Some Results on the AKLT Model on General Graphs

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To my family

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Abstract

The subject of this dissertation is the minimal AKLT model [3], a special, well-behaved isotropic model defined on any general graph. This model was first constructed in order to study the Heisenberg model [16] which is conjectured by Haldane [14] to have a unique gapped ground state when the spin is an integer. This model can be constructed on any arbitrary graph, and a natural question is which graphs can this model be shown to have a unique ground state with a stable spectral gap and which can be shown to have a degenerate or Néel ordered ground state. There are few such examples: the one-dimensional chain [3] is known to have a unique stably gapped ground state, the hexagonal chain [26] is known to have a unique gapped ground state, the hexagonal lattice is known to have a unique ground state with exponential decay of correlations [19], and a spectral gap [25]. The Cayley trees of degree $2, 3, 4$ are known to have unique ground states with exponential decay of correlations and those of degree $d \geq 5$ are known to have a Néel ordered degenerate ground state space [3], [10]. Numerical evidence exists for the spectral gap of the square lattice [30]. Beyond these not much is known. In this thesis we extend previous results in multiple directions to several infinite classes of graphs: we show that quasi-one-dimensional graphs that are bipartite have a unique ground state with exponential decay of correlations and a stable spectral gap. We show that under certain conditions, trees constructed from repeated bipartite graphs have degenerate, long-range ordered ground states, as well as trees with a sufficiently high minimum splitting number. We also prove that the hexagonal lattice model has a stable spectral gap using techniques unrelated to the first two proofs. Each of these uses completely different styles of techniques which show the richness of the model. Lastly in the conclusion, we provide some refined conjectures for the macroscopic behavior of this model on general lattices.

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CHAPTER 1

Introduction

1.1. Quantum Spin Systems

1.1.1. Spin. The property known as quantum spin has a deep connection to classical angular momentum and representation theory, and is subject to not a little phenomenological controversy. However the mathematical framework brooks no controversy, and so we introduce it first stripped of its physical interpretation. The state space of a quantum spin is defined to be a finite-dimensional, complex Hilbert space $\mathcal{H}$; experiments now over a century old led to the interpretation that these complex vectors represent the microscopic states of measurable particles. It is a standard exercise in analysis [18] that all finite dimensional complex Hilbert spaces are isomorphic if and only if they have the same dimension, so we identify $\mathcal{H} = \mathbb{C}^d$ for some integer $d \geq 1$. Spin is known from the spin-statistics theorem [50] to be quantized into half-integers, where a particle with spin $s \in \frac{1}{2}\mathbb{Z}$ has Hilbert space dimension $d = 2s + 1$. Finite dimensional Hilbert spaces do not cover all possible physically relevant systems (e.g. the harmonic oscillator) but very often can be used for models of subsystems with finitely many degrees of freedom. In the parlance of quantum computation, a quantum system with Hilbert space of dimension 2 is called a **qubit** (a portmanteau of "quantum" and "bit") and by analogy a quantum system with $d \geq 3$ dimensions is called a **qudit**.

A d-dimensional complex Hilbert space can be identified with the pair $(\mathbb{C}^d, \langle \cdot, \cdot \rangle)$ where $\mathbb{C}^d$ is the d-dimensional complex vector space and the inner product of two vectors $u, v \in \mathbb{C}^d$ is given by the standard inner product

$$\langle u, v \rangle = \sum_{i=1}^{d} \overline{u_i} v_i \tag{1.1}$$

This product gives us a notion of orthogonality and brings with it all the tools of Hilbert space theory which we can use to analyze a system.

1.1.2. Observables. We identify the **algebra of observables** (often denoted $\mathcal{A}$) on a Hilbert space $\mathcal{H}$ as the set of all bounded linear operators $\mathcal{A} := \mathcal{B}(\mathcal{H})$; generally physicists refer to the observables as the self-adjoint elements of $\mathcal{B}(\mathcal{H})$. For a d-dimensional Hilbert space, the algebra of observables is the set $\mathcal{B}(\mathcal{H}) = M_d(\mathbb{C})$ of all $d \times d$ matrices with elements in $\mathbb{C}$. The **self-adjoint** elements of $M_d(\mathbb{C})$ are those for which the conjugate transpose $A^* := \overline{A^T}$ is the identity map (i.e. $A^* = A$). The spectrum of a self-adjoint operator, denoted $\mathrm{spec}(A)$, is a subset of the real numbers $\mathbb{R}$; physically, these correspond

to values of measurement outcomes. In general, the observable space $\mathcal{B}(\mathcal{H})$ can itself be equipped with an inner product called the **Hilbert-Schmidt inner product**

$$\langle A, B\rangle_{\text{HS}} = \text{tr}(A^* B) \tag{1.2}$$

When equipped with an inner product, it is convenient to have a basis of the observable algebra; for instance if $\mathcal{H} = \mathbb{C}^2$ the Pauli matrices defined as

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.3}$$

form an orthonormal basis under $\frac{1}{2}\langle \cdot, \cdot \rangle_{\text{HS}}$. That the Pauli matrices also correspond to the generators of the 2-dimensional $\mathfrak{su}(2)$ representation is a happy coincidence which will help with some simple computations.

Every inner-product also induces a norm on the space (another standard exercise [18]); the norm induced by the Hilbert-Schmidt norm is distinct from the standard **operator norm**:

$$||A|| = \sup_{\psi \neq 0} \frac{||A\psi||}{||\psi||} \tag{1.4}$$

A **compact operator** is an operator T such that if X is a bounded set then $\overline{T(X)}$ is compact. An operator is a **Hilbert-Schmidt operator** if

$$||A||^2_{\text{H.S.}} = \sum_{i=1} ||Ae_i||^2 \tag{1.5}$$

for any orthonormal basis $\{e_i\}$ of $\mathcal{H}$. Every Hilbert-Schmidt operator is compact [18].

1.1.3. States. While the Hilbert space $\mathcal{H}$ may contain all the vectors associated to a quantum state, the states themselves obey restrictions which allow them to be interpreted as encoding probability distributions over measurement outcomes. Thus a **state** on an observable algebra $\mathcal{A}$ is a normalized, positive linear functional, meaning that $\omega : \mathcal{A} \to \mathbb{C}$ is linear and satisfies

$$\omega(A^* A) \geq 0, \quad \text{for all } A \in \mathcal{A}, \quad \text{and } \omega(\mathbb{1}) = 1 \tag{1.6}$$

The evaluation $\omega(A)$ is called the **expectation value** of an observable A in the state ω. Every unit vector in $\mathcal{H}$, meaning, every vector such that $||\psi|| = 1$ has a corresponding state given by $\omega_\psi(A) = \langle \psi, A\psi \rangle$. Not all states can be written this way; ones that can are called **vector states**. A more general expression is $\omega_\psi = \text{tr}(P_\psi A)$ where $P_\psi = \langle \psi, \phi \rangle \psi$ is called the **orthogonal projection** onto ψ.

The set of states $\mathcal{S}$ on $\mathcal{A}$ is convex, and the extreme points are called the **pure states**, and states which are not pure are called **mixed states**; in finite dimensions, the pure states are the vector states; moreover

every state can be written as a convex combination of vector states, meaning there are $t_1, ..., t_n \geq 0$ and unit vectors $\psi_1, ..., \psi_n$ such that $\omega = \sum_{i=1}^n t_i \omega_{\psi_i}$. It then follows that there is a non-negative matrix in $\rho \in M_d$ with $\omega(A) = \mathrm{tr}(\rho A)$ where $\rho = \sum_{i=1}^n t_i P_{\psi_i}$ and $\mathrm{tr}(\rho \mathbb{1}) = \sum_{i=1}^n t_i = 1$. Matrices of this form are called **density matrices** and can be interpreted as encoding a probability distribution over measurement outcomes.

1.1.4. Infinite volume limits: C* algebras, quasi-local observables, and states. While we will not directly talk about particles with infinite degrees of freedom, in order to talk about states of infinitely particles with finitely many degrees of freedom, we will need to introduce a more general framework. In order to do this, we notice four important properties of our observable algebra $\mathcal{B}(\mathcal{H})$:

i. **Completeness**: It is complete with respect to the metric topology induced by the operator norm.

ii. **Submultiplicativity of the norm**: $||AB|| \leq ||A||||B||$ which means the operator product is continuous in the operator norm topology.

iii. **$*$-algebraic structure**: The $^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an anti-linear involution, and an algebra anti-morphism, meaning $(AB)^* = B^* A^*$

iv. **C* Property**: $||A^*|| = ||A||$ and $||A^* A|| = ||A||^2$

The first two properties are the definition of a **Banach algebra**, and the last two properties make it a **C*-algebra**. If a C* algebra $\mathcal{A}$ has an identity element $\mathbb{1} \in \mathcal{A}$, called the unit, it is called **unital**. Every C* algebra referred to in this thesis is unital.

A **state** ω on a C* algebra is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ that is non-negative (meaning $\omega(A^* A) \geq 0$) and normalized $\omega(\mathbb{1}) = 1$. In finite dimensions every state corresponds to a density matrix; in infinite dimensional systems, the positive trace-class operators such that $\mathrm{tr}(\rho) = 1$ define states via the formula $\omega(A) = \mathrm{tr}(\rho A)$ on $\mathcal{B}(\mathcal{H})$ but do not exhaust them.

We denote the set of self-adjoint elements of $\mathcal{A}$ as

$$\mathcal{A}_{\mathrm{sa}} = \{A \in \mathcal{A} \mid A = A^*\} \tag{1.7}$$

An operator $A \in \mathcal{A}$ is positive (denoted $A \geq 0$) if there exists $B \in \mathcal{A}$ such that $A = B^* B$; we define a partial ordering on $\mathcal{A}_{\mathrm{sa}}$ by saying $A \geq B$ if and only if $A - B \geq 0$.

There is a deep theory [5] of the relationship between C* algebras and states, entropy measures, and dynamics; however we will be mostly concerned with statics, so our coverage of dynamics will be just enough to define ground states.

Moreover thus far we have only defined systems of single particles, while we will be interested in many-body systems. Let the set of particles be identified with a set Λ where for each $x \in \Lambda$ we have an

associated finite-dimensional Hilbert space of dimension $d_x \geq 2$; then the full Hilbert space dimension is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^{d_x} \tag{1.8}$$

and the associated algebra of observables is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} M^{d_x}(\mathbb{C}) \tag{1.9}$$

Often for subsets $\Lambda_1 \subseteq \Lambda$ we identify $\mathcal{A}_{\Lambda_1}$ by identifying $A \in \mathcal{A}_{\Lambda_1}$ with $A \otimes \mathbb{1}_{\Lambda\setminus\Lambda_1} \in \mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda\setminus\Lambda_1}$.
For an infinite system we will assume that our total set Γ is a lattice, meaning a countable metric space (Γ, d); the usual example of $\Gamma = \mathbb{Z}^\nu$ with the normal graph distance d. As before each site $x \in \Gamma$ has an associated finite-dimensional Hilbert space $\mathcal{H}_x = \mathbb{C}^{d_x}$ and observable algebra $\mathcal{A}_x = M_{d_x}(\mathbb{C})$ and for any finite subset of $\Lambda \subset \Gamma$ we have the Hilbert space and observable algebra from equations 1.8 and 1.9.
From the inclusion identification we have that $\mathcal{A}_{\Lambda_1} \subseteq \mathcal{A}_{\Lambda_2}$ when $\Lambda_1 \subseteq \Lambda_2$ so a natural object is the **local observable algebra**, which is the union over all finite subsets of Γ

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\Subset\Gamma} \mathcal{A}_\Lambda \tag{1.10}$$

where the notation $\Lambda \Subset \Gamma$ means $\Lambda \subset \Gamma$ and $|\Lambda| < \infty$. We want to work with a C* algebra so we take the norm-completion of $\mathcal{A}_\Gamma^{\mathrm{loc}}$ to get the **quasi-local observable algebra**:

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}^{||\cdot||} \tag{1.11}$$

1.1.5. Hamiltonians and ground states. The **Hamiltonian** is an operator $H = H^* \in \mathcal{A}_\Lambda$ (where Λ is finite) that generates the dynamics. This means that for any pure state $\psi_0 \in \mathcal{H}_\Lambda$ we have that for any time $t \in \mathbb{R}$ our state evolves according to the Schrödinger equation:

$$i\frac{d}{dt}\psi(t) = H\psi(t) \quad \text{with } \psi(0) = \psi_0 \tag{1.12}$$

which has solutions given by

$$\psi(t) = U_t\psi_0, \quad U_t = e^{-itH} \tag{1.13}$$

The self-adjointness of H insures U_t is unitary for all t and $U_t^* = U_{-t}$ and that $U_tU_s = U_{t+s}$, so that the set $\{U_t | t \in \mathbb{R}\}$ is a one-parameter group of unitaries. One can also describe the evolution of observables via the Heisenberg equation

$$\frac{d}{dt}A(t) = i[H, A(t)] \quad \text{with } A(0) = A \tag{1.14}$$

One can see this is equivalent to evolving the observable by the same unitary as $A(t) = U_t^* A U_t$. Expectation values $\omega(H)$ quantify the energy of our system in the state ω. Because the Hamiltonian is self-adjoint, expectation values are real since its spectrum is real.

Any Hamiltonian can in general be written as a sum over interaction terms $\Phi : \mathcal{P}(\Lambda) \to \mathcal{A}_\Lambda$ where $\mathcal{P}$ denotes the power set of Λ with the property that for each $X \in \mathcal{P}(\Lambda)$ we have $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X)^* = \Phi(X)$. For any $Z \subset \Lambda$ the Hamiltonian given by the interaction Φ is defined as

$$H_Z = \sum_{X \in \mathcal{P}(Z)} \Phi(X) \tag{1.15}$$

For the finite-volume case, we can define the finite-volume dynamics by the set of automorphisms $\tau_t^\Lambda : \mathcal{A} \to \mathcal{A}$ defined as:

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad \text{for any } A \in \mathcal{A}_\Lambda, \text{ and } t \in \mathbb{R} \tag{1.16}$$

The set $\{\tau_t^\Lambda\}$, labeled by $\Lambda \in \mathcal{P}_0(\Gamma)$ where $\mathcal{P}_0 = \{\Lambda \Subset \Gamma\}$ is the set of all finite subsets, is a one-parameter group of $*$-automorphisms on $\mathcal{A}_\Lambda$. Under suitable assumptions about the decay of interactions, dynamics can be guaranteed via the use of a Lieb-Robinson bound; for our purposes we do not need the general theory, but merely appeal to it in order to justify the following: from semi-group theory, the strongly-continuous one-parameter group of $*$-automorphisms $\{\tau_t\}_{t \in \mathbb{R}}$ is generated by a closed operator δ; from a Lieb-Robinson bound one can show the existence of the following limit for all $A \in \mathcal{A}_\Gamma^{\text{loc}}$:

$$\delta(A) = \lim_{\Lambda \to \Gamma} [H_\Lambda, A] \tag{1.17}$$

where the limit is taken over an increasing sequence of volumes. One can define δ as the closure of this operator, which is generally unbounded; one also has that $\mathcal{A}_\Gamma^{\text{loc}}$ is a core (or essential domain), meaning $\mathcal{A}_\Gamma^{\text{loc}} \subset \text{dom}(\delta)$ and $\overline{\delta_{\mathcal{A}_\Gamma^{\text{loc}}}} = \delta$, or in other words the closure of the restriction of δ to $\mathcal{A}_\Gamma^{\text{loc}}$ is δ.

Note this solves the equation

$$\frac{d}{dt}\tau_t(A) = i\delta(\tau_t(A)) = i\tau_t(\delta(A)) \tag{1.18}$$

and so it makes sense to write $\tau_t = e^{it\delta}$.

The **ground state** of a finite-volume Hamiltonian is any state ω_0 which minimizes the energy, meaning:

$$\omega_0(H_\Lambda) = \min\{\omega(H_\Lambda) | \omega \in \mathcal{S}_\Lambda\} \tag{1.19}$$

where $\mathcal{S}_\Lambda$ is the set of states on $\mathcal{A}_\Lambda$. In finite-dimension, ground states are states with density matrices that have range that is a subspace of the eigenspace corresponding to the smallest eigenvalue of H_Λ. A ground state is called **unique** if this eigenvalue is simple, meaning the eigenspace is one-dimensional.

Let $E_0(\Lambda) = \min(\text{spec}(H_\Lambda))$ where $\mathcal{H}_\Lambda$ is finite-dimensional. We call $E_0(\Lambda)$ the **ground state energy** and any state with $\omega(H_\Lambda) = E_0(\Lambda)$ is a **ground state**.

We provide a well-known argument for the following alternative characterization of ground states as those for which:

$$\omega(A^*[H_\Lambda, A]) \geq 0, \quad \text{for all } A \in \mathcal{A}_\Lambda \tag{1.20}$$

which is inherited by limits of sequences of ground states of finite-volume systems for Hamiltonians with sufficient decay of interactions. To any sequence of finite volumes $\{\Lambda_n\}$ with $\Lambda_n \to \Gamma$ there will correspond possibly many sequences of ground states ω_{Λ_n}; by the Banach-Alaoglu theorem [18], one can see these sequences have an accumulation point and convergent subsequence in the sense of the weak limits. The introduction of C* algebras is due to the following fact: there is no simple definition of a limit of an infinite tensor product of Hilbert spaces $\mathcal{H}_{\Lambda_n}$, nor of the Hamiltonians H_{Λ_n}; however the limiting states ω_{Λ_n} on $\mathcal{A}_\Gamma^{\text{loc}}$ and by the existence of a unique continuous extension, onto $\mathcal{A}_\Gamma$; thus we have the following generalization of ground states for infinite-volumes.

