$ are all odd. Then $n_{i+1} - n_i$ is even for $1 \le i \le k - 1$. Moreover, observe that $\chi(\frac{n_1+1}{2}) = \cdots = \chi(\frac{n_k+1}{2})$. Therefore $\frac{n_{i+1}+1}{2} - \frac{n_i+1}{2} = \frac{n_{i+1}-n_i}{2} \in G$ for $1 \le i \le k - 1$, so $\frac{n_1+1}{2}, \ldots, \frac{n_k+1}{2}$ is a k-term G-diffsequence, a contradiction. If we assume instead that $n_1, \ldots, n_k$ are even, then we can reach a contradiction by a similar argument, which completes the proof. $\square$

The following result gives several bounds on differences of fractional parts, and it is central to the proof of Theorem 4.2.1.

LEMMA 4.2.2. *The following bounds hold.*

(i) Suppose $x_2 - x_1 = \frac{g_i}{2}$ with $g_i \in G$ even. If $x_1 = \lfloor m_1\phi \rfloor$ and $x_2 = \lfloor m_2\phi \rfloor$ for some integers m_1 and m_2, then

$$\{m_2\phi\} - \{m_1\phi\} < \frac{\phi^{-3i+1} - \phi}{4} < -0.38.$$

(ii) Suppose $x_2 - x_1 = \frac{g_i+1}{2}$ with $g_i \in G$ odd. If $x_1 = \lfloor m_1\phi \rfloor$ and $x_2 + 1 = \lfloor m_2\phi \rfloor$ for some integers m_1 and m_2, then

$$\{m_2\phi\} - \{m_1\phi\} < \frac{3\phi - 6}{4} < -0.28.$$

(iii) Suppose $x_2 - x_1 = \frac{g_i-1}{2}$ with $g_i \in G$ odd and $i > 1$. If $x_1 + 1 = \lfloor m_1\phi \rfloor$ and $x_2 = \lfloor m_2\phi \rfloor$ for some integers m_1 and m_2, then

$$\{m_2\phi\} - \{m_1\phi\} < \frac{-5\phi + 6}{4} < -0.52.$$

Proof. For (i), we have

$$m_2\phi - m_1\phi - \{m_2\phi\} + \{m_1\phi\} = x_2 - x_1 = \frac{g_i}{2} = \frac{f_{3i}}{4},$$

and after rearranging and applying Corollary 4.1.1, we have

$$m_2 - m_1 = \frac{f_{3i}}{4\phi} + \frac{\{m_2\phi\} - \{m_1\phi\}}{\phi} = \frac{f_{3i-1}}{4} - \frac{\phi^{-3i}}{4} + \frac{\{m_2\phi\} - \{m_1\phi\}}{\phi}.$$

Since g_i is even, it follows from the definition of g_i that i is even, so $f_{3i-1} \equiv 1 \pmod 4$. Observe also that $\left|\frac{\{m_2\phi\}-\{m_1\phi\}}{\phi}\right| < 1$. Because $m_2 - m_1$ is an integer, it follows that either $\{m_2\phi\} - \{m_1\phi\} = \phi(\frac{\phi^{-3i}-1}{4})$ or $\{m_2\phi\} - \{m_1\phi\} = \phi(1 - \frac{1}{4} + \frac{\phi^{-3i}}{4})$. However, the latter case is impossible since we must have $|\{m_2\phi\} - \{m_1\phi\}| < 1$, but $\phi(1 - \frac{1}{4} + \frac{\phi^{-3i}}{4}) > 1$ for all i. The minimum value of i is 2, so we have

$$\{m_2\phi\} - \{m_1\phi\} = \frac{\phi^{-3i+1} - \phi}{4} \leq \frac{\phi^{-5} - \phi}{4} < -0.38$$

for all i.

The proofs of the remaining parts are similar. For (ii), since g_i is odd, $g_i = \frac{f_{3i}}{2}$ with i odd. Then we have

$$m_2\phi - m_1\phi - \{m_2\phi\} + \{m_1\phi\} = x_2 + 1 - x_1 = \frac{g_i + 3}{2} = \frac{f_{3i}}{4} + \frac{3}{2}.$$

Rearranging and applying Corollary 4.1.1 gives

$$m_2 - m_1 = \frac{f_{3i-1}}{4} + \frac{\phi^{-3i}}{4} + \frac{3}{2\phi} + \frac{\{m_2\phi\} - \{m_1\phi\}}{\phi}.$$

Here, note that $f_{3i-1} \equiv 1 \pmod 4$ and $1 < \frac{1}{4} + \frac{3}{2\phi} < 2$. By a similar argument as above, it follows that either $\{m_2\phi\} - \{m_1\phi\} = -\phi\left(\frac{3}{2\phi} - \frac{3}{4} + \frac{\phi^{-3i}}{4}\right) = \frac{3\phi-6}{4} - \frac{\phi^{-3i+1}}{4}$ or $\{m_2\phi\} - \{m_1\phi\} = \phi\left(1 - \left(\frac{3}{2\phi} - \frac{3}{4} + \frac{\phi^{-3i}}{4}\right)\right) = \frac{7\phi-6}{4} - \frac{\phi^{-3i+1}}{4}$. Since $i \geq 1$, it follows that $\frac{7\phi-6}{4} - \frac{\phi^{-3i+1}}{4} > 1$ and the latter case is impossible. Therefore $\{m_2\phi\} - \{m_1\phi\} = \frac{3\phi-6}{4} - \frac{\phi^{-3i+1}}{4} < \frac{3\phi-6}{4} < -0.28$ for all i.

For (iii), we again have $g_i = \frac{f_{3i}}{2}$ with i odd. Then we have

$$m_2\phi - m_1\phi - \{m_2\phi\} + \{m_1\phi\} = x_2 - (x_1 + 1) = \frac{g_i - 3}{2} = \frac{f_{3i}}{4} - \frac{3}{2}.$$

Rearranging and applying Corollary 4.1.1 gives

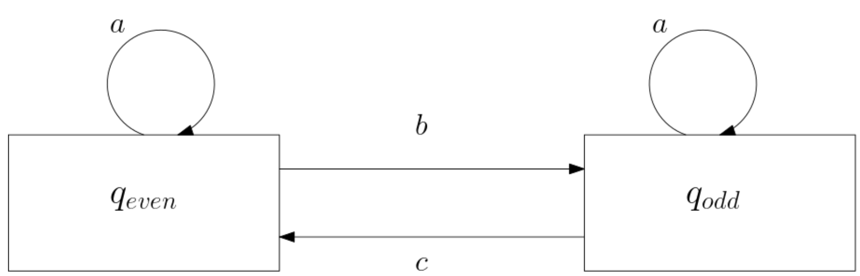
$$m_2 - m_1 = \frac{f_{3i-1}}{4} + \frac{\phi^{-3i}}{4} - \frac{3}{2\phi} + \frac{\{m_2\phi\} - \{m_1\phi\}}{\phi}.$$

Here we have $-1 < \frac{1}{4} - \frac{3}{2\phi} < 0$, so it follows that either $\{m_2\phi\} - \{m_1\phi\} = -\phi\left(\frac{1}{4} - \frac{3}{2\phi} + \frac{\phi^{-3i}}{4}\right) = \frac{6-\phi}{4} - \frac{\phi^{-3i+1}}{4} > 1$ since $i > 1$, or $\{m_2\phi\} - \{m_1\phi\} = \phi\left(-1 - \left(\frac{1}{4} - \frac{3}{2\phi} + \frac{\phi^{-3i}}{4}\right)\right) = \frac{6-5\phi}{4} - \frac{\phi^{-3i+1}}{4}$. The former case is impossible, so $\{m_2\phi\} - \{m_1\phi\} < \frac{6-5\phi}{4} < -0.52$ for all i. $\square$

We are now equipped to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. By Lemma 4.2.1, it is sufficient to find a 2-coloring of $\mathbb{Z}^+$ that contains no monochromatic 4-term G-diffsequences. We will show that the coloring $\chi(n) = S(n)$, where $S(n)$ is the n-th symbol in the word S, satisfies this property.

Throughout this proof, we assume that y_1, y_2, y_3, y_4 is a 4-term G-diffsequence, and we will show a contradiction. First, consider the following finite state machine.



Suppose $S(y_1) = S(y_2) = S(y_3) = S(y_4)$ with $y_{i+1} - y_i = g_{j_i}$ for some $g_{j_i} \in G$ and $i = 1, 2, 3$. For each diffsequence, we have a sequence of states q_1, q_2, q_3, q_4, and a sequence of transitions t_1, t_2, t_3. If y_i is even, then we set $q_i := q_{even}$, and if y_i is odd, we set $q_i := q_{odd}$. The transitions t_i are determined by the transition arrow that takes q_i to q_{i+1}, so that, for example, if $q_1 = q_{even}$ and $q_2 = q_{odd}$, then $t_1 = b$. For each y_i, set

$$x_i := \begin{cases} \frac{y_i}{2} & \text{if } y_i \text{ is even },\\ \frac{y_i+1}{2} & \text{if } y_i \text{ is odd }. \end{cases}$$

By Lemma 4.1.2, for each i there is a unique integer m_i that satisfies

$$x_i = \begin{cases} \lfloor m_i\phi \rfloor & \text{if } y_i \text{ is even and } S(y_i) = 0, \\ \lfloor m_i\phi \rfloor - 1 & \text{if } y_i \text{ is odd and } S(y_i) = 0, \\ \lfloor m_i\phi \rfloor + 1 & \text{if } y_i \text{ is even and } S(y_i) = 1, \\ \lfloor m_i\phi \rfloor & \text{if } y_i \text{ is odd and } S(y_i) = 1. \end{cases}$$

We are done if we show that $\{m_4\phi\} - \{m_1\phi\} < -1$, which is a contradiction. By Lemma 4.2.2, for all i we have that

$$\{m_{i+1}\phi\} - \{m_i\phi\} < \begin{cases} -0.38 & \text{if } t_i = a, \\ -0.28 & \text{if } t_i = b, \\ -0.52 & \text{if } t_i = c. \end{cases}$$

By examining all the combinations of values for t_1, t_2, t_3, we see that $\{m_4\phi\} - \{m_1\phi\} < -1$ unless $t_1 = t_2 = t_3 = b$. But this case is impossible since consecutive transitions cannot both be b, and the proof is complete.

□

4.2.2. Proof of Theorem 4.2.2. The proof of Theorem 4.2.2 is similar to the proof of Theorem 4.2.1. We will show that the 2-coloring induced by $T = \nu(F_\infty)$ has no monochromatic 5-term arithmetic progressions whose gaps are in F. The following lemma is a technical result which is essential for the computations in the proof of Theorem 4.2.2.

Lemma 4.2.3. *Let f_n be the n-th Fibonacci number, and suppose $\epsilon \in \{-4, 0, 4\}$ and $n \geq 13$. Then the following identities hold:*

(i)

$$\left\{ \frac{f_n + \epsilon}{2\phi - 1} \right\} = \frac{2(-\phi)^{-n}}{5} + c_{n,\epsilon},$$

where

$$c_{n,\epsilon} = \begin{cases} \frac{4+(2\sqrt{5}-5)\epsilon}{10} & \textit{if } n \equiv 0 \pmod 4, \\ \frac{4+(4\sqrt{5}-5)\epsilon}{20} & \textit{if } n \equiv 1 \pmod 4 \textit{ and } \epsilon \in \{0,4\}, \\ \frac{11-4\sqrt{5}}{5} & \textit{if } n \equiv 1 \pmod 4 \textit{ and } \epsilon = -4, \\ \frac{6+(2\sqrt{5}-5)\epsilon}{10} & \textit{if } n \equiv 2 \pmod 4, \\ \frac{8+(2\sqrt{5}-5)\epsilon}{10} & \textit{if } n \equiv 3 \pmod 4 \textit{ and } \epsilon \in \{0,4\}, \\ \frac{9-4\sqrt{5}}{5} & \textit{if } n \equiv 3 \pmod 4 \textit{ and } \epsilon = -4. \end{cases}$$

(ii) $\left\lfloor \frac{f_n+\epsilon}{2\phi-1} \right\rfloor$ *is even if and only if one of the following cases holds:*

- $\epsilon = 0$ *and* $n \equiv 0,1,2,3,5,10 \pmod{12}$,
- $\epsilon = 4$ *and* $n \equiv 0,2,3,9,10 \pmod{12}$,
- $\epsilon = -4$ *and* $n \equiv 0,1,2,5,7,10,11 \pmod{12}$.

