Graphical Categories and Representations of Quantum Groups

By

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Abstract

We review examples of Witten-Reshetikhin-Turaev quantum invariants, Hecke algebra, BMW algebra, web categories, and relations between them. We define web categories for the quantum orthogonal group in detail.

We also review recursive formulas for the highest-weight projectors in the web categories. We present the triple clasp formulas for G_2 in detail.

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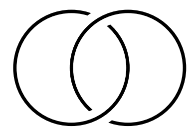
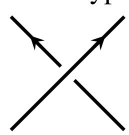
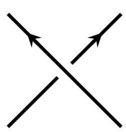
CHAPTER 1

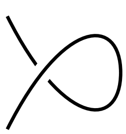
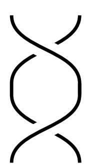
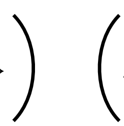
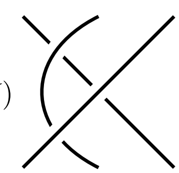
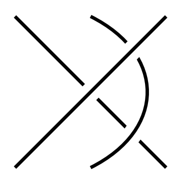
Introduction

A knot is an embedding of a circle in the 3-dimensional space, and a link is a collection of non-intersecting knots which may be linked together. Alternatively, a link is an embedding of a disjoint union of finitely many circles in the 3-dimensional space, and a knot can be seen as a link with one component. Two links are topologically the same when one link can be transformed to the other one by continuous distortion of the ambient space. In other words, we study the embeddings of circles up to ambient isotopy.

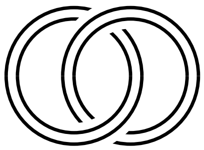
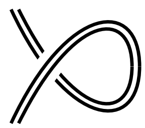
When the ambient space is $\mathbb{R}^3$, a link can be projected onto a plane $\mathbb{R}^2$. The projection is known as a link diagram. For example, the link diagram of a Hopf link can be drawn as .

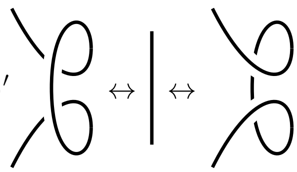
When a link is drawn on a plane, there are two types of crossings once we assign an orientation to each component of the link: a positive crossing  which follows the right hand rule, and a negative crossing  which follows the left hand rule.

Reidemeister [**51**] and Alexander-Briggs [**1**] showed that two link diagrams describe the same link up to isotopy of $\mathbb{R}^3$, if and only if they can be related by a sequence of the three Reidemeister moves:

(*RI*)  $\leftrightarrow$ , (*RII*)  $\leftrightarrow$ , (*RIII*)  $\leftrightarrow$ .

A link invariant is a quantity assigned to each link, which remains unchanged for links that are the same up to ambient isotopy. The link invariants associated to two link diagrams connected by Reidemeister moves yield the same answer.

A framed link is a link where annuli are embedded in the 3-dimensional space instead of circles. A link diagram can be seen as a diagram of a framed link by replacing each segment of the link diagram with a ribbon lying on the plane. Since the framing of the so obtained framed link is parallel to $\mathbb{R}^2$, it is called the blackboard framing. For example, a framed Hopf link with the blackboard framing can be drawn as . Since  $=$  $\neq$ , the Reidemeister move (RI) no longer holds for framed links. Instead, one can replace (RI) by its modification

$(RI)'$  .

An invariant for framed links is a quantity assigned to each framed link, which remains unchanged for framed links that are the same up to ambient isotopy. If the link diagrams of two framed links are connected by a sequence of Reidemeister moves $(R1)'$, $(R2)$, and $(R3)$, then the invariants associated to the framed links are the same. In this thesis, links refer to framed links, and link invariants refer to invariants of framed links.

The discovery of the Jones polynomial in the early 1980's [**25**] triggered mathematical developments in areas including knot theory and quantum algebra. The Jones polynomial is a link invariant, written as a Laurent polynomial in one variable. Witten [**64**] showed that the Jones polynomial of a given link can be obtained by considering Chern–Simons topological quantum field theory. It was discovered by Reshetikhin and Turaev [**52**] that the Jones polynomial can be defined by using the braiding structure in the Ribbon category, which is universally constructed for any simple Lie algebra $\mathfrak{g}$, generalizing the Jones polynomial to a family of quantum link invariants. When q is at a root of unity, the ribbon categories also give invariants of a three-manifold by coloring the link along which Dehn surgery [**14**] is performed. These quantum invariants are known as Witten-Reshetikhin-Turaev invariants.

The Ribbon category related to the Jones polynomial can be presented as the Temperley-Lieb category [**60**]. Half a decade earlier Rummer-Teller-Weyl found a description of morphisms between tensor products of the vector representation of $SL_2(\mathbb{C})$ in terms of cup and cap diagrams [**56**, Equation 3]. The q-analogue of their result is that the Temperley-Lieb Category is monoidally equivalent to the full monoidal subcategory of $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$ generated by the q-analogue of the vector representation. Hence we can use diagrams and graphical calculations in the Temperley-Lieb category to study the representation theory of $U_q(\mathfrak{sl}_2)$.

The web category of $\mathfrak{g}_2$ was first introduced by Kuperberg [**35**] to compute the quantum link invariants for in the $\mathfrak{g}_2$ case. The definition of web categories was later generalized to include all the rank two simple Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{sp}_4$, or $\mathfrak{g}_2$ [**36**]. It was shown in the paper that the web category of $\mathfrak{g}$ is monoidally equivalent to the category of fundamental representations of the quantum group $U_q(\mathfrak{g})$, generalizing the relation between the Temperley-Lieb Category and the representations of $U_q(\mathfrak{sl}_2)$.

Bearing the goal of giving graphically presented, generators and relations descriptions of the monoidal categories of fundamental representations of quantum groups, the definition of web categories was later extended to types A [**11**] and C [**8**].

Khovanov developed a homology theory for links that categorifies the Jones polynomial [**30**], which provides a link invariant with more information than the Jones polynomial. Web categories are used for the categorification of quantum link invariants in the $\mathfrak{sl}_3$ case [**31**] and $\mathfrak{sl}_n$ case [**42**, **50**, **54**].

In this thesis, we review examples of quantum link invariants in different Lie types, the graphically presented algebras and categories related to quantum invariants and representations of quantum groups, and their relations to web categories. We define the web categories for the quantum orthogonal group, based on joint work with Bodish [**10**]. We also demonstrate how to use web categories to study the representations of quantum groups, by giving a graphical expression for any irreducible representation of the quantum group $U_q(\mathfrak{g}_2)$, based on joint work with Bodish [**9**].

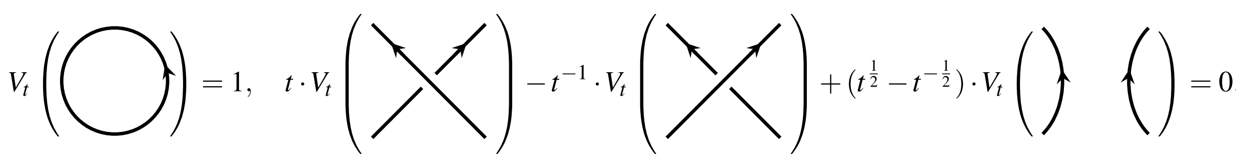
CHAPTER 2

Quantum invariants and Skein categories

2.1. Jones polynomial and Temperley–Lieb category

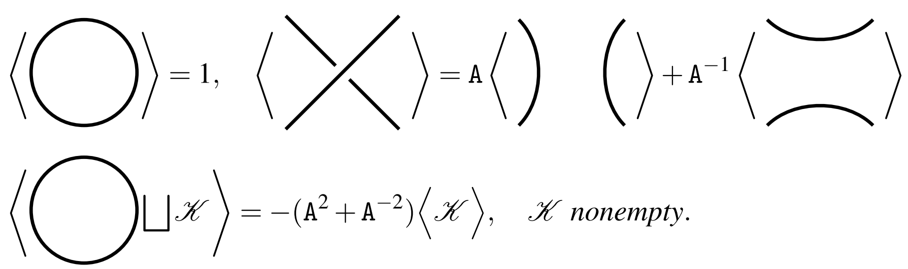
2.1.1. Jones Polynomial. The Jones polynomial is the very first example of quantum invariants for links and knots. It can be computed via the following definition given by Jones.

DEFINITION 2.1.1. *[**25**] The Jones polynomial $V_t(\mathscr{K})$ associated to a link or knot $\mathscr{K}$ is a Laurent polynomial in $t^{\frac{1}{2}}$, satisfying the following relations:*

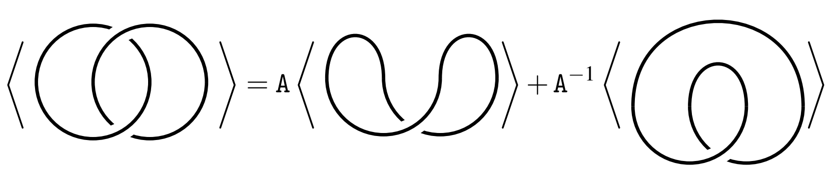


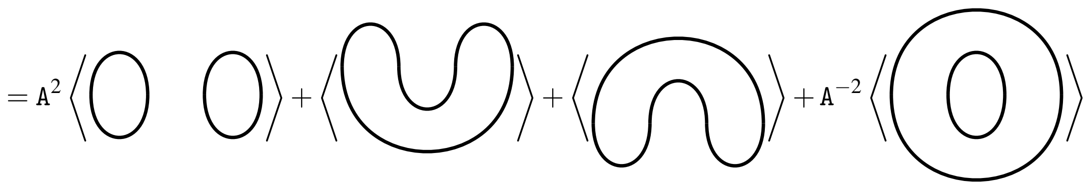
2.1.2. Bracket polynomial. The bracket polynomial defined by Kauffman, also known as the Kauffman bracket, is a reinterpretation of the Jones polynomial.

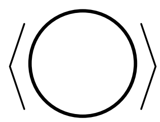
DEFINITION 2.1.2. *[**28**] The bracket polynomial $\langle \mathscr{K} \rangle$ of a link or knot $\mathscr{K}$ is a Laurent polynomial in* $\mathtt{A}$*, which satisfies the following relations:*



EXAMPLE 2.1.1. *We compute the bracket polynomial of a Hopf link.*





$$= \left(-\mathtt{A}^2(\mathtt{A}^2+\mathtt{A}^{-2})+1+1+-\mathtt{A}^{-2}(\mathtt{A}^2+\mathtt{A}^{-2})\right)$$  $$= -\mathtt{A}^4-\mathtt{A}^{-4}$$

DEFINITION 2.1.3. *Given a link or knot $\mathscr{K}$, the writhe of $\mathscr{K}$ is defined as the number of positive crossings of $\mathscr{K}$ minus the number of negative crossings of $\mathscr{K}$. Denote the writhe of $\mathscr{K}$ by $wr(\mathscr{K})$.*

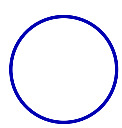
The Jones polynomial can be seen as a normalized bracket polynomial in the following sense.

THEOREM 2.1.1. *[28] Given a link or knot $\mathscr{K}$, the relation between the Jones polynomial and the bracket polynomial is the following*

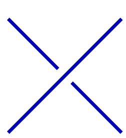
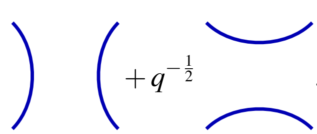
$$V_{\mathtt{A}^{-4}}(\mathscr{K}) = (-\mathtt{A})^{-3wr(\mathscr{K})} \cdot \langle \mathscr{K} \rangle .$$

2.1.3. Temperley–Lieb category. Using the skein relations from the bracket polynomial, we can define a graphical category know as the Temperley–Lieb category.

DEFINITION 2.1.4. *[60] The Temperley–Lieb category **TL** is a pivotal $\mathbb{Z}(q^{\pm})$-linear category whose object is a tuple of dots, and whose morphism is a linear combination of planar matchings between two tuples of dots, modulo the tensor ideal generated by the following relation:*

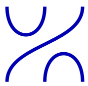
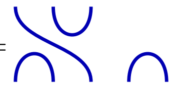
 $$= -(q+q^{-1}).$$

***TL** can be made into a $\mathbb{Z}(q^{\pm\frac{1}{2}})$-linear braided tensor category by defining the braiding:*

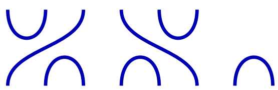
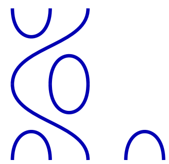
 $$= q^{\frac{1}{2}}$$ 

REMARK 2.1.1. *The tensor product of objects in the Temperley–Lieb category* ***TL*** *is concatenation of tuples of dots, for example:* $\bullet\bullet \otimes \bullet\bullet\bullet = \bullet\bullet\bullet\bullet\bullet$.

The tensor product of morphisms is horizontal concatenation. The composition of morphisms is vertical stacking. For example, consider the following morphisms in the Temperley-Lieb category

$f =$  $\in End_{\boldsymbol{TL}}(\bullet^{\otimes 3})$ *and* $g =$  $\in Hom_{\boldsymbol{TL}}(\bullet^{\otimes 5}, \bullet^{\otimes 3})$,

we know that

$f \otimes g =$  *and* $f \circ g =$  $= -(q+q^{-1})$  .

PROPOSITION 2.1.1. *A link or knot $\mathscr{K}$ evaluated as a morphism in the Temperley–Lieb category from the empty word to the empty word yields a scalar $\langle \mathscr{K} \rangle_{\boldsymbol{TL}}$. Let $q = \mathtt{A}^2$, $\langle \mathscr{K} \rangle_{\boldsymbol{TL}}$ is related to the bracket polynomial by the following relation*

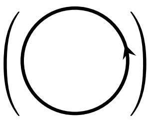
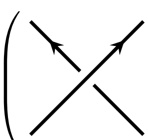
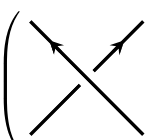
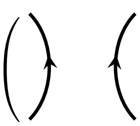
$$\langle \mathscr{K} \rangle_{\boldsymbol{TL}} = -(q+q^{-1})\langle \mathscr{K} \rangle.$$

PROOF. Set $q = \mathtt{A}^2$, then $\mathscr{K}$ is evaluated in the Temperley–Lieb category by the same skein relations as the ones applied to compute the bracket polynomial, until the end where an unknot is evaluated as $-(q+q^{-1})$ in the Temperley-Lieb category whereas $\langle unknot \rangle = 1$. □

2.2. Type A quantum link invariant and skein category, and Hecke algebra

2.2.1. HOMFLY-PT polynomial. The Jones polynomial is the quantum invariant for links and knots in the $\mathfrak{sl}_2$ case. A generalization to the $\mathfrak{sl}_n$ case is known as the HOMFLY-PT polynomial.

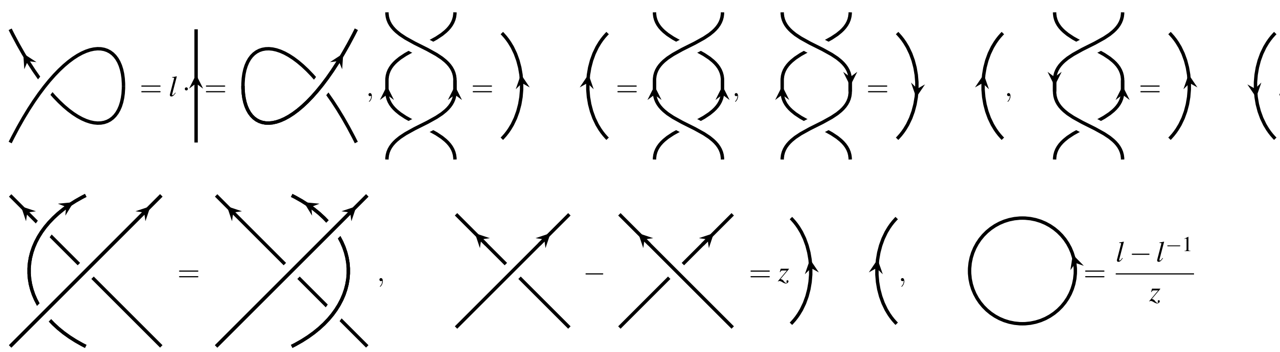
DEFINITION 2.2.1. *[**18**, **49**] The HOMFLY-PT polynomial $P_{x,y,z}(\mathscr{K})$ associated to a link or knot $\mathscr{K}$ is a Laurent polynomial in x, y, and z, satisfying the following relations:*

$P_{x,y,z}\Big($  $\Big) = 1$, $\quad x \cdot P_{x,y,z}\Big($  $\Big) + y \cdot P_{x,y,z}\Big($  $\Big) + z \cdot P_{x,y,z}\Big($  $\Big) = 0.$

REMARK 2.2.1. *When $x = -t^{-1}$, $y = t$, and $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, $P_{x,y,z}(\mathscr{K}) = V_t(\mathscr{K})$*

2.2.2. HOMFLY-PT skein category. Using the language of category theory, one can take the skein relations from the definition of HOMFLY-PT polynomial for knots and links to generate relations on tangles.

DEFINITION 2.2.2. *[62] The HOMFLY-PT skein category $\boldsymbol{OS}(z,l)$ is a $\mathbb{Z}[z^{\pm},l^{\pm}]$-linear pivotal category whose objects are generated by $\{\uparrow,\downarrow\}$, and whose morphism is a linear combination of matchings (framed oriented tangles) between two words in the letters $\uparrow$ and $\downarrow$, modulo the tensor ideal generated by the following relations:*



REMARK 2.2.2. *A link or knot $\mathscr{K}$ evaluated as a morphism in the HOMFLY-PT skein category from the empty word to the empty word yields a scalar $\langle \mathscr{K} \rangle_A$. $\langle \mathscr{K} \rangle_A$ is related to the HOMFLY-PT polynomial by the following relation*

$$l^{-wr(\mathscr{K})} \langle \mathscr{K} \rangle_A = \frac{l-l^{-1}}{z} P_{l,-l^{-1},-z}(\mathscr{K}).$$

2.2.3. Hecke algebra.

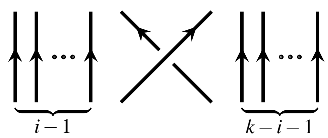
DEFINITION 2.2.1. *The k-strand Hecke algebra of type A, denoted by $H_k(q)$, is the unital associative $\mathbb{Z}[q^{\pm}]$-algebra generated by T_i for $1 \le i \le k-1$, with relations:*

(1) $T_i^2 = (q-q^{-1}) \cdot T_i + 1$,
(2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ *for* $1 \le i \le k-2$,
(3) $T_i T_j = T_j T_i$ *for* $|i-j| > 1$.

PROPOSITION 2.2.1. *There is an isomorphism between the Hecke algebra and the endomorphism space in the HOMFLY-PT skein category:*

$$F : H_k(q) \longrightarrow End_{\boldsymbol{OS}(q-q^{-1},l)}(\uparrow^{\otimes k})$$

$$T_i \mapsto$$

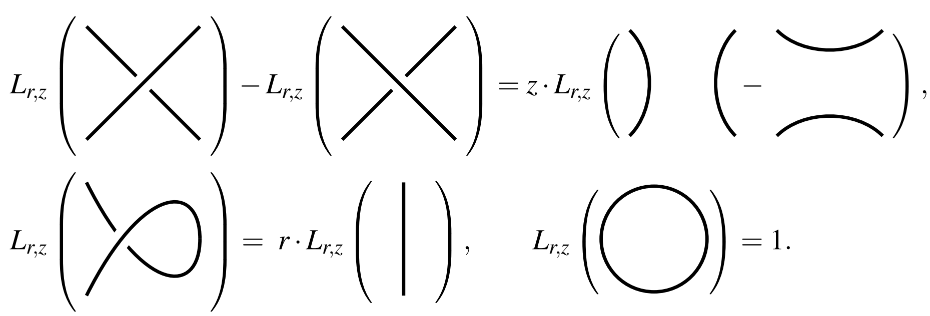


PROOF. One can verify that the relations in Definition 2.2.1 are satisfied by applying relations in Definition 2.2.2 when $z = q - q^{-1}$. $\square$

2.3. Type B,C, and D quantum link invariant and skein category, and BMW algebra

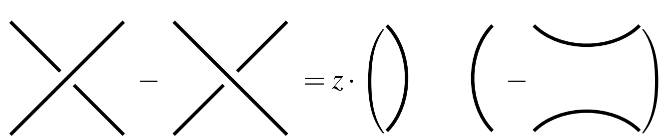
2.3.1. Kauffman polynomial. Further generalization of the Jones polynomial to the type B,C, and D cases is known as the Kauffman polynomial.

DEFINITION 2.3.1. *[29] The Kauffman polynomial $L_{r,z}(\mathscr{K})$ associated to a link or knot $\mathscr{K}$ is a rational function in r and z, satisfying the following relations:*

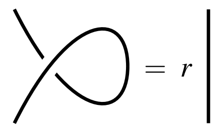


2.3.2. BMW category. Again, one can define a category whose morphisms are given by linear combinations of tangles, which satisfy the skein relations from the Kauffman polynomial.

DEFINITION 2.3.2. *Define the BMW skein category, **BMW**(r,z), to be the $\mathbb{Z}[r^{\pm},z^{\pm}]$-linear braided pivotal category with generating object $\bullet$, such that*



and



.

REMARK 2.3.1. *A link or knot $\mathscr{K}$ evaluated as a morphism in the BMW skein category from the empty word to the empty word yields a scalar $\langle \mathscr{K} \rangle_{BCD}$. $\langle \mathscr{K} \rangle_{BCD}$ is related to the Kauffman polynomial by the following relation*

$$\langle \mathscr{K} \rangle_{BCD} = \left(1 + \frac{r - r^{-1}}{z}\right) L_{r,z}(\mathscr{K}).$$

2.3.3. BMW algebra.

DEFINITION 2.3.1. *[5, 45] The k-strand BMW algebra $BMW_k(r,z)$ is the unital associative $\mathbb{Z}[r^{\pm}, z^{\pm}]$-algebra generated by e_i, g_i, g_i^{-1} for $1 \leq i \leq k-1$, with relations:*

(1) $g_i - g_i^{-1} = z(1 - e_i), \quad g_i g_i^{-1} = 1 = g_i^{-1} g_i,$

(2) $e_i^2 = \left(1 + \frac{r - r^{-1}}{z}\right) e_i,$

(3) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ *for* $1 \leq i \leq k-2,$

(4) $g_i g_j = g_j g_i$ *for* $|i - j| > 1,$

(5) $e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1} \quad$ *for* $1 \leq i \leq k-2,$

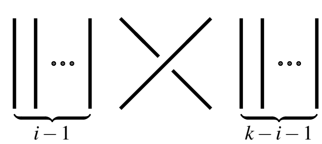
(6) $g_i g_{i+1} e_i = e_{i+1} e_i, \quad g_{i+1} g_i e_{i+1} = e_i e_{i+1} \quad$ *for* $1 \leq i \leq k-2,$

(7) $e_i g_i = g_i e_i = r^{-1} e_i,$

(8) $e_i g_{i+1} e_i = r e_i, \quad e_{i+1} g_i e_{i+1} = r e_{i+1} \quad$ *for* $1 \leq i \leq k-2.$

PROPOSITION 2.3.1. *There is an isomorphism between the BMW algebra and the endomorphism space in the BMW skein category:*

$$F : BMW_k(r,z) \longrightarrow End_{\boldsymbol{BMW}(r,z)}(\bullet^{\otimes k})$$

$g_i \mapsto$ 

$e_i \mapsto$ 

CHAPTER 3

Web categories

3.1. Quantum groups and their representation categories

We recall the definition of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$, as well as the representation category and fundamental representation category of $U_q(\mathfrak{g})$.

DEFINITION 3.1.1. *Define the quantum integer* $[n]_v := \dfrac{v^n - v^{-n}}{v - v^{-1}}$. *Denote* $[n]_v! := [1]_v[2]_v[3]_v \ldots [n]_v$. *Denote* $\begin{bmatrix} n \\ k \end{bmatrix}_v := \dfrac{[n]_v!}{[k]_v![n-k]_v!}$. *When* $v = q$, *write* $[n] := [n]_q$.

DEFINITION 3.1.2. *[**23**, Section 4.3]*

Let $\mathfrak{g}$ *be a semisimple Lie algebra, over* $\mathbb{C}$, *with associated root system* Φ, *viewed as a subset of the weight lattice* X. *Fix a choice of simple roots* $\Pi \subset \Phi$. *The Weyl group* W *acts on* $\mathbb{Z}\Phi$. *Write* $(-,-)$ *to denote the unique* W *invariant symmetric bilinear form on* $\mathbb{Z}\Phi$, *normalized such that* $(\alpha, \alpha) = 2$ *whenever* α *is a short root. Write*

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)} \in X \quad \textit{and} \quad q_\alpha := q^{(\alpha,\alpha)/2} \in \mathbb{C}(q) \quad \textit{for all } \alpha \in \Pi.$$

Define $U_q(\mathfrak{g})$ *as the associative* $\mathbb{F}$*-algebra generated by*

$$E_\alpha, F_\alpha, K_\alpha^{\pm 1},\ \alpha \in \Pi$$

subject to relations $(R1)$*-*$(R6)$ *[**23**, Section 4.3].*

The algebra $U_q(\mathfrak{g})$ *is a Hopf algebra with comultiplication* Δ, *antipode* S, *and counit* ε *defined on generators as follows:*

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \quad \Delta(F_\alpha) = 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}, \quad \Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \tag{3.1}$$

$$S(E_\alpha) = -K_\alpha^{-1} E_\alpha, \quad S(F_\alpha) = -F_\alpha K_\alpha, \quad S(K_\alpha) = K_\alpha^{-1}, \tag{3.2}$$

$$\varepsilon(E_\alpha)=0, \quad \varepsilon(F_\alpha)=0, \quad \textit{and} \quad \varepsilon(K_\alpha)=1. \tag{3.3}$$

The irreducible, finite dimensional, type-**1** representations of $U_q(\mathfrak{g})$ are in bijection with the finite dimensional irreducible representations of $\mathfrak{g}(\mathbb{C})$. The dominant weights, X_+, are the $Z_{\geq 0}$ span of the fundamental weights ϖ_i. For each $\lambda \in X_+$ we write $V(\lambda)$ for the $U_q(\mathfrak{g})$ module which corresponds to the $\mathfrak{g}$ representation with highest weight λ.

The algebra $U_q(\mathfrak{g})$ is a Hopf algebra, so its representation category is a monoidal category. We are only interested in type-**1** $U_q(\mathfrak{g})$ modules, that is modules such that $\{K_\alpha : \alpha \in \Pi\}$ act diagonalizably with eigenvalues in $+q^m$ for $m \in \mathbb{Z}$. It is not hard to see that the condition of being type-**1** is closed under taking tensor product.

NOTATION 3.1.1. *We write* $\mathbf{Rep}(U_q(\mathfrak{g}))$ *for the monoidal category of finite dimensional type-**1** $U_q(\mathfrak{g})$ modules.*

The category $\mathbf{Rep}(U_q(\mathfrak{g}))$ is completely reducible [**23**, Theorem 5.17]. Moreover, we can determine how a module in $\mathbf{Rep}(U_q(\mathfrak{g}))$ decomposes by looking at its weight space decomposition.

The modules $V(\lambda)$ are type-**1**. Also, we have

$$V(\lambda)\otimes V(\mu) \cong \bigoplus_{\nu\in X_+} V(\nu)^{\oplus m_\nu^{\lambda,\mu}},$$

where the integers $m_\nu^{\lambda,\mu}$ are the same as those describing the tensor product decomposition of the analogous $\mathfrak{g}(\mathbb{C})$ modules. So the tensor product of type-**1** modules are also type-**1**.

DEFINITION 3.1.1 ([**23**, Section 5.1]). *A module* $W \in \mathbf{Rep}(U_q(\mathfrak{g}))$ *decomposes as a direct sum*

$$W = \oplus_{\mu\in X} W_\mu,$$

where

$$W_\mu = \{w \in W \mid K_\alpha w = q^{(\alpha,\mu)}w, \alpha \in \Pi\}.$$

We will call this direct sum decomposition the ***weight space decomposition*** *of W, say that W_μ is the μ* ***weight space*** *of W, and call $w \in W_\mu$ a* ***weight vector*** *of weight μ. We say that*

$$\mathrm{wt}\, W := \{\mu \mid W_\mu \neq 0\}$$

*is the set of **weights** of W.*

NOTATION 3.1.2. *Let W be a module in $\mathbf{Rep}(U_q(\mathfrak{g}))$. For each $\lambda \in X_+$ there are non-negative integers $m_\lambda(W)$ such that*

$$W \cong \bigoplus_{\lambda \in X_+} V(\lambda)^{\oplus m_\lambda(W)}.$$

We write $[W : V(\lambda)] := m_\lambda(W)$ in this case.

DEFINITION 3.1.2. *The category of **fundamental representations**, $\mathbf{Fund}(U_q(\mathfrak{g}))$ is the full monoidal subcategory of $\mathbf{Rep}(U_q(\mathfrak{g}))$ generated by the objects $V(\varpi_i)$.*

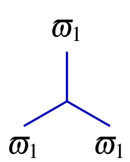
REMARK 3.1.1. *The objects in the category $\mathbf{Fund}(U_q(\mathfrak{g}))$ are all isomorphic to iterated tensor products of fundamental representations. This includes the empty tensor product, which we take to be the trivial module, denoted by $\mathbf{1}$. The category is $\mathbb{C}(q)$-linear additive, but is not closed under taking direct summands.*

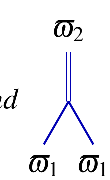
3.2. Web categories for G_2

We now recall the definition of the first web category $\mathbf{Web}_q(\mathfrak{g}_2)$, initially invented to compute the quantum invariants for links and knots in the G_2 case by introducing trivalent graphs [**35**].

3.2.1. Definition of $\mathfrak{g}_2$ Webs.

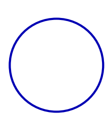
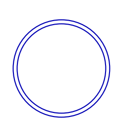
DEFINITION 3.2.1. *[**35, 36**] The category $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ is the strict pivotal $\mathbb{C}(q)$-linear category, whose objects are generated by self-dual objects ϖ_1 and ϖ_2, and whose morphisms are generated by the following two trivalent vertices:*

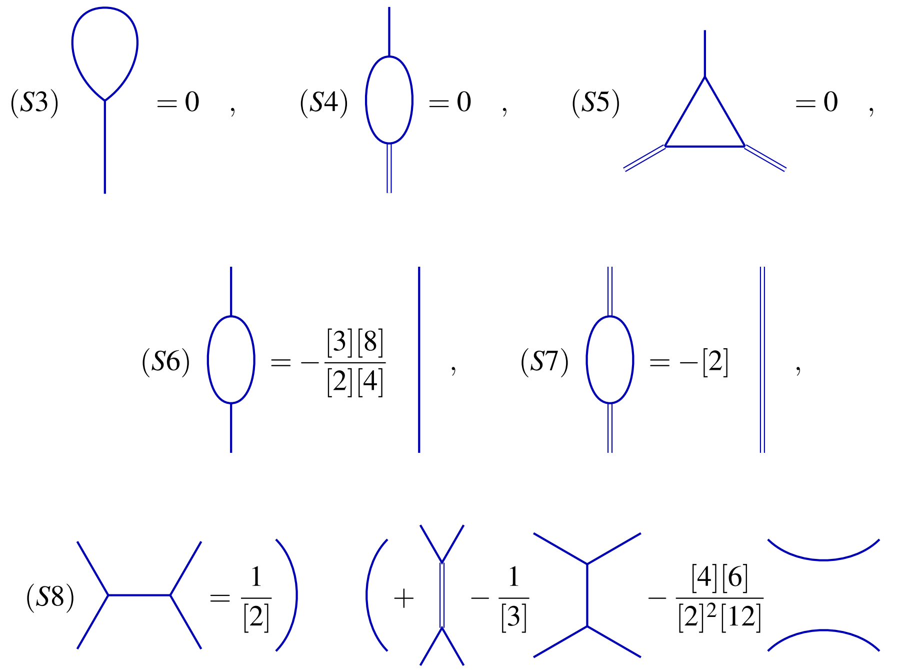


$\in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\varpi_1 \otimes \varpi_1, \varpi_1)$ *and*


$\in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\varpi_1 \otimes \varpi_1, \varpi_2)$,

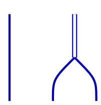
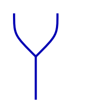
modulo the tensor-ideal generated by the following relations:

$(S1)$  $= \dfrac{[2][7][12]}{[4][6]}$, $(S2)$  $= \dfrac{[7][8][15]}{[3][4][5]}$,

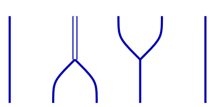


The tensor product of objects in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ is concatenation of words. Tensor product of morphisms is horizontal concatenation. Composition of morphisms is vertical stacking.

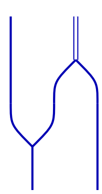
EXAMPLE 3.2.1. *Let*

$f =$  $\in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\varpi_1^{\otimes 3}, \varpi_1 \otimes \varpi_2)$, $\quad g =$  $\in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\varpi_1^{\otimes 2}, \varpi_1^{\otimes 3})$.

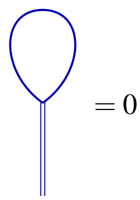
Then

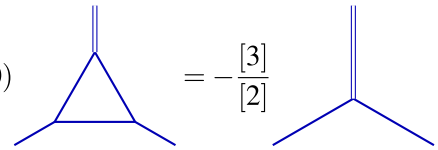
$f \otimes g =$  $\in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\varpi_1^{\otimes 5}, \varpi_1 \otimes \varpi_2 \otimes \varpi_1^{\otimes 3})$

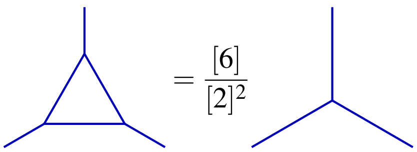
and

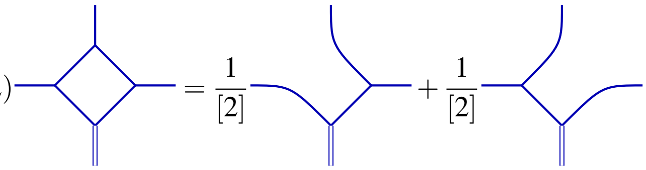
$f \circ g =$  $\in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\varpi_1^{\otimes 2}, \varpi_1 \otimes \varpi_2)$.

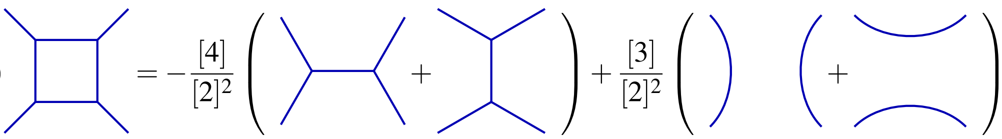
LEMMA 3.2.1. *The following relations follow from the skein relations given in Definition 3.2.1:*

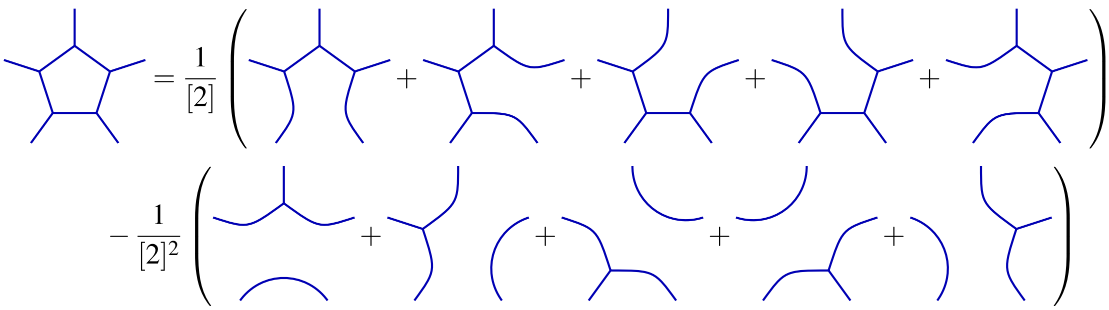
(S9) 

(S10) 

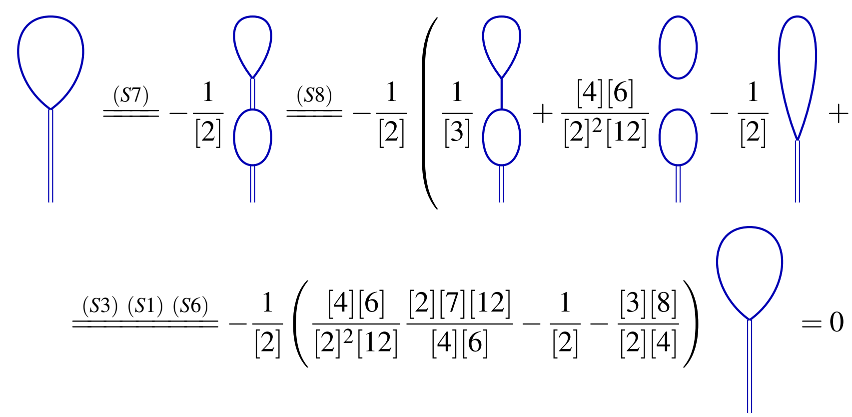
(S11) 

(S12) 

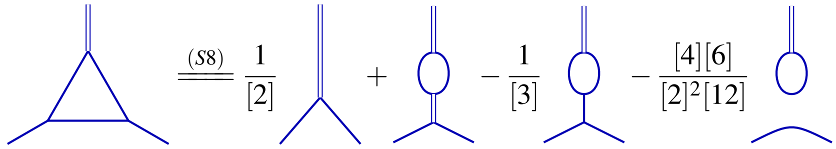
(S13) 

(S14) 

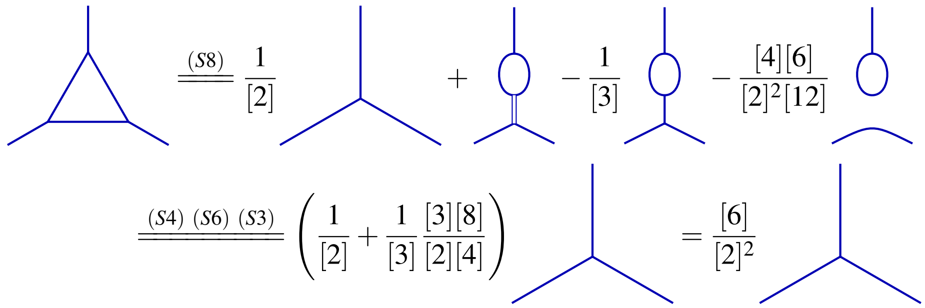
PROOF.

(S9) : 

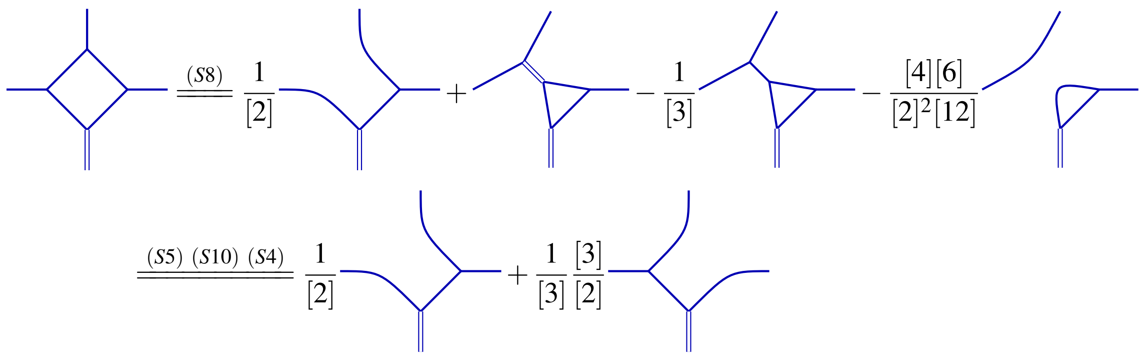
(S10) :



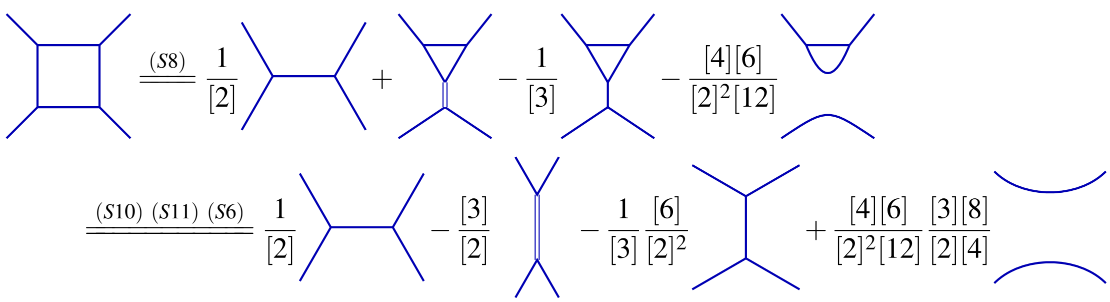
(S11) :



(S12) :

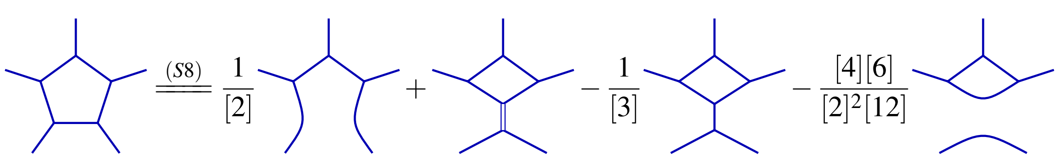


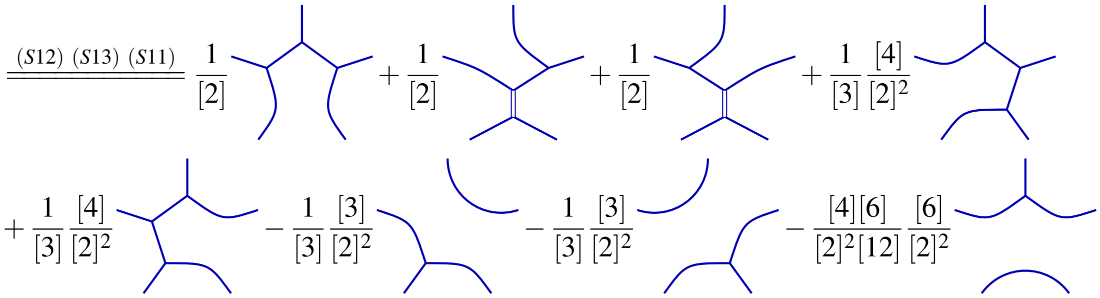
(S13) :



Then by (S8), we can replace the internal double edge and get (S13).

(S14) :





Then use (S8) to get rid of the internal double edges, and we obtain (S14).

□

3.2.2. Equivalence between $\mathrm{Kar}(\mathbf{Web}_q(\mathfrak{g}_2))$ **and** $\mathbf{Rep}(U_q(\mathfrak{g}_2))$**.** We recall the results of **[36]** which describe the relation between $\mathfrak{g}_2$ webs and representations of the quantum group associated to $\mathfrak{g}_2$. We will only work over the field $\mathbb{C}(q)$ where q is either an indeterminant or a generic element of $\mathbb{C}^\times$.

NOTATION 3.2.1. *Let Φ be the root system of type $\mathfrak{g}_2$ with Weyl group W and simple roots α_1 and α_2, where α_1 is the short root. It follows that the positive roots are*

$$\Phi_+ = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}.$$

Equip $\mathbb{Z}\Phi$ with the W invariant symmetric form determined by

$$(\alpha_1,\alpha_1) = 2, \quad (\alpha_1,\alpha_2) = -3 = (\alpha_2,\alpha_1), \quad \textit{and} \quad (\alpha_2,\alpha_2) = 6.$$

We write X for the integral weight lattice and X_+ for the dominant integral weights. The fundamental weights are $\varpi_1 = 2\alpha_1+\alpha_2$ and $\varpi_2 = 3\alpha_1+2\alpha_2$. We may use the notation (a,b) for $a\varpi_1+b\varpi_2$, in particular $X_+ = \{(a,b) \mid a,b \geq 0\}$.

DEFINITION 3.2.2. *Let $\lambda,\mu \in X_+$. We define $\mu \leq \lambda$ if $\lambda-\mu$ is a non-negative linear combination of positive roots. We also write $\mu < \lambda$ if $\mu \leq \lambda$ and $\mu \neq \lambda$.*

DEFINITION 3.2.3. *Let $\underline{\mathrm{w}}$ be an object in $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. Then $\underline{\mathrm{w}} = w_1 w_2 \dots w_n$ for $w_i \in \{\varpi_1,\varpi_2\}$. We define*

$$V(\underline{\mathrm{w}}) := V(w_1)\otimes V(w_2)\otimes \dots \otimes V(w_n).$$

REMARK 3.2.1. *Note that*

$$V(\underline{\mathrm{w}}) \cong \bigoplus_{\mu \in X_+} V(\mu)^{\oplus m_\mu^{\mathrm{w}}}.$$

The integers $m_\mu^{\mathrm{w}} := m_\mu(V(\underline{\mathrm{w}})) = [V(\underline{\mathrm{w}}) : V(\mu)]$ *are the same as those describing the tensor product decomposition of the analogous* $\mathfrak{g}_2(\mathbb{C})$ *modules.*

NOTATION 3.2.2. *Given an object* $\underline{\mathrm{w}} = w_1 w_2 \dots w_n$*, we write*

$$\mathrm{wt}\,\underline{\mathrm{w}} = \sum_{i=1}^{n} \mathrm{wt}\, w_i.$$

Note that $\mathrm{wt}\,\underline{\mathrm{w}} \in X_+$ *for all* $\underline{\mathrm{w}}$.

THEOREM 3.2.1 ([**36**, Theorem 5.1]). *There is an essentially surjective monoidal functor*

$$\Phi : \boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2) \to \mathbf{Fund}(\mathrm{U_q}(\mathfrak{g}_2))$$

such that $\Phi(\varpi) = V(\varpi)$ *for* $\varpi \in \{\varpi_1, \varpi_2\}$.

THEOREM 3.2.2 ([**36**, Theorem 6.10]). *Let* $\underline{\mathrm{w}}$ *and* $\underline{\mathrm{u}}$ *be objects in* $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$*. Then*

$$\dim \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}}) = \dim \mathrm{Hom}_{\mathbf{Fund}(\mathrm{U_q}(\mathfrak{g}_2))}(V(\underline{\mathrm{w}}), V(\underline{\mathrm{u}})),$$

and it follows that the functor Φ *is an equivalence of monoidal categories.*

Recall that given a category $\mathscr{C}$, the Karoubi envelope of $\mathscr{C}$, is the category with objects: pairs (X, e), where X is an object in $\mathscr{C}$ and $e \in \mathrm{End}_{\mathscr{C}}(X)$ is an idempotent, and morphisms: triples $(e', f, e) : (X, e) \to (Y, e')$, where $f : X \to Y$ is a morphism in $\mathscr{C}$ so that $e' \circ f \circ e = f$. Given a $\mathbb{C}(q)$-linear category $\mathscr{C}$, the additive envelope of $\mathscr{C}$ is the category with objects formal direct sums of objects in $\mathscr{C}$ and morphisms matrices of morphisms in $\mathscr{C}$.

DEFINITION 3.2.4. *Let* $\mathscr{C}$ *be a* $\mathbb{C}(q)$*-linear category. Define the* Karoubi completion *of* $\mathscr{C}$ *to be the additive envelope of the Karoubi envelope of* $\mathscr{C}$.

COROLLARY 3.2.1. *The functor* Φ *induces an equivalence of monoidal categories*

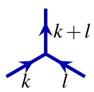
$$\mathrm{Kar}(\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)) \to \mathbf{Rep}(\mathrm{U_q}(\mathfrak{g}_2))$$

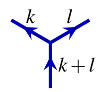
such that $(\underline{\mathrm{w}},e)\mapsto \operatorname{im}\Phi(e)$ *and* $(e',f,e):(\underline{\mathrm{w}},e)\to(\underline{\mathrm{w}},e')\mapsto\Phi(e'\circ f\circ e)$.

PROOF. Since every object in $\mathbf{Rep}(\mathrm{U_q}(\mathfrak{g}_2))$ is a direct sum of direct summands of objects in $\mathbf{Fund}(\mathrm{U_q}(\mathfrak{g}_2))$, this follows from Φ being an equivalence. $\square$

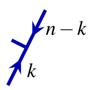
3.3. Web categories in type A

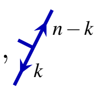
DEFINITION 3.3.1. *[**11**] Define the pivotal $\mathbb{C}(q)$-linear category $\boldsymbol{Web}_q(\mathfrak{sl}_n)$ whose objects are generated by the objects $k^{\pm}, k\in\{1,2,...,n-1\}$, and whose morphisms are generated by the following four types of vertices:*



$\in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{sl}_n)}(k\otimes l,k+l)$, 

$\in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{sl}_n)}(k+l,k\otimes l)$,

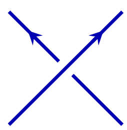
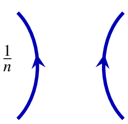
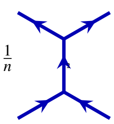


$\in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{sl}_n)}(k,(n-k)^-)$, 

$\in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{sl}_n)}(k^-,n-k)$,

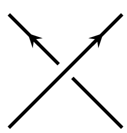
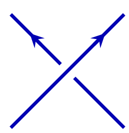
*for $k\in\mathbb{Z}_{\geq 0}$, modulo the tensor ideal generated by the relations (2.3)-(2.10) from [**11**].*

PROPOSITION 3.3.1. *[**11**] The braiding $\beta_{1,1}\in Hom_{\boldsymbol{Web}_q(\mathfrak{sl}_n)}(1\otimes 1,1\otimes 1)$ is given by the following*

 $=q^{1-\frac{1}{n}}$  $-q^{-\frac{1}{n}}$  .

3.3.1. Functor from the type A web categories to the HOMFLY-PT skein category.

PROPOSITION 3.3.2. *The assignments* $\uparrow\mapsto 1$ *and*

 $\mapsto q^{\frac{1}{n}}$ 

determines a pivotal braided monoidal functor

$$\eta:\boldsymbol{OS}(q-q^{-1},q^n)\longrightarrow \boldsymbol{Web}_q(\mathfrak{sl}_n).$$

The functor is full when we restrict $\boldsymbol{Web}_q(\mathfrak{sl}_n)$ to its full subcategory whose objects are generated by 1.

PROOF. One can verify that the relations in Definition 2.2.2 are satisfied, by resolving crossings into trivalent graphs using Proposition 3.3.1, and then apply relations in Definition 3.3.1. $\square$

3.3.2. Equivalence between the type A web categories and the representation categories of quantum groups in type A.

THEOREM 3.3.2. *[11] There is a functor*

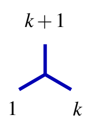
$$\Phi : \boldsymbol{Web}_q(\mathfrak{sl}_n) \to \mathbf{Fund}(U_q(\mathfrak{sl}_n))$$

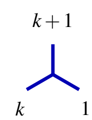
sending k to V_{ω_k}, which is an equivalence of $\mathbb{C}(q)$-linear pivotal categories.