Theorem 1.1.1. [35]

i. Let ω be a state of system defined on finite-dimensional Hilbert space $\mathcal{H}$ with Hamiltonian H, and suppose that the range of the density matrix ρ of ω is a subspace of the eigenspace of H with smallest eigenvalue E_0. Then

$$\omega(A^*[H_\Lambda, A]) \geq 0, \quad \text{for all } A \in \mathcal{A}_\Lambda \tag{1.21}$$

and every state on $\mathcal{B}(\mathcal{H})$ satisfying 1.21 is a ground state.

ii. Let $\{\Lambda_n\}$ be an exhaustive sequence of finite volumes in Γ. For each $n \geq 1$ let ω_n be a ground state of H_{Λ_n}. If

$$\omega_n(A) \to \omega(A) \tag{1.22}$$

for all $A \in \mathcal{A}_\Gamma^{\text{loc}}$ and

$$\delta_{\Lambda_n}(A) := [H_{\Lambda_n}, A] \to \delta(A) \tag{1.23}$$

for all $A \in \mathcal{A}_\Gamma^{\text{loc}}$ in norm, then ω is a state on $\mathcal{A}_\Gamma^{\text{loc}}$with

$$\omega(A^*\delta(A)) \geq 0 \tag{1.24}$$

for all $A \in \mathcal{A}_\Gamma^{\text{loc}}$

PROOF. i. Note that $\omega(A^*[H,A]) = \mathrm{tr}(\rho A^*[H,A]) = \mathrm{tr}(\rho A^* H A) - \mathrm{tr}(H\rho A^* A) = \mathrm{tr}(\rho A^*(H - E_0\mathbb{1})A) \geq 0$ since $H\rho = E_0\rho$ and $H_\Lambda \geq E_0\mathbb{1}$

Going the other way, if $\psi \neq 0$ and $\psi \in \mathrm{ran}(\rho)$ with $H\psi \neq E_0\psi$, consider a state ψ_0 with $H\psi_0 = E_o\psi_0$ and define the observable $A = |\psi_0\rangle\langle\psi|$ and we have that $\omega(A^*[H,A]) < 0$.

ii. Note that $\omega(A^*\delta(A)) - \omega_{\Lambda_n}(A^*\delta_{\Lambda_n}(A)) = \omega(A^*(\delta(A) - \delta_{\Lambda_n}(A))) + (\omega - \omega_{\Lambda_n})(A^*(\delta_{\Lambda_n}(A) - \delta(A))) + (\omega - \omega_{\Lambda_n})(A^*\delta(A))$ The limit of the first two terms vanishes because of 1.23 and because $||\omega|| = 1$, and for the last term we notice that for every $\epsilon > 0$ there is a $B \in \mathcal{A}_\Gamma^{\mathrm{loc}}$ with $||A^*\delta(A) - B|| \leq \epsilon$ from which we get that

$$\lim_{n\to\infty} \sup |(\omega - \omega_{\Lambda_n})(A^*\delta(A))| \leq 2\epsilon \tag{1.25}$$

so the third term goes to 0 as well.

□

The upshot of this is that there are many sequences H_{Λ_n} where 1.23 occurs for a fixed generator δ; the limits of ground states of each such sequence of finite-volume Hamiltonians will satisfy 1.21 and so any such state will be considered a ground state of the system $(\mathcal{A}_\Gamma, \{\tau_t = e^{it\delta} | t \in \mathbb{R}\}$. Thus we define the following:

DEFINITION 1.1.2. Let $\{\tau_t = e^{it\delta} | t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of automorphism of a C* algebra $\mathcal{A}$. A state ω on $\mathcal{A}$ is a **ground state** for τ_t if

$$\omega(A^*\delta(A)) \geq 0 \tag{1.26}$$

for all $A \in C$ where C is a core for δ.

We now proceed to discuss the notion of a spectral gap for an infinite volume ground state. To do this we will need the Gelfand-Naimark-Segal representation of a state on a general C*-algebra, which we will state but not prove.

The GNS construction is a way of associating to each state ω on a C* algebra a canonical representation on a Hilbert space that is unique up to unitary equivalence. For infinite volume systems, we can get a representation of the ground state of quantum systems for which the dynamics are generated by a densely defined self-adjoint operator called the GNS Hamiltonian.

The form of the **GNS construction** associated to a state ω is a triple consisting of the following:

i. a Hilbert space $\mathcal{H}_\omega$;

ii. a representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ of the underlying C* algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$;

iii. a vector $\Omega_\omega \in \mathcal{H}_\omega$ that is cyclic for π_ω, and such that $\omega(A) = \langle\Omega_\omega, \pi_\omega(A)\Omega_\omega\rangle$ for all $A \in \mathcal{A}$.

A vector Ω_ω is **cyclic** if $\overline{\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\}} = \mathcal{H}_\omega$; that is, the range of it under the representation π_ω is dense in $\mathcal{A}$. This construction is unique up to unitary equivalence meaning that if both $(\mathcal{H}_1, \pi_1, \Omega_1)$ and $(\mathcal{H}_2, \pi_2, \Omega_2)$ both satisfy the 3 properties listed above, then:

$$\Omega_1 = U\Omega_2 \quad \text{and} \quad \pi_2(A) = U\pi_1(A)U^* \tag{1.27}$$

for some unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$; if the latter equation holds for π_1, π_2 then these are **unitarily equivalent** representations. A full proof of these facts can be found in [5]. One important corollary of these facts is the following:

Corollary 1.1.1. [35] Let ω be a state on a C*-algebra and let α be an automorphism of $\mathcal{A}$ such that $\omega \circ \alpha = \omega$. Then there exists a unique unitary $U_\alpha \in B(\mathcal{H}_\omega)$ such that

$$\pi_\omega(\alpha(A)) = U_\alpha \pi_\omega(A) U_\alpha^* \tag{1.28}$$

for all $A \in \mathcal{A}$, where $\mathcal{H}_\omega$ is the GNS Hilbert space and where $U_\alpha\Omega = \Omega$.

Proof. Follows from the GNS construction and the unitary equivalence of the representations π_ω and $\pi_\omega \circ \alpha$ □

This has an important application: the existence of the GNS Hamiltonian in infinite-volume.
Let ω be a ground state, from the above corollary we have for each $t \in \mathbb{R}$ a unique unitary $U_t \in \mathcal{B}(\mathcal{H}_\omega)$ implementing τ_t, and thus

$$U_t\Omega_\omega = \Omega_\omega \quad \text{and} \quad U_t^*\pi_\omega(A)U_t = \pi_\omega(\tau_t(A)) \tag{1.29}$$

for all $A \in \mathcal{A}_\Gamma$. The semigroup property of τ_t carries to U_t. Moreover we note that

$$||U_t^*\pi_\omega(A)\Omega_\omega - \pi_\omega(A)\Omega_\omega|| = ||U_t^*\pi_\omega(A)U_tU_t^*\Omega_\omega - \pi_\omega(A)\Omega_\omega|| \tag{1.30}$$

$$= ||\pi_\omega(\tau_t(A))\Omega_\omega - \pi_\omega(A)\Omega_\omega|| \tag{1.31}$$

$$= ||\pi_\omega(\tau_t(A) - (A))\Omega_\omega|| \tag{1.32}$$

the last vanishes as $t \to 0$ because of the strong continuity of τ. From Stone's theorem [5], we conclude that there exists a densely defined self-adjoint operator H_ω acting on a dense subset of $\mathcal{H}_\omega$ which implements the time evolution unitaries U_t, meaning:

$$U_t = e^{-itH_\omega} \tag{1.33}$$

This operator H_ω is called the **GNS Hamiltonian** and is the Hamiltonian to which we refer when talking about whether a ground state is gapped in the infinite volume limit.

One simple consequence of this is that from Definition 1.1.2 we have

$$0 \leq \omega(A^*\delta(A)) = -i\frac{d}{dt}\omega(A^*\tau_t(A))|_{t=0} \tag{1.34}$$

$$= -i\frac{d}{dt}\langle\Omega_\omega, \pi_\omega(A^*\tau_t(A))\Omega_\omega\rangle|_{t=0} \tag{1.35}$$

$$= -i\frac{d}{dt}\langle\pi_\omega(A)\Omega_\omega, U_t^*\pi_\omega(A)U_t\Omega_\omega\rangle|_{t=0} \tag{1.36}$$

$$= \langle\pi_\omega(A)\Omega_\omega, H_\omega\pi_\omega(A)\Omega_\omega\rangle \tag{1.37}$$

so $0 \leq H_\omega$; in the GNS representation the ground state is then an eigenvector of the Hamiltonian H_ω with eigenvalue 0 and from what we just calculated we have that $\inf \operatorname{spec}(H_\omega) = 0$

1.2. Spectral gaps of infinite volume states

As is common with infinite-dimensional generalizations, in the finite volume setting, there are concepts which are trivial or not well-defined which become unambiguous in the infinite-volume setting. For instance the concept of a gapped excitation spectrum is somewhat trivial for a finite quantum system, since the set of eigenvalues is a finite set so there is always a gap between the 0 eigenvalue and the next non-zero eigenvalue. Moreover, there is no simple relationship between the degeneracy of the ground state eigenvalue and the degeneracy of the number of ground states in the thermodynamic limit; the AKLT model on the chain $\mathbb{Z}$ is a good example of this, as we will see 2.1.3. However in the thermodynamic limit, we have for any well-defined state ω we have the excitation spectrum is the spectrum of the GNS Hamiltonian H_ω, which is non-negative and the ground state is given by an eigenvector of eigenvalue 0. In the thermodynamic limit we can then define the **spectral gap** of the Hamiltonian as

$$\gamma := \sup\{\delta > 0 \mid (0, \delta) \cap \operatorname{spec}(H_\omega) = \emptyset\} \tag{1.1}$$

with the convention of $\gamma = 0$ if the right hand side is the empty set. If the ground state of the GNS Hamiltonian is unique then the above is equivalent to the property that

$$\gamma \cdot \omega(A^*A) \leq \omega(A^*\delta(A)) \tag{1.2}$$

for all $A \in D(\delta)$ with $\omega(A) = 0$.

There are many methods for proving or computing the spectral gap of a state [9], [21], [33]. Each has its benefits and must be adjusted to the system at hand. We will give an overview of the martingale method for unique frustration-free matrix product ground states later, which has some wide applications in 1-dimensional systems we will be interested in, compared to the others.

1.2.1. Two big name theorems on continuous symmetry breaking. One of the natural questions is whether symmetries of the Hamiltonian pass to the ground states, and in what cases. When

the ground state is unique and translation invariant and the Hamiltonian satisfies some decay conditions, we have the Goldstone theorem:

Theorem 1.2.1. [35] Let $(\mathcal{A}_{\mathbb{Z}^\nu}, \tau_t)$ be a C* dynamical system over $\mathbb{Z}^\nu$ with τ_t generated by a suitably decaying interaction (see [5]). The let α_s be a continuous symmetry of the dynamics, meaning that $\alpha_s \circ \tau_t = \tau_i \circ \alpha_s$ for all $s, t \in \mathbb{R}$. If ω is a translation invariant ground state for which there is a $\gamma > 0$ such that

$$\gamma \cdot \omega(A^* A) \leq \omega(A^* \delta(A)) \tag{1.3}$$

for all $A \in \mathcal{A}_\Gamma^{\text{loc}}$ with $\omega(A) = 0$. Then ω is invariant with respect to α_s, meaning

$$\omega(\alpha_s(A)) = \omega(A) \tag{1.4}$$

[23] In words this theorem is saying that gapped, unique, translation-invariant ground states do not permit continuous symmetry breaking. The existence of continuous symmetry breaking and the criteria that rule it out is one of the motivating ideas behind this thesis.
In certain cases, both classical and quantum, a version of what is called the Mermin-Wagner theorem can be proved. First we define a state that obeys the Kubo-Martin-Schwinger (KMS) condition:

Definition 1.2.2. Let $\tau_t(A)$ be defined as above and define its unique analytic continuation for $t \in \mathbb{C}$. Then a state ω obeys the β-KMS condition if $\omega(A\tau_{i\beta}(B)) = \omega(BA)$

Many consequences of this condition exist and is equivalent to ω being a Gibbs state for inverse temperature β, for instance. However Mermin-Wagner gives a restriction on symmetries of these states in one and two dimensions

Theorem 1.2.3. [5] Let $\{\tau_t = e^{it\delta} | t \in \mathbb{R}\}$ be a strongly continuous one-parameter group of automorphism of a C* algebra $\mathcal{A}$. If ω is a β-KMS state for τ_t and we have an automorphism that satisfies the following two properties:

i. α is **approximately inner**, meaning that there is a sequence (U_n) of unitaries such that

$$\lim_{n\to\infty} ||\tau(A) - U_n^* A U_N|| = 0 \tag{1.5}$$

for all $A \in \mathcal{A}$

ii. Either $||\delta(U_n)||$ are uniformly bounded; or all β-KMS states are α^2 invariant and $||U_n^* \delta(U_n) + U_n \delta(U_n^*)||$ are uniformly bounded.

Then we have that all β-KMS states are α^2 invariant for all $\beta \in [0, \infty)$

Note that $\beta = \infty$ is the zero-temperature state, in which case we have obvious counterexamples such as quantum ferromagnets. The informal way to state the above theorem is that there can be no continuous symmetry breaking at finite (inverse-) temperature; however this statement comes with caveats, as it is not true in arbitrary dimensions. In the context of spin systems, the α automorphism will very often be the spin-flip operation, so its square is the identity.

1.2.2. Frustration-free systems. In order to get better results on when we can prove systems have unique gaps and obey the conditions of Goldstone's theorem (and other symmetry breaking theorems) we need the concept of frustration-free systems, which has a somewhat simple definition.
We call an interaction $\Phi : \mathcal{P}_0(\Gamma) \to \mathcal{A}_{\text{loc}}$ **frustration-free** if for all $\Lambda \in \mathcal{P}_0(\Gamma)$ we have

$$\inf \operatorname{spec} \left(\sum_{X \subset \Lambda} \Phi(X) \right) = \sum_{X \in \Lambda} \inf \operatorname{spec}(\Phi(X)) \tag{1.6}$$

In words, this means that the total energy is minimized by taking the simultaneously satisfiable conditions of minimizing on every subset; this may not always be possible. However for those systems for which it is (and which are believed to be in some sense dense in the set of all states), a lot can be known.
In more explicit terms this is saying that the finite-volume Hamiltonians $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$ and each of their subterms $\Phi(X)$ have a common eigenvector in their ground state space.

1.3. Angular momentum and SU(2) representation theory

The theory of angular momentum in quantum mechanics is mathematically identical to the representation of $\mathfrak{su}(2)$ as a Lie algebra. We define a **representation** as a linear map $\rho : G \to GL(n)$ for some $n \geq 1$ such that $\rho(g)\rho(h) = \rho(gh)$; we will in general be strictly interested in the representations of $\mathfrak{su}(2)$ which is generated by three elements T_1, T_2, T_3 that satisfy the commutation relations

$$[T^i, T^j] = i \sum_{k=1}^{3} \varepsilon_{ijk} T^k \tag{1.1}$$

where ε_{ijk} is the Levi-Civita symbol defined by

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \tag{1.2}$$

and

$$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1 \tag{1.3}$$

and $\varepsilon_{ijk} = 0$ for all other triples $i, j, k \in \{1, 2, 3\}$.
As an example, the **fundamental representation** of $\mathfrak{su}(2)$ is a map $\rho : \mathfrak{su}(2) \to M_2(\mathbb{C})$ which by linearity extends to all of M_2 and necessarily means $\rho(\mathbb{1}) = \mathbb{1}$; in the fundamental representation we

have $\rho(T^i) = S_i = \sigma_i$ where σ_i are the Pauli matrices from 1.3. From now on we will use S_i to refer to a general representative of the generators of $\mathfrak{su}(2)$, with dimension understood by context or superscript S_i^J, and we use

$$\mathbf{S} = (S_1, S_2, S_3) \tag{1.4}$$

to be the vector of generating matrices. We can define the **Casimir element** of the representation as $\mathbf{S}^2 = (S_1)^2 + (S_2)^2 + (S_3)^2$; we know that since the S_i's are self adjoint that $\mathbf{S}^2 \geq 0$ and that $[\mathbf{S}^2, S_i] = 0$ for all i. The eigenvalues of $\mathbf{S}^2$ are $J(J+1)$ for some integer $J \geq 0$ and we can denote the eigenvalues of S_3 by M. Letting $\mathcal{H} = \mathbb{C}^d$ where $d = \dim S_i$ is the dimension of the representation, one can decompose

$$\mathcal{H} = \bigoplus_{J,M} \mathcal{H}_{J,M} \tag{1.5}$$

where if $\phi \in \mathcal{H}_{J,M}$ then ϕ is an eigenvector of $\mathbf{S}^2$ with eigenvalue $J(J+1)$ and is an eigenvector of S_3 with eigenvalue M. It can be shown [50] that $J \in \frac{1}{2}\mathbb{Z}$ and that given J, M takes one of $2J+1$ values from $[-J, J+1, ..., J-1, J]$.

1.3.1. Addition of angular momenta. Angular momenta can be added but the theory involved is nontrivial; we provide a description of the Clebsch-Gordan theory involved in the breakdown in 1.5 when we take two neighboring Hilbert spaces of (possibly differing) dimension. Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be two Hilbert spaces with associated angular momenta $\mathbf{S}_1$ and $\mathbf{S}_2$; we can define the total angular momentum by $\mathbf{S} = \mathbf{S}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{S}_2$ which satisfy the commutation relations for $\mathfrak{su}(2)$ and we call $J(J+1)$ the eigenvalue of $\mathbf{S}^2$. The range of values is given by Clebsch-Gordan theory [7] for J is the set $\{|J_1 - J_2|, |J_1 - J_2| + 1, ..., J_1 + J_2 - 1, J_1 + J_2\}$. Thus $\mathcal{H} \cong \mathbb{C}^{J_1} \otimes \mathbb{C}^{J_2} \cong \bigoplus_{i \geq |J_1 - J_2|}^{J_1+J_2} \mathbb{C}^i$ where each $\mathbb{C}^i$ is an invariant subspace of the action of $\mathfrak{su}(2)$.

1.4. Quantum channels and matrix norms

Much of the theory of one-dimensional quantum systems is characterized by studying quantum channels, which have a rich structure. We give a brief overview of their structure and properties of interest.

In general we will be interested in the properties of linear mappings between matrix spaces $T : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$; in many cases we will consider the case of $d_1 = d_2$. There is a rich theory of quantum channels which can be found in [54]; we will be largely focused on the spectral theory of such operators, which is only viable when $d_1 = d_2$.