Proof. By Lemma 4.1.1, we have

$$\frac{f_n+\epsilon}{2\phi-1} = \frac{f_n+\epsilon}{\sqrt{5}} = \frac{\phi^n - (-\phi)^{-n}}{5} + \frac{\epsilon}{\sqrt{5}} = \frac{\ell_n}{5} - \frac{2(-\phi)^{-n}}{5} + \frac{\epsilon}{\sqrt{5}}. \tag{4.1}$$

Note that the Lucas numbers ℓ_n are periodic modulo 5 with period 4, so $\frac{\ell_n}{5} = m + \frac{r}{5}$ for some $m \in \mathbb{Z}$ and $r \in \{1,2,3,4\}$, with r depending only on the value of n modulo 4. Therefore, if n is sufficiently large, then $\left\{\frac{f_n+\epsilon}{2\phi-1}\right\} = -\frac{2(-\phi)^{-n}}{5} + \left\{\frac{r}{5} + \frac{\epsilon}{\sqrt{5}}\right\}$. Moreover, for $n \geq 13$, we have $\left|\frac{2(-\phi)^{-n}}{5}\right| < .001$, and some straightforward calculations give part (i).

For part (ii), we take the floor of both sides of Equation 4.1. If we write $\ell_n = 10m + r$ for $m \in \mathbb{Z}$ with $0 \leq r \leq 9$, we observe that

$$\left\lfloor \frac{f_n+\epsilon}{2\phi-1} \right\rfloor = \left\lfloor \frac{10m+r}{5} - \frac{2(-\phi)^{-n}}{5} + \frac{\epsilon}{\sqrt{5}} \right\rfloor = 2k + \left\lfloor \frac{r}{5} - \frac{2(-\phi)^{-n}}{5} + \frac{\epsilon}{\sqrt{5}} \right\rfloor,$$

and we see that the parity of $\left\lfloor \frac{f_n+\epsilon}{2\phi-1} \right\rfloor$ is dependent only on ϵ and the value of r. (The term $\frac{2(-\phi)^{-n}}{5}$ is negligible since $n \geq 13$ and the Lucas numbers are nonzero modulo 5.) The Lucas numbers are periodic modulo 10 with period 12, and a straightforward check of all the possible values of ϵ and $n \pmod{12}$ gives the result. □

We now prove Theorem 4.2.2.

Proof of Theorem 4.2.2. Suppose towards contradiction that T contains a 5-term arithmetic progression x_1, x_2, x_3, x_4, x_5 with common difference $f_n \in F$ and $T(x_1) = \cdots = T(x_5)$. We first consider the case where $T(x_i) = 1$ for all i. By Lemma 4.1.2, for all i there exists a positive integer m_i such that $x_i = 2\lfloor m_i\phi \rfloor - m_i$.

Therefore for $i = 1, 2, 3, 4$ we have

$$m_{i+1} - m_i = 2(\lfloor m_{i+1}\phi \rfloor - \lfloor m_i\phi \rfloor) + x_i - x_{i+1} = 2(\lfloor m_{i+1}\phi \rfloor - \lfloor m_i\phi \rfloor) - f_n,$$

so $m_{i+1} - m_i + f$ is even, hence $m_{i+1} - m_i$ and f_n have the same parity. After simplifying further, we obtain

$$m_{i+1} - m_i = 2(m_{i+1}\phi - \{m_{i+1}\phi\} - m_i\phi + \{m_i\phi\}) - f_n,$$

which implies

$$m_{i+1} - m_i = \frac{f_n}{2\phi - 1} + \frac{2}{2\phi - 1}(\{m_{i+1}\phi\} - \{m_i\phi\}).$$

Since $\left|\frac{2}{2\phi-1}(\{m_{i+1}\phi\} - \{m_i\phi\})\right| < 1$, it follows that either $m_{i+1} - m_i = \lfloor \frac{f_n}{2\phi-1} \rfloor$ or $m_{i+1} - m_i = \lceil \frac{f_n}{2\phi-1} \rceil$. By the parity argument above, $m_{i+1} - m_i$ is equal to the value in $\{\lfloor \frac{f_n}{2\phi-1} \rfloor, \lceil \frac{f_n}{2\phi-1} \rceil\}$ that has the same parity as f_n. Therefore we have

$$\{m_{i+1}\phi\} - \{m_i\phi\} = \begin{cases} \frac{2\phi-1}{2}\left(1 - \left\{\frac{f_n}{2\phi-1}\right\}\right) & \text{if } m_{i+1} - m_i = \lceil \frac{f_n}{2\phi-1} \rceil, \\ \frac{2\phi-1}{2}\left(-\left\{\frac{f_n}{2\phi-1}\right\}\right) & \text{if } m_{i+1} - m_i = \lfloor \frac{f_n}{2\phi-1} \rfloor. \end{cases} \tag{4.2}$$

The Fibonacci number f_n is even if and only if n is a multiple of 3, and so by using Equation 4.2 and Lemma 4.2.3, we can now calculate the differences $\{m_{i+1}\phi\} - \{m_i\phi\}$. Note that these differences are dependent only on f_n, so they are equal for all i. If the absolute value of these differences is at least $\frac{1}{4}$, then there are no 5-term arithmetic progressions with $T(x_i) = 1$ for all i. We give the values of $\{m_{i+1}\phi\} - \{m_i\phi\}$, rounded to three decimal places, of f_n below for $1 \leq n \leq 12$:

n	f_n	$\{m_{i+1}\phi\} - \{m_i\phi\}$
1,2	1	.618
3	2	-1
4	3	-.382
5	5	.854
6	8	.472
7	13	-.910
8	21	-.438
9	34	.889
10	55	.451
11	89	-.897
12	144	-.446

For $n \geq 13$, using Equation 4.2 and Lemma 4.2.3, we see that $\{m_{i+1}\phi\} - \{m_i\phi\}$ is approximately $\frac{-1}{\sqrt{5}}$, $\frac{2}{\sqrt{5}}$, $\frac{1}{\sqrt{5}}$, or $\frac{-2}{\sqrt{5}}$ when n is congruent to 0, 1, 2, or 3 modulo 4, respectively. We see that $|\{m_{i+1}\phi\} - \{m_i\phi\}| > \frac{1}{3}$ for all n, and so in fact there are not even any 4-term arithmetic progressions x_1, x_2, x_3, x_4 with gaps in F with $T(x_i) = 1$ for $i = 1, 2, 3, 4$.

We now move to the case $T(x_i) = 0$. First, observe that the string 11 never appears in the word F_∞, hence the string 000 never appears in T. Moreover, each 0 in T is adjacent to another 0. Consequently, if $T(x) = 0$, then either $T(x-2) = 1$ or $T(x+2) = 1$. For each x_i, choose $y_i \in \{x_i - 2, x_i + 2\}$ such that $T(y_i) = 1$. Therefore, if $x_{i+1} - x_i = f_n$ for all i, then $y_{i+1} - y_i \in \{f_n - 4, f_n, f_n + 4\}$ for all i. By Lemma 4.1.2, for all i there exists an m_i such that $y_i = 2\lfloor m_i\phi \rfloor - m_i$.

Our analysis is now similar as above. Notice again that $m_{i+1} - m_i$ and f_n must have the same parity since

$$m_{i+1} - m_i = 2(\lfloor m_{i+1}\phi \rfloor - \lfloor m_i\phi \rfloor) + y_i - y_{i+1} = 2(\lfloor m_{i+1}\phi \rfloor - \lfloor m_i\phi \rfloor) - (f_n + \epsilon_i),$$

and $\epsilon_i \in \{-4, 0, 4\}$.

Rearranging and using a similar argument as above, we have $m_{i+1}-m_i = \lfloor \frac{f+\epsilon_i}{2\phi-1} \rfloor$ or $m_{i+1}-m_i = \lceil \frac{f+\epsilon_i}{2\phi-1} \rceil$, where $\epsilon_i \in \{-4, 0, 4\}$. Therefore we have

$$\{m_{i+1}\phi\} - \{m_i\phi\} = \begin{cases} \frac{2\phi-1}{2}\left(1 - \left\{\frac{f+\epsilon_i}{2\phi-1}\right\}\right) & \text{if } m_{i+1} - m_i = \lceil \frac{f+\epsilon_i}{2\phi-1} \rceil, \\ \frac{2\phi-1}{2}\left(-\left\{\frac{f+\epsilon_i}{2\phi-1}\right\}\right) & \text{if } m_{i+1} - m_i = \lfloor \frac{f+\epsilon_i}{2\phi-1} \rfloor. \end{cases} \tag{4.3}$$

We again calculate the first several values of $\{m_{i+1}\phi\} - \{m_i\phi\}$ when $\epsilon = \pm 4$ below.

n	$f_n + 4$	$\{m_{i+1}\phi\} - \{m_i\phi\}$
1,2	5	.854
3	6	-.764
4	7	-.146
5	9	1.090
6	12	.708
7	17	-.674
8	25	-.202
9	38	-1.111
10	59	.687
11	93	-.661
12	148	-.210

n	$f_n - 4$	$\{m_{i+1}\phi\} - \{m_i\phi\}$
1,2	-3	.382
3	-2	1
4	-1	-.618
5	1	.618
6	4	.236
7	9	1.090
8	17	-.674
9	30	.652
10	51	.215
11	85	1.103
12	140	-.682

If $\epsilon_i = 4$, then by Lemma 4.2.3, for $n \geq 13$, the values of $\{m_{i+1}\phi\} - \{m_i\phi\}$ are approximately (within .001 of) one of the values

$$\frac{2(2-\sqrt{5})}{\sqrt{5}} \approx -.211, \frac{2(1-\sqrt{5})}{\sqrt{5}} \approx -1.106, \frac{2(3-\sqrt{5})}{\sqrt{5}} \approx .683, \frac{3-2\sqrt{5}}{\sqrt{5}} \approx -.658,$$

depending on whether $n \equiv 0, 1, 2,$ or 3 modulo 4, respectively. Similarly, if $\epsilon_i = -4$, then the values of $\{m_{i+1}\phi\} - \{m_i\phi\}$ are approximately (within .001 of) one of the values

$$\frac{-2(3-\sqrt{5})}{\sqrt{5}} \approx -.683, \frac{-3+2\sqrt{5}}{\sqrt{5}} \approx .658, \frac{-2(2-\sqrt{5})}{\sqrt{5}} \approx .211, \frac{-2(1-\sqrt{5})}{\sqrt{5}} \approx 1.106,$$

again depending on whether $n \equiv 0, 1, 2,$ or 3 modulo 4, respectively.

Let $d_i := \{m_{i+1}\phi\} - \{m_i\phi\}$. If $|d_i| \geq 1$ for any i, then we are done immediately. First, one can show that $d_i > 0$ if and only if $\epsilon_i = 0$ and $n \equiv 1, 2 \pmod 4$, or $\epsilon_i = 4$ and $n \equiv 2 \pmod 4$ or $n = 5$, or $\epsilon_i = -4$ and $n \not\equiv 0 \pmod 4$.

Note also that $|d_i| \geq 1$ when $\epsilon_i = 4$ and $n \equiv 1 \pmod 4$ and when $\epsilon_i = -4$ and $n \equiv 3 \pmod 4$. Therefore these two cases are impossible, and considering the remaining possibilities of $n \pmod 4$ and ϵ_i, we see that d_i always has the same sign regardless of ϵ_i.

Observe that if $\epsilon_i = \pm 4$, then $\epsilon_{i+1} = 0$ or $\epsilon_{i+1} = \mp 4$. Using the approximations for d_i for large n and considering all possible sequences of ϵ_i for $i = 1, 2, 3, 4$, we have $|d_1 + d_2 + d_3 + d_4| \geq 1$ in all cases (in fact, $|d_1 + d_2 + d_3| \geq 1$ unless $n = 4$), which concludes the proof.

□

4.3. Experimental Results and Further Questions

In this section we give some results on $\Delta(D, \ell; k)$ and $n(AP_D, \ell; k)$ for different sets D and discuss some open questions. Our primary method for computing these values is the SAT solver CaDiCaL [**20**].

For each number $\Delta(D, \ell; k)$ (or $n(AP_D, \ell; k)$, we construct formulas ϕ_n that are satisfiable if and only if $\Delta(D, \ell; k) > n$ (or $n(AP_D, \ell; k) > n$). These formulas are constructed in essentially the same way as our formulas for Rado numbers in Chapter 3. The set of variables in each ϕ_n is $\{v_i^c : 1 \leq i \leq n,\ 1 \leq c \leq r\}$. The variable v_i^c is assigned true if and only if integer i is colored color c. We again use positive, negative, and optional clauses. Positive and optional clauses are exactly the same as we saw in Section 3.2. The only small difference is that now negative clauses ensure that there are no monochromatic D-diffsequences (or arithmetic progressions with common difference in D). If $x_1, \ldots, x_k$ is a k-term D-diffsequence (or arithmetic progression with common difference in D), then we include a clause of the form

$$\bar{v}_{x_1}^c \vee \bar{v}_{x_2}^c \vee \cdots \vee \bar{v}_{x_k}^c$$

for all colors c and all k-term diffsequences (or arithmetic progressions with common difference in D) with $1 \leq x_1 \leq \cdots \leq x_k \leq n$.