REMARK 3.3.1. *The composition of two functors $\Phi \circ \eta$ induces a full functor from $\boldsymbol{OS}(q-q^{-1}, q^n)$ to $\mathbf{Fund}(U_q(\mathfrak{sl}_n))$, which is not faithful.*

3.4. Web categories in type C

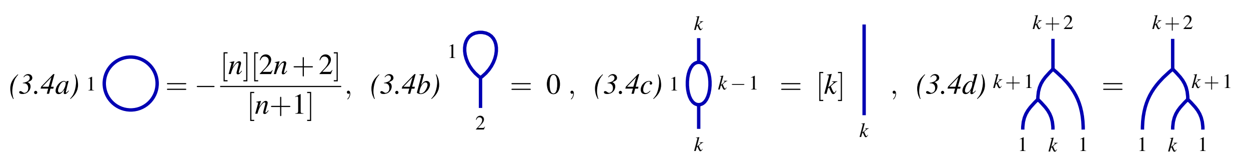
DEFINITION 3.4.1. *[8] Define the pivotal $\mathbb{C}(q)$-linear category $\boldsymbol{Web}_q(\mathfrak{sp}_{2n})$ whose objects are generated by the self-dual objects $k \in \{1, 2, ..., n\}$, and whose morphisms are generated by the following trivalent vertices:*

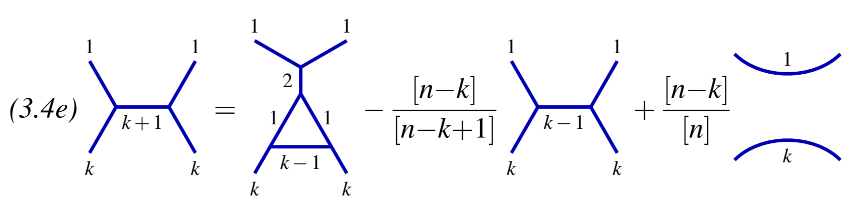


$\in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{sp}_{2n})}(1 \otimes k, k+1)$,


$\in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{sp}_{2n})}(k \otimes 1, k+1)$

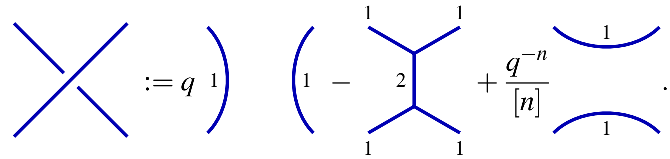
for $k \in \mathbb{Z}_{\geq 0}$, modulo the tensor ideal generated by id_k for $k > n$, and the following relations.



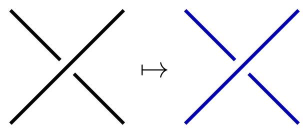


3.4.1. Functor from the type C web categories to the BMW category.

DEFINITION 3.4.2. *[8] The braiding* $\beta_{1,1} \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{sp}_{2\mathrm{n}})}(1 \otimes 1, 1 \otimes 1)$ *is given by the following*



PROPOSITION 3.4.1. *[8] The assignments* $\bullet \mapsto 1$ *and*



determines a pivotal braided monoidal functor

$$\eta : \boldsymbol{BMW}(-q^{2n+1}, q - q^{-1}) \longrightarrow \boldsymbol{Web}_q(\mathfrak{sp}_{2n}).$$

The functor is full when we restrict $\boldsymbol{Web}_q(\mathfrak{sp}_{2n})$ *to its full subcategory whose objects are generated by* 1.

3.4.2. Equivalence between the type C web categories and the representation categories of quantum groups in type C.

THEOREM 3.4.3. *[8] There is a functor*

$$\Phi : \boldsymbol{Web}_q(\mathfrak{sp}_{2n}) \to \mathbf{Fund}(U_q(\mathfrak{sp}_{2n}))$$

sending k to V_{ω_k}*, which is an equivalence of* $\mathbb{C}(q)$*-linear pivotal categories.*

REMARK 3.4.1. *The composition of two functors* $\Phi \circ \eta$ *induces a full functor from* $\boldsymbol{BMW}(-q^{2n+1}, q - q^{-1})$ *to* $\mathbf{Fund}(U_q(\mathfrak{sp}_{2n}))$*, which is not faithful.*

3.5. Web categories for the quantum orthogonal group

3.5.1. Results and idea of the proof.

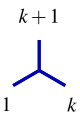
3.5.1.1. *Results.*

NOTATION 3.5.1. *Let* $\mathbb{F} := \mathbb{C}(q)$ *and* $\mathbb{A} := \mathbb{C}[q]_{(q-1)\mathbb{C}[q]} \subset \mathbb{F}$*. Note that* $\mathbb{C} \cong \mathbb{A}/(q-1)\mathbb{A}$*.*

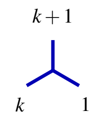
REMARK 3.5.1. *If v is a power of q, in particular if* $v = q$ *or* q^2*, then quantum integers* $[n]_v := \dfrac{v^n - v^{-n}}{v - v^{-1}}$ *and quantum binomials* $[n]_v! := [1]_v[2]_v[3]_v \dots [n]_v$ *lie in* $\mathbb{A}$*. Therefore we can consider their image in* $\mathbb{C}$ *or*

$\mathbb{F}$. *We do not make new notation for this, but instead leave it up to context whether a particular expression involving quantum integers is in* $\mathbb{F}, \mathbb{A}$, *or* $\mathbb{C}$.

DEFINITION 3.5.1. *Let* $\boldsymbol{R} \in \{\mathbb{F}, \mathbb{A}, \mathbb{C}\}$. *Define the pivotal* $\boldsymbol{R}$*-linear category* $\boldsymbol{Web_R}(\mathrm{O(m)})$ *whose objects are generated by the self-dual objects* $n \in \mathbb{Z}_{\geq 0}$, *and whose morphisms are generated by the following trivalent vertices:*



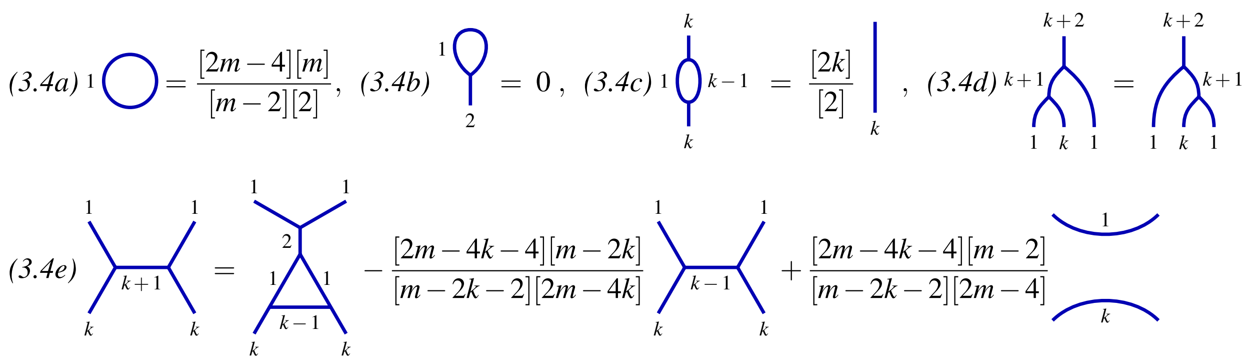
$\in \mathrm{Hom}_{\boldsymbol{Web_R}(\mathrm{O(m)})}(1 \otimes k, k+1)$,



$\in \mathrm{Hom}_{\boldsymbol{Web_R}(\mathrm{O(m)})}(k \otimes 1, k+1)$

for $k \in \mathbb{Z}_{\geq 0}$, *modulo the tensor ideal generated by* id_k *for* $k > m$, *and the following relations.*

(3.4)



REMARK 3.5.2. *We will use the convention that strands labelled zero can be erased and strands labelled* $k < 0$ *are equal to zero.*

REMARK 3.5.3. *The presentation of* $\boldsymbol{Web_R}(\mathrm{O(m)})$ *we give in Definition 3.5.1 is practically the same as the presentation of* $\boldsymbol{Web}(\mathfrak{sp}_{2n})$ *in [**8**, Definition 1.1], but with different coefficients.*

REMARK 3.5.4. *When* $\boldsymbol{R} = \mathbb{C}$, *i.e.* $q = 1$, *the coefficients in Relation (3.4e) are all* ± 1.

There are also simplifications in the representation category when $q = 1$. For example, the braiding[1] becomes symmetric, meaning it is equal to its own inverse. In the symmetric case, there is a standard definition of the exterior power of a representation. But in the braided case things become more complicated [**3**].

[1]This can be defined as the map $P \circ f \circ \Theta$ [**23**, Theorem 7.8], where P is the tensor flip map, f is as in [**23**, Section 7.9], and Θ is the quasi-R matrix [**23**, Section 7.2].

In Section 3.5.3.3 we carefully define the q-analogue of the exterior powers of the defining representation. Write $\Lambda^k_{\mathbb{C}}$ for the usual k-th exterior power of the defining representation and write $\Lambda^k_{\mathbb{F}}$ to denote the q-analogue. For $\mathbf{R} \in \{\mathbb{F}, \mathbb{C}\}$, the monoidal category $\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$ is defined to be the full monoidal subcategory of $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-mod generated by $\Lambda^k_{\mathbf{R}}$ for $k = 0, \dots, m$.

The main theorem of this article is the following.

THEOREM 3.5.2. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. There is a functor*

$$\Phi_{\boldsymbol{R}} : \boldsymbol{Web_R}(\mathrm{O(m)}) \to \boldsymbol{Fund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$$

sending k to $\Lambda^k_{\boldsymbol{R}}$ which is an equivalence of $\boldsymbol{R}$-linear pivotal categories.

REMARK 3.5.5. *Instead of working with $O(m)$, we use $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$*[2]*, which is a $\mathbb{Z}/2$ extension of the universal enveloping algebra of $\mathfrak{so}_{\mathrm{m}}$. Every $O(m)$ representation can be made into a module for $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$ and vice-versa. Moreover, a $\mathbb{C}$-linear map between such representations is an $O(m)$ intertwinter if and only if it is a $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$ intertwiner. In particular, the category $\boldsymbol{Fund}(\mathrm{U}_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}))$ is isomorphic to the full monoidal subcategory of $\boldsymbol{Rep}(O(m))$ generated by the exterior powers of $\mathbb{C}^m$. We choose to use $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$, instead of $O(m)$, since it is easier to see how to relate its representations to those of $U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})$.*

REMARK 3.5.6. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. Given an $\boldsymbol{R}$-linear monoidal category $\mathscr{C}$ we can build an additive monoidal category, denoted Add($\mathscr{C}$), where the objects are formal direct sums of objects in the original category and morphisms are matrices of morphisms in the original category. Given an $\boldsymbol{R}$-linear additive monoidal category $\mathscr{A}$, we can build an additive monoidal category, called the Karoubi envelope of $\mathscr{A}$ and denoted Kar($\mathscr{A}$), which is closed under taking direct summands. Objects in Kar($\mathscr{A}$) are pairs (X, e), where X is an object in $\mathscr{A}$ and $e \in \mathrm{End}_{\mathscr{A}}(X)$ is an idempotent. Morphisms in Kar($\mathscr{A}$) are defined by*

$$\mathrm{Hom}_{Kar(\mathscr{A})}((X,e),(Y,f)) := f\,\mathrm{Hom}_{\mathscr{A}}(X,Y)e.$$

*Finite dimensional representations of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ are completely reducible. Moreover, every finite dimensional irreducible type-**1** representation of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$) is a direct summand of some tensor product of $\Lambda^k_{\boldsymbol{R}}$'s.*

[2]This is just notation for an associative algebra which acts like the enveloping algebra of a non-existent $\mathfrak{o}_{\mathrm{m}}$, and should not be taken literally. The Lie algebra of $O(m)$ is simply $\mathfrak{so}_{\mathrm{m}}$.

Thus, we have an equivalence of $\boldsymbol{R}$-linear additive monoidal categories

$$Kar(Add(\boldsymbol{Fund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})))) \cong \boldsymbol{Rep}(U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})),$$

where by $\boldsymbol{Rep}(U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$ we mean the category of finite dimensional type-$\boldsymbol{1}$ representations of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$. Since Add and Kar are universal constructions, we can interpret Theorem 3.5.2 as a presentation of the monoidal category $\boldsymbol{Rep}(U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$.

EXAMPLE 3.5.1. *Let $m = 1$. Consider the one dimensional vector space $V_{\mathbb{C}}$, spanned by basis vector v and equipped with symmetric form $(v, v) = 1$. We have $O(1) \cong \mathbb{Z}/2$, where the generator $\sigma \in \mathbb{Z}/2$ acts as $\sigma(v) = -v$. Since $\Lambda^k(V_{\mathbb{C}}) = 0$ for $k > 1$, we see that $\boldsymbol{Fund}(U_{\mathbb{C}}(\mathfrak{o}_1))$ is the monoidal category generated by $V_{\mathbb{C}}$. Write $\mathbb{C}$ to denote the trivial $O(1)$-module. It is immediate that*

$$V_{\mathbb{C}}^{\otimes d} \cong \begin{cases} V_{\mathbb{C}} & \text{if } d \text{ is odd} \\ \mathbb{C} & \text{if } d \text{ is even.} \end{cases},$$

and the isomorphisms are induced by $v^{\otimes d} \mapsto v$ and $v^{\otimes d} \mapsto 1$ respectively.

Since $m = 1$, we set $\mathrm{id}_k = 0$ in $\boldsymbol{Web}_{\mathbb{C}}(\mathrm{O(m)})$ for all $k > 1$. One easily checks that the only defining relations which are not of the form $0 = 0$ are Relation (3.4a), which says that the circle labelled 1 evaluates to $1 \in \mathbb{C}$, and Relation (3.4e) which says that the identity of $1 \otimes 1$ is equal to the cup-cap.

*The $\mathbb{C}$-linear version of the $n = 2$ case of [**27**, Exercise 4.13(i)] says that if $\mathscr{C}$ is an $\mathbb{C}$-linear additive monoidal category which is closed under taking direct summands, then monoidal functors $\boldsymbol{Rep}(\mathbb{C}[\mathbb{Z}/2]) \to \mathscr{C}$ are in bijection with objects $X \in \mathscr{C}$, equipped with an isomorphism $\alpha : X \otimes X \to \boldsymbol{1}_{\mathscr{C}}$, such that $\alpha \otimes \mathrm{id} = \mathrm{id} \otimes \alpha$. We leave it as an exercise to prove the $m = 1$, $\boldsymbol{R} = \mathbb{C}$, case of Theorem 3.5.2 by hand, and then use Remark 3.5.6 to deduce the universal property described above. Hint: for the deduction step, use the pivotal structure on $\boldsymbol{Web}_{\mathbb{C}}(\mathrm{O(m)})$ to rewrite the identity equals cup-cap relation so it resembles $\alpha \otimes \mathrm{id} = \mathrm{id} \otimes \alpha$.*

REMARK 3.5.7. *What about the $m = 1$ case over $\mathbb{F}$? It is easy to verify that $\mathbb{F} \otimes \boldsymbol{Web}_{\mathbb{C}}(O(1)) \cong \boldsymbol{Web}_{\mathbb{F}}(O(1))$. Since $SO(1) \subset O(1)$ is the trivial group, $\mathfrak{so}_1$ is the trivial Lie algebra. We are lead to define $U_{\mathbb{F}}(\mathfrak{so}_1)^3 = \mathbb{F}$ and $U_{\mathbb{F}}(\mathfrak{o}_1) = \mathbb{F}[\mathbb{Z}/2]$. The same analysis in Example 3.5.1 works here to show that there is a monoidal equivalence $\boldsymbol{Web}_{\mathbb{F}}(O(1)) \cong \boldsymbol{Fund}(U_{\mathbb{F}}(\mathfrak{o}_1))$.*

[3]Since $U_q(\mathfrak{g})$ is generated by E_α, F_α, and $K_\alpha^{\pm 1}$ for $\alpha \in \Pi$, and since there are no simple roots for $\mathfrak{so}_1$, we think of $U_{\mathbb{F}}(\mathfrak{so}_1)$ as the $\mathbb{F}$-algebra with no generators.

EXAMPLE 3.5.2. *Let $m = 2$. In this case, the Lie algebra of $SO(2) \cong S^1$ is abelian and therefore is not semisimple. In particular, its enveloping algebra does not have a Serre presentation which can be q-deformed as usual. We take the following approach. We* define *$U_{\mathbb{F}}(\mathfrak{so}_2) := \mathbb{F}[K^{\pm 1}]$ and $V_{\mathbb{F}} := \mathbb{F} \cdot a_1 \oplus \mathbb{F} \cdot b_1$, with $K \cdot a_1 = q^2 a_1$ and $K \cdot b_1 = q^{-2} b_1$. There is then an involutive algebra automorphism of $U_{\mathbb{F}}(\mathfrak{so}_2)$, denoted σ, which acts by $\sigma(K) = K^{-1}$. Defining $\sigma(a_1) = b_1$ and $\sigma(b_1) = a_1$ induces an action on $V_{\mathbb{F}}$ by the algebra $U_{\mathbb{F}}(\mathfrak{o}_2) := \mathbb{F}[K^{\pm 1}]\langle \sigma \mid \sigma^2 = 1, \quad \sigma K^{\pm 1} \sigma = \sigma(K^{\pm 1})\rangle$. The $U_{\mathbb{F}}(\mathfrak{o}_2)$ module $\Lambda^2_{\mathbb{F}}$, which is spanned by $a_1 b_1$, has $K^{\pm 1}$ in its kernel and σ acts as -1. The category **Fund**$(U_{\mathbb{F}}(\mathfrak{o}_2))$ is a q-analogue of the category of representations of $O(\mathbb{C}^2)$, generated by $\mathbb{C}^2$ and* det. *Similar to when $m = 1$, we have an equivalence of pivotal $\mathbb{F}$-linear categories **Web**$_{\mathbb{F}}(O(2)) \cong \mathbb{F} \otimes$**Web**$_{\mathbb{C}}(O(2))$. However, the braiding on **Web**$_{\mathbb{F}}(O(2))$ is non-trivial, so this is not an equivalence of braided categories.*

REMARK 3.5.8. *The $m = 1, 2$ cases of our main theorem are somewhat hidden in the body of this paper. We make a few comments along the way for how things change, but for the sake of readability we mostly explain things when $m \geq 3$, so $\mathfrak{so}_\mathrm{m}$ is semisimple and therefore we have a more uniform notation. Regardless, the main results still hold for $m = 1, 2$, and the careful reader will be able to see what needs to be changed.*

3.5.1.2. *Idea of the proof.* Citing classical results about invariant theory, Lehrer-Zhang prove [**40**, Theorem 4.8] that there is an equivalence between the Brauer category [**40**, Definition 2.4], modulo the antisymmetrizing idempotent on $m+1$ strands, and the full monoidal subcategory of **Rep**$(O(\mathbb{C}^m))$ generated by $\mathbb{C}^m$. Since $\Lambda^k_{\mathbb{C}}$ is a direct summand of $(\mathbb{C}^m)^{\otimes k}$, for $k = 0, 1, \ldots, m$, one might hope to reduce the proof of our main theorem, when $\mathbf{R} = \mathbb{C}$, to a calculation verifying that the antisymmetrizing idempotent $\frac{1}{(m+1)!} \sum_{w \in S_{m+1}} (-1)^{\ell(w)} w$ is zero in **Web**$_{\mathbb{C}}(\mathrm{O(m)})$. We carry out this calculation in Proposition 3.5.12.

The idea of the proof of Theorem 3.5.2 is roughly as follows. First prove the result for $\mathbf{R} = \mathbb{C}$, using the ideas outlined above. Then, using the well-known result that the braiding endomorphism for the tensor square of the vector representation generates the endomorphism rings of arbitrary tensor powers of the vector representation, we prove that $\Phi_{\mathbb{F}}$ is full. Finally, we carefully argue that everything we defined actually makes sense over the ring $\mathbb{A}$. The $\mathbb{A}$ versions of our categories and functors can then be specialized to $\mathbb{C}$ or $\mathbb{F}$. Since $\mathbb{A}$ is a local ring and a principal ideal domain, basic facts about finitely generated modules over a PID allow us to deduce our functor is faithful when $\mathbf{R} = \mathbb{F}$ from knowing it is full over $\mathbb{F}$ and an equivalence over $\mathbb{C}$.

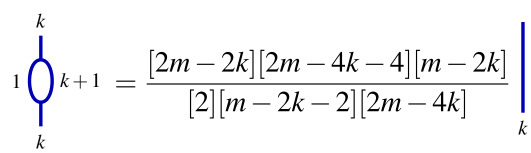
Let us comment on why we do not just prove Theorem 3.5.2 directly for $\mathbb{F}$ the same way we do for $\mathbb{C}$. There is a q-analogue of Lehrer-Zhang's result [**40**, Theorem 8.2], in which the Brauer category is replaced with the BMW category. However, just as the definition of the q-analogue of the exterior powers is not trivial, it is not so easy to explicitly describe the q-analogue of the antisymmetrizer in the BMW category. Lehrer-Zhang only discuss it abstractly [**40**, Theorem 8.2(iii)], using the theory of cellular algebras. The abstract description is sufficient to prove their result, but several years earlier Tuba-Wenzl gave a recursive formula for this idempotent by relating it to the q-antisymmetrizer in the Hecke algebra [**61**, Equation 7.12]. There is also work on explicitly describing the q-antisymmetrizer in the BMW category: when $m = 3$ in [**39**, Equation 7.8], and for all $m \geq 1$ in [**15**, **21**, **22**]. These descriptions are not very easy to compute with. An instance of this is that we have not yet found how to use the relations in $\mathbf{Web}_{\mathbb{F}}(\mathrm{O(m)})$ to show the q-antisymmetrizer on $m+1$ strands is zero in the web category for $O(m)$, even though this is implied by Theorem 3.5.2.

3.5.2. Web category for quantum orthogonal group. In this section we will use the generators and relations for $\mathbf{Web}_{\mathbf{R}}(\mathrm{O(m)})$ to derive some further relations and to establish a connection to the Birman-Mirakami-Wenzl algebras.

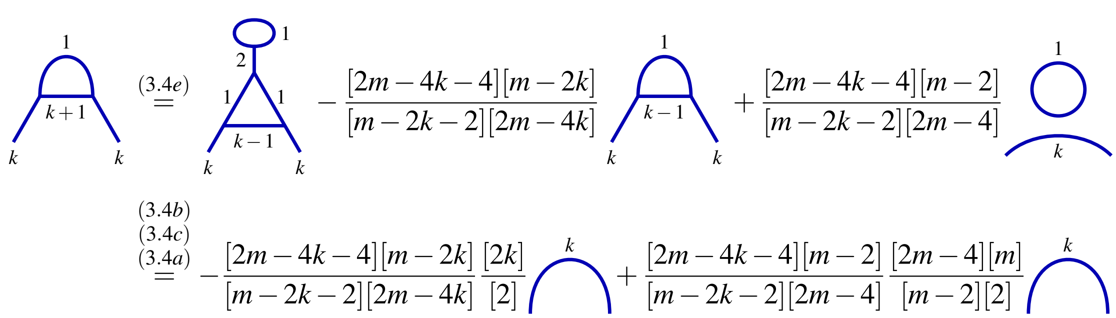
3.5.2.1. *Further relations.*

LEMMA 3.5.1.

(3.5)



PROOF.



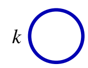
□

REMARK 3.5.9. *When $\boldsymbol{R}=\mathbb{C}$, the coefficient in Equation* (3.5) *becomes* $(m-k)$.

Using the previous Lemma, it is not hard to derive the following relation generalizing Equation (3.4a)

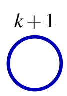
LEMMA 3.5.2.

(3.6)

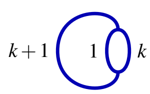


$$= \frac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}$$

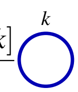
PROOF. We prove the claim by induction on k. The base case, $k=1$, follows from Equation (3.4a) . Assuming Equation (3.6) holds for k, we find



$$\overset{(3.4c)}{=} \frac{[2]}{[2k+2]}$$



$$\overset{(3.5)}{=} \frac{[2]}{[2k+2]} \frac{[2m-2k][2m-4k-4][m-2k]}{[2][m-2k-2][2m-4k]}$$



$$\overset{(3.6)}{=} \frac{[2]}{[2k+2]} \frac{[2m-2k][2m-4k-4][m-2k]}{[2][m-2k-2][2m-4k]} \frac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}$$

$$= \frac{[2m-2k][2m-4k-4][m]}{[2k+2][m-2k-2][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}$$

$$= \frac{[2m-4(k+1)][m]}{[m-2(k+1)][2m]} \begin{bmatrix} m \\ k+1 \end{bmatrix}_{q^2}.$$

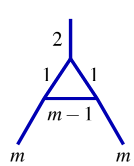
$\square$

REMARK 3.5.10. *When $\boldsymbol{R}=\mathbb{C}$, the coefficient in Equation* (3.6) *becomes* $\binom{m}{k}$.

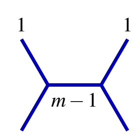
REMARK 3.5.11. *Note that* $[2m-4k]/[m-2k] = [2]_{q^{m-2k}}$. *Therefore, if* $m=2k$, *then* $[2m-4k]/[m-2k]=2$.

The following relations are a simplification of Equation (3.4e) when $k=m$.

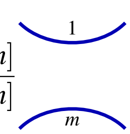
LEMMA 3.5.3.



$$=0 \qquad \textit{and}$$



$$= \frac{[m-2][2m]}{[2m-4][m]}$$

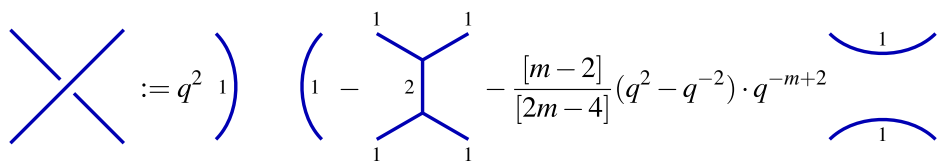


PROOF. When $k = m$, the left hand side of Equation (3.4e) is zero, since a strand carries the label $m+1$. Now, postcompose Equation (3.4e), when $k = m$, with a trivalent vertex $1 \otimes 1 \to 2$ and then simplify to derive the triangle equals 0 relation. This triangle is a subdiagram of the first term on the right hand side of Equation (3.4e), so that term is also zero. It is then easy to derive the identity equals merge-split relation in the statement of the Lemma. $\square$

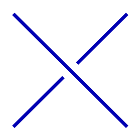
3.5.2.2. *The braiding.*

DEFINITION 3.5.3. *Let* $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. *We define* ${}^{\boldsymbol{R}}\beta_{1,1} \in \mathrm{Hom}_{\boldsymbol{Web_R}(\mathrm{O(m)})}(1 \otimes 1, 1 \otimes 1)$ *as*

(3.7)

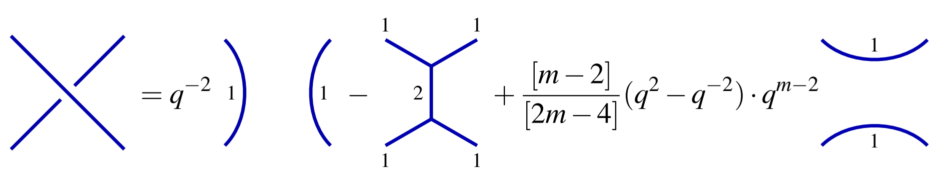


NOTATION 3.5.2. *We will write the* 90 *degree rotation of* ${}^{\boldsymbol{R}}\beta_{1,1}$ *diagrammatically as follows.*

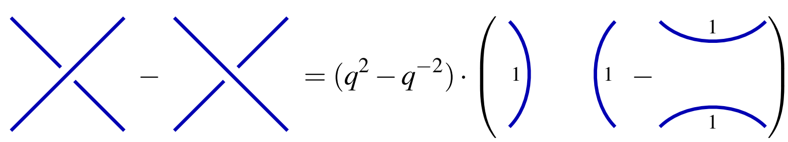


PROPOSITION 3.5.1.

(3.8)



(3.9)



PROOF. Equation (3.8) follows from rotating the diagrams on both sides of Equation (3.7), and then apply Equation (3.4e) when k=1. Equation (3.9) follows from Equation (3.7) and Equation (3.8). $\square$

REMARK 3.5.12. *We have the identities*

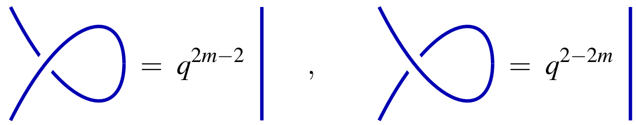
$$-\frac{[m-2]}{[2m-4]}(q^2 - q^{-2}) \cdot q^{-m+2} = q^{-2} - \frac{[2m-8][m-2]}{[m-4][2m-4]}$$

and

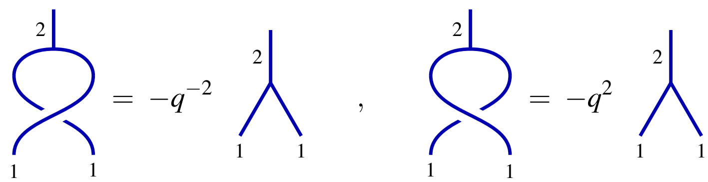
$$\frac{[m-2]}{[2m-4]}(q^2-q^{-2})\cdot q^{m-2}=q^2-\frac{[2m-8][m-2]}{[m-4][2m-4]}.$$

PROPOSITION 3.5.2. *The following relations hold in* $\textbf{\textit{Web}}_{\textbf{\textit{R}}}(\mathrm{O(m)})$.

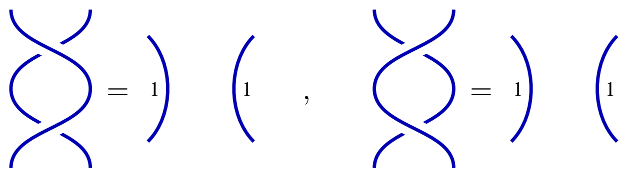
(3.10)



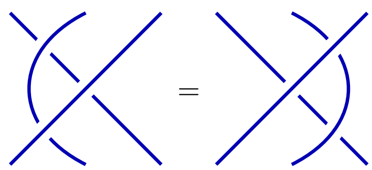
(3.11)



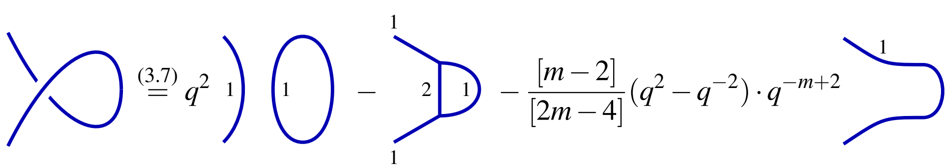
(3.12)



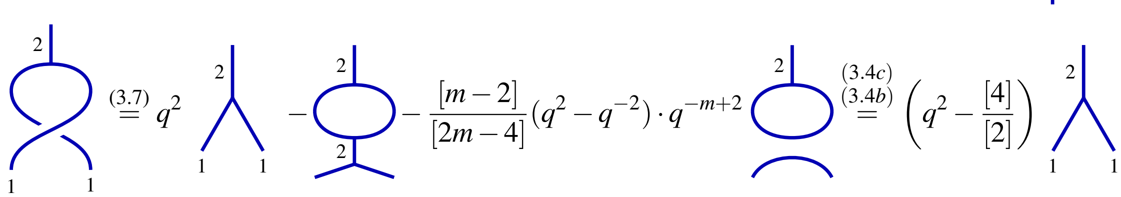
(3.13)

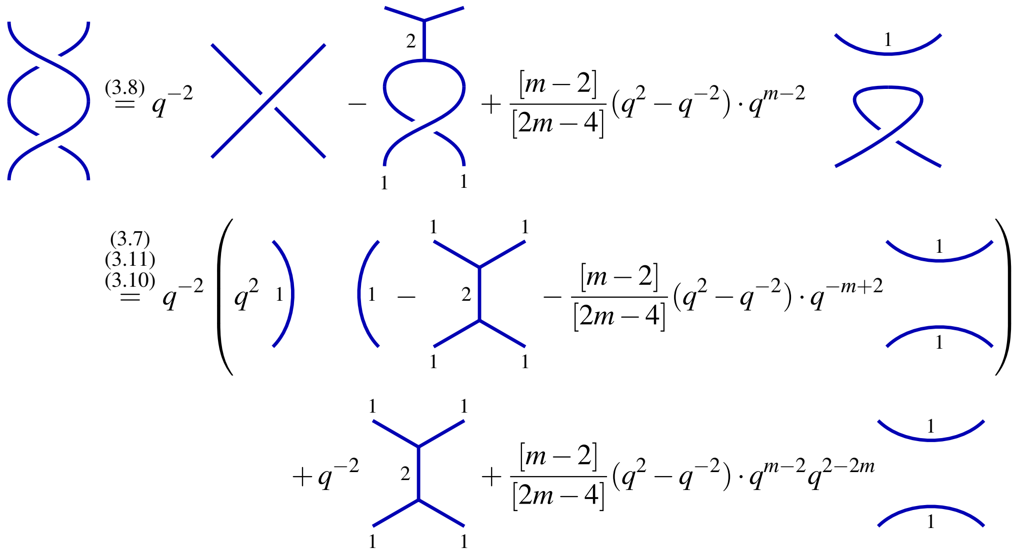


PROOF.



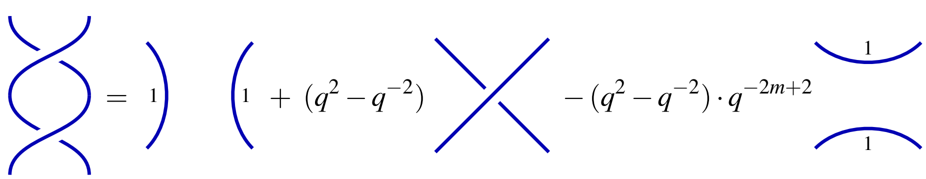
$$\overset{(3.4a)}{\underset{(3.5)}{=}}\left(q^2\frac{[2m-4][m]}{[m-2][2]}-\frac{[2m-2][2m-8][m-2]}{[2][m-4][2m-4]}-\frac{[m-2]}{[2m-4]}(q^2-q^{-2})\cdot q^{-m+2}\right)$$ 





The same argument to verify the Reidemeister *III* braid relation in the proof of [**8**, Proposition 5.7] also works in $\mathbf{Web_R}(\mathrm{O(m)})$, so we leave the verification of Equation (3.13) as an exercise. ☐

COROLLARY 3.5.4.

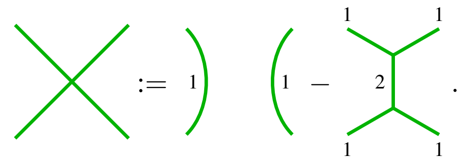


PROOF. Compose Equation (3.9) with the braiding ${}^{\mathbf{R}}\beta_{1,1}$, then apply Equations (3.10) and (3.12). ☐

NOTATION 3.5.3. *As noted in Remark 3.5.4, Remark 3.5.9, and Remark 3.5.10, upon specialization to* $\mathbb{C}$ *there is a drastic simplification in the coefficients of the defining relations. In order to make clear to the reader which calculations hold for any* $\boldsymbol{R} \in \{\mathbb{F}, \mathbb{A}, \mathbb{C}\}$ *and which are special to* $\boldsymbol{Web}_{\mathbb{C}}(\mathrm{O(m)})$*, we will color the diagrams in* $\boldsymbol{Web}_{\mathbb{C}}(\mathrm{O(m)})$ *green. Thus, a blue diagram is interpreted in* $\boldsymbol{Web_R}(\mathrm{O(m)})$ *for some* $\boldsymbol{R} \in \{\mathbb{F}, \mathbb{A}, \mathbb{C}\}$*, depending on context, while a green diagram is always interpreted in* $\boldsymbol{Web}_{\mathbb{C}}(\mathrm{O(m)})$.

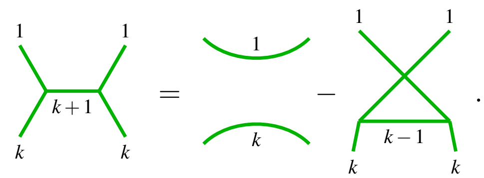
NOTATION 3.5.4. *When* $\boldsymbol{R} = \mathbb{C}$*, Equation* (3.9) *implies* ${}^{\boldsymbol{R}}\beta_{1,1} = {}^{\boldsymbol{R}}\beta_{1,1}^{-1}$*, so in our green diagrammatic calculus for* $\boldsymbol{Web}_{\mathbb{C}}(\mathrm{O(m)})$ *we do not distinguish between the over-crossing and the under-crossing. Thus,* *the formula for the braiding becomes*

(3.14)

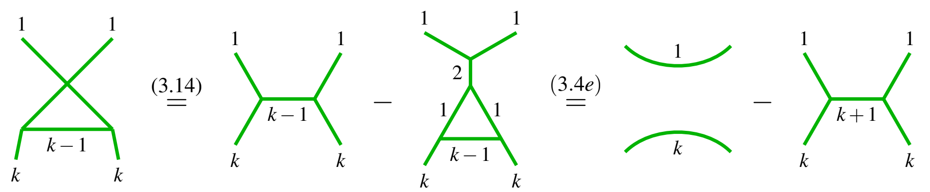


LEMMA 3.5.4. *When $\boldsymbol{R} = \mathbb{C}$, we have*

(3.15)



PROOF. This follows from



$\square$

3.5.2.3. *Finite generation.* In order to make certain arguments relating the $O(m)$ web category over $\mathbb{F}$ and over $\mathbb{C}$, we will need to know that the homomorphism spaces in $\mathbf{Web}_{\mathbb{A}}(\mathrm{O(m)})$ are finitely generated. We first show that the webs with all boundary labels 1 can be rewritten in terms of the braiding along with cups and caps.

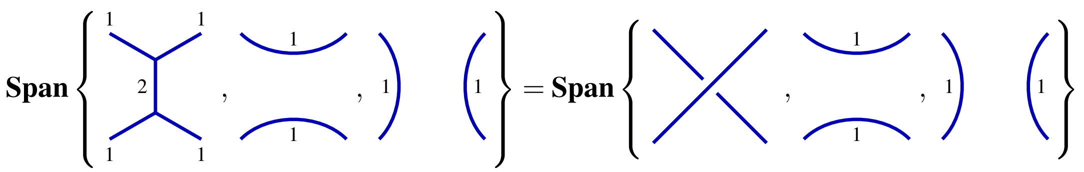
DEFINITION 3.5.5. *Define the standard web category $\boldsymbol{StdWeb}_{\boldsymbol{R}}(\mathrm{O(m)})$ as the full monoidal subcategory of $\boldsymbol{Web}_{\boldsymbol{R}}(\mathrm{O(m)})$ generated by the object 1.*

DEFINITION 3.5.6. *Define the braiding standard web category $\boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$ as the pivotal subcategory of $\boldsymbol{Web}_{\boldsymbol{R}}(\mathrm{O(m)})$, where the objects in $\boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$ are tensor products of the self-dual object 1, and morphisms in $\boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$ are generated by the braiding ${}^{\boldsymbol{R}}\beta_{1,1}$.*

REMARK 3.5.7. *Let $a, b \in \mathbb{Z}_{\geq 0}$, and write w_0 for the longest element in the symmetric group S_{a+b}. Define $\beta_{1^{\otimes a}, 1^{\otimes b}}$ to be the diagram in $\boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$ which is the positive braid lift of the minimal length* *element in the coset* $w_0 \cdot (S_a \times S_b) \in S_{a+b}/(S_a \times S_b)$*. Using that* $\boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$ *is pivotal, Equation* (3.12)*, and Equation* (3.13)*, a standard argument shows that this family of maps satisfy naturality and the hexagon axioms, see e.g. [**26**, Example 2.1], and thus make the category* $\boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$ *a braided category.*

PROPOSITION 3.5.3. $\boldsymbol{StdWeb}_{\boldsymbol{R}}(\mathrm{O(m)}) = \boldsymbol{StdWeb}^{\beta}_{\boldsymbol{R}}(\mathrm{O(m)})$

PROOF. We need to show that any morphism in $\mathbf{StdWeb_R}(\mathrm{O(m)})$ is also a morphism in $\mathbf{StdWeb}^{\beta}_{\mathbf{R}}(\mathrm{O(m)})$. We will prove this for web diagrams in $\mathbf{StdWeb_R}(\mathrm{O(m)})$. Then the desired result is immediate for linear combinations of web diagrams. By Equation (3.7), we have

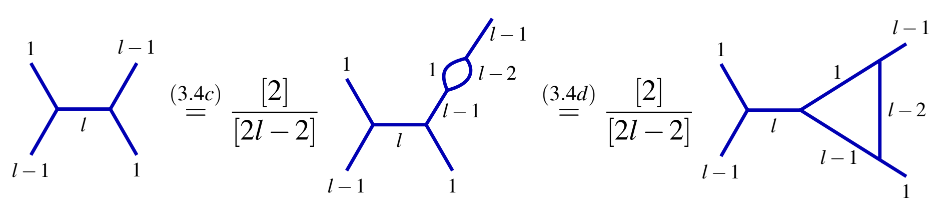


so it suffices to show that an arbitrary web diagram in $\mathbf{StdWeb_R}(\mathrm{O(m)})$ can be rewritten as a linear combination of diagrams with strands only labelled 1 and 2.

Fix a diagram f in $\mathbf{StdWeb_R}(\mathrm{O(m)})$. Suppose that the largest label on a strand in f is l. If $l \geq m+1$, then $f = 0$, and if $l \leq 2$, then we are done. Assume that $3 \leq l \leq m$. Fix a point in this l labelled strand, then choose a direction and traverse the strand away from this point in that direction. Since f is in $\mathbf{StdWeb_R}(\mathrm{O(m)})$, the l labelled strand cannot extend to the boundary. So either we return to this point, or we meet a trivalent vertex. Since l is the largest label of a strand in f, and the trivalent vertex must be a generator from Definition 3.5.1, this trivalent vertex has labels $1, l-1$, and l.

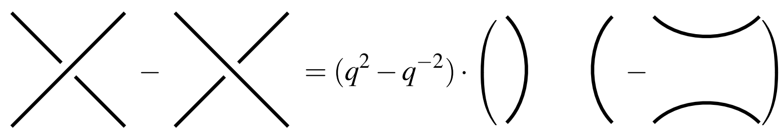
If we meet a trivalent vertex, then traversing the l labelled strand in the other direction we also meet a trivalent vertex with labels $1, l-1$, and l. On the other hand, if the strand is closed, then we can use Equation (3.4c), with $k = l$, to introduce two trivalent vertices with labels $1, l-1$, and l, up to an invertible scalar in $\mathbb{A}$. In either case, the l labelled strand is a segment between two trivalent vertices with labels $1, l-1$, and l.

We can either apply Equation (3.4e), with $k+1 = l$, or apply the following relation

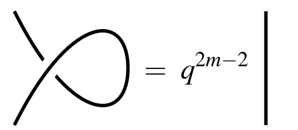


and then apply Equation (3.4e), with $k+1=l$. Thus, we can write f as a linear combination of web diagrams, each of which has one fewer strand with label l. By induction, we can remove all strands with label l, so f is a linear combination of diagrams with largest strand label less than or equal to $l-1$. Using induction again, we find that f is a linear combination of web diagrams with only 1 and 2 labelled strands. $\square$

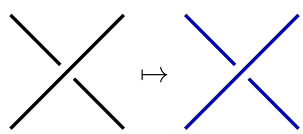
DEFINITION 3.5.1. *Define the BMW* category, $\boldsymbol{BMW_R}(\mathrm{O(m)})$, *to be the free* $\boldsymbol{R}$*-linear braided pivotal category*[4] *with generating object* $\bullet$ *which is self-dual of dimension* $\frac{[2m-4][m]}{[m-2][2]}$ *such that*



and



PROPOSITION 3.5.4. *The assignments* $\bullet \mapsto 1$ *and*



determines a full pivotal braided monoidal functor

$$\eta_{\boldsymbol{R}} : \boldsymbol{BMW_R}(\mathrm{O(m)}) \longrightarrow \boldsymbol{StdWeb_R}(\mathrm{O(m)}) .$$

PROOF. It is clear that 1 is self dual with dimension $\frac{[2m-4][m]}{[m-2][2]}$. Combining Remark 3.5.7 with Lemma 3.5.3 we also see that $\mathbf{StdWeb_R}(\mathrm{O(m)})$ is braided. Thanks to Equation (3.9) and Equation (3.10), the claim follows from the universal mapping property of $\mathbf{BMW_R}(\mathrm{O(m)})$. The image of $\eta_{\mathbf{R}}$ is $\mathbf{StdWeb}^{\beta}_{\mathbf{R}}(\mathrm{O(m)}) = \mathbf{StdWeb_R}(\mathrm{O(m)})$, so $\eta_{\mathbf{R}}$ is full. $\square$

LEMMA 3.5.5. *Homomorphism spaces in* $\boldsymbol{BMW}_{\mathbb{A}}(\mathrm{O(m)})$ *are finitely generated* $\mathbb{A}$*-modules.*

PROOF. This is standard, for example see [**44**, Theorem 3]. $\square$

[4] This means we can draw positive crossings to represent the braiding of $\bullet$ with itself, as well as draw cups and caps coming from pivotal structure.

PROPOSITION 3.5.5. *Homomorphism spaces in* $\textbf{\textit{StdWeb}}_{\mathbb{A}}(\mathrm{O(m)})$ *are finitely generated* $\mathbb{A}$*-modules.*

PROOF. Since $\eta_{\mathbb{A}}$ is full, this follows from Lemma 3.5.5. $\square$

3.5.3. Representation theory of the quantum orthogonal group.

3.5.3.1. *Quantum orthogonal algebra.* Write $X(\mathfrak{so}_{\mathrm{m}}) \subset \oplus_{i=1}^{n} \mathbb{Z}\frac{\varepsilon_i}{2}$ for the weight lattice of $\mathfrak{so}_{\mathrm{m}}$, where $m = 2n$ if m is even, and $m = 2n+1$ if m is odd. We enumerate the simple roots for $\mathfrak{so}_{2n}$ (i.e. type D_n) as

$$\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n\},$$

and for $\mathfrak{so}_{2n+1}$ (i.e. type B_n) as

$$\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n\}.$$

The pairing $(-,-)$ for $\mathfrak{so}_{2n}$ is defined as $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$ and for $\mathfrak{so}_{2n+1}$ is defined as $(\varepsilon_i, \varepsilon_j) = 2\delta_{i,j}$. The fundamental weights for $\mathfrak{so}_{\mathrm{m}}$ are:

$$\varpi_1 = \varepsilon_1, \varpi_2 = \varepsilon_1 + \varepsilon_2, \ldots, \varpi_{n-2} = \varepsilon_1 + \cdots + \varepsilon_{n-2},$$

$$\varpi_{n-1} = \frac{\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n}{2}, \quad \text{and} \quad \varpi_n = \frac{\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n}{2},$$

if $m = 2n$, and

$$\varpi_1 = \varepsilon_1, \varpi_2 = \varepsilon_1 + \varepsilon_2, \ldots, \varpi_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1},$$

and

$$\varpi_n = \frac{\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n}{2},$$

if $m = 2n+1$. The dominant weights, $X_+(\mathfrak{so}_{\mathrm{m}})$, are the $\mathbb{Z}_{\geq 0}$ span of the fundamental weights.

NOTATION 3.5.5. *In order to make the statements of our results uniform, we need to compensate for the different conventions for* $(-,-)$ *if m is even or odd. To this end, we will write*

$$U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}}) := \begin{cases} U_q(\mathfrak{so}_m), & \text{if } m \text{ is odd}, \quad \text{and} \\ U_{q^2}(\mathfrak{so}_m), & \text{if } m \text{ is even}. \end{cases}$$

DEFINITION 3.5.2. *For* $\mathbf{a} \in X(\mathfrak{so}_\mathrm{m})$ *and* $V \in U_\mathbb{F}(\mathfrak{so}_\mathrm{m})$*-mod, we set*

$$V[\mathbf{a}] := \{v \in V \mid K_\alpha v = q_\alpha^{(\alpha^\vee, \mathbf{a})} v, \text{ for all } \alpha \in \Pi\}.$$

If $v \in V[\mathbf{a}]$*, then we say that* v *is a* weight vector *of* weight $\mathbf{a}$.

DEFINITION 3.5.3. *Suppose that* V *is a finite dimensional* $U_\mathbb{F}(\mathfrak{so}_\mathrm{m})$*-module such that*

$$V = \oplus_{\mathbf{a} \in X(\mathfrak{so}_\mathrm{m})} V[\mathbf{a}],$$

then we say that V *is a* type-**1** $U_\mathbb{F}(\mathfrak{so}_\mathrm{m})$*-module.*

For each $\mathbf{a} \in X_+(\mathfrak{so}_\mathrm{m})$, there is an irreducible type-**1** $U_\mathbb{F}(\mathfrak{so}_\mathrm{m})$-module with highest weight $\mathbf{a}$, and highest weight vector $v_\mathbf{a}^+$, which we will denote by $L_\mathbb{F}(\mathbf{a})$. Moreover, each finite dimensional irreducible type-**1** $U_\mathbb{F}(\mathfrak{so}_\mathrm{m})$-module is isomorphic to $L_\mathbb{F}(\mathbf{a})$ for some $\mathbf{a} \in X_+(\mathfrak{so}_\mathrm{m})$ [**23**, Theorem 5.10].

DEFINITION 3.5.4. *We define Lusztig's divided powers algebra, denoted* $U_\mathbb{A}(\mathfrak{g})$*, as the* $\mathbb{A}$*-subalgebra in* $U_q(\mathfrak{g})$ *generated by* $K_\alpha^{\pm 1}$, $E_\alpha^{(n)} := E_\alpha^n/[n]_{q_\alpha}!$*, and* $F_\alpha^{(n)} := F_\alpha^n/[n]_{q_\alpha}!$*, for all* $\alpha \in \Pi$ *and* $n \in \mathbb{Z}_{\geq 0}$.

DEFINITION 3.5.5. *Suppose that* V *is a free finitely generate* $\mathbb{A}$*-module with an action of* $U_\mathbb{A}(\mathfrak{so}_\mathrm{m})$ *such that the* K_α *action on* V *is diagonalizable over* $\mathbb{A}$ *with all eigenvalues positive powers of* q*, for all* $\alpha \in \Pi$*. Then we say that* V *is a type-**1*** $U_\mathbb{A}(\mathfrak{so}_\mathrm{m})$*-module.*

NOTATION 3.5.6. *Let* $U_\mathbb{C}(\mathfrak{so}_\mathrm{m})$ *denote the usual enveloping algebra of* $\mathfrak{so}_m$*. Upon specialization to* $\mathbb{C}$*, we have instead to consider the elements* h_α *in the Cartan subalgebra. For* $V \in U_\mathbb{C}(\mathfrak{so}_\mathrm{m})$*-mod and* $\mathbf{a} \in X(\mathfrak{so}_\mathrm{m})$ *we have*

$$V[\mathbf{a}] := \{v \in V \mid h_\alpha v = \mathbf{a}(h_\alpha)v, \text{ for all } \alpha \in \Pi\}.$$

This is the classical notion of weight vector. For convenience, we will refer to finite dimensional $U_\mathbb{C}(\mathfrak{so}_\mathrm{m})$*-modules as type-**1** modules.*

NOTATION 3.5.7. *Let* $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$*. Suppose that* V *is a type-**1*** $U_{\boldsymbol{R}}(\mathfrak{so}_\mathrm{m})$*-module. If* $V[\mathbf{a}] \neq 0$*, then we say that* $\mathbf{a}$ *is a* weight *of* V.