The **spectrum** of a linear operator is the set

$$\text{spec}(T) := \{\lambda \in \mathbb{C} | (\lambda \mathbb{1} - T) \text{ is not invertible}\} \tag{1.1}$$

In finite-dimensional cases this is the set

$$\text{spec}(T) = \{\lambda \in \mathbb{C} \mid \exists M \in M_d(\mathbb{C}) \text{ with } TM = \lambda M\} \tag{1.2}$$

We will often refer to an M satisfying the eigenvalue equation in 1.2 as an eigenvector, even though it is a matrix, as we can see $M_d(\mathbb{C})$ as a vector space; if we give this vector space a Hilbert space structure it will be via the Hilbert-Schmidt norm. Thus the spectrum of T is the set of eigenvalues of the operator's $d^2 \times d^2$ matrix representation [54] Section 2.3. We have a Hermitian basis of operators so that $T^* = \hat{T}^T$ and so we have that $\text{spec}(T) = \overline{\text{spec}(T)}$ and thus the eigenvalues are either real or come in complex conjugate pairs. For each $\lambda \in \mathbb{R}$ we have a Hermitian eigenvector such that $TX = \lambda X$. The **spectral radius** of an operator is defined as

$$\rho(T) := \sup\{|\lambda| \mid \lambda \in \text{spec}(T)\} \tag{1.3}$$

In finite dimensions this is simply the largest eigenvalue. The notion of the largest eigenvalue gives a norm structure to the matrix algebra [17]. A function $||\cdot|| : M_d \to \mathbb{R}$ is a **matrix norm** if, for all $A, B \in M_d(\mathbb{C})$ we have:

i. **Non-negativity**: $||A|| \geq 0$

ii. **Positivity** or **Non-degeneracy**: $||A|| = 0 \iff A = 0$

iii. **Homogeneity**:$||cA|| = |c| \cdot ||A||$ for all $c \in \mathbb{C}$

iv. **Triangle Inequality**: $||A + B|| \leq ||A|| + ||B||$

v. **Submultiplicativity**: $||AB|| \leq ||A|| \cdot ||B||$

There are several examples of matrix norms:

i. The $\mathbf{l_1}$**-norm**:

$$||A||_{l_1} := \sum_{i,j=1}^{n} |a_{ij}| \tag{1.4}$$

ii. The $\mathbf{l_2}$**-norm** or **Hilbert-Schmidt norm**:

$$||A||_2 := |\text{tr}(AA^*)|^{1/2} \tag{1.5}$$

iii. The $\mathbf{l_\infty}$**-norm**:

$$||A||_{l_\infty} := \max_{1 \leq i,j \leq n} |a_{ij}| \tag{1.6}$$

iv. The **operator norm**:

$$||A||_{\text{op}} := \max_{||x||=1} ||Ax|| \tag{1.7}$$

where $||x||$ is the familiar vector 2-norm.

v. The **spectral norm**:

$$||A||_s = \sigma_1(A) \tag{1.8}$$

where σ_1 is the largest singular value of A.

Each of these norms can be used with varying success and are related to one another via inequalities that are largely dimension dependent; the choice of matrix norm is often made via its relationship to the eigenvalues. It follows from Gelfand's theorem [42] that for *any* matrix norm $||\cdot||$ that

$$\rho(A) = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}} \tag{1.9}$$

Thus in a limiting sense all matrix-norms are determined by the spectrum of the matrix. In order to know more about the spectrum of the matrix we need to know more about the matrix's structure, so we restrict to the case of completely positive trace-preserving maps between matrix spaces.

1.5. Completely positive trace-preserving maps

The following theorem is very useful for the study of positive maps. We say a linear map $T : M_{d_1}(\mathbb{C}) \to M_{d_2}(\mathbb{C})$ is **positive** if $A \geq 0$ implies $T(A) \geq 0$.

Theorem 1.5.1. [54] If T is a positive map on $M_d(\mathbb{C})$, then its spectral radius is bounded as

$$\rho(T) \leq ||T(\mathbb{1})||_\infty \tag{1.1}$$

If T is trace-preserving, then there is an eigenvalue $\lambda = 1$ and $\rho(T) = 1$, and $\text{spec}(T) \subseteq \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq 1\}$

Proof. From the Russo-Dye theorem [46] we know that $||T(X)||_\infty \leq ||T(\mathbb{1})||_\infty ||X||_\infty$ so we have that

$$|\lambda| ||X||_\infty = ||T(X)||_\infty \leq ||T(\mathbb{1})||_\infty ||X||_\infty \tag{1.2}$$

so we have 1.1. If T is unital then $T(\mathbb{1}) = \mathbb{1}$ and so $\rho(T) = 1$; if T is trace-preserving then T^* is unital and since $\text{spec}(T) = \text{spec}(T^*)$ we have that $\rho(T) = 1$. □

We will often be able to encode physical information (say correlations of our system) into the iteration of a positive operator, so we characterize in which cases the limits $\lim_{n\to\infty} T^n$ is well-defined.

We call a linear map $T : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ **irreducible** if for all Hermitian projectors $P \in M_d(\mathbb{C})$ such that $T(PM_d(\mathbb{C})P) \subseteq PM_d(\mathbb{C})P$ we have that P is either 0 or the identity $\mathbb{1}$.

This property has nontrivial consequences for the spectrum of T, namely if a map T is irreducible then the spectral radius $\rho(T)$ is a non-degenerate eigenvalue and the corresponding left- and right-eigenvectors are

positive-definite, meaning $T(X) = \rho(T)X > 0$ and $T^*(Y) = \rho Y > 0$. Moreover there is a positive semi-definite $X \in M_d(\mathbb{C})$ such that $T(X) = \rho(T)X$. We are most interested in when the largest eigenvalue is unique.

We call an operator T **primitive** if it has trivial peripheral spectrum, meaning that there is only one eigenvector with $T(X) = \lambda_0 X$ where λ_0 is the largest eigenvalue, and where X is positive semi-definite, and so for any eigenvector $T(Y) = \lambda Y$ with $Y \neq X$ we have that $|\lambda| < 1$.

We provide the following alternative characterizations:

Theorem 1.5.2. [54] T is said to be primitive if one of the following equivalent conditions holds:

i. There exists $n \in \mathbb{N}$ such that for all density matrices $\rho \in M_d(\mathbb{C})$ we have that $T^n(\rho) > 0$.

ii. T^k is irreducible for ever $k \in \mathbb{N}$ and the limit $T_0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{k:|\lambda_k|=1} (\lambda_k T)^n$ is irreducible.

iii. For all density matrices ρ the limit $\lim_{k\to\infty} T^k(\rho)$ exists and is independent of ρ; moreover the limit is a positive density matrix ρ_∞

iv. T has trivial peripheral spectrum and the corresponding unique eigenvector $T(\rho) = \rho$ that is positive-definite.

The proof can be found in [54], but requires some ergodic theory which is outside the scope of this thesis. However, the conditions will give some idea of how to verify this for some quasi-1-dimensional AKLT models.

1.6. Quantum ferromagnetism and antiferromagnetism

Very often we probe whether or not spins in a ground state are correlated at distance in $\mathbb{Z}$, and we often do this by appealing to the spin generators S_x^i at a site $x \in \mathbb{Z}$; when we have that $\omega(S_x^i) \neq 0$ we call this state **spontaneously magnetized**; when

$$\omega(S_x^i \cdot S_y^i) \geq \epsilon > 0 \tag{1.1}$$

where ϵ is independent of x, y, we call such a state **ferromagnetically ordered**, and when

$$\omega((-1)^{|x-y|} S_x^i \cdot S_y^i) \geq \epsilon > 0 \tag{1.2}$$

where ϵ is independent of x, y, we call such a state **antiferromagnetically ordered** or **Néel ordered**. Both of these situations are examples of detectable **long range order**, which in general is non-decaying correlation of observables at distance.

Much progress in statistical mechanics has been gained by studying simple toy models for certain physical properties of interest. In particular, the phenomenon of quantum magnetism has been studied via the

quantum Heisenberg model which is given by the Hamiltonian

$$H_{\text{Heis}} = J \cdot \sum_{x \in \mathbb{Z}} S_x \cdot S_{x+1} \tag{1.3}$$

where for simplicity we consider J constant. This model has some surprising properties in one dimension: namely that when $J < 0$ we have a pair of ground states that exhibit ferromagnetism; however for $J > 0$ the correlative behavior depends on whether the spins are whole-integer or half-integer [50]. Moreover the ground states for the antiferromagnet are conjectured by Haldane [14] to be gapless when the spin is half-integer, and gapped and unique when whole-integer. This conjecture was surprising for the time, and led directly to the development of the AKLT model, which is a particular choice of biquadratic extension of the above Hamiltonian of the form

$$H = \sum_{x \in \mathbb{Z}} J_2(S_x \cdot S_{x+1})^2 + J_1(S_x \cdot S_{x+1}) \tag{1.4}$$

As it turns out, the only thing that determines the overall physics is the ratio $\frac{\alpha}{\beta}$ so the above Hamiltonians can be parametrized by a single variable $\tan(\theta) = \frac{J_1}{J_2}$. The **AKLT model** is a choice of $\tan(\theta) = \frac{1}{3}$ which gives a frustration-free model; all the models in this class are **isotropic** or **SU(2)-invariant** (meaning that if all particles are rotated, the expectations of observables remain the same). The AKLT model is also (presumably) in the antiferromagnetic phase along with the spin-1 Heisenberg chain. A full characterization of the states parametrized by θ can be found in [50], this parametrization was used to map out the different phases of quantum matter in one dimension; these phases were then extensively studied [40], [39] and the AKLT model was found to be in the nontrivial Haldane phase [49] as well as the integer spin Heisenberg chains [51] (provided they are gapped, for which there is no rigorous proof). The conjectured phase diagram for the spin-1 chains is given in Figure 1.1

The unique place of the AKLT model in the spin-1 diagram allows for many important properties to be explored, and is often credited as the jumping-off point of the study of topological phases. As we will see, the AKLT model has many nice properties, each of which can be generalized in a different way to obtain different physically relevant results. However, the AKLT model generalizes readily to any general graph, and results about these graphs are very few.

The AKLT model is often a toy model to study antiferromagnetism, and it was conjectured [3] to be Néel ordered when the dimension of the regular lattice and its degree are high enough. Note this is in the face of the fact that the Hamiltonian is isotropic, thus the continuous SU(2) symmetry is spontaneously broken, and this phenomenon is called **spontaneous symmetry breaking**.

While we have the Mermin-Wagner theorem 1.2.3, this only applies to Hamiltonians with bounded energy, and the classical Hamiltonian associated to the AKLT model has unbounded energy, so simply ruling out Néel order in two or fewer dimensions in the AKLT model via the Mermin-Wagner theorem

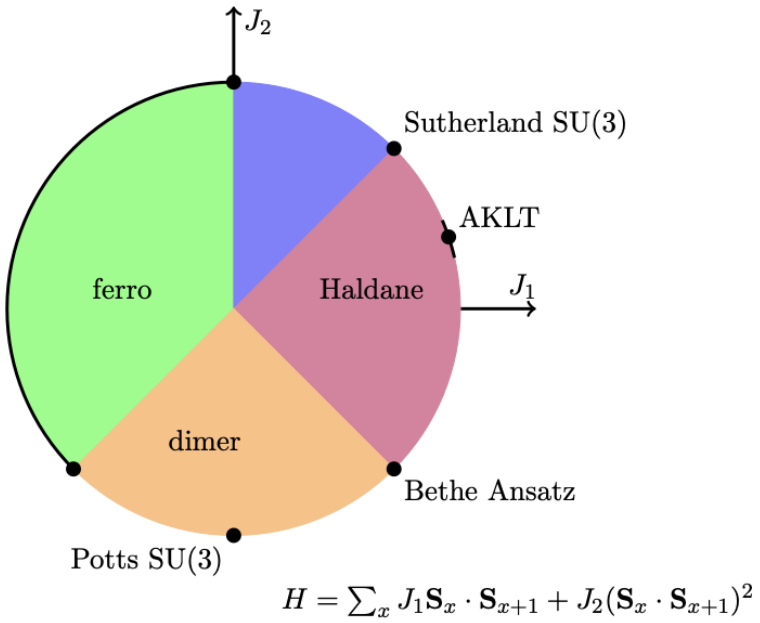


FIGURE 1.1. Conjectured phase diagram for the depicted Hamiltonian in terms of the variables J_1, J_2, from [35]

is not possible. Thus, as it stands, apart from Cayley trees, it is unknown on which lattices the minimal AKLT ground state as defined in [3] exhibits long range order. We expand this list, but fail to prove it for any of the familiar lattices such as the cubic; however we give a simple conjecture that applies to higher-dimensional lattices.

1.7. Outline of Results

In **Chapter 2** we show that an appropriately defined quasi-one-dimensional graph has a unique, stably gapped AKLT ground state.

In **Chapter 3** we give a verifiable condition for trees and treelike graphs that implies that the AKLT ground states on these lattices are degenerate and exhibit continuous symmetry breaking. We provide a conjecture for how this indicates higher-dimensional behavior.

In **Chapter 4** we provide a proof that the ground state of the AKLT model on the hexagonal lattice has a stable spectral gap, meaning that it is stable under sufficiently bounded perturbations, by showing it has two properties called ground state indistinguishability and local topological quantum order. Note: In the finalization process of this chapter, we found a bound used in [19] was not sufficient for one subcase of polymer enumeration bounds; these insufficient numbers are displayed in red and will be adjusted before publication.

In **Chapter 5** we provide a discussion of these results and an outlook for future research into these models.

CHAPTER 2

The AKLT Model in Quasi-One-Dimensional Systems

Introduction

The AKLT chain is a spin-1 model whose ground state has many interesting properties [3], [20] that have been generalized [9], [19] in many different and interesting ways. Much as the Ising model is a toy model for testing phenomena in classical statistical mechanics, the AKLT model and its generalizations are often toy models for understanding basic characteristics of quantum statistical mechanical systems. The AKLT model has been generalized to SU(n) symmetry [3], [13], [12], as well as other symmetry groups [52], [48], [47], along with many related systems [8], [34] with similar symmetries and ground state properties. The valence-bond solid picture which led to the now generally-used finitely-correlated state formulation [9] has a simple generalization in the case of the SU(2) AKLT model; that of a one-dimensional chain constructed out of identical copies of a finite bipartite graph; we call these models quasi-one-dimensional AKLT models and show they have many of the nice properties of the original AKLT chain.

This chapter is organized as follows:

In **Section 2.1** we follow [35] and introduce the AKLT Hamiltonian on a chain and illustrate the matrix product state formulation for the ground state expectations, the valence-bond solid picture, and the finitely-correlated state picture.

In **Section 2.2** and **Section 2.3** are repetitions of the arguments of [35] in showing that unique, frustration-free matrix product ground states with a primitive transfer operator have a spectral gap above the ground state. Readers familiar with [35] can skip this section as it is largely identical.

In **Section 2.4** is a repetition of the arguments of [36] showing that these finitely correlated states have local topological quantum order [6] and thus their ground states are stable under sufficiently local perturbations. Readers familiar with [36] can skip this section as it is largely identical.

In **Section 2.5** we prove that AKLT models on chains formed by stitching together identical copies of finite bipartite graphs have a unique frustration-free ground state with no continuous symmetry breaking, exponential decay of correlations, and a spectral gap which is stable under sufficiently decaying perturbations.

2.1. The AKLT Chain

The AKLT Hamiltonian [3] on a finite chain $[1,n] \subset \mathbb{Z}$ is defined as

$$H_{[1,n]} = \sum_{i\in[1,n]} \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{6}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 = \sum_{i\in[1,n]} P^{(2)}_{i,i+1} \tag{2.1}$$

where $\mathbf{S}_i = (S_i^{(1)}, S_i^{(2)}, S_i^{(3)})$ is the vector of 3-dimensional Pauli matrices corresponding to the spin-1 degree of freedom at each site, and $P^{(2)}$ is the projector onto the spin-2 subspace of neighboring spin-1's (see section on Clebsch-Gordan). Note then that for $n = 2$ that $\ker(H_{[1,2]}) = D^{(0)} \oplus D^{(1)}$ and so $\dim(\ker H_{[1,2]}) = 4$; we next show that $\dim(\ker H_{[1,n]}) = 4$ for all $n \geq 2$ and in fact the ground state energy is 0 for all finite chains and the AKLT Hamiltonian is frustration free.

We will use three different approaches to prove various desirable properties of the ground state.

2.1.1. Matrix product state formulation. From the Clebsch-Gordan decomposition of the tensor product of neighboring spin-1's

$$D^{(1)} \otimes D^{(1/2)} \cong D^{(1/2)} \oplus D^{(3/1)} \tag{2.2}$$

there is an isometry $V : \mathbb{C}^2 \to \mathbb{C}^3 \otimes \mathbb{C}^2$ which is unique up to a phase factor satisfying the intertwining property:

$$VD^{(1/2)}(g) = (D^{(1)}(g) \otimes D^{(1/2)}(g))V \tag{2.3}$$

for all $g \in \mathrm{SU}(2)$ One can compute the coefficients of this intertwiner using the Clebsch-Gordan coefficients (see earlier chapter):

$$V|m\rangle = \sum_{\substack{m_1,m_2 \\ m_1+m_2=m}} \langle 1, 1/2; m_1, m_2|1/2, m\rangle |1, 1/2; m_1, m_2\rangle \tag{2.4}$$

satisfying that $V^*V = \mathbb{1}$ Now we define the ground state $\psi^{(n)}_{\alpha\beta} \in \mathcal{H}_{[1,n]}$ in terms of the length of the chain and the boundary conditions $\alpha, \beta \in \mathbb{C}^2$ as

$$\psi^{(n)}_{\alpha\beta} = (\mathbb{1}_3^{\otimes n} \otimes \langle\beta|)(\mathbb{1}_3^{\otimes n-1} \otimes V)...(\mathbb{1}_3 \otimes V)V|\alpha\rangle \tag{2.5}$$

and thus by the intertwining property

$$\begin{aligned}(D^{(1)})^{\otimes n}\psi^{(n)}_{\alpha\beta} &= (\mathbb{1}_3^{\otimes n} \otimes \langle D^{(1/2)}\beta|)(D^(1)^{\otimes n} \otimes D^{(1/2)}V)...V|\alpha\rangle \\ &= (\mathbb{1}_3^{\otimes n} \otimes \langle D^{(1/2)}\beta|)(\mathbb{1}_3^{\otimes n-1} \otimes V)...(\mathbb{1}_3 \otimes V)V|D^{(1/2)}\alpha\rangle\end{aligned}$$

and so SU(2) acts on the space $\{\psi^{(n)}_{\alpha\beta} \mid \alpha, \beta \in \mathbb{C}^2\}$ by the representation $D^{(1/2)} \otimes D^{(1/2)} \cong D^{(1)} \oplus D^{(0)}$ and thus

$$P^{(2)}_{i,i+1}\psi^2\alpha\beta = 0 \tag{2.6}$$

for all α, β. The matrix product state description will allow us to state that in fact $\text{span}(\{\psi^{(n)}_{\alpha\beta} \mid \alpha, \beta \in \mathbb{C}^2\} - \ker H_{[1,n]})$. Let $|1\rangle$, $|0\rangle$, and $|-1\rangle$ be the standard basis of $\mathbb{C}^3$ and define

$$v_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 \end{bmatrix}, \qquad v_0 = \begin{bmatrix} -\sqrt{\frac{1}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} \end{bmatrix}, \qquad v_{-1} = \begin{bmatrix} 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 \end{bmatrix} \tag{2.7}$$

Then one can see by cyclicity of trace that

$$\langle i_1, ..., i_n|\psi^{(n)}_{\alpha\beta} = \langle\beta|v_{i_n}...v_{i_1}|\alpha\rangle = \text{Tr}|\alpha\rangle\langle\beta|v_{i_n}...v_{i_1} \tag{2.8}$$

and so

$$\psi^{(n)}_{\alpha\beta} = \sum_{i_1,...,i_n} \text{Tr}[|\alpha\rangle\langle\beta|v_{i_n}...v_{i_1}]|i_1, ..., i_n\rangle \tag{2.9}$$

and so these vectors can be extended linearly to a map on $\psi^{(n)} : M_2(\mathbb{C}) \to \mathcal{H}_{[1,n]}$ as

$$\psi^{(n)}(B) = \sum_{i_1,...,i_n} \text{Tr}[Bv_{i_n}...v_{i_1}]|i_1, ..., i_n\rangle \tag{2.10}$$

for all $B \in M_2(\mathbb{C})$ and thus $\text{span}\{\psi^{(n)}_{\alpha\beta}|\alpha, \beta \in \mathbb{C}^2\} = \{\psi^{(n)}(B)|B \in M_2(\mathbb{C})\}$
We now have enough to show the following proposition.