The formula ϕ_n is the conjunction of all positive, negative, and optional clauses for the given parameters n, D, k, r.

We first consider the set of Lucas numbers $L = \{2, 1, 3, 4, 7, 11, \dots\}$ and $doa(L)$. It is easily shown that no Lucas number is a multiple of 5, so $\Delta(L, 2; 5) = \infty$ since coloring each integer its congruence class mod 5 avoids 2-term L-diffsequences. The following result gives a slight improvement and shows $doa(L) \leq 3$.

Proposition 4.3.1. *Let L be the set of Lucas numbers. Then $\Delta(L, 3; 4) = \infty$. In particular, L is not 4-accessible and $doa(L) \leq 3$.*

Proof. Define a coloring $\chi : \mathbb{Z}^+ \to [4]$ as follows:

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \pmod 8, \\ 2 & \text{if } n \equiv 2, 4 \pmod 8, \\ 3 & \text{if } n \equiv 3, 5 \pmod 8, \\ 4 & \text{if } n \equiv 0, 6 \pmod 8. \end{cases}$$

We suppose towards contradiction that x_1, x_2, x_3 is a 3-term L-diffsequence with $\chi(x_1) = \chi(x_2) = \chi(x_3)$. Observe that the Lucas numbers are periodic modulo 8 and congruent to

$$2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2, 1, \dots \pmod 8,$$

so no Lucas number is congruent to 0 or 6 modulo 8. Thus by the definition of χ, one can check that we must have $x_2 - x_1 \equiv 2 \pmod 8$. But then we must have that $x_3 - x_2$ is congruent to either 0 or 6 modulo 8, which is a contradiction. □

Table 4.1 gives some additional computed values of $\Delta(L, \ell; k)$.

We next study the set $P = \{2, 3, 5, 7, 10, 12, 17, 22, \dots\}$, the set of nonzero Perrin (or "skiponacci") numbers p_n, which are given by $p_1 = 3$, $p_2 = 0$, $p_3 = 2$, and $p_n = p_{n-2} + p_{n-3}$ for $n \geq 4$. Table 4.2 gives some values of $\Delta(P, \ell; k)$. The most difficult computation was the upper bound $\Delta(P, 3; 5) \leq 107$, which required over 5 hours using CaDiCaL. For this computation we produced

TABLE 4.1. Table of numbers $\Delta(L, \ell; k)$

$k \backslash \ell$	2	3	4	5	6	7
2	3	5	7	13	15	21
3	4	13	22	51		
4	5	∞				
5	∞					

TABLE 4.2. Table of numbers $\Delta(P, \ell; k)$

$k \backslash \ell$	2	3	4	5	6	7
2	5	9	13	19	23	31
3	7	17	28	43		
4	13	35	81			
5	18	107				
6	25					
7	> 5000					

a certificate of unsatisfiability for the upper bound; this used over 3.3GB of memory. In contrast, CaDiCaL gives the lower bound $\Delta(P, 3; 5) \geq 107$ in merely two seconds, outputting the 5-coloring of [106] shown in Table 4.3.

Color	Color class
1	{1, 5, 8, 12, 16, 23, 27, 31, 38, 42, 46, 53, 57, 64, 68, 72, 79, 83, 87, 94, 100}
2	{2, 13, 20, 24, 28, 35, 39, 43, 50, 54, 61, 65, 69, 76, 80, 84, 91, 95, 99, 102, 106}
3	{ 3, 4, 11, 15, 19, 26, 30, 34, 41, 45, 49, 56, 60, 67, 71, 75, 82, 86, 90, 97, 98}
4	{6, 7, 14, 18, 22, 29, 33, 37, 44, 48, 52, 55, 59, 63, 70, 74, 78, 85, 89, 93, 101, 105}
5	{9, 10, 17, 21, 25, 32, 36, 40, 47, 51, 58, 62, 66, 73, 77, 81, 88, 92, 96, 103, 104}

TABLE 4.3. 5-coloring of [106] avoiding 3-term P-diffsequences.

Figure 4.1 displays a 7-coloring of [5000] avoiding 2-term P-diffsequences. The precise numerical color classes are given in the Appendix.

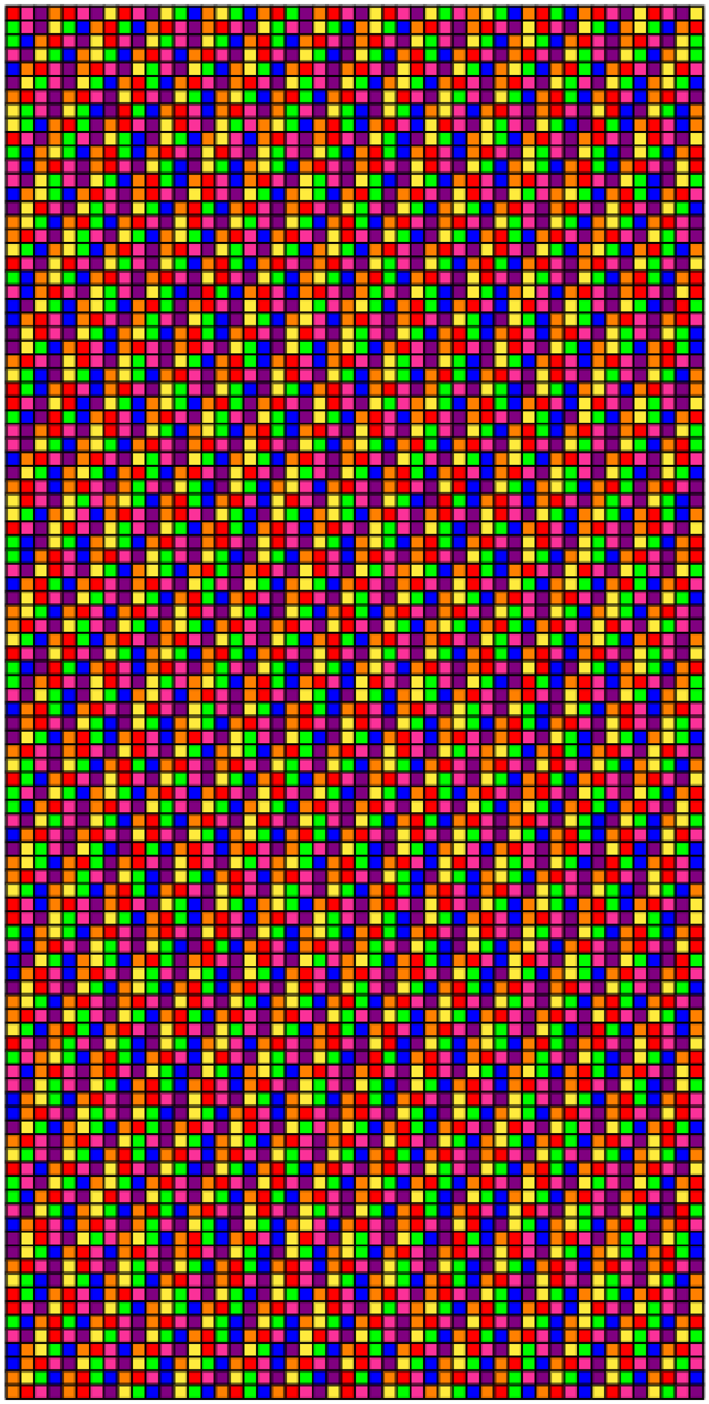


FIGURE 4.1. Read left to right, top to bottom: 7-coloring of [5000] avoiding 2 term P-diffsequences.

CHAPTER 5

Additional Ramsey-type Numbers and Research Directions

In this chapter we give more results that were obtained from our computational endeavors. This work is a further illustration of our SAT methods and shows their applications not just to arithmetic Ramsey theory, but a more diverse array of combinatorial problems.

5.1. Ramsey Numbers for Book and Wheel Graphs

Our computations using SAT solvers yielded new lower bounds for Ramsey numbers involving the book graphs B_n and wheel graphs W_n. The *book graph* B_n is the graph $K_2 + \overline{K_n}$ (here $+$ denotes the join operation). The numbers $R(B_m, B_n)$ are known for several families of m, n, but many cases are unknown (see [**114**]). The upper bounds $R(B_4, B_5) \leq 19$ and $R(B_3, B_6) \leq 19$ were established by Lidický and Pfender in [**97**] using semidefinite programming. The *wheel graph* is the graph $K_1 + C_{n-1}$. In [**97**] it is also shown that $R(W_5, W_7) \leq 16$, and the previously known best lower bound was $R(W_5, W_7) \geq 13$. Theorem 5.1.1 shows that the bounds for the book graphs are tight and improves the lower bound for $R(W_5, W_7)$ by two.

We used a simple SAT encoding to produce formulas $F_n(G_1, G_2)$ for graphs G_1, G_2 that are satisfiable if and only if $R(G_1, G_2) > n$. This encoding has a variable x_e for each edge of K_n which is set to true if and only if e is included in the graph.

Encoding C. *The Ramsey number $R(G_1, G_2)$ is at most n if the formula $F_n(G_1, G_2)$ is unsatisfiable, where $(V, E) = K_n$ and*

$$F_n(G_1, G_2) := \left(\bigwedge_{H \subset K_n, H \cong G_1} \left(\bigvee_{e \in E(H)} \bar{x}_e \right) \right) \wedge \left(\bigwedge_{H \subset K_n, H \cong G_2} \left(\bigvee_{e \in E(H)} x_e \right) \right).$$

Moreover, if $F_n(G_1, G_2)$ is satisfiable, then $R(G_1, G_2) \geq n + 1$.

Proof of Theorem 5.1.1. The following adjacency list describes a graph of order 18 that contains no B_4 or $\overline{B_5}$:

v	neighbors of v
1	2, 8, 9, 12, 14, 15, 17, 18
2	1, 5, 7, 10, 12, 14, 16, 17
3	4, 5, 7, 8, 9, 12, 13, 17
4	3, 6, 7, 10, 11, 12, 15, 16
5	2, 3, 7, 8, 9, 11, 16, 18
6	4, 8, 9, 11, 13, 14, 15, 16
7	2, 3, 4, 5, 13, 14, 15, 18
8	1, 3, 5, 6, 11, 14, 15, 17
9	1, 3, 5, 6, 12, 13, 16, 18
10	2, 4, 11, 12, 13, 16, 17, 18
11	4, 5, 6, 8, 10, 12, 14, 18
12	1, 2, 3, 4, 9, 10, 11, 14
13	3, 6, 7, 9, 10, 14, 17, 18
14	1, 2, 6, 7, 8, 11, 12, 13
15	1, 4, 6, 7, 8, 16, 17, 18
16	2, 4, 5, 6, 9, 10, 15, 17
17	1, 2, 3, 8, 10, 13, 15, 16
18	1, 5, 7, 9, 10, 11, 13, 15

The following adjacency list describes a graph of order 18 that contains no B_3 or $\overline{B}_6$:

v	neighbors of v
1	2, 3, 6, 7, 10, 11, 18
2	1, 6, 8, 9, 10, 12, 16
3	1, 7, 12, 14, 15, 16, 18
4	5, 6, 7, 10, 12, 13, 17
5	4, 6, 9, 13, 14, 16, 18
6	1, 2, 4, 5, 11, 12, 14
7	1, 3, 4, 8, 10, 14, 17
8	2, 7, 11, 13, 14, 16, 17
9	2, 5, 10, 14, 15, 17, 18
10	1, 2, 4, 7, 9, 13, 15
11	1, 6, 8, 13, 15, 17, 18
12	2, 3, 4, 6, 15, 16, 17
13	4, 5, 8, 10, 11, 15, 16
14	3, 5, 6, 7, 8, 9, 15
15	3, 9, 10, 11, 12, 13, 14
16	2, 3, 5, 8, 12, 13, 18
17	4, 7, 8, 9, 11, 12, 18
18	1, 3, 5, 9, 11, 16, 17

The following adjacency list describes a graph of order 14 that contains no W_5 or $\overline{W_7}$:

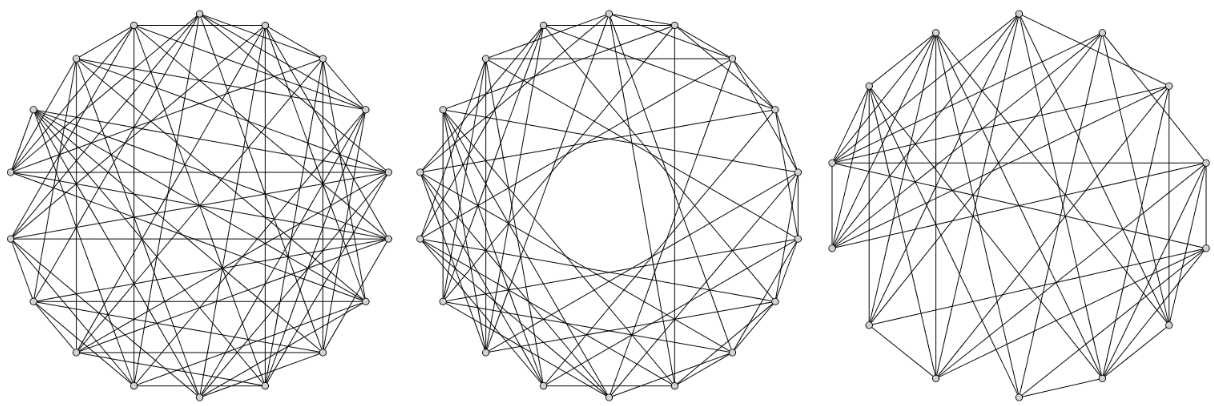


FIGURE 5.1. From left to right, a graph that contains no B_4 or $\overline{B_5}$, a graph that contains no B_3 or $\overline{B_6}$, a graph that contains no W_5 or $\overline{W_7}$.

v	neighbors of v
1	3, 6, 7, 9, 10, 11, 12
2	4, 6, 8, 10, 11, 12
3	1, 6, 7, 9, 11, 12
4	2, 5, 6, 7, 9, 11, 12
5	4, 7, 8, 9, 10, 12
6	1, 2, 3, 4, 7, 14
7	1, 3, 4, 5, 6, 8, 13, 14
8	2, 5, 7, 13, 14
9	1, 3, 4, 5, 10, 13, 14
10	1, 2, 5, 9, 13, 14
11	1, 2, 3, 4, 12, 13, 14
12	1, 2, 3, 4, 5, 11, 13, 14
13	7, 8, 9, 10, 11, 12, 14
14	6, 7, 8, 9, 10, 11, 12, 13

Figure 5.1 gives drawings of (isomorphic copies of) each of the three graphs. All three graphs were found using SATCH. SATCH and CADICAL were unable to find a graph on 15 vertices containing no copies of W_5 or $\overline{W_7}$ in over twelve days of computation time.

□

5.2. Results for Extremal Problems

Though this dissertation has focused on problems in Ramsey theory, we note that our SAT methods have application to other areas of combinatorics as well. This section largely consists of experimental data rather than theoretical results, but it may be helpful in future work. In particular, we are interested in extremal problems in graph theory and geometry.

Our main method is to use *cardinality constraints* in Boolean formulas to find graphs or point configurations that avoid certain properties. We use the notation $\leq_r (x_1, \ldots, x_n)$ to denote the constraint "at most r of the variables in $\{x_1, \ldots x_n\}$ are assigned to true." There are many ways to encode this constraint using Boolean variables (see [**133**]), but here we use the *sequential counter encoding* from [**125**].

Encoding D ([**125**]). *The following clauses encode the constraint* $\leq_r (x_1, \ldots, x_n)$.

$$\bigwedge_{k=1}^{r} \bigwedge_{j=k}^{n+k-r-2} \overline{e}_j^k \vee e_{j+1}^k,$$
$$\bigwedge_{k=0}^{r} \bigwedge_{j=k}^{n+k-r-1} \overline{e}_j^k \vee e_{j+1}^{k+1} \vee \overline{x}_{j+1}.$$

We used this encoding for its relative simplicity; other encodings may be faster for the problems we consider, but we did not explore them.

A fundamental result in extremal graph theory is *Turán's theorem*, which gives a tight bound on the number of edges in a graph on n vertices that does not contain a copy of K_r. More generally, we define the *Turán number* $ex(n, G)$ of a graph G as the largest number of edges in a graph on n vertices that does not contain G as a subgraph. In this language, Turán's theorem [**129**] says $ex(n, K_r) = \left(1 - \frac{1}{r-1} + o(1)\right) \frac{n^2}{2}$. Another famous result is the *Kővári-Sós-Turán theorem* , which says $ex(n, K_{s,t}) = O(n^{2-\frac{1}{t}})$ for complete bipartite graphs $K_{s,t}$ with $2 \leq s \leq t$. This bound is asymptotically tight when $t \leq 3$ (see [**66**]), but this is not known for $t \geq 4$. Many values of $ex(n, K_{2,2}) = ex(n, C_4)$ have been computed (see OEIS entry A006855 and the references there), but to our knowledge little has been done in computing $ex(n, K_{s,t})$ for higher values of s, t. We computed several of these values using the following encoding.

ENCODING E. *There is a graph of order n with E edges that does not contain a copy of G if and only if the following formula is satisfiable.*

$$F_{n,E}(G) := \left(\bigwedge_{H \subset K_n, H \cong G} \left(\bigvee_{e \in E(H)} \bar{x}_e \right) \right) \wedge \leq_E (x_e)_{e \in E(K_n)} \wedge \leq_{\binom{n}{2}-E} (\bar{x}_e)_{e \in E(K_n)},$$

where $\leq_r (x_1, \ldots, x_n)$ is the cardinality constraint described in Encoding D.

Here each edge has a variable x_e that is assigned true if and only if the edge is present in our graph. The first set of clauses forbids a copy of G, the second constraint forbids more than E edges, and the last constraint forbids more than $\binom{n}{2} - E$ nonedges. Using this encoding, we computed data for $ex(n, G)$ for $G \in \{K_{2,3}, K_{3,3}, K_{4,4}\}$.

TABLE 5.1. Table of values of $ex(n, G)$ for $G \in \{K_{2,3}, K_{3,3}, K_{4,4}\}$.

n	$ex(n, K_{2,3})$
5	7
6	10
7	12
8	16
9	19
10	22
11	25

n	$ex(n, K_{3,3})$
6	12
7	16
8	19
9	24
10	30
11	34

n	$ex(n, K_{4,4})$
8	24
9	30
10	35
11	≥ 41

Another natural modification of Ramsey problems is to look not for monochromatic subgraphs in edge colorings of K_n, but subgraphs with a given coloring pattern. This has been studied, for instance, in [**29, 37**]. A corollary of the main results of [**37**] is that for sufficiently large n, there exists a smallest number $e = \alpha(n, r)$ such that 2-coloring the edges of K_n with at least e red edges and e blue edges induces a monochromatic K_r with edges to some vertex the other color. In other words, if there are sufficiently many red and blue edges, either the red subgraph or the blue subgraph contains an induced $K_{1,r}$. We investigated the parameter $\alpha(n, r)$ using SAT solvers using an encoding similar to Encoding E. We give the results in Table 5.2. The results suggest that $\alpha(n, 3) = n + 1$ for $n \geq 10$, which agrees with the known Turán number $ex(n, K_{1,3}) = n + 1$ (see [**96**] for a generalization of this result). Notably, coloring the Paley graph of order 9 red and all other edges blue gives an equal number of edges of each color and avoids induced $K_{1,3}$. For $r = 4$, there is a critical graph with at least $\lfloor n/2 \rfloor$ edges of each color for up to $n = 20$.

Table 5.2. Table of values of $\alpha(n, 3)$ for small n.

n	$\alpha(n, 3)$
8	doesn't exist
9	doesn't exist
10	11
11	12
12	13
13	14

A *Sidon set* is a subset of an abelian group that contains no solution to the equation $x + y = z + w$ where $\{x, y\} \neq \{z, w\}$. Another Ramsey-type number is the *Sidon-Ramsey number* $SR(k)$ introduced in [**95**], which is the minimum n such that there is no k-coloring of $[n]$ where each color class is a Sidon set. The authors of [**50**] improved the asymptotics of $SR(n)$ and studied the numbers $sr(n_1, \ldots, n_d)$, the largest number of colors k such that there is no k-coloring of $\prod_{i=1}^{d}[n_i]$ where each color class (viewed as a subset of $\mathbb{Z}^d$) is a Sidon set. We computed several values of the $d = 2$ case using SAT solvers, and we give the values in Table 5.3.

Table 5.3. Table of numbers $sr(m, n)$.

m \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	1	1	2	2	2	2	2	3	3	3	3	3	3	4
3		1	2	2	2	3	3	3	3	4					
4			2	3	3	3	3	4							
5				3	3	4									
6					4										

The next problems we examine have a slightly different flavor than what we have seen. An old conjecture of Erdős [**47**] stated that any collection of more than 2^n points in $\mathbb{R}^n$ determines an obtuse angle, and this was proven by Danzer and Grünbaum in [**38**]. Harangi et al. went on to study sets $S \subseteq \mathbb{R}^n$ that do not contain three points making a given angle α and investigated how large the Hausdorff dimension of S can be. Bennett asked similar questions over finite fields rather than $\mathbb{R}^n$ and determined asymptotic bounds for sets of $\mathbb{F}_q^n$ that necessarily contain *i*) right angles and *ii*) right angles whose vertex is at the origin. In finite fields with q odd, a triple (x, y, z) with x, y, z distinct forms a right angle with vertex y if $2\langle x - y, z - y\rangle = 0$, an acute angle if $2\langle x - y, z - y\rangle \in Q$, where Q is the set of quadratic residues in $\mathbb{F}_q$, and an obtuse angle otherwise.

Omar recently generalized Bennett's result to acute and obtuse angles [**110**]. Our results here give some explicit values of maximal sizes of $\mathbb{F}_q^n$ that avoid right, acute, and obtuse angles.

Our most complete result is the case of right angles with vertex at the origin. We let $\rho_{n,q}^0$ denote the largest size of a subset of $\mathbb{F}_q^n$ that does not contain two distinct orthogonal vectors. We give a table of values in Table 5.4.

TABLE 5.4. Largest size of a subset of $\mathbb{F}_q^n$ that does not contain two orthogonal vectors.

$n \backslash q$	2	3	5	7	11	13	17	19	23	29
2	2	4	10	24	60	74	130	180	264	394
3	3	7	29							
4	4	11								
5	5	19								
6	6									
7	8									
8	9									

We have an exact formula for $\rho_{2,q}^0$ when $n = 2$.

PROPOSITION 5.2.1. *For q prime,* $\rho_{2,q}^0 = \begin{cases} \frac{q^2-1}{2} & q \equiv 3 \pmod 4, \\ \frac{q^2-1}{2} - q + 3 & q \equiv 1 \pmod 4. \end{cases}$

PROOF. Let L_q be the set of lines in $\mathbb{F}_q^2$ through the origin. One can check that $|L_q| = q + 1$ since there are q distinct lines spanned by elements of the form $(1, t)$, $t \in \mathbb{F}_q$ together with the line spanned by $(0, 1)$. The lines spanned by $(0, 1)$ and $(1, 0)$ are orthogonal, and then two lines $(1, t_1)$ and $(1, t_2)$ are orthogonal if and only if $t_1 t_2 = -1$.

Recall that -1 is a square in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod 4$. Therefore if $q \equiv 1 \pmod 4$, then there are $\frac{q-1}{2}$ pairs of orthogonal lines and 2 self-orthogonal lines spanned by $(1, t_1)$ and $(1, t_2)$, where $t_1^2 = t_2^2 = -1$. If $q \equiv 3 \pmod 4$, then there are $\frac{q+1}{2}$ pairs of orthogonal lines.

To show lower bounds for $\rho_{2,q}^0$, we exhibit sets S that do not contain any orthogonal points. If $q \equiv 3 \pmod 4$, then for each pair of orthogonal lines, we include the $q - 1$ nonzero points on one of those lines in S, for a total of $\frac{(q-1)(q+1)}{2} = \frac{q^2-1}{2}$. None of these points are orthogonal because if a point (a, b) lies on some line ℓ through the origin, then the only points orthogonal to (a, b) lie on the line ℓ' orthogonal to ℓ, and there are no points on ℓ' in S.

Similarly, for the case where $q \equiv 1 \pmod 4$, for every pair of orthogonal lines we include the $q-1$ nonzero points in S, and then from the self-orthogonal lines we include 1 point from each of them for a total of $\frac{(q-1)(q-1)}{2} + 2 = \frac{q^2-1}{2} - q + 3$.

To show the upper bound for the case $q \equiv 3 \pmod 4$, by the pigeonhole principle, if $|S| > \frac{q^2-1}{2}$, then there is one pair of orthogonal lines that contains at least q points in S. Since the origin can never be in S, by the pigeonhole principle again, there must be two points in S that are on orthogonal lines, so these points are orthogonal, a contradiction.