DEFINITION 3.5.6. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Write $\chi(\mathfrak{so}_{\mathrm{m}})$ for the free $\mathbb{Z}$-module with basis $\{e^{\mathbf{a}}\}_{\mathbf{a}\in X(\mathfrak{so}_{\mathrm{m}})}$. The* formal character *of a type-**1** representation V is the expression*

$$\mathrm{ch}(V) := \sum_{\mathbf{a}\in X(\mathfrak{so}_{\mathrm{m}})} \dim_{\boldsymbol{R}} V[\mathbf{a}] \cdot e^{\mathbf{a}} \in \chi(\mathfrak{so}_{\mathrm{m}}).$$

For each $\mathbf{a} \in X^{+}(\mathfrak{so}_{\mathrm{m}})$ there is an irreducible $U_{\mathbb{C}}(\mathfrak{so}_{\mathrm{m}})$-module with highest weight $\mathbf{a}$, and highest weight vector $v_{\mathbf{a}}^{+}$, which we denote by $L_{\mathbb{C}}(\mathbf{a})$. Each finite dimensional irreducible representation of $U_{\mathbb{C}}(\mathfrak{so}_{\mathrm{m}})$ is isomorphic to $L_{\mathbb{C}}(\mathbf{a})$ for some $\mathbf{a} \in X_{+}(\mathfrak{so}_{\mathrm{m}})$.

LEMMA 3.5.6. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. The characters $\{\mathrm{ch}(L_{\boldsymbol{R}}(\mathbf{a}))\}_{\mathbf{a}\in X_{+}(\mathfrak{so}_{\mathrm{m}})}$ are a basis for the span of formal characters of all type-**1** representations.*

PROOF. Use that if $L_{\mathbf{R}}(\mathbf{a})[\mathbf{b}] \neq 0$, then $(\mathbf{a}-\mathbf{b}) \in \mathbb{Z}_{\geq 0}\Phi_{+}$. $\square$

LEMMA 3.5.7. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. Every type-**1** $U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})$-module is completely reducible.*

PROOF. This is Weyl's theorem on complete reducibility, when $\mathbf{R} = \mathbb{C}$, and [**23**, Theorem 5.17], when $\mathbf{R} = \mathbb{F}$ $\square$

LEMMA 3.5.8. *If V is a type-**1** $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$-module, then*

$$\mathbb{C}\otimes V \cong \bigoplus_{\mathbf{a}\in X_{+}(\mathfrak{so}_{\mathrm{m}})} L_{\mathbb{C}}(\mathbf{a})^{\oplus m_{\mathbf{a}}}, \quad \mathbb{F}\otimes V \cong \bigoplus_{\mathbf{a}\in X_{+}(\mathfrak{so}_{\mathrm{m}})} L_{\mathbb{F}}(\mathbf{a})^{\oplus n_{\mathbf{a}}}, \quad \text{and } m_{\mathbf{a}} = n_{\mathbf{a}}.$$

PROOF. Follows from Lemma 3.5.7, Lemma 3.5.6, and that $\dim_{\mathbb{C}} L_{\mathbb{C}}(\mathbf{a})[\mathbf{b}] = \dim_{\mathbb{F}} L_{\mathbb{F}}(\mathbf{a})[\mathbf{b}]$ for all $\mathbf{a} \in X_{+}(\mathfrak{so}_{\mathrm{m}})$ and $\mathbf{b} \in X(\mathfrak{so}_{\mathrm{m}})$ [**23**, Theorem 5.15]. $\square$

LEMMA 3.5.9. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. If $\mathbf{a}, \mathbf{b} \in X_{+}(\mathfrak{so}_{\mathrm{m}})$, then $\mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})}(L_{\boldsymbol{R}}(\mathbf{a}), L_{\boldsymbol{R}}(\mathbf{b})) = 0$ if $\mathbf{a} \neq \mathbf{b}$, and $\mathrm{End}_{U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})}(L_{\boldsymbol{R}}(\mathbf{a})) = \boldsymbol{R} \cdot \mathrm{id}_{L_{\boldsymbol{R}}(\mathbf{a})}$.*

PROOF. Follows from Schur's lemma and standard theory about highest weight vectors. $\square$

When $m = 2n$ there is an order 2 automorphism σ of the Dynkin diagram, swapping the simple roots $\alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_{n}$ and $\alpha_{n} = \varepsilon_{n-1} + \varepsilon_{n}$. This induces an automorphism of $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ such that

$$\sigma(E_{\alpha_{n-1}}) = E_{\alpha_{n}},\ \sigma(F_{\alpha_{n-1}}) = F_{\alpha_{n}},\ \sigma(K_{\alpha_{n-1}}) = K_{\alpha_{n}},$$

$$\sigma(E_{\alpha_n}) = E_{\alpha_{n-1}},\ \sigma(F_{\alpha_n}) = F_{\alpha_{n-1}},\ \sigma(K_{\alpha_n}) = K_{\alpha_{n-1}},$$

and σ fixes all the other generators for $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$.

If $m = 2n+1$, there are no Dynkin diagram automorphisms. In this case we write σ to denote the identity automorphism of $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$.

DEFINITION 3.5.8. *[**38**, Section 8.1.2] Let $U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})$ be the associative algebra generated by $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ and σ, such that $\sigma^2 = 1$ and $\sigma X \sigma^{-1} = \sigma(X)$, for $X \in U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$.*

The algebra $U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})$ is a Hopf algebra with Δ, S, ε defined on elements of $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ as in Definition 3.1.2, along with

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad S(\sigma) = \sigma^{-1}, \text{ and } \quad \varepsilon(\sigma) = 1.$$

The automorphism σ preserves $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}}) \subset U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$, so we define $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$ as the algebra generated by $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$ and σ, such that $\sigma X \sigma^{-1} = \sigma(X)$, for $X \in U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$. Note that $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$ is the unital $\mathbb{A}$ subalgebra of $U_K(\mathfrak{o}_{\mathrm{m}})$ generated by σ, $K_\alpha^{\pm 1}$, $E_\alpha^{(n)}$, and $F_\alpha^{(n)}$, for all $n \in \mathbb{Z}_{\geq 0}$, $\alpha \in \Pi$.

DEFINITION 3.5.9. *Define $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$ as the universal enveloping algebra of $\mathfrak{so}_m(\mathbb{C})$, denoted $U(\mathfrak{so}_{\mathrm{m}}(\mathbb{C}))$, augmented by the algebra automorphism, which we will denote by σ, determined by the non-trivial Dynkin diagram automorphism when m is even, and the identity automorphism when m is odd.*

For any $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-module which restricts to a type-**1** $U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$-module, we can use the same notion of weight spaces as in Definition 3.5.3 and Notation 3.5.6, and such a module will be a direct sum of its weight spaces. Note that the equation $\sigma K_\alpha \sigma^{-1} = \sigma(K_\alpha)$ implies that σ acts on weight spaces. The induced action on weights is such that σ acts on $X(\mathfrak{so}_{\mathrm{m}})$ trivially if $m = 2n+1$, and σ swaps ϖ_{n-1} and ϖ_n if $m = 2n$.

REMARK 3.5.13. *Any finite dimensional representation of $O(\mathbb{C}^m)$ is a finite dimensional representation of $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$ such that the weights are contained in $\oplus_{i=1}^n \mathbb{Z}\varepsilon_i$, and vice-versa. Moreover, a linear map between such representations commutes with the actions of $O(\mathbb{C}^m)$ if and only if the map commutes with $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$. Such representations are exactly the $O(\mathbb{C}^m)$ modules which occur as submodules of $(\mathbb{C}^m)^{\otimes d}$ for some $d \geq 0$.*

DEFINITION 3.5.7. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. A $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module such that its restriction to $U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})$ is type-**1** with weights contained in $\oplus_{i=1}^n \mathbb{Z}\varepsilon_i$, will be referred to as a* type-**1** *$U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module*[5].

[5]An example of a $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})$-module which is not of this form would be the induction, from $U_{\mathbb{C}}(\mathfrak{so}_{2n+1})$, of the irreducible spinor module $V(\varpi_k)$, i.e. $U_{\mathbb{C}}(\mathfrak{o}_{2n+1}) \otimes_{U_{\mathbb{C}}(\mathfrak{so}_{2n+1})} V(\varpi_n)$.

DEFINITION 3.5.8. *Let $\boldsymbol{R}[\mathbb{Z}/2]$ denote the group algebra of $\mathbb{Z}/2$ over $\boldsymbol{R}$. There is an algebra homomorphism*

$$U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}) \to \boldsymbol{R}[\mathbb{Z}/2]$$

with $U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})$ in the kernel and such that $\sigma \mapsto -1 \in \mathbb{Z}/2$. Composing this homomorphism with the sign representation of $\mathbb{Z}/2$, we obtain a one dimensional $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module, denoted $\det_{\boldsymbol{R}}$.

REMARK 3.5.14. *The module* $\det_{\boldsymbol{R}}$ *restricts to the trivial $U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})$ module, and therefore is a type-**1** $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module.*

There is a classification of finite dimensional irreducible type-**1** $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-modules. If $\mathbf{a} \in \oplus_{i=1}^{n}\mathbb{Z}\varepsilon_i \cap X_+(\mathfrak{so}_{\mathrm{m}})$, then $\mathbf{a} = \sum_{i=1}^{n} \mathbf{a}_i \varepsilon_i$ such that $\mathbf{a}_1 \geq \cdots \geq \mathbf{a}_{n-1} \geq |\mathbf{a}_n|$, and $\mathbf{a}_n = |\mathbf{a}_n|$ if $m = 2n+1$, while $\mathbf{a}_n$ is any integer if $m = 2n$. For such an $\mathbf{a}$, we obtain a representation $L_{\mathbf{R}}(\mathbf{a})$ of $U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$, which we can then induce to $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$. The induced module $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}) \otimes_{U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})} L_{\mathbf{R}}(\mathbf{a})$ is isomorphic to $L_{\mathbf{R}}(\mathbf{a}) \oplus L_{\mathbf{R}}(\sigma(\mathbf{a}))$ as $U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$-modules, by the map $1 \otimes \ell \mapsto (\ell, 0)$ and $\sigma \otimes \ell \mapsto (0, \ell)$. The action of $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$ is determined by

$$\sigma \cdot (\ell, \ell') = (\ell', \ell) \quad \text{and} \quad X \cdot (\ell, \ell') = (X \cdot \ell, \sigma(X) \cdot \ell').$$

If $m = 2n+1$, or $m = 2n$ and $\mathbf{a}_n = 0$, then the induced module decomposes into a direct sum of two irreducible $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-modules corresponding to the $+1$ and -1 eigenspaces of σ. We write $L_{\mathbf{R}}(\mathbf{a}, +1)$ and $L_{\mathbf{R}}(\mathbf{a}, -1)$ for these representations. If $m = 2n$ and $\mathbf{a}_n \neq 0$, then the induced module is irreducible and is isomorphic to $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}) \otimes_{U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})} L(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, -\mathbf{a}_n)$.

PROPOSITION 3.5.6. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. The following is a complete and irredundant list of irreducible type-**1** representations of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$. For $m = 2n+1$:*

$$L_{\boldsymbol{R}}(\mathbf{a}, +1) \quad \textit{and} \quad L_{\boldsymbol{R}}(\mathbf{a}, -1), \quad \textit{such that} \quad \mathbf{a}_1 \geq \cdots \geq \mathbf{a}_n \geq 0,$$

and for $m = 2n$:

$$U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}) \otimes_{U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})} L(\mathbf{a}) \quad \textit{such that} \quad \mathbf{a}_1 \geq \cdots \geq \mathbf{a}_n > 0,$$

$$L_{\boldsymbol{R}}(\mathbf{a}, +1) \quad \textit{and} \quad L_{\boldsymbol{R}}(\mathbf{a}, -1), \quad \textit{such that} \quad \mathbf{a}_1 \geq \cdots \geq \mathbf{a}_n = 0.$$

PROOF. Use that σ is central when $m = 2n+1$. For $m = 2n$, observe that σ preserves the space of vectors annihilated by E_α's and acts on weights by $(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_n) \mapsto (\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, -\mathbf{a}_n)$. For more details, see [**20**, Section 5.5.5]. $\square$

LEMMA 3.5.10. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. If S and T are two irreducible type-**1** $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-modules from the list of irreducibles in Proposition 3.5.6, then*

$$\operatorname{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}(S,T) = \begin{cases} 0 & \text{if } S \neq T, \quad \text{and} \\ \boldsymbol{R} \cdot \mathrm{id}_S & \text{if } S = T. \end{cases}$$

PROOF. This follows by looking first at $\operatorname{Hom}_{U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})}(\mathbf{Res}(S), \mathbf{Res}(T))$, then analyzing which of these maps commute with σ. We leave it to the reader to complete the case-by-case analysis. $\square$

If W is a type-**1** $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-module, then $W^* := \operatorname{Hom}_{\mathbf{R}}(W, \mathbf{R})$ is an $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-module via the antipode, denoted S in Definition 3.5.8. We say W is *self-dual* if $W \cong W^*$ as $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-modules.

LEMMA 3.5.11. *The type-**1** irreducible representations of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ are self-dual.*

PROOF. The irreducible representations of $U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$ are self-dual. We leave it as an exercise to the reader to verify that inducing a self-dual module from $U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$ to $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$ results in a self-dual module. Since a direct summand of a self-dual module is self-dual, the claim follows from Proposition 3.5.6. $\square$

LEMMA 3.5.12. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. Every type-**1** $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module is completely reducible.*

PROOF. Let W be a type-**1** $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-module and let $S \subset W$ be a $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-submodule, it suffices to show that S is a direct summand of W. We adapt the argument in [**2**, Theorems 3.1, 9.2]. By Lemma 3.5.7 there is an idempotent $e_S \in \operatorname{End}_{U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})}(W)$ with image S. Note that σ induces a linear endomorphism of W which preserves S. One can check that $e'_S := \frac{1}{2}(e_S + \sigma \circ e_S \circ \sigma)$ is an endomorphism of W which commutes with $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$, has image contained in S, and acts as the identity on S. Thus, $e'_S \in \operatorname{End}_{U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})}(W)$ is an idempotent with image S. $\square$

REMARK 3.5.15. *It now follows that every irreducible type-**1** $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module is self-dual.*

Let $\mathbf{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. For a type-$\mathbf{1}$ $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-module W, we can restrict to obtain a type-$\mathbf{1}$ $U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$-module. In particular, for $\mathbf{a} \in X(\mathfrak{so}_m)$, we have $W[\mathbf{a}]$. For $\varepsilon \in \{\pm 1\}$ and $\mathbf{a} \in X(\mathfrak{so}_{\mathrm{m}})$ such that $\sigma(\mathbf{a}) = \mathbf{a}$, define

$$W[\mathbf{a}, \varepsilon] := \{w \in W[\mathbf{a}] \mid \sigma(w) = \varepsilon \cdot w\}.$$

Note that if $\sigma(\mathbf{a}) \neq \mathbf{a}$, then σ acts on $W[\mathbf{a}] \oplus W[\sigma(\mathbf{a})]$ as $\dim W[\mathbf{a}]$ copies of the regular representation of $\langle \sigma \rangle \cong \mathbb{Z}/2$.

DEFINITION 3.5.9. *Let $\chi(O(m))$ denote the free $\mathbb{Z}$-module with basis $\{e^{(\mathbf{a},\varepsilon)}\}_{\substack{\mathbf{a} \in X(\mathfrak{o}_{\mathrm{m}}), \varepsilon = \pm 1 \\ \sigma(\mathbf{a}) = \mathbf{a}}} \cup \{e^{\mathbf{a}}\}_{\substack{\mathbf{a} \in X(\mathfrak{o}_{\mathrm{m}}) \\ \sigma(\mathbf{a}) \neq \mathbf{a}}}$. We define the* formal character *of W to be the expressions*

$$\mathrm{ch}(W) := \sum_{\substack{\mathbf{a} \in X(\mathfrak{so}_{\mathrm{m}}) \\ \sigma(\mathbf{a}) = \mathbf{a}}} \dim W[\mathbf{a}, \varepsilon] e^{(\mathbf{a},\varepsilon)} + \sum_{\substack{\mathbf{a} \in X(\mathfrak{so}_{\mathrm{m}}) \\ \sigma(\mathbf{a}) \neq \mathbf{a}}} \dim W[\mathbf{a}] e^{\mathbf{a}} \in \chi(O(m)).$$

LEMMA 3.5.13. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. The formal characters of the irreducible representations in Proposition 3.5.6 form a basis for the span of formal characters of all type-**1** representations.*

PROOF. Use Lemma 3.5.6 and keep track of ± 1 eigenspaces of σ. $\square$

PROPOSITION 3.5.7. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. The character of a type-**1** $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module determines the isomorphism class of the representation.*

PROOF. Use Proposition 3.5.6, Lemma 3.5.13, and Lemma 3.5.12. $\square$

LEMMA 3.5.14. *Suppose that V, W are type-**1** $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$-modules. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})}(\mathbb{C} \otimes V, \mathbb{C} \otimes W) = \dim_{\mathbb{F}} \mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\mathbb{F} \otimes V, \mathbb{F} \otimes W).$$

PROOF. If U is a type-$\mathbf{1}$ $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$-module and $\mathbf{a} \in X(\mathfrak{so}_{\mathrm{m}})$ such that $\sigma(\mathbf{a}) = \mathbf{a}$, then

$$\dim_{\mathbb{C}} \mathbb{C} \otimes U[\mathbf{a}, \varepsilon] = \mathrm{rk}_{\mathbb{A}} U[\mathbf{a}, \varepsilon] = \dim_{\mathbb{F}} \mathbb{F} \otimes U[\mathbf{a}, \varepsilon].$$

The result then follows from Proposition 3.5.7 and Lemma 3.5.10 $\square$

LEMMA 3.5.15.

$$\mathbb{C} \otimes U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})/(K_\alpha - 1, \alpha \in \Pi) \cong U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}) \quad \textit{and} \quad \mathbb{F} \otimes U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}) \cong U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}).$$

PROOF. The first isomorphism follows from [**12**, Proposition 9.2.3]. The second isomorphism is clear. □

LEMMA 3.5.16. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. If V and W are type-**1** representations, then*

$$\mathrm{Hom}_{\boldsymbol{R}\otimes U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(\boldsymbol{R}\otimes V, \boldsymbol{R}\otimes W) = \mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}(\boldsymbol{R}\otimes V, \boldsymbol{R}\otimes W).$$

PROOF. The action of $\mathbb{C}\otimes U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$ on V and W factors through $U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}) \cong \mathbb{C}\otimes U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})/(K_\alpha - 1, \alpha \in \Pi)$. The claim then follows from Lemma 3.5.15 □

LEMMA 3.5.17. *Suppose that V, W are type-**1** $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$-modules. Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. Then there is an $\boldsymbol{R}$-linear map*

$$\mathfrak{b}_{\boldsymbol{R}} : \boldsymbol{R}\otimes \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(V,W) \to \mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}(\boldsymbol{R}\otimes_{\mathbb{A}} V, \boldsymbol{R}\otimes_{\mathbb{A}} W),$$
$$1\otimes f \mapsto \big(1\otimes v \mapsto 1\otimes f(v)\big).$$

PROOF. Since V and W are finitely generated free $\mathbb{A}$-modules, so is $\mathrm{Hom}_{\mathbb{A}}(V,W)$. Therefore,

$$\mathbf{R}\otimes \mathrm{Hom}_{\mathbb{A}}(V,W) \cong \mathrm{Hom}_{\mathbf{R}\otimes\mathbb{A}}(\mathbf{R}\otimes V, \mathbf{R}\otimes W).$$

We obtain a map

$$\mathfrak{B}_{\mathbf{R}} : \mathbf{R}\otimes \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(V,W) \to \mathbf{R}\otimes \mathrm{Hom}_{\mathbb{A}}(V,W) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{R}\otimes\mathbb{A}}(\mathbf{R}\otimes V, \mathbf{R}\otimes W),$$

and it is routine to verify that the image is contained in $\mathrm{Hom}_{\mathbf{R}\otimes U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(\mathbf{R}\otimes V, \mathbf{R}\otimes W)$. The claim then follows from Lemma 3.5.16. □

REMARK 3.5.16. *In general, if $f : \mathbb{A}^m \to \mathbb{A}^n$ is injective, then $\mathbb{F}\otimes f : \mathbb{F}\otimes\mathbb{A}^m \to \mathbb{F}\otimes\mathbb{A}^n$ is also injective. However, this may fail for $\mathbb{C}\otimes_{\mathbb{A}}(-)$. For example the endomorphism of $\mathbb{A}$ given by multiplication by $q-1$ is injective, but becomes zero after applying the functor $\mathbb{C}\otimes_{\mathbb{A}}(-)$.*

3.5.3.2. *Quantum vector representation.* The natural vector representation of the orthogonal group has a type-**1** q-analogue.

NOTATION 3.5.8. *Fix $m \in \mathbb{Z}_{\geq 1}$. Let n be such that $m = 2n$, if m is even, and $m = 2n+1$, if m is odd.*

DEFINITION 3.5.10. *Let $V_{\mathbb{F}}$ be the $U_{\mathbb{F}}(\mathfrak{so}_m)$-module with basis:*

$$\begin{cases} a_1, a_2, \ldots, a_n, u, b_n, \ldots, b_2, b_1 & \text{if } m = 2n+1 \\ a_1, a_2, \ldots, a_n, b_n, \ldots, b_2, b_1 & \text{if } m = 2n, \end{cases}$$

such that for $i = 1, \ldots, n-1$

$$F_i \cdot a_i = a_{i+1}, \quad F_i \cdot b_{i+1} = b_i,$$

$$E_i \cdot a_{i+1} = a_i, \quad E_i \cdot b_i = b_{i+1},$$

$$\begin{cases} F_n \cdot a_n = u, \quad F_n \cdot u = (q+q^{-1})b_n, \quad E_n \cdot u = (q+q^{-1})a_n, \quad E_n \cdot b_n = u, & \text{if } m = 2n+1, \\ F_n \cdot a_n = b_n, \quad E_n \cdot b_n = a_n, \qquad \text{if } m = 2n, \end{cases}$$

and

$$K_\alpha v = (q^2)^{(\alpha, \mathrm{wt}\, v)}, \quad \text{where } \mathrm{wt}(a_i) = \varepsilon_i, \ \mathrm{wt}(u) = 0, \text{ and } \mathrm{wt}(b_i) = -\varepsilon_i.$$

NOTATION 3.5.9. *Write $V_{\mathbb{A}}$ to denote the $\mathbb{A}$ span of the given basis for $V_{\mathbb{F}}$. This is a free $\mathbb{A}$-module of rank m.*

LEMMA 3.5.18. *The algebra $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$ preserves the $\mathbb{A}$-module $V_{\mathbb{A}}$.*

PROOF. All the higher divided power operators: $E_k^{(d)}$ and $F_k^{(d)}$, for $k = 1, \ldots, n$, and $d \geq 2$, act as zero on $V_{\mathbb{F}}$, except if $m = 2n+1$, when $F_n^{(2)} a_n = b_n$ and $E_n^{(2)} b_n = a_n$. $\square$

REMARK 3.5.17. *The algebra $U_{\mathbb{C}}(\mathfrak{so}_{\mathrm{m}}) \cong \mathbb{C} \otimes U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})/(K_\alpha - 1)$ acts on $V_{\mathbb{C}} := \mathbb{C} \otimes V_{\mathbb{A}}$.*

REMARK 3.5.18. *In the notation of Section 3.1.2, we have $V_{\boldsymbol{R}} \cong L_{\boldsymbol{R}}(\varpi_1)$, for $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$.*

LEMMA 3.5.19. *Setting $\sigma \cdot a_1 = (-1)^m a_1$, induces an action of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ on $V_{\boldsymbol{R}}$, for $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$.*

PROOF. Let $v \in V_{\mathbf{R}}$. Then there is $X_v \in U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$ such that $v = X_v \cdot a_1$. Define $\sigma \cdot v = \sigma(X_v) \cdot (\sigma \cdot a_1)$. This determines a well-defined $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$ action if $\sigma \cdot (\sigma \cdot v) = v$ and $\sigma \cdot (X \cdot (\sigma \cdot v)) = \sigma(X) \cdot v$, for all $X \in U_{\mathbf{R}}(\mathfrak{so}_{\mathrm{m}})$ and $v \in V_{\mathbf{R}}$. We check the first equality:

$$\sigma \cdot (\sigma \cdot v) = \sigma \cdot (\sigma(X_v) \cdot (\sigma \cdot a_1)) = (-1)^m \sigma \cdot (\sigma(X_v) \cdot a_1) = (-1)^m \sigma(\sigma(X_v)) \cdot (\sigma \cdot a_1) = X_v \cdot a_1 = v,$$

and the second equality:

$$\begin{aligned}\sigma\cdot(X\cdot(\sigma\cdot v)) &= \sigma\cdot(X\cdot(\sigma(X_v)\cdot(\sigma\cdot a_1))) = (-1)^m\sigma\cdot(X\cdot(\sigma(X_v)\cdot a_1))\\ &= (-1)^m\sigma\cdot(\sigma^2(X)\cdot(\sigma(X_v)\cdot a_1)) = (-1)^m\sigma\cdot((\sigma^2(X)\sigma(X_v))\cdot a_1)\\ &= (-1)^m\sigma\cdot(\sigma\big(\sigma(X)X_v\big)\cdot a_1) = (-1)^m((\sigma(X)X_v)\cdot(\sigma\cdot a_1))\\ &= (\sigma(X)X_v)\cdot a_1 = \sigma(X)\cdot(X_v\cdot a_1) = \sigma(X)\cdot v.\end{aligned}$$

□

REMARK 3.5.19. *We have the following explicit description of σ's action on $V_{\boldsymbol{R}}$.*

$$\begin{cases}\sigma\cdot a_i = -a_i,\ \textit{for } i\le n, \quad \sigma\cdot u = -u, \quad \textit{and} \quad \sigma\cdot b_i = -b_i,\ \textit{for } i\le n, & \textit{if } m = 2n+1,\\ \sigma\cdot a_i = a_i,\ \textit{for } i< n, \quad \sigma\cdot a_n = b_n,\ \sigma\cdot b_n = a_n, \quad \textit{and} \quad \sigma\cdot b_i = b_i,\ \textit{for } i<n, & \textit{if } m = 2n.\end{cases}$$

REMARK 3.5.20. *In the notation of Proposition 3.5.6, the representation $V_{\boldsymbol{R}}$ is isomorphic to $L_{\boldsymbol{R}}(\varpi_1,-1)$, if m is odd, and $L_{\boldsymbol{R}}(\varpi_1,+1)$, if m is even. The reason for the choice of sign becomes apparent in the next section. It is to ensure that $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ acts on the exterior algebra by algebra automorphisms, making the algebra structure maps $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module homomorphisms, and that $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ acts on the top exterior power as* $\det_{\boldsymbol{R}}$.

3.5.3.3. *Quantum exterior algebra.* The usual exterior algebra $\Lambda^\bullet(V_{\mathbb{C}})$ is defined as the quotient of the tensor algebra of $V_{\mathbb{C}}$ by the two sided ideal generated by the symmetric tensors $S^2(V_{\mathbb{C}})$. As a module over the special orthogonal group $SO(V_{\mathbb{C}})$, we find that $S^2(V_{\mathbb{C}})$ contains a copy of the trivial module, corresponding to the symmetric form preserved by $SO(V_{\mathbb{C}})$. The complement of the trivial module in $S^2(V_{\mathbb{C}})$ is the irreducible module $L_{\mathbb{C}}(2\varpi_1)$.

From this perspective, we see that to define a q-analogue of the exterior algebra, we first need to find the q-analogue of the symmetric square. Moreover, this can be done by decomposing $V_{\mathbb{F}}^{\otimes 2}$ into irreducible submodules and defining the symmetric square to be the submodule $L_{\mathbb{F}}(2\varpi_1)\oplus L_{\mathbb{F}}(0)$.

LEMMA 3.5.20. *The $\mathbb{F}$-span of*

$$a_i\otimes a_i, \quad b_i\otimes b_i,$$

$$a_i\otimes a_j + q^{-2}a_j\otimes a_i \quad i<j,$$

$$b_j \otimes b_i + q^{-2} b_i \otimes b_j \quad i < j,$$

$$a_i \otimes b_j + q^{-2} b_j \otimes a_i \quad i \neq j,$$

$$a_i \otimes u + q^{-2} u \otimes a_i, \quad u \otimes b_i + q^{-2} b_i \otimes u,$$

$$a_i \otimes b_i + q^{-2} b_{i+1} \otimes a_{i+1} + q^{-2} a_{i+1} \otimes b_{i+1} + q^{-4} b_i \otimes a_i \quad i < n,$$

$$a_n \otimes b_n + q^{-4} b_n \otimes a_n + q^{-1} u \otimes u \quad \textit{if } m \textit{ is odd},$$

is the $U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})$ *submodule of* $V_{\mathbb{F}}^{\otimes 2}$ *generated by* $a_1 \otimes a_1$.

The $\mathbb{F}$*-span of*

$$\frac{(-q^2)^n}{q+q^{-1}} u \otimes u + \sum_{i=1}^{n} \left((-q^2)^{i-1} a_i \otimes b_i - (-q^2)^{2n-i} b_i \otimes a_i \right) \quad \textit{if } m \textit{ is odd},$$

or

$$\sum_{i=1}^{n} \left((-q^2)^{i-1} a_i \otimes b_i + (-q^2)^{2n-i-1} b_i \otimes a_i \right) \quad \textit{if } m \textit{ is even}.$$

is the unique copy of the trivial submodule of $V_{\mathbb{F}}^{\otimes 2}$.

Thus, the $\mathbb{F}$*-span of these vectors taken together is*

$$L_{\mathbb{F}}(2\varpi_1) \oplus L_{\mathbb{F}}(0).$$

PROOF. We leave it to the reader to use Definition 3.5.10 and Equation (3.1) to check this claim. □

REMARK 3.5.21. *The braiding endomorphism of* $V_{\mathbb{F}}^{\otimes 2}$ *acts on* $L_{\mathbb{F}}(2\varpi_1)$ *as* q^2*, on* $L_{\mathbb{F}}(0)$ *as* q^{2-2m}*, and on* $L_{\mathbb{F}}(\varpi_2)$ *as* $-q^{-2}$ *[**38**, Equation 6.12]. It follows that* $L_{\mathbb{F}}(2) \oplus L_{\mathbb{F}}(0)$ *is equal to the subspace of "positive" eigenvectors for the braiding. This subspace is also referred to as* S_q^2*, in [**3**].*

DEFINITION 3.5.11. *Define* $\Lambda_{\mathbb{A}}^{\bullet}$ *to be the associative* $\mathbb{A}$*-algebra generated by the elements*

$$a_1, \ldots, a_n, u, b_n, \ldots, b_1$$

subject to the following relations:

$$a_i^2 = 0 \quad b_i^2 = 0,$$

$$a_j a_i = -q^2 a_i a_j \quad i < j,$$

$$b_i b_j = -q^2 b_j b_i \quad i < j,$$

$$b_j a_i = -q^2 a_i b_j \quad i \neq j,$$

$$u a_i = -q^2 a_i u, \quad b_i u = -q^2 u b_i,$$

$$b_{i+1} a_{i+1} = -a_{i+1} b_{i+1} - \left(q^{-2} b_i a_i + q^2 a_i b_i\right) \quad i < n,$$

$$b_n a_n = -q^4 a_n b_n - q^3 u^2 \quad \textit{if } m \textit{ is odd},$$

$$u = 0 \quad \textit{if } m \textit{ is even},$$

$$\frac{(-q^2)^n}{q+q^{-1}} u^2 + \sum_{i=1}^{n} \left((-q^2)^{i-1} a_i b_i - (-q^2)^{2n-i} b_i a_i\right) = 0 \quad \textit{if } m \textit{ is odd},$$

and

$$\sum_{i=1}^{n} \left((-q^2)^{i-1} a_i b_i + (-q^2)^{2n-i-1} b_i a_i\right) = 0 \quad \textit{if } m \textit{ is even}.$$

Let $\Lambda_{\mathbb{A}}^k$ be the $\mathbb{A}$ submodule spanned by monomials of degree k.

LEMMA 3.5.21. *If $m = 2n$ is even, then the relations*

$$b_i a_i = -a_i b_i - \sum_{k=1}^{i-1} (-q^2)^{-k+1} (q^2 - q^{-2}) a_{i-k} b_{i-k} \quad i=1, \ldots, n,$$

are equivalent to the relations

$$b_{i+1} a_{i+1} = -a_{i+1} b_{i+1} - \left(q^{-2} b_i a_i + q^2 a_i b_i\right) \quad i = 1, \ldots, n-1$$

and

$$\sum_{i=1}^{n} \left((-q^2)^{i-1} a_i b_i + (-q^2)^{2n-i-1} b_i a_i\right) = 0.$$

If $m = 2n+1$ is odd, then the relations

$$b_i a_i = -a_i b_i - \sum_{k=1}^{i-1} (-q^2)^{-k+1} (q^2 - q^{-2}) a_{i-k} b_{i-k} \quad i=1, \ldots, n$$

and

$$u^2 = q \sum_{k=1}^{n} (-q^2)^{-k} (q^2 - q^{-2}) a_{n+1-k} b_{n+1-k},$$

are equivalent to the relations

$$\frac{(-q^2)^n}{q+q^{-1}} u^2 + \sum_{i=1}^{n} \left((-q^2)^{i-1} a_i b_i - (-q^2)^{2n-i} b_i a_i\right) = 0,$$

$$b_{i+1}a_{i+1} = -a_{i+1}b_{i+1} - \left(q^{-2}b_ia_i + q^2a_ib_i\right) \quad i = 1,\ldots,n-1,$$

and

$$b_na_n = -q^4a_nb_n - q^3u^2.$$

PROOF. We provide an example calculation in each case. Generalizing these to the general case is left to the reader.

Suppose $m = 2\cdot 2$. We are tasked with showing that

$$b_1a_1 = -a_1b_1 \quad \text{and} \quad b_2a_2 = -a_2b_2 - (q^2 - q^{-2})a_1b_1 \tag{3.16}$$

is equivalent to

$$b_2a_2 = -a_2b_2 - (q^{-2}b_1a_1 + q^2a_1b_1) \quad \text{and} \quad a_1b_1 + q^4b_1a_1 - q^2a_2b_2 - q^2b_2a_2 = 0. \tag{3.17}$$

Assume Equation (3.16), then

$$b_2a_2 + a_2b_2 + q^{-2}b_1a_1 + q^2a_1b_1 = -a_2b_2 - (q^2 - q^{-2})a_1b_1 + a_2b_2 - q^{-2}a_1b_1 + q^2a_1b_1 = 0$$

and

$$a_1b_1 + q^4b_1a_1 - q^2a_2b_2 - q^2b_2a_2 = a_1b_1 - q^4a_1b_1 - q^2a_2b_2 - q^2\left(-a_2b_2 - (q^2 - q^{-2})a_1b_1\right).$$

Assume Equation (3.17). First, we rewrite the second relation as

$$b_1a_1 = -q^{-4}a_1b_1 + q^{-2}a_2b_2 + q^{-2}b_2a_2,$$

so the first relation becomes

$$b_2a_2 = -a_2b_2 - (q^{-2}b_1a_1 + q^2a_1b_1) = -a_2b_2 - q^{-2}\left(-q^{-4}a_1b_1 + q^{-2}a_2b_2 + q^{-2}b_2a_2\right) - q^2a_1b_1$$

which we rewrite as

$$q^{-2}(q^2 + q^{-2})b_2a_2 = -q^{-2}(q^2 + q^{-2})a_2b_2 - q^{-2}(q^4 - q^{-4})a_1b_1.$$

Since $q^2+q^{-2} \in \mathbb{A}^\times$ and $q^2+q^{-2} = (q^4-q^{-4})/(q^2-q^{-2})$, this implies

$$b_2a_2 = -a_2b_2 - (q^2-q^{-2})a_1b_1.$$

Now, rewriting the second relation in Equation (3.17) again we find

$$b_1a_1 = -q^{-4}a_1b_1 + q^{-2}a_2b_2 + q^{-2}\left(-a_2b_2 - (q^2-q^{-2})a_1b_1\right) = -a_1b_1.$$

Suppose $m = 2\cdot 1+1$. We need to show that

$$b_1a_1 = -a_1b_1 \quad \text{and} \quad u^2 = -q^{-1}(q^2-q^{-2})a_1b_1 \tag{3.18}$$

is equivalent to

$$\frac{-q^2}{q+q^{-1}}u^2 + a_1b_1 + q^2b_1a_1 = 0 \quad \text{and} \quad b_1a_1 = -q^4a_1b_1 - q^3u^2. \tag{3.19}$$

Assume Equation (3.18). Then

$$\begin{aligned}\frac{-q^2}{q+q^{-1}}u^2 + a_1b_1 + q^2b_1a_1 &= \frac{-q^2}{q+q^{-1}}\left(-q^{-1}(q^2-q^{-2})a_1b_1\right) + a_1b_1 - q^2a_1b_1 \\ &= q(q-q^{-1})a_1b_1 + a_1b_1 - q^2a_1b_1 = 0\end{aligned}$$

and

$$b_1a_1 + q^4a_1b_1 + q^3u^2 = -a_1b_1 + q^4a_1b_1 + q^3\left(-q^{-1}(q^2-q^{-2})a_1b_1\right) = 0.$$

Assume Equation (3.19). Then we can rewrite the first relation as

$$u^2 = q^{-2}(q+q^{-1})a_1b_1 + (q+q^{-1})b_1a_1$$

so the second relation becomes

$$b_1a_1 = -q^4a_1b_1 - q^3u^2 = -q^4a_1b_1 - q^3\left(q^{-2}(q+q^{-1})a_1b_1 + (q+q^{-1})b_1a_1\right)$$

which we can rewrite as

$$(q^4+q^2+1)b_1a_1 = (-q^4-q^2-1)a_1b_1.$$

Since $q^4+q^2+1 \in \mathbb{A}^\times$, it follows that $b_1a_1 = -a_1b_1$. Thus, we can rewrite the first relation in Equation (3.19) again as

$$u^2 = q^{-2}(q+q^{-1})a_1b_1 - (q+q^{-1})a_1b_1 = -q^{-1}(q^2-q^{-2})a_1b_1.$$

$\square$

COROLLARY 3.5.1. *The algebra $\Lambda^\bullet_{\mathbb{A}}$ is the associative $\mathbb{A}$-algebra generated by the elements*

$$a_1,\dots,a_n,u,b_n,\dots,b_1$$

subject to the following relations:

$$a_i^2 = 0 \quad b_i^2 = 0,$$

$$a_ja_i = -q^2a_ia_j \quad i<j,$$

$$b_ib_j = -q^2b_jb_i \quad i<j,$$

$$b_ja_i = -q^2a_ib_j \quad i\neq j,$$

$$ua_i = -q^2a_iu, \quad b_iu = -q^2ub_i,$$

$$b_ia_i = -a_ib_i - \sum_{k=1}^{i-1}(-q^2)^{-k+1}(q^2-q^{-2})a_{i-k}b_{i-k},$$

$$u = 0 \quad \textit{if } m \textit{ is even},$$

$$u^2 = q\sum_{k=1}^{n}(-q^2)^{-k}(q^2-q^{-2})a_{n+1-k}b_{n+1-k} \quad \textit{if } m \textit{ is odd}.$$

PROOF. This follows immediately from Definition 3.5.11 and Lemma 3.5.21. $\square$

DEFINITION 3.5.12. *It will be convenient to write the vectors in $V_{\mathbb{A}}$ as follows:*

$$v_i := a_i, \quad \textit{for} \quad i=1,\dots,n,$$

$$v_{m-i+1} := b_i, \quad \textit{for} \quad i=1,\dots,n,$$

and if $m=2n+1$, then

$$v_{n+1} := u.$$

There is an involution of $\{1,\dots,m\}$ *defined by* $i \mapsto i' := m-i+1$. *We will write* $v_{i'} := v_{m-i+1}$. *Let* $S \subset \{1,\dots,m\}$. *If* $S = \{s_1,\dots,s_k\}$ *such that* $s_1 < \dots < s_k$, *set* $v_S := v_{s_1} \cdots v_{s_k}$. *We extend the involution* $i \mapsto i'$ *to the set of subsets of* $\{1,\dots,m\}$ *by* $S \mapsto S' := \{s'_1,\dots,s'_k\}$. *Note that* $v_{S'} := v_{s'_k} \cdots v_{s'_1}$.

THEOREM 3.5.10. *The set* $\{v_S\}_{S\subset\{1,\dots,m\}}$ *forms a basis of* $\Lambda^{\bullet}_{\mathbb{A}}$. *In particular,* $\{v_S\}_{\substack{S\subset\{1,\dots,m\}\\ |S|=k}}$ *forms a basis of* $\Lambda^k_{\mathbb{A}}$.

PROOF. This is a standard application of Bergman's diamond lemma [**4**, Theorem 1.2]. Note that if $1 < \cdots < m$, then the lexicographic order on monomials in v_i satisfies the hypotheses of the diamond lemma, and the irreducible monomials are the elements of $\{v_S\}_{S\subset\{1,\dots,m\}}$. Therefore, it suffices to show that all the ambiguities in the defining relations are resolvable.

The following are all the overlap ambiguities in the defining relations of $\Lambda^{\bullet}_{\mathbb{A}}$:

$$a_xa_xa_x,\quad a_ja_ja_i,\quad b_xb_xb_x,\quad b_ib_ib_j,\quad b_xb_xa_y,\quad b_xb_xu,\quad a_ja_ia_i,\quad a_ja_ia_k,$$

$$b_ib_jb_j,\quad b_ib_ja_i,\quad b_ib_ju,\quad b_ib_ja_j,\quad b_xa_ya_y,\quad b_xa_ya_z,$$

$$ua_xa_x,\quad ua_ja_i,\quad b_xua_y,\quad b_xua_x,\quad b_xuu,\quad b_xa_xa_x,\quad b_ja_ja_i,\quad uua_x,\quad \text{and}\quad uuu,$$

where $1 \le x,y,z,i,j,k \le n$, $x \ne y$, $z < y$, and $k < i < j$.

We provide an example calculation to verify the resolution of an ambiguity, and leave the rest as an exercise. To simplify notation, write $\xi := (q^2 - q^{-2})$. On the one hand, we have

$$\begin{aligned}
(uu)a_i &= q\sum_{k=1}^{n}(-q^2)^{-(n-k+1)}\xi a_kb_ka_i\\
&= q\sum_{1\le k<i}(-q^2)^{-n+k}\xi a_ka_ib_k - q\sum_{1\le k<i}(-q^2)^{-n+k+1}\xi^2 a_ka_ib_k + q\sum_{i<k\le n}(-q^2)^{-n+k+1}\xi a_ia_kb_k\\
&= q\sum_{1\le k<i}(-q^2)^{-n+k}\xi\left(1-(-q^2)\xi\right)a_ka_ib_k + q\sum_{i<k\le n}(-q^2)^{-n+k+1}\xi a_ia_kb_k\\
&= q\sum_{1\le k<i}(-q^2)^{-n+k+2}\xi a_ka_ib_k + q\sum_{i<k\le n}(-q^2)^{-n+k+1}\xi a_ia_kb_k,
\end{aligned}$$

and on the other

$$\begin{aligned} u(ua_i) = (-q^2)ua_iu(-q^2)^2a_iu^2 &= q\sum_{k=1}^{n}(-q^2)^{-n+k+1}\xi a_ia_kb_k \\ &= q\sum_{1\le k<i}(-q^2)^{-n+k+2}\xi a_ka_ib_k + q\sum_{i<k\le n}(-q^2)^{-n+k+1}\xi a_ia_kb_k. \end{aligned}$$

□

REMARK 3.5.22. *Write $\Lambda^\bullet_{\boldsymbol{R}}$ to denote the associative $\boldsymbol{R}$-algebra with generators and relations as in Definition 3.5.11. The basis $\{v_S\}_{S\subset\{1,\dots,m\}}$ of $\Lambda^\bullet_{\mathbb{A}}$ is also a basis for $\Lambda^\bullet_{\boldsymbol{R}}$, when $\boldsymbol{R}\in\{\mathbb{C},\mathbb{F}\}$. Define isomorphism $\iota_k : \Lambda^k_{\boldsymbol{R}} \to \boldsymbol{R}\otimes\Lambda^k_{\mathbb{A}}$ by $v_S \mapsto 1\otimes v_S$.*

3.5.3.4. *Fundamental category for orthogonal groups.* For simple Lie algebras, the fundamental category is the monoidal category generated by irreducible representations with highest weight a fundamental weight. We define an analogue of this category for $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$, when $\mathbf{R}\in\{\mathbb{F},\mathbb{A},\mathbb{C}\}$. The first step is to show that for $k=0,1,\dots,m$, $\Lambda^k_{\mathbf{R}}$ are non-isomorphic, irreducible, self-dual $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-modules.

LEMMA 3.5.22. *The action of $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ on the tensor algebra of $V_{\mathbb{F}}$ descends to an action of $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ on $\Lambda^\bullet_{\mathbb{F}}$. Moreover, the multiplication for $\Lambda^\bullet_{\mathbb{F}}$ is $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ equivariant.*

PROOF. This follows from observing that $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$ preserves the defining relations. See the discussion in [**37**, Sections 3.2, 4.1] for the symmetric analogue. □

DEFINITION 3.5.13. *Let $\operatorname{wt} v_S := \sum_{i\in S}\operatorname{wt} v_i$, where $\operatorname{wt} v_i$ is as defined in Definition 3.5.10. Then $K_\alpha\cdot v_S = (q^2)^{(\alpha,\operatorname{wt} v_S)}$.*

REMARK 3.5.23. *The modules $\Lambda^k_{\mathbb{F}}$ are finite dimensional type-**1** representations of $U_{\mathbb{F}}(\mathfrak{so}_{\mathrm{m}})$, in particular we have*

$$\Lambda^k_{\mathbb{F}} = \bigoplus_{\mathbf{a}\in X(\mathfrak{so}_{\mathrm{m}})}\Lambda^k_{\mathbb{F}}[\mathbf{a}].$$

In fact, this remains true over $\mathbb{A}$, since $\Lambda^\bullet_{\mathbb{A}}$ is spanned by $\{v_S\}_{S\subset\{1,\dots,m\}}$ and each v_S is a weight vector.

LEMMA 3.5.23. *We have the following equality of formal characters:*

$$\sum_{\mathbf{a}\in X(\mathfrak{so}_{\mathrm{m}})}\dim\Lambda^k_{\mathbb{F}}[\mathbf{a}]e^{\mathbf{a}} = \sum_{\mathbf{a}\in X(\mathfrak{so}_{\mathrm{m}})}\dim\Lambda^k_{\mathbb{C}}[\mathbf{a}]e^{\mathbf{a}}.$$

PROOF. Follows from observing that $v_S \in \Lambda_{\mathbf{R}}^{|S|}[\mathrm{wt}\, v_S]$ for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$. $\square$

LEMMA 3.5.24. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. We have the following isomorphisms of $U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})$-modules. If $m = 2n+1$, then*

$$\Lambda_{\boldsymbol{R}}^1 \cong L_{\boldsymbol{R}}(\varpi_1), \quad \Lambda_{\boldsymbol{R}}^2 \cong L_{\boldsymbol{R}}(\varpi_2), \quad \ldots, \quad \Lambda_{\boldsymbol{R}}^{n-1} \cong L_{\boldsymbol{R}}(\varpi_{n-1}),$$

$$\text{and} \quad \Lambda_{\boldsymbol{R}}^n \cong L_{\boldsymbol{R}}(2\varpi_n).$$

If $m = 2n$, then

$$\Lambda_{\boldsymbol{R}}^1 \cong L_{\boldsymbol{R}}(\varpi_1), \quad \Lambda_{\boldsymbol{R}}^2 \cong L_{\boldsymbol{R}}(\varpi_2), \quad \ldots, \quad \Lambda_{\boldsymbol{R}}^{n-2} \cong L_{\boldsymbol{R}}(\varpi_{n-2}),$$

$$\Lambda_{\boldsymbol{R}}^{n-1} \cong L_{\boldsymbol{R}}(\varpi_{n-1} + \varpi_n), \quad \text{and} \quad \Lambda_{\boldsymbol{R}}^n \cong L_{\boldsymbol{R}}(2\varpi_{n-1}) \oplus L_{\boldsymbol{R}}(2\varpi_n).$$

In either case $\Lambda_{\boldsymbol{R}}^m \cong L_{\boldsymbol{R}}(0)$

PROOF. Thanks to Lemma 3.5.7, it suffices to compute characters, and the result follows for $\mathbf{R} = \mathbb{C}$ from [**19**, Sections 19.2, 19.4], and then for $\mathbf{R} = \mathbb{F}$, from Lemma 3.5.23. $\square$

LEMMA 3.5.25. *The algebra $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$ preserves the lattice $\Lambda_{\mathbb{A}}^\bullet \subset \Lambda_{\mathbb{F}}^\bullet$.*

PROOF. Since $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$ preserves $V_{\mathbb{A}}^{\otimes d} \subset V_{\mathbb{F}}^{\otimes d}$ for all $d \geq 0$, it suffices to show that $U_{\mathbb{A}}(\mathfrak{so}_{\mathrm{m}})$ preserves the $\mathbb{A}$-span of the defining relations for $\Lambda_{\mathbb{A}}^\bullet$. This then immediately reduces to verifying that $E_\alpha^{(2)}$ and $F_\alpha^{(2)}$ preserve the $\mathbb{A}$-span of the defining relations. For example, using that $\Delta(F) = 1 \otimes F + F \otimes K^{-1}$, we find:

$$F_1^{(2)} \cdot a_1 \otimes a_1 = \frac{1}{q^2+q^{-2}} F_1 \cdot (a_1 \otimes a_2 + q^{-2} a_2 \otimes a_1) = \frac{1}{q^2+q^{-2}} (q^2 a_2 \otimes a_2 + q^{-2} a_2 \otimes a_2) = a_2 \otimes a_2.$$

The remaining calculations we leave to the reader. $\square$

LEMMA 3.5.26. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. The operator $\sigma \in U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ acts via the coproduct on $V_{\boldsymbol{R}}^{\otimes d}$, for all $d \geq 0$, and preserves the defining relations of $\Lambda_{\boldsymbol{R}}^\bullet$. Thus, there is an action of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ on $\Lambda_{\boldsymbol{R}}^\bullet$, and the algebra structure maps are $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ equivariant.*

PROOF. Use the description of σ's action on $V_{\mathbf{R}}$ in Remark 3.5.19 to verify that σ preserves the relations in Definition 3.5.11 of $\Lambda_{\mathbf{R}}^\bullet$. $\square$

PROPOSITION 3.5.8. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. The $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module $\Lambda_{\boldsymbol{R}}^k$ is self-dual and irreducible for $k = 0, \ldots, m$, if $\Lambda_{\boldsymbol{R}}^i \cong \Lambda_{\boldsymbol{R}}^j$, then $i = j$, we have the following isomorphisms of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-modules*[6].