Proposition 2.1.1. [35]For all $n \geq 2$

$$\{\psi^{(n)}(B)|B \in M_2(\mathbb{C})\} \subseteq \ker H_{[1,n]} \tag{2.11}$$

Proof. We calculate for arbitrary $1 \leq i \leq n$

$$\langle\psi^{(n)}_{\alpha\beta}P^{(2)}_{i,i+1}\psi^{(n)}_{\alpha\beta}\rangle = \sum_{\substack{i_1,...,i_n \\ j_1,...,j_n}} \overline{\text{Tr}[Bv_{i_n}...v_{i_1}]}\text{Tr}[Bv_{j_n}...v_{j_1}]\langle i_1, ..., i_n|\mathbb{1}_{[1,i-1]} \otimes P^{(2)} \otimes \mathbb{1}_{[i+2,n]}|j_1...j_n\rangle \tag{2.12}$$

$$= \sum_{\substack{i_1,...,i_n \\ j_1,...,j_n}} \langle\psi^{(2)}(v_{i_{i-1}}...v_{i_1}P^{(2)}\psi^{(2)}(v_{i_{i-1}...v_{i_1}Bv_{i_n}...v_{i_{i+2}}})) \tag{2.13}$$

$$= 0 \tag{2.14}$$

from 2.6 □

Now we prove the reverse inclusion.

Proposition 2.1.2. [35] For all $n \geq 2$

$$\{\psi^{(n)}(B)|B \in M_2(\mathbb{C})\} = \ker H_{[1,n]} \tag{2.15}$$

Proof. First we show that

$$\dim\{\psi^{(2)}(B) \mid B \in M_2(\mathbb{C})\} = 4 \tag{2.16}$$

Now recall that

$$(D^{(1)} \otimes D^{(1)})\psi^{(2)}(B) = \psi^{(2)}((D^{(1/2)})^* B D^{(1/2)}) \tag{2.17}$$

and so $\psi^{(2)}(\mathbb{1})$ is a singlet and $\psi^{(2)}(\sigma^i)$ is a triplet representation of SU(2). Note that

$$\begin{aligned}\psi^{(2)}(\mathbb{1}) &= \mathrm{Tr}(v_{-1}v_1)|1,-1\rangle + \mathrm{Tr}(v_0^2)|00\rangle + \mathrm{Tr}(v_1 v_{-1})|-1,1\rangle \\ &= -\frac{2}{3}(|1,-1\rangle + |0,0\rangle + |-1,1\rangle)\end{aligned}$$

Thus our singlet representation is a non-zero vector. One can also verify this for $\psi^{(2)}(\sigma^i)$ and find a non-zero vector. Thus also $B \mapsto \psi^{(2)}(B)$ is injective.

We will show by induction that the maps $\psi^{(n)} : M_2(\mathbb{C}) \to \mathcal{H}_{[1,n]}$ are all injective for $n \geq 2$. Note that injectivity implies the existence of $c_n > 0$ such that $||\psi^{(n)}(B)||^2 \geq c_n \mathrm{Tr} B^* B$ We can compute the norm above explicitly from 2.14 and get

$$\begin{aligned}||\psi^{(n+1)}(B)||^2 &= \sum_{i_{n+1}} \sum_{i_1,\ldots,i_n} |\mathrm{Tr} B v_{i_{n+1}} v_{i_n} \ldots v_{i_1}|^2 \\ &= \sum_{i_{n+1}} ||\psi^{(n)}(B v_{i_{n+1}})||^2 \\ &\geq c_n \sum_{i_{n+1}} \mathrm{Tr}(B v_{i_{n+1}})^* B v_{i_{n+1}} \\ &= c_n \mathrm{Tr}\left[\sum_{i_{n+1}} v_{i_{n+1}} v_{i_{n+1}}^*\right] B^* B \\ &= c_n \mathrm{Tr} B^* B\end{aligned}$$

where the last equality follows from 2.7, and thus $\psi^{(n+1)}$ is injective.

Next we use the valence bond solid picture to show that the ground state space has a particular intersection property; namely if

$$\mathcal{G}_n = \{\psi^{(n)}(B) \mid B \in M_2(\mathbb{C})\} \tag{2.18}$$

First we show that

$$\ker H_{[1,3]} = (\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2) = \mathcal{G}_3 \tag{2.19}$$

We already have that $\mathcal{G}_3 \subset \ker H_{[1,3]}$ from 2.15 and we know that $\psi^{(3)}$ is injective and so $\dim\{\mathcal{G}_3\} = 4$ Thus we can prove the equality if $\dim H_{[1,3]} \leq 4$. Note that $\mathcal{G}_2 \otimes \mathbb{C}^3$ can be decomposed into irreducible representations of SU(2). For $\phi \in \ker H_{[1,3]}$ we have $\phi \in \mathcal{G}_2 \otimes \mathbb{C}^3$ and $\mathbb{1} \otimes P^{(2)}\phi = 0$; from before we know that $\mathcal{G}_2 = W_0 \oplus W_1$ where W_0 and W_1 are singlet and triplet representations of SU(2) respectively. Thus in the decomposition of $\mathcal{G}_2 \otimes \mathbb{C}^3$ one has two triplets and one singlet and one spin-2 representation. Moreover the highest weight vector in the spin-2 representation is $\xi_{22} = |1,0,1\rangle - |0,1,1\rangle$ and the highest weight spin-1 vector is $\xi_{11} = |1,-1,1\rangle - |0,0,1\rangle + |-1,1,1\rangle$. Note also that $(\mathbb{1} \otimes P^{(2)})\xi_{22} \neq 0$ and since this representation is unique inside $\mathcal{G}_2 \otimes \mathbb{C}^3$, this vector must be orthogonal to $\ker H_{[1,3]}$. Similarly $(\mathbb{1} \otimes P^{(2)})\xi_{11} \neq 0$. Thus since there are only a single singlet and triplet left, one has that $\dim \ker H_{[1,3]} \leq 4$ and we have the stated equality for $n = 3$ This is the base case; next we need the inductive step; let $l, r \geq 0$ and $m \geq 2$. We show that

$$\mathcal{G}_{l+m} \otimes \mathcal{H}_{[1,r]} \cap (\mathcal{H}_{[1,l]} \otimes \mathcal{G}_{m+r}) = \mathcal{G}_{l+m+r} \tag{2.20}$$

Define the indices $\mathbf{i} = (i_1, ..., i_l)$,$\mathbf{j} = (j_1, ..., j_m)$ and

$$\mathbf{k} = (k_1, ..., k_r) \tag{2.21}$$

with associated product $v_{\mathbf{i}} = v_{i_l} ... v_{i_1}$. Note we have $C_{\mathbf{i}}, D_{\mathbf{k}} \in M_2(\mathbb{C})$ such that for all $\phi \in \mathcal{G}_{l+m} \otimes \mathcal{H}_{[1,r]} \cap (\mathcal{H}_{[1,l]} \otimes \mathcal{G}_{m+r})$ we have

$$\phi = \sum_{\mathbf{i}} |\mathbf{i}\rangle \otimes \psi^{(m+r)}(C_{\mathbf{i}}) = \sum_{\mathbf{k}} \psi^{(l+m)}(D_{\mathbf{k}}) \otimes |\mathbf{k}\rangle \tag{2.22}$$

and find

$$0 = \mathrm{Tr} C_{\mathbf{i}} v_{\mathbf{j}} v_{\mathbf{k}} - \mathrm{Tr} D_{\mathbf{k}} v_{\mathbf{i}} v_{\mathbf{j}} = \mathrm{Tr}\left[v_{\mathbf{k}} C_{\mathbf{i}} - D_{\mathbf{k}} v_{\mathbf{i}}\right] v_{\mathbf{j}} \tag{2.23}$$

and thus for all $\mathbf{i}, \mathbf{k}$ we have $\psi^{(m)}(v_{bk} C_{\mathbf{i}} - D_{\mathbf{k}} v_{\mathbf{i}}) = 0$; from the above $\psi^{(m)}$ is injective and so $v_{bk} C_{\mathbf{i}} - D_{\mathbf{k}} v_{\mathbf{i}} = 0$ for all $\mathbf{i}, \mathbf{k}$. Note this implies that

$$\sum_{\mathbf{k}} (v_{\mathbf{k}}^* v_{\mathbf{k}}) C_{\mathbf{i}} = \left(\sum_{\mathbf{k}} v_{\mathbf{k}}^* D_{\mathbf{k}}\right) v_{\mathbf{i}} \tag{2.24}$$

and since the v_i's form an isometry we have

$$C_{\mathbf{i}} = B v_{\mathbf{i}} \tag{2.25}$$

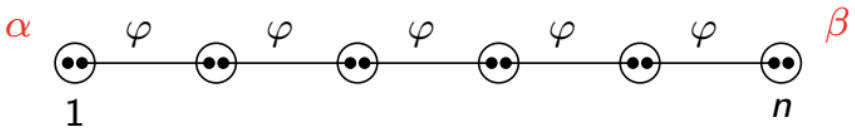


FIGURE 2.1. Valence bond solid picture of the AKLT Chain on n sites, courtesy of Amanda Young

where $B = \sum_{\mathbf{k}} v_{\mathbf{k}}^* D_{\mathbf{k}}$ and so

$$\phi = \psi^{(l+m+r)}(B) \tag{2.26}$$

and so we have 2.20.

The foregoing results mean that

$$\begin{aligned}
\ker H_{[1,n]} &= \bigcap_{x=1}^{n-1} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_2 \otimes \mathcal{H}_{[c+2,n]} \\
&= \bigcap_{x=1}^{n-1} \mathcal{H}_{[1,x-1]} \otimes ((\mathcal{G}_2 \otimes \mathbb{C}^3) \cap (\mathbb{C}^3 \otimes \mathcal{G}_2)) \otimes \mathcal{H}_{[c+3,n]} \\
&= \bigcap_{x=1}^{n-1} \mathcal{H}_{[1,x-1]} \otimes \mathcal{G}_3 \otimes \mathcal{H}_{[c+3,n]} \\
&\vdots \\
&= \mathcal{G}_n
\end{aligned}$$

using 2.20 and 2.15. □

One can visualize the above property as the valence bond representation of the state as in Figure 2.1. These two formulations of the state are useful for uniqueness, but another interpretation as a finitely correlated state allows one to calculate correlations in the ground state easily.

From the definition

$$\psi_{\alpha\beta}^{(n)} = \sum_{i_1,\dots,i_n} \mathrm{Tr}[|\alpha\rangle\langle\beta|v_{i_n}\dots v_{i_1}]|i_1,\dots,i_n\rangle \tag{2.27}$$

one has for $A_1,\dots,A_n \in M_3(\mathbb{C})$ that

$$\begin{aligned}
&\langle\psi_{\alpha\beta}^{(n)}|A_1 \otimes \dots \otimes A_n \psi_{\alpha\beta}^{(n)}\rangle \\
&= \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_n}} (A_1)_{i_1 j_1}\dots(A_n)_{i_n j_n} \overline{\langle\beta|v_{i_n}\dots v_{i_1}|\alpha\rangle}\langle\beta|v_{j_n}\dots v_{j_1}|\alpha\rangle \\
&= \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_n}} (A_1)_{i_1 j_1}\dots(A_n)_{i_n j_n} \langle\alpha|v_{i_1}^*\dots v_{i_n}^*|\beta\rangle\langle\beta|v_{j_n}\dots v_{j_1}|\alpha\rangle
\end{aligned}$$

$$= \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_n}} (A_1)_{i_1 j_1} \dots (A_n)_{i_{n-1} j_{n-1}} \langle \alpha | v_{i_1}^* \dots v_{i_{n-1}}^* \left[\sum_{i_n, j_n} (A_n)_{i_n j_n} v_{i_n}^* |\beta\rangle\langle\beta| v_{j_n} \right] \cdot v_{j_{n-1}} \dots v_{j_1} |\alpha\rangle$$

One can define the map $\mathbb{E}_A : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ for all $A \in M_3(\mathbb{C})$ by

$$\mathbb{E}_A(B) = \sum_{ij} A_{ij} v_i^* B v_j = V^*(A \otimes B)V \tag{2.28}$$

Note that the form of $\mathbb{E}$ implies it is a completely positive map [54]. This map allows us to express the expectations above in a simpler form as

$$\begin{aligned} \langle \psi_{\alpha\beta}^{(n)} | A_1 \otimes \dots \otimes A_n \psi_{\alpha\beta}^{(n)} \rangle &= \sum_{\substack{i_1,\dots,i_n \\ j_1,\dots,j_n}} (A_1)_{i_1 j_1} \dots (A_n)_{i_{n-1} j_{n-1}} \langle \alpha | v_{i_1}^* \dots v_{i_n}^* \left[\mathbb{E}_{A_n}(|\beta\rangle\langle\beta|) \right] v_{j_{n-1}} \dots v_{j_1} |\alpha\rangle \\ &= \langle \alpha | \mathbb{E}_{A_1} \circ \dots \circ \mathbb{E}_{A_n}(|\beta\rangle\langle\beta|) | \alpha \rangle \end{aligned}$$

One can then compute the thermodynamic limit readily by adding $l+1$ sites to the left and r sites to the right of the interval $[1, n]$ and calculate the expectation of an observable $A = \mathbb{1}^{\otimes l} \otimes A_1 \otimes \dots \otimes A_n \otimes \mathbb{1}^{\otimes r} \in \mathcal{A}_{[-l+1, n+r]}$ in the state $\psi_{\alpha\beta}^{l+n+r}$ as

$$\begin{aligned} \langle \psi_{\alpha\beta}^{(l+n+r)} | A \psi_{\alpha\beta}^{(l+n+r)} \rangle &= \langle \alpha | \mathbb{E}_{\mathbb{1}}^l \circ \mathbb{E}_{A_1} \circ \dots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^r(|\beta\rangle\langle\beta|) | \alpha \rangle \\ &= \mathrm{Tr} |\alpha\rangle\langle\alpha| \mathbb{E}_{\mathbb{1}}^l \circ \mathbb{E}_{A_1} \circ \dots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^r(|\beta\rangle\langle\beta|) \\ &= \mathrm{Tr}((\mathbb{E}_{\mathbb{1}}^T)^l(|\alpha\rangle\langle\alpha|) \mathbb{E}_{A_1} \circ \dots \circ \mathbb{E}_{A_n} \circ \mathbb{E}_{\mathbb{1}}^r(|\beta\rangle\langle\beta|) \end{aligned}$$

where $\mathbb{E}_{\mathbb{1}}^T$ is the transpose with respect to the Hilbert-Schmidt inner product on $M_2(\mathbb{C})$. This map is called the transfer operator and its spectral properties determine the infinite volume limits and thus correlative behavior. One has the following diagonalization which we prove in the following section

$$\mathbb{E}_{\mathbb{1}}(\mathbb{1}) = \mathbb{1}, \qquad \mathbb{E}_{\mathbb{1}}(\sigma^i) = -\frac{1}{3}\sigma^i \tag{2.29}$$

Note that since the identity and Pauli's span $M_2(\mathbb{C})$ we have

$$B = \frac{1}{2}(\mathrm{Tr}(B))\mathbb{1} + \frac{1}{2}\sum_{i=1}^{3}(\mathrm{Tr}B\sigma^i)\sigma^i \tag{2.30}$$

we then express

$$\mathbb{E}_{\mathbb{1}}(B) = \frac{1}{2}(\mathrm{Tr}B)\mathbb{1} - \frac{1}{3}\left[B - \frac{1}{2}(\mathrm{Tr}B)\mathbb{1}\right] \tag{2.31}$$

and thus by iteration

$$\mathbb{E}_{\mathbb{1}}^p(|\beta\rangle\langle\beta|) = \frac{1}{2}||\beta||^2\mathbb{1} + \left(-\frac{1}{3}\right)^p \left[|\beta\rangle\langle\beta| - \frac{1}{2}||\beta||^2\mathbb{1}\right] \tag{2.32}$$

Which implies

$$\lim_{l\to\infty,r\to\infty}\langle\alpha|\mathbb{E}_{\mathbb{1}}^{l}\circ\mathbb{E}_{A_1}\circ...\circ\mathbb{E}_{A_n}\circ\mathbb{E}_{\mathbb{1}}^{r}(|\beta\rangle\langle\beta|)|\alpha\rangle=\frac{||\alpha||^2||\beta||^2}{2}\mathrm{Tr}\left(\frac{1}{2}\mathbb{1}\right)\mathbb{E}_{A_1}\circ...\circ\mathbb{E}_{A_n}(\mathbb{1}) \tag{2.33}$$

$$=\omega(A_1\otimes...\otimes A_n) \tag{2.34}$$

uniquely defines a translation invariant pure state on the quasi-local observable algebra $\mathcal{A}_{\mathbb{Z}}$ by extension from pure tensor observables.