When $q \equiv 1 \pmod 4$, we note that there can be a maximum of one point on each self-orthogonal line, and the claim follows by a similar contradiction as before. □

We do the same for acute and obtuse angles with vertex at the origin. Let $\alpha^0_{n,q}$ (respectively $\omega^0_{n,q}$) denote largest size of subset of $\mathbb{F}_q^n$ that does not contain two distinct vectors that form an acute (respectively obtuse) angle. We give values of $\alpha^0_{n,q}$ and $\omega^0_{n,q}$ in Tables 5.5 and 5.6.

TABLE 5.5. Table of values of $\alpha^0_{n,q}$

$n \backslash q$	3	5	7	11	13	17	19	23
2	5	7	10	17	25	33	30	≥ 45
3	7	15	28					
4	11							
5	≥ 18							

TABLE 5.6. Table of values of $\omega^0_{n,q}$

$n \backslash q$	3	5	7	11	13	17
2	5	7	10	17	25	33
3	7	13				
4	11					

We can also do the same analysis when we allow the vertex to be any point. Let $(\rho_{n,q}\ \alpha_{n,q}, \omega_{n,q})$ denote the largest size of subset of $\mathbb{F}_q^n$ that does not contain a (right, acute, obtuse) angle. We give the values in Tables 5.7, 5.8, and 5.9.

5.3. Future Directions

We conclude by mentioning some open problems and directions for future research.

TABLE 5.7. Table of values of $\rho_{n,q}$

$n \backslash q$	3	5	7	11
2	3	5	7	11
3	5			

TABLE 5.8. Table of values of $\alpha_{n,q}$

$n \backslash q$	3	5	7
2	4	5	4
3	4		

TABLE 5.9. Table of values of $\omega_{n,q}$

$n \backslash q$	3	5	7
2	4	5	4
3	6		
4	< 10		

5.3.1. Problems on Nullstellensatz certificates. Theorem 2.3.3 gives upper bounds for polynomial encodings of Ramsey-type numbers in terms of Builder-Painter games. An obvious related question is to study the corresponding lower bounds for these certificates. One way of obtaining lower bounds is through *designs* of degree d, which are certain maps from the set polynomials of degree at most d to the base field K (for a precise definition, see [**26**]). The existence of a design of degree d is equivalent to the nonexistence of a Nullstellensatz certificate of degree at most d. It is plausible that the underlying combinatorial structure of Ramsey-type problems could be leveraged to construct designs.

Another observation that we have made is that the certificate degree bounds in terms of Builder-Painter games are not tight in general. In some sense this means that the Nullstellensatz is more powerful as a proof system than Builder-Painter games. If $d(r, s, n)$ is the smallest degree of a Nullstellensatz certificate for the polynomial system in Theorem 2.3.1 that has a solution if and only if $R(r, s) > n$, then we have

$$d(r, r, n) \leq \tilde{R}(r, r; n) - 1 \leq \binom{R(r, r)}{2}.$$

We know that these inequalities are strict already in the case $n = 6$, $r = 3$. We suspect they are strict in general, and it would be interesting to give a more precise description of how these numbers relate.

PROBLEM 5.3.1. *Investigate lower bounds and the tightness of upper bounds for the minimal degrees of Nullstellensatz certificates for the polynomial systems in Theorem 2.3.3.*

5.3.2. Problems on Rado Numbers. Perhaps one of the most interesting open questions mentioned already is Conjecture 1.2.1, *Rado's boundedness conjecture.* The work of Fox and Kleitman in [**54**] shows that 24 colors are sufficient to avoid solutions to nonregular three variable homogeneous linear equations, but our experimental data suggests that as few as 4 colors may be enough. An interesting problem would be to optimize this bound, and this may be an incremental step towards Rado's boundedness conjecture for $m \geq 4$ variable equations.

PROBLEM 5.3.2. *What is the smallest number of colors k required to avoid monochromatic solutions to all nonregular three variable linear homogeneous equations? Is $k = 24$ the best possible?*

Lemma 3.3.3 gives colorings that give lower bounds for the generalized Schur numbers $S(k, m)$. These bounds are tight for all m when $k \leq 3$. However, for $k = 4$, this fails for $m = 3$. The (ordinary) Schur number $S(4, 3) = R_4(x_1 + x_2 = x_3) = S(4) = 45$, but the bound given in Lemma 3.3.3 gives only $S(4, 3) \geq 41$. Theorem 3.6.1 shows that the bound becomes tight again for $S(4, 4), S(4, 5)$, and $S(4, 6)$, suggesting that this may hold for $m \geq 7$ as well. It seems plausible that for all k, the 3.3.3 holds for sufficiently large m, though this is difficult to verify experimentally for large k.

Our proof of $R_3(x + (m - 2)y = z) = m^3 - m^2 - m - 1$ in Theorem 3.4.1 used a set of 685 polynomials for the set S in Lemma 3.4.1, but the result in [**22**] showed $R_3(x_1 + x_2 + x_3 + (m - 4)x_4 = x_5) = m^3 - m^2 - m - 1$ using similar methods on a set of only 30 polynomials. Reducing the number of polynomials in these sets would give a better understanding on the reasons for when the bound from Lemma 3.3.3 is tight, and the size of these sets also gives an upper bound for the restricted online Rado numbers as in Corollary 3.4.1.

Moreover, it is not so clear what the best way to generate these sets S is. The procedure detailed in Section 3.10 required much experimentation, and it would be useful to have an algorithm that guarantees that a set of polynomials will be sufficient to prove a given upper bound for a specified Rado number.

PROBLEM 5.3.3. *Investigate methods to generate sets of polynomials S to use in Lemma 3.4.1. Specifically, study the sets used in proving bounds for $S(3,m)$. Determine when the best known lower bounds for $S(k,m)$ are tight.*

Theorem 3.6.3 gave the degree of regularity for $dor(\mathcal{E}_{a,b}) = dor(a(x+y) = bz)$ when $a \leq 5$ or $b \leq 2$. The essential part of the proof was using Lemma 3.5.1 to show that the interesting cases happen when $a < 2b$ and $b < 4a$. For *all* of these cases it turned out that $\mathcal{E}_{a,b}$ was 3-regular, and it is reasonable to conjecture the following.

CONJECTURE 5.3.1. *The equation $a(x+y) = bz$ is 3-regular if and only if $a < 2b$ and $b < 4a$.*

A similar result does *not* hold in general for the equations $ax + by = cz$. For example, the equation $x + 3y = 9z$ is not 3-regular by Lemma 3.5.3, but it also does not satisfy the inequalities in Lemma 3.5.1. However, it seems worthwhile to investigate when the bounds given by Lemma 3.5.1 are "tight" in the sense that they characterize *exactly* when an equation is k-regular. This might be a step towards generalizing Theorem 1.2.5 to characterize 3-regular equations, and it seems likely that both algebraic and modular conditions are a part of such a characterization.

5.3.3. Problems on Diffsequences. The proofs of Theorem 4.2.1 and Theorem 4.2.2 give new bounds on $doa(F)$ and $doa(AP_F)$ by showing, respectively, that $\Delta(F,4;4) = \infty$ and $n(AP_F,5;2) = \infty$, where F is the set of Fibonacci numbers. It is known from [**7**] that $\Delta(F,2;4) = 9$, and a SAT solver easily shows $n(AP_F,3;2) = 17$. However, we were unable to compute the values $\Delta(F,3;4)$ and $n(AP_F,4;2)$.

Using a greedy algorithm based on CaDiCaL's "lucky" heuristic, which is essentially greedy assignment combined with unit propagation, we were able to find a 2-coloring of [50000] that does not contain any 3-term G-diffsequences, which implies the bound $\Delta(G,3;2) > 50000$. The coloring used in the proof of Lemma 4.2.1 then gives the bound $\Delta(F,3;2) > 100000$. Figure 5.2 gives a 4-coloring of [5000] with no 3-term F-diffsequence. Moreover, with a SAT solver we were able to show $n(AP_F,4;2) > 8000$. We see there are large gaps between $\Delta(F,3;4)$ and $\Delta(F,4;4)$ as well as $n(AP_F, 3; 2)$ and $n(AP_F, 4; 2)$. Similarly, there is also large gap between $\Delta(P, 2; 6) = 25$ and $\Delta(P, 2; 7)$, which is at least 5000. We feel there is sufficient evidence to make the following conjecture.

CONJECTURE 5.3.2. $\Delta(F, 3; 4) = n(AP_F, 4; 2) = \Delta(P, 2; 7) = \infty$.

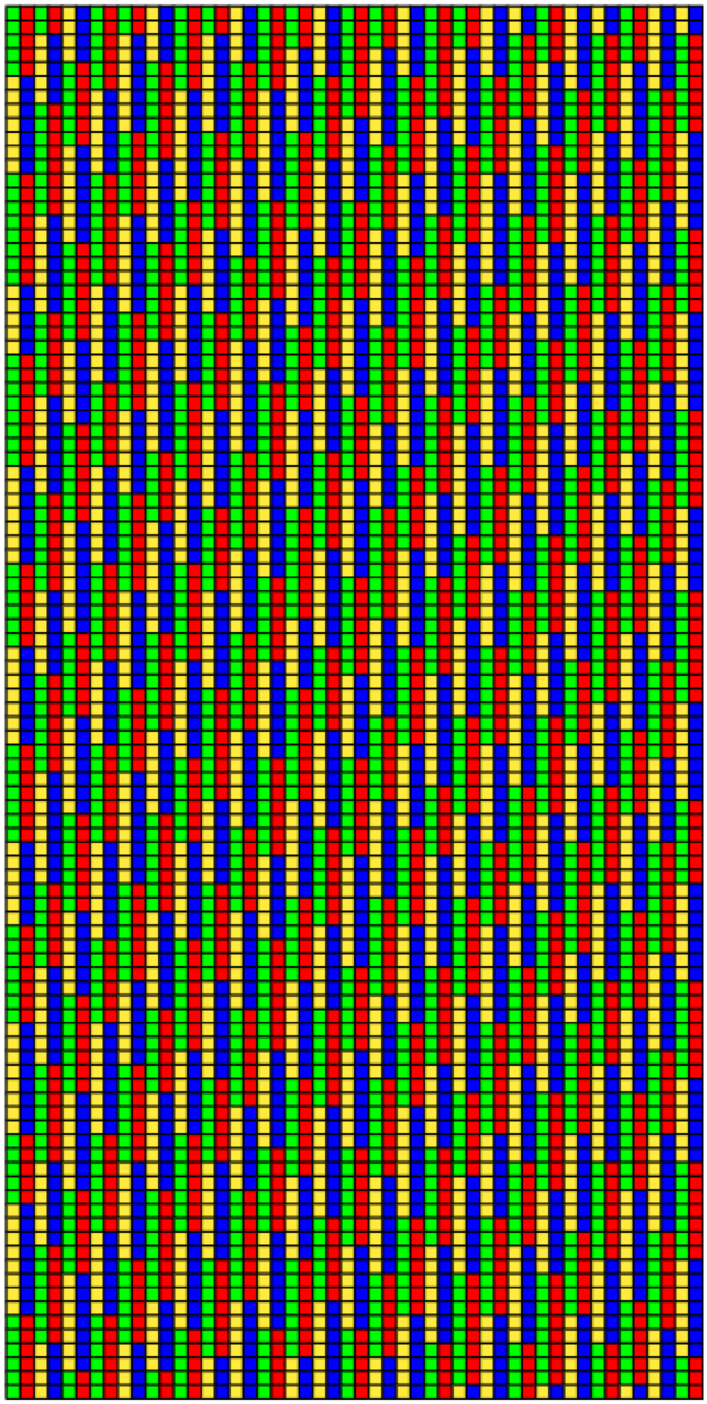


FIGURE 5.2. Read left to right, top to bottom: 4-coloring of [5000] avoiding 3-term F-diffsequences.

APPENDIX A

Rado Number Tables

A.1. 2-color Rado Numbers

This section displays our largest table of Rado numbers from Theorem 3.6.1; it contains 3500 entries.