If $m = 2n+1$, then

$$\Lambda_{\boldsymbol{R}}^i \cong L_{\boldsymbol{R}}(\varpi_i, (-1)^i), \quad \Lambda_{\boldsymbol{R}}^{m-i} \cong L_{\boldsymbol{R}}(\varpi_i, -(-1)^i), \quad \textit{for } i = 0, 1, \ldots, n-1,$$

$$\Lambda_{\boldsymbol{R}}^n \cong L_{\boldsymbol{R}}(2\varpi_n, (-1)^n), \quad \textit{and} \quad \Lambda_{\boldsymbol{R}}^{n+1} \cong L_{\boldsymbol{R}}(2\varpi_n, -(-1)^n).$$

If $m = 2n$, then

$$\Lambda_{\boldsymbol{R}}^i \cong L_{\boldsymbol{R}}(\varpi_i, +1), \quad \Lambda_{\boldsymbol{R}}^{m-i} \cong L_{\boldsymbol{R}}(\varpi_i, -1), \quad \textit{for } i = 0, 1, \ldots, n-2,$$

$$\Lambda_{\boldsymbol{R}}^{n-1} \cong L_{\boldsymbol{R}}(\varpi_{n-1} + \varpi_n, +1), \quad \Lambda_{\boldsymbol{R}}^{n+1} \cong L_{\boldsymbol{R}}(\varpi_{n-1} + \varpi_n, -1), \quad \textit{and}$$

$$\Lambda_{\boldsymbol{R}}^n \cong U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}) \otimes_{U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})} L_{\boldsymbol{R}}(2\varpi_{n-1}).$$

PROOF. Self duality is from Lemma 3.5.11. Thanks to Proposition 3.5.7, and Lemma 3.5.24, the remaining claims follow once we show that σ acts on $v_1 v_1 \ldots v_i$ by the prescribed eigenvalue in the statement of the Proposition. We will argue this for $v_1 v_2 \ldots v_m$, where σ always acts by -1, leaving the other cases to the reader. Using the coproduct from Definition 3.5.8, we find $\sigma(v_1 \ldots v_m) = \sigma(v_1) \ldots \sigma(v_m)$. From Remark 3.5.19 we see that for $m = 2n+1$,

$$\sigma(v_1 v_2 \ldots v_m) = (-v_1)(-v_2) \ldots (-v_m) = -v_1 v_2 \ldots v_m.$$

For $m = 2n$,

$$\sigma(v_1 v_2 \ldots v_m) = v_1 \ldots v_{n-1} v_{n+1} v_n v_{n+2} \ldots v_{2n} = -v_1 v_2 \ldots v_m,$$

where the last equality follows from Lemma 3.5.21 and the defining relations of $\Lambda_{\mathbf{R}}^{\bullet}$. □

REMARK 3.5.24. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. There is an isomorphism of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-modules $\Lambda_{\boldsymbol{R}}^m \cong \det_{\boldsymbol{R}}$, and isomorphisms of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-modules $\Lambda_{\boldsymbol{R}}^i \cong \det_{\boldsymbol{R}} \otimes \Lambda_{\boldsymbol{R}}^{m-i}$.*

REMARK 3.5.25. *Since $\mathbb{A}$ is not a field, it does not make sense to ask for $\Lambda_{\mathbb{A}}^i$ to be irreducible. However, we will show, in Lemma 3.5.32, that $\Lambda_{\mathbb{A}}^i$ is a self-dual $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$-module, essentially by proving that there is an isomorphism $\Lambda_{\mathbb{F}}^i \to (\Lambda_{\mathbb{F}}^i)^*$ which preserves $\Lambda_{\mathbb{A}}^i$ and $(\Lambda_{\mathbb{A}}^i)^*$.*

[6] In the notation of Proposition 3.5.6. Writing $\varpi_0 = 0$.

LEMMA 3.5.27. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$, then we have the following tensor product decompositions.*

$$\Lambda_{\boldsymbol{R}}^{k} \otimes V_{\boldsymbol{R}} \cong \begin{cases} L_{\boldsymbol{R}}(\varpi_1 + \varpi_k, +1) \oplus \Lambda_{\boldsymbol{R}}^{k+1} \oplus \Lambda_{\boldsymbol{R}}^{k-1}, & \textit{if} \quad k \leq m-1, \quad \textit{and} \\ \Lambda_{\boldsymbol{R}}^{m-1}, & \textit{if} \quad k = m. \end{cases}$$

PROOF. Standard result (up to tensoring with $\det_{\mathbf{R}}$). See [**34**] and [**61**, Equation 6.1]. □

DEFINITION 3.5.14. *Let $\boldsymbol{R} \in \{\mathbb{F}, \mathbb{A}, \mathbb{C}\}$. Define the monoidal category $\boldsymbol{Fund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$ to be the full monoidal subcategory of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-modules generated by $\Lambda_{\boldsymbol{R}}^{k}$ for $k = 0, \ldots, m$. Define $\boldsymbol{StdFund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$ as the full monoidal subcategory of $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-modules generated by $\Lambda_{\boldsymbol{R}}^{1} = V_{\boldsymbol{R}}$.*

Let $\gamma = (\gamma_1, \ldots, \gamma_s)$, such that $0 \leq \gamma_i \leq m$, for $i = 1, \ldots, s$. We write $\Lambda_{\mathbf{R}}^{\gamma} := \Lambda_{\mathbf{R}}^{\gamma_1} \otimes \cdots \otimes \Lambda_{\mathbf{R}}^{\gamma_s}$. Objects in $\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$ are all of the form $\Lambda_{\mathbf{R}}^{\gamma}$ for some γ.

PROPOSITION 3.5.9. *Let $\gamma = (\gamma_1, \ldots, \gamma_s)$ and $\delta = (\delta_1, \ldots, \delta_t)$ such that $0 \leq \gamma_i \leq m$, for $i = 1, \ldots, s$, and $0 \leq \delta_j \leq m$, for $j = 1, \ldots, t$. Then* $\dim_{\mathbb{C}} \mathrm{Hom}_{U_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda_{\mathbb{C}}^{\gamma}, \Lambda_{\mathbb{C}}^{\delta}) = \dim_{\mathbb{F}} \mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda_{\mathbb{F}}^{\gamma}, \Lambda_{\mathbb{F}}^{\delta})$.

PROOF. The $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$-module $\Lambda_{\mathbb{A}}^{k}$ is type-**1**, for $k = 0, \ldots, m$, see Definition 3.5.7 and Remark 3.5.23. Tensor products of type-**1** modules are type **1**, so $\Lambda_{\mathbb{A}}^{\gamma}$ is a type-**1** $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$-module for all γ. The result then follows from Lemma 3.5.14. □

LEMMA 3.5.28. *Homomorphism spaces in $\boldsymbol{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$ are free and finitely generated $\mathbb{A}$-modules.*

PROOF. Since $\Lambda_{\mathbb{A}}^{\gamma}$ and $\Lambda_{\mathbb{A}}^{\delta}$ are both free and finitely generated $\mathbb{A}$-modules, see Theorem 3.5.10, the $\mathbb{A}$-module $\mathrm{Hom}_{\mathbb{A}}(\Lambda_{\mathbb{A}}^{\gamma}, \Lambda_{\mathbb{A}}^{\delta})$ is free and finitely generated over $\mathbb{A}$. Since $\mathbb{A}$ is a PID, the claim follows from observing that $\mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda_{\mathbb{A}}^{\gamma}, \Lambda_{\mathbb{A}}^{\delta}) \subset \mathrm{Hom}_{\mathbb{A}}(\Lambda_{\mathbb{A}}^{\gamma}, \Lambda_{\mathbb{A}}^{\delta})$. □

We also define an auxiliary $\mathbf{R}$-linear monoidal category $\mathbf{R} \otimes \mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$, which has the same objects as $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$, but with morphisms

$$\mathrm{Hom}_{\mathbf{R} \otimes \mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))}(\Lambda_{\mathbb{A}}^{\gamma}, \Lambda_{\mathbb{A}}^{\delta}) := \mathbf{R} \otimes \mathrm{Hom}_{\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))}(\Lambda_{\mathbb{A}}^{\gamma}, \Lambda_{\mathbb{A}}^{\delta}).$$

We have identifications $\iota_{\gamma} := \iota_{\gamma_1} \otimes \cdots \otimes \iota_{\gamma_s} : \Lambda_{\mathbf{R}}^{\gamma} \to \mathbf{R} \otimes \Lambda_{\mathbb{A}}^{\gamma}$, see Remark 3.5.22. Using $\mathfrak{b}_{\mathbf{R}}$ from Lemma 3.5.17, we define a monoidal functor

$$\mathfrak{B}_{\mathbf{R}} : \mathbf{R} \otimes \mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})) \to \mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})),$$

on objects as $\Lambda^\gamma_{\mathbb{A}} \mapsto \Lambda^\gamma_{\mathbf{R}}$, and on morphisms by sending $r \otimes f \in \mathrm{Hom}_{\mathbf{R}\otimes\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))}(\Lambda^\gamma_{\mathbb{A}}, \Lambda^\delta_{\mathbb{A}})$ to

$$\iota_\delta^{-1} \circ \mathfrak{b}_{\mathbf{R}}(r \otimes f) \circ \iota_\gamma \in \mathrm{Hom}_{\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))}(\Lambda^\gamma_{\mathbf{R}}, \Lambda^\delta_{\mathbf{R}}).$$

REMARK 3.5.26. *The notation makes $\mathfrak{B}_{\boldsymbol{R}}$ appear more complicated than it is. Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. The $\{v_S\}$ basis for $\Lambda^\bullet_{\boldsymbol{R}}$ gives rise to a basis for $\Lambda^\gamma_{\boldsymbol{R}}$, for all γ, and therefore a basis for $\mathrm{Hom}_{\boldsymbol{R}}(\Lambda^\gamma_{\boldsymbol{R}}, \Lambda^\delta_{\boldsymbol{R}})$ for all γ, δ. For $f \in \mathrm{Hom}_{\boldsymbol{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))}(\Lambda^\gamma_{\mathbb{A}}, \Lambda^\delta_{\mathbb{A}})$, we can use this basis to view f as a matrix with entries in $\mathbb{A}$. Then for $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$, $\mathfrak{B}_{\boldsymbol{R}}(1 \otimes f)$ is the same matrix, but with the entries interpreted as elements of $\boldsymbol{R}$.*

One of our main goals is to derive various relations among morphisms in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$. However, it will be easier to work in $\mathbf{Fund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$, so the following lemma is useful.

LEMMA 3.5.29. *The functor $\mathfrak{B}_{\mathbb{F}}$ is faithful.*

PROOF. Follows from injectivity of $\mathfrak{b}_{\mathbb{F}}$, see Remark 3.5.16. □

3.5.3.5. *Generating intertwiners for tensor products of exterior powers.*

DEFINITION 3.5.15. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$ and define*

$${}^{\boldsymbol{R}}m^{i+j}_{i,j} : \Lambda^i_{\boldsymbol{R}} \otimes \Lambda^j_{\boldsymbol{R}} \longrightarrow \Lambda^{i+j}_{\boldsymbol{R}}$$

by $x \otimes y \mapsto xy$.

REMARK 3.5.27. *The map ${}^{\boldsymbol{R}}m^{i+j}_{i,j}$ is a $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-linear transformation such that*

$$v_{\{1,\ldots,i\}} \otimes v_{\{i+1,\ldots,i+j\}} \mapsto v_{\{1,\ldots,i,i+1,\ldots,i+j\}}.$$

In particular, when $i + j \leq m$, the map is non-zero and therefore is surjective. Moreover, $\mathfrak{B}_{\boldsymbol{R}}({}^{\mathbb{A}}m^{i+j}_{i,j}) = {}^{\boldsymbol{R}}m^{i+j}_{i,j}$.

Let $\mathbf{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$ and let X be a free $\mathbf{R}$-module with basis $B_X = \{b_1, \ldots, b_d\}$. Then $X^* := \mathrm{Hom}_{\mathbf{R}}(X, \mathbf{R})$ has basis $\{b_1^*, \ldots, b_d^*\}$, where $b_i^*(b_j) = \delta_{i,j}$. Consider the elements $C \in X \otimes X^*$ defined by

$$C = \sum_{i=1}^{d} b_i \otimes b_i^*.$$

There are **R**-linear maps

$$\mathrm{coev}: \mathbf{R} \to X \otimes X^*, \quad 1 \mapsto C, \quad \text{and} \quad \mathrm{ev}: X^* \otimes X \to \mathbf{R}, \quad f \otimes x \mapsto f(x).$$

It is easy to verify that

$$(\mathrm{id}_X \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes \mathrm{id}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{ev} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}) = \mathrm{id}_{X^*}. \tag{3.20}$$

REMARK 3.5.28. *We can regard $\boldsymbol{R}$ as the trivial $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module via the counit, denoted ε in Definition 3.5.8. Also, If X is a $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module, which is free over $\boldsymbol{R}$, then X^* is as well, via the antipode, denoted S in Definition 3.5.8. One easily checks that the maps ev and coev are $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-module maps, where $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ acts on $X \otimes X^*$ via the coproduct.*

DEFINITION 3.5.16. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Define*

$${}^{\boldsymbol{R}}\varphi_1 : V_{\boldsymbol{R}} \longrightarrow (V_{\boldsymbol{R}})^*$$

to be the unique $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$-linear map such that

$$v_1 \mapsto v_m^*.$$

This is easily seen to be an isomorphism of $\boldsymbol{R}$-modules. Also, we have $\mathfrak{B}_{\boldsymbol{R}}({}^{\mathbb{A}}\varphi_1) = {}^{\boldsymbol{R}}\varphi_1$.

We define

$${}^{\boldsymbol{R}}e_1 : V_{\boldsymbol{R}} \otimes V_{\boldsymbol{R}} \longrightarrow \boldsymbol{R} \quad \text{by} \quad ev \circ ({}^{\boldsymbol{R}}\varphi_1 \otimes \mathrm{id}),$$

and

$${}^{\boldsymbol{R}}c_1 : \boldsymbol{R} \longrightarrow V_{\boldsymbol{R}} \otimes V_{\boldsymbol{R}} \quad \text{by} \quad (\mathrm{id} \otimes ({}^{\boldsymbol{R}}\varphi_1)^{-1}) \circ coev.$$

Note that for $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$, we have $\mathfrak{B}_{\boldsymbol{R}}({}^{\mathbb{A}}c_k) = {}^{\boldsymbol{R}}c_k$ and $\mathfrak{B}_{\boldsymbol{R}}({}^{\mathbb{A}}e_k) = {}^{\boldsymbol{R}}e_k$.

REMARK 3.5.29. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Recall that $V_{\boldsymbol{R}}$ is generated over $U_{\boldsymbol{R}}(\mathfrak{so}_{\mathrm{m}})^{<0}$ by a_1 while $V_{\boldsymbol{R}}^*$ is similarly generated by b_1^*. It then follows from explicit calculation using: $U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})$ equivariance of ${}^{\boldsymbol{R}}\varphi_1$, the formulas for the antipode in Definition 3.1.2, and the description of the action on $V_{\boldsymbol{R}}$ in Definition 3.5.10, that if $m = 2n+1$ is odd, then*

$${}^{\boldsymbol{R}}\varphi_1(a_i) = (-q^2)^{i-1} b_i^*, \quad {}^{\boldsymbol{R}}\varphi_1(u) = (-q^2)^{n-1}[2]_q u^*, \quad \text{and} \quad {}^{\boldsymbol{R}}\varphi_1(b_i) = (-q^2)^{m-i} a_i^*,$$

and if $m = 2n$ is even, then

$$^{\boldsymbol{R}}\varphi_1(a_i) = (-q^2)^{i-1} b_i^* \quad \textit{and} \quad ^{\boldsymbol{R}}\varphi_1(b_i) = (-q^2)^{m-i-1} a_i^*.$$

DEFINITION 3.5.17. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Define $^{\boldsymbol{R}}c_k \in \mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}(\boldsymbol{R}, \Lambda^k_{\boldsymbol{R}} \otimes \Lambda^k_{\boldsymbol{R}})$ inductively by*

$$^{\boldsymbol{R}}c_k := \frac{[2]}{[2k]} \cdot (^{\boldsymbol{R}}m^k_{k-1,1} \otimes {}^{\boldsymbol{R}}m^k_{1,k-1}) \circ (\mathrm{id}_{\Lambda^{k-1}_{\boldsymbol{R}}} \otimes {}^{\boldsymbol{R}}c_1 \otimes \mathrm{id}_{\Lambda^{k-1}_{\boldsymbol{R}}}) \circ {}^{\boldsymbol{R}}c_{k-1}.$$

LEMMA 3.5.30. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Then $^{\boldsymbol{R}}c_k \neq 0$.*

PROOF. Since $^{\mathbf{R}}c_k = \mathfrak{B}_{\mathbf{R}}(^{\mathbb{A}}c_k)$ for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$, it suffices to show $^{\mathbb{C}}c_k \neq 0$.

Let $\pi_k \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^k_{\mathbb{C}} \otimes \Lambda^k_{\mathbb{C}}, \mathbb{C})$ be the projection to $v_1 \dots v_k \otimes v_{m-k+1} \dots v_m$, with respect to the basis $\{v_S \otimes v_T\}_{\substack{S,T \subset \{1,\dots,m\} \\ |S|=|T|=k}}$. It suffices to show that $\pi_k \circ {}^{\mathbb{C}}c_k \neq 0$.

Write $(^{\mathbb{C}}\varphi_1)^{-1}(v_j^*) = t_j v_{j'}$, where $j' = m-k+1$. Note that $t_j \in \mathbb{C}^\times$ for $j = 1, \dots, m$. Then

$$\begin{aligned}
\pi_k \circ {}^{\mathbb{C}}c_k(1) &= \frac{1}{k!} \sum_{w \in S_k} \pi_k \left(v_{w(1)} \dots v_{w(k)} \otimes {}^{\mathbb{C}}\varphi_1^{-1}(v_{w(k)}) \dots {}^{\mathbb{C}}\varphi_1^{-1}(v_{w(1)}) \right) \\
&= \frac{t_1 \dots t_k}{k!} \cdot \sum_{w \in S_k} \pi_k \left(v_{w(1)} \dots v_{w(k)} \otimes v_{w(k)'} \dots v_{w(1)'} \right) \\
&= \frac{t_1 \dots t_k}{k!} \cdot \sum_{w \in S_k} \pi_k \left((-1)^{\ell(w)} v_1 \dots v_k \otimes (-1)^{\ell(w)} v_{m-k-1} \dots v_m \right) \\
&= t_1 \dots t_k \neq 0.
\end{aligned}$$

□

DEFINITION 3.5.18. *Let*

$$^{\boldsymbol{R}}\psi_k := (ev \otimes \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}}) \circ (\mathrm{id}_{(\Lambda^k_{\boldsymbol{R}})^*} \otimes {}^{\boldsymbol{R}}c_k)$$

Since $ev \in \mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}((\Lambda^k_{\boldsymbol{R}})^* \otimes \Lambda^k_{\boldsymbol{R}}, \boldsymbol{R})$, it follows that $^{\boldsymbol{R}}\psi_k \in \mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}((\Lambda^k_{\boldsymbol{R}})^*, \Lambda^k_{\boldsymbol{R}})$.

LEMMA 3.5.31. *We have the following equality in $\mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}(\boldsymbol{R}, \Lambda^k_{\boldsymbol{R}} \otimes \Lambda^k_{\boldsymbol{R}})$*

$$^{\boldsymbol{R}}c_k = (\mathrm{id}_{\Lambda^k_{\boldsymbol{R}}} \otimes {}^{\boldsymbol{R}}\psi_k) \circ coev$$

PROOF. Follows from Equation 3.20. □

LEMMA 3.5.32. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Then $^{\boldsymbol{R}}\psi_k$ is an isomorphism.*

PROOF. Thanks to Proposition 3.5.8, for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$, ${}^{\mathbf{R}}\psi_k$ is an isomorphism if and only if ${}^{\mathbf{R}}\psi_k \neq 0$. We know from weight considerations that ${}^{\mathbb{A}}\psi_k(v^*_{\{m-k+1,\dots,m\}}) = \xi_k \cdot v_{\{1,\dots,k\}}$ for some $\xi_k \in \mathbb{A}$, and ${}^{\mathbb{A}}\psi_k$ is an isomorphism if and only if $\xi_k \in \mathbb{A}^\times$ if and only if ξ_k does not map to zero under $\mathbb{A} \to \mathbb{C}$. Since $\mathfrak{B}_{\mathbf{R}}({}^{\mathbb{A}}\psi_k) = {}^{\mathbf{R}}\psi_k$ and $\mathfrak{B}_{\mathbb{F}}$ is faithful, the claim will follow if we show ${}^{\mathbb{C}}\psi_k \neq 0$. This follows from Lemma 3.5.30 and Equation 3.20. □

DEFINITION 3.5.11. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. Define ${}^{\boldsymbol{R}}\varphi_k := ({}^{\boldsymbol{R}}\psi_k)^{-1}$ and*

$$ {}^{\boldsymbol{R}}e_k := ev \circ ({}^{\boldsymbol{R}}\varphi_k \otimes \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}}) \in \mathrm{Hom}_{U_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda^k_{\boldsymbol{R}} \otimes \Lambda^k_{\boldsymbol{R}}, \boldsymbol{R}). $$

REMARK 3.5.30. *When $k = 1$, the previous definition agrees with Definition 3.5.16.*

LEMMA 3.5.33. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. The following equality of morphisms holds.*

$$ (\mathrm{id}_{\Lambda^k_{\boldsymbol{R}}} \otimes {}^{\boldsymbol{R}}e_k) \circ ({}^{\boldsymbol{R}}c_k \otimes \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}}) = \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}} \quad \textit{and} \quad ({}^{\boldsymbol{R}}e_k \otimes \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}}) \circ (\mathrm{id}_{\Lambda^k_{\boldsymbol{R}}} \otimes {}^{\boldsymbol{R}}c_k) = \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}}. $$

PROOF. Using Equation (3.20) this follows from the definition of ${}^{\mathbf{R}}e_k$ and ${}^{\mathbf{R}}c_k$ along with the interchange law for monoidal categories[7] □

PROPOSITION 3.5.10. *The category $\boldsymbol{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$ can be described by a planar diagrammatic calculus with unoriented strands such that isotopic diagrams represent equal morphisms, and the unoriented cups and caps labelled by k are equal to ${}^{\mathbb{A}}c_k$ and ${}^{\mathbb{A}}e_k$ respectively.*

PROOF. Using the pivotal structure from [**59**], which is such that the Frobenius-Schur indicator of each irreducible is $+1$, it is known that $\mathbf{Fund}(U_{\mathbb{F}}(\mathfrak{so}_m))$ can be described by an unoriented diagrammatic calculus [**58**]. For more details, see the discussion in [**8**, Section 2.2 and Theorem 5.1]. Viewing $\mathbf{Fund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$ as a subcategory of $\mathbf{Fund}(U_{\mathbb{F}}(\mathfrak{so}_m))$, it follows that $\mathbf{Fund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$ can be described by an unoriented diagrammatic calculus with respect to some cups and caps

$$ \mathrm{cup}_k \in \mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\mathbb{F}, \Lambda^k_{\mathbb{F}} \otimes \Lambda^k_{\mathbb{F}}) \quad \text{and} \quad \mathrm{cap}_k \in \mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda^k_{\mathbb{F}} \otimes \Lambda^k_{\mathbb{F}}, \mathbb{F}). $$

Note that we can re-scale cup_k by any $\lambda_k \in \mathbb{F}^\times$ as long as we also re-scale cap_k by λ_k^{-1}.

[7]If $\mathscr{C}$ is a monoidal category, then for $\alpha \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ and $\alpha' \in \mathrm{Hom}_{\mathscr{C}}(X',Y')$, we have $(\alpha \otimes \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \otimes \alpha') = (\mathrm{id}_Y \otimes \alpha') \circ (\alpha \otimes \mathrm{id}_{X'})$.

We know that $\mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\mathbb{F}, \Lambda^k_{\mathbb{F}} \otimes \Lambda^k_{\mathbb{F}})$ is one dimensional. Also, both cup_k and ${}^{\mathbb{F}}c_k$ are non-zero elements of $\mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\mathbb{F}, \Lambda^k_{\mathbb{F}} \otimes \Lambda^k_{\mathbb{F}})$. Therefore, there is $\lambda_k \in \mathbb{F}^{\times}$ such that ${}^{\mathbb{F}}c_k = \lambda_k \cdot \mathrm{cup}_k$. Using the hypothesis that we can describe $\mathbf{Fund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$ graphically with cup_k and cap_k, and Lemma 3.5.33 we find

$$\begin{aligned}
{}^{\mathbb{F}}e_k &= {}^{\mathbb{F}}e_k \circ \Big(\mathrm{id}_{\Lambda^k_{\mathbb{F}}} \otimes \big((\mathrm{id}_{\Lambda^k_{\mathbb{F}}} \otimes \lambda_k^{-1} \cdot \mathrm{cap}_k) \circ (\lambda_k \cdot \mathrm{cup}_k \otimes \mathrm{id}_{\Lambda^k_{\mathbb{F}}})\big)\Big) \\
&= \lambda_k^{-1} \cdot \mathrm{cap}_k \circ \Big(\big(({}^{\mathbb{F}}e_k \otimes \mathrm{id}_{\Lambda^k_{\mathbb{F}}}) \circ (\mathrm{id}_{\Lambda^k_{\mathbb{F}}} \otimes \lambda_k \cdot \mathrm{cup}_k)\big) \otimes \mathrm{id}_{\Lambda^k_{\mathbb{F}}}\Big) \\
&= \lambda_k^{-1} \cdot \mathrm{cap}_k \circ \Big(\big(({}^{\mathbb{F}}e_k \otimes \mathrm{id}_{\Lambda^k_{\mathbb{F}}}) \circ (\mathrm{id}_{\Lambda^k_{\mathbb{F}}} \otimes {}^{\mathbb{F}}c_k)\big) \otimes \mathrm{id}_{\Lambda^k_{\mathbb{F}}}\Big) \\
&= \lambda_k^{-1} \cdot \mathrm{cap}_k.
\end{aligned}$$

So without loss of generality we may assume that $\mathrm{cap}_k = {}^{\mathbb{F}}e_k$ and $\mathrm{cup}_k = {}^{\mathbb{F}}c_k$.

Note that,

$$\mathfrak{B}_{\mathbb{F}}({}^{\mathbb{A}}e_k) = {}^{\mathbb{F}}e_k \quad \text{and} \quad \mathfrak{B}_{\mathbb{F}}({}^{\mathbb{A}}c_k) = {}^{\mathbb{F}}c_k.$$

Given any graphical equation involving coupons labelled by morphisms in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$, cups, and caps, we can then apply $\mathfrak{B}_{\mathbb{F}}(-)$ and deduce, from the above discussion, an equation of morphisms in $\mathbf{Fund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$. It follows from Lemma 3.5.29 that this equation of morphisms is true in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$.

□

DEFINITION 3.5.19. *Define* ${}^{\mathbb{A}}m^{i,j}_{i+j}$ *using the graphical calculus for morphisms as the* 180 *degree rotation of* ${}^{\mathbb{A}}m^{i+j}_{j,i}$. *For* $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$, *define* ${}^{\boldsymbol{R}}m^{i,j}_{i+j} := \mathfrak{B}_{\boldsymbol{R}}({}^{\mathbb{A}}m^{i,j}_{i+j})$. *By Proposition 3.5.10 it does not matter whether we rotate clockwise or counterclockwise.*

LEMMA 3.5.34. *Let* $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. *For* $k = 1, \ldots, m$,

$${}^{\boldsymbol{R}}m^{k}_{1,k-1} \circ {}^{\boldsymbol{R}}m^{1,k-1}_{k} = \frac{[2k]}{[2]} \mathrm{id}_{\Lambda^k_{\boldsymbol{R}}} .$$

PROOF. For $\mathbf{R} = \mathbb{A}$, this follows by using the graphical calculus for $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$ and Definition 3.5.17. Then applying $\mathfrak{B}_{\mathbf{R}}$ yields the result for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$. □

LEMMA 3.5.35. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. For $k = 1, \ldots, m$,*

$$ {}^{\boldsymbol{R}}e_k \circ {}^{\boldsymbol{R}}c_k = \frac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} \cdot \mathrm{id}_{\Lambda^0_{\boldsymbol{R}}} . $$

PROOF. We give a sketch, for more details see [**8**, Section 2.2].

First, we observe that $\mathrm{End}_{\mathbb{A}}(\Lambda^0_{\mathbb{A}}) = \mathbb{A} \cdot \mathrm{id}_{\Lambda^0_{\mathbb{A}}}$, and every $\mathbb{A}$-linear endomorphism of the trivial module commutes with $U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})$, so $\mathrm{End}_{U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda^0_{\mathbb{A}}) = \mathbb{A} \cdot \mathrm{id}_{\Lambda^0_{\mathbb{A}}}$. It follows that ${}^{\mathbb{A}}e_k \circ {}^{\mathbb{A}}c_k = d_k(m) \cdot \mathrm{id}_{\Lambda^0_{\mathbb{A}}}$, for some $d_k(m) \in \mathbb{A}$. Thus,

$$ {}^{\mathbb{C}}e_k \circ {}^{\mathbb{C}}c_k = \overline{d_k(m)} \cdot \mathrm{id}_{\Lambda^0_{\mathbb{C}}} \quad \text{and} \quad {}^{\mathbb{F}}e_k \circ {}^{\mathbb{F}}c_k = d_k(m) \cdot \mathrm{id}_{\Lambda^0_{\mathbb{F}}}, $$

where $\overline{d_k(m)}$ denotes the image of $d_k(m)$ under $\mathbb{A} \to \mathbb{C}$. So it suffices to show the claim for $\mathbf{R} = \mathbb{F}$.

Taking the trace of the identity of an object, with respect to our chosen pivotal structure, gives the quantum dimension. The quantum Weyl dimension formula states that for our chosen pivotal structure on $\mathbf{Rep}(U_q(\mathfrak{so}_{\mathrm{m}}))$

$$ \mathrm{qdim}(L_{\mathbb{F}}(\mathbf{a})) := \mathrm{trace}_q\left(\mathrm{id}_{L_{\mathbb{F}}(\mathbf{a})}\right) = (-1)^{(\rho^\vee, 2\mathbf{a})} \prod_{\alpha \in \Phi_+} \frac{[(\alpha^\vee, \mathbf{a}+\rho)]_{v_\alpha}}{[(\alpha^\vee, \mathbf{a})]_{v_\alpha}} , $$

where $v = q$, if m is odd, and $v = q^2$, if m is even. On the other hand, the trace of $\mathrm{id}_{\Lambda^k_{\mathbb{F}}}$, with respect to our chosen pivotal structure, is exactly the coefficient $d_k(m)$.

We leave it as an exercise to use the quantum dimension formula, along with Remark 3.5.11, to derive the dimension formula in the statement of the Lemma. For a hint, look at proof of [**8**, Proposition 2.2]. ☐

The last intertwiner we will consider is the braiding isomorphism $\beta_{\mathrm{V}_{\mathbb{F}}, \mathrm{V}_{\mathbb{F}}} : V_{\mathbb{F}} \otimes V_{\mathbb{F}} \to V_{\mathbb{F}} \otimes V_{\mathbb{F}}$. To this end, we observe that $V_{\mathbb{F}} \otimes V_{\mathbb{F}}$ is a direct sum of three non-isomorphic irreducible representations[8]: $L_{\mathbb{F}}(2\varpi_1, +1)$, which is characterized as containing $v_1 \otimes v_1$, $L_{\mathbb{F}}(\varpi_2, +1)$, which is isomorphic to $\Lambda^2_{\mathbb{F}}$, and $V_{\mathbb{F}}(0, +1)$, the trivial module. Write $\pi_{(2)}$, $\pi_{(1,1)}$, and π_0 for the projections in $\mathrm{End}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(V_{\mathbb{F}}^{\otimes 2})$ with image $L_{\mathbb{F}}(2\varpi_1, +1)$, $L_{\mathbb{F}}(\varpi_2, +1)$, and $L_{\mathbb{F}}(0, +1)$ respectively. Then it follows from [**38**, Equation 6.12] that

$$ \beta_{\mathrm{V}_{\mathbb{F}}, \mathrm{V}_{\mathbb{F}}} = q^2 \pi_{(2)} - q^{-2} \pi_{(1,1)} + q^{2-2m} \pi_{(0)} \quad \text{and} \quad \beta^{-1}{}_{\mathrm{V}_{\mathbb{F}}, \mathrm{V}_{\mathbb{F}}} = q^{-2} \pi_{(2)} - q^2 \pi_{(1,1)} + q^{-2+2m} \pi_{(0)}. $$

[8]Except when $m = 1$, and we have $V_{\mathbb{F}}(2) = 0 = V_{\mathbb{F}}(1,1)$. In this case $v_1 \otimes v_1$ generates $V_{\mathbb{F}}(0)$.

LEMMA 3.5.36. *We can express the braiding and its inverse in terms of our previously defined morphisms as*

$$\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}} = q^2 \cdot \mathrm{id}_{V_{\mathbb{F}}\otimes V_{\mathbb{F}}} - {}^{\mathbb{F}}m_2^{1,1} \circ {}^{\mathbb{F}}m_{1,1}^2 - \frac{[m-2]}{[2m-4]}(q^2-q^{-2})q^{-m+2} \cdot {}^{\mathbb{F}}c_1 \circ {}^{\mathbb{F}}e_1$$

and

$$\beta^{-1}{}_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}} = q^{-2} \cdot \mathrm{id}_{V_{\mathbb{F}}\otimes V_{\mathbb{F}}} - {}^{\mathbb{F}}m_2^{1,1} \circ {}^{\mathbb{F}}m_{1,1}^2 + \frac{[m-2]}{[2m-4]}(q^2-q^{-2})q^{m-2} \cdot {}^{\mathbb{F}}c_1 \circ {}^{\mathbb{F}}e_1.$$

PROOF. First, we note that

$$\pi_{(1,1)} = \frac{[2]}{[4]} \cdot {}^{\mathbb{F}}m_2^{1,1} \circ {}^{\mathbb{F}}m_{1,1}^2 \quad \text{and} \quad \pi_{(0)} = \frac{[m-2][2m]}{[2m-4][m][m]_{q^2}} \cdot {}^{\mathbb{F}}c_1 \circ {}^{\mathbb{F}}e_1.$$

Since $\mathrm{id}_{V_{\mathbb{F}}\otimes V_{\mathbb{F}}} = \pi_{(2)} + \pi_{(1,1)} + \pi_0$, it follows that

$$\pi_{(2)} = \mathrm{id}_{V_{\mathbb{F}}\otimes V_{\mathbb{F}}} - \frac{[2]}{[4]} \cdot {}^{\mathbb{F}}m_2^{1,1} \circ {}^{\mathbb{F}}m_{1,1}^2 - \frac{[m-2][2m]}{[2m-4][m][m]_{q^2}} \cdot {}^{\mathbb{F}}c_1 \circ {}^{\mathbb{F}}e_1.$$

Thus,

$$\begin{aligned}
\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}} &= q^2\,\mathrm{id}_{V_{\mathbb{F}}\otimes V_{\mathbb{F}}} + \left(-q^2\frac{[2]}{[4]} - q^{-2}\frac{[2]}{[4]}\right) \cdot {}^{\mathbb{F}}m_2^{1,1} \circ {}^{\mathbb{F}}m_{1,1}^2 \\
&\quad + \left(-q^2\frac{[m-2][2m]}{[2m-4][m][m]_{q^2}} + q^{2-2m}\frac{[m-2][2m]}{[2m-4][m][m]_{q^2}}\right) \cdot {}^{\mathbb{F}}c_1 \circ {}^{\mathbb{F}}e_1 \\
&= q^2 \cdot \mathrm{id}_{V_{\mathbb{F}}\otimes V_{\mathbb{F}}} - {}^{\mathbb{F}}m_2^{1,1} \circ {}^{\mathbb{F}}m_{1,1}^2 - \frac{[m-2]}{[2m-4]}(q^2-q^{-2})q^{-m+2} \cdot {}^{\mathbb{F}}c_1 \circ {}^{\mathbb{F}}e_1.
\end{aligned}$$

The same argument is used to derive the formula for $\beta^{-1}{}_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}}$. □

DEFINITION 3.5.20. *Let* $\boldsymbol{R} \in \{\mathbb{C},\mathbb{A},\mathbb{F}\}$*, we define*

$$\beta_{\mathrm{V}_{\boldsymbol{R}},\mathrm{V}_{\boldsymbol{R}}} := q^2 \cdot \mathrm{id}_{V_{\boldsymbol{R}}\otimes V_{\boldsymbol{R}}} - {}^{\boldsymbol{R}}m_2^{1,1} \circ {}^{\boldsymbol{R}}m_{1,1}^2 - \frac{[m-2]}{[2m-4]}(q^2-q^{-2})q^{-m+2} \cdot {}^{\boldsymbol{R}}c_1 \circ {}^{\boldsymbol{R}}e_1$$

and

$$\beta^{-1}{}_{\mathrm{V}_{\boldsymbol{R}},\mathrm{V}_{\boldsymbol{R}}} := q^{-2} \cdot \mathrm{id}_{V_{\boldsymbol{R}}\otimes V_{\boldsymbol{R}}} - {}^{\boldsymbol{R}}m_2^{1,1} \circ {}^{\boldsymbol{R}}m_{1,1}^2 + \frac{[m-2]}{[2m-4]}(q^2-q^{-2})q^{m-2} \cdot {}^{\boldsymbol{R}}c_1 \circ {}^{\boldsymbol{R}}e_1.$$

LEMMA 3.5.37. *Let* $\boldsymbol{R} \in \{\mathbb{C},\mathbb{A},\mathbb{F}\}$. *The* 90 *degree rotation of* $\beta_{\mathrm{V}_{\boldsymbol{R}},\mathrm{V}_{\boldsymbol{R}}}$ *is* $\beta^{-1}{}_{\mathrm{V}_{\boldsymbol{R}},\mathrm{V}_{\boldsymbol{R}}}$.

PROOF. The claim is equivalent to

$$(\mathrm{id}_{V_{\mathbf{R}}} \otimes {}^{\mathbf{R}}e_1) \circ (\beta_{\mathrm{V}_{\mathbf{R}},\mathrm{V}_{\mathbf{R}}} \otimes \mathrm{id}_{V_{\mathbf{R}}}) \circ (\mathrm{id}_{V_{\mathbf{R}}} \otimes \beta_{\mathrm{V}_{\mathbf{R}},\mathrm{V}_{\mathbf{R}}}) = ({}^{\mathbf{R}}e_1 \otimes \mathrm{id}_{V_{\mathbf{R}}}).$$

By Lemma 3.5.29 it suffices to show this is true for $\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}}$. The Hexagon equation implies that $\beta_{V_{\mathbb{F}}\otimes V_{\mathbb{F}},V_{\mathbb{F}}} = (\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}}\otimes \mathrm{id}_{V_{\mathbb{F}}})\circ(\mathrm{id}_{V_{\mathbb{F}}}\otimes\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}})$. Since we are in a strict braided monoidal category, we also have $\beta_{\mathbb{F},V_{\mathbb{F}}} = \mathrm{id}_{V_{\mathbb{F}}}$. Also, by naturality of the braiding we have $(\mathrm{id}_{V_{\mathbb{F}}}\otimes {}^{\mathbb{F}}e_1)\circ\beta_{V_{\mathbb{F}}\otimes V_{\mathbb{F}},V_{\mathbb{F}}} = \beta_{\mathbb{F},V_{\mathbb{F}}}\circ({}^{\mathbb{F}}e_1\otimes\mathrm{id}_{V_{\mathbb{F}}})$. Thus,

$$\begin{aligned}(\mathrm{id}_{V_{\mathbb{F}}}\otimes^{\mathbb{F}}e_1)\circ(\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}}\otimes\mathrm{id}_{V_{\mathbb{F}}})\circ(\mathrm{id}_{V_{\mathbb{F}}}\otimes\beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}}) &= (\mathrm{id}_{V_{\mathbb{F}}}\otimes^{\mathbb{F}}e_1)\circ\beta_{V_{\mathbb{F}}\otimes V_{\mathbb{F}},V_{\mathbb{F}}}\\ &= \beta_{\mathbb{F},V_{\mathbb{F}}}\circ({}^{\mathbb{F}}e_1\otimes\mathrm{id}_{V_{\mathbb{F}}})\\ &= \mathrm{id}_{V_{\mathbb{F}}}\circ({}^{\mathbb{F}}e_1\otimes\mathrm{id}_{V_{\mathbb{F}}})\\ &= ({}^{\mathbb{F}}e_1\otimes\mathrm{id}_{V_{\mathbb{F}}}).\end{aligned}$$

□

Unsurprisingly, the braiding when $q=1$ is just the tensor flip map.

LEMMA 3.5.38. *The map $\beta_{\mathrm{V}_{\mathbb{C}},\mathrm{V}_{\mathbb{C}}}$ acts on $V_{\mathbb{C}}^{\otimes 2}$ by $v\otimes w\mapsto w\otimes v$.*

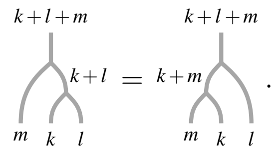
PROOF. Let s denote the tensor flip map. Since $q-1=0$ in $\mathbb{C}$, we see that Definition 3.5.20 simplifies to $\beta_{\mathrm{V}_{\mathbb{C}},\mathrm{V}_{\mathbb{C}}} = \mathrm{id}_{V_{\mathbb{C}}\otimes V_{\mathbb{C}}} - {}^{\mathbf{R}}m_2^{1,1}\circ{}^{\mathbf{R}}m_{1,1}^{2}$. Since ${}^{\mathbb{C}}m_2^{1,1}\circ{}^{\mathbb{C}}m_{1,1}^{2}$ factors through $\Lambda^2_{\mathbb{C}}$ and squares to 2, we have $\frac{1}{2}{}^{\mathbb{C}}m_2^{1,1}\circ{}^{\mathbb{C}}m_{1,1}^{2} = \frac{1}{2}(\mathrm{id}_{V_{\mathbb{C}}\otimes V_{\mathbb{C}}}-s)$, the anti-symmetrizing idempotent. Thus, $\beta_{\mathrm{V}_{\mathbb{C}},\mathrm{V}_{\mathbb{C}}} = \mathrm{id}_{V_{\mathbb{C}}\otimes V_{\mathbb{C}}} - (\mathrm{id}_{V_{\mathbb{C}}\otimes V_{\mathbb{C}}}-s) = s$. □

3.5.4. Existence of the functor. Let $\mathbf{R}\in\{\mathbb{F},\mathbb{A},\mathbb{C}\}$. We will prove that there is a pivotal functor $\Phi_{\mathbf{R}}$: $\mathbf{Web}_{\mathbf{R}}(\mathrm{O}(\mathrm{m}))\longrightarrow\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$. Since $\mathbf{Web}_{\mathbf{R}}(\mathrm{O}(\mathrm{m}))$ is a generators and relations category, it suffices to define where generators go and check relations. Thus, the majority of this section is devoted to deriving various relations among morphisms in $\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$. For later arguments, it is important for us to have canonical identifications between $\mathbf{R}\otimes\Phi_{\mathbb{A}}$ and $\Phi_{\mathbf{R}}$. To make this precise, we construct $\Phi_{\mathbb{A}}$ first, then define $\Phi_{\mathbf{R}} := \mathfrak{B}_{\mathbf{R}}\circ(\mathbf{R}\otimes\Phi_{\mathbb{A}})$.

3.5.4.1. *Deriving relations.* In this section, all graphical calculations are in the category $\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$, as opposed to $\mathbf{Web}_{\mathbf{R}}(\mathrm{O}(\mathrm{m}))$, and trivalent graphs represent the multiplication maps $\Lambda^i_{\mathbf{R}}\otimes\Lambda^j_{\mathbf{R}}\to\Lambda^{i+j}_{\mathbf{R}}$. That this is valid for $\mathbf{R}\in\{\mathbb{C},\mathbb{A},\mathbb{F}\}$ is justified by Proposition 3.5.10. To make this clear, we will use grey diagrams in this section. We also assume that we are working in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$, unless we explicitly say that we are working over $\mathbb{F}$ or $\mathbb{C}$.

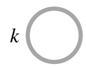
First, since $\Lambda^\bullet_{\mathbb{A}}$ is an associative graded algebra, we have the following.

(3.21)



Lemma 3.5.35 says that

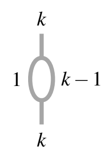
(3.22)



$$= \frac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2},$$

and from Lemma 3.5.34 we find,

(3.23)

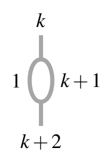


$$= \frac{[2k]}{[2]}$$

.

Since $\mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda_{\mathbb{A}}^{k+2}, \Lambda_{\mathbb{A}}^{k}) = 0$, we also have

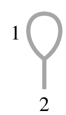
(3.24)



$$= 0.$$

Let $k = 0$ in Equation (3.24), we have

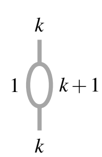
(3.25)



$$= 0.$$

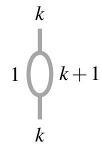
LEMMA 3.5.39.

(3.26)



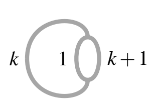
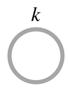
$$= \frac{[2m-2k][2m-4k-4][m-2k]}{[2][m-2k-2][2m-4k]}$$

PROOF. We temporarily work over $\mathbb{F}$. Since $\mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_\mathrm{m})}(\Lambda^k_{\mathbb{F}}, \Lambda^k_{\mathbb{F}})$ is 1-dimensional, there exists $\alpha \in \mathbb{F}$ such that

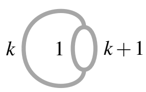
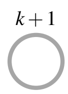
 $= \alpha$  .

If we show Equation (3.26) is true over $\mathbb{F}$, then in particular $\alpha \in \mathbb{A}$, so Lemma 3.5.29 implies the equation also holds in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_\mathrm{m}))$.

Observe that

 $= \alpha$  $= \alpha \dfrac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}$.

On the other hand,

 $= \dfrac{[2k+2]}{[2]}$  $= \dfrac{[2k+2]}{[2]} \dfrac{[2m-4k-4][m]}{[m-2k-2][2m]} \begin{bmatrix} m \\ k+1 \end{bmatrix}_{q^2}$.

Thus,

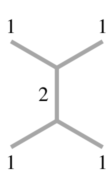
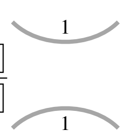
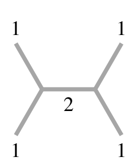
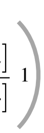
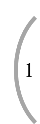
$$\alpha \frac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} \cdot \mathrm{id}_{\Lambda^0_{\mathbb{F}}} = \frac{[2k+2]}{[2]} \frac{[2m-4k-4][m]}{[m-2k-2][2m]} \begin{bmatrix} m \\ k+1 \end{bmatrix}_{q^2} \cdot \mathrm{id}_{\Lambda^0_{\mathbb{F}}},$$

and since $\mathrm{id}_{\Lambda^0_{\mathbb{F}}} \neq 0$, we can compare coefficients and solve for α from

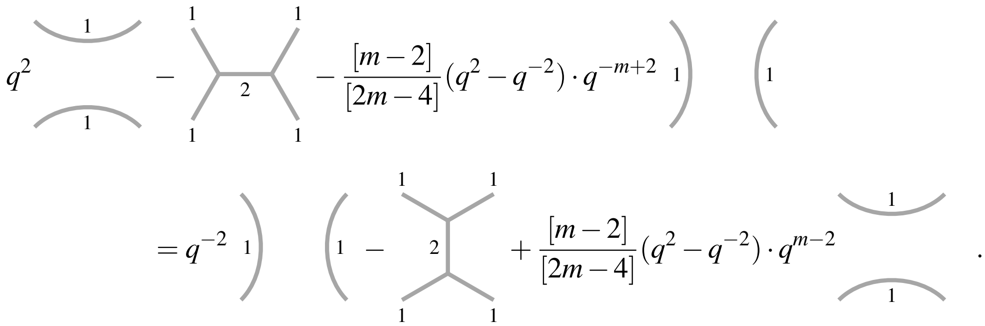
$$\alpha \frac{[2m-4k][m]}{[m-2k][2m]} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} = \frac{[2k+2]}{[2]} \frac{[2m-4k-4][m]}{[m-2k-2][2m]} \begin{bmatrix} m \\ k+1 \end{bmatrix}_{q^2}.$$

□

LEMMA 3.5.40.

(3.27)  $+ \dfrac{[2m-8][m-2]}{[m-4][2m-4]}$  $=$  $+ \dfrac{[2m-8][m-2]}{[m-4][2m-4]}$  

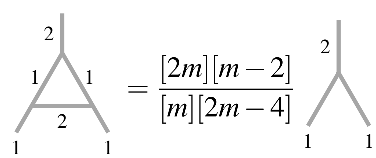
PROOF. By Definition 3.5.20 and Lemma 3.5.37, we know that the 90 degree rotation of $\beta_{\mathrm{V}_{\mathbf{R}},\mathrm{V}_{\mathbf{R}}}$ is the same as $\beta^{-1}{}_{\mathrm{V}_{\mathbf{R}},\mathrm{V}_{\mathbf{R}}}$, so



We obtain Equation (3.27) by combining terms and using the identities in Remark 3.5.12. □

LEMMA 3.5.41.

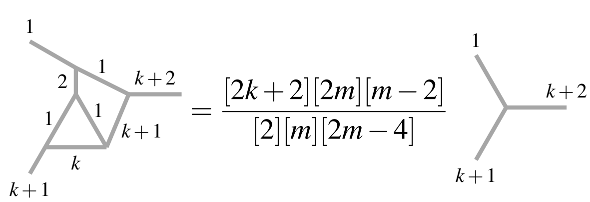
(3.28)



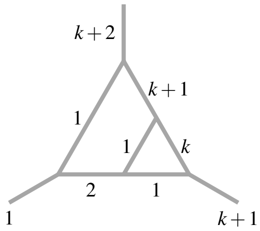
PROOF. This is immediate from Equation (3.27), Equation (3.23), and Equation (3.25). □

LEMMA 3.5.42.

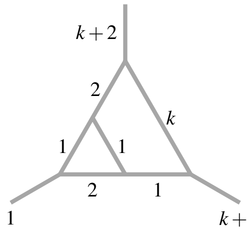
(3.29)



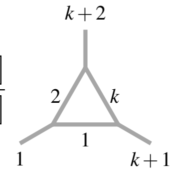
PROOF.