Note that 2.32 implies that

$$||\mathbb{E}_{\mathbb{1}}^{p}-\frac{1}{2}(\mathrm{Tr}B)\mathbb{1}||\leq\frac{2}{3^p} \tag{2.35}$$

and so letting $A_i=\mathbb{1}$ for all $i\neq 1,n$ we can calculate the two-point correlator directly as

$$\begin{aligned}|\omega(A_1\otimes\mathbb{1}\otimes...\otimes\mathbb{1}\otimes A_n)-\omega(A_1)\omega(A_n)|&=\left|\frac{1}{2}\mathrm{Tr}\mathbb{E}_{A_1}\circ(\mathbb{E}_{\mathbb{1}}^{n-2})-\frac{1}{2}(\mathrm{Tr}B)\mathbb{1})\circ\mathbb{E}_{A_n}\right|\\&\leq||A_1||||A_n||\frac{C}{3^n}\end{aligned}$$

and so the infinite-volume state has exponential decay of correlations. However we have not shown that ω is unique.

Proposition 2.1.3. [35] ω defined above is the unique state on $\mathcal{A}_{\mathbb{Z}}$ such that $\omega(P_{x,x+1}^{(2)})=0$ for all $x\in\mathbb{Z}$

Proof. Every state η on $\mathcal{A}_{\mathbb{Z}}$ is uniquely determined by its restrictions to $\mathcal{A}_{[a,b]}$ where $a<b$. Denote by $\rho_{[a,b]}$ the density matrices of η restricted to $\mathcal{A}_{[a,b]}$. Then if $\eta(P_{x,x+1}^{(2)})=0$ for $x=a,...,b-1$ implies that $\mathrm{ran}\rho_{[a,b]}\subset\mathcal{G}_{b-a+1}$. Then 2.34 and Proposition 2.1.2 imply that for all $a_1<b_1\in\mathbb{Z}, A_{a_1},...,A_{b_1}\in M_3(\mathbb{C})$ we have

$$\begin{aligned}\eta(A_{a_1}\otimes...\otimes A_{b_1})&=\lim_{\substack{a\to-\infty\\b\to\infty}}\mathrm{Tr}\rho_{[a,b]}\mathbb{1}_{[a,a_1-1]}\otimes A_{a_1}\otimes...\otimes A_{b_1}\otimes\mathbb{1}_{[b_1+1,b]}\\&=\omega(A_{a_1}\otimes...\otimes A_{b_1})\end{aligned}$$

and thus ω is unique. □

Thus we have that the AKLT model on the 1-dimensional chain is unique and has exponential decay of correlations.

2.2. Spectral Gap of Unique, Frustration-Free Matrix Product Ground States

We follow the proof of [35] to show that unique, frustration free Hamiltonians with a matrix product ground state have a spectral gap, using a method known as the martingale method.

to each site x on the half-infinite lattice $\mathbb{N}$ we associate a spin $s_x \in \mathbb{N}$ with $\mathcal{H}_x = \mathbb{C}^d$, meaning the dimension is independent of x, so the spin degrees of freedom are translation invariant; defining our finite-volume Hamiltonian as

$$H_{[1,L]} = \sum_{x=1}^{L-1} h_{x,x+1} \tag{2.1}$$

where $0 \leq h_{x,x+1} = h \in \mathcal{A}_{[x,x+1]}$ and $\ker(h) \neq \emptyset$ for all $L \geq 2$. Letting $\mathcal{G} = \ker h \subset \mathbb{C}^d \otimes \mathbb{C}^d$. The frustration free property amounts to

$$\bigcap_{x=1}^{L-1} \mathbb{C}^d \otimes ... \otimes \mathbb{C}^d \otimes \mathcal{G} \otimes \mathbb{C}^d \otimes ... \otimes \mathbb{C}^d \neq \{0\} \tag{2.2}$$

where $\mathcal{G}$ occurs at the xth index, and $L \geq 2$. Any state such that $\omega(h_{x,x+1}) = 0$ for all $1 \leq x \leq L-1$ has weak-$*$ limit points, and any such limiting state satisfies $\omega(h_{x,x+1}) = 0$ for all $x \in \mathbb{N}$; these are by the definition 1.19 ground states of the infinite-volume system; any linear combination of them is also a ground state, so the set of pure ground states is also non-empty. Now suppose that there exists h such that $\mathcal{G} = \ker h$ satisfies 2.2 and suppose ω is a pure state on $\mathcal{A}_{\mathbb{Z}}$ such that $\omega(h_{x,x+1}) = 0$ for all $x \geq 1$. Then given a basis $\{|i\rangle\}$ of $\mathbb{C}^d$, we have a finite-dimensional Hilbert space $\mathcal{K}$ and a unit vector $\Omega \in \mathcal{K}$ and a set of operators $V_1, ..., V_d \in \mathcal{B}(\mathcal{K})$:

i. $\mathcal{K}$ is the closed linear span of $\{V_{i_1} \cdot V_{i_n} \Omega\}$

ii. $\sum_{\eta=1}^{d} V_\eta^* V_\eta$

iii. For all $n \geq 1$ and index sets $\{i_a\}_{a=0}^d$ and $\{j_a\}_{a=0}^d$ we have

$$\omega(|i_1, ..., i_n\rangle\langle j_1, ..., j_n|) = \langle V_{i_1} ... V_{i_1} \Omega, V_{j_n} ... V_{j_1} \Omega \rangle \tag{2.3}$$

iv. For every $\psi = \sum_{i,j=1}^{d} \psi_{i,j} |i, j\rangle \in \mathcal{G}^\perp$ we have

$$\sum_{i,j=1}^{d} \overline{\psi_{i,j}} V_j V_i = 0 \tag{2.4}$$

v. The induced operator

$$\mathbb{E}(X) = \sum_{i=1}^{d} V_i^* X V_i \tag{2.5}$$

is completely-positive, trace preserving, and unital with unique trace-one fixed point $X = \mathbb{1}$.

Since the V_i's are assumed to be finite matrices this implies that the eigenvalue $1 \in \text{spec}(\mathbb{E})$ is simple, and since the transpose also has simple eigenvalue 1 ([54]), there is a unique density matrix such that $\mathbb{E}^T(\rho) = \rho$ which is positive, and so $\text{Tr}(\rho \mathbb{E}(B)) = \text{Tr}(\rho B)$. Substituting ρ for $|\Omega\rangle\langle\Omega|$ the expectations

$\omega(A_1 \otimes ... \otimes A_n)$ are translation invariant and

$$\omega(\mathbb{1} \otimes A_1 \otimes ... \otimes A_n) = \omega(A_1 \otimes ... \otimes A_n) \tag{2.6}$$

Thus finite chain expressions $\omega(A_1 \otimes ... \otimes A_n) = \mathrm{Tr}(\rho \mathbb{E}_{A_1} \circ ... \circ \mathbb{E}_{A_n}(\mathbb{1}))$ defines a unique state on $\mathcal{A}_{\mathbb{Z}}$ which is also translation invariant.

One of the basic assumptions in order to obtain a gap is that the transfer operator $\mathbb{E}$ be primitive 1.5.2 [9]. Since ρ has trivial kernel, we can define a non-degenerate inner-product on $M_k(\mathbb{C})$ by

$$\langle A, B \rangle_\rho = \mathrm{Tr}(\rho A^* B) \tag{2.7}$$

with induced norm $||\cdot||_\rho$. Letting $\rho_{\min} = \min \mathrm{spec}(\rho) > 0$ since ρ is finite-dimensional; then $||\cdot||_\rho$ is equivalent to the Hilbert-Schmidt norm, with explicit constant given by $\rho_{\min}$

$$||A||_2^2 = \mathrm{Tr}[A^* A] \leq \mathrm{Tr}[\frac{\rho}{\rho_{\min}} A^* A] = \frac{1}{\rho_{\min}} ||A||_\rho^2 \tag{2.8}$$

The primitivity of $\mathbb{E}$ implies that $||\mathbb{E}^n - |\mathbb{1}\rangle\langle\rho|| \leq C\lambda^n$ for some constant $C > 0$ and $\lambda \in (0, 1)$ and from the unitality of $\mathbb{E}$ we get that the sequence $|\mathbb{E}^{n+1} - |\mathbb{1}\rangle\langle\rho|| = |\mathbb{E} \circ (\mathbb{E}^n - |\mathbb{1}\rangle\langle\rho)|| \leq |\mathbb{E}^n - |\mathbb{1}\rangle\langle\rho||$ is monotonically decreasing in n. From before we had the set of maps $\psi^n(\cdot)$ we used 2.10 in the AKLT model, we define a set of maps $\Gamma_n : \mathcal{M}_k(\mathbb{C}) \to \mathcal{H}_{[1,n]}$ via

$$\Gamma_n(B) = \sum_{i_1,...,i_n} \mathrm{Tr}[B v_{i_n} ... v_{i_1}] |i_1, ..., i_n\rangle, \quad B \in \mathcal{M}_k(\mathbb{C}) \tag{2.9}$$

The form of the above Γ's is that of a **matrix product state**, meaning the coefficients of the state are given by taking the trace over an appropriate matrix product. These states are much easier to deal with in general, as we will see.

Lemma 2.2.0.1. [35]

$$|\langle \Gamma_n(B), \Gamma_n(C) \rangle - \langle B, C \rangle_\rho| \leq \frac{k}{\rho_{\min}} |\mathbb{E}^n - |\mathbb{1}\rangle\langle\rho||| B||_\rho ||C||_\rho \tag{2.10}$$

The proof can be found in [35] Lemma (11.4). Defining the sequence

$$b(c) = \frac{k|\mathbb{E}^n - |\mathbb{1}\rangle\langle\rho||}{\rho_{\min}} \tag{2.11}$$

One has that

$$||B||_\rho \sqrt{1 - b(n)} \leq ||\Gamma_n(B)|| \leq ||B||_\rho \sqrt{1 + b(n)} \tag{2.12}$$

PROOF. From 2.2.0.1 we immediately have

$$|||\Gamma_n(B)||^2 - ||B||_\rho^2| \leq b(n)||B||_\rho^n \tag{2.13}$$

If $B = 0$ the proof is trivial; otherwise dividing we have

$$-b(n) \leq \frac{||\Gamma_n(B)||^2}{||B||_\rho^2} - 1 \leq b(n) \tag{2.14}$$

from which the result follows. □

We also have the following bound from matrix norm inequalities and 2.8

$$\sum_{j=1}^{k} ||B^* \alpha_j|| \leq \sqrt{\frac{k}{\rho_{\min}}} ||B||_\rho \tag{2.15}$$

We can then produce the useful bound

$$\sum_{j=1}^{k} ||B^* \alpha_j|| \leq \sqrt{\frac{k}{\rho_{\min}}} ||B||_\rho \tag{2.16}$$

which shows that there exists an n_0 such that for all $n \geq n_0$ the map Γ_n is injective.

Now in one dimension, we will split up three intervals ordered from left to right of lengths l, m, r. When $m \geq n_0$ we have that Γ_{m+r} and Γ_{m+l}, so there exist matrices $B_\phi(k_1, ..., k_r), B_\psi(i_1, ..., i_l) \in M_k(\mathbb{C})$ where

$$\varphi = \sum_{k_1,...,k_r} \Gamma_{l+m}(B_\varphi(k_1, ..., k_r)) \otimes |k_1, ..., k_r\rangle \tag{2.17}$$

and

$$\psi = \sum_{k_1,...,k_r} \Gamma_{l+m}(B_\psi(i_1, ..., i_r)) \otimes |i_1, ..., i_r\rangle \tag{2.18}$$

We can then define:

$$C_\varphi = \sum_{k_1,...,k_r} B_\varphi(k_1, ..., k_r) \rho v_{k_1}^* ... v_{k_r}^* \rho^{-1} \tag{2.19}$$

and

$$D_\psi = \sum_{i_1,...,i_r} v_{i_1}^* ... v_{i_l} B_\psi(i_1, ..., i_l) \tag{2.20}$$

We can define $\mathbf{i} = (i_1, ..., i_l), \mathbf{j} = (j_1, ..., j_m), \mathbf{k} = (k_1, ..., k_r)$ and where $v_\mathbf{i} = \prod_k v_{i_k}$ and similarly for $v_\mathbf{i}^*$

We then have the following fact:

Proposition 2.2.1. [35] Let m be such that $b(m) < 1$. Let $l, r \geq 0$; then for every $\varphi \in \mathcal{G}_{l+m} \otimes (\mathbb{C}^d)^{\otimes r}$ and $\psi \in (\mathbb{C}^d)^{\otimes l} \otimes \mathcal{G}_{m+r}$, then we have

$$|\langle \varphi, \psi \rangle - \langle C_\varphi, D_\psi \rangle_\rho| \leq \frac{b(m)}{1 - b(m)} ||\varphi|| \, ||\psi|| \tag{2.21}$$

The proof can be found in [35]. Using this, we can then use the commutation property, which can be extended (in limited cases) to higher dimensional systems [2]. First, we suppose that $\mathbb{E}$ is primitive with a unique density matrix ρ with eigenvalue 1 of $\mathbb{E}^T$, and assume $\ker(\rho) = \{0\}$, and let G_n be the orthogonal projections onto $\mathcal{G}_n$.

Proposition 2.2.2. Commutation Property [35] For all $m \geq 1, l, r \geq 0$ we have

$$|(G_{l+m} \otimes \mathbb{1}^{\otimes r})(\mathbb{1}^{\otimes l} \otimes G_{m+r}) - G_{l+m+r}| \leq \frac{b(m)}{(1 - b(m))^2} \tag{2.22}$$

Proof. Since G_{l+m+r} projects onto a subspace of the ranges of both $G_{l+m} \otimes \mathbb{1}^{\otimes r}$ and $\mathbb{1}^{\otimes l} \otimes G_{m+r}$, then we have the identity

$$(G_{l+m} \otimes \mathbb{1}^{\otimes r})(\mathbb{1}^{\otimes l} \otimes G_{m+r}) - G_{l+m+r} = (G_{l+m} \otimes \mathbb{1}^{\otimes r} - G_{l+m+r})(\mathbb{1}^{\otimes l} \otimes G_{m+r}) - G_{l+m+r}) \tag{2.23}$$

Now suppose that $\varphi \in \mathcal{G}_{l+m} \otimes (\mathbb{C})^{\otimes r}, \psi \in \mathbb{C}^{\otimes l} \otimes \mathcal{G}_{m+r}$ such that $G_{l+m+r}\varphi = G_{l+m+r}\psi = 0$ then

$$|\langle \varphi, \psi \rangle| \leq \frac{b(m)}{(1 - b(m))^2} ||\varphi|| \, ||\psi|| \tag{2.24}$$

The left-hand side can be estimated via equation 2.2.1; since $\varphi \in \mathcal{G}_{l+m} \otimes (\mathbb{C})^{\otimes r}$ we know there is a $A \in M_k(\mathbb{C})$ where $B_\varphi(\mathbf{k}) = Av_{\mathbf{k}}$ from which we evaluate

$$C_\varphi = \sum_{\mathbf{k}} A v_{\mathbf{k}} \rho v_{\mathbf{k}}^* \rho^{-1} = A(\mathbb{E}^T)^r(\rho)\rho^{-1} = A \tag{2.25}$$

Since $G_{l+m+r}\psi = 0$ which implies $\langle \Gamma_{l+m+r}(A), \psi \rangle = 0$ for all A. Thus we have

$$|\langle A, D_\psi \rangle_\rho| \leq \frac{b(m)}{1 - b(m)} ||\psi|| \, ||A||_\rho \tag{2.26}$$

so

$$||D_\psi||_\rho \leq \frac{b(m)}{1 - b(m)} ||\psi|| \tag{2.27}$$

and similarly $||C_\varphi||_\rho \leq \frac{b(m)}{1-b(m)} ||\varphi||$ From these last two inequalities we have the result. □

This is called the commutation property because it gives a bound on the commutators of the ground state projections:

$$||[G_{l+m} \otimes \mathbb{1}^{\otimes r}, \mathbb{1}^{\otimes l} \otimes G_{m+r}]|| \leq |(G_{l+m} \otimes \mathbb{1}^{\otimes r})(\mathbb{1}^{\otimes l} \otimes G_{m+r}) - G_{l+m+r}|$$

$$+|G_{l+m+r} - (\mathbb{1}^{\otimes l} \otimes G_{m+r})(G_{l+m} \otimes \mathbb{1}^{\otimes r}| \leq \frac{2b(m)}{1-b(m)}$$

This property has important consequences for the existence of the spectral gap and the area law for entanglement entropy. [15].

2.3. Martingale Method for Spectral Gap from the Commutation Property

For a frustration-free ground state the total ground state space is the intersection of all the ground state spaces for finite chains; thus we can produce a decreasing sequence $\mathcal{G}_N \subset ... \subset \mathcal{G}_1 \subset \mathcal{H}_N$, where $H_n \in \mathcal{B}(\mathcal{H}_N)$ and $\mathcal{G}_n = \ker(H_n)$. We then define the sequence:

$$h_n = \begin{cases} H_1 & n = 1 \\ H_n - H_{n-1} & 2 \leq n \leq N \end{cases} \tag{2.1}$$

and G_n and g_n the orthogonal projections onto $\ker(H_n)$ and $\ker(h_n)$ respectively. Then define

$$E_n = \begin{cases} \mathbb{1} - G_1 & n = 0 \\ G_n - G_{n+1} & 1 \leq n \leq N-1 \\ G_N & n = N \end{cases} \tag{2.2}$$

We can see that E_n are mutually orthogonal projections such that $\sum_{n=0}^{N} E_n = \mathbb{1}$. In order to use martingale method we need a few assumptions.