TABLE A.1. 2-color Rado numbers $R_2(ax + by = cz)$, $1 \leq a \leq b \leq 20$, $1 \leq c \leq 20$, $gcd(a, b, c) = 1$.

a	b	c	$R_2(ax+by=cz)$	a	b	c	$R_2(ax+by=cz)$	a	b	c	$R_2(ax+by=cz)$
1	1	1	5	1	1	2	1	1	1	3	9
1	1	4	10	1	1	5	15	1	1	6	21
1	1	7	28	1	1	8	36	1	1	9	45
1	1	10	55	1	1	11	66	1	1	12	78
1	1	13	91	1	1	14	105	1	1	15	120
1	1	16	136	1	1	17	153	1	1	18	171
1	1	19	190	1	1	20	210	1	2	1	11
1	2	2	4	1	2	3	1	1	2	4	4
1	2	5	15	1	2	6	4	1	2	7	14
1	2	8	16	1	2	9	14	1	2	10	26
1	2	11	20	1	2	12	20	1	2	13	39
1	2	14	28	1	2	15	41	1	2	16	40
1	2	17	68	1	2	18	44	1	2	19	49
1	2	20	50	1	3	1	19	1	3	2	9
1	3	3	9	1	3	4	1	1	3	5	15
1	3	6	9	1	3	7	4	1	3	8	18
1	3	9	15	1	3	10	25	1	3	11	33

1	3	12	9	1	3	13	52	1	3	14	26
1	3	15	36	1	3	16	36	1	3	17	42
1	3	18	34	1	3	19	43	1	3	20	60
1	4	1	29	1	4	2	8	1	4	3	14
1	4	4	16	1	4	5	1	1	4	6	12
1	4	7	14	1	4	8	16	1	4	9	27
1	4	10	14	1	4	11	33	1	4	12	24
1	4	13	9	1	4	14	28	1	4	15	36
1	4	16	32	1	4	17	57	1	4	18	52
1	4	19	57	1	4	20	16	1	5	1	41
1	5	2	15	1	5	3	15	1	5	4	15
1	5	5	25	1	5	6	1	1	5	7	18
1	5	8	20	1	5	9	16	1	5	10	25
1	5	11	33	1	5	12	30	1	5	13	65
1	5	14	35	1	5	15	25	1	5	16	40
1	5	17	68	1	5	18	32	1	5	19	57
1	5	20	30	1	6	1	55	1	6	2	18
1	6	3	9	1	6	4	14	1	6	5	18
1	6	6	36	1	6	7	1	1	6	8	16
1	6	9	12	1	6	10	20	1	6	11	24
1	6	12	36	1	6	13	39	1	6	14	22
1	6	15	35	1	6	16	32	1	6	17	43
1	6	18	36	1	6	19	57	1	6	20	40
1	7	1	71	1	7	2	18	1	7	3	39
1	7	4	15	1	7	5	20	1	7	6	28
1	7	7	49	1	7	8	1	1	7	9	27
1	7	10	25	1	7	11	16	1	7	12	33
1	7	13	39	1	7	14	49	1	7	15	60
1	7	16	34	1	7	17	68	1	7	18	63

1	7	19	76
1	8	2	24
1	8	5	21
1	8	8	64
1	8	11	26
1	8	14	42
1	8	17	68
1	8	20	52
1	9	3	18
1	9	6	18
1	9	9	81
1	9	12	18
1	9	15	21
1	9	18	81
1	10	1	131
1	10	4	30
1	10	7	27
1	10	10	100
1	10	13	30
1	10	16	32
1	10	19	65
1	11	2	44
1	11	5	38
1	11	8	33
1	11	11	121
1	11	14	35
1	11	17	50
1	11	20	85
1	12	3	24

1	7	20	52
1	8	3	20
1	8	6	20
1	8	9	1
1	8	12	24
1	8	15	50
1	8	18	16
1	9	1	109
1	9	4	20
1	9	7	20
1	9	10	1
1	9	13	52
1	9	16	72
1	9	19	57
1	10	2	36
1	10	5	25
1	10	8	25
1	10	11	1
1	10	14	28
1	10	17	36
1	10	20	100
1	11	3	22
1	11	6	33
1	11	9	21
1	11	12	1
1	11	15	45
1	11	18	35
1	12	1	181
1	12	4	16

1	8	1	89
1	8	4	16
1	8	7	32
1	8	10	20
1	8	13	65
1	8	16	64
1	8	19	44
1	9	2	27
1	9	5	27
1	9	8	45
1	9	11	33
1	9	14	28
1	9	17	68
1	9	20	54
1	10	3	35
1	10	6	26
1	10	9	50
1	10	12	24
1	10	15	35
1	10	18	54
1	11	1	155
1	11	4	22
1	11	7	33
1	11	10	66
1	11	13	40
1	11	16	44
1	11	19	76
1	12	2	48
1	12	5	41

1	12	6	36	1	12	7	16	1	12	8	40
1	12	9	27	1	12	10	34	1	12	11	72
1	12	12	144	1	12	13	1	1	12	14	24
1	12	15	33	1	12	16	32	1	12	17	53
1	12	18	42	1	12	19	36	1	12	20	48
1	13	1	209	1	13	2	52	1	13	3	32
1	13	4	52	1	13	5	65	1	13	6	40
1	13	7	39	1	13	8	65	1	13	9	34
1	13	10	52	1	13	11	39	1	13	12	91
1	13	13	169	1	13	14	1	1	13	15	60
1	13	16	40	1	13	17	65	1	13	18	71
1	13	19	54	1	13	20	51	1	14	1	239
1	14	2	64	1	14	3	32	1	14	4	36
1	14	5	28	1	14	6	42	1	14	7	49
1	14	8	18	1	14	9	25	1	14	10	36
1	14	11	38	1	14	12	35	1	14	13	98
1	14	14	196	1	14	15	1	1	14	16	28
1	14	17	41	1	14	18	30	1	14	19	46
1	14	20	48	1	15	1	271	1	15	2	68
1	15	3	45	1	15	4	45	1	15	5	45
1	15	6	24	1	15	7	50	1	15	8	60
1	15	9	45	1	15	10	25	1	15	11	48
1	15	12	30	1	15	13	48	1	15	14	120
1	15	15	225	1	15	16	1	1	15	17	68
1	15	18	30	1	15	19	80	1	15	20	45
1	16	1	305	1	16	2	80	1	16	3	64
1	16	4	48	1	16	5	52	1	16	6	48
1	16	7	72	1	16	8	64	1	16	9	50
1	16	10	38	1	16	11	55	1	16	12	40

1	16	13	52
1	16	16	256
1	16	19	57
1	17	2	85
1	17	5	61
1	17	8	68
1	17	11	48
1	17	14	42
1	17	17	289
1	17	20	50
1	18	3	45
1	18	6	54
1	18	9	81
1	18	12	42
1	18	15	57
1	18	18	324
1	19	1	419
1	19	4	41
1	19	7	41
1	19	10	57
1	19	13	57
1	19	16	51
1	19	19	361
1	20	2	120
1	20	5	25
1	20	8	60
1	20	11	62
1	20	14	42
1	20	17	60

1	16	14	36
1	16	17	1
1	16	20	40
1	17	3	44
1	17	6	48
1	17	9	68
1	17	12	68
1	17	15	68
1	17	18	1
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1	19	17	80
1	19	20	1
1	20	3	50
1	20	6	32
1	20	9	75
1	20	12	60
1	20	15	40
1	20	18	40

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1	16	18	36
1	17	1	341
1	17	4	102
1	17	7	50
1	17	10	68
1	17	13	62
1	17	16	153
1	17	19	68
1	18	2	100
1	18	5	70
1	18	8	63
1	18	11	63
1	18	14	48
1	18	17	162
1	18	20	44
1	19	3	102
1	19	6	60
1	19	9	68
1	19	12	76
1	19	15	57
1	19	18	190
1	20	1	461
1	20	4	60
1	20	7	50
1	20	10	100
1	20	13	63
1	20	16	44
1	20	19	200

1	20	20	400
2	2	5	15
2	2	11	66
2	2	17	153
2	3	2	13
2	3	5	1
2	3	8	16
2	3	11	20
2	3	14	26
2	3	17	39
2	3	20	52
2	4	5	18
2	4	11	18
2	4	17	68
2	5	2	31
2	5	5	25
2	5	8	24
2	5	11	20
2	5	14	20
2	5	17	34
2	5	20	25
2	6	5	20
2	6	11	33
2	6	17	37
2	7	2	57
2	7	5	35
2	7	8	17
2	7	11	33
2	7	14	49

2	2	1	34
2	2	7	28
2	2	13	91
2	2	19	190
2	3	3	9
2	3	6	9
2	3	9	9
2	3	12	18
2	3	15	12
2	3	18	30
2	4	1	76
2	4	7	14
2	4	13	39
2	4	19	49
2	5	3	24
2	5	6	15
2	5	9	27
2	5	12	24
2	5	15	30
2	5	18	52
2	6	1	134
2	6	7	12
2	6	13	52
2	6	19	45
2	7	3	28
2	7	6	17
2	7	9	1
2	7	12	28
2	7	15	50

2	2	3	10
2	2	9	45
2	2	15	120
2	3	1	53
2	3	4	10
2	3	7	9
2	3	10	14
2	3	13	65
2	3	16	36
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2	4	9	10
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2	5	16	32
2	5	19	34
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2	6	9	9
2	6	15	30
2	7	1	169
2	7	4	20
2	7	7	49
2	7	10	20
2	7	13	24
2	7	16	16

2	7	17	45	2	7	18	16	2	7	19	49
2	7	20	52	2	8	1	208	2	8	3	32
2	8	5	16	2	8	7	20	2	8	9	27
2	8	11	33	2	8	13	24	2	8	15	36
2	8	17	51	2	8	19	57	2	9	1	251
2	9	2	91	2	9	3	27	2	9	4	27
2	9	5	29	2	9	6	18	2	9	7	45
2	9	8	24	2	9	9	81	2	9	10	38
2	9	11	1	2	9	12	24	2	9	13	41
2	9	14	30	2	9	15	30	2	9	16	54
2	9	17	51	2	9	18	81	2	9	19	52
2	9	20	40	2	10	1	298	2	10	3	20
2	10	5	25	2	10	7	16	2	10	9	20
2	10	11	38	2	10	13	65	2	10	15	25
2	10	17	68	2	10	19	57	2	11	1	349
2	11	2	133	2	11	3	63	2	11	4	44
2	11	5	43	2	11	6	35	2	11	7	27
2	11	8	29	2	11	9	66	2	11	10	29
2	11	11	121	2	11	12	34	2	11	13	1
2	11	14	33	2	11	15	60	2	11	16	33
2	11	17	34	2	11	18	56	2	11	19	51
2	11	20	77	2	12	1	404	2	12	3	36
2	12	5	32	2	12	7	24	2	12	9	24
2	12	11	26	2	12	13	39	2	12	15	30
2	12	17	34	2	12	19	57	2	13	1	463
2	13	2	183	2	13	3	65	2	13	4	52
2	13	5	27	2	13	6	26	2	13	7	52
2	13	8	52	2	13	9	39	2	13	10	65
2	13	11	91	2	13	12	40	2	13	13	169

2	13	14	39	2	13	15	1	2	13	16	72
2	13	17	68	2	13	18	26	2	13	19	57
2	13	20	52	2	14	1	526	2	14	3	70
2	14	5	70	2	14	7	49	2	14	9	34
2	14	11	32	2	14	13	48	2	14	15	60
2	14	17	68	2	14	19	76	2	15	1	593
2	15	2	241	2	15	3	45	2	15	4	72
2	15	5	35	2	15	6	45	2	15	7	66
2	15	8	65	2	15	9	45	2	15	10	30
2	15	11	50	2	15	12	24	2	15	13	120
2	15	14	40	2	15	15	225	2	15	16	60
2	15	17	1	2	15	18	32	2	15	19	60
2	15	20	30	2	16	1	664	2	16	3	40
2	16	5	62	2	16	7	72	2	16	9	16
2	16	11	42	2	16	13	74	2	16	15	42
2	16	17	68	2	16	19	26	2	17	1	739
2	17	2	307	2	17	3	111	2	17	4	85
2	17	5	72	2	17	6	63	2	17	7	68
2	17	8	65	2	17	9	57	2	17	10	60
2	17	11	48	2	17	12	53	2	17	13	68
2	17	14	49	2	17	15	153	2	17	16	68
2	17	17	289	2	17	18	68	2	17	19	1
2	17	20	68	2	18	1	818	2	18	3	56
2	18	5	36	2	18	7	70	2	18	9	81
2	18	11	42	2	18	13	65	2	18	15	36
2	18	17	80	2	18	19	57	2	19	1	901
2	19	2	381	2	19	3	59	2	19	4	111
2	19	5	95	2	19	6	57	2	19	7	57
2	19	8	50	2	19	9	60	2	19	10	57

2	19	11	57
2	19	14	29
2	19	17	190
2	19	20	57
2	20	5	40
2	20	11	40
2	20	17	44
3	3	2	15
3	3	7	28
3	3	11	66
3	3	16	136
3	3	20	210
3	4	3	21
3	4	6	9
3	4	9	18
3	4	12	16
3	4	15	30
3	4	18	33
3	5	1	197
3	5	4	15
3	5	7	18
3	5	10	25
3	5	13	39
3	5	16	40
3	5	19	36
3	6	2	48
3	6	7	18
3	6	11	16
3	6	16	36