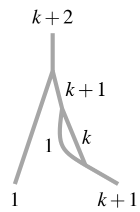
$\overset{(3.21)}{=}$



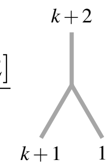
$\overset{(3.28)}{=} \dfrac{[2m][m-2]}{[m][2m-4]}$



$\overset{(3.21)}{=} \dfrac{[2m][m-2]}{[m][2m-4]}$

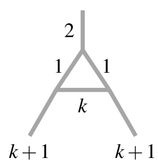


$\overset{(3.23)}{=} \dfrac{[2k+2][2m][m-2]}{[2][m][2m-4]}$



□

LEMMA 3.5.43. *For $k+1<m$,*

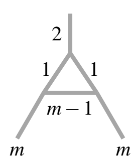


$$\neq 0.$$

PROOF. Since this triangle is part of the left hand side of Equation (3.29), and the right hand side of Equation (3.29) is a non-zero scalar multiple of the map $m_{1,k+1}^{k+2}$, which, as long as $k+2 \leq m$, is non-zero by Remark 3.5.27. □

LEMMA 3.5.44.

(3.30)



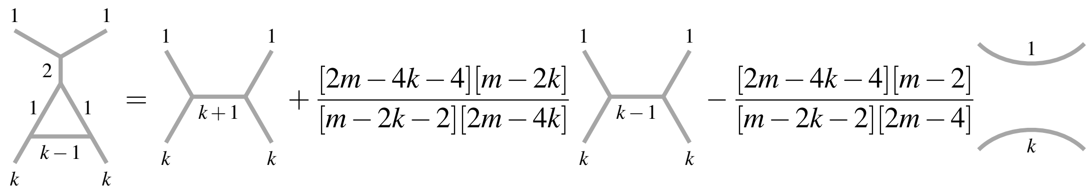
$$= 0$$

PROOF. We know that $\Lambda_{\mathbb{F}}^m \otimes \Lambda_{\mathbb{F}}^m \cong \det_{\mathbb{F}} \otimes \det_{\mathbb{F}} \cong \Lambda_{\mathbb{F}}^0$. Also, Proposition 3.5.8 implies that

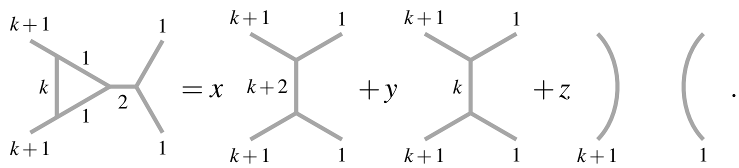
$$\dim_{\mathbb{F}} \operatorname{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda_{\mathbb{F}}^0, \Lambda_{\mathbb{F}}^2) = 0.$$

This means that Equation (3.30) holds after applying $\mathfrak{B}_{\mathbb{F}}$, and the claim follows from Lemma 3.5.29. □

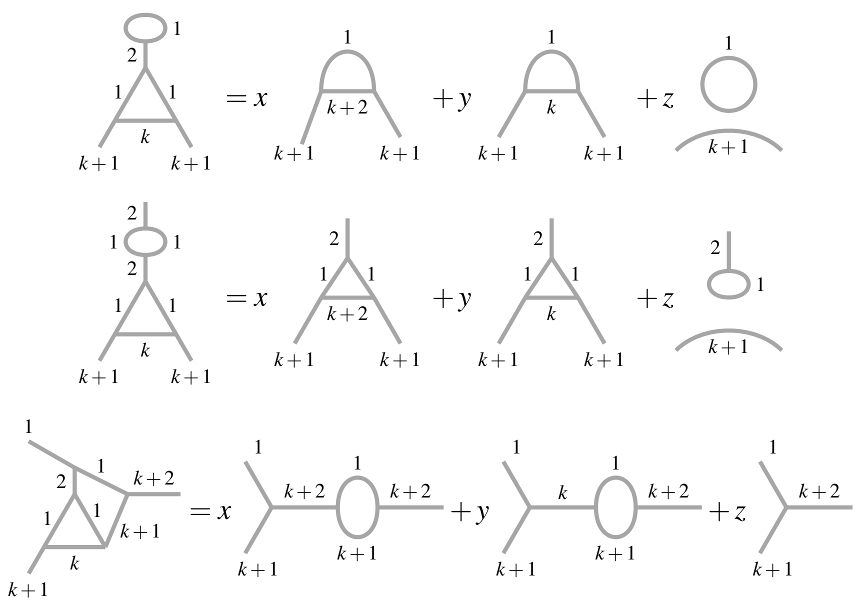
PROPOSITION 3.5.11. *Suppose that $k \leq m$. Then the following equation holds in **Fund**$(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$.*

(3.31)


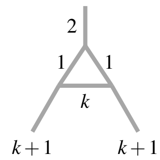
PROOF. We prove Equation (3.31) by induction. We verified the base case, that Equation (3.31) holds when $k = 1$, in Lemma 3.27. Suppose Equation (3.31) holds in **Fund**$(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$ for $k \geq 1$, then we will show that it is true for $k+1$. Since $\Lambda_{\mathbb{A}}^{k+1} = 0$ when $k+1 > m$, we may assume $k+1 \leq m$, in order for Equation (3.31) to be non-trivial in **Fund**$(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$. Suppose that $k+1 < m$. We temporarily work over $\mathbb{F}$. It follows from Lemma 3.5.27 that $\Lambda_{\mathbb{F}}^{k+1} \otimes V_{\mathbb{F}} \cong \Lambda_{\mathbb{F}}^{k+2} \oplus \Lambda_{\mathbb{F}}^{k} \oplus L_{\mathbb{F}}(\varpi_1 + \varpi_k, +1)$. Therefore, the merge-split to $k+2$, the merge-split to k, and $\mathrm{id}_{\Lambda_{\mathbb{F}}^{k+1} \otimes V_{\mathbb{F}}}$ form a basis for $\mathrm{End}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(\Lambda_{\mathbb{F}}^{i} \otimes V_{\mathbb{F}})$. In particular, there are $x, y, z \in \mathbb{F}$ such that

(3.32)


Now, in order to find a linear system of equations about x,y,z, we first attach caps or trivalent vertices, in three different ways, to each term of Equation (3.32).



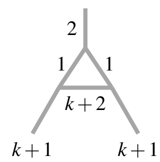
In order to simplify the second equation, we need to relate the two triangles on the right hand side. Using results in [**34**], one can check that since $k+1<m$, $\dim_{\mathbb{F}} \mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_m)}(\Lambda_{\mathbb{F}}^{k+1}\otimes\Lambda_{\mathbb{F}}^{2},\Lambda_{\mathbb{F}}^{k+1})=1$. By Lemma 3.5.43

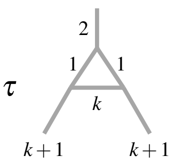


$\neq 0$.

So there exists $\tau\in\mathbb{F}$ such that

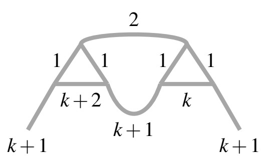
(3.33)

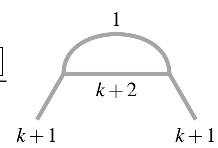


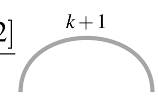
$= \tau$


.

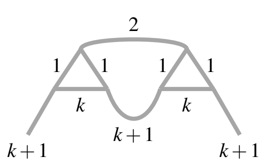
On one hand, we know:

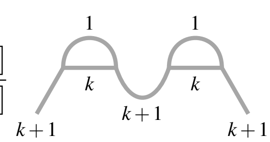


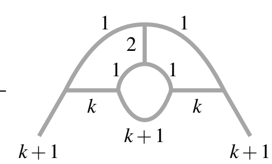
$\overset{(3.29)}{=} \dfrac{[2k+2][2m][m-2]}{[2][m][2m-4]}$


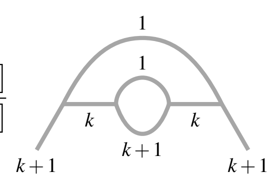
$\overset{(3.26)}{=} \dfrac{[2k+2][2m][m-2][2m-2k-2][2m-4k-8][m-2k-2]}{[2][m][2m-4][2][m-2k-4][2m-4k-4]}$


On the other hand, we know that



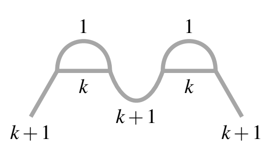
$\overset{(3.27)}{=} -\dfrac{[2m-8][m-2]}{[m-4][2m-4]}$


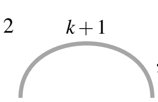
$+$


$+\dfrac{[2m-8][m-2]}{[m-4][2m-4]}$


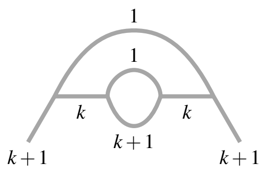
,

where

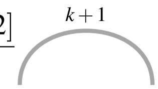


$\overset{(3.23)}{=} \left(\dfrac{[2k+2]}{[2]}\right)^2$


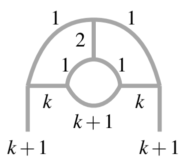
,



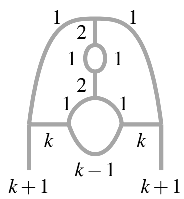
$$\overset{(3.26)}{\underset{(3.23)}{=}} \frac{[2m-2k][2m-4k-4][m-2k]}{[2][m-2k-2][2m-4k]}\frac{[2k+2]}{[2]}$$

 .

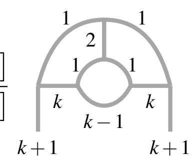
By the inductive hypothesis,



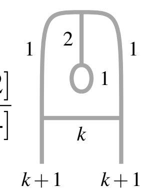
$$\overset{(3.31)}{=}$$



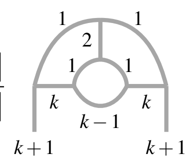
$$-\frac{[2m-4k-4][m-2k]}{[m-2k-2][2m-4k]}$$



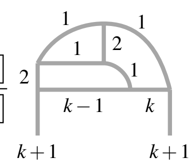
$$+\frac{[2m-4k-4][m-2]}{[m-2k-2][2m-4]}$$



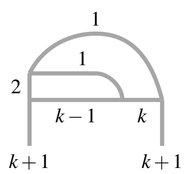
$$\overset{(3.23)}{\underset{(3.25)}{=}} \frac{[2m-4k+4][m-2k]}{[m-2k+2][2m-4k]}$$



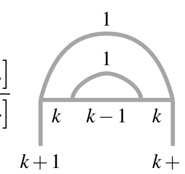
$$\overset{(3.21)}{=} \frac{[2m-4k+4][m-2k]}{[m-2k+2][2m-4k]}$$



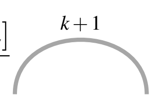
$$\overset{(3.28)}{=} \frac{[2m-4k+4][m-2k]}{[m-2k+2][2m-4k]}\frac{[2m][m-2]}{[m][2m-4]}$$



$$\overset{(3.21)}{=} \frac{[2m-4k+4][m-2k][2m][m-2]}{[m-2k+2][2m-4k][m][2m-4]}$$



$$\overset{(3.23)}{=} \frac{[2m-4k+4][m-2k][2m][m-2]}{[m-2k+2][2m-4k][m][2m-4]}\frac{[2k]}{[2]}\frac{[2k+2]}{[2]}$$

 .

In conclusion,

$$\tau = \frac{[2m-4k][m-2k-2]}{[m-2k][2m-4k-4]}.$$

So by applying Equations (3.22), (3.23), (3.24), (3.25), (3.26), (3.29), and (3.33), we have the following system of linear equations.

$$\begin{aligned}
0 &= \frac{[2m-2k-2][2m-4k-8][m-2k-2]}{[2][m-2k-4][2m-4k-4]}x + \frac{[2k+2]}{[2]}y + \frac{[2m-4][m]}{[m-2][2]}z, \\
\frac{[4]}{[2]} &= \frac{[2m-4k][m-2k-2]}{[m-2k][2m-4k-4]}x + y, \quad \text{and} \\
\frac{[2k+2][2m][m-2]}{[2][m][2m-4]} &= \frac{[2k+4]}{[2]}x + z.
\end{aligned}$$

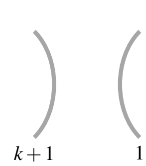
The values

$$
\begin{aligned}
x &= 1\\
y &= \frac{[2m-4k-8][m-2k-2]}{[m-2k-4][2m-4k-4]}\\
z &= -\frac{[2m-4k-8][m-2]}{[m-2k-4][2m-4]}
\end{aligned}
$$

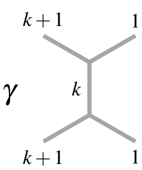
satisfy the equations, and therefore are a unique set of solutions. Since $x, y, z \in \mathbb{A}$, it follows from Lemma 3.5.29 that Equation (3.31) holds in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$ when $k+1 < m$.

Now, suppose that $k+1 = m$. Lemma 3.5.27 implies that $\Lambda_{\mathbb{F}}^{k+1} \otimes V_{\mathbb{F}} \cong \Lambda_{\mathbb{F}}^{k}$. Thus, there exists $\gamma \in \mathbb{F}$ such that

(3.34)

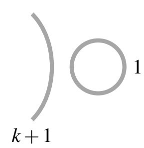


$$= \gamma$$

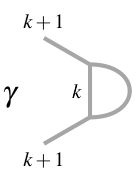


,

so



$$= \gamma$$

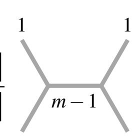


.

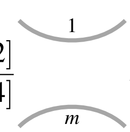
Using Equations (3.22) and (3.23), we get $\frac{[2m-4][m]}{[m-2][2]} \cdot \mathrm{id}_{\Lambda_{\mathbb{F}}^{k+1}} = \gamma \cdot \frac{[2k+2]}{[2]} \cdot \mathrm{id}_{\Lambda_{\mathbb{F}}^{k+1}}$, so $\gamma = \frac{[2m-4][m]}{[m-2][2m]} \in \mathbb{A}$. It follows from Lemma 3.5.29 that Equation (3.34) also holds in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))$.

On the other hand, consider Equation (3.31) when $k = m$. Lemma 3.5.44 implies that the left hand side of Equation (3.31) is zero. Also, the first term on the right hand side of Equation (3.31) has a label $m+1$, so is also zero. Thus, Equation (3.31) becomes

$$0 = 0 + \frac{[2m-4m-4][m-2m]}{[m-2m-2][2m-4m]}$$



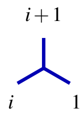
$$-\frac{[2m-4m-4][m-2]}{[m-2m-2][2m-4]}$$

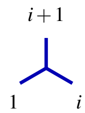


,

which agrees with Equation (3.34). $\square$

THEOREM 3.5.12. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{A}, \mathbb{F}\}$. There is a pivotal functor $\Phi_{\boldsymbol{R}} : \boldsymbol{Web_R}(\mathrm{O(m)}) \to \boldsymbol{Fund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_\mathrm{m}))$ such that,*



$\mapsto {}^{\boldsymbol{R}}m^{i+1}_{i,1}$,


$\mapsto {}^{\boldsymbol{R}}m^{i+1}_{1,i}$, *and* $\Phi_{\boldsymbol{R}}({}^{\boldsymbol{R}}\beta_{1,1}) = \beta_{\mathrm{V}_{\boldsymbol{R}},\mathrm{V}_{\boldsymbol{R}}}$

Moreover, we have canonical identifications $\Phi_{\boldsymbol{R}} = \mathfrak{B}_{\boldsymbol{R}} \circ (\boldsymbol{R} \otimes \Phi_{\mathbb{A}})$.

PROOF. Suppose we have $\Phi_{\mathbb{A}}$ as in the statement of the theorem. Then for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$, we define $\Phi_{\mathbf{R}}$ as the composition

$$\mathbf{Web_R}(\mathrm{O(m)}) = \mathbf{R} \otimes \mathbf{Web}_{\mathbb{A}}(\mathrm{O(m)}) \xrightarrow{\mathbf{R}\otimes\Phi_{\mathbb{A}}} \mathbf{R} \otimes \mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_\mathrm{m})) \xrightarrow{\mathfrak{B}_\mathbf{R}} \mathbf{Fund}(\mathrm{U}_\mathbf{R}(\mathfrak{o}_\mathrm{m}))\,.$$

It is then easy to see that it suffices to prove the result over $\mathbb{A}$. To this end, we just need to check the defining relations in $\mathbf{Web}_{\mathbb{A}}(\mathrm{O(m)})$, see Equation (3.4), are satisfied in $\mathbf{Fund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_\mathrm{m}))$. This follows from Equations (3.22), (3.23), (3.25), (3.21), and (3.31).

Having established everything else in the statement of the theorem, the equality $\Phi_\mathbf{R}({}^\mathbf{R}\beta_{1,1}) = \beta_{\mathrm{V}_\mathbf{R},\mathrm{V}_\mathbf{R}}$ follows from comparing Definition 3.5.20 and Definition 3.5.3. □

3.5.4.2. *Compatability with classical invariant theory.* Let $\mathbb{C}^m$ be a vector space with basis $\{v_1, \ldots, v_m\}$ and bilinear form $(v_i, v_j) = \delta_{i,j}$. We write $O(\mathbb{C}^m)$ for the subgroup of $GL(\mathbb{C}^m)$ preserving $(-,-)$.

We want to identify $\mathbf{StdFund}(\mathrm{U}_\mathbb{C}(\mathfrak{o}_\mathrm{m}))$ with the full monoidal subcategory of $\mathbf{Rep}(O(\mathbb{C}^m))$ generated by $\mathbb{C}^m$. To this end, consider the $\mathbb{C}$-basis for $V_\mathbb{C}$:

$$\begin{cases} (\sqrt{-1})^{i-1}\left(\frac{a_i+b_i}{2}\right), \quad (\sqrt{-1})^{i-1}\left(\frac{a_i-b_i}{2}\right), \quad i = 1, \ldots, n, \quad \text{if } m = 2n \\ (\sqrt{-1})^{i-1}\left(\frac{a_i+b_i}{2}\right), \quad (\sqrt{-1})^{n-1}\frac{u}{\sqrt{2}}, \quad \text{and} \quad (\sqrt{-1})^{i-1}\left(\frac{a_i-b_i}{2}\right), \quad i = 1, \ldots, n, \quad \text{if } m = 2n+1. \end{cases}$$

This basis gives an identification $\mathbb{C}^m = V_\mathbb{C}$ under which the form $(v_i, v_j) = \delta_{i,j}$ on $\mathbb{C}^m$ agrees with the form $(v, w) = {}^\mathbb{C}e_1(v \otimes w)$ on $V_\mathbb{C}$.

Let $\mathscr{B}(m)$ be the Brauer category as defined in [**40**, Definition 2.4 and Theorem 2,6]. Lehrer-Zhang prove there is a unique monoidal functor $F : \mathscr{B}(m) \to \mathbf{Rep}(O(\mathbb{C}^m)) = \mathbf{Rep}(\mathrm{U}_\mathbb{C}(\mathfrak{o}_\mathrm{m}))$ such that the crossing diagram maps to the tensor flip map, and the cup and cap diagrams map to the natural homomorphisms constructed with $(-,-)$ [**40**, Theorem 3.4].

On the other hand, we have constructed a monoidal functor $\Phi_{\mathbb{C}}|_{\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})} : \mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)}) \to \mathbf{StdFund}(\mathrm{U}_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}))$. Using the canonical identification $\mathbf{BMW}_{\mathbb{C}}(\mathrm{O(m)}) = \mathscr{B}(m)$, Proposition 3.5.4 gives a monoidal functor $\eta_{\mathbb{C}} : \mathscr{B}(m) \to \mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})$. By definition, $F \circ \eta_{\mathbb{C}}$ sends ${}^{\mathbb{C}}\beta_{1,1}$ to the tensor flip map, and by Lemma 3.5.38 $\Phi_{\mathbb{C}}$ acts the same way. Thus, after identifying $\mathbf{StdFund}(\mathrm{U}_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}))$ with the full monoidal subcategory of $\mathbf{Rep}(O(\mathbb{C}^m))$ generated by $V_{\mathbb{C}}$, we have $F = \Phi_{\mathbb{C}}|_{\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})} \circ \eta_{\mathbb{C}}$.

3.5.5. Proof of the equivalence.

3.5.5.1. *Reduction to standard subcategories.* Let $\mathbf{R} = \mathbb{F}$ or $\mathbb{C}$. In Theorem 3.5.12 we showed the existence of a pivotal functor $\Phi_{\mathbf{R}} : \mathbf{Web}_{\mathbf{R}}(\mathrm{O(m)}) \to \mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$. Our goal is to show that the functor $\Phi_{\mathbf{R}}$ is an equivalence. Essential sujectivity is immediate from the definitions, but we need to work to show $\Phi_{\mathbf{R}}$ is full and faithful. The first step is to reduce to showing that $\Phi_{\mathbf{R}}|_{\mathbf{StdWeb}_{\mathbf{R}}(\mathrm{O(m)})} : \mathbf{StdWeb}_{\mathbf{R}}(\mathrm{O(m)}) \to \mathbf{StdFund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$ is full and faithful.

LEMMA 3.5.45. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. If $\Phi_{\boldsymbol{R}}|_{\boldsymbol{StdWeb}_{\boldsymbol{R}}(\mathrm{O(m)})}$ is full, then $\Phi_{\boldsymbol{R}}$ is full. If $\Phi_{\boldsymbol{R}}|_{\boldsymbol{StdWeb}_{\boldsymbol{R}}(\mathrm{O(m)})}$ is faithful, then $\Phi_{\boldsymbol{R}}$ is faithful.*

PROOF. Using the merge and split trivalent vertices, it is easy to see that each generating object k in $\mathbf{Web}_{\mathbf{R}}(\mathrm{O(m)})$ is a direct summand of $1^{\otimes k}$. The claim then follows from [**8**, Lemma 5.5]. $\square$

3.5.5.2. *Fullness.* It is well known that if $\mathbf{R} = \mathbb{C}$ or $\mathbb{F}$, then the Brauer algebra, respectively the BMW algebra, is Schur-Weyl dual to $U_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}})$ acting on tensor powers of $V_{\mathbf{R}}$. An adjunction argument, see [**8**, Theorem 5.8], then yields fullness of $\Phi_{\mathbf{R}}|_{\mathbf{StdWeb}_{\mathbf{R}}(\mathrm{O(m)})}$. We give precise citations in the proof below.

THEOREM 3.5.13. *Let $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$. The functor $\Phi_{\boldsymbol{R}}|_{\boldsymbol{StdWeb}_{\boldsymbol{R}}(\mathrm{O(m)})}$ is full.*

PROOF. We first argue for $\mathbf{R} = \mathbb{C}$. We know that F is full [**40**, Theorem 4.8]. Also, $F = \Phi_{\mathbb{C}}|_{\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})} \circ \eta_{\mathbb{C}}$, so it follows that $\Phi_{\mathbb{C}}|_{\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})}$ is full.

Suppose $\mathbf{R} = \mathbb{F}$. Then by [**38**, Theorem 8.5] the operators $\mathrm{id} \otimes \beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}} \otimes \mathrm{id}$ generate $\mathrm{End}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(V_{\mathbf{R}}^{\otimes d})$, for all $d \in \mathbb{Z}_{\geq 0}$. Since $\Phi_{\mathbb{F}}$ is monoidal and $\Phi_{\mathbb{F}}({}^{\mathbb{F}}\beta_{1,1}) = \beta_{\mathrm{V}_{\mathbb{F}},\mathrm{V}_{\mathbb{F}}}$, it follows that the map

$$\Phi_{\mathbb{F}} : \mathrm{End}_{\mathbf{StdWeb}_{\mathbb{F}}(\mathrm{O(m)})}(1^{\otimes d}) \to \mathrm{End}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(V_{\mathbb{F}}^{\otimes d})$$

is surjective for all $d \in \mathbb{Z}_{\geq 0}$. Since $\Phi_{\mathbb{F}}$ is pivotal, we can use adjunction in $\mathbf{StdWeb}_{\mathbb{F}}(\mathrm{O(m)})$ and $\mathbf{StdFund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$ to deduce that

$$\Phi_{\mathbb{F}} : \mathrm{Hom}_{\mathbf{StdWeb}_{\mathbb{F}}(\mathrm{O(m)})}(1^{\otimes b}, 1^{\otimes c}) \to \mathrm{Hom}_{U_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}})}(V_{\mathbb{F}}^{\otimes b}, V_{\mathbb{F}}^{\otimes c})$$

is surjective for all $b, c \in \mathbb{Z}_{\geq 0}$ such that $b+c$ is even. Since the homomorphism spaces are zero when $b+c$ is odd, it follows that $\Phi_{\mathbb{F}}|_{\mathbf{StdWeb}_{\mathbb{F}}(\mathrm{O(m)})}$ is full. $\square$

3.5.5.3. *Faithfulness.* Our goal is to show that a functor is faithful, so we will necessarily have to analyze the kernel of a functor. The kernel of a monoidal functor is a monoidal ideal, so we recall some Lemmas about the interactions between monoidal functors and monoidal ideals.

NOTATION 3.5.10. *Let $\mathscr{C}$ be an $\boldsymbol{R}$-linear monoidal category and let x be a homomorphism in $\mathscr{C}$. Write $\langle x \rangle$ to denote the monoidal ideal generated by x in $\mathscr{C}$. Given a morphism $y \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, we write $y \in \langle x \rangle$ if $y \in \mathrm{Hom}_{\langle x \rangle}(X,Y)$.*

LEMMA 3.5.46. *Suppose that $G : \mathscr{C} \to \mathscr{D}$ is an $\boldsymbol{R}$-linear monoidal functor and x is a morphism in $\mathscr{C}$. Then if $y \in \langle x \rangle$, then $G(y) \in \langle G(x) \rangle$.*

PROOF. Follows from observing that G preserves linear combinations, tensor products, and compositions of morphisms. $\square$

Classical invariant theory gives us a description of the kernel of the functor $F : \mathscr{B}(m) \to \mathbf{Rep}(O(\mathbb{C}^m))$.

DEFINITION 3.5.14. *Given an element $w \in S_k$, we naturally get an element $w \in \mathrm{End}_{\mathscr{B}(m)}(1^{\otimes k})$. Let $a_k := \frac{1}{k!}\sum_{w \in S_k}(-1)^{\ell(w)} w \in \mathrm{End}_{\mathscr{B}(m)}(1^{\otimes k})$. We will represent the elements a_k and $\eta_{\mathbb{C}}(a_k)$ graphically by a box labelled by k.*

THEOREM 3.5.15. *The kernel of the functor*

$$F : \mathscr{B}(m) \longrightarrow \boldsymbol{StdFund}(\mathrm{U}_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}))$$

is the monoidal ideal generated by a_{m+1}.

PROOF. This is [**40**, Theorem 4.8(ii)]. $\square$

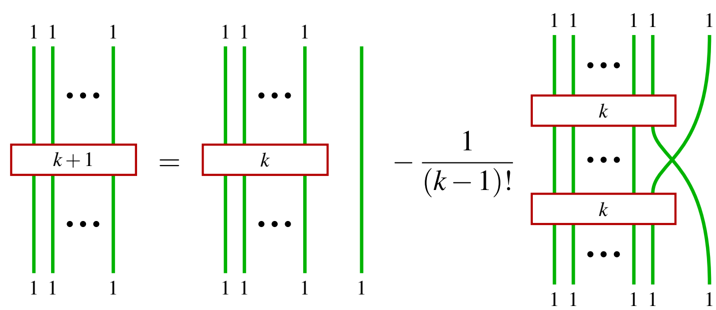
LEMMA 3.5.47. *The kernel of the functor $\Phi_{\mathbb{C}}|_{\boldsymbol{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})}$ is the monoidal ideal of $\boldsymbol{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})$ generated by $\eta_{\mathbb{C}}(a_{m+1})$.*

PROOF. Since $\Phi_{\mathbb{C}}(\eta_{\mathbb{C}}(a_{m+1})) = F(a_{m+1}) = 0$, we have $\langle \eta_{\mathbb{C}}(a_{m+1}) \rangle \subset \ker \Phi_{\mathbb{C}}$. To show the reverse inclusion, let $f \in \ker \Phi_{\mathbb{C}}$. Since $\eta_{\mathbb{C}}$ is full, see Proposition 3.5.4, there is some $\tilde{f}$ in $\mathscr{B}(m)$ such that $\eta_{\mathbb{C}}(\tilde{f}) = f$. Thus, $F(\tilde{f}) = \Phi_{\mathbb{C}} \circ \eta_{\mathbb{C}}(\tilde{f}) = \Phi_{\mathbb{C}}(f) = 0$, so $\tilde{f}$ is in the kernel of F. Theorem 3.5.15 implies $\tilde{f} \in \langle a_{m+1} \rangle$, and by Lemma 3.5.46 we have $f = \eta_{\mathbb{C}}(\tilde{f}) \in \langle \eta_{\mathbb{C}}(a_{m+1}) \rangle$. $\square$

We will show that $\Phi_{\mathbb{C}}|_{\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})}$ is faithful. Since this is equivalent to the kernel of $\Phi_{\mathbb{C}}$ being $\langle 0 \rangle$, we want to argue that $\eta_{\mathbb{C}}(a_{m+1})$ is already equal to zero in $\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})$. This is implied by a diagrammatic calculation in $\mathbf{Web}_{\mathbb{C}}(\mathrm{O(m)})$, which relies on the following Lemma.

LEMMA 3.5.48. *The following equation holds in* $\boldsymbol{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})$.

(3.35)

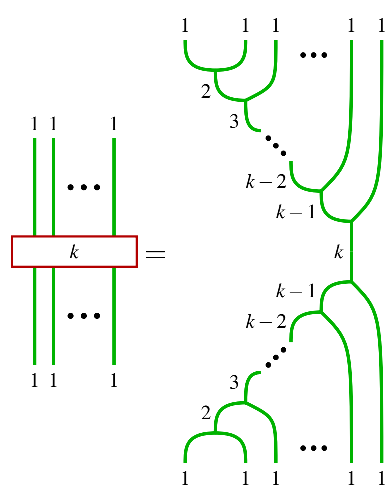


PROOF. Apply $\eta_{\mathbb{C}}$ to the analogous equation in $\mathscr{B}(m)$, which holds by [**40**, Lemma 2.11(1)]. $\square$

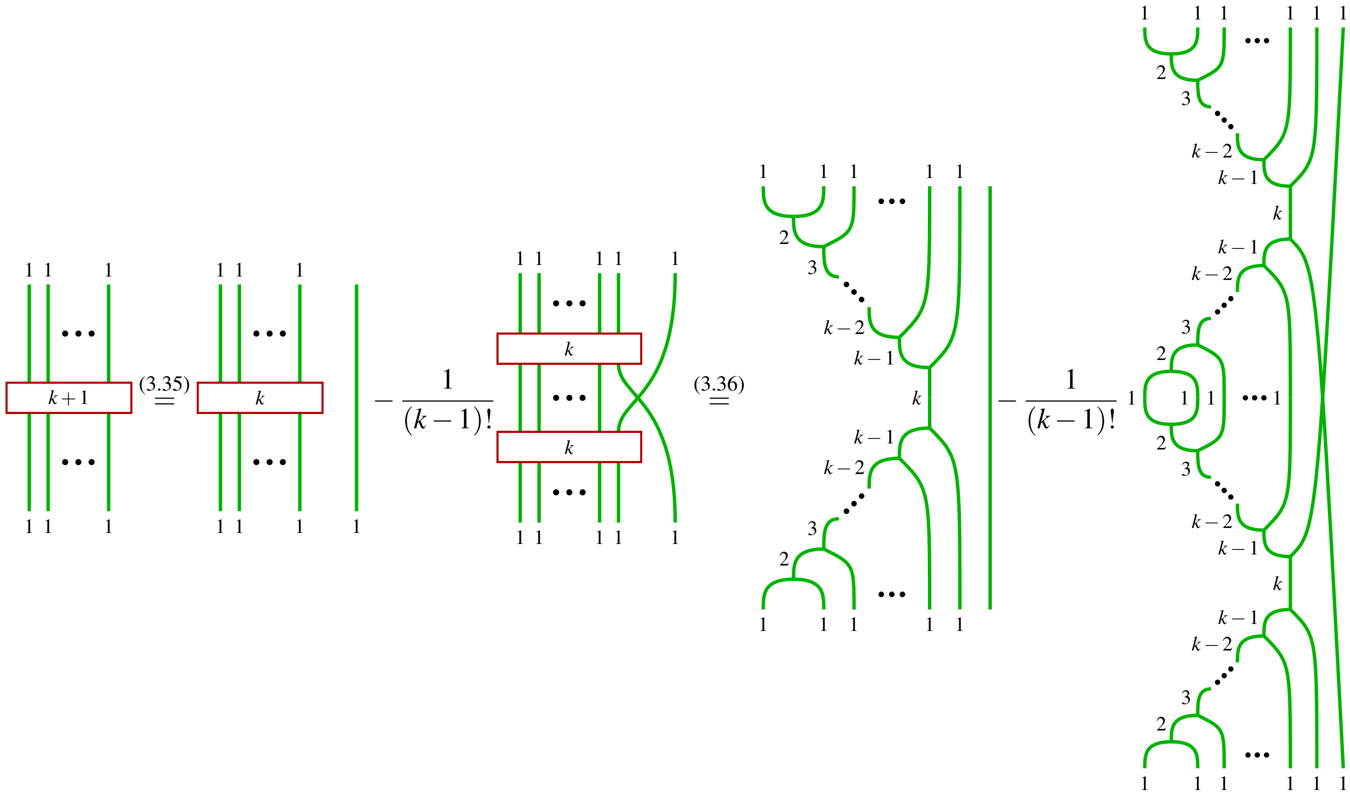
Next, we perform the calculation in $\mathbf{Web}_{\mathbb{C}}(\mathrm{O(m)})$ required to deduce $\eta_{\mathbb{C}}(a_{m+1}) = 0$.

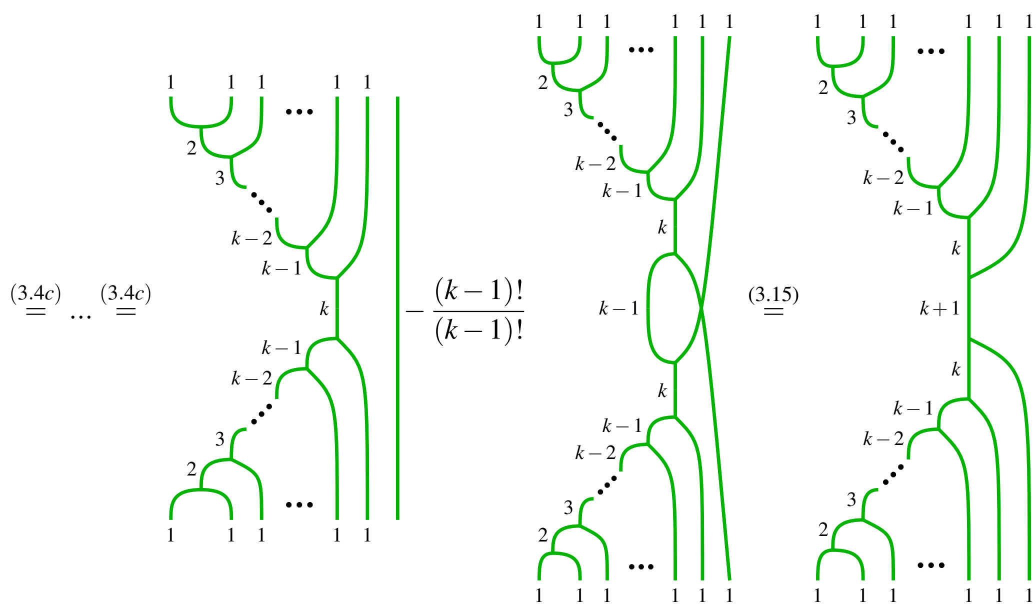
PROPOSITION 3.5.12. *Let $k \in \mathbb{Z}_{\geq 0}$. The following equality holds in* $\boldsymbol{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})$.

(3.36)



PROOF. We use proof by induction. The base case, $k = 1$, is trivial. Suppose (3.36) holds for $k \in \mathbb{Z}_{\geq 1}$. The following graphical calculation proves that Equation (3.36) holds for $k+1$.





$\square$

COROLLARY 3.5.2. *The functor*

$$\Phi_{\mathbb{C}}|_{\textbf{\textit{StdWeb}}_{\mathbb{C}}(\mathrm{O(m)})} : \textbf{\textit{StdWeb}}_{\mathbb{C}}(\mathrm{O(m)}) \longrightarrow \textbf{\textit{StdFund}}(\mathrm{U}_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}))$$

is faithful.

PROOF. Since strands labelled by $m+1$ are equal to zero in $\mathbf{StdWeb}_{\mathbb{C}}(\mathrm{O(m)})$, Proposition 3.5.12 implies that $\eta_{\mathbb{C}}(a_{m+1}) = 0$. We then deduce from Lemma 3.5.47 that $\ker \Phi_{\mathbb{C}} = \langle 0 \rangle$, i.e. $\Phi_{\mathbb{C}}$ is faithful. $\square$

Before we prove that $\Phi_{\mathbb{F}}|_{\mathbf{StdWeb}_{\mathbb{F}}(\mathrm{O(m)})}$ is faithful, we state two technical lemmas.

LEMMA 3.5.49. *Let W, F be $\mathbb{A}$-modules and assume that W is finitely generated over $\mathbb{A}$. Let $f : W \longrightarrow F$ be an $\mathbb{A}$-module homomorphism. Suppose that $\boldsymbol{R} \otimes f : \boldsymbol{R} \otimes W \longrightarrow \boldsymbol{R} \otimes F$ is surjective for $\boldsymbol{R} \in \{\mathbb{F}, \mathbb{C}\}$, and that $\dim_{\mathbb{C}}(\mathbb{C} \otimes F) = \dim_{\mathbb{F}}(\mathbb{F} \otimes F)$. If $\mathbb{C} \otimes f$ is injective, then $\mathbb{F} \otimes f$ is injective.*

PROOF. Since $\mathbb{A}$ is a principal ideal domain and W is finitely generated, it follows that $W \cong \mathbb{A}^{\dim_{\mathbb{F}}(\mathbb{F} \otimes W)} \oplus T$ where $\mathbb{F} \otimes T = 0$. Since $\mathbb{A}$ is also a local ring with residue field $\mathbb{C}$, it follows that there are $r_1, \ldots, r_d \in \mathbb{Z}_{\geq 1}$ such that $T \cong \oplus_{i=1}^{d} (\mathbb{A}/M^{r_i})$ and $d = \dim_{\mathbb{C}}(\mathbb{C} \otimes W) - \dim_{\mathbb{F}}(\mathbb{F} \otimes W)$. In particular,

$$\dim_{\mathbb{C}}(\mathbb{C} \otimes W) - \dim_{\mathbb{F}}(\mathbb{F} \otimes W) \geq 0.$$

Suppose $\mathbb{C} \otimes f$ is injective. Then $\mathbb{C} \otimes f$ is an isomorphism, so

$$\dim_{\mathbb{C}}(\mathbb{C} \otimes W) = \dim_{\mathbb{C}}(\mathbb{C} \otimes F),$$

and $\mathbb{F} \otimes f$ is surjective, so

$$\dim_{\mathbb{F}}(\mathbb{F} \otimes W) \geq \dim_{\mathbb{F}}(\mathbb{F} \otimes F).$$

We further assumed that $\dim_{\mathbb{C}}(\mathbb{C} \otimes F) = \dim_{\mathbb{F}}(\mathbb{F} \otimes F)$. It follows that

$$\dim_{\mathbb{C}}(\mathbb{C} \otimes W) - \dim_{\mathbb{F}}(\mathbb{F} \otimes W) \leq \dim_{\mathbb{C}}(\mathbb{C} \otimes F) - \dim_{\mathbb{F}}(\mathbb{F} \otimes F) = 0.$$

Hence,

$$\dim_{\mathbb{F}}(\mathbb{F} \otimes W) = \dim_{\mathbb{C}}(\mathbb{C} \otimes W) = \dim_{\mathbb{C}}(\mathbb{C} \otimes F) = \dim_{\mathbb{F}}(\mathbb{F} \otimes F).$$

Surjectivity of $\mathbb{F} \otimes f$ implies that $\mathbb{F} \otimes f$ is an isomorphism. $\square$

LEMMA 3.5.50. *Let W, F be $\mathbb{A}$-modules and assume that W is finitely generated over $\mathbb{A}$ and F is free and finitely generated over $\mathbb{A}$. Let $f : W \to F$ be an $\mathbb{A}$-module homomorphism. Assume further that for $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$, there are vector spaces $F_{\boldsymbol{R}}$ and linear maps $b_{\boldsymbol{R}} : \boldsymbol{R} \otimes F \to F_{\boldsymbol{R}}$, such that $b_{\mathbb{F}}$ is injective. Suppose that $b_{\boldsymbol{R}} \circ (\boldsymbol{R} \otimes f) : \boldsymbol{R} \otimes W \to F_{\boldsymbol{R}}$ is surjective for $\boldsymbol{R} \in \{\mathbb{C}, \mathbb{F}\}$, and that $\dim_{\mathbb{C}} F_{\mathbb{C}} = \dim_{\mathbb{F}} F_{\mathbb{F}}$. If $b_{\mathbb{C}} \circ (\mathbb{C} \otimes f)$ is injective, then $b_{\mathbb{F}} \circ (\mathbb{F} \otimes f)$ is injective.*

PROOF. Let $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$. Since $b_{\mathbf{R}} \circ (\mathbf{R} \otimes f)$ is surjective, it follows that $b_{\mathbf{R}}$ is surjective so $\dim_{\mathbf{R}} F_{\mathbf{R}} \leq \dim_{\mathbf{R}}(\mathbf{R} \otimes F)$. Since $b_{\mathbb{F}}$ is injective, it follows that $\dim_{\mathbb{F}}(\mathbb{F} \otimes F) \leq \dim_{\mathbb{F}} F_{\mathbb{F}}$. Thus, $b_{\mathbb{F}}$ is an isomorphism and $\dim_{\mathbb{F}}(\mathbb{F} \otimes F) = \dim_{\mathbb{F}} F_{\mathbb{F}}$.

Using that F is free over $\mathbb{A}$, we find

$$\dim_{\mathbb{C}}(\mathbb{C} \otimes F) = \mathrm{rk}_{\mathbb{A}} F = \dim_{\mathbb{F}}(\mathbb{F} \otimes F) = \dim_{\mathbb{F}} F_{\mathbb{F}} = \dim_{\mathbb{C}} F_{\mathbb{C}}.$$

Thus, surjectivity of $b_{\mathbb{C}}$ implies $b_{\mathbb{C}}$ is an isomorphism.

Suppose $b_{\mathbb{C}} \circ (\mathbb{C} \otimes f)$ is injective. Since $b_{\mathbf{R}}$ is an isomorphism for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$, the claim follows from Lemma 3.5.49. $\square$

THEOREM 3.5.16. *The functor*

$$\Phi_{\mathbb{F}}|_{\boldsymbol{StdWeb}_{\mathbb{F}}(\mathrm{O(m)})} : \boldsymbol{StdWeb}_{\mathbb{F}}(\mathrm{O(m)}) \longrightarrow \boldsymbol{StdFund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))$$

is faithful.

PROOF. For $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$ and $d, e \in \mathbb{Z}_{\geq 0}$ we have induced maps

$$\Phi_{\mathbf{R}} : \mathrm{Hom}_{\mathbf{StdWeb_R}(\mathrm{O(m)})}(1^{\otimes d}, 1^{\otimes e}) \to \mathrm{Hom}_{\mathbf{StdFund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))}(V_{\mathbf{R}}^{\otimes d}, V_{\mathbf{R}}^{\otimes e}).$$

It suffices to show that $\Phi_{\mathbb{F}}$ is injective for all $d, e \in \mathbb{Z}_{\geq 0}$. Recall from Theorem 3.5.12 that $\Phi_{\mathbf{R}} = \mathfrak{B}_{\mathbf{R}} \circ (\mathbf{R} \otimes \Phi_{\mathbb{A}})$. So we will use Lemma 3.5.50 when $W = \mathrm{Hom}_{\mathbf{StdWeb}_{\mathbb{A}}(\mathrm{O(m)})}(1^{\otimes d}, 1^{\otimes e})$, $F = \mathrm{Hom}_{\mathbf{StdFund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))}(V_{\mathbb{A}}^{\otimes d}, V_{\mathbb{A}}^{\otimes e})$, $f = \Phi_{\mathbb{A}}$, $F_{\mathbf{R}} = \mathrm{Hom}_{\mathbf{StdFund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))}(V_{\mathbf{R}}^{\otimes d}, V_{\mathbf{R}}^{\otimes e})$, and $b_{\mathbf{R}} = \mathfrak{B}_{\mathbf{R}}$.

Proposition 3.5.5 says that $\mathrm{Hom}_{\mathbf{StdWeb}_{\mathbb{A}}(\mathrm{O(m)})}(1^{\otimes d}, 1^{\otimes e})$ is a finitely generated $\mathbb{A}$-module, and Lemma 3.5.28 implies that $\mathrm{Hom}_{\mathbf{StdFund}(\mathrm{U}_{\mathbb{A}}(\mathfrak{o}_{\mathrm{m}}))}(V_{\mathbb{A}}^{\otimes d}, V_{\mathbb{A}}^{\otimes e})$ is a free and finitely generated $\mathbb{A}$-module. Lemma 3.5.29 implies that $\mathfrak{B}_{\mathbb{F}}$ is injective.

Theorem 3.5.13 implies that $\Phi_{\mathbf{R}} = \mathfrak{B}_{\mathbf{R}} \circ (\mathbf{R} \otimes \Phi_{\mathbb{A}})$ is surjective for $\mathbf{R} \in \{\mathbb{C}, \mathbb{F}\}$. Proposition 3.5.9 says that $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbf{StdFund}(\mathrm{U}_{\mathbb{F}}(\mathfrak{o}_{\mathrm{m}}))}(V_{\mathbb{F}}^{\otimes d}, V_{\mathbb{F}}^{\otimes e}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{StdFund}(\mathrm{U}_{\mathbb{C}}(\mathfrak{o}_{\mathrm{m}}))}(V_{\mathbb{C}}^{\otimes d}, V_{\mathbb{C}}^{\otimes e})$.

By Corollary 3.5.2 we know that $\Phi_{\mathbb{C}}$ is injective. Thus, Lemma 3.5.50 implies that $\Phi_{\mathbb{F}}$ is injective. □

3.5.5.4. *Main theorem.* We now prove Theorem 3.5.2

PROOF. Thanks to Lemma 3.5.45, it follows from Theorem 3.5.13, Corollary 3.5.2, and Theorem 3.5.16 that $\Phi_{\mathbf{R}}$ is full and faithful. Since the objects of $\mathbf{Fund}(\mathrm{U}_{\mathbf{R}}(\mathfrak{o}_{\mathrm{m}}))$ are tensor products of $\Lambda_{\mathbf{R}}^{k}$, for $k \in \{0, 1, \ldots, m\}$, and $\Phi_{\mathbf{R}}(k) = \Lambda_{\mathbf{R}}^{k}$, it follows that $\Phi_{\mathbf{R}}$ is essentially surjective. Hence, $\Phi_{\mathbf{R}}$ is an equivalence of $\mathbf{R}$-linear pivotal categories. □

REMARK 3.5.31. *Since* $\boldsymbol{Fund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$ *is a ribbon category, with braiding* $\beta_{\mathrm{V}_{\boldsymbol{R}}, \mathrm{V}_{\boldsymbol{R}}}$, *we can use the equivalence* $\Phi_{\boldsymbol{R}}$ *to define a braiding on* $\boldsymbol{Web_R}(\mathrm{O(m)})$. *We know that* $\Phi_{\boldsymbol{R}}({}^{\boldsymbol{R}}\beta_{1,1}) = \beta_{\mathrm{V}_{\boldsymbol{R}}, \mathrm{V}_{\boldsymbol{R}}}$ *and* $\Phi_{\boldsymbol{R}}({}^{\boldsymbol{R}}\beta_{1,1}^{-1}) = \beta^{-1}{}_{\mathrm{V}_{\boldsymbol{R}}, \mathrm{V}_{\boldsymbol{R}}}$. *Naturality of the braiding on* $\boldsymbol{Fund}(\mathrm{U}_{\boldsymbol{R}}(\mathfrak{o}_{\mathrm{m}}))$ *allows us to define a braiding on* $\boldsymbol{Web_R}(\mathrm{O(m)})$, *as in [**8**, Section 5.9]. The functor* $\Phi_{\boldsymbol{R}}$ *can then be treated as an equivalence of braided pivotal (in fact ribbon) categories.*

CHAPTER 4

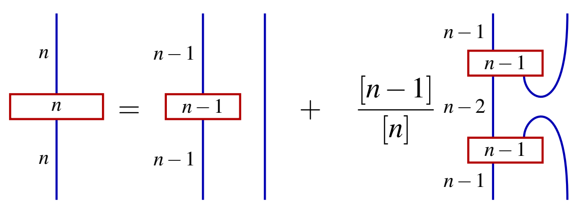
Clasps

4.1. Introduction and history of highest weight projectors

4.1.1. History of Clasp Formulas. Let $V \in \mathbf{Rep}(U_q(\mathfrak{sl}_2))$ denote the q-analogue of the vector representation of $SL_2(\mathbb{C})$. For each $n \in \mathbb{Z}_{\geq 0}$, there is an irreducible representation $V(n)$, which is a direct summand of $V^{\otimes n}$ and which is not a direct summand of $V^{\otimes m}$ for $m < n$. Note that $V(1) \cong V$. So for each n, there is an idempotent in the Temperley-Lieb category which can be viewed as the idempotent in $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(V^{\otimes n})$ with image $V(n)$. The condition that $V(n)$ is not a summand of $V^{\otimes m}$ for $m < n$ implies that composing a projector with any cap diagram will result in zero.

These idempotents are usually called Jones-Wenzl projectors, as they were first considered by Jones [**24**, Section 4.2], and the following explicit inductive formula was first given by Wenzl [**63**].

(4.1)



Here we use the notation $[m]$ to denote the quantum integer $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$, for each $m \in \mathbb{Z}$. Our convention is that a red box with label n is the morphism in the Temperley-Lieb category which corresponds to the idempotent with image $V(n)$.

The Jones-Wenzl projectors and the recursive formula in Equation (4.1) describing them have been proven useful in link homology [**13**], Soergel bimodules [**16**], and the theory of subfactors and planar algebras [**48**]. The present work is concerned with generalizing Equation (4.1) from $\mathfrak{sl}_2$ to the Lie algebra $\mathfrak{g}_2$. However, many things we say in the introduction make sense for all semisimple Lie algebras.

Fix a finite dimensional semisimple Lie algebra $\mathfrak{g}$. There is an associated quantum enveloping algebra $U_q(\mathfrak{g})$, which is a $\mathbb{C}(q)$ algebra defined by generators and relations which "quantize" the Serre presentation of the usual enveloping algebra [**23**, Chapter 4]. The finite dimensional irreducible type-**1** representations[1] are in bijection with the finite dimensional irreducible representations of $\mathfrak{g}$, i.e. for each dominant integral weight λ there is a finite dimensional irreducible module of $U_q(\mathfrak{g})$, which we denote by $V(\lambda)$. We will abuse notation and write $\mathbf{Rep}(U_q(\mathfrak{g}))$ to refer to the category of finite dimensional type-**1** representations of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra [**23**, Section 4.8], and it turns out that $\mathbf{Rep}(U_q(\mathfrak{g}))$ is closed under taking tensor products. Furthermore, since we are working over $\mathbb{C}(q)$, where q is an indeterminant or a generic element of $\mathbb{C}$, the category $\mathbf{Rep}(U_q(\mathfrak{g}))$ is a semisimple tensor category, and the Grothendieck ring of $\mathbf{Rep}(U_q(\mathfrak{g}))$ is isomorphic to the Grothendieck ring of the category of finite dimensional representations of $\mathfrak{g}$.