Theorem 2.3.1. [35] If the following three conditions are met:

i. There is a constant $\gamma > 0$ such that $h_n \geq \gamma(\mathbb{1} - g_n)$ for $1 \leq n \leq N$

ii. We have integers $l \geq 0$ and $r \geq 1$ such that $E_k g_n = g_n E_k$ if $k \notin [n-l, n+r]$

iii. There is $\varepsilon \in [0, \sqrt{l+r}]$ such that $E_n g_{n+1} E_n \leq \varepsilon^2 E_n$

Then we have that given $\psi \in \mathcal{H}_N$ with $G_N \psi = 0$ then

$$\langle \psi, H_N \psi \rangle \geq \frac{\gamma}{2} \left(1 - \varepsilon\sqrt{1+r+l}\right)^2 ||\psi||^2 \tag{2.3}$$

Proof. We have by assumption that $E_N \psi = G_N \psi = 0$ so

$$||\psi||^2 = \sum_{n=0}^{N-1} ||E_n \psi||^2 \tag{2.4}$$

so then for all n we have

$$||E_n\psi||^2 = \langle \psi, (\mathbb{1} - g_{n+1}) E_n \psi \rangle + \langle \psi, g_{n+1} E_n \psi \rangle = \langle \psi, (\mathbb{1} - g_{n+1}) E_n \psi \rangle + \langle \psi, \left(\sum_{m=0}^{N-1} E_k \right) g_{n+1} E_n \psi \rangle \tag{2.5}$$

and from the fact that the E_n's form a resolution of the identity we have

$$||E_n\psi||^2 \leq \langle\psi, (\mathbb{1} - g_{n+1})E_n\psi\rangle + \langle\psi, \left(\sum_{k=n-l}^{n+r} E_m\right) g_{n+1}E_n\psi\rangle \tag{2.6}$$

We also know that

$$|\langle\varphi_1, \varphi_2\rangle| \leq \frac{1}{2c}||\varphi_1||^2 + \frac{c}{2}||\varphi_2||^2 \tag{2.7}$$

for $c > 0$. Then we have that

$$||E_n\psi||^2 \leq \frac{1}{2c_1}\langle\psi, (\mathbb{1} - g_{n+1})\psi\rangle + \frac{c_1}{2}\langle\psi, E_n\psi\rangle + \frac{1}{2c_2}\langle\psi, E_n g_{n+1}E_n\psi\rangle + \frac{c_2}{2}\langle\psi, \left(\sum_{k=n-l}^{n+r} E_k\right)^2\psi\rangle \tag{2.8}$$

From our first and third assumption we can manipulate to get

$$\left(1 - \frac{c_1}{2} - \frac{\varepsilon^2}{2c_2}\right)||E_n\psi||^2 - \frac{c_2}{2}\sum_{k=n-l}^{n+r}||E_k\psi||^2 \leq \frac{1}{2c_1\gamma}\langle\psi, h_{n+1}\psi\rangle \tag{2.9}$$

Now taking the sum over $0 \leq n \leq N-1$ to get

$$\langle\psi, H_N\psi\rangle \geq 2c_1\gamma\left[1 - \frac{c_1}{2} - \frac{\varepsilon^2}{2c_2} - \frac{c_2(1+l+r)}{2}\right]||\psi||^2 \tag{2.10}$$

We can then maximize this by choosing $c_1 = 1 - \varepsilon\sqrt{1+l+r}$ and $c_2 = \frac{\varepsilon}{\sqrt{1+l+r}}$ yields the desired result. □

Now we use the proof to show that the AKLT chain is gapped.

Theorem 2.3.2. The AKLT Hamiltonian $\sum_{x\in\mathbb{Z}} P^{(2)}_{x,x+1}$ has a gap above the ground state.

Proof. We have that

$$h_n = \sum_{x=(n-1)m+1}^{(n+1)m-1} P^{(2)}_{x,x+1} \tag{2.11}$$

So we have

$$H_n = \sum_{k=1}^{n} h_k \tag{2.12}$$

so

$$H_{[1,(n+1)m]} \leq H_n \leq 2H_{[1,(n+1)m]} \tag{2.13}$$

and

$$E_n = G_{(n+1)m} \otimes \mathbb{1}^{\otimes(N-n)m} - G_{(n+2)m} \otimes \mathbb{1}^{\otimes(N-n-1)m} \tag{2.14}$$

and finally

$$g_n = \mathbb{1}^{\otimes(n-1)m} \otimes G_{2m} \otimes \mathbb{1}^{\otimes(N-n)m} \tag{2.15}$$

We have that the first assumption of theorem 2.3.1 is given by the spectral gap of $H_{[(n-1)m+1,(n+1)m]}$; since m is fixed and the model is translation invariant this gap is positive and independent of n. Letting $l = 0$ and $r = 1$ we have the second assumption. The last assumption is guaranteed by the commutation property, since

$$||g_{n+1}E_n|| = || \left(\mathbb{1}^{\otimes(n-1)m} \otimes G_{2m}\right) \left(G_{(n+1)m} \otimes \mathbb{1}^{\otimes m} - G_{(n+2)m}\right) \leq \frac{b(m)}{(1-b(m))^2} \tag{2.16}$$

Then for any two orthogonal projections we have

$$||GE|| \leq \varepsilon \iff EGE \leq \varepsilon^2 E \tag{2.17}$$

Noticing then that the AKLT chain satisfies all three of these assumptions we have a nonzero spectral gap above the infinite volume ground state. □

The main upshot of all of this is that when we can guarantee that the model is frustration-free with a unique matrix-product ground state and that the transfer operator has simple eigenvalue one, we can find a non-zero spectral gap. In the case of the quasi-one dimensional AKLT models, we have the first two are guaranteed by construction, so the main work is showing that the transfer operator has simple eigenvalue 1. From the previous result we can also prove a somewhat trivial (in one dimension) extension of this result for states with such a primitive transfer operator: that the spectral gap is stable under sufficiently local perturbations.

2.4. LTQO for Finitely correlated states

Local topological quantum order is a condition from which the stability of the spectral gap can be derived. An important consequence of the above is that one-dimensional states with such properties satisfy a form of the LTQO condition 4.0.2, which we will state but not prove. In order for the definition of LTQO to make sense we need the notion of an indistinguishability radius.

Definition 2.4.1. Let $\Omega : \mathbb{R} \to [0, \infty)$ be a non-increasing function. The **indistinguishability radius** of H_Λ at $x \in \Lambda$, denoted by $r_x^\Omega(\Lambda)$ is the largest integer $r_x^\Omega(\Lambda) \leq \text{diam}(\Lambda)$ such that for all integers $0 \leq k \leq n \leq r_x^\Omega(\Lambda)$ and all observables $A \in \mathcal{A}_{b_x^\Lambda(k)}$ we have

$$||P_{b_x^\Lambda}(n)AP_{b_x^\Lambda(n)} - \omega(A)P_{b_x^\Lambda(n)}|| \leq |b_x^\Omega(k)||\,||A||\Omega(n-k) \tag{2.1}$$

LTQO is then informally defined as being the property that for a fixed $x \in \Gamma$ the indistinguishability radius $r_x^{\Omega}(\Lambda) \to \infty$ as system size increases and the distance of x to the boundary diverges. The rate at which this occurs determines whether this has consequences for the stability of the spectral gap. Explicitly, we use a less strict version of the following satisfactory condition for LTQO.

Theorem 2.4.2. [37] In one dimension, if $\omega : \mathcal{A}_\Gamma$ is a unique frustration-free ground state then ω has local topological quantum order (LTQO) if (but not only if) for finite-volume projections P_n corresponding to finite volumes Λ_n we have for $A \in \mathcal{A}_\Gamma$ with $\mathrm{supp}(A) \subset \Lambda_k$

$$||P_n A P_n - \omega(A) P_n|| \leq ||A|| \cdot |\Lambda_n| \cdot G(n-k) \tag{2.2}$$

where $G(m) \leq Ce^{-\lambda m}$ where $C > 0$ and $\lambda > 0$.

This convergence does not have to be exponential in general for LTQO to be satisfied [36].
We can show that for a finitely-correlated state that $G(n-k) \leq e^{-\lambda(n-k)}$ for some $\lambda > 0$. We give the full proof for a finite-rank matrix product state with unique gapped ground state, which comes almost wholesale from [36].
Assume we have a state space of the same on-site Hilbert space, so that $\mathcal{H}_x = \mathbb{C}^d$ for all $x \in \mathbb{Z}$, and assume that the matrix product state is translation invariant and has a primitive transfer operator $\mathbb{E} : M_k \to M_k$. For our case, this Hilbert space will be that of the finite patch Λ, which has bond dimension $|\partial\Lambda|/2$. We have an orthonormal basis $\mathcal{H}_x$ and a density matrix $\rho \in M_k$ for which $\mathbb{E}$ with $\mathbb{E}(B) := \sum_{i=1}^d v_i^* B v_i$; the primitive assumption guarantees that ρ is invertible [54] and the existence of a $\lambda \in [0, 1)$ and $c > 0$ such that

$$||\mathbb{E}^n - \mathbb{E}^\infty|| \leq c\lambda^n \tag{2.3}$$

where $\mathbb{E}^\infty(B) = \mathrm{Tr}(\rho B)\mathbb{1}$ For the resulting matrix product state defined as a map on $B \in M_k$ as

$$\Gamma_{[l,r]}(B) = \sum_{i_1,\ldots,i_r=1}^{d} \mathrm{Tr}(B v_{i_l} \ldots v_{i_r}) |i_l \ldots i_r\rangle \tag{2.4}$$

we have a finite-range frustration-free interaction on $\mathcal{A}_{\mathbb{Z}}^{\mathrm{loc}}$ so that the ground state space for each of the corresponding local Hamiltonians coincides, meaning $\ker H_{[l,r]} = \mathrm{ran}\Gamma_{[l,r]}$. Moreover, the assumptions guarantee that this model has a unique ground state in the thermodynamic limit $\omega : \mathcal{A}_{\mathbb{Z}} \to \mathbb{C}$ defined by

$$\omega(A) = \mathrm{Tr}(\rho \mathbb{E}_A(\mathbb{1})) \tag{2.5}$$

where for each $A \in \mathcal{A}_{[a,b]}$ and $B \in M_k$ we have

$$\mathbb{E}_A(B) = V_{[a,b]}^* A \otimes B V_{[a,b]} \tag{2.6}$$

and

$$V_{[a,b]} = \sum_{i_a,\ldots,i_b=1}^{d} |i_a \ldots i_b\rangle \otimes v_{i_a} \ldots v_{i_b} \tag{2.7}$$

For any finite interval $[a,b]$ it is easy to verify that

$$V^*_{[a,b]} V_{[a,b]} = \mathbb{E}^{b-a+1}(\mathbb{1}) = \mathbb{1} \tag{2.8}$$

and the state ω is consistent under the identification $A \mapsto A' = \mathbb{1}^{\otimes n} \otimes A \otimes \mathbb{1}^{\otimes m} \in \mathcal{A}_{[a-n,b+m]}$. We can then establish a ground-state indistinguishability result in the following way.

$$||P_{[l,r]} A P_{[l,r]} - \omega(A) P_{[l,r]}|| = \sup_{\Gamma_{[l,r]}(B), \Gamma_{[l,r]}(C)} \frac{|\langle \Gamma_{[l,r]}(B), (A_\omega(A)\mathbb{1})\Gamma_{[l,r]}(C)\rangle|}{||\Gamma_{[l,r]}(B)||\, ||\Gamma_{[l,r]}(C)||} \tag{2.9}$$

We can bound the right-hand side of this in the following way.

Lemma 2.4.2.1. [35] Fix $[a,b] \subseteq [l,r]$ and let $\Gamma_{[l,r]} : M_k \to \mathcal{H}_{[l,r]}$ be a translation-invariant matrix product state with primitive transfer operator $\mathbb{E}$ with $\mathbb{E}(\mathbb{1}) = \mathbb{1}$ and a unique density matrix ρ with $\mathbb{E}^t(\rho) = \rho$. Then

$$|\langle \Gamma_{[l,r]}(B), A\Gamma_{[l,r]}(C)\rangle - \omega(A)\langle B, C\rangle_\rho \leq c(\mathrm{Tr}(\rho^{-1})\lambda^{a-l} + \lambda^{r-b})||A||\, ||B||_\rho ||C||_\rho \tag{2.10}$$

where $\langle B, C\rangle_\rho = \mathrm{Tr}(\rho B^* C)$ and ω is the unique infinite volume state given by

$$\omega(A) = \mathrm{Tr}(\rho \mathbb{E}_A(\mathbb{1})) \tag{2.11}$$

Proof. Fixing the orthonormal basis $\mathcal{B}$ of $\mathbb{C}^k$ we rewrite the trace as

$$\langle \Gamma_{[l,r]}(B), A\Gamma_{[l,r]}(C)\rangle = \sum_{\substack{i_l,\ldots,i_r \\ j_l,\ldots,j_r}} \mathrm{Tr}(v^*_{i_r} \ldots v^*_{i_l} B^*)\mathrm{Tr}(C v_{j_l} \ldots v_{j_r})\langle i_l \ldots i_r|A|j_l \ldots j_r\rangle \tag{2.12}$$

$$= \sum_{\alpha,\beta\in\mathcal{B}} \sum_{\substack{i_l,\ldots,i_r \\ j_l,\ldots,j_r}} \langle \alpha v^*_{i_r} \ldots v^*_{i_l} B^* \alpha\rangle \langle \beta, C v_{j_l} \ldots v_{j_r} \beta\rangle \langle i_l \ldots i_r|A|j_l \ldots j_r\rangle \tag{2.13}$$

$$= \sum_{\alpha,\beta\in\mathcal{B}} \langle \alpha \mathbb{E}^{r-b} \circ \mathbb{E}_A \circ \mathbb{E}^{a-l}(B^*|\alpha\rangle\langle\beta|C)\beta\rangle \tag{2.14}$$

where $\mathbb{E}^0 = \mathbb{1}$. Letting $\mathbb{E}^\infty(B) = \mathrm{Tr}(\rho B)\mathbb{1}$ we have

$$\mathbb{E}^{r-b} \circ \mathbb{E}_A \circ \mathbb{E}^{a-l} = \mathbb{E}^\infty \circ \mathbb{E}_A \circ \mathbb{E}^\infty + (\mathbb{E}^{r-b} - \mathbb{E}^\infty) \circ \mathbb{E}_A \circ \mathbb{E}^\infty + \mathbb{E}^{r-b} \circ \mathbb{E}_A \circ (\mathbb{E}^{a-l} - \mathbb{E}^\infty) \tag{2.15}$$

so we can split the sum up in 2.14 to get three pieces, each of which will be handled separately. From the definition of $\mathbb{E}^\infty$ and the orthonormality $\mathcal{B}$, one finds that

$$\omega(A)\langle B, C\rangle_\rho = \sum_{\alpha,\beta\in\mathcal{B}} \langle \alpha, \mathbb{E}^\infty \circ \mathbb{E}_A \circ \mathbb{E}^\infty(B^*|\alpha\rangle\langle\beta|C)\beta\rangle \tag{2.16}$$

For the second term we have $\sum_{\alpha,\beta\in\mathcal{B}}\langle \alpha, (\mathbb{E}^{r-b} - \mathbb{E}^\infty)\circ\mathbb{E}_A\circ\mathbb{E}^\infty(B^*|\alpha\rangle\langle\beta|C)\beta\rangle = \mathrm{Tr}(C\rho B^*(\mathbb{E}^{r-b}-\mathbb{E}^\infty)\circ \mathbb{E}_A(\mathbb{1}))$ We have that $||\mathbb{E}_A|| \leq ||A||$ via the isometry given by $V_{[a,b]}$. Then by Hölder's inequality:

$$\begin{aligned}
|\mathrm{Tr}(C\rho B^*(\mathbb{E}^{r-b} - \mathbb{E}^\infty)\circ\mathbb{E}_A(\mathbb{1}))| &\leq ||(\mathbb{E}^{r-b}-\mathbb{E}^\infty)\circ\mathbb{E}_A(\mathbb{1})||\ ||C\rho B^*||_1 \\
&\leq c\lambda^{r-b}||A||\ ||C\rho^{1/2}||_2||\rho^{1/2}B^*||_2 \\
&= c\lambda^{r-b}||A||\ ||C||_\rho||B||_\rho
\end{aligned}$$

For the last term, we have

$$|\langle \alpha, \mathbb{E}^{r-b}\circ\mathbb{E}_A\circ(\mathbb{E}^{a-l}-\mathbb{E}^\infty)(B^*|\alpha\rangle\langle\beta|C)\beta\rangle| \leq c\lambda^{a-l}||A||\ ||B^*|alpha||\ ||C|\beta\rangle|| \tag{2.17}$$

Assuming $\mathcal{B}$ diagonalizes ρ so that $\rho|\alpha\rangle = \rho_\alpha|\alpha\rangle$ we have

$$\sum_{\alpha\in\mathcal{B}} ||B^*|\alpha\rangle|| = \sum_\alpha \rho_\alpha^{-1/2}||B^*\rho^{1/2}|\alpha\rangle|| \leq \sqrt{\mathrm{Tr}(\rho^{-1})}||B||_\rho \tag{2.18}$$

We can similarly sum over β to get

$$\sum_{\alpha,\beta\in\mathcal{B}} |\langle \alpha, \mathbb{E}^{r-b}\circ\mathbb{E}_A\circ(\mathbb{E}^{a-l}-\mathbb{E}^\infty)(B^*|\alpha\rangle\langle\beta|C)\beta\rangle| \leq c\mathrm{Tr}(\rho^{-1})\lambda^{a-l}||A||\ ||B||_\rho||C||_\rho \tag{2.19}$$

from which the result follows. □

Notice that if $A = \mathbb{1}$ the operator inside of equation 2.14 becomes $\mathbb{E}^{r-l+1} = \mathbb{E}^\infty + (\mathbb{E}^{r-l+1} - \mathbb{E}^\infty)$ and we have that

$$|\langle \Gamma_{[l,r]}(B), \Gamma_{[l,r]}(C)\rangle - \langle B, C\rangle_\rho| \leq c\mathrm{Tr}(\rho^{-1}\lambda^{r-l+1}||B||_\rho||C||_\rho \tag{2.20}$$

By choosing $B = C$ we have injectivity for $\Gamma_{[l,r]}$ when r and l are sufficiently far apart; specifically we have

$$1 - c\mathrm{Tr}(\rho^{-1})\lambda^{r-l+1} \leq \frac{||\Gamma_{[r,l]}(B)||^2}{||B||_\rho^2} \leq 1 + c\mathrm{Tr}(\rho^{-1})\lambda^{r-l+1} \tag{2.21}$$