2	19	12	46
2	19	15	43
2	19	18	37
2	20	1	988
2	20	7	32
2	20	13	54
2	20	19	64
3	3	4	12
3	3	8	32
3	3	13	91
3	3	17	153
3	4	1	151
3	4	4	16
3	4	7	1
3	4	10	20
3	4	13	24
3	4	16	16
3	4	19	44
3	5	2	40
3	5	5	25
3	5	8	1
3	5	11	33
3	5	14	28
3	5	17	52
3	5	20	25
3	6	4	12
3	6	8	18
3	6	13	39
3	6	17	68

2	19	13	52
2	19	16	76
2	19	19	361
2	20	3	80
2	20	9	104
2	20	15	60
3	3	1	111
3	3	5	15
3	3	10	50
3	3	14	105
3	3	19	190
3	4	2	35
3	4	5	17
3	4	8	16
3	4	11	33
3	4	14	16
3	4	17	68
3	4	20	60
3	5	3	31
3	5	6	15
3	5	9	18
3	5	12	18
3	5	15	25
3	5	18	26
3	6	1	249
3	6	5	21
3	6	10	20
3	6	14	28
3	6	19	45

3	6	20	48
3	7	3	57
3	7	6	18
3	7	9	15
3	7	12	18
3	7	15	24
3	7	18	45
3	8	1	371
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11	13	9	106
11	13	12	57
11	13	15	91
11	13	18	52
11	14	1	6889
11	14	4	282
11	14	7	160
11	14	10	98

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10	20	7	120
10	20	13	100
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11	11	4	341
11	11	7	198
11	11	10	143
11	11	14	154
11	11	17	187
11	11	20	220
11	12	3	471
11	12	6	145
11	12	9	82
11	12	12	144
11	12	15	51
11	12	18	44
11	13	1	6349
11	13	4	238
11	13	7	119
11	13	10	105
11	13	13	169
11	13	16	65
11	13	19	65
11	14	2	1514
11	14	5	146
11	14	8	115
11	14	11	211

11	14	12	74
11	14	15	63
11	14	18	61
11	15	1	7451
11	15	4	345
11	15	7	189
11	15	10	108
11	15	13	60
11	15	16	69
11	15	19	120
11	16	2	2417
11	16	5	385
11	16	8	145
11	16	11	273
11	16	14	76
11	16	17	87
11	16	20	65
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11	17	6	254
11	17	9	162
11	17	12	91
11	17	15	107
11	17	18	81
11	18	1	9269
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11	18	7	289
11	18	10	133
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11	18	16	108
11	14	13	75
11	14	16	56
11	14	19	86
11	15	2	937
11	15	5	280
11	15	8	157
11	15	11	241
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11	15	17	60
11	15	20	56
11	16	3	498
11	16	6	192
11	16	9	104
11	16	12	64
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11	16	18	47
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11	17	4	319
11	17	7	153
11	17	10	170
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11	17	16	95
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11	18	8	148
11	18	11	343
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11	18	17	71
11	14	14	196
11	14	17	105
11	14	20	52
11	15	3	753
11	15	6	187
11	15	9	135
11	15	12	85
11	15	15	225
11	15	18	78
11	16	1	8035
11	16	4	532
11	16	7	200
11	16	10	123
11	16	13	120
11	16	16	256
11	16	19	82
11	17	2	1112
11	17	5	317
11	17	8	132
11	17	11	307
11	17	14	74
11	17	17	289
11	17	20	69
11	18	3	985
11	18	6	204
11	18	9	145
11	18	12	108
11	18	15	87
11	18	18	324

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11	19	2	1247
11	19	5	220
11	19	8	284
11	19	11	381
11	19	14	97
11	19	17	88
11	19	20	55
11	20	3	757
11	20	6	284
11	20	9	313
11	20	12	120
11	20	15	92
11	20	18	88
12	12	1	6924
12	12	11	168
12	12	19	228
12	13	3	649
12	13	6	236
12	13	9	93
12	13	12	183
12	13	15	41
12	13	18	65
12	14	1	8126
12	14	7	170
12	14	13	40
12	14	19	64
12	15	4	447
12	15	8	93

11	18	20	82
11	19	3	704
11	19	6	178
11	19	9	132
11	19	12	98
11	19	15	71
11	19	18	95
11	20	1	10591
11	20	4	665
11	20	7	200
11	20	10	151
11	20	13	134
11	20	16	88
11	20	19	107
12	12	5	348
12	12	13	156
12	13	1	7513
12	13	4	334
12	13	7	158
12	13	10	58
12	13	13	169
12	13	16	65
12	13	19	81
12	14	3	638
12	14	9	94
12	14	15	54
12	15	1	8763
12	15	5	219
12	15	10	57

11	19	1	9919
11	19	4	429
11	19	7	161
11	19	10	91
11	19	13	95
11	19	16	124
11	19	19	361
11	20	2	2133
11	20	5	320
11	20	8	187
11	20	11	421
11	20	14	98
11	20	17	107
11	20	20	400
12	12	7	252
12	12	17	204
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12	13	5	169
12	13	8	129
12	13	11	90
12	13	14	97
12	13	17	50
12	13	20	48
12	14	5	306
12	14	11	80
12	14	17	64
12	15	2	1941
12	15	7	66
12	15	11	87

12	15	13	78
12	15	17	54
12	16	1	9424
12	16	7	32
12	16	13	48
12	16	19	48
12	17	3	829
12	17	6	325
12	17	9	122
12	17	12	307
12	17	15	91
12	17	18	68
12	18	1	10818
12	18	11	72
12	18	19	96
12	19	3	1071
12	19	6	423
12	19	9	199
12	19	12	381
12	19	15	98
12	19	18	68
12	20	1	12308
12	20	7	172
12	20	13	128
12	20	19	48
13	13	3	1027
13	13	6	377
13	13	9	247
13	13	12	195

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12	15	19	69
12	16	3	848
12	16	9	80
12	16	15	64
12	17	1	10109
12	17	4	768
12	17	7	290
12	17	10	178
12	17	13	121
12	17	16	73
12	17	19	81
12	18	5	108
12	18	13	120
12	19	1	11551
12	19	4	673
12	19	7	338
12	19	10	179
12	19	13	145
12	19	16	75
12	19	19	361
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12	20	9	92
12	20	15	40
13	13	1	8801
13	13	4	559
13	13	7	325
13	13	10	221
13	13	14	182

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12	16	5	272
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12	16	17	96
12	17	2	2885
12	17	5	456
12	17	8	175
12	17	11	102
12	17	14	98
12	17	17	289
12	17	20	136
12	18	7	90
12	18	17	72
12	19	2	3427
12	19	5	458
12	19	8	264
12	19	11	125
12	19	14	121
12	19	17	90
12	19	20	95
12	20	5	340
12	20	11	156
12	20	17	84
13	13	2	1105
13	13	5	442
13	13	8	286
13	13	11	208
13	13	15	195

13	13	16	208	13	13	17	221	13	13	18	234
13	13	19	247	13	13	20	260	13	14	1	9491
13	14	2	1762	13	14	3	635	13	14	4	412
13	14	5	349	13	14	6	182	13	14	7	197
13	14	8	153	13	14	9	93	13	14	10	136
13	14	11	98	13	14	12	116	13	14	13	211
13	14	14	196	13	14	15	105	13	14	16	63
13	14	17	107	13	14	18	65	13	14	19	67
13	14	20	72	13	15	1	10207	13	15	2	1470
13	15	3	733	13	15	4	326	13	15	5	365
13	15	6	270	13	15	7	154	13	15	8	156
13	15	9	131	13	15	10	117	13	15	11	150
13	15	12	90	13	15	13	241	13	15	14	78
13	15	15	225	13	15	16	74	13	15	17	120
13	15	18	45	13	15	19	82	13	15	20	65
13	16	1	10949	13	16	2	3125	13	16	3	1095
13	16	4	901	13	16	5	406	13	16	6	277
13	16	7	168	13	16	8	197	13	16	9	212
13	16	10	147	13	16	11	120	13	16	12	127
13	16	13	273	13	16	14	78	13	16	15	124
13	16	16	256	13	16	17	84	13	16	18	64
13	16	19	136	13	16	20	70	13	17	1	11717
13	17	2	1471	13	17	3	742	13	17	4	483
13	17	5	247	13	17	6	234	13	17	7	266
13	17	8	196	13	17	9	208	13	17	10	136
13	17	11	132	13	17	12	102	13	17	13	307
13	17	14	104	13	17	15	78	13	17	16	136
13	17	17	289	13	17	18	95	13	17	19	85
13	17	20	91	13	18	1	12511	13	18	2	2752

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13	18	6	380	13	18	7	182	13	18	8	247
13	18	9	197	13	18	10	157	13	18	11	176
13	18	12	117	13	18	13	343	13	18	14	113
13	18	15	95	13	18	16	91	13	18	17	104
13	18	18	324	13	18	19	85	13	18	20	69
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13	19	4	570	13	19	5	477	13	19	6	337
13	19	7	301	13	19	8	171	13	19	9	202
13	19	10	197	13	19	11	162	13	19	12	134
13	19	13	381	13	19	14	106	13	19	15	146
13	19	16	78	13	19	17	128	13	19	18	106
13	19	19	361	13	19	20	104	13	20	1	14177
13	20	2	3026	13	20	3	793	13	20	4	759
13	20	5	433	13	20	6	320	13	20	7	459
13	20	8	184	13	20	9	182	13	20	10	197
13	20	11	120	13	20	12	125	13	20	13	421
13	20	14	117	13	20	15	91	13	20	16	107
13	20	17	116	13	20	18	88	13	20	19	112
13	20	20	400	14	14	1	10990	14	14	3	1316
14	14	5	560	14	14	9	308	14	14	11	252
14	14	13	224	14	14	15	210	14	14	17	238
14	14	19	266	14	15	1	11789	14	15	2	2248
14	15	3	1055	14	15	4	407	14	15	5	335
14	15	6	322	14	15	7	272	14	15	8	181
14	15	9	128	14	15	10	111	14	15	11	138
14	15	12	95	14	15	13	133	14	15	14	241
14	15	15	225	14	15	16	120	14	15	17	75
14	15	18	60	14	15	19	72	14	15	20	52

14	16	1	12616	14	16	3	520	14	16	5	208
14	16	7	290	14	16	9	64	14	16	11	128
14	16	13	120	14	16	15	16	14	16	17	60
14	16	19	60	14	17	1	13471	14	17	2	2621
14	17	3	1388	14	17	4	586	14	17	5	475
14	17	6	313	14	17	7	325	14	17	8	228
14	17	9	226	14	17	10	181	14	17	11	147
14	17	12	153	14	17	13	119	14	17	14	307
14	17	15	144	14	17	16	85	14	17	17	289
14	17	18	82	14	17	19	102	14	17	20	153
14	18	1	14354	14	18	3	1154	14	18	5	496
14	18	7	362	14	18	9	226	14	18	11	88
14	18	13	180	14	18	15	92	14	18	17	84
14	18	19	78	14	19	1	15265	14	19	2	3066
14	19	3	674	14	19	4	686	14	19	5	655
14	19	6	278	14	19	7	401	14	19	8	258
14	19	9	198	14	19	10	215	14	19	11	132
14	19	12	118	14	19	13	131	14	19	14	381
14	19	15	126	14	19	16	98	14	19	17	118
14	19	18	95	14	19	19	361	14	19	20	95
14	20	1	16204	14	20	3	1488	14	20	5	480
14	20	7	442	14	20	9	204	14	20	11	196
14	20	13	150	14	20	15	100	14	20	17	54
14	20	19	64	15	15	1	13515	15	15	2	1695
15	15	4	855	15	15	7	495	15	15	8	435
15	15	11	315	15	15	13	270	15	15	14	255
15	15	16	240	15	15	17	255	15	15	19	285
15	16	1	14431	15	16	2	3767	15	16	3	1365
15	16	4	1189	15	16	5	582	15	16	6	258