The recursion in Equation (4.1) is expressed in terms of the Temperley-Lieb category, which describes the full monoidal subcategory of $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$ generated by V. The generalization of this subcategory to arbitrary quantum groups is $\mathbf{Fund}(\mathfrak{g})$ in Definition 3.1.2.

We denote the set of fundamental weights of $\mathfrak{g}$ by $\{\varpi_i\}$. Let λ be a dominant integral weight. Then we can write $\lambda = \sum n_i \varpi_i$, where $n_i \in \mathbb{Z}_{\geq 0}$. There is a partial order on all weights, where $\mu \leq \lambda$ when $\lambda - \mu$ is a $\mathbb{Z}_{\geq 0}$-linear combination of positive roots. With respect to this partial order, the irreducible representation $V(\lambda)$ has highest weight λ. Also, $V(\lambda)$ is a direct summand of the tensor product $\bigotimes_i V(\varpi_i)^{\otimes n_i}$. Thus, there are projection and inclusion maps $\bigotimes_i V(\varpi_i)^{\otimes n_i} \to V(\lambda) \to \bigotimes_i V(\varpi_i)^{\otimes n_i}$ such that the composition is an idempotent C_λ in $\mathbf{Fund}(U_q(\mathfrak{g}))$. We are interested in finding explicit descriptions of these idempotents, generalizing Formula (4.1).

Unless λ is a fundamental weight or zero, $V(\lambda)$ will not be an object in $\mathbf{Fund}(U_q(\mathfrak{g}))$. However, C_λ is a morphism in $\mathbf{Fund}(U_q(\mathfrak{g}))$ and we think of it as a replacement for $V(\lambda)$. Analogous to how $V(\lambda)$ is characterized as the irreducible representation with highest weight λ, the morphism C_λ is characterized as the non-zero idempotent endomorphism of $\bigotimes_i V(\varpi_i)^{\otimes n_i}$ such that if $f : \bigotimes_i V(\varpi_i)^{\otimes n_i} \to V(\varpi_{i_1}) \otimes \cdots \otimes V(\varpi_{i_r})$ is a morphism in $\mathbf{Fund}(U_q(\mathfrak{g}))$, and $\sum_{k=1}^{r} \varpi_{i_k} < \lambda$, then $f \circ C_\lambda = 0$.

Kuperberg [**36**] introduced the terminology *clasp* to refer to an idempotent projecting to the highest weight irreducible summand of a tensor product of fundamental representations, viewed as a morphism in $\mathbf{Web}_q(\mathfrak{g})$. If the highest weight of this irreducible summand is λ, then we will call this idempotent a λ-clasp.

[1]This means that for all simple roots α, the element K_α acts on the μ weight space of any representation by $+q^{(\alpha,\mu)}$ [**23**, Section 5.2]. We will only consider type-**1** representations in this paper.

To generalize the Jones-Wenzl recursion, a first step is to find recursive formulas of clasps in the rank two cases.

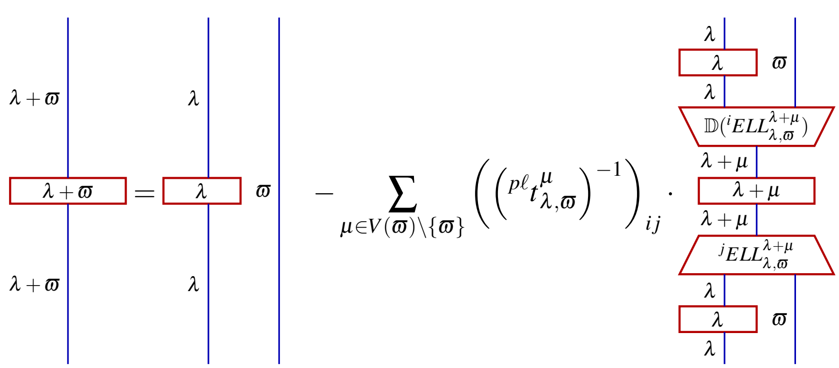
In the $\mathfrak{sl}_3$ case, a recursive formula was given by Ohtsuki and Yamada [**46**, Definition 2.4], where they called a clasp a “magic element”. Later, Dongseok Kim found other recursive formulas for the $\mathfrak{sl}_3$ case as well [**32**, Theorem 3.3].

In [**17**, Conjecture 3.16], Elias made his type A clasp conjecture, which implies a recursive description of each $\mathfrak{sl}_n$ clasp using the language of $\mathfrak{sl}_n$ webs. Also, [**17**, Theorem 2.57] provides a basis for all homomorphism spaces between fundamental representations for $\mathfrak{sl}_n$. These bases have a particularly nice form which reduces the validity of the type A clasp conjecture to an explicit calculation, which is hard to carry out for an arbitrarily large n. In [**43**], Martin and Spencer proved the type A clasp conjecture with cell modules.

Since $\mathfrak{sp}_4$ is rank two, there are two simple roots: one short α_1 and one long α_2. We write ϖ_1 and ϖ_2 for the corresponding fundamental weights. In the $\mathfrak{sp}_4$ case, Kim gave recursive clasp formulas for the $a\varpi_1$-clasp [**32**, Corollary 4.3] and the $b\varpi_2$-clasp [**32**, Corollary 4.5]. However, an inductive formula for the $\mathfrak{sp}_4$ $a\varpi_1 + b\varpi_2$-clasp remained unknown until recently, when Bodish derived formulas generalizing Elias’s type A clasp conjecture to type C_2 [**6**, Theorem 1.5].

In the $\mathfrak{g}_2$ case, little was known before the present work. Attempts at getting the $\mathfrak{g}_2$-clasp formulas have been made, including a few base-case calculations by Sakamoto and Yonezawa [**57**, Section 5]. In this paper, we give triple clasp expansions for the $\mathfrak{g}_2$ λ-clasps for all dominant integral weights λ. Our results are summarized in the following theorem.

THEOREM 4.1.1. *Let λ be a dominant integral weight for $\mathfrak{g}_2$. Then the $\lambda + \varpi$ clasp is given by the following recursive formula.*



Here a red box labelled by χ denotes the χ-clasp, i.e. the morphism in $\boldsymbol{Web}_q(\mathfrak{g}_2)$ which corresponds to the idempotent with image $V(\chi)$. The diagrams $\mathbb{D}({}^iELL_{\lambda,\varpi}^{\lambda+\mu})$ and ${}^jELL_{\lambda,\varpi}^{\lambda+\mu}$ are given explicitly in Formula (4.6) *and Formula* (4.7) *in Section 4.3.1. The coefficients ${}^{p\ell}t_{\lambda,\varpi}^{\mu}$ are given explicitly by Equation* (4.8) *to Equation* (4.29) *in Section 4.3.1 and Equation* (A.1) *to Equation* (A.3) *in Appendix A.1.*

4.1.2. Connection to the Clasp Conjecture. Let $\mathbb{F}$ be a field. Consider objects X and S in an additive $\mathbb{F}$-linear Karoubian category with duality $\mathbb{D}$, i.e. a contravariant endofunctor with $\mathbb{D}^2 \cong \mathrm{id}$, such that $\mathrm{End}(S) = \mathbb{F} \cdot \mathrm{id}_S$, $\mathbb{D}(X) = X$, and $\mathbb{D}(S) = S$. Given $\pi : X \to S$, we obtain a map $\iota = \mathbb{D}(\pi) : S \to X$ and

$$\pi \circ \iota = \kappa \, \mathrm{id}_S,$$

for some $\kappa \in \mathbb{F}$. If $\kappa \neq 0$, then S is isomorphic to the image of the idempotent

$$e = \frac{1}{\kappa} \iota \circ \pi.$$

In [**17**, Definition 3.8], the coefficient κ, computed in the $\mathfrak{sl}_n$ web category, is called a local intersection form. We carry out analogous calculations in the $\mathfrak{g}_2$ web category and find an analogue of Elias's clasp conjecture holds for G_2.

In fact, we expect that something in general will hold. Let $\mathfrak{g}$ be a simple Lie algebra and let $U_q(\mathfrak{g})$ be the associated quantum group. Let W denote the Weyl group associated to $\mathfrak{g}$. For $V \in \mathbf{Rep}(U_q(\mathfrak{g}))$ we will write $\mathrm{wt}V$ to denote the set of all weights μ such that the μ weight space of V is non-zero. Fix a fundamental weight ϖ. For each $\mu \in \mathrm{wt}V(\varpi)$, such that μ is in the same W orbit as ϖ, there should be a clasped *elementary light ladder map*[2]

$$LL_{\lambda,\varpi}^{\lambda+\mu} : V(\lambda) \otimes V(\varpi) \to V(\lambda+\mu).$$

DEFINITION 4.1.1. *For an extremal weight μ in a fundamental representation (i.e. a weight in the W orbit of ϖ_i for some i) we write d_μ to denote the minimal length element $w \in W$ so that $w(\mu)$ is dominant. We also define Φ_μ to be the set of positive roots which are sent to negative roots by d_μ.*

[2]In examples, this is most easily defined using web categories. But with some care should make sense in general, even without having a generators and relations presentation of $\mathbf{Fund}(\mathfrak{g})$. The main feature should be that the map is the composition of projectors and neutral maps with some fixed map from $V(\lambda_{min}) \otimes V(\varpi) \to V(\lambda_{min}+\mu)$, where λ_{min} is the smallest dominant weight so that $V(\lambda) \otimes V(\varpi)$ contains a copy of $V(\lambda+\mu)$.

CONJECTURE 4.1.1. *If we denote by $\mathbb{D}$ the duality*[3] *on* $\mathbf{Fund}(\mathfrak{g})$*, and write*

$$LL^{\lambda+\mu}_{\lambda,\varpi} \circ \mathbb{D}(LL^{\lambda+\mu}_{\lambda,\varpi}) = \kappa^{\mu}_{\lambda,\varpi}\,\mathrm{id}_{V(\lambda+\mu)},$$

then

$$\kappa^{\mu}_{\lambda,\varpi} = \pm \prod_{\alpha\in\Phi_\mu} \frac{[(\alpha^\vee,\lambda+\rho)]_{q^{\ell(\alpha)}}}{[(\alpha^\vee,\lambda+\mu+\rho)]_{q^{\ell(\alpha)}}}. \tag{4.2}$$

Here ρ is the sum of the fundamental weights and $l(\alpha) = (\alpha,\alpha)/2$.

REMARK 4.1.1. *The conjecture is proven to be true in types A_n [**43**] and in type C_2 [**6**].*

The following Proposition is an elementary consequence of our main theorem.

PROPOSITION 4.1.1. *The conjecture is true for $\mathfrak{g}_2$.*

PROOF. See Corollary 4.3.1. □

REMARK 4.1.2. *We also expect there to be a more general form of the conjecture which describes what happens for $\mu \in V(\varpi)$ which are not in the extremal Weyl orbit. The work in this paper and [**6**] could give enough data to guess the answer when $V(\varpi)_\mu$ is one dimensional, but we have not yet carried this out. We also hope the general form of the conjecture will give rise to a product formula which computes the elementary divisors of the matrix of local intersection forms when* $\dim V(\varpi)_\mu > 1$.

4.1.3. Witten-Reshetikhin-Turaev invariants via skein-theory. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. In order to define an analogue of the Jones polynomial for $\mathfrak{g}$, Reshetikhin and Turaev defined a link invariant using the category $\mathbf{Rep}(U_q(\mathfrak{g}))$ [**53**]. Their construction gives a knot invariant for every type-**1** representation of $U_q(\mathfrak{g})$. More generally, one can label each component of a link with an object in $\mathbf{Rep}(U_q(\mathfrak{g}))$ and their construction gives the colored quantum link invariant.

Kuperberg's original motivation for studying $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ was to compute the quantum link invariant associated to $\mathfrak{g}_2$. Originally, he gave a diagrammatic method to compute the $\mathfrak{g}_2$ link invariant when each component is colored by the first fundamental representation [**35**]. Soon after, using $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, he gave diagrammatic tools for computing the $\mathfrak{g}_2$ link invariant colored by both fundamental representations [**36**,

[3]Again, this is most easily defined in terms of webs, in which case it is just flipping the diagram upside down.

Section 4]. In this paper, we give explicit formulas for idempotents projecting to each irreducible $U_q(\mathfrak{g}_2)$ module. Combined with Kuperberg's earlier work this gives a diagrammatic approach to computing the quantum invariant of a link with components colored by any irreducible.

Reshetikhin-Turaev's paper about their link invariant was intended as a prequel to their work which gave an associated 3-manifold invariant [**52**]. The first step one takes to make sense of their 3-manifold invariant is to leave behind representation theory of $U_q(\mathfrak{g})$ for generic q and work instead with q specialized to a root of unity.

Let $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}_2)$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathrm{U_q}(\mathfrak{g}_2)$ generated by $\frac{E_\alpha^{(a)}}{[a]_{q^{\ell(\alpha)}}!}$, $\frac{F_\alpha^{(a)}}{[a]_{q^{\ell(\alpha)}}!}$, and $K_\alpha^{\pm 1}$, for all simple roots α, and all $a \in \mathbb{Z}_{\geq 0}$. When ξ is a root of unity in $\mathbb{C}$, we can study the relation between $\mathbb{C} \otimes_{q=\xi} \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, and the category of tilting modules of $\mathbb{C} \otimes_{q=\xi} U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}_2)$. It is possible to adapt the approach from [**7**], which itself is based on [**17**], to prove that the Karoubi envelope of $\mathbb{C} \otimes_{q=\xi} \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ is equivalent to the category of tilting modules as long as $[2]_\xi, [3]_\xi \neq 0$. The same result is work in progress of Victor Ostrik and Noah Snyder, but they propose a slightly different approach.

When ξ is a root of unity of order greater than 5, the generators of the negligible ideal in the category of $U_\xi(\mathfrak{g}_2)$ tilting modules are (identity morphisms of) certain irreducible tilting modules with quantum dimension zero. Irreducible tilting modules are also Weyl modules, so these generating objects correspond to clasps in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. Moreover, the objects which survive in the negligible quotient are the irreducible Weyl modules with non-zero quantum dimension. Once the equivalence between the web category and the category of tilting modules is established, one can give a generators and relations presentation of the associated modular tensor category[4] using the $\mathfrak{g}_2$ triple clasp formulas for the negligible clasps. Combined with our description of the clasps corresponding to the irreducible Weyl modules with non-zero dimension, this gives an explicit way to compute the quantum $\mathfrak{g}_2$ 3-manifold invariant [**52**].

On the other hand, topologists have tried to understand the quantum 3-manifold invariants with graphical categories. A construction of the quantum $\mathfrak{sl}_2$ 3-manifold invariant using the Temperley-Lieb category was given by Lickorish [**41**]. This work was generalized to the $\mathfrak{sl}_3$ case by Ohtsuki and Yamada [**46**] with $\mathfrak{sl}_3$ webs. A self-contained proof of invariance under Kirby moves [**33**] using the graphical category was given in both cases. One can now give similar constructions and proofs in the $\mathfrak{g}_2$ case by using our clasp formulas.

[4]In the case that ξ is a root of unity that actually gives rise to a modular category as the negligible quotient [**55**].

4.2. Clasps and light ladders for G_2

4.2.1. Definition of clasps.

LEMMA 4.2.1. *Let $D \in \mathrm{End}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}})$ such that $\Phi(D)$ acts as zero on $V(\mathrm{wt}\,\underline{\mathrm{w}}) \overset{\oplus}{\subset} V(\underline{\mathrm{w}})$. Then we can write D as linear combination*

$$D = \sum_i A_i \circ B_i,$$

where $B_i \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}}_i)$ and $A_i \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{u}}_i, \underline{\mathrm{w}})$ for some $\underline{\mathrm{u}}_i$ with $\mathrm{wt}\,\underline{\mathrm{u}}_i < \mathrm{wt}\,\underline{\mathrm{w}}$.

PROOF. Since

$$V(\underline{\mathrm{w}}) = V(\mathrm{wt}\,\underline{\mathrm{w}}) \bigoplus_{\mu < \mathrm{wt}\,\underline{\mathrm{w}}} V(\mu)^{\oplus m_\mu^{\mathrm{w}}},$$

our hypothesis on $\Phi(D)$ implies that we can write $\Phi(D) = \sum_i \iota_i \circ \pi_i$, where for each i there is some $\mu_i < \mathrm{wt}\,\underline{\mathrm{w}}$ such that π_i is a projection $V(\underline{\mathrm{w}}) \to V(\mu_i)$ and ι_i is an inclusion $V(\mu_i) \to V(\underline{\mathrm{w}})$.

For each μ_i fix an object $\underline{\mathrm{u}}_i$ in $\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ with $\mathrm{wt}\,\underline{\mathrm{u}}_i = \mu_i$. There is a projection $\gamma_i : V(\underline{\mathrm{u}}_i) \to V(\mu_i)$ and inclusion $\gamma^i : V(\mu_i) \to V(\underline{\mathrm{u}}_i)$ so that $\gamma_i \circ \gamma^i = \mathrm{id}_{V(\mu_i)}$. Then we can write

$$\iota_i \circ \pi_i = \iota_i \circ \mathrm{id}_{V(\mu_i)} \circ \pi_i = \iota_i \circ \gamma_i \circ \gamma^i \circ \pi_i.$$

Thus, $\iota_i \circ \gamma_i \in \mathrm{Hom}_{\mathrm{U_q}(\mathfrak{g}_2)}(V(\underline{\mathrm{u}}_i), V(\underline{\mathrm{w}}))$ and $\gamma^i \circ \pi_i \in \mathrm{Hom}_{\mathrm{U_q}(\mathfrak{g}_2)}(V(\underline{\mathrm{w}}), V(\underline{\mathrm{u}}_i))$. The desired result now follows from Φ being an equivalence. □

DEFINITION 4.2.1. *The **neutral coefficient** of a diagram $D \in \mathrm{End}(\underline{\mathrm{w}})$ is the scalar by which $\Phi(D)$ acts on the one dimensional weight space $V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}$. We write $\Phi(D)|_{V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}} = N_D \cdot \mathrm{id}$.*

LEMMA 4.2.2. *Let $D \in \mathrm{End}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}})$. Then we can express D as a linear combination of diagrams*

$$D = N_D \cdot \mathrm{id}_{\underline{\mathrm{w}}} + \sum_i A_i \circ B_i,$$

where $B_i \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}}_i)$ and $A_i \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{u}}_i, \underline{\mathrm{w}})$ for some $\underline{\mathrm{u}}_i$ with $\mathrm{wt}\,\underline{\mathrm{u}}_i < \mathrm{wt}\,\underline{\mathrm{w}}$.

PROOF. Consider $\Phi(D) - N_D\,\mathrm{id}_{V(\underline{\mathrm{w}})} \in \mathrm{End}(V(\underline{\mathrm{w}}))$. This endomorphism has $V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}$ in its kernel and therefore also acts as zero on $V(\mathrm{wt}\,\underline{\mathrm{w}}) \overset{\oplus}{\subset} V(\underline{\mathrm{w}})$. The desired result now follows from Lemma 4.2.1. □

DEFINITION 4.2.2. *Let* $\underline{\mathrm{w}} \in \boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$*. A diagrammatic* $\underline{\mathrm{w}}$*-**clasp** is a morphism* $C_{\underline{\mathrm{w}}} \in \mathrm{End}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}})$ *which satisfies the following conditions:*

(1) $C_{\underline{\mathrm{w}}} \neq 0$*,*

(2) $C_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}}$*, and*

(3) *If* $D \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}})$ *and* $\mathrm{wt}\,\underline{\mathrm{u}} < \mathrm{wt}\,\underline{\mathrm{w}}$*, then* $D \circ C_{\underline{\mathrm{w}}} = 0$*.*

REMARK 4.2.1. *Note that we only use the terminology clasp to refer to idempotents. This is consistent with Kuperberg's original use of the term [**36**], but less general than Elias's [**17**, Definition 1.12]. In Section 4.2.2, we will define* generalized clasps*, which will agree with Elias's notion of clasp.*

LEMMA 4.2.3. *If the* $\underline{\mathrm{w}}$ *clasp exists, then it is unique and* $N_{C_{\underline{\mathrm{w}}}} = 1$*.*

PROOF. Suppose $C_{\underline{\mathrm{w}}}$ and $C'_{\underline{\mathrm{w}}}$ are both $\underline{\mathrm{w}}$-clasps. By Lemma 4.2.2 we can write $C_{\underline{\mathrm{w}}} = N_{C_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A_i \circ B_i$ and $C'_{\underline{\mathrm{w}}} = N_{C'_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A'_i \circ B'_i$. As a consequence of the definition of clasps, we find

$$C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} = \Big(N_{C_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A_i \circ B_i\Big) \circ C_{\underline{\mathrm{w}}} = N_{C_{\underline{\mathrm{w}}}} C_{\underline{\mathrm{w}}}$$

and

$$C'_{\underline{\mathrm{w}}} = C'_{\underline{\mathrm{w}}} \circ C'_{\underline{\mathrm{w}}} = \Big(N_{C'_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A'_i \circ B'_i\Big) \circ C'_{\underline{\mathrm{w}}} = N_{C'_{\underline{\mathrm{w}}}} C'_{\underline{\mathrm{w}}}.$$

Since $\underline{\mathrm{w}}$-clasps are non-zero elements of the vector space $\mathrm{End}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}})$, it follows that $N_{C_{\underline{\mathrm{w}}}} = 1 = N_{C'_{\underline{\mathrm{w}}}}$. Thus,

$$\begin{aligned} C'_{\underline{\mathrm{w}}} = N_{C_{\underline{\mathrm{w}}}} C'_{\underline{\mathrm{w}}} = \Big(N_{C_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A_i \circ B_i\Big) \circ C'_{\underline{\mathrm{w}}} &= C_{\underline{\mathrm{w}}} \circ C'_{\underline{\mathrm{w}}} \\ &= C_{\underline{\mathrm{w}}} \circ \Big(N_{C'_{\underline{\mathrm{w}}}}\,\mathrm{id} + \sum_i A'_i \circ B'_i\Big) = N_{C'_{\underline{\mathrm{w}}}} C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}}. \end{aligned}$$

$\square$

DEFINITION 4.2.3. *Let* $\pi_{\underline{\mathrm{w}}} \in \mathrm{End}_{\mathrm{U}_{\mathrm{q}}(\mathfrak{g}_2)}(V(\underline{\mathrm{w}}))$ *be the idempotent endomorphism with image* $V(\mathrm{wt}\,\underline{\mathrm{w}})$*. The endomorphism* $\Phi^{-1}(\pi_{\underline{\mathrm{w}}})$ *in* $\mathrm{End}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}})$ *is the **algebraic*** $\underline{\mathrm{w}}$***-clasp**.

LEMMA 4.2.4. *The algebraic clasp is a clasp, and* $\Phi^{-1}(\pi_{\underline{\mathrm{w}}}) = C_{\underline{\mathrm{w}}}$*.*

PROOF. Since the algebraic clasp is non-zero and idempotent, we just need to argue that the algebraic clasp satisfies the third condition in the definition of clasp. Fix $\underline{\mathrm{u}}$ such that $\mathrm{wt}\,\underline{\mathrm{u}} < \mathrm{wt}\,\underline{\mathrm{w}}$ and let

$D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$. The module $V(\mathrm{wt}\,\underline{w})$ is not isomorphic to any summand of $V(\underline{u})$, so we know that $\mathrm{Hom}_{U_q(\mathfrak{g}_2)}(V(\mathrm{wt}\,\underline{w}), V(\underline{u})) = 0$. Therefore, $\Phi(D) \circ \pi_{\underline{w}} = 0$, and by Theorem 3.2.2 we may conclude that $D \circ \left(\Phi^{-1}(\pi_{\underline{w}})\right) = 0$. □

LEMMA 4.2.5. *Let $\underline{w}$ be an object in $\boldsymbol{Web}_q(\mathfrak{g}_2)$, then there is a unique diagrammatic $\underline{w}$-clasp $C_{\underline{w}}$ and $\Phi(C_{\underline{w}})$ is an idempotent endomorphism of $V(\underline{w})$ projecting to $V(\mathrm{wt}\,\underline{w})$.*

PROOF. From Lemma 4.2.4 we see that clasps exist and map under Φ to the projector for $V(\mathrm{wt}\,\underline{w})$. Uniqueness follows from Lemma 4.2.3. □

LEMMA 4.2.6. *Let $E \in \mathrm{End}_{\boldsymbol{Web}_q(\mathfrak{g}_2)}(\underline{w})$ be a non-zero endomorphism so that $E^2 = E$. If $C_{\underline{u}} \circ D \circ E = 0$ for all $D \in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$ such that $\mathrm{wt}\,\underline{u} < \mathrm{wt}\,\underline{w}$, then $E = C_{\underline{w}}$.*

PROOF. Since we assume E is non-zero and idempotent, we just need to show that E satisfies the third condition in Definition 4.2.2. Fix $\underline{u}$ such that $\mathrm{wt}\,\underline{u} < \mathrm{wt}\,\underline{w}$. Suppose inductively that $B \circ E = 0$ for all $B \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{v})$ where $\mathrm{wt}\,\underline{v} < \mathrm{wt}\,\underline{u}$. By Lemma 4.2.2 we can write $C_{\underline{u}} = \mathrm{id} + \sum_i A_i \circ B_i$ where each A_i has domain $\underline{v}_i$ such that $\mathrm{wt}\,\underline{v}_i < \mathrm{wt}\,\underline{u}$. If $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$, then

$$D \circ E = \mathrm{id} \circ D \circ E = C_{\underline{u}} \circ D \circ E - \sum_i A_i \circ B_i \circ D \circ E = -\sum_i A_i \circ B_i \circ D \circ E. \tag{4.3}$$

Since $B_i \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{u}, \underline{v}_i)$ and $D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{u})$, we have $B_i \circ D \in \mathrm{Hom}_{\mathbf{Web}_q(\mathfrak{g}_2)}(\underline{w}, \underline{v}_i)$. The induction hypothesis applies, so $(B_i \circ D) \circ E = 0$, and Equation 4.3 implies $D \circ E = 0$. □

LEMMA 4.2.7 (Clasp Schur's Lemma). *Let $\underline{u}, \underline{v} \in \boldsymbol{Web}_q(\mathfrak{g}_2)$ and let $D \in \mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{g}_2)}(\underline{u}, \underline{v})$.*

(1) *If $\mathrm{wt}\,\underline{u} \neq \mathrm{wt}\,\underline{v}$, then $C_{\underline{v}} \circ D \circ C_{\underline{u}} = 0$.*
(2) *If $\underline{u} = \underline{v}$, then $C_{\underline{v}} \circ D \circ C_{\underline{u}} = N_D \cdot C_{\underline{u}}$.*

PROOF. By Corollary 3.2.1, we find: $\dim \mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_q(\mathfrak{g}_2)}(C_{\underline{u}}, C_{\underline{v}}) = \delta_{\mathrm{wt}\,\underline{u}, \mathrm{wt}\,\underline{v}}$. Thus, we can deduce the following.

(1) If $\mathrm{wt}\,\underline{u} \neq \mathrm{wt}\,\underline{v}$, then $\mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_q(\mathfrak{g}_2)}(C_{\underline{u}}, C_{\underline{v}}) = 0$.
(2) If $\underline{u} = \underline{v}$, then $\mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_q(\mathfrak{g}_2)}(C_{\underline{u}}, C_{\underline{v}}) = \mathbb{C}(q) \cdot C_{\underline{u}}$.

□

LEMMA 4.2.8. *Let* $\underline{\mathrm{w}},\underline{\mathrm{u}}$, *and* $\underline{\mathrm{v}} \in \boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. *If* $V(\mathrm{wt}\,\underline{\mathrm{v}})$ *is not a direct summand of* $V(\mathrm{wt}\,\underline{\mathrm{w}}) \otimes V(\mathrm{wt}\,\underline{\mathrm{u}})$, *then*

$$C_{\underline{\mathrm{v}}} \circ D \circ (C_{\underline{\mathrm{w}}} \otimes C_{\underline{\mathrm{u}}}) = 0$$

for all $D \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}} \otimes \underline{\mathrm{u}}, \underline{\mathrm{v}})$.

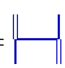
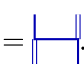
PROOF. Corollary 3.2.1 implies that $\dim \mathrm{Hom}_{\mathrm{Kar}(\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2))}(C_{\underline{\mathrm{w}}} \otimes C_{\underline{\mathrm{u}}}, C_{\underline{\mathrm{v}}}) = 0$, when $V(\mathrm{wt}\,\underline{\mathrm{v}})$ is not a direct summand of $V(\mathrm{wt}\,\underline{\mathrm{w}}) \otimes V(\mathrm{wt}\,\underline{\mathrm{u}})$. $\square$

LEMMA 4.2.9 (Clasp absorption). *Let* $\underline{\mathrm{w}} = \underline{\mathrm{x}} \otimes \underline{\mathrm{y}} \otimes \underline{\mathrm{z}}$ *in* $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$, *then*

$$(\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \circ (\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}).$$

PROOF. Since $V(\mathrm{wt}\,\underline{\mathrm{w}})$ appears with multiplicity one in $V(\underline{\mathrm{w}})$, it follows that $\pi_{\underline{\mathrm{w}}}$ is a central idempotent in $\mathrm{End}_{\mathrm{U_q}(\mathfrak{g}_2)}(V(\underline{\mathrm{w}}))$. Therefore, $(\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \circ (\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}})$ is also an idempotent and $C_{\underline{\mathrm{u}}} \circ D \circ (\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} = 0$ for all $D \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}})$ such that $\mathrm{wt}\,\underline{\mathrm{u}} < \mathrm{wt}\,\underline{\mathrm{w}}$. Thus, by Lemma 4.2.6 it suffices to show that $(\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}} \neq 0$. This is deduced from observing that the morphism $\Phi((\mathrm{id}_{\underline{\mathrm{x}}} \otimes C_{\underline{\mathrm{y}}} \otimes \mathrm{id}_{\underline{\mathrm{z}}}) \circ C_{\underline{\mathrm{w}}})$ acts on $V(\underline{\mathrm{w}})_{\mathrm{wt}\,\underline{\mathrm{w}}}$ as multiplication by 1. $\square$

4.2.2. Neutral diagrams and generalized clasps.

DEFINITION 4.2.4. *We will write* $\mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} :=$  *and* $\mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} :=$ . *These are the* ***basic neutral diagrams***.

LEMMA 4.2.10 (Neutral absorption). *If* $\underline{\mathrm{w}} = \underline{\mathrm{w}}_1\, \varpi_1\varpi_2\, \underline{\mathrm{w}}_2$ *and* $\underline{\mathrm{w}}' = \underline{\mathrm{w}}_1\, \varpi_2\varpi_1\, \underline{\mathrm{w}}_2$, *then*

$$(\mathrm{id}_{\underline{\mathrm{w}}_1} \otimes \mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} \otimes \mathrm{id}_{\underline{\mathrm{w}}_2}) \circ C_{\underline{\mathrm{w}}} \circ (\mathrm{id}_{\underline{\mathrm{w}}_1} \otimes \mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} \otimes \mathrm{id}_{\underline{\mathrm{w}}_2}) = C_{\underline{\mathrm{w}}'}$$

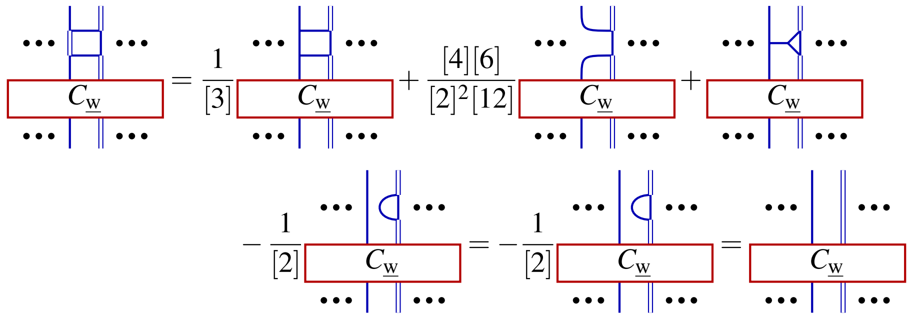
in $\mathrm{End}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}')$.

PROOF. Write $\mathsf{H}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} := \mathrm{id}_{\underline{\mathrm{w}}_1} \otimes \mathsf{H}^{\varpi_2\varpi_1}_{\varpi_1\varpi_2} \otimes \mathrm{id}_{\underline{\mathrm{w}}_2}$ and $\mathsf{H}^{\underline{\mathrm{w}}}_{\underline{\mathrm{w}}'} := \mathrm{id}_{\underline{\mathrm{w}}_1} \otimes \mathsf{H}^{\varpi_1\varpi_2}_{\varpi_2\varpi_1} \otimes \mathrm{id}_{\underline{\mathrm{w}}_2}$. By Lemma 4.2.3 we only need to show that $(\mathsf{H}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}}) \circ C_{\underline{\mathrm{w}}} \circ (\mathsf{H}^{\underline{\mathrm{w}}}_{\underline{\mathrm{w}}'})$ satisfies the defining properties of a clasp.

Let $D \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}', \underline{\mathrm{u}})$ where $\mathrm{wt}\,\underline{\mathrm{u}} < \mathrm{wt}\,\underline{\mathrm{w}}'$. Then $D \circ \mathsf{H}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{u}})$ and $\mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{w}}' > \mathrm{wt}\,\underline{\mathrm{u}}$, so $D \circ \mathsf{H}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} = 0$. So $\mathsf{H}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}^{\underline{\mathrm{w}}}_{\underline{\mathrm{w}}'}$ satisfies the third condition in the definition of clasps.

The following calculation shows that $\mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}}$.

(4.4)



So we have

$$(\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}) \circ (\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}) = \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ (\mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}}) \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}$$

$$= \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}} = \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}.$$

This tells us that $\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}$ satisfies the second condition in the definition of clasps.

What's more, $\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}} \neq 0$. Otherwise $C_{\underline{\mathrm{w}}} = \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}} \circ (\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}) \circ \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} = 0$, which is a contradiction. ☐

DEFINITION 4.2.5. *A **neutral diagram** $\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \in \mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{w}}')$ is a composition of tensor products of identity diagrams and basic neutral diagrams. A **reduced neutral diagram**, is a neutral diagram such that $\mathsf{H}_{\varpi_2\varpi_1}^{\varpi_1\varpi_2} \circ \mathsf{H}_{\varpi_1\varpi_2}^{\varpi_2\varpi_1}$ or $\mathsf{H}_{\varpi_1\varpi_2}^{\varpi_2\varpi_1} \circ \mathsf{H}_{\varpi_2\varpi_1}^{\varpi_1\varpi_2}$ do not occur as subdiagrams of $\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'}$.*

LEMMA 4.2.11. *Fix $\underline{\mathrm{w}}$ and $\underline{\mathrm{w}}'$.*

(1) *There is a neutral diagram $\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'}$ if and only if $\mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{w}}'$.*

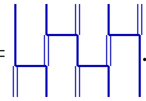
(2) *If $\mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{w}}'$, then there is a reduced neutral diagram in $\mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{w}}')$.*

(3) *Reduced neutral diagrams are unique.*

(4) *Suppose ${}^1\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'}$ and ${}^2\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'}$ are two neutral diagrams. Then ${}^1\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}} = {}^2\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} \circ C_{\underline{\mathrm{w}}}$.*

PROOF. Omitted. ☐

NOTATION 4.2.1. *Suppose $\mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{w}}'$, then we will write $\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'}$ for the reduced neutral diagram in* $\mathrm{Hom}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}, \underline{\mathrm{w}}')$.

EXAMPLE 4.2.1. *Consider* $\underline{\mathrm{w}} = \varpi_2\varpi_1\varpi_2\varpi_1\varpi_1$, $\underline{\mathrm{w}}' = \varpi_1\varpi_1\varpi_2\varpi_1\varpi_2$. *We know that* $\mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{w}}' = (3,2)$. *The reduced neutral diagram is* $\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'} =$ .

DEFINITION 4.2.6. *Given a diagram D in* $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ *we will write* $\mathbb{D}(D)$ *for the diagram obtained by flipping D upside down. Note that* $\mathbb{D}\big(\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{w}}'}\big) = \mathsf{H}_{\underline{\mathrm{w}}'}^{\underline{\mathrm{w}}}$.

DEFINITION 4.2.7. *Given* $\underline{\mathrm{x}}, \underline{\mathrm{y}}$ *so that* $\mathrm{wt}\,\underline{\mathrm{x}} = \mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{y}}$, *we define the* ***generalized clasp*** $C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}} := \mathsf{H}_{\underline{\mathrm{w}}}^{\underline{\mathrm{y}}} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{H}_{\underline{\mathrm{x}}}^{\underline{\mathrm{w}}}$. *From Lemma 4.2.11 it follows that if* $\mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{y}}}$ *and* $\mathsf{N}_{\underline{\mathrm{x}}}^{\underline{\mathrm{w}}}$ *are any neutral diagrams, then* $C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}} = \mathsf{N}_{\underline{\mathrm{w}}}^{\underline{\mathrm{y}}} \circ C_{\underline{\mathrm{w}}} \circ \mathsf{N}_{\underline{\mathrm{x}}}^{\underline{\mathrm{w}}}$.

PROPOSITION 4.2.1. *The generalized clasps satisfy the following properties:*

(1) $C_{\underline{\mathrm{x}}}^{\underline{\mathrm{x}}} = C_{\underline{\mathrm{x}}}$,
(2) $C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}} \circ \mathsf{H}_{\underline{\mathrm{z}}}^{\underline{\mathrm{x}}} = C_{\underline{\mathrm{z}}}^{\underline{\mathrm{y}}}$,
(3) $\mathsf{H}_{\underline{\mathrm{y}}}^{\underline{\mathrm{z}}} \circ C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}} = C_{\underline{\mathrm{x}}}^{\underline{\mathrm{z}}}$,
(4) $C_{\underline{\mathrm{y}}}^{\underline{\mathrm{z}}} \circ C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}} = C_{\underline{\mathrm{x}}}^{\underline{\mathrm{z}}}$, *and*
(5) $\mathbb{D}(C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}}) = C_{\underline{\mathrm{y}}}^{\underline{\mathrm{x}}}$.

PROOF. Exercise for the reader. For hints, see [**17**, Proposition 3.2]. $\square$

4.2.3. Elementary Light Ladders.

NOTATION 4.2.2. *We write* $f_1^{(k)} := \dfrac{f_1^k}{[k]_q!}$ *and* $f_2^{(k)} := \dfrac{f_2^k}{[k]_{q^3}!}$, *where* $[k]_q! := [k][k-1]\dots[2][1]$ *and* $[k]_{q^3}! := [k]_{q^3}[k-1]_{q^3}\dots[2]_{q^3}[1]_{q^3}$. *Note that* $[k]_{q^3} = \frac{[3k]}{[3]}$.

For each fundamental weight $\varpi \in \{\varpi_1, \varpi_2\}$ we choose a basis $\{{}^i v_{\mu,\varpi}\}_{i=1,\dots,\dim V(\varpi)_\mu}$ for all weight spaces $V(\varpi)_\mu$. Our convention will be to not record the superscript i in ${}^i v_{\mu,\varpi}$ when the weight space is multiplicity one. Explicitly, we choose the following basis of $V(\varpi_1)$:

$$v_{(1,0),\varpi_1} = v_1, \quad v_{(-1,1),\varpi_1} = f_1 v_1, \quad v_{(2,-1),\varpi_1} = f_2 f_1 v_1, \quad v_{(0,0),\varpi_1} = f_1 f_2 f_1 v_1,$$

$$v_{(-2,1),\varpi_1} = f_1^{(2)} f_2 f_1 v_1, \quad v_{(1,-1),\varpi_1} = f_2 f_1^{(2)} f_2 f_1 v_1, \quad \text{and} \quad v_{(-1,0),\varpi_1} = f_1 f_2 f_1^{(2)} f_2 f_1 v_1,$$

and the following basis of $V(\varpi_2)$:

$$v_{(0,1),\varpi_2} = v_2, \quad v_{(3,-1),\varpi_2} = f_2 v_2, \quad v_{(1,0),\varpi_2} = f_1 f_2 v_2, \quad v_{(-1,1),\varpi_2} = f_1^{(2)} f_2 f_1 v_2,$$

$$v_{(2,-1),\varpi_2} = f_2 f_1^{(2)} f_2 v_2, \quad v_{(-3,2),\varpi_2} = f_1^{(3)} f_2 v_2,$$

$$^1 v_{(0,0),\varpi_2} = f_1 f_2 f_1^{(2)} f_2 v_2, \quad ^2 v_{(0,0),\varpi_2} = f_2 f_1^{(3)} f_2 v_2,$$

$$v_{(3,-2),\varpi_2} = f_2^{(2)} f_1^{(3)} f_2 v_2, \quad v_{(-2,1),\varpi_2} = f_1^{(2)} f_2 f_1^{(2)} f_2 v_2$$

$$v_{(1,-1),\varpi_2} = f_1 f_2^{(2)} f_1^{(3)} f_2 v_2, \quad v_{(-1,0),\varpi_2} = f_1^{(2)} f_2^{(2)} f_1^{(3)} f_2 v_2,$$

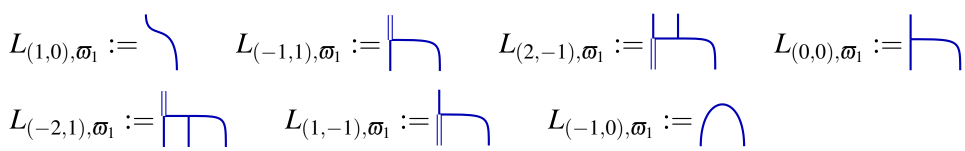
$$v_{(-3,1),\varpi_2} = f_1^{(3)} f_2^{(2)} f_1^{(3)} f_2 v_2, \quad \text{and} \quad v_{(0,-1),\varpi_2} = f_2 f_1^{(3)} f_2^{(2)} f_1^{(3)} f_2 v_2.$$

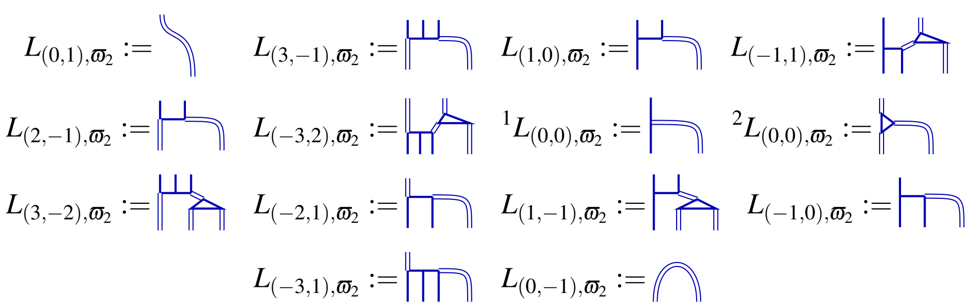
REMARK 4.2.2. *The following relation holds in $V(\varpi_2)$:*

$$f_1 f_2^{(2)} f_1^{(3)} f_2 v_2 = f_2 f_1^{(2)} f_2 f_1^{(2)} f_2 v_2.$$

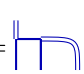
Thus, there are two ways to present the vector $v_{(1,-1),\varpi_2}$.

DEFINITION 4.2.8. *For each vector $^i v_{\mu,\varpi} \in V(\varpi)$, we associate a diagram in $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ denoted $^i L_{\mu,\varpi}$. Our convention will be to not record the superscript i in $^i L_{\mu,\varpi}$ when the weight space is multiplicity one.*





*The diagram $^i L_{\mu,\varpi}$ is a morphism from $^i \underline{\mathrm{x}}_{\mu,\varpi} \otimes \varpi \to ^i \underline{\mathrm{y}}_{\mu,\varpi}$. We will refer to $^i \underline{\mathrm{x}}_{\mu,\varpi}$ as the **in strand** of $^i L_{\mu,\varpi}$ and $^i \underline{\mathrm{y}}_{\mu,\varpi}$ as the **out strand** of $^i L_{\mu,\varpi}$. Note that $\mu = \mathrm{wt}(^i \underline{\mathrm{y}}_{\mu,\varpi}) - \mathrm{wt}(^i \underline{\mathrm{x}}_{\mu,\varpi})$.*

EXAMPLE 4.2.2. *For $L_{(-2,1),\varpi_2} :=$*  *, we have $\underline{\mathrm{x}}_{(-2,1),\varpi_2} = \varpi_1 \varpi_1$, $\underline{\mathrm{y}}_{(-2,1),\varpi_2} = \varpi_2$, and $(-2,1) =$ $\mathrm{wt}(\underline{\mathrm{y}}_{(-2,1),\varpi_2}) - \mathrm{wt}(\underline{\mathrm{x}}_{(-2,1),\varpi_2})$.*

NOTATION 4.2.3. *If W is a subspace of a $\mathrm{U_q}(\mathfrak{g}_2)$ module, we will write*

$$Ker_{(\mathrm{a,b})}(W) := \{w \in W \ : \ e_1^{a+1} w = 0 = e_2^{b+1} w\}.$$

LEMMA 4.2.12. *Let $a, b \in \mathbb{Z}_{\geq 0}$. Fix a fundamental weight ϖ and let $\mu \in \mathrm{wt}\, V(\varpi)$. Then*

$$[V(a,b) \otimes V(\varpi) : V((a,b)+\mu)] = \dim Ker_{(\mathrm{a,b})}(V(\varpi)_\mu). \tag{4.5}$$

PROOF. Follows from [**47**, Theorem 2.1]. □

LEMMA 4.2.13. *The following are equivalent:*

(1) ${}^i v_{\mu,\varpi} \in Ker_{(\mathrm{a,b})}(V(\varpi)_\mu)$, *and*
(2) *There is $(c,d) \in \mathbb{N} \times \mathbb{N}$ such that $\mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi}) + (c,d) = (a,b)$.*

PROOF. The lemma can be deduced from the following claim: the weight of the in strand for ${}^i L_{\mu,\varpi}$, $\mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi})$, is equal to the minimal (a,b) so that ${}^i v_{\mu,\varpi} \in \mathrm{Ker}_{(\mathrm{a,b})}(V(\varpi)_\mu)$. The claim is verified from the vector to diagram correspondence ${}^i v_{\mu,\varpi} \mapsto {}^i L_{\mu,\varpi}$, along with Equation (4.5), and the description of action of e_1 and e_2 on the vectors in each fundamental representation. Computing $e_k \cdot {}^i v_{\mu,\varpi}$ is left as an exercise, the most interesting case is the zero weight space for the second fundamental representation. □

EXAMPLE 4.2.3. *When $a \geq 3$ and $b \geq 2$, $[V(a,b) \otimes V(\varpi_2) : V((a,b)+\mu)] = 1$ when $\mu \neq (0,0)$, and $[V(a,b) \otimes V(\varpi_2) : V(a,b)] = 2$. The reader should compare this with the observation that for each ${}^i L_{\mu,\varpi_2}$ the number of ϖ_1 colored in strands is less than or equal to 3 and the number of ϖ_2 colored in strands is less than or equal to 2.*

For each dominant integral weight $\lambda = a\varpi_1 + b\varpi_2 \in X_+$, we choose a distinguished object $\underline{\mathrm{u}}_\lambda \in \mathbf{Web}_\mathrm{q}(\mathfrak{g}_2)$ such that $\mathrm{wt}\, \underline{\mathrm{u}}_\lambda = \lambda$.

EXAMPLE 4.2.4. *We must have $\underline{\mathrm{u}}_{(2,0)} = \varpi_1\varpi_1$ and for $\underline{\mathrm{u}}_{(1,1)}$ we choose one of $\varpi_1\varpi_2$ or $\varpi_2\varpi_1$.*

DEFINITION 4.2.9. *Let $\underline{\mathrm{w}}$ be an object in $\boldsymbol{Web}_\mathrm{q}(\mathfrak{g}_2)$ and let $\lambda = \mathrm{wt}\, \underline{\mathrm{w}}$. Suppose that ${}^i v_{\mu,\varpi} \in Ker_\lambda(V(\varpi)_\mu)$, so in particular $\lambda + \mu \in X_+$. Let $(c,d) := \lambda - \mathrm{wt}({}^i\underline{\mathrm{x}}_{\mu,\varpi})$, then by Lemma 4.2.13 we have $(c,d) \in \mathbb{N} \times \mathbb{N}$, so there is a reduced neutral diagram*

$$\mathsf{H}_{\underline{\mathrm{w}}}^{\underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{x}}_{\mu,\varpi})} : \underline{\mathrm{w}} \to \underline{s}_{(c,d)} \otimes ({}^i\underline{\mathrm{x}}_{\mu,\varpi}).$$

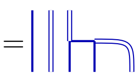
Since $(c,d)+\mathrm{wt}({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})=\lambda+\mu$, *there is also a reduced neutral diagram*

$$\mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{s}_{(c,d)}\otimes({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})}:\underline{s}_{(c,d)}\otimes({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})\to\underline{\mathrm{u}}_{\lambda+\mu}\,.$$

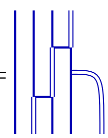
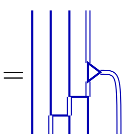
We define the ***elementary light ladder diagram*** *to be*

$${}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}:=\mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{s}_{(c,d)}\otimes({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})}\circ\left((\mathrm{id}_{\underline{s}_{(c,d)}})\otimes{}^{i}L_{\mu,\varpi}\right)\circ\left(\mathsf{H}^{\underline{s}_{(c,d)}\otimes({}^{i}\underline{\mathrm{x}}_{\mu,\varpi})}_{\underline{\mathrm{w}}}\otimes\mathrm{id}_{\varpi}\right).$$

EXAMPLE 4.2.5. *Consider* $\underline{\mathrm{w}}=\varpi_1\varpi_2\varpi_1\varpi_1$ *and* $\varpi=\varpi_2$. *We know that* $\lambda=\mathrm{wt}\,\underline{\mathrm{w}}=(3,1)$. *When* $\mu=(-2,1)$, *so* $\lambda+\mu=(1,2)$, *choose* $\underline{\mathrm{u}}_{\lambda+\mu}=\varpi_1\varpi_2\varpi_2$. *Then*

$${}^{1}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}=$$  .

When $\mu=(0,0)$, $\lambda+\mu=(3,1)$, *choose* $\underline{\mathrm{u}}_{\lambda+\mu}=\varpi_1\varpi_1\varpi_1\varpi_2$. *Then*

$${}^{1}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}=$$  *and* $${}^{2}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}=$$  .