With this we can prove the indistinguishability result

Theorem 2.4.3. [35] Let $[a,b] \subseteq [l,r]$ be fixed and let $\Gamma_{[l,r]} : M_k \to \mathcal{H}_{[l,r]}$ be a translation-invariant matrix product state with primitive transfer operator $\mathbb{E}$. Then for any $A \in \mathcal{A}_{[a,b]}$,

$$||P_{[l,r]}AP_{[l,r]} - \omega(A)P_{[l,r]}|| \leq C(r-l+1)\left[\mathrm{Tr}(\rho^{-1})(\lambda^{r-l+1} + \lambda^{a-l}) + \lambda^{r-b}\right]||A|| \tag{2.22}$$

where ω is as before and $C(n) = c(1 - c\mathrm{Tr}(\rho^{-1})\lambda^n)^{-1}$

Proof. Let $B, C \in M_k$ be nonzero and fixed; from the above 2.10 and 2.20 we have

$$|\langle \Gamma_{[l,r]}(B), (A - \omega(A)\mathbb{1})\Gamma_{[l,r]}(C)\rangle \leq |\langle \Gamma_{[l,r]}(B), A\Gamma_{[l,r]}(C)\rangle - \omega(A)\langle B, C\rangle_\rho| \tag{2.23}$$

$$+|\omega(A)||\langle \Gamma_{[l,r]}(B), \Gamma_{[l,r]}(C)\rangle - \langle B, C\rangle_\rho| \tag{2.24}$$

$$\leq c\left[\mathrm{Tr}(\rho^{-1})(\lambda^{r-l+1} - \lambda^{a-l}) + \lambda^{r-b}\right]||A||\,||B||_\rho||C||_\rho \tag{2.25}$$

We can then invert the bounds from 2.21 and using 2.9 get

$$\frac{||B||_\rho}{||\Gamma_{[l,r]}(B)||} \leq \frac{1}{\sqrt{1 - c\mathrm{Tr}(\rho^{-1})\lambda^{r-l+1}}} \tag{2.26}$$

For the indistinguishability radius, let $x \in \mathbb{Z}$ we have that $b_x^{\Lambda'}(n) = [x-n, x+n]$ for all $n \leq d(x, \partial\Lambda')$. Thus if $A \in \mathcal{A}_{[x-m,x+m]}$ for $m \leq n$, we have

$$||P_{b_x^{\Lambda'}} - \omega(A)P_{b_x^{\Lambda}(n)}|| \leq C(n-m)(2\mathrm{Tr}(\rho^{-1}) + 1)\lambda^{n-m}||A|| \tag{2.27}$$

Defining the ground state expectations $\omega(A) = \mathrm{Tr}(AP_{\Lambda'})/\mathrm{Tr}(P_{\Lambda'})$, then we can obtain the indistinguishability radius by bounding the difference

$$||P_{b_x^{\Lambda'}(n)}AP_{b_x^{\Lambda'}(n)} - \omega_{\Lambda'}(A)P_{b_x^{\Lambda'}(n)}|| \leq ||P_{b_x^{\Lambda'}(n)}AP_{b_x^{\Lambda'}(n)} - \omega(A)P_{b_x^{\Lambda'}(n)}|| + |\omega_{\Lambda'}(A) - \omega(A)| \tag{2.28}$$

From which we get the indistinguishability radius of $r_x^\Omega(\Lambda') \geq d(x, \partial\Lambda)$ where $\Omega(n) = 2C(n)(2\mathrm{Tr}(\rho^{-1}) + 1)\lambda^n$. For sufficiently large Λ' the injectivity implies that $\omega_{\Lambda'}(A) = k^{-2}\sum_{i=1}^{k^2}\langle \Gamma_{\Lambda'}(B_i), A\Gamma_{\Lambda'}(B_i)\rangle$ for an orthonormal basis $\{\Gamma_{\Lambda'}(B_i)\}$ of $\mathrm{ran}(\Gamma_{\Lambda'})$. For any normalized state $\Gamma_{\Lambda'}(B)$ the arguments from 2.25 and 2.26 with the assumption $B = C$ to get

$$|\langle \Gamma_{\Lambda'}(B), A\Gamma_{\Lambda'}(B)\rangle - \omega(A)| \leq C(n-m)(2\mathrm{Tr}(\rho^{-1}) + 1)\lambda^{n-m}||A|| \tag{2.29}$$

And thus we get the result from the following bound:

$$|\omega_{\Lambda'}(A) - \omega(A)| \leq \frac{1}{k^2}\sum_{i=1}^{k^2}|\langle \Gamma_{\Lambda'}(B_i), A\Gamma_{\Lambda'}(B_i)\rangle - \omega(A)| \leq C(n-m)(2\mathrm{Tr}(\rho^{-1}) + 1)\lambda^{n-m}||A|| \tag{2.30}$$

□

Note we have the following corollary which provides local topological quantum order.

Corollary 2.4.1. [35] Let $[a,b] \subseteq [l,r]$ be fixed and let $\Gamma_{[l,r]} : M_k \to \mathcal{H}_{[l,r]}$ be a translation-invariant matrix product state with primitive transfer operator $\mathbb{E}$. Then for any $A \in \mathcal{A}_{[a,b]}$,

$$||P_\Lambda A P_\Lambda - \omega(A)P_\Lambda|| \leq C(|\Lambda|)(2\mathrm{Tr}(\rho^{-1}) + 1)\lambda^{d(X,\partial\Lambda)}||A|| \tag{2.31}$$

From the results of Bravyi, Hastings, and Michalakis [6] as well as Nachtergaele, Sims, and Young [37] we have that this implies that the spectral gap is stable.

Corollary 2.4.2 (Spectral Gap Stability [37]). Assume there exist $\gamma > 0$, and $N_0 \geq 1$ such that $\mathrm{spec}\, H_{\Lambda_N} \cap (0,\gamma) = \emptyset$, for all $N \geq N_0$, and that there are constants $a > 0$, and $\theta \in (0,1]$ such that

$$\|V_{n,\tilde{x}}\| \leq e^{-an^\theta}, \text{ for all } n \geq 1,\ \tilde{x} \in \tilde{\Gamma}\,. \tag{2.32}$$

Then, for all $\gamma_0 \in (0,\gamma)$, there exists $s(\gamma_0) > 0$ and a positive, decreasing sequence $\epsilon_N \to 0$, such that

$$\mathrm{spec}\, H^V_{\Lambda_N}(s) \subseteq [-\epsilon_N, \epsilon_N] \cup [\gamma_0 + \epsilon_N, \infty) = \emptyset, \text{ for all } N \text{ large enough, and } |s| < s(\gamma_0). \tag{2.33}$$

We thus have this for any of these frustration-free matrix product states with a primitive transfer operator and a unique infinite-volume state.

2.5. Quasi-One-Dimensional AKLT models: general results

In this section we provide a particular construction of a quasi-one-dimensional model from a finite bipartite graph Λ.

Definition 2.5.1. Let Λ be a finite bipartite graph; choose two sets of vertices $\mathcal{V}_{\partial\Lambda_l}, \mathcal{V}_{\partial\Lambda_r} \subseteq \mathcal{V}_\Lambda$ such that $|\mathcal{V}_{\partial\Lambda_l}| = |\mathcal{V}_{\partial\Lambda_r}|$. For each $x \in \mathbb{Z}$ associate a copy Λ^x of Λ and define an edge set $\mathcal{E}_{\partial\Lambda^x} = \{(x_k, y_k) \,|\, x_k \in \mathcal{V}_{\partial\Lambda_l^x},\ y_k \in \mathcal{V}_{\Lambda_r^{x-1}}\}$. In the case that $\Gamma = (\bigcup_{x\in\mathbb{Z}} \mathcal{V}_{\Lambda^x}, \bigcup_{x\in\mathbb{Z}} \mathcal{E}_{\Lambda^x} \cup \mathcal{E}_{\partial\Lambda^x}$ is bipartite we call Γ the **quasi-one-dimensional lattice** generated by Λ and $\mathcal{E}_{\partial\Lambda}$.

Proposition 2.5.1. The lattice Γ is bipartite if $\Lambda = \Lambda_A \cup \Lambda_B$ is bipartite and one of the following conditions is satisfied:

i. For all $(x,y) = e \in \mathcal{E}_{\partial\Lambda}$ we have that $x \in A$ and $y \in B$ or $x \in B$ and $y \in A$

ii. We have that $x \in A$ and $y \in A$ for all $(x,y) = e \in \mathcal{E}_{\partial\Lambda}$, or $x \in B$ and $y \in B$ for all $(x,y) = e \in \mathcal{E}_{\partial\Lambda}$

Proof. It is easy to see that in the first case the concatenated graph is bipartite with the same coloring $\Lambda = \Lambda_A \cup \Lambda_B$ as for Λ. For the second case, we have that $\Lambda_A^x = (\Lambda_B)^x$ if $x \in 2\mathbb{Z}$ and $\Lambda_A^x = (\Lambda_A)^x$ otherwise provides a bipartition of Γ. □

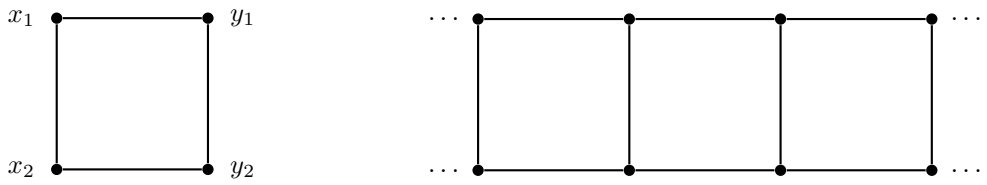


FIGURE 2.2. A finite bipartite graph Λ, a choice of $\mathcal{V}_{\Lambda_l} = \{x_1, x_2\}$ and $\mathcal{V}_{\Lambda_r} = \{y_1, y_2\}$ on the left and the quasi-one-dimensional graph Γ constructed from it on the right

We show this infinite-volume expectations for AKLT models defined on such quasi-one-dimensional graphs are unique by using the classical expression for the ground state expectations.

2.5.1. Ground States and Expectations of AKLT Models. As in [19], we use the Weyl representation of the Lie algebra $\mathfrak{su}(2)$ acting on homogeneous polynomials of two variables to give a description of the ground state space of the AKLT Hamiltonian. We introduce the Hilbert space representing a particle of spin $s \in \mathbb{N}/2$

$$\mathcal{H}^{(s)} = \left\{ \sum_{k=0}^{2s} \lambda_k u^k v^{2s-k} : \lambda_k \in \mathbb{C} \right\}$$

where $u = \cos(\theta/2)e^{i\phi/2}$ and $v = \sin(\theta/2)e^{-i\phi/2}$ are functions of $0 \leq \phi < 2\pi$, $0 \leq \theta < \pi$. The inner product for all Ψ, $\Phi \in \mathcal{H}^{(s)}$ is

$$\langle \Psi, \Phi \rangle = \int d\Omega \overline{\Psi(\theta, \phi)} \Phi(\theta, \phi), \qquad d\Omega = \frac{1}{4\pi} \sin(\theta) d\phi d\theta \tag{2.1}$$

where $d\Omega$ is the measure associated with the spherical variable

$$\Omega = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \tag{2.2}$$

Note that $\mathcal{H}^{(s)} \subseteq L^2(d\Omega)$ for all $s \in \mathbb{N}/2$, the latter of which is independent of s. The spin-s matrices in $\mathcal{B}(\mathcal{H}^{(s)})$ are given by the Weyl representation

$$S^3 = \frac{1}{2}\left(v\partial_v - u\partial_u\right), \qquad S^- = u\partial_v, \qquad S^+ = v\partial_u\,. \tag{2.3}$$

For each $x \in \mathcal{V}_\Gamma$ we label the corresponding variables by u_x, v_x, θ_x, ϕ_x, and $d\Omega_x$. The inner product on $\mathcal{H}_\Lambda$ for any finite Λ is given by

$$\langle \Phi, \Psi \rangle = \int d\Omega^\Lambda \overline{\Phi(\theta, \phi)} \Psi(\theta, \phi) \qquad \forall \Phi, \Psi \in \mathcal{H}_\Lambda \tag{2.4}$$

where $d\Omega^\Lambda$ is the product measure associated with $\{d\Omega_x : x \in \mathcal{V}_\Lambda\}$ and $\Phi(\theta, \phi)$ and $\Psi(\theta, \phi)$ are functions of the variables $\theta = (\theta_x)_{x \in \mathcal{V}_\Lambda}$ and $\phi = (\phi_x)_{x \in \mathcal{V}_\Lambda}$. Considering (2.3), the AKLT interaction P_e for any

$e = (x, y) \in \mathcal{E}_\Gamma$ is the orthogonal projection onto the subspace

$$\text{span}\left\{(u_x\partial_{v_x} + u_y\partial_{v_y})^k v_x^3 v_y^3 : 0 \le k \le 6\right\} \cong \mathcal{H}^{(3)}\,. \tag{2.5}$$

We consider AKLT models on simply connected graphs $G = (\mathcal{V}_G, \mathcal{E}_G)$, for which the construction above easily generalizes. Namely, the AKLT model on G is defined by setting $\mathcal{H}_x = \mathcal{H}^{(s_x)}$ for all $x \in \mathcal{V}_G$ where $s_x = \deg_G(x)/2$, and then replacing P_e with P_e^G for any $e = (x, y) \in \mathcal{E}_G$ where P_e^G is the orthogonal projection onto the subspace of maximal total spin $s_x + s_y$. In terms of the Weyl representation, this is the subspace as in (2.5) obtained by replacing $v_x^3 \mapsto v_x^{2s_x}$, $v_y^3 \mapsto v_y^{2s_y}$ and $6 \mapsto s_x + s_y$. Regardless of the simply connected graph G, the ground state space of the AKLT model for any finite subgraph $\Lambda \subseteq G$ is given by the following result.

Theorem 2.5.1 (Ground State Space [19]). *Let $G = (\mathcal{V}_G, \mathcal{E}_G)$ be any simply connected graph. For any finite subgraph $\Lambda = (\mathcal{V}_\Lambda, \mathcal{E}_\Lambda) \subseteq G$, the ground state space of the AKLT Hamiltonian $H_\Lambda = \sum_{e\in\mathcal{E}_\Lambda} P_e$ is*

$$\mathcal{G}_\Lambda = \left\{\Psi(f) = f \prod_{(x,y)\in\mathcal{E}_\Lambda} (u_x v_y - v_y u_x) : f \in \mathcal{H}_{\partial\Lambda}^{\text{gss}}\right\} \tag{2.6}$$

where

$$\mathcal{H}_{\partial\Lambda}^{\text{gss}} = \bigotimes_{x\in\partial\Lambda} \mathcal{H}^{(d_x/2)}, \qquad d_x = |\{(x, y) \in \mathcal{E}_G : (x, y) \notin \mathcal{E}_\Lambda\}| \tag{2.7}$$

This result follows from first proving that for any edge $e = (x, y) \in \Gamma$,

$$\ker P_e = \left\{(v_x u_y - u_x v_y)g \in \mathcal{H}_x \otimes \mathcal{H}_y : g \in \mathcal{H}^{(s_x-1/2)} \otimes \mathcal{H}^{(s_y-1/2)}\right\}.$$

Since $\mathcal{H}_x = \mathcal{H}^{(s_x)}$ is the set of all homogeneous polynomials of degree $2s_x = \deg_G(x)$ in two variables, it follows that every interaction P_e^G, $e = (x, y) \in \mathcal{E}_\Lambda$, is simultaneously minimized by any vector $\psi \in \mathcal{H}_\Lambda$ that has a singlet factor $u_x v_y - v_x u_y$ across each edge. Since the polynomials over $\mathbb{C}$ form a unique factorization domain, this produces (2.6). For complete details, see [19].

It was shown [4] that for any finite subgraph, Λ, of a simply connected graph, G, and any $A \in \mathcal{A}_\Lambda$, there exists a polynomial $A(\mathbf{\Omega})$ in the variables $(u_x, v_x, \overline{u_x}, \overline{v_x})_{x\in\text{supp}(A)}$, called the *symbol of* A, such that the matrix element of A for any vectors $\Psi, \Phi \in \mathcal{H}_\Lambda$ is

$$\langle\Psi, A\Phi\rangle = \int d\Omega^\Lambda \overline{\Psi(\theta, \phi)}\Phi(\theta, \phi)A(\mathbf{\Omega})\,. \tag{2.8}$$

This is a consequence of using that $\{\partial_u^k \partial_v^l u^{k+j} v^{l-j} : k, l \in \mathbb{N}_0,\ -k \leq j \leq l\}$ is a spanning set of $\mathcal{B}(\mathcal{H}^{(s)})$ for any s, and then applying the identity

$$\left\langle \Phi, \partial_u^k \partial_v^l u^{k+j} v^{l-j} \Psi \right\rangle = C_{k,l} \left\langle u^k v^l \Phi, u^{k+j} v^{l-j} \Psi \right\rangle, \quad C_{k,l} = \frac{(2s+l+k+1)!}{(2s+1)!}.$$

See [19, 27] for more details. We note that the symbol $A(\mathbf{\Omega})$ is not necessarily unique, but that our calculations will not depend on the particular choice.

By the expression 2.8 we have

$$\langle \Psi, A\Phi \rangle = \int d\Omega^{\Lambda} \overline{\Psi(\theta, \phi)} \Phi(\theta, \phi) A(\mathbf{\Omega}). \tag{2.9}$$

and that the ground states are of the form

$$\Psi = \Phi \prod_{(x,y) \in \mathcal{E}} (u_x v_y - u_x v_y) \tag{2.10}$$

where $\Phi = \mathcal{H}_{gss}^{\partial\Lambda}$.

The key is noting [19] that the expression

$$\int d\Omega \prod_{(i,j) \in \mathcal{E}} |u_i v_j - v_i u_j|^2 |\Phi|^2 = \int d\Omega \prod_{(i,j) \in \mathcal{E}} \left[\frac{1 - \Omega_i \cdot \Omega_j}{2} \right] \cdot f(\Omega) \tag{2.11}$$

where $d\Omega = \prod_{i \in \Lambda} d\Omega_i$ where we have used that

$$|u_i v_j - v_i u_j|^2 = \frac{1}{2}(1 - \Omega_i \cdot \Omega_j) \tag{2.12}$$

and $f(\Omega) = |\Phi|^2$. Similarly for

$$\langle \Psi, A\Psi \rangle = \int d\Omega A(\Omega) \prod_{(i,j) \in \mathcal{E}} \left[\frac{1 - \Omega_i \cdot \Omega_j}{2} f(\Omega) \right] \tag{2.13}$$

Combining the two we get

$$\omega(A) = \frac{\langle \Psi, A\Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\int d\Omega A(\Omega) \prod_{(i,j) \in \mathcal{E}} (1 - \Omega_i \cdot \Omega_j) f(\Omega)}{\int d\Omega \prod_{(i,j) \in \mathcal{E}} (1 - \Omega_i \cdot \Omega_j) f(\Omega)} \tag{2.14}$$

This can then be seen as the expectation value of the classical model with Hamiltonian

$$H = -\sum_{e \in \mathcal{E}} \ln(1 - \Omega_i \cdot \Omega_j) \tag{2.15}$$

[4] of the classical observable $A(\Omega)$, where $\Omega_i \in S^2$; this can be seen as an $O(3)$ model with similar macroscopic behavior as the Heisenberg model. The ground state expectations of quantum observables A of the AKLT model are the Gibbs state expectations of associated classical symbols $A(\Omega)$ of this classical model at inverse temperature $\beta = 1$; because of this, one would like to use Mermin-Wagner for

this associated classical system, however the Hamiltonian fails the hypothesis that $||\delta(A_n)||$ are uniformly bounded. However, we can get around this somewhat easily in quasi-one-dimensional systems, since these singularities of the energy are probability zero. First we have some background for the general setting of a classical expectation on a one-dimensional chain; this section is following the setup and conventions of [38].