15	16	7	279	15	16	8	257	15	16	9	184
15	16	10	115	15	16	11	182	15	16	12	126
15	16	13	177	15	16	14	142	15	16	15	273
15	16	16	256	15	16	17	136	15	16	18	64
15	16	19	72	15	16	20	45	15	17	1	15377
15	17	2	2176	15	17	3	1535	15	17	4	497
15	17	5	434	15	17	6	430	15	17	7	201
15	17	8	221	15	17	9	230	15	17	10	168
15	17	11	121	15	17	12	102	15	17	13	157
15	17	14	96	15	17	15	307	15	17	16	90
15	17	17	289	15	17	18	68	15	17	19	153
15	17	20	61	15	18	1	16353	15	18	2	2979
15	18	4	468	15	18	5	531	15	18	7	279
15	18	8	153	15	18	10	111	15	18	11	63
15	18	13	147	15	18	14	117	15	18	16	75
15	18	17	108	15	18	19	117	15	18	20	42
15	19	1	17359	15	19	2	2177	15	19	3	1905
15	19	4	693	15	19	5	687	15	19	6	500
15	19	7	239	15	19	8	277	15	19	9	213
15	19	10	158	15	19	11	238	15	19	12	140
15	19	13	172	15	19	14	111	15	19	15	381
15	19	16	118	15	19	17	91	15	19	18	76
15	19	19	361	15	19	20	72	15	20	1	18395
15	20	2	3460	15	20	3	1645	15	20	4	755
15	20	6	260	15	20	7	105	15	20	8	145
15	20	9	130	15	20	11	170	15	20	12	80
15	20	13	70	15	20	14	40	15	20	16	60
15	20	17	120	15	20	18	50	15	20	19	55
16	16	1	16400	16	16	3	1968	16	16	5	832

16	16	7	592
16	16	13	320
16	16	19	304
16	17	3	673
16	17	6	368
16	17	9	153
16	17	12	119
16	17	15	152
16	17	18	153
16	18	1	18514
16	18	7	310
16	18	13	146
16	18	19	68
16	19	3	2065
16	19	6	490
16	19	9	331
16	19	12	185
16	19	15	133
16	19	18	106
16	20	1	20756
16	20	7	152
16	20	13	176
16	20	19	108
17	17	3	2227
17	17	6	833
17	17	9	561
17	17	12	425
17	17	15	340
17	17	19	323

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16	16	15	288
16	17	1	17441
16	17	4	1577
16	17	7	240
16	17	10	214
16	17	13	170
16	17	16	307
16	17	19	98
16	18	3	1734
16	18	9	290
16	18	15	116
16	19	1	19619
16	19	4	1907
16	19	7	228
16	19	10	190
16	19	13	183
16	19	16	381
16	19	19	361
16	20	3	788
16	20	9	84
16	20	15	52
17	17	1	19669
17	17	4	1241
17	17	7	714
17	17	10	493
17	17	13	391
17	17	16	323
17	17	20	340

16	16	11	384
16	16	17	272
16	17	2	4473
16	17	5	660
16	17	8	325
16	17	11	144
16	17	14	105
16	17	17	289
16	17	20	89
16	18	5	638
16	18	11	102
16	18	17	66
16	19	2	5243
16	19	5	445
16	19	8	401
16	19	11	157
16	19	14	100
16	19	17	157
16	19	20	74
16	20	5	516
16	20	11	176
16	20	17	116
17	17	2	2465
17	17	5	986
17	17	8	629
17	17	11	459
17	17	14	357
17	17	18	306
17	18	1	20843

17	18	2	4112	17	18	3	2783	17	18	4	855
17	18	5	528	17	18	6	632	17	18	7	272
17	18	8	290	17	18	9	325	17	18	10	225
17	18	11	215	17	18	12	171	17	18	13	199
17	18	14	114	17	18	15	119	17	18	16	170
17	18	17	343	17	18	18	324	17	18	19	171
17	18	20	68	17	19	1	22051	17	19	2	3078
17	19	3	1442	17	19	4	698	17	19	5	784
17	19	6	340	17	19	7	462	17	19	8	323
17	19	9	221	17	19	10	245	17	19	11	275
17	19	12	139	17	19	13	254	17	19	14	147
17	19	15	187	17	19	16	121	17	19	17	381
17	19	18	102	17	19	19	361	17	19	20	97
17	20	1	23293	17	20	2	5484	17	20	3	2331
17	20	4	1247	17	20	5	667	17	20	6	550
17	20	7	463	17	20	8	395	17	20	9	325
17	20	10	325	17	20	11	204	17	20	12	221
17	20	13	273	17	20	14	204	17	20	15	147
17	20	16	130	17	20	17	421	17	20	18	136
17	20	19	111	17	20	20	400	18	18	1	23346
18	18	5	1170	18	18	7	846	18	18	11	540
18	18	13	450	18	18	17	360	18	18	19	342
18	19	1	24661	18	19	2	4681	18	19	3	3313
18	19	4	1013	18	19	5	901	18	19	6	787
18	19	7	278	18	19	8	387	18	19	9	401
18	19	10	251	18	19	11	290	18	19	12	182
18	19	13	215	18	19	14	151	18	19	15	164
18	19	16	127	18	19	17	189	18	19	18	381
18	19	19	361	18	19	20	190	18	20	1	26012

18	20	3	2396	18	20	5	802	18	20	7	216
18	20	9	442	18	20	11	118	18	20	13	96
18	20	15	96	18	20	17	124	18	20	19	68
19	19	1	27455	19	19	2	3439	19	19	3	3135
19	19	4	1729	19	19	5	1387	19	19	6	1159
19	19	7	988	19	19	8	874	19	19	9	779
19	19	10	703	19	19	11	627	19	19	12	589
19	19	13	532	19	19	14	494	19	19	15	475
19	19	16	437	19	19	17	418	19	19	18	399
19	19	20	380	19	20	1	28919	19	20	2	7097
19	20	3	1954	19	20	4	1189	19	20	5	916
19	20	6	630	19	20	7	543	19	20	8	480
19	20	9	211	19	20	10	401	19	20	11	210
19	20	12	230	19	20	13	152	19	20	14	203
19	20	15	131	19	20	16	150	19	20	17	161
19	20	18	209	19	20	19	421	19	20	20	400
20	20	1	32020	20	20	3	3620	20	20	7	1160
20	20	9	900	20	20	11	740	20	20	13	620
20	20	17	480	20	20	19	440				

APPENDIX B

Diffsequence-Avoiding Colorings

Here we give the numerical data for various colorings used that avoid certain sequences.

B.1. 7-coloring of [5000] Avoiding 2-term P-Diffsequences (Figure 4.1)

Color 1: {1, 5, 9, 20, 24, 28, 39, 43, 47, 58, 62, 66, 81, 85, 89, 100, 104, 108, 119, 123, 127, 142, 146, 161, 165, 169, 180, 184, 188, 203, 207, 222, 226, 230, 241, 245, 249, 260, 264, 268, 283, 287, 302, 306, 310, 321, 325, 329, 340, 344, 348, 359, 363, 367, 386, 390, 394, 405, 409, 413, 424, 428, 432, 443, 447, 451, 470, 474, 489, 493, 497, 508, 512, 516, 527, 531, 535, 550, 554, 558, 573, 577, 581, 592, 596, 600, 611, 615, 619, 634, 638, 642, 657, 661, 665, 676, 680, 684, 695, 699, 703, 714, 718, 722, 737, 741, 745, 756, 760, 764, 775, 779, 783, 798, 802, 817, 821, 825, 836, 840, 844, 859, 863, 878, 882, 886, 897, 901, 905, 916, 920, 924, 935, 939, 943, 958, 962, 966, 977, 981, 985, 996, 1000, 1004, 1015, 1019, 1023, 1042, 1046, 1050, 1061, 1065, 1069, 1080, 1084, 1088, 1099, 1103, 1107, 1126, 1130, 1145, 1149, 1153, 1164, 1168, 1172, 1183, 1187, 1191, 1206, 1210, 1214, 1229, 1233, 1237, 1248, 1252, 1256, 1267, 1271, 1275, 1290, 1294, 1298, 1313, 1317, 1321, 1332, 1336, 1340, 1351, 1355, 1359, 1374, 1378, 1397, 1401, 1405, 1416, 1420, 1424, 1435, 1439, 1443, 1454, 1458, 1462, 1477, 1481, 1485, 1500, 1504, 1508, 1519, 1523, 1527, 1538, 1542, 1546, 1561, 1565, 1569, 1580, 1584, 1588, 1599, 1603, 1607, 1622, 1626, 1641, 1645, 1649, 1660, 1664, 1668, 1679, 1683, 1687, 1702, 1706, 1725, 1729, 1733, 1744, 1748, 1752, 1763, 1767, 1771, 1782, 1786, 1790, 1809, 1813, 1828, 1832, 1836, 1847, 1851, 1855, 1866, 1870, 1874, 1889, 1893, 1897, 1912, 1916, 1920, 1931, 1935, 1939, 1950, 1954, 1958, 1973, 1977, 1981, 1996, 2000, 2004, 2015, 2019, 2023, 2034, 2038, 2042, 2053, 2057, 2061, 2076, 2080, 2084, 2095, 2099, 2103, 2114, 2118, 2122, 2137, 2141, 2156, 2160, 2164, 2175, 2179, 2183, 2198, 2202, 2217, 2221, 2225, 2236, 2240, 2244, 2255, 2259, 2263, 2274, 2278, 2282, 2297, 2301, 2305, 2316, 2320, 2324, 2335, 2339, 2343, 2354, 2358, 2362, 2381, 2385, 2389, 2400, 2404, 2408, 2419, 2423, 2427, 2438, 2442, 2446, 2465, 2469, 2484, 2488, 2492, 2503, 2507, 2511, 2522, 2526, 2530, 2545, 2549, 2553, 2568, 2572, 2576, 2587, 2591, 2595, 2606, 2610, 2614, 2629, 2633, 2637, 2652, 2656, 2660, 2671, 2675, 2679, 2690, 2694, 2698, 2709, 2713, 2717, 2732, 2736, 2740, 2751, 2755, 2759, 2770, 2774, 2778, 2789, 2793, 2797, 2812, 2816, 2820, 2831, 2835, 2839, 2850, 2854, 2858, 2873, 2877, 2881, 2892, 2896, 2900, 2911, 2915, 2919, 2930, 2934, 2938, 2953, 2957, 2961, 2972, 2976, 2980, 2991, 2995, 2999, 3010, 3014, 3018, 3037, 3041, 3045, 3056, 3060, 3064, 3075, 3079, 3083, 3094, 3098, 3102, 3117, 3121, 3125, 3140, 3144, 3148, 3159, 3163, 3167, 3178, 3182, 3186, 3201, 3205, 3209, 3224, 3228, 3232, 3243, 3247, 3251, 3262, 3266, 3270, 3285, 3289, 3293, 3308, 3312, 3316, 3327, 3331, 3335, 3346, 3350, 3354, 3365, 3369, 3373, 3392, 3396, 3400, 3411, 3415, 3419, 3430, 3434, 3438, 3449, 3453, 3457, 3472, 3476, 3480, 3495, 3499, 3503, 3514, 3518, 3522, 3533, 3537, 3541, 3556, 3560, 3564, 3575, 3579, 3583, 3594, 3598, 3602, 3613, 3617, 3621, 3636, 3640, 3644, 3655, 3659, 3663, 3674, 3678, 3682, 3693, 3697, 3701, 3720, 3724, 3728, 3739, 3743, 3747, 3758, 3762, 3766, 3777, 3781, 3785, 3800, 3804, 3808, 3823, 3827, 3831, 3842, 3846, 3850, 3861, 3865, 3869, 3884, 3888, 3892, 3907, 3911, 3915, 3926, 3930, 3934, 3945, 3949, 3953, 3968, 3972, 3976, 3991, 3995, 3999, 4010, 4014, 4018, 4029, 4033, 4037, 4048, 4052, 4056, 4071, 4075, 4079, 4090, 4094, 4098, 4109, 4113, 4117, 4128, 4132, 4136, 4151, 4155, 4159, 4170, 4174, 4178, 4189, 4193, 4197, 4212, 4216, 4220, 4231, 4235, 4239, 4250, 4254, 4258, 4269, 4273, 4277, 4292, 4296, 4300, 4311, 4315, 4319, 4330, 4334, 4338, 4349, 4353, 4357, 4376, 4380, 4384, 4395, 4399, 4403, 4414, 4418, 4422, 4433, 4437, 4441, 4456, 4460, 4464, 4479, 4483, 4487, 4498, 4502, 4506, 4517, 4521, 4525, 4540, 4544, 4548, 4563, 4567, 4571, 4582, 4586, 4590, 4601, 4605, 4609, 4624, 4628, 4632, 4647, 4651, 4655, 4666, 4670, 4674, 4685, 4689, 4693, 4704, 4708, 4712, 4727, 4731, 4735, 4746, 4750, 4754, 4765, 4769, 4773, 4784, 4788, 4792, 4807, 4811, 4815, 4826, 4830, 4834, 4845, 4849, 4853, 4868, 4872, 4876, 4887, 4891, 4895, 4906, 4910, 4914, 4925, 4929, 4933, 4948, 4952, 4956, 4967, 4971, 4975, 4986, 4990, 4994}

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B.2. 4-coloring of [5000] Avoiding 3-term F-Diffsequences (Figure 5.2)

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