DEFINITION 4.2.10. *Let* $\underline{\mathrm{w}}\in\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$. *Write* $\lambda=\mathrm{wt}\,\underline{\mathrm{w}}$, *and suppose that* ${}^{i}v_{\mu,\varpi}\in Ker_{\lambda}(V(\varpi)_{\mu})$. *We define the* ***(clasped) light ladder diagram*** *to be the following diagram:*

$${}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}:=C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ({}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})\circ(C_{\underline{\mathrm{w}}}\otimes\mathrm{id}_{\varpi}).$$

LEMMA 4.2.14. *Let* $\mathsf{N}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}}:\underline{\mathrm{w}}\to\underline{\mathrm{w}}'$ *be a neutral diagram. Then*

$${}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi}\circ(N^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}}\otimes\mathrm{id}_{\varpi})\circ C_{\underline{\mathrm{w}}}\otimes\mathrm{id}_{\varpi}={}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\,.$$

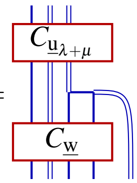
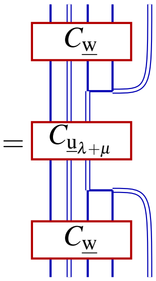
PROOF. Follows from Lemma 4.2.11. □

DEFINITION 4.2.11. *Suppose that* ${}^{i}v_{\mu,\varpi},{}^{j}v_{\mu,\varpi}\in Ker_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_{\mu})$ *(we allow for* $i=j$*). We define the* ***(clasped) double ladder diagram*** *to be the following diagram in* $\mathrm{End}_{\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}\otimes\varpi)$*:*

$${}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}:=(\mathbb{D}({}^{i}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}))\circ({}^{j}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}).$$

In the case that $V(\varpi)_{\mu}$ *is one dimensional, we will drop the superscripts* ij *in double ladders and drop superscript* i *in (clasped) light ladders.*

EXAMPLE 4.2.6. *Consider* $\underline{\mathrm{w}} = \varpi_1\varpi_2\varpi_1\varpi_1$, $\varpi = \varpi_2$. *When* $\mu = (-2,1)$, *choose* $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_2\varpi_2$, *then*

$LL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}} =$  *and* $\mathbb{LL}_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}} =$  .

REMARK 4.2.3. *Using the definition of the elementary light ladder, and basic properties of clasps, we can expand the clasped light ladder:*

$$\begin{aligned} {}^{i}LL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}} &:= C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{i}ELL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}}) \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \\ &= C^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})} \circ \left((\mathrm{id}_{\underline{s}_{(c,d)}}) \otimes {}^{i}L_{\mu,\varpi} \right) \circ \left(C_{\underline{\mathrm{w}}}^{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{x}}_{\mu,\varpi})} \otimes \mathrm{id}_{\varpi} \right). \end{aligned}$$

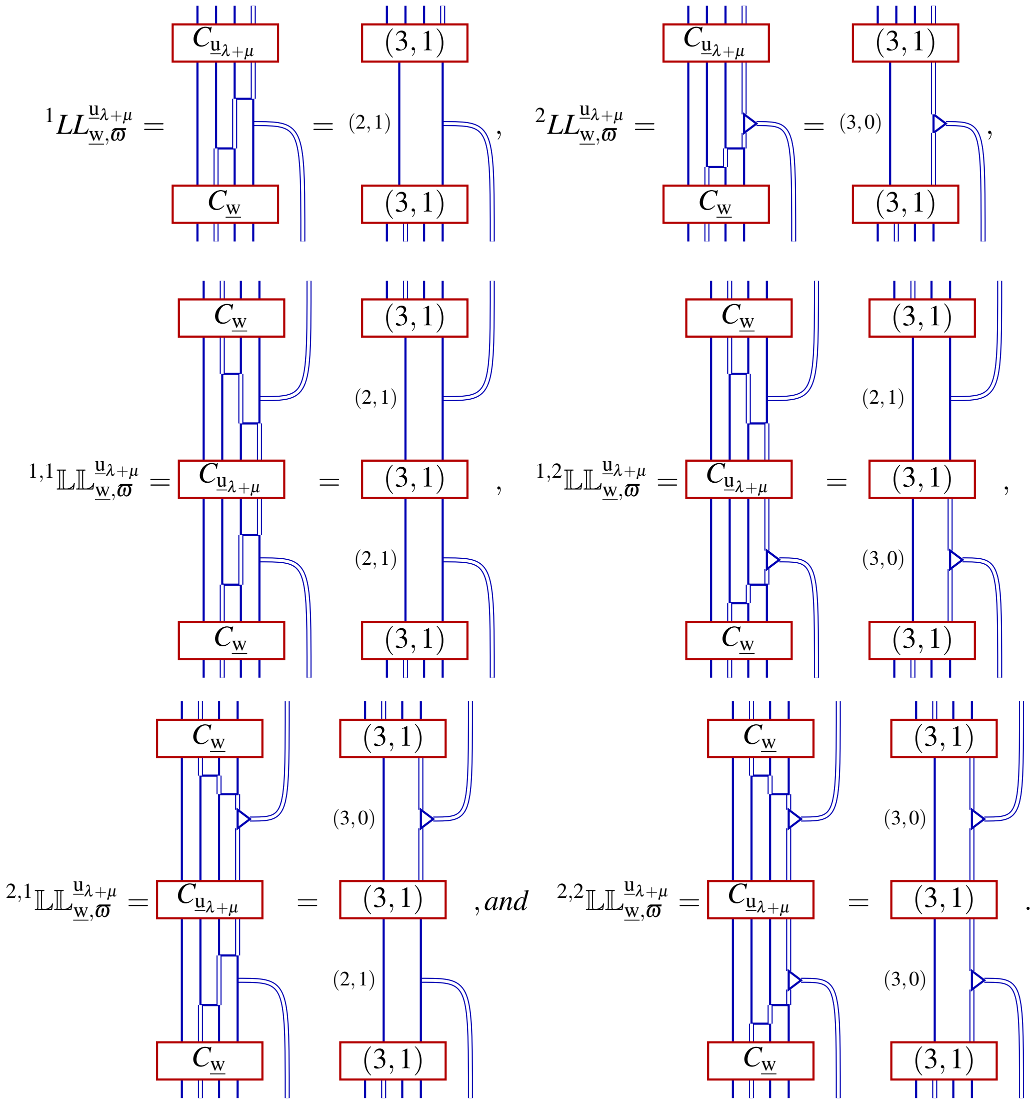
We can similarly expand the clasped double ladder:

$$\begin{aligned} {}^{ij}\mathbb{LL}_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}} &:= (\mathbb{D}({}^{i}LL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}})) \circ ({}^{j}LL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}}) \\ &= (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \circ (\mathbb{D}({}^{i}ELL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}})) \circ C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{j}ELL_{\underline{\mathrm{w}},\varpi}^{\underline{\mathrm{u}}_{\lambda+\mu}}) \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \\ &= \left(C^{\underline{\mathrm{w}}}_{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{x}}_{\mu,\varpi})} \otimes \mathrm{id}_{\varpi} \right) \circ \left((\mathrm{id}_{\underline{s}_{(c,d)}}) \otimes \mathbb{D}({}^{i}L_{\mu,\varpi}) \right) \circ C^{\underline{s}_{(c,d)} \otimes ({}^{i}\underline{\mathrm{y}}_{\mu,\varpi})}_{\underline{s}_{(e,f)} \otimes ({}^{j}\underline{\mathrm{y}}_{\mu,\varpi})} \\ &\circ \left((\mathrm{id}_{\underline{s}_{(e,f)}}) \otimes {}^{j}L_{\mu,\varpi} \right) \circ \left(C_{\underline{\mathrm{w}}}^{\underline{s}_{(e,f)} \otimes ({}^{j}\underline{\mathrm{x}}_{\mu,\varpi})} \otimes \mathrm{id}_{\varpi} \right). \end{aligned}$$

This more complicated looking expanded formula, is actually simpler when viewed in terms of the graphical calculus, as we illustrate in Example 4.2.7.

NOTATION 4.2.4. *When* $\mathrm{wt}\,\underline{\mathrm{x}} = \mathrm{wt}\,\underline{\mathrm{y}} = (a,b)$, *we will use an* (a,b) *labelled box in* $\mathrm{Hom}_{\boldsymbol{Web}_q(\mathfrak{g}_2)}(\underline{\mathrm{x}},\underline{\mathrm{y}})$ *to denote* $C_{\underline{\mathrm{x}}}^{\underline{\mathrm{y}}}$.

EXAMPLE 4.2.7. *Let* $\underline{\mathrm{w}} = \varpi_1\varpi_2\varpi_1\varpi_1, \varpi = \varpi_2$. *When* $\mu = (0,0)$, *choose* $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_1\varpi_1\varpi_2$, *then (using Lemma 4.2.10) we have*



REMARK 4.2.4. *As* (a,b) *varies, so does* $Ker_{(\mathrm{a,b})}(V(\varpi))$. *However, the vector* $v_\varpi = v_{\varpi,1}$ *is always contained in* $Ker_{(\mathrm{a,b})}(V(\varpi))$. *Moreover, the associated elementary light ladder is just a composition of neutral diagrams, so the associated (clasped) double ladder can be simplified to*

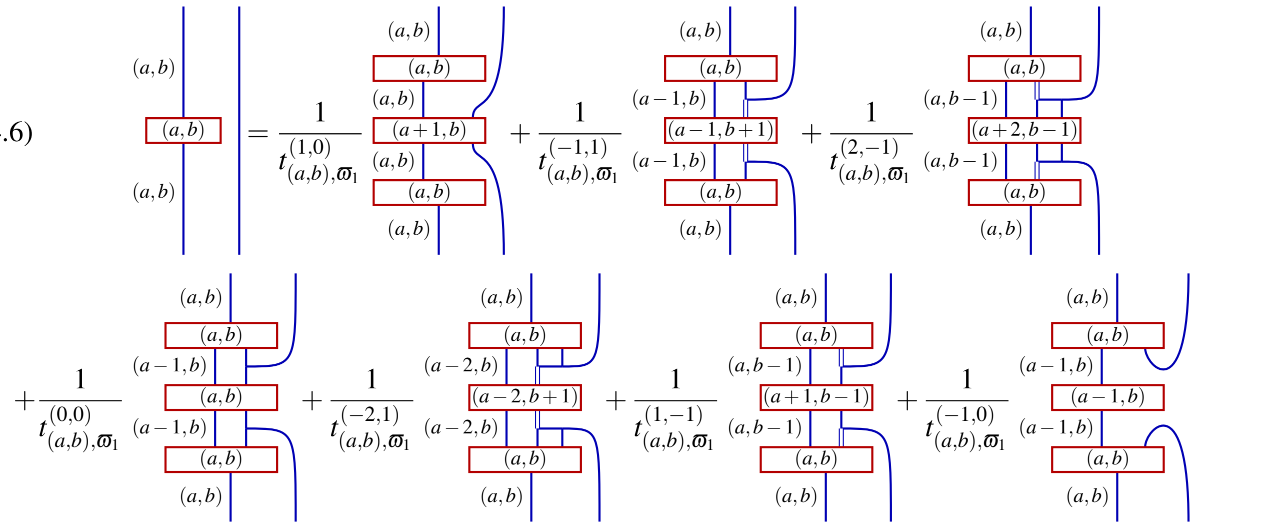
$$\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\varpi}}_{\underline{\mathrm{w}},\varpi} = C_{\underline{\mathrm{w}}\otimes\varpi}.$$

Thus, the (clasped) double ladder associated to the highest weight vector in $V(\varpi)$ *is itself a clasp.*

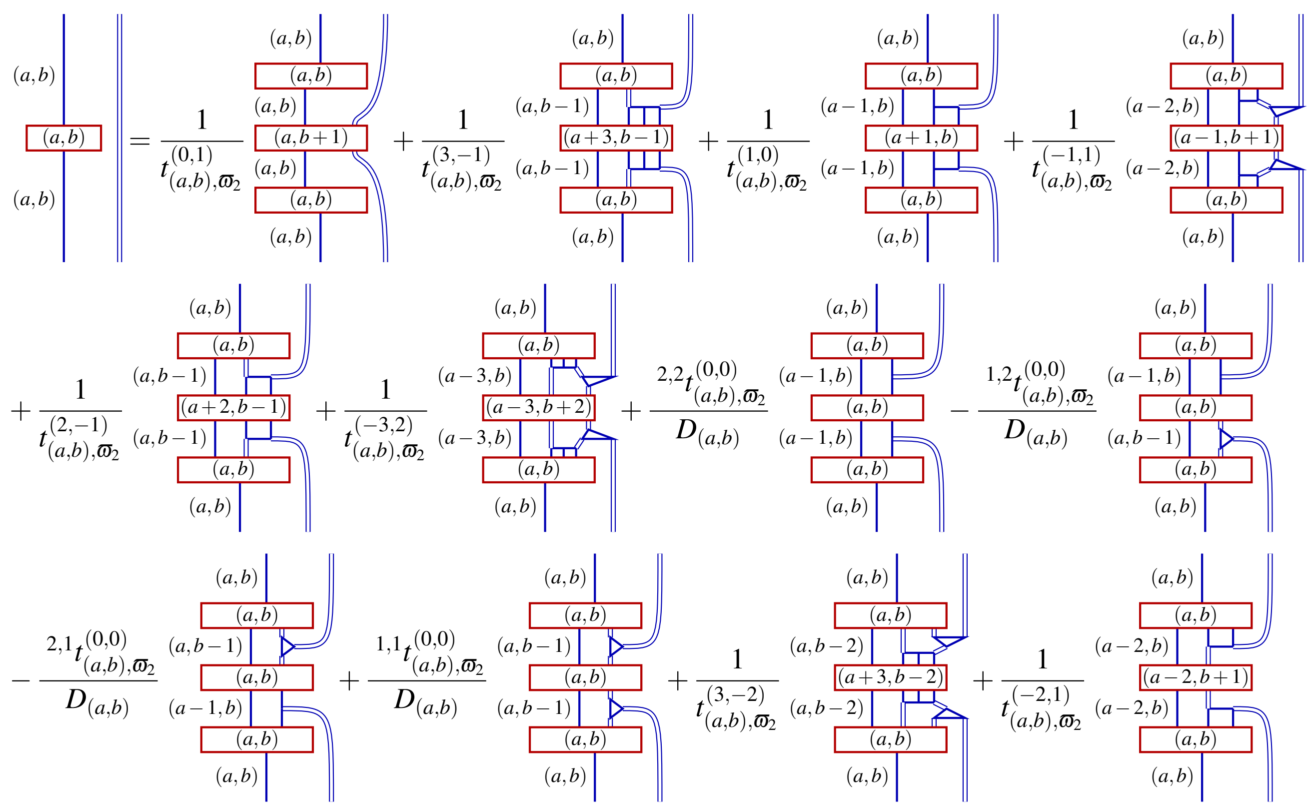
4.3. Triple Clasp Formula for G_2

4.3.1. Formulas.

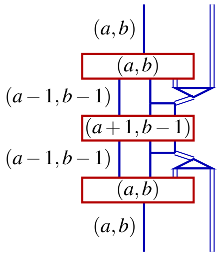
(4.6)



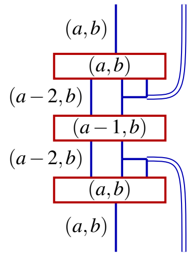
(4.7)



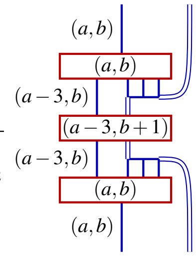
$$+\frac{1}{t^{(1,-1)}_{(a,b),\varpi_2}}$$



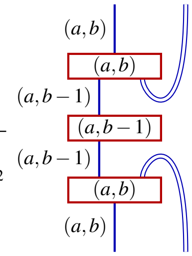
$$+\frac{1}{t^{(-1,0)}_{(a,b),\varpi_2}}$$



$$+\frac{1}{t^{(-3,1)}_{(a,b),\varpi_2}}$$



$$+\frac{1}{t^{(0,-1)}_{(a,b),\varpi_2}}$$



(4.8) $$t^{(1,0)}_{(a,b),\varpi_1}=1$$

(4.9) $$t^{(-1,1)}_{(a,b),\varpi_1}=-\frac{[a+1]}{[a]}$$

(4.10) $$t^{(2,-1)}_{(a,b),\varpi_1}=\frac{[3b+3][a+3b+4]}{[3b][a+3b+3]}$$

(4.11) $$t^{(0,0)}_{(a,b),\varpi_1}=-\frac{[a+2][a+3b+5][2a+3b+6]}{[2][a][a+3b+3][2a+3b+4]}$$

(4.12) $$t^{(-2,1)}_{(a,b),\varpi_1}=\frac{[a+1][2a+3b+5][3a+3b+6]}{[a-1][2a+3b+4][3a+3b+3]}$$

(4.13) $$t^{(1,-1)}_{(a,b),\varpi_1}=-\frac{[3b+3][a+3b+4][2a+3b+5][3a+6b+9]}{[3b][a+3b+2][2a+3b+4][3a+6b+6]}$$

(4.14) $$t^{(-1,0)}_{(a,b),\varpi_1}=\frac{[a+1][a+3b+4][2a+3b+5][3a+3b+6][3a+6b+9]}{[a][a+3b+3][2a+3b+3][3a+3b+3][3a+6b+6]}$$

(4.15) $$t^{(0,1)}_{(a,b),\varpi_2}=1$$

(4.16) $$t^{(3,-1)}_{(a,b),\varpi_2}=-\frac{[3b+3]}{[3b]}$$

(4.17) $$t^{(1,0)}_{(a,b),\varpi_2}=\frac{[a+3][a+3b+6]}{[3][a][a+3b+3]}$$

(4.18) $$t^{(-1,1)}_{(a,b),\varpi_2}=-\frac{[a+1][a+2][2a+3b+7]}{[3][a-1][a][2a+3b+4]}$$

(4.19) $$t^{(2,-1)}_{(a,b),\varpi_2}=\frac{[3b+3][a+3b+4][a+3b+5][2a+3b+7]}{[3][3b][a+3b+2][a+3b+3][2a+3b+4]}$$

$$t^{(-3,2)}_{(a,b),\varpi_2}=\frac{[a+1][3a+3b+6]}{[a-2][3a+3b+3]}$$

$$t^{(3,-2)}_{(a,b),\varpi_2} = \frac{[3b+3][a+3b+4][3a+6b+9]}{[3b-3][a+3b+1][3a+6b+6]} \tag{4.20}$$

$$t^{(-2,1)}_{(a,b),\varpi_2} = \frac{[a+1][a+3b+6][2a+3b+5][2a+3b+6][3a+3b+6]}{[3][a-1][a+3b+3][2a+3b+3][2a+3b+4][3a+3b+3]} \tag{4.21}$$

$$t^{(1,-1)}_{(a,b),\varpi_2} = -\frac{[a+3][3b+3][a+3b+4][2a+3b+5][2a+3b+6][3a+6b+9]}{[3][a][3b][a+3b+2][2a+3b+3][2a+3b+4][3a+6b+6]} \tag{4.22}$$

$$t^{(-1,0)}_{(a,b),\varpi_2} = \frac{[a+1][a+2][a+3b+4][a+3b+5][2a+3b+5][3a+3b+6][3a+6b+9]}{[3][a-1][a][a+3b+2][a+3b+3][2a+3b+3][3a+3b+3][3a+6b+6]} \tag{4.23}$$

$$t^{(-3,1)}_{(a,b),\varpi_2} = -\frac{[a+1][2a+3b+5][3a+3b+6][3a+6b+9]}{[a-2][2a+3b+2][3a+3b][3a+6b+6]} \tag{4.24}$$

$$t^{(0,-1)}_{(a,b),\varpi_2} = \frac{[3b+3][a+3b+4][2a+3b+5][3a+3b+6][3a+6b+9]}{[3b][a+3b+1][2a+3b+2][3a+3b+3][3a+6b+3]} \tag{4.25}$$

$$\left({}^{p,\ell}t^{(0,0)}_{(a,b),\varpi_2} \right) = \begin{pmatrix} {}^{1,1}t^{(0,0)}_{(a,b),\varpi_2} & {}^{1,2}t^{(0,0)}_{(a,b),\varpi_2} \\ {}^{2,1}t^{(0,0)}_{(a,b),\varpi_2} & {}^{2,2}t^{(0,0)}_{(a,b),\varpi_2} \end{pmatrix} \tag{4.26}$$

$$\mathscr{D}_{(a,b)} := \det \left({}^{p,\ell}t^{(0,0)}_{(a,b),\varpi_2} \right) \tag{4.27}$$

Moreover,

$${}^{1,2}t^{(0,0)}_{(a,b),\varpi_2} = {}^{2,1}t^{(0,0)}_{(a,b),\varpi_2}, \tag{4.28}$$

$$\mathscr{D}_{(a,b)} = \frac{[4][6][a+2][3b+6][a+3b+5][2a+3b+6][3a+3b+9][3a+6b+12]}{[2][3][12][a][3b][a+3b+3][2a+3b+4][3a+3b+3][3a+6b+6]}, \tag{4.29}$$

and the entries of the matrix can be computed from the relations in Appendix A.1.

4.3.2. Verifying the clasp conjecture. Before proving our main theorem, we will prove the following, which implies the clasp conjecture in type G_2.

COROLLARY 4.3.1. *Fix $\lambda = a\varpi_1 + b\varpi_2$ with $a, b \in \mathbb{Z}_{\geq 0}$. Let $\varpi \in \{\varpi_1, \varpi_2\}$ be a fundamental weight, and let $\mu \in W \cdot \varpi$ be a weight in the Weyl group orbit of ϖ. Then*

$$t^{\mu}_{\lambda,\varpi} = \pm \prod_{\alpha \in \Phi_\mu} \frac{[(\alpha^\vee, \lambda + \rho)]_{\ell^{\ell(\alpha)}}}{[(\alpha^\vee, \lambda + \varpi + \rho)]_{\ell^{\ell(\alpha)}}}.$$

PROOF. In type G_2, The W invariant bilinear pairing on $\mathbb{Z}\Phi$ is determined by $(\alpha_1,\alpha_1)=2$ and $(\alpha_2,\alpha_2)=6$. In particular, $l(\alpha_1)=1$ and $l(\alpha_2)=3$. We set $\alpha^\vee=2\alpha/(\alpha,\alpha)$. The positive roots are

$$\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2, \tag{4.30}$$

the corresponding coroots are

$$\alpha_1^\vee, \alpha_1^\vee+\alpha_2^\vee, 2\alpha_1^\vee+3\alpha_2^\vee, \alpha_1^\vee+2\alpha_2^\vee, \alpha_1^\vee+3\alpha_2^\vee, \alpha_2^\vee. \tag{4.31}$$

To simplify notation, we will write $s_i := s_{\alpha_i}$. It is not hard to see that

$$d_{(1,0)}=1, \qquad d_{(0,1)}=1, \qquad d_{(-2,1)}=s_1s_2s_1, \qquad d_{(3,-2)}=s_2s_1s_2, \tag{4.32}$$

$$d_{(-1,1)}=s_1, \qquad d_{(3,-1)}=s_2, \qquad d_{(1,-1)}=s_1s_2s_1s_2, \qquad d_{(-3,1)}=s_2s_1s_2s_1, \tag{4.33}$$

$$d_{(2,-1)}=s_1s_2, \qquad d_{(-3,2)}=s_2s_1, \quad d_{(-1,0)}=s_1s_2s_1s_2s_1, \quad \text{and} \quad d_{(0,-1)}=s_2s_1s_2s_1s_2. \tag{4.34}$$

The claim then follows from the formulas for $t^{\mu}_{\lambda,\varpi}$ in Section 4.3.1. One verifies this by using that if $w=s_{\beta_1}s_{\beta_2}\dots s_{\beta_n}$, then

$$\{\alpha\in\Phi_+ \,:\, w\alpha\in\Phi_-\}=\{\beta_n, s_{\beta_n}(\beta_{n-1}), s_{\beta_n}s_{\beta_{n-1}}(\beta_{n-2}),\dots s_{\beta_n}s_{\beta_{n-1}}\dots s_{\beta_2}(\beta_1)\},$$

along with the quantum number identity $[n]_{q^3}=[3n]/[3]$. □

4.3.3. Proof of Triple Clasp Formula. Suppose that $V(\mathrm{wt}\,\underline{\mathrm{u}})$ is a summand of $V(\mathrm{wt}\,\underline{\mathrm{w}})\otimes V(\varpi)$, and that $(a,b)=\lambda=\mathrm{wt}\,\underline{\mathrm{w}}$ and $(m,n)=\mu=\mathrm{wt}\,\underline{\mathrm{u}}-\mathrm{wt}\,\underline{\mathrm{w}}$. Then we will write

$${}^{p\ell}t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} := {}^{p\ell}t^{(m,n)}_{(a,b),\varpi}.$$

Our convention is that the $p\ell$ superscript is neglected when $\dim V(\varpi)_\mu=1$. By definition the elements ${}^{p\ell}t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}$ only depend on the weights $\mathrm{wt}\,\underline{\mathrm{w}}$ and $\mathrm{wt}\,\underline{\mathrm{u}}$, not the words $\underline{\mathrm{w}}$ and $\underline{\mathrm{u}}$.

We will write $({}^{p\ell}t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi})$ to denote the matrix of scalars ${}^{p\ell}t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}$, for $v_p, v_\ell\in\mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)$.

LEMMA 4.3.1. *The matrix $({}^{p\ell}t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi})$ is invertible over $\mathbb{C}(q)$.*

PROOF. From Section 4.3.1, using Equation (4.8) to Equation (4.25), and Equation (4.29), one can check that the determinant of this matrix is invertible in $\mathbb{C}(q)$. $\square$

DEFINITION 4.3.1. *Let* $\underline{\mathrm{w}} \in \boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$*, write* $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$*, and let* $\varpi \in \{\varpi_1, \varpi_2\}$*. We define the* ***triple clasp*** *to be the following inductively defined diagram:*

$$T_{\underline{\mathrm{w}}\otimes\varpi} := T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi} - \sum_{\substack{\mu\in \mathrm{wt}\,V(\varpi)\setminus\{\varpi\} \\ v_i, v_j \in Ker_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}} (({}^{p\ell}t^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1})_{ij} \cdot {}^{ij}\mathbb{T}\mathbb{T}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi},$$

where

$${}^{ij}\mathbb{T}\mathbb{T}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} := (T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \circ (\mathbb{D}({}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})) \circ T_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{j}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \circ (T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}).$$

DEFINITION 4.3.2. *By Lemma 4.2.7 there is a scalar* ${}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}$ *such that*

$${}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) = {}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}}.$$

We call the matrix

$$({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}),$$

such that $v_p, v_\ell \in Ker_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)$*, a* ***local intersection form matrix****.*

LEMMA 4.3.2. *If* $\mathrm{wt}\,\underline{\mathrm{w}} = \mathrm{wt}\,\underline{\mathrm{w}}'$*, then*

$${}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = {}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi}.$$

PROOF. First observe that

$$\begin{aligned}
{}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}} &= {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \\
&= {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \circ \mathbb{D}({}^{\ell}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \\
&= {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \circ (\mathsf{H}^{\underline{\mathrm{w}}}_{\underline{\mathrm{w}}'} \otimes \mathrm{id}_{\varpi}) \circ (C_{\underline{\mathrm{w}}'} \otimes \mathrm{id}_{\varpi}) \circ (\mathsf{H}^{\underline{\mathrm{w}}'}_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \circ \mathbb{D}({}^{\ell}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \\
&= {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi} \circ (C_{\underline{\mathrm{w}}'} \otimes \mathrm{id}_{\varpi}) \circ \mathbb{D}({}^{\ell}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi}) \\
&= {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi} \circ \mathbb{D}({}^{\ell}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi}) \\
&= {}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}}',\varpi} \cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}}.
\end{aligned}$$

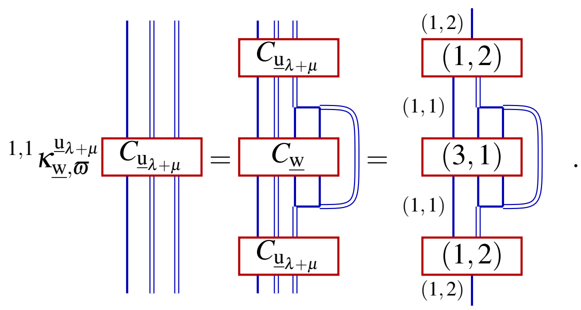
The claim follows from comparing neutral coefficients. $\square$

NOTATION 4.3.1. *We will write*

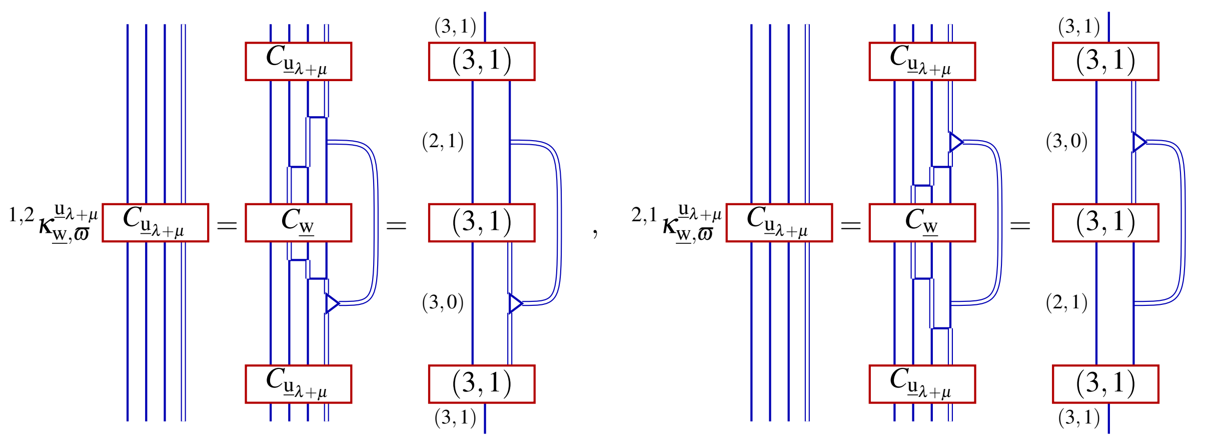
$$^{p\ell}\kappa^{\mu}_{\lambda,\varpi} := {}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi},$$

where $\mathrm{wt}\,\underline{\mathrm{w}} = (a,b)$.

EXAMPLE 4.3.1. *Consider* $\underline{\mathrm{w}} = \varpi_1\varpi_2\varpi_1\varpi_1$ *and* $\varpi = \varpi_2$. *When* $\mu = (-2,1)$, *choose* $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_2\varpi_2$. *Then*



When $\mu = (0,0)$, *choose* $\underline{\mathrm{u}}_{\lambda+\mu} = \varpi_1\varpi_1\varpi_1\varpi_2$. *Then we have the following:*



REMARK 4.3.1. *When* $\mu = (0,0)$, *by taking the quantum trace, we know that* ${}^{1,2}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = {}^{2,1}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}$. *So the local intersection form matrix* $\left({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\right)$ *is symmetric.*

NOTATION 4.3.2. *Fix the following set of formal variables*

$$\mathscr{X} := \left\{x^{(c,d)}_{(a,b),\varpi} \mid a,b,c,d \in \mathbb{Z}_{\geq 0} \text{ and } \varpi \in \{\varpi_1,\varpi_2\}\right\}.$$

We will consider elements in the ring $\mathscr{A} := \mathbb{C}(q)[x^{\pm 1} \mid x \in \mathscr{X}]$.

Suppose that $(a,b) = \lambda = \mathrm{wt}\,\underline{\mathrm{w}}$ *and* $(m,n) = \mu = \mathrm{wt}\,\underline{\mathrm{u}}_{\lambda+\mu} - \mathrm{wt}\,\underline{\mathrm{w}}$*, then we will write*

$$ {}^{p\ell}\rho^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} := {}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi} \in \mathscr{A}, $$

where ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}$ *is the recursive relation described by Equation* (A.4) *to Equation* (A.25)*, in Appendix A.2.*

We also write ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(\kappa)$ *to denote the right hand side of the recursive relation with each* ${}^{p\ell}x^{\mu}_{\lambda,\varpi}$ *replaced with* ${}^{p\ell}\kappa^{\mu}_{\lambda,\varpi}$*. Similarly, we write* ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(t)$ *to denote the right hand side of the recursive relation with each* ${}^{p\ell}x^{\mu}_{\lambda,\varpi}$ *replaced by* ${}^{p\ell}t^{\mu}_{\lambda,\varpi}$*.*

Our convention is that the $p\ell$ *superscript is neglected when* $\dim V(\varpi)_\mu = 1$.

EXAMPLE 4.3.2. *Consider* $\underline{\mathrm{w}}$ *and* $\underline{\mathrm{u}}_{\lambda+\mu}$ *in* $\boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ *such that* $\mathrm{wt}\,\underline{\mathrm{w}} = (a,b)$ *and* $\mathrm{wt}\,\underline{\mathrm{u}}_{\lambda+\mu} = (a+1,b)$*. Also, let* $\varpi = \varpi_2$*. By Equation* (A.13)*:*

$$ \rho^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi_2} = \rho^{(1,0)}_{(a,b),\varpi_2} = \frac{[7]}{[3]} - \frac{1}{x^{(-1,1)}_{(a-1,b),\varpi_1}} x^{(3,-1)}_{(a-2,b+1),\varpi_2} - \frac{1}{x^{(2,-1)}_{(a-1,b),\varpi_1}}. $$

REMARK 4.3.2. *Since the recursive relations in Appendix A.2 are elements of* $\mathscr{A}$*, there is no question whether a particular element in* $\mathscr{X}$ *appearing in a relation is invertible or not. Thanks to Lemma 4.3.1 the elements* ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(t)$ *are also always well defined. This is not obviously true for* ${}^{p\ell}\rho^{(m,n)}_{(a,b),\varpi}(\kappa)$*. However we will prove that* $\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = t^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}$.

THEOREM 4.3.1. *If* $\underline{\mathrm{w}} \in \boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$*,* $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$*, and* $\varpi \in \{1,2\}$*, then*

$$ T_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \quad \text{and} \quad \kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = t^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}. $$

We will prove Theorem 4.3.1 by induction. To simplify the arguments, we will break the various steps of the proof into smaller lemmas about the following statements. In what follows we write $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$.

$$ S_1(\underline{\mathrm{w}}) := \left(T_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}} \right) $$

$$ S_1'(\underline{\mathrm{w}},\varpi) := \left({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = {}^{p\ell}t^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}, \text{for all } \mu \in \mathrm{wt}\,V(\varpi) \text{ and for all } v_{\mu,p}, v_{\mu,\ell} \in \mathrm{Ker}_\lambda(V(\varpi)_\mu) \right) $$

$$ S_2(\underline{\mathrm{w}},\varpi) := \left(C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi \in \mathrm{span} \bigcup_{\substack{\mu \in \mathrm{wt}\,V(\varpi) \\ v_p, v_\ell \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)}} \{ {}^{p\ell}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \} \right) $$

$$S_2'(\underline{\mathrm{w}},\varpi) := \left(\{ {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \}_{v_p \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)} \text{ is a linearly independent set, for all } \mu \in \mathrm{wt}\, V(\varpi) \right)$$

$$S_3(\underline{\mathrm{w}},\varpi) := \left(\{ {}^{p}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \}_{v_p \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)} \text{ is a basis for } \mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi, C_{\underline{\mathrm{u}}_{\lambda+\mu}}), \right.$$
$$\left. \text{for all } \mu \in \mathrm{wt}\, V(\varpi) \right)$$

$$S_4(\underline{\mathrm{w}},\varpi) := \left({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = {}^{p\ell}\rho^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}(\kappa), \text{for all } \mu \in \mathrm{wt}\, V(\varpi) \right)$$

$$S_5(\underline{\mathrm{w}},\varpi) := \left(T_{\underline{\mathrm{w}}\otimes\varpi} = C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi - \sum_{\substack{\mu \in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\} \\ v_i, v_j \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)}} ({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij} \cdot {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} \right)$$

$$S_6(\underline{\mathrm{w}}) := \left(T^2_{\underline{\mathrm{w}}} = T_{\underline{\mathrm{w}}} \right)$$

$$S_6'(\underline{\mathrm{w}},\varpi) := \left(C_{\underline{\mathrm{u}}} \circ D \circ T_{\underline{\mathrm{w}}\otimes\varpi} = 0, \text{for } \underline{\mathrm{u}} \text{ such that } (\mathrm{wt}\,\underline{\mathrm{u}} - \lambda) \in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\} \text{ and for all possible diagram } D \right)$$

LEMMA 4.3.3. *If $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$, then $\lambda+\mu \leq \lambda+\varpi$.*

PROOF. Suppose $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$. Then $V(\varpi)_\mu \neq 0$. It follows that $\mu \in \varpi + \mathbb{Z}_{\leq 0}\Phi_+$, so $\varpi - \mu \geq 0$. □

LEMMA 4.3.4. *If $S_1(\underline{\mathrm{x}})$ for all $\underline{\mathrm{x}}$ such that* $\mathrm{wt}\,\underline{\mathrm{x}} \leq \mathrm{wt}(\underline{\mathrm{w}}\otimes\varpi)$*, then* $S_2(\underline{\mathrm{w}},\varpi)$.

PROOF. Write $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$. Since $T_{\underline{\mathrm{w}}\otimes\varpi} = C_{\underline{\mathrm{w}}\otimes\varpi}$,

$$C_{\underline{\mathrm{w}}\otimes\varpi} = T_{\underline{\mathrm{w}}\otimes\varpi} = T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi - \sum_{\substack{\mu \in \mathrm{wt}\, V(\varpi)\setminus\{\varpi\} \\ v_i, v_j \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)}} ({}^{p\ell}t^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij} \cdot {}^{ij}\mathbb{TT}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

Also, we have $\mathrm{wt}\,\underline{\mathrm{w}} < \mathrm{wt}(\underline{\mathrm{w}}\otimes\varpi)$ and $\mathrm{wt}\,\underline{\mathrm{u}}_{\lambda+\mu} \leq \mathrm{wt}(\underline{\mathrm{w}}\otimes\varpi)$, for all μ such that $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$, so $T_{\underline{\mathrm{w}}} = C_{\underline{\mathrm{w}}}$ and $T_{\underline{\mathrm{u}}_{\lambda+\mu}} = C_{\underline{\mathrm{u}}_{\lambda+\mu}}$. Therefore,

$$\begin{aligned} {}^{ij}\mathbb{TT}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} &:= (T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \circ (\mathbb{D}({}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})) \circ T_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{j}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \circ (T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \\ &= (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \circ (\mathbb{D}({}^{i}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})) \circ C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ ({}^{j}ELL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}) \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi) \\ &= {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}. \end{aligned}$$

The claim follows from observing that $C_{\underline{\mathrm{w}}\otimes\varpi} = \mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\varpi}}_{\underline{\mathrm{w}},\varpi}$, which is a consequence of Lemmas 4.2.9 and 4.2.10 (i.e. clasp absorption and neutral absorption).

$\square$

LEMMA 4.3.5. *If $S_1'(\underline{\mathrm{w}},\varpi)$, then $S_2'(\underline{\mathrm{w}},\varpi)$.*

PROOF. Write $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$. For each μ such that $V(\lambda+\mu)$ is a summand of $V(\lambda)\otimes V(\varpi)$, consider the linear relation

$$\sum_p \xi_p \cdot {}^p LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = 0.$$

We can precompose the relation with $\mathbb{D}({}^\ell LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})$ for all $v_\ell \in \mathrm{Ker}_\lambda(V(\varpi)_\mu)$ to obtain a family of relations

$$\sum_p \xi_p \cdot {}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = 0.$$

By our hypothesis, we obtain

$$\sum_p \xi_p \cdot {}^{p\ell}t^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi} = 0,$$

and it follows from Lemma 4.3.1 that each $\xi_p = 0$. $\square$

LEMMA 4.3.6. *If $S_2'(\underline{\mathrm{w}},\varpi)$, then $S_3(\underline{\mathrm{w}},\varpi)$.*

PROOF. Let $\mu = \mathrm{wt}\,\underline{\mathrm{u}} - \mathrm{wt}\,\underline{\mathrm{w}}$. By combining Corollary 3.2.1 with Equation (4.5), we may deduce the following

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{Kar}\,\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_\varpi, C_{\underline{\mathrm{u}}}) &= \dim \mathrm{Hom}_{\mathrm{U_q}(\mathfrak{g}_2)}(V(\mathrm{wt}\,\underline{\mathrm{w}})\otimes V(\varpi), V(\mathrm{wt}\,\underline{\mathrm{u}})) \\ &= \dim \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu).\end{aligned}$$

The claim follows by observing that a linearly independent set with cardinality equal to the dimension of the vector space must be a spanning set. $\square$

LEMMA 4.3.7. *If $S_4(\underline{\mathrm{w}},\varpi)$, and $S_1'(\underline{\mathrm{x}},\psi)$ whenever $\mathrm{wt}(\underline{\mathrm{x}}\otimes\psi) \leq \mathrm{wt}\,\underline{\mathrm{w}}$, then $S_1'(\underline{\mathrm{w}},\varpi)$.*

PROOF. The right hand side of the equation ${}^{p\ell}\kappa^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} = {}^{p\ell}\rho^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}(\kappa)$ only involves terms ${}^{ij}\kappa^{\underline{\mathrm{y}}}_{\underline{\mathrm{x}},\psi}$ such that $\mathrm{wt}(\underline{\mathrm{x}}\otimes\psi) \leq \mathrm{wt}\,\underline{\mathrm{w}}$. If we write ${}^{p\ell}\rho^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}(t)$ to denote the same formula with each ${}^{ij}\kappa^{\underline{\mathrm{y}}}_{\underline{\mathrm{x}},\psi}$ replaced by ${}^{ij}t^{\underline{\mathrm{y}}}_{\underline{\mathrm{x}},\psi}$, then our hypotheses imply that

$${}^{p\ell}\kappa^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} = {}^{p\ell}\rho^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}(t).$$

Thus, to show that $S_1'(\underline{\mathrm{w}},\varpi)$ holds we must verify the following equality of rational functions in $\mathbb{C}(q)$:

$$^{p\ell}t^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} = {}^{p\ell}\rho^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}(t),$$

which we verified using the SAGE code included with the source file of the arXiv version of [**9**]. $\square$

EXAMPLE 4.3.3. *We take verification of Equation* (A.13) *as an example. In order to verify that*

$$t^{(1,0)}_{(a,b),\varpi_2} = \rho^{(1,0)}_{(a,b),\varpi_2}(t) = \frac{[7]}{[3]} - \frac{1}{t^{(-1,1)}_{(a-1,b),\varpi_1}} t^{(3,-1)}_{(a-2,b+1),\varpi_2} - \frac{1}{t^{(2,-1)}_{(a-1,b),\varpi_1}},$$

we first write the $t^{(s,t)}_{(x,y),\varpi}$'s explicitly using Equations (4.17), (4.9), (4.16), *and* (4.10) *to obtain*

$$\frac{[a+3][a+3b+6]}{[3][a][a+3b+3]} = \frac{[7]}{[3]} - \frac{1}{\left(-\dfrac{[a]}{[a-1]}\right)}\left(-\frac{[3b+6]}{[3b+3]}\right) - \frac{1}{\dfrac{[3b+3][a+3b+3]}{[3b][a+3b+2]}}.$$

We can rewrite this as:

$$\frac{(q^{a+3}-q^{-a-3})(q^{a+3b+6}-q^{-a-3b-6})(q-q^{-1})}{(q^3-q^{-3})(q^a-q^{-a})(q^{a+3b+3}-q^{-a-3b-3})} = \tag{4.35}$$

$$\frac{q^7-q^{-7}}{q^3-q^{-3}} - \frac{(q^{a-1}-q^{-a+1})(q^{3b+6}-q^{-3b-6})}{(q^a-q^{-a})(q^{(3b+3)-q^{-3b-3}})} - \frac{(q^{3b}-q^{-3b})(q^{a+3b+2}-q^{-a-3b-2})}{(q^{3b+3}-q^{-3b-3})(q^{a+3b+3}-q^{-a-3b-3})}. \tag{4.36}$$

Making the substitutions $A = q^a$ and $B = q^b$, we obtain:

$$\frac{(Aq^3-A^{-1}q^{-3})(AB^3q^6-A^{-1}B^{-3}q^{-6})(q-q^{-1})}{(q^3-q^{-3})(A-A^{-1})(AB^3q^3-A^{-1}B^{-3}q^{-3})} = \tag{4.37}$$

$$\frac{q^7-q^{-7}}{q^3-q^{-3}} - \frac{(Aq^{-1}-A^{-1}q)(B^3q^6-B^{-3}q^{-6})}{(A-A^{-1})(B^3q^3-B^{-3}q^{-3})} - \frac{(B^3-B^{-3})(AB^3q^2-A^{-1}B^{-3}q^{-2})}{(B^3q^3-B^{-3}q^{-3})(AB^3q^3-A^{-1}B^{-3}q^{-3})}. \tag{4.38}$$

Then we can use .simplify_full() in SAGE to simplify the rational function of A, B, and q, which is given by the difference of the left hand side and right hand side of the above equation. The result computed by SAGE is equal to 0, which tells us that Equation (A.13) *holds.*

LEMMA 4.3.8. *If $S_1(\underline{\mathrm{x}})$ for all $\underline{\mathrm{x}}$ such that* $\mathrm{wt}\,\underline{\mathrm{x}} < \mathrm{wt}(\underline{\mathrm{w}}\otimes\varpi)$*, and $S_1'(\underline{\mathrm{w}},\varpi)$, then $S_5(\underline{\mathrm{w}},\varpi)$.*

PROOF. By the definition of $T_{\underline{\mathrm{w}}\otimes\varpi}$ we find

$$T_{\underline{\mathrm{w}}\otimes\varpi} := T_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi} - \sum_{\substack{\mu\in \mathrm{wt}\,V(\varpi)\setminus\{\varpi\} \\ v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}} \left({}^{p\ell}t^{\underline{\mathrm{u}}_{\mathrm{wt}\,\underline{\mathrm{w}}+\mu}}_{\underline{\mathrm{w}},\varpi}\right)^{-1}_{ij} \cdot {}^{ij}\mathbb{TT}^{\underline{\mathrm{u}}_{\mathrm{wt}\,\underline{\mathrm{w}}+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

Then by our hypotheses, we deduce that

$$T_{\underline{\mathrm{w}}\otimes\varpi} = C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi} - \sum_{\substack{\mu\in \mathrm{wt}\,V(\varpi)\setminus\{\varpi\} \\ v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)}} \left({}^{p\ell}\kappa^{\underline{\mathrm{u}}_{\mathrm{wt}\,\underline{\mathrm{w}}+\mu}}_{\underline{\mathrm{w}},\varpi}\right)^{-1}_{ij} \cdot {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\mathrm{wt}\,\underline{\mathrm{w}}+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

□

LEMMA 4.3.9. *If $S_5(\underline{\mathrm{w}},\varpi)$, then $S_6(\underline{\mathrm{w}}\otimes\varpi)$.*

PROOF. By Lemma 4.2.7 we deduce the following multiplication formula for double ladders:

$${}^{p\ell}\mathbb{LL}^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} \circ {}^{rs}\mathbb{LL}^{\underline{\mathrm{v}}}_{\underline{\mathrm{w}},\varpi} = \delta_{\mathrm{wt}\,\underline{\mathrm{u}},\mathrm{wt}\,\underline{\mathrm{v}}}\, {}^{\ell r}\kappa^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi} \cdot {}^{ps}\mathbb{LL}^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}},\varpi}.$$

Using the expression for $T_{\underline{\mathrm{w}}\otimes\varpi}$ from $S_5(\underline{\mathrm{w}},\varpi)$ and the above formula, one can explicitly compute to verify that $T_{\underline{\mathrm{w}},\varpi}$ is idempotent.

□

LEMMA 4.3.10. *If $S_5(\underline{\mathrm{w}},\varpi)$ and $S_3(\underline{\mathrm{w}},\varpi)$, then $S_6'(\underline{\mathrm{w}},\varpi)$.*

PROOF. Write $\lambda = \mathrm{wt}\,\underline{\mathrm{w}}$. Let $\mu \in \mathrm{wt}\,V(\varpi)\setminus\{\varpi\}$ and let $\underline{\mathrm{u}} \in \mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ such that $\mathrm{wt}\,\underline{\mathrm{u}} = \lambda+\mu$. Let $D \in \mathrm{Hom}_{\mathbf{Web}_{\mathrm{q}}(\mathfrak{g}_2)}(\underline{\mathrm{w}}\otimes\varpi,\underline{\mathrm{u}})$. Consider the neutral diagram $\mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{u}}} : \underline{\mathrm{u}} \to \underline{\mathrm{u}}_{\lambda+\mu}$ and write $D' = \mathsf{H}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{u}}} \circ C_{\underline{\mathrm{u}}} \circ D$.

Combining that clasps are idempotent with Lemma 4.2.7 we find

$$C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ D' \circ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\nu}}_{\underline{\mathrm{w}},\varpi} = \delta_{\mu,\nu} \cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ D' \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) \circ {}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

By $S_3(\underline{\mathrm{w}},\varpi)$ there are scalars ξ_k such that

$$C_{\underline{\mathrm{u}}_{\lambda+\mu}} \circ D' \circ (C_{\underline{\mathrm{w}}} \otimes \mathrm{id}_{\varpi}) = \sum_{v_k\in \mathrm{Ker}_{\mathrm{wt}\,\underline{\mathrm{w}}}(V(\varpi)_\mu)} \xi_k \cdot {}^{k}LL^{\underline{\mathrm{u}}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}.$$

Thus, using $S_5(\underline{\mathrm{w}},\varpi)$ we can rewrite $C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ T_{\underline{\mathrm{w}}\otimes\varpi}$ as

$$\begin{aligned}
&C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ(C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})-\\
&\sum_{v_i,v_j\in \mathrm{Ker}_{\lambda}(V(\varpi)_{\mu})}({}^{p\ell}\kappa^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\cdot C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ(C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})\circ {}^{ij}\mathbb{LL}^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&=C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ(C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})-\sum_{v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{w}}}(V(\varpi)_{\mu})}\sum_k({}^{p\ell}\kappa^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\,\xi_k\cdot {}^{k}LL^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\circ {}^{ij}\mathbb{LL}^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&=C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ(C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})-\sum_{v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{w}}}(V(\varpi)_{\mu})}\sum_k({}^{p\ell}\kappa^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi})^{-1}_{ij}\,\xi_k\,{}^{ki}\kappa^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\cdot {}^{j}LL^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&=C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ(C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})-\sum_{v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{w}}}(V(\varpi)_{\mu})}\sum_k\xi_k\delta_{k,j}\cdot {}^{j}LL^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&=C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'\circ(C_{\underline{\mathrm{w}}}\otimes \mathrm{id}_{\varpi})-\sum_{v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{w}}}(V(\varpi)_{\mu})}\xi_j\cdot {}^{j}LL^{\mathrm{u}_{\lambda+\mu}}_{\underline{\mathrm{w}},\varpi}\\
&=0.
\end{aligned}$$

Using Lemmas 4.2.9 and 4.2.10 it is not hard to see that $C_{\underline{\mathrm{u}}}\circ \mathsf{H}^{\underline{\mathrm{u}}}_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ C_{\underline{\mathrm{u}}_{\lambda+\mu}}\circ D'=C_{\underline{\mathrm{u}}}\circ D$, and it follows that $C_{\underline{\mathrm{u}}}\circ D\circ T_{\underline{\mathrm{w}}\otimes\varpi}=0$. $\square$

LEMMA 4.3.11. *Let $\underline{\mathrm{w}}\in \boldsymbol{Web}_{\mathrm{q}}(\mathfrak{g}_2)$ and let ϖ be a fundamental weight. If $S_2(\underline{\mathrm{x}},\psi)$ and $S_2'(\underline{\mathrm{x}},\psi)$ whenever* $\mathrm{wt}(\underline{\mathrm{x}}\otimes\psi)\le \mathrm{wt}\,\underline{\mathrm{w}}$, *then $S_4(\underline{\mathrm{w}},\varpi)$.*

PROOF. Consider $\underline{\mathrm{x}},\psi$ such that $\mathrm{wt}(\underline{\mathrm{x}}\otimes\psi)\le \mathrm{wt}\,\underline{\mathrm{w}}$. By $S_2(\underline{\mathrm{x}},\psi)$ we obtain the following.

$$C_{\underline{\mathrm{x}}}\otimes \mathrm{id}_{\psi}=\sum_{\substack{\mu\in \mathrm{wt}\,V(\psi)\\ v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{x}}}(V(\psi)_{\mu})}}{}^{ij}\xi^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\cdot {}^{ij}\mathbb{LL}^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}$$

Postcomposing with ${}^{p}LL^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}$ and using Lemma 4.2.7 results in the next sequence of equalities.

$$\begin{aligned}
{}^{p}LL^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}&=\sum_{v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{x}}}(V(\psi)_{\mu})}{}^{ij}\xi^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\cdot {}^{p}LL^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\circ {}^{ij}\mathbb{LL}^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\\
&=\sum_{v_i,v_j\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{x}}}(V(\psi)_{\mu})}{}^{ij}\xi^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\cdot {}^{pi}\kappa^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\cdot {}^{j}LL^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}
\end{aligned}$$

By $S_2'(\underline{\mathrm{x}},\psi)$ it follows that

$$\sum_{v_i\in \mathrm{Ker}_{\mathrm{wt}\underline{\mathrm{x}}}(V(\psi)_{\mu})}{}^{ij}\xi^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}\cdot {}^{pi}\kappa^{\mathrm{u}_{\mathrm{wt}\underline{\mathrm{x}}+\mu}}_{\underline{\mathrm{x}},\psi}=\delta_{jp}.$$

From Definition 4.2.10, we can use that clasps are idempotent to write

$$LL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}} \circ \mathbb{D}(LL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}}) = C_{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}} \circ ELL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}} \circ (C_{\underline{x}} \otimes \mathrm{id}_{\psi}) \circ \mathbb{D}(ELL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}}) \circ C_{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}}.$$

By Definition 4.2.9, $ELL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}}$ is a neutral map. Therefore, Lemma 4.2.11, Lemma 4.2.10, and Lemma 4.2.9, along with clasps being idempotent, implies

$$LL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}} \circ \mathbb{D}(LL_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}}) = C_{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}}.$$

It follows that $\kappa_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}} = 1$. Thus, $\xi_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\psi}} = 1$ and

$$C_{\underline{x}\otimes\psi} = C_{\underline{x}} \otimes \mathrm{id}_{\psi} - \sum_{\substack{\mu \in \mathrm{wt}\,V(\psi)\setminus\{\psi\} \\ v_i, v_j \in \mathrm{Ker}_{\mathrm{wt}\,\underline{x}}(V(\psi)_{\mu})}} ({}^{p\ell}\kappa_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\mu}})^{-1}_{ij} \cdot {}^{ij}\mathbb{LL}_{\underline{x},\psi}^{\underline{u}_{\mathrm{wt}\,\underline{x}+\mu}}. \tag{4.39}$$

Observe that

$$\begin{aligned} {}^{p\ell}\kappa_{\underline{w},\varpi}^{\underline{u}} C_{\underline{u}} &= {}^{p}LL_{\underline{w},\varpi}^{\underline{u}} \circ \mathbb{D}({}^{\ell}LL_{\underline{w},\varpi}^{\underline{u}}) \\ &= C_{\underline{u}} \circ {}^{p}ELL_{\underline{w},\varpi}^{\underline{u}} \circ (C_{\underline{w}} \otimes \mathrm{id}_{\varpi}) \circ \mathbb{D}({}^{\ell}ELL_{\underline{w},\varpi}^{\underline{u}}) \circ C_{\underline{u}}. \end{aligned}$$

Then use Equation (4.39) for $\underline{w} = \underline{x} \otimes \psi$ to rewrite the $C_{\underline{w}}$ term on the right hand side. This new sum will reduce to a scalar multiple of $C_{\underline{u}}$ by repeatedly applying graphical reductions or by replacing another clasp, necessarily of the form $C_{\underline{y}\otimes\varpi}$ for some $\underline{y}, \varpi$ such that $\mathrm{wt}(\underline{y}\otimes\varpi) \le \mathrm{wt}\,\underline{w}$, using Equation (4.39). The exact form of the coefficient is determined via the calculations in Appendix A.4 of the arXiv version of [**9**], where it is shown to be equal to ${}^{p\ell}\rho_{\underline{w},\varpi}^{\underline{u}}(\kappa)$. Therefore, ${}^{p\ell}\kappa_{\underline{w},\varpi}^{\underline{u}} C_{\underline{u}} = {}^{p\ell}\rho_{\underline{w},\varpi}^{\underline{u}}(\kappa) C_{\underline{u}}$ and the desired result follows from looking at the neutral coefficient of each map. □

EXAMPLE 4.3.4. *The above argument is best illustrated by example. Consider* $\underline{w}$ *with* $\mathrm{wt}\,\underline{w} = (a,b)$ *and assume* $S_2(\underline{x},\psi)$ *and* $S_2'(\underline{x},\psi)$ *whenever* $\mathrm{wt}(\underline{x}\otimes\psi) \le \mathrm{wt}\,\underline{w}$. *Note that*

$$\rho_{(a,b),\varpi_1}^{(-1,1)} := -[2] - \frac{1}{x_{(a-1,b),\varpi_1}^{(-1,1)}}.$$

We will show that $\kappa_{\underline{w},\varpi_1}^{\underline{u}_{(a-1,b+1)}} = \rho_{(a,b),\varpi_1}^{(-1,1)}(\kappa)$. *Let* $\mathrm{wt}(\underline{v}\otimes\varpi_1) = \mathrm{wt}\,\underline{w}$. *It follows from Lemma 4.3.2 that* $\kappa_{\underline{w},\varpi_1}^{\underline{u}_{(a-1,b+1)}} = \kappa_{\underline{v}\otimes\varpi_1,\varpi_1}^{\underline{u}_{(a-1,b+1)}}$. *By definition we have that* $\kappa_{\underline{v}\otimes\varpi_1,\varpi_1}^{\underline{u}_{(a-1,b+1)}} C_{\underline{u}_{(a-1,b+1)}}$ *is equal to*

$$C_{\underline{u}_{(a-1,b+1)}} \circ ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v}\otimes\varpi_1,\varpi_1} \circ \left(C_{\underline{v}\otimes\varpi_1}\otimes \mathrm{id}_{\varpi_1}\right)\circ \mathbb{D}(ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v}\otimes\varpi_1,\varpi_1})\circ C_{\underline{u}_{(a-1,b+1)}}.$$

As in the first half of the proof of Lemma 4.3.11, our hypotheses allow us to write

$$C_{\underline{v}\otimes\varpi_1} = C_{\underline{v}}\otimes \mathrm{id}_{\varpi_1} - \sum_{\substack{\mu\in \mathrm{wt}\,V(\varpi_1)\setminus\{\varpi_1\} \\ v_i,v_j\in Ker_{\mathrm{wt}\,\underline{v}}(V(\varpi_1)_\mu)}} ({}^{p\ell}\kappa^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1})^{-1}_{ij}\cdot {}^{ij}\mathbb{LL}^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1}.$$

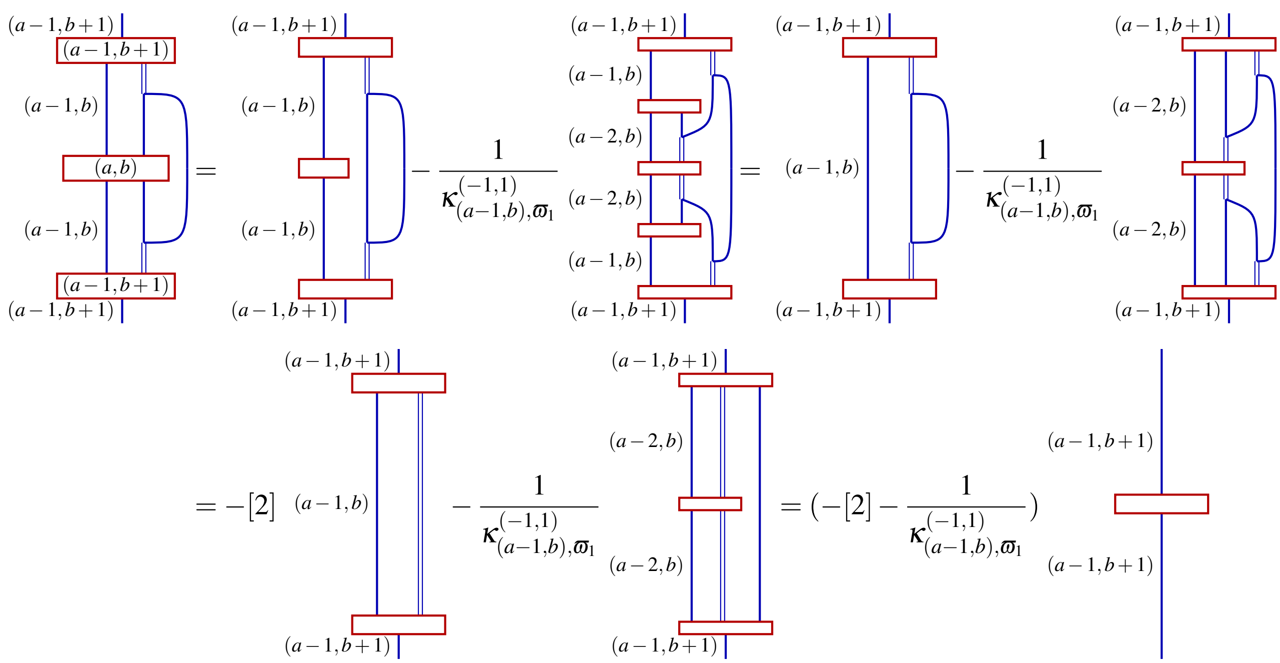
Using Lemma 4.2.8, we observe that if $\mu\neq(-1,1)$, then

$$C_{\underline{u}_{(a-1,b+1)}} \circ ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v}\otimes\varpi_1,\varpi_1} \circ \left(({}^{p\ell}\kappa^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1})^{-1}_{ij}\cdot {}^{ij}\mathbb{LL}^{\underline{u}_{\mathrm{wt}\,\underline{v}+\mu}}_{\underline{v},\varpi_1}\otimes \mathrm{id}_{\varpi_1}\right)\circ \mathbb{D}(ELL^{\underline{u}_{(a-1,b+1)}}_{\underline{v}\otimes\varpi_1,\varpi_1})\circ C_{\underline{u}_{(a-1,b+1)}}$$

is zero. Finally, applying web relations (and properties of clasps) we find

$$\kappa^{(-1,1)}_{(a,b),\varpi_1} C_{\underline{u}_{(a-1,b+1)}} = \left(-[2]-\frac{1}{\kappa^{(-1,1)}_{(a-1,b),\varpi_1}}\right)C_{\underline{u}_{(a-1,b+1)}}.$$

We conclude with a schematic of the graphical calculations involved.



Finally, we combine the previous lemmas to deduce the result of our main theorem.

PROOF OF THEOREM 4.3.1. We will prove the result by induction on $\mathrm{wt}\,\underline{\mathrm{w}}$ with respect to $\leq$. The base case follows from observing that $T_\emptyset$ is 1 times the empty diagram, which agrees with $C_\emptyset$. Assume that $S_1(\underline{\mathrm{x}})$ holds for all $\underline{\mathrm{x}}$ such that $\mathrm{wt}\,\underline{\mathrm{x}} < \mathrm{wt}\,\underline{\mathrm{w}}$ and assume that $S_1'(\underline{\mathrm{y}}, \psi)$ holds for all $\underline{\mathrm{y}}, \psi$ such that $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) < \mathrm{wt}\,\underline{\mathrm{w}}$. We will show $S_1(\underline{\mathrm{w}})$ and $S_1'(\underline{\mathrm{w}}', \varpi)$, where $\underline{\mathrm{w}} = \underline{\mathrm{w}}' \otimes \varpi$.

Consider $\underline{\mathrm{y}}, \psi$ such that $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) < \mathrm{wt}\,\underline{\mathrm{w}}$. Then $S_1(\underline{\mathrm{x}})$ holds whenever $\mathrm{wt}\,\underline{\mathrm{x}} \leq \mathrm{wt}(\underline{\mathrm{y}} \otimes \psi)$, and by Lemma 4.3.4 we deduce $S_2(\underline{\mathrm{y}}, \psi)$. Thus, $S_2(\underline{\mathrm{y}}, \psi)$ holds for all $\underline{\mathrm{y}}, \psi$ such that $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) < \mathrm{wt}\,\underline{\mathrm{w}}$.

If $\underline{\mathrm{y}}, \psi$ is such that $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) < \mathrm{wt}\,\underline{\mathrm{w}}$, then our inductive hypothesis also says that $S_1'(\underline{\mathrm{y}}, \psi)$ holds. Along with Lemmas 4.3.5 and 4.3.6, this implies $S_2'(\underline{\mathrm{y}}, \psi)$ and $S_3(\underline{\mathrm{y}}, \psi)$. Hence, $S_2'(\underline{\mathrm{y}}, \psi)$ and $S_3(\underline{\mathrm{y}}, \psi)$ holds for all $\underline{\mathrm{y}}, \psi$ such that $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) < \mathrm{wt}\,\underline{\mathrm{w}}$.

For $\underline{\mathrm{x}}$ such that $\mathrm{wt}\,\underline{\mathrm{x}} < \mathrm{wt}\,\underline{\mathrm{w}}$, we have $S_2(\underline{\mathrm{y}}, \psi)$ and $S_2'(\underline{\mathrm{y}}, \psi)$ whenever $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) \leq \mathrm{wt}\,\underline{\mathrm{x}}$. So from Lemma 4.3.11 we deduce $S_4(\underline{\mathrm{x}}, \varpi)$ for all $\underline{\mathrm{x}}$ such that $\mathrm{wt}\,\underline{\mathrm{x}} < \mathrm{wt}\,\underline{\mathrm{w}}$ and for arbitrary ϖ.

If $\underline{\mathrm{w}} = \underline{\mathrm{w}}' \otimes \varpi$, then $\mathrm{wt}\,\underline{\mathrm{w}}' < \mathrm{wt}\,\underline{\mathrm{w}}$, so $S_4(\underline{\mathrm{w}}', \varpi)$. Also, if $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) \leq \mathrm{wt}\,\underline{\mathrm{w}}'$, then $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) < \mathrm{wt}\,\underline{\mathrm{w}}$ so $S_1'(\underline{\mathrm{y}}, \psi)$ holds whenever $\mathrm{wt}(\underline{\mathrm{y}} \otimes \psi) \leq \mathrm{wt}\,\underline{\mathrm{w}}'$. Thus, Lemma 4.3.7 implies $S_1'(\underline{\mathrm{w}}', \varpi)$.

At this point, we know that $S_1(\underline{\mathrm{x}})$ whenever $\mathrm{wt}\,\underline{\mathrm{x}} < \mathrm{wt}(\underline{\mathrm{w}}' \otimes \varpi)$ and that $S_1'(\underline{\mathrm{w}}', \varpi)$ holds. Therefore, Lemma 4.3.8 implies that $S_5(\underline{\mathrm{w}}', \varpi)$ holds too. Then from Lemma 4.3.9 we deduce $S_6(\underline{\mathrm{w}}' \otimes \varpi)$ is true.

Moreover, since $S_1'(\underline{\mathrm{w}}', \varpi)$ is true, Lemmas 4.3.5 and 4.3.6 together imply $S_3(\underline{\mathrm{w}}', \varpi)$. Therefore, we can use Lemma 4.3.10 to deduce $S_6'(\underline{\mathrm{w}}', \varpi)$.

If we show that $T_{\underline{\mathrm{w}}' \otimes \varpi} \neq 0$, then Definition 4.2.2 and Lemma 4.2.3 will tell us that $S_6(\underline{\mathrm{w}}' \otimes \varpi)$ and $S_6'(\underline{\mathrm{w}}', \varpi)$ imply $S_1(\underline{\mathrm{w}}' \otimes \varpi)$, so we are then done by induction. To see that $T_{\underline{\mathrm{w}}' \otimes \varpi}$ is not 0, we apply Φ from Theorem 3.2.1 and evaluate on a weight vector in $V(\underline{\mathrm{w}}' \otimes \varpi)_{\mathrm{wt}\,\underline{\mathrm{w}}' + \varpi}$. Using $S_5(\underline{\mathrm{w}}', \varpi)$, along with the observations that for all $\underline{\mathrm{u}}$ such that $\mathrm{wt}\,\underline{\mathrm{u}} - \mathrm{wt}\,\underline{\mathrm{w}}' = \mu \in \mathrm{wt}\,V(\varpi) \backslash \{\varpi\}$, the map $\Phi\big({}^{ij}\mathbb{LL}^{\underline{\mathrm{u}}}_{\underline{\mathrm{w}}', \varpi}\big)$ acts as zero on $V(\underline{\mathrm{w}}' \otimes \varpi)_{\mathrm{wt}\,\underline{\mathrm{w}}' + \varpi}$ (since these maps factor through representations which do not have $\mathrm{wt}\,\underline{\mathrm{w}}' + \varpi$ as a weight) and $\Phi\big(C_{\underline{\mathrm{w}}'} \otimes \mathrm{id}_\varpi\big)$ acts on $V(\underline{\mathrm{w}}' \otimes \varpi)_{\mathrm{wt}\,\underline{\mathrm{w}}' + \varpi}$ as multiplication by 1, we deduce that $T_{\underline{\mathrm{w}}' \otimes \varpi}$ is non-zero. $\square$

APPENDIX A

Appendix for G_2 clasp expansions

A.1. Relations for computing the ${}^{p\ell}t^{(0,0)}_{(a,b),\varpi_2}$ coefficients

(A.1)

$$\frac{1}{[2]}\,{}^{2,2}t^{(0,0)}_{(a,b),\varpi_2} + {}^{1,2}t^{(0,0)}_{(a,b),\varpi_2} = -\frac{[3][a+2][3a+6b+9]}{[2][3b][a+3b+3][2a+3b+4][3a+3b+3][3a+6b+6]}\cdot$$
$$\left([3a+3b+6][2a+3b+5]+[a+3b+4][3b+3]+\frac{[3a+3b+6][3b+3][2]^2}{[3]}+[a+4]-[a-2]\right)$$

(A.2)

$${}^{1,1}t^{(0,0)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]}+\frac{[2][a+2][a+3b+5][2a+3b+2]}{[3][a+1][a+3b+4][2a+3b+4]}+\frac{[a-2][3b+6][2a+3b+2][3a+3b]}{[a][3b+3][2a+3b+3][3a+3b+3]}$$
$$+\frac{[a-1][3b+6][a+3b+5][a+3b+6][2a+3b+6]}{[3][a][3b+3][a+3b+3][a+3b+4][2a+3b+3]}+\frac{[3b][a+3b+1][2a+3b+2][3a+6b+3]}{[3b+3][a+3b+3][2a+3b+3][3a+6b+6]}$$
$$+\frac{[a+2][a+3][3b][a+3b+2][2a+3b+6]}{[3][a][a+1][3b+3][a+3b+3][2a+3b+3]}$$

(A.3)

$$\frac{{}^{2,2}t^{(0,0)}_{(a,b),\varpi_2}}{\mathscr{D}_{(a,b)}}\left(\frac{[4][6]^2}{[2][3]^2[12]}\right)^2+2\cdot\frac{{}^{1,2}t^{(0,0)}_{(a,b),\varpi_2}}{\mathscr{D}_{(a,b)}}\frac{[4]^2[6]^3}{[2]^2[3]^2[12]^2}+\frac{{}^{1,1}t^{(0,0)}_{(a,b),\varpi_2}}{\mathscr{D}_{(a,b)}}\left(\frac{[4][6]}{[2][12]}\right)^2$$
$$=-\frac{[6][8][15]}{[3][5][12]}+\frac{[a+3][3b][a+3b+2][a+3b+3][2a+3b+4]}{[3][a+2][3b+3][a+3b+4][a+3b+5][2a+3b+7]}$$
$$+\frac{[a-1][a][3b+6][a+3b+6][2a+3b+4]}{[3][a+1][a+2][3b+3][a+3b+5][2a+3b+7]}+\frac{[2][a][a+3b+3][2a+3b+8]}{[3][a+1][a+3b+4][2a+3b+6]}$$
$$+\frac{[a+4][3b][2a+3b+8][3a+3b+12]}{[a+2][3b+3][2a+3b+7][3a+3b+9]}+\frac{[3b+6][a+3b+7][2a+3b+8][3a+6b+15]}{[3b+3][a+3b+5][2a+3b+7][3a+6b+12]}$$

A.2. Recursions for the coefficients

We write $x_{(a,b)}^{(c,d)} := x_{(a,b),\varpi_1}^{(c,d)}$ and $y_{(a,b)}^{(c,d)} := x_{(a,b),\varpi_2}^{(c,d)}$.

$$\rho_{(a,b),\varpi_1}^{(1,0)} = 1 \tag{A.4}$$

$$\rho_{(a,b),\varpi_1}^{(-1,1)} = -[2] - \frac{1}{x_{(a-1,b)}^{(-1,1)}} \tag{A.5}$$

$$\rho_{(a,b),\varpi_1}^{(2,-1)} = \frac{[7]}{[3]} - \frac{1}{y_{(a,b-1)}^{(3,-1)}} x_{(a+3,b-2)}^{(-1,1)} - \frac{1}{y_{(a,b-1)}^{(1,0)}} \frac{1}{[3]^2} \tag{A.6}$$

$$\rho_{(a,b),\varpi_1}^{(0,0)} = -\frac{[3][8]}{[2][4]} - \frac{x_{(a-2,b+1)}^{(2,-1)}}{x_{(a-1,b)}^{(-1,1)}} - \frac{x_{(a+1,b-1)}^{(-1,1)}}{x_{(a-1,b)}^{(2,-1)}} - \frac{1}{[2]^2 x_{(a-1,b)}^{(0,0)}} \tag{A.7}$$

$$\begin{aligned}\rho_{(a,b),\varpi_1}^{(-2,1)} = {} & -\frac{[3][8]}{[2][4]} x_{(a-1,b)}^{(-1,1)} - \frac{1}{x_{(a-1,b)}^{(-1,1)}} \left(\frac{1}{x_{(a-2,b)}^{(-1,1)}} \right)^2 x_{(a-2,b+1)}^{(0,0)} - \frac{1}{x_{(a-1,b)}^{(0,0)}} \left(\frac{[3]}{[2]} + \frac{1}{x_{(a-2,b)}^{(-1,1)}} \right)^2 x_{(a-1,b)}^{(-1,1)} \\ & - \frac{1}{x_{(a-1,b)}^{(-2,1)}} \left(-\frac{[3]}{[2]} + \frac{1}{x_{(a-2,b)}^{(-1,1)}} \frac{1}{x_{(a-3,b)}^{(-1,1)}} \frac{1}{[2]} \right)^2\end{aligned} \tag{A.8}$$

$$\begin{aligned}\rho_{(a,b),\varpi_1}^{(1,-1)} = {} & -\frac{[6][8][15]}{[3][5][12]} - \frac{1}{y_{(a,b-1)}^{(3,-1)}} x_{(a+3,b-2)}^{(-2,1)} - \frac{1}{y_{(a,b-1)}^{(1,0)}} \left(\frac{[2]}{[3]} \right)^2 x_{(a+1,b-1)}^{(0,0)} - \frac{1}{y_{(a,b-1)}^{(2,-1)}} \left(\frac{1}{[3]} \right)^2 x_{(a+2,b-2)}^{(-1,1)} \\ & - \frac{1}{y_{(a,b-1)}^{(-1,1)}} \left(\frac{1}{[3]} \right)^2 x_{(a-1,b)}^{(2,-1)} - \frac{1}{{}_{1,1}y_{(a,b-1)}^{(0,0)}} \left(\frac{[4][6]^2}{[2][3]^2[12]} \right)^2 - \frac{1}{{}_{2,2}y_{(a,b-1)}^{(0,0)}} \left(\frac{[4][6]}{[2][12]} \right)^2 \\ & + \left(\frac{1}{{}_{1,2}y_{(a,b-1)}^{(0,0)}} + \frac{1}{{}_{2,1}y_{(a,b-1)}^{(0,0)}} \right) \frac{[4][6]^2}{[2][3]^2[12]} \frac{[4][6]}{[2][12]}\end{aligned} \tag{A.9}$$

$$\begin{aligned}\rho_{(a,b),\varpi_1}^{(-1,0)} = {} & \frac{[2][7][12]}{[4][6]} - \frac{1}{x_{(a-1,b)}^{(-1,1)}} x_{(a-2,b+1)}^{(1,-1)} - \frac{1}{x_{(a-1,b)}^{(2,-1)}} x_{(a+1,b-1)}^{(-2,1)} - 1 - \frac{1}{x_{(a-1,b)}^{(-2,1)}} x_{(a-3,b+1)}^{(2,-1)} \\ & - \frac{1}{x_{(a-1,b)}^{(1,-1)}} x_{(a,b-1)}^{(-1,1)} - \frac{1}{x_{(a-1,b)}^{(-1,0)}}\end{aligned} \tag{A.10}$$

$$\rho^{(0,1)}_{(a,b),\varpi_2} = 1 \tag{A.11}$$

$$\rho^{(3,-1)}_{(a,b),\varpi_2} = -\frac{[6]}{[3]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} \tag{A.12}$$

$$\rho^{(1,0)}_{(a,b),\varpi_2} = \frac{[7]}{[3]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} y^{(3,-1)}_{(a-2,b+1)} - \frac{1}{x^{(2,-1)}_{(a-1,b)}} \tag{A.13}$$

$$\rho^{(-1,1)}_{(a,b),\varpi_2} = -\frac{[7]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-2,b)}}\frac{[7]}{[3]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}}\left(\frac{1}{x^{(-1,1)}_{(a-2,b)}}\right)^2 y^{(1,0)}_{(a-2,b+1)} - \frac{1}{x^{(0,0)}_{(a-1,b)}}\left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-2,b)}}\right)^2 \tag{A.14}$$

(A.15)

$$\rho^{(2,-1)}_{(a,b),\varpi_2} = \frac{[8][10]}{[3]^2[5]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} y^{(-1,1)}_{(a+3,b-2)} - \frac{1}{y^{(1,0)}_{(a,b-1)}}\frac{[2]^2}{[3]^2} y^{(1,0)}_{(a+1,b-1)} - \frac{1}{y^{(-1,1)}_{(a,b-1)}}\frac{1}{[3]^2} y^{(3,-1)}_{(a-1,b)} - \frac{1}{y^{(2,-1)}_{(a,b-1)}}\frac{1}{[3]^2}$$

(A.16)

$$\rho^{(-3,2)}_{(a,b),\varpi_2} = \frac{[7]}{[3]}\left(-\frac{[3]}{[2]}x^{(-1,1)}_{(a-2,b)} + \frac{1}{x^{(-1,1)}_{(a-2,b)}\left(x^{(-1,1)}_{(a-3,b)}\right)^2}\left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\right)\right) - \frac{y^{(-1,1)}_{(a-2,b+1)}}{x^{(-1,1)}_{(a-1,b)}\left(x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}\right)^2}$$

$$- \frac{1}{x^{(-2,1)}_{(a-1,b)}}\left(-\frac{[3]}{[2]}x^{(-1,1)}_{(a-2,b)} + \frac{1}{x^{(-1,1)}_{(a-2,b)}}\left(\frac{1}{x^{(-1,1)}_{(a-3,b)}}\right)^2\left(\frac{[3]}{[2]} + \frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\right)\right)^2$$

(A.17)

$${}^{1,1}\rho^{(0,0)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} y^{(2,-1)}_{(a-2,b+1)} - \frac{1}{x^{(2,-1)}_{(a-1,b)}} y^{(-1,1)}_{(a+1,b-1)} - \frac{1}{x^{(0,0)}_{(a-1,b)}} y^{(1,0)}_{(a-1,b)} - \frac{1}{x^{(-2,1)}_{(a-1,b)}} y^{(3,-1)}_{(a-3,b+1)} - \frac{1}{x^{(1,-1)}_{(a-1,b)}}$$

(A.18)

$${}^{2,2}\rho^{(0,0)}_{(a,b),\varpi_2} = -\frac{[4][6]^2[18]}{[3][9][12]} - \frac{1}{y^{(3,-1)}_{(a,b-1)}} y^{(-3,2)}_{(a+3,b-2)} - \frac{1}{y^{(1,0)}_{(a,b-1)}} y^{(-1,1)}_{(a+1,b-1)} - \frac{1}{y^{(-1,1)}_{(a,b-1)}} y^{(1,0)}_{(a-1,b)} - \frac{1}{y^{(-3,2)}_{(a,b-1)}} y^{(3,-1)}_{(a-3,b+1)}$$

$$- \frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\left(\frac{[4][6]}{[2][12]}\right)^2 - \left(\frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}} + \frac{{}^{2,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\right)\frac{[4]^2[6]^2}{[2][12]^2} - \frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\left(\frac{[4][6]}{[12]}\right)^2$$

(A.19)

$$
{}^{1,2}\rho^{(0,0)}_{(a,b),\varpi_2} = \frac{[4][6]^2[18]}{[2][3][9][12]} + \frac{y^{(-1,1)}_{(a+1,b-1)}}{y^{(1,0)}_{(a,b-1)}}\left(\frac{[4]}{[3]} + \frac{1}{x^{(-1,1)}_{(a-1,b-1)}}\right) + \frac{y^{(1,0)}_{(a-1,b)}}{y^{(-1,1)}_{(a,b-1)}}\left(\frac{[4]}{[3]} - \frac{[2]}{[3]x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}}\right)
$$

$$
+ \frac{y^{(3,-1)}_{(a-3,b+1)}}{y^{(-3,2)}_{(a,b-1)}}\left(\frac{[4]}{[3]} + \frac{1}{[3]x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}x^{(-1,1)}_{(a-3,b-1)}}\right) + \frac{{}^{2,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\left(\frac{[4][6]}{[2][12]}\right)^2 + \frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\frac{[4]^2[6]^2}{[2][12]^2}
$$

$$
- \frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\frac{[4][6]}{[2][12]}\left(\frac{1}{[2]} + \frac{1}{[3]x^{(-1,1)}_{(a-1,b-1)}} - \frac{[4][6]}{[2]^2[12]}\right) - \frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\frac{[4][6]}{[12]}\left(\frac{1}{[2]} + \frac{1}{[3]x^{(-1,1)}_{(a-1,b-1)}} - \frac{[4][6]}{[2]^2[12]}\right)
$$

(A.20)

$$
\rho^{(3,-2)}_{(a,b),\varpi_2} = -\frac{[4][6]^2[18]}{[3][9][12]}y^{(3,-1)}_{(a,b-1)} - \frac{1}{y^{(3,-1)}_{(a,b-1)}}\left([3]^2\ {}^{1,1}y^{(0,0)}_{(a+3,b-2)} - \frac{2\cdot[3]}{y^{(3,-1)}_{(a,b-2)}}\ {}^{1,2}y^{(0,0)}_{(a+3,b-2)} + \frac{{}^{2,2}y^{(0,0)}_{(a+3,b-2)}}{\left(y^{(3,-1)}_{(a,b-2)}\right)^2}\right)
$$

$$
-\frac{1}{y^{(1,0)}_{(a,b-1)}}y^{(2,-1)}_{(a+1,b-1)} - \frac{1}{y^{(2,-1)}_{(a,b-1)}}\left(y^{(3,-1)}_{(a,b-1)}\right)^2 y^{(1,0)}_{(a+2,b-2)} - \frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\left(\frac{[4][6]}{[2][12]}\right)^2 y^{(3,-1)}_{(a,b-1)}
$$

$$
-2\cdot\frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\frac{[4][6]}{[2][12]}\left(\frac{[4][6]}{[12]} + y^{(3,-1)}_{(a,b-1)}\right)y^{(3,-1)}_{(a,b-1)} - \frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\left(\frac{[4][6]}{[12]} + y^{(3,-1)}_{(a,b-1)}\right)^2 y^{(3,-1)}_{(a,b-1)}
$$

$$
-\frac{1}{y^{(3,-2)}_{(a,b-1)}}\left(\frac{[4][6]}{[12]} - \frac{[6]}{[3]} + \frac{1}{y^{(3,-1)}_{(a,b-2)}}\left(\frac{[4][6]^2}{[3][12]} - 1 + \frac{1}{y^{(3,-1)}_{(a,b-3)}}\frac{[4][6]}{[12]}\right)\right)^2
$$

(A.21)

$$
\rho^{(-2,1)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]}x^{(-1,1)}_{(a-1,b)} - \frac{1}{x^{(-1,1)}_{(a-1,b)}}\left(\frac{{}^{1,1}y^{(0,0)}_{(a-2,b+1)}}{\left(x^{(-1,1)}_{(a-2,b)}\right)^2} - \frac{2\cdot\left({}^{1,2}y^{(0,0)}_{(a-2,b+1)}\right)}{x^{(-1,1)}_{(a-2,b)}} + {}^{2,2}y^{(0,0)}_{(a-2,b+1)}\right)
$$

$$
-\frac{1}{x^{(2,-1)}_{(a-1,b)}}y^{(-3,2)}_{(a+1,b-1)} - \frac{1}{x^{(0,0)}_{(a-1,b)}}\frac{y^{(-1,1)}_{(a-1,b)}}{\left(x^{(-1,1)}_{(a-2,b)}\right)^2} - \frac{1}{x^{(-2,1)}_{(a-1,b)}}\frac{y^{(1,0)}_{(a-3,b+1)}}{\left(x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}\right)^2} - \frac{1}{x^{(-1,0)}_{(a-1,b)}}\left(x^{(-1,1)}_{(a-1,b)}\right)^2
$$

(A.22)

$$\rho^{(1,-1)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]} x^{(2,-1)}_{(a-1,b)} - \frac{1}{x^{(-1,1)}_{(a-1,b)}} y^{(3,-2)}_{(a-2,b+1)}$$

$$- \frac{1}{x^{(2,-1)}_{(a-1,b)}} \Bigg(\Bigg(\frac{[2]^2}{[3]} - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]^2} + \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \Bigg)^2 {}^{1,1}y^{(0,0)}_{(a+1,b-1)} + \Bigg(\frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \Bigg)^2 {}^{2,2}y^{(0,0)}_{(a+1,b-1)}$$

$$- 2 \cdot \Bigg(\frac{[2]^2}{[3]} - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]^2} + \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \Bigg) \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} {}^{1,2}y^{(0,0)}_{(a+1,b-1)} \Bigg)$$

$$- \frac{1}{x^{(0,0)}_{(a-1,b)}} \Bigg(\frac{[2]}{[3]} + \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \Bigg(\frac{1}{[2]} + \frac{1}{x^{(-1,1)}_{(a,b-2)}} \frac{1}{x^{(-1,1)}_{(a-1,b-2)}} \Bigg(- \frac{[3]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \Bigg) \Bigg)$$

$$- \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]} \Bigg(\frac{1}{[2][3]} - \frac{[3]}{[2]} - \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \Bigg) \Bigg)^2 y^{(2,-1)}_{(a-1,b)}$$

$$- \frac{1}{x^{(1,-1)}_{(a-1,b)}} \Bigg(- \frac{[4]}{[3]} - \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \frac{1}{x^{(-1,1)}_{(a,b-2)}} - \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{[2]}{[3]^2} \Bigg)^2 y^{(1,0)}_{(a,b-1)}$$

$$- \frac{1}{x^{(-1,0)}_{(a-1,b)}} \Bigg(- \frac{[4]}{[3]} - \frac{1}{y^{(3,-1)}_{(a-1,b-1)}} \Bigg(\frac{1}{[3]} + \frac{1}{x^{(-1,1)}_{(a+1,b-2)}} \frac{1}{x^{(-1,1)}_{(a,b-2)}} \frac{1}{x^{(-1,1)}_{(a-1,b-2)}} \Bigg(\frac{[4]}{[3]} + \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \Bigg) \Bigg)$$

$$+ \frac{1}{y^{(1,0)}_{(a-1,b-1)}} \frac{1}{[3]} \Bigg(\frac{[4]}{[3]} + \frac{1}{x^{(-1,1)}_{(a-2,b-1)}} \Bigg) \Bigg)^2 y^{(3,-1)}_{(a-2,b)}$$

(A.23)

$$\rho^{(-1,0)}_{(a,b),\varpi_2} = -\frac{[6][8][15]}{[3][5][12]} x^{(0,0)}_{(a-1,b)} \quad - \frac{1}{x^{(-1,1)}_{(a-1,b)}} \frac{1}{\Big(x^{(-1,1)}_{(a-2,b)}\Big)^2} y^{(1,-1)}_{(a-2,b+1)} \quad - \frac{1}{x^{(2,-1)}_{(a-1,b)}} \Bigg(\frac{1}{[2]} + \frac{1}{x^{(2,-1)}_{(a-2,b)}} \Bigg(\frac{[5]}{[2][3]}$$

$$+ \frac{1}{y^{(3,-1)}_{(a-2,b-1)}} \Bigg(\frac{1}{[2]^2} + \frac{1}{[2]^2 x^{(-1,1)}_{(a,b-2)} x^{(-1,1)}_{(a-1,b-2)}} \Bigg) + \frac{1}{[2][3]^2 y^{(1,0)}_{(a-2,b-1)}} \Bigg) \Bigg)^2 y^{(-2,1)}_{(a+1,b-1)} \quad - \frac{1}{x^{(0,0)}_{(a-1,b)}} \Bigg\{ \Bigg(\Bigg(- \frac{[3]}{[2]}$$

$$- \frac{1}{x^{(-1,1)}_{(a-2,b)}} \Bigg(\frac{[2]^2}{[3]} - \frac{1}{[3]^2 y^{(1,0)}_{(a-3,b)}} \Bigg) - \frac{1}{x^{(-1,1)}_{(a-1,b-1)}} \Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)} y^{(3,-1)}_{(a-3,b)}} - \frac{1}{x^{(2,-1)}_{(a-2,b)}} \Bigg) - \frac{1}{[2]^2 x^{(0,0)}_{(a-2,b)}} \Bigg)^2 {}^{1,1}y^{(0,0)}_{(a-1,b)}$$

$$+2\cdot\Bigg(-\frac{[3]}{[2]}-\frac{1}{x^{(-1,1)}_{(a-2,b)}}\bigg(\frac{[2]^2}{[3]}-\frac{1}{[3]^2y^{(1,0)}_{(a-3,b)}}\bigg)-\frac{1}{x^{(-1,1)}_{(a-1,b-1)}}\bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\bigg)-\frac{1}{[2]^2x^{(0,0)}_{(a-2,b)}}\Bigg)\cdot$$

$$\bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\bigg)\,{}^{1,2}y^{(0,0)}_{(a-1,b)}+\bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\bigg)^2\,{}^{2,2}y^{(0,0)}_{(a-1,b)}\Bigg\}-\frac{1}{x^{(-2,1)}_{(a-1,b)}}\Bigg(\frac{[3]}{[2]}-$$

$$\frac{1}{x^{(-1,1)}_{(a-2,b)}}\Bigg(\frac{[2]}{[3]x^{(-1,1)}_{(a-3,b)}}+\frac{1}{[3]y^{(1,0)}_{(a-3,b)}x^{(-1,1)}_{(a-3,b)}}\bigg(\frac{[4]}{[3]}+\frac{1}{x^{(-1,1)}_{(a-4,b)}}\bigg)\Bigg)-\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}\bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\bigg)\Bigg)^2y^{(2,-1)}_{(a-3,b+1)}$$

$$-\frac{1}{x^{(1,-1)}_{(a-1,b)}}\Bigg(-\frac{[3]}{[2]}+\frac{1}{x^{(2,-1)}_{(a-2,b)}}\bigg(\frac{[5]}{[2][3]}+\frac{1}{y^{(3,-1)}_{(a-2,b-1)}x^{(-1,1)}_{(a,b-2)}x^{(-1,1)}_{(a-1,b-2)}}+\frac{[2]}{[3]^2y^{(1,0)}_{(a-2,b-1)}}\bigg)+\bigg(\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}$$

$$+\frac{1}{x^{(2,-1)}_{(a-2,b)}x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}}\bigg)\Bigg(\frac{[4]}{[2]^2}-\frac{1}{x^{(2,-1)}_{(a-3,b)}}\bigg(\frac{[5]}{[2][3]}+\frac{1}{y^{(3,-1)}_{(a-3,b-1)}}\bigg(\frac{1}{[2]^2}+\frac{1}{[2]^2x^{(-1,1)}_{(a-1,b-2)}x^{(-1,1)}_{(a-2,b-2)}}\bigg)$$

$$+\frac{1}{[2][3]^2y^{(1,0)}_{(a-3,b-1)}}\bigg)\Bigg)\Bigg)^2y^{(-1,1)}_{(a,b-1)}\quad-\frac{1}{x^{(-1,0)}_{(a-1,b)}}\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}}\bigg(\frac{[4]}{[3]}+\frac{[2]}{[3]^2y^{(1,0)}_{(a-3,b)}}\bigg)+\bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}y^{(3,-1)}_{(a-3,b)}}$$

$$-\frac{1}{x^{(2,-1)}_{(a-2,b)}}\bigg)\frac{1}{x^{(-1,1)}_{(a-1,b-1)}x^{(-1,1)}_{(a-2,b-1)}}-\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}\Bigg)^2y^{(1,0)}_{(a-2,b)}$$

(A.24) $$\rho^{(-3,1)}_{(a,b),\varpi_2}=-\frac{[6][8][15]}{[3][5][12]}x^{(-2,1)}_{(a-1,b)}-\frac{1}{x^{(-1,1)}_{(a-1,b)}}\frac{y^{(-1,0)}_{(a-2,b+1)}}{\Big(x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}\Big)^2}-\frac{1}{x^{(0,0)}_{(a-1,b)}}\Bigg(-\frac{[3]}{[2]}$$

$$+\frac{1}{x^{(-1,1)}_{(a-2,b)}x^{(-1,1)}_{(a-3,b)}}\Bigg(\frac{[2]}{[3]}+\frac{1}{[3]y^{(1,0)}_{(a-3,b)}}\bigg(\frac{[4]}{[3]}+\frac{1}{x^{(-1,1)}_{(a-4,b)}}\bigg)\Bigg)+\frac{1}{[2]x^{(0,0)}_{(a-2,b)}}\bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\bigg)\Bigg)^2y^{(-2,1)}_{(a-1,b)}$$

$$-\frac{1}{x^{(-2,1)}_{(a-1,b)}}\Bigg\{\Bigg(\bigg(\frac{[3]}{[2]x^{(-1,1)}_{(a-3,b)}}-\frac{1}{x^{(-1,1)}_{(a-2,b)}\Big(x^{(-1,1)}_{(a-3,b)}\Big)^2}\bigg)\Bigg(-\frac{[3]}{[2]}-\frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\bigg(\frac{[2]^2}{[3]}+\frac{1}{y^{(3,-1)}_{(a-5,b+1)}x^{(-1,1)}_{(a-3,b)}}$$

$$-\frac{1}{[3]^2y^{(1,0)}_{(a-5,b+1)}}\bigg)+\frac{1}{x^{(2,-1)}_{(a-4,b+1)}x^{(-1,1)}_{(a-3,b)}}-\frac{1}{[2]^2x^{(0,0)}_{(a-4,b+1)}}\Bigg)+\frac{1}{x^{(0,0)}_{(a-2,b)}}\bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\bigg)^2\frac{1}{x^{(-1,1)}_{(a-3,b)}}$$

$$-\frac{1}{x^{(-2,1)}_{(a-2,b)}}\bigg(\frac{[3]}{[2]}-\frac{1}{[2]x^{(-1,1)}_{(a-3,b)}x^{(-1,1)}_{(a-4,b)}}\bigg)^2\Bigg)^2\,{}^{1,1}y^{(0,0)}_{(a-3,b+1)}\quad+\;2\cdot\Bigg(-\frac{[3]}{[2]}-\frac{1}{x^{(-1,1)}_{(a-2,b)}\Big(x^{(-1,1)}_{(a-3,b)}\Big)^2}\cdot$$

$$\Bigg(\frac{1}{x^{(-1,1)}_{(a-4,b+1)}y^{(3,-1)}_{(a-5,b+1)}}-\frac{1}{x^{(2,-1)}_{(a-4,b+1)}}\Bigg)-\frac{1}{x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)^2\Bigg)\cdot\Bigg(\frac{[3]}{[2]x^{(-1,1)}_{(a-3,b)}}-\frac{1}{x^{(-1,1)}_{(a-2,b)}\Big(x^{(-1,1)}_{(a-3,b)}\Big)^2}\cdot$$

$$\Bigg(-\frac{[3]}{[2]}-\frac{1}{x^{(-1,1)}_{(a-4,b+1)}}\Bigg(\frac{[2]^2}{[3]}+\frac{1}{y^{(3,-1)}_{(a-5,b+1)}x^{(-1,1)}_{(a-3,b)}}-\frac{1}{[3]^2y^{(1,0)}_{(a-5,b+1)}}\Bigg)+\frac{1}{x^{(2,-1)}_{(a-4,b+1)}x^{(-1,1)}_{(a-3,b)}}-\frac{1}{[2]^2x^{(0,0)}_{(a-4,b+1)}}\Bigg)$$

$$+\frac{1}{x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)^2\frac{1}{x^{(-1,1)}_{(a-3,b)}}-\frac{1}{x^{(-2,1)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}-\frac{1}{[2]x^{(-1,1)}_{(a-3,b)}x^{(-1,1)}_{(a-4,b)}}\Bigg)^2\Bigg)\ {}^{1,2}y^{(0,0)}_{(a-3,b+1)}+\Bigg(-\frac{[3]}{[2]}-$$

$$\frac{1}{x^{(-1,1)}_{(a-2,b)}\Big(x^{(-1,1)}_{(a-3,b)}\Big)^2}\Bigg(\frac{1}{x^{(-1,1)}_{(a-4,b+1)}y^{(3,-1)}_{(a-5,b+1)}}-\frac{1}{x^{(2,-1)}_{(a-4,b+1)}}\Bigg)-\frac{1}{x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)^2\Bigg)^2\ {}^{2,2}y^{(0,0)}_{(a-3,b+1)}\Bigg\}$$

$$-\frac{1}{x^{(1,-1)}_{(a-1,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)\Bigg(\frac{[4]}{[2]^2}-\frac{1}{x^{(2,-1)}_{(a-3,b)}}\Bigg(\frac{[5]}{[2][3]}-\frac{1}{[2]y^{(3,-1)}_{(a-3,b-1)}x^{(-1,1)}_{(a-1,b-2)}}$$

$$+\frac{1}{[2][3]^2y^{(1,0)}_{(a-3,b-1)}}\Bigg)\Bigg)\Bigg)^2y^{(-3,2)}_{(a,b-1)}-\frac{1}{x^{(-1,0)}_{(a-1,b)}}\Bigg(\frac{1}{x^{(-1,1)}_{(a-2,b)}}\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg(-\frac{[3]}{[2]}+\frac{1}{x^{(2,-1)}_{(a-4,b+1)}x^{(-1,1)}_{(a-3,b)}}\Bigg(-\frac{[5]}{[3]}$$

$$+\frac{1}{y^{(3,-1)}_{(a-4,b)}x^{(-1,1)}_{(a-2,b-1)}}-\frac{1}{[3]^2y^{(1,0)}_{(a-4,b)}}\Bigg)+\frac{1}{[2]x^{(0,0)}_{(a-4,b+1)}}\Bigg(\frac{[4]}{[2]^2}-\frac{1}{x^{(2,-1)}_{(a-5,b+1)}}\Bigg(\frac{[5]}{[2][3]}-\frac{1}{[2]y^{(3,-1)}_{(a-5,b)}x^{(-1,1)}_{(a-3,b-1)}}+$$

$$\frac{1}{[2][3]^2y^{(1,0)}_{(a-5,b)}}\Bigg)\Bigg)\Bigg)-\frac{1}{x^{(0,0)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}+\frac{1}{x^{(-1,1)}_{(a-3,b)}}\Bigg)\frac{1}{x^{(-1,1)}_{(a-3,b)}}+\frac{1}{x^{(-2,1)}_{(a-2,b)}}\Bigg(\frac{[3]}{[2]}-\frac{1}{[2]x^{(-1,1)}_{(a-3,b)}x^{(-1,1)}_{(a-4,b)}}\Bigg)\Bigg)^2y^{(-1,1)}_{(a-2,b)}$$

(A.25)
$$\rho^{(0,-1)}_{(a,b),\varpi_2}=\frac{[7][8][15]}{[3][4][5]}-\frac{1}{y^{(3,-1)}_{(a,b-1)}}y^{(-3,1)}_{(a+3,b-2)}-\frac{1}{y^{(1,0)}_{(a,b-1)}}y^{(-1,0)}_{(a+1,b-1)}-\frac{1}{y^{(-1,1)}_{(a,b-1)}}y^{(1,-1)}_{(a-1,b)}$$

$$-\frac{1}{y^{(2,-1)}_{(a,b-1)}}y^{(-2,1)}_{(a+2,b-2)}-\frac{1}{y^{(-3,2)}_{(a,b-1)}}y^{(3,-2)}_{(a-3,b+1)}-\frac{{}^{2,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\ {}^{1,1}y^{(0,0)}_{(a,b-1)}+\frac{{}^{1,2}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\ {}^{1,2}y^{(0,0)}_{(a,b-1)}$$

$$+\frac{{}^{2,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\ {}^{2,1}y^{(0,0)}_{(a,b-1)}-\frac{{}^{1,1}y^{(0,0)}_{(a,b-1)}}{\mathscr{D}_{(a,b-1)}}\ {}^{2,2}y^{(0,0)}_{(a,b-1)}-\frac{1}{y^{(3,-2)}_{(a,b-1)}}y^{(-3,2)}_{(a+3,b-3)}-\frac{1}{y^{(-2,1)}_{(a,b-1)}}y^{(2,-1)}_{(a-2,b)}$$

$$-\frac{1}{y^{(1,-1)}_{(a,b-1)}}y^{(-1,1)}_{(a+1,b-2)}-\frac{1}{y^{(-1,0)}_{(a,b-1)}}y^{(1,0)}_{(a-1,b-1)}-\frac{1}{y^{(-3,1)}_{(a,b-1)}}y^{(3,-1)}_{(a-3,b)}-\frac{1}{y^{(0,-1)}_{(a,b-1)}}y^{(0,1)}_{(a,b-2)}$$

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