To each site x on the integers $\mathbb{Z}$ assign the same compact configuration space K equipped with a fixed Borel probability measure dx, called the **phase space** of the point x. For any subset $X \subset \mathbb{Z}$, let $K^X = \prod_{x\in X} K_x$ be the set of **configurations** on X; in words the elements of K^X are the functions $f_X : X \to K$. In general we will write dx for the product measure on K^X where understood (the appropriate space will appear in the bounds of integration).

Definition 2.5.2. A classical **state** on $K^{\mathbb{Z}}$ is a probability measure on $K^{\mathbb{Z}}$ that is absolutely continuous with respect to the measure dx; it is given by a family of density distributions satisfying:

i. $\omega(A) = \int_A dx\mu_X(x)$ for measurable $A \subset K^X$,

ii. $\int_{K^X} dx\mu_X(x) = 1$,

iii. $\mu_X(x) = \int_{K^{X'\setminus X}} dy\mu_{X'}(x \times y)$ if $X \subset X'$.

In one dimension we can construct states with correlation length n in the following way. Let ρ_i^n be a family of non-negative integrable functions on $K^{[i,n+i-1]}$ with $i \in \mathbb{Z}$ satisfying

$$\int_{K^{[x,n+x-1]}} dx\rho_i^N(x) = 1 \tag{2.16}$$

and if $Y \subset [i, n+i-1] \cap [j, j+n-1]$ be nonempty, let $Y_a = Y^c \cap ([a, a+n-1])$, then

$$\int_{K^{Y_i}} dx\rho_i^n(x_Y \times x) = \int_{K^{Y_j}} dx\rho_j^n(x_Y \times x) \tag{2.17}$$

Then letting

$$\rho_i^{n-1}(y) = \int_{K^{\{i+n-1\}}} dx\rho_i^n(x \times y) \tag{2.18}$$

We can then specify a unique state by giving all density distributions μ_X in the following way. Supposing $X = [a, b]$ and $b - a \geq n$ we have

$$\mu^n_{[a,b]}(x) = \begin{cases} \frac{\prod_{i=a}^{b-n+1} \rho_i^n(x_{[i,i+n-1]})}{\prod_{i=a}^{b-n} \rho_{i+1}^{n-1}(x_{[i+1,i+n-1]})} & \rho_i^{n-1}(\cdot) \neq 0 \\ 0 & \rho_l^{n-1}(\cdot) = 0 \text{ for any } l \end{cases} \tag{2.19}$$

One can then extend this to any bounded set $X = \{i_1, ..., i_k\}$ where $i_1 < ... < i_k$ as

$$\mu_X^n(y) = \int_{K^{[i_1,i_k]\setminus X}} dx\mu^n_{[i_1,i_k]}(x \times y) \tag{2.20}$$

This family of distributions $(\mu_X^n)_{X\subset X}$ defines a state ω [38]. We can define the set of translations for $a \in \mathbb{Z}$ as $(\tau_a x)_i = x_{i+a}$ for $i \in \mathbb{Z}$ so that $\rho_i^n \circ \tau_a = \rho_{i+a}^n$. Now let $\mathcal{S}_1$ be the set of all translation invariant product states, and use 2.20 to define the sets $\mathcal{S}_n$ as the set of all states constructed via 2.20, which we will call **quasi-product states**, and let $\mathcal{S}$ be the set of all translation-invariant states absolutely continuous with respect to dx. Note that to any translation invariant state ω we can associate a quasi-product state with the same n-point correlation functions. Such n-point approximations allow for quantities like entropy density to be easily defined.

We now begin to define equilibrium states and argue for their existence and uniqueness in the infinite volume setting.

Let Ω be a measurable subset of $K^n := K^{[1,n]}$ and call $\xi \in K^{\mathbb{Z}}$ an **allowed configuration** if for all $a \in \mathbb{Z}$ we have $(\tau_a \xi)_{[1,n]} \in \Omega$, and call the set of allowed configurations $\mathcal{U}$, and we say an **allowed state** is one whose probability densities are supported on $\mathcal{U}$. An **energy density** h is given by a non-negative function defined on Ω, and so for an interval $[a, b]$ be have the local Hamiltonian

$$H_{[a,b]}(\xi) = \sum_{a-1\le i\le b-n} h((\tau_i \xi)_{[1,n]}) \tag{2.21}$$

We can now easily define a transfer operator for this classical system.

Definition 2.5.3. Let $S \subseteq \Omega$ be measurable, and define

$$L_S\phi(x_2, ..., x_n) = \int_X dx_1 k_S(x_1, ..., x_n)\phi(x_1, ..., x_n) \tag{2.22}$$

where $k_S = \chi_S \exp[-\beta h]$ where χ_S is the familiar characteristic function on S. We can define $\mathcal{L}_S = L_S^{n-1}$

The above transfer matrices have the following properties:

i. $L_S^*(x_1, ..., x_{n-1}) = \int_K dx_n k_S(x_1, ..., x_n)\phi(x_2, ..., x_n)$

ii.

$$\mathcal{L}_S\phi(y_1, ..., y_{n-1}) = \int_{K^{n-1}} dx_1 ... dx_{n-1} k_S(x_{n-1,y_1,...,y_{n-1}}) \cdot k_S(x_{n-2}, x_{n-1}, y_1, ..., y_n) \cdot ... \cdot k(x_1, ..., x_n, y_1)\phi(x_1, ..., x_{n-1})$$

iii. $\mathcal{L}_S$ is Hilbert-Schmidt and thus compact.

Now we make the following definition:

Definition 2.5.4. Let (G, μ) be a probability space and $\mathcal{L}$ an integral operator on $L^2(G, \mu)$ defined by a non-negative kernel $k : G \times G \to \mathbb{R}$. Then $\mathcal{L}$ is called **irreducible** if for all measurable $A \subseteq G$ such that $0 < \mu(A) < 1$ we have $\int_{G\setminus A} d\mu(y) \int_A d\mu(x)k(x, y) > 0$

We then have the following theorem [38]

Theorem 2.5.5. [38] Given Hamiltonian h and Ω such that the spectral radius $\varrho(\mathcal{L}_\Omega) \neq 0$ and $\mathcal{L}_\Omega$ is irreducible, then the system has a unique thermodynamically stable state ω defined by the density ϱ^n and ω is such that $S = \mathrm{supp}(\varrho^n) = \Omega$.

We can now prove the following:

Theorem 2.5.6. The infinite-volume ground state expectations $\omega_\Gamma(A)$ for the AKLT model defined on the quasi-one-dimensional lattice defined by a bipartite graph Λ and appropriate choice of boundary $\mathcal{E}_{\partial\Lambda}$ are unique.

Proof. Note our configuration space is now $K^\Lambda = \prod_{x\in\mathcal{V}_\Lambda} S_x^2$ which is compact. The ground state integral 2.14 has a corresponding transfer operator

$$L_{K^\Lambda}\phi(\Omega) = \frac{\int_{K^{\Lambda\setminus\partial\Lambda}} d\Omega \prod_{(i,j)\in\mathcal{E}_\Lambda}(1-\Omega_i\cdot\Omega_j)\phi(\Omega)}{\int_{K^{\Lambda\setminus\partial\Lambda}} d\Omega \prod_{(i,j)\in\mathcal{E}_\Lambda}(1-\Omega_i\cdot\Omega_j)} \tag{2.23}$$

where $d\Omega$ is the product measure on $K^{\Lambda\setminus\partial\Lambda}$. This integral operator has a positive kernel

$$k_\Lambda = \frac{\prod_{(i,j)\in\mathcal{E}_\Lambda}(1-\Omega_i\cdot\Omega_j)}{\int_{K^{\Lambda\setminus\partial\Lambda}} d\Omega \prod_{(i,j)\in\mathcal{E}_\Lambda}(1-\Omega_i\cdot\Omega_j)} \tag{2.24}$$

and the set of zeros of this operator is a set of measure 0 and so is irreducible by 2.5.4. Thus the infinite volume limit $\omega(A)$ is unique and well-defined for $A \in \mathcal{A}_{\mathrm{loc}}$. □

Corollary 2.5.1. The transfer operator $\mathbb{E}_\Lambda$ of the quasi-one-dimensional lattice generated by a finite bipartite graph Λ is primitive.

Proof. This follows directly from the fact that since the infinite volume limit exists, the iteration $\lim_{n\to\infty}\mathbb{E}_\Lambda^n(\rho) = \rho_\infty$ is independent of ρ and is necessarily unique since $\mathbb{E}$ is finite-dimensional, so by 1.5.2 it must be primitive. □

Corollary 2.5.2. The quasi-one-dimensional AKLT model generated by a finite bipartite graph Λ has a unique, stably gapped ground state with exponential decay of correlations.

Proof. We gain this by successively applying the fact that the quasi-one-dimensional models are, by definition, translation-invariant, are frustration-free from Theorem 3.1.1, have an expression as a matrix-product state via Equation 2.14, with a primitive transfer operator by Corollary 2.5.1; they thus have a spectral gap above the ground state by Theorem 2.3.1 and this gap is stable by Theorem 2.4.2. □

CHAPTER 3

AKLT Model on Trees and Treelike Graphs: Proofs of Néel Order

Introduction

The AKLT model [3] is the first example of a proven Haldane phase [14] and of a matrix product state (MPS) [43] and consequently of a tensor network state (TNS) [41]. However, basic questions about its behavior in higher dimensions remain unanswered. The ground state has the highly unusual property [4] that expectations of observables can be given by a classical partition function of a system in the same number of variables. This led AKLT to conjecture that it does not have a unique ground state and that the ground states have long-range order (LRO) for high-dimensional lattices and high-degree graphs. This was later shown for the Cayley trees (i.e. Bethe lattices) in [10], with some evidence for further results [44], but otherwise a proof has remained elusive for general lattices. We extend the result of [10] to include various trees and tree-like graphs that obey certain conditions. Our examples split into three cases: Cayley-like tree-like graphs generated by a finite subgraph, for which we have a simple condition; arbitrary trees with a prescribed growth rate of their volume; and bilayer Cayley trees. This chapter is organized as follows:

In **Section 3.1** we introduce the AKLT Hamiltonian, prove it is unique up to boundary conditions, give our definition of our infinite-volume ground states, and the graphical language with which we will describe our trees and tree-like graphs.

In **Section 3.2** we introduce the graphical language and compute the transfer operator for a single site; we compute explicitly the transfer function and give relevant properties and bounds which will be instrumental for our proof.

In **Section 3.3** we prove the result of [10] that Cayley trees of degree 5 or more have degenerate ground state space in infinite volume in two ways which generalize in two separate ways as well.

In **Section 3.4** we define the transfer operator on any finite graph with one outgoing index and finitely many ingoing indices. We define our tree-like graphs and show using the graphical picture a condition for a non-unique ground state.

In **Section 3.5** we prove for a type of irregular tree that a geometric mean condition on the degree of the vertices is enough to ensure a non-unique ground state. We give various examples.

In **Section 3.6** we discuss the bilayer Cayley trees and show the degree $d = 5$ bilayer Cayley tree does not have a unique ground state.

3.1. The AKLT Hamiltonian and ground state on a general tree

Let $T = (\mathcal{V}_T, \mathcal{E}_T)$ be an infinite tree with vertex set $\mathcal{V}_T$ and edge set $\mathcal{E}_T$ and distinguished root $\overline{0} \in \mathcal{V}_T$, with the property that there are no vertices such that $\deg(v) = 1$ and $|v - \overline{0}| < \infty$, meaning the leaves are infinitely far away. Define the sequence of vertex sets $\{\mathcal{V}_T^n\}_{n\geq 0}$ with $\mathcal{V}_T^n = \{x \in \mathcal{V}_T : |x - \overline{0}| = n\}$ where $|\cdot|$ denotes the familiar graph distance; we call $\mathcal{V}_T^n$ the nth layer of the graph T, and call the corresponding n-layer tree $T_n = (\bigcup_{i\leq n} \mathcal{V}_T^i, \bigcup_{i\leq n} \mathcal{E}_T^i)$ where $\mathcal{E}_T^n = \{e \in \mathcal{E}_T : e \cap \mathcal{V}_T^n \neq \emptyset\}$. We define the AKLT Hamiltonian on T via the formal Hamiltonian

$$H = \sum_{x\in\mathcal{V}_T} P^{(\deg(x)/2)} \tag{3.1}$$

introduced in [3]. This Hamiltonian is a sum of projectors, and we show that $\ker(H) \neq \emptyset$ and that the vectors $\psi \in \ker(H)$ are unique up to a choice of boundary condition; moreover, we show the Hamiltonian is frustration-free.

3.1.1. The Weyl Representation and Uniqueness of Finite-Volume States. We follow the proof of [19] to construct the AKLT ground state on an arbitrary graph. Consider the representation of $\mathfrak{su}(2)$ on the space of homogeneous polynomials of degree $(\deg(x) + \deg(y))/2 = 2S_e$, denoted

$$\mathcal{S} = \operatorname{span}\{u^{S_e-k}v^k \mid 0 \leq k \leq 2S_e\} \tag{3.2}$$

$\mathcal{S}$ is a $2S_e + 1$ dimensional vector space; note that our spin operators correspond to

$$\begin{aligned} S^+ &= S^x + iS^y = u\frac{\partial}{\partial v} \\ S^- &= S^x - iS^y = v\frac{\partial}{\partial u} \\ S^z &= \frac{1}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right) \end{aligned}$$

And note that the eigenstates of S^z are the polynomials $u^j v^{S_e-j}$ for $0 \leq j \leq 2S_e$; we define an inner product so that the polynomials

$$u^j v^{2S_e-j}\left[\binom{2S_e}{j}(2S_e+1)\right]^{1/2} \tag{3.3}$$

are an orthonormal basis. This gives a $2S_e+1$-dimensional irreducible representation of $\mathfrak{su}(2)$. The inner product can be made explicit via a change of basis where

$$u = e^{i\phi/2}\cos(\theta/2) \tag{3.4}$$

and

$$v = e^{-i\phi/2}\sin(\theta/2) \tag{3.5}$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Then we can define

$$\langle \psi, \phi \rangle = \int d\Omega \overline{\psi(u,v)}\phi(u,v) \tag{3.6}$$

where

$$d\Omega = (4\pi)^{-1}\sin\theta d\theta d\phi \tag{3.7}$$

It was shown by [4] that for each local observable A there exists a unique polynomial in θ and ϕ labeled $A(\Omega)$ called the symbol such that

$$\langle \psi, A\phi \rangle = \int d\Omega \overline{\psi(u,v)}\phi(u,v)A(\Omega) \tag{3.8}$$

At each site $x \in \mathcal{V}$ we associate the variables u_x and v_x and the spin-S_x subspace is given by the span of homogeneous polynomials of degree S_x. We then show the following.

Theorem 3.1.1. [19] Let $\psi \in \mathcal{H}$ such that $H\psi = 0$; then there exists a unique ϕ such that ϕ is a polynomial in u_x, v_x for $x \in \partial\Lambda$ and such that

$$\psi = \phi \prod_{(x,y)\in\mathcal{E}} (u_x v_y - u_x v_y) \tag{3.9}$$

Conversely if the last equation holds then $H\psi = 0$

Proof. We first prove two lemmata. For any edge (x, y) denote the Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$ and label the subspaces

$$S_1 = \{(u_x v_y - u_y v_x)\phi(u_x, v_x, u_y, v_y)\} \tag{3.10}$$

$$S_2 = \{\psi(u_x, v_x, u_y, v_y) | P_{x,y}^{2S_e}\psi = 0\} \tag{3.11}$$

Lemma 3.1.1.1. [19] $S_1 \subseteq S_2$

Proof. Define $T^\alpha = S_x^\alpha + S_y^\alpha$ for $\alpha = x, y, z$. Then

$$T^\alpha(u_x v_y - u_y v_x) = 0 \tag{3.12}$$

for all α and so

$$T^\alpha \psi = (u_x v_y - v_x u_y) T^\alpha \phi \tag{3.13}$$

and thus

$$T^2 \psi = (u_x v_y - v_x u_y) T^2 \phi \tag{3.14}$$

We note that ϕ can be written as a sum of eigenvectors of T^2 so that $\phi = \sum_j \phi_j$ where $T^2 \phi_j = j(j+1)\phi_j$ and thus

$$T^2 (u_x v_y - v_x u_y)\phi_j = j(j+1)(u_x v_y - v_x u_y)\phi_j \tag{3.15}$$

and so $(u_x v_y - v_x u_y)\phi_j$ is an eigenstate of T^2; by assumption $\phi_{2S_e} = 0$ because it is in the ground state of P^{2S_e} and so $\psi \in S_2$ so we know $S_1 \subseteq S_2$.
Next note that $\dim S_2 = (2S_e+1)^2 - (4S_e+1) = (2S_e)^2$ because we have every eigenspace $(2S_e+1)^2$ minus the ones corresponding to highest spin $4S_e+1$. Note also that S_1 is $(2S_e)^2$ dimensional and thus $S_1 = S_2$ □

Lemma 3.1.1.2. [19] If $\psi \in \mathcal{H} = \bigotimes_k \mathcal{H}_k$ satisfies $P^{2S_e}_{(x,y)}\psi = 0$ then $\psi = (u_x v_y - u_y v_x)\phi$; the converse also holds.

Proof. The space of all ψ satisfying

$$P^{2S_e}_{(x,y)}\psi = 0 \tag{3.16}$$

is $S_2 \otimes (\bigotimes_{k \neq x,y} \mathcal{H}_k)$ and similarly all ψ such that $\psi = (u_x v_y - u_y v_x)\phi$ is

$$S_1 \otimes (\bigotimes_{k \neq x,y} \mathcal{H}_k) \tag{3.17}$$

and thus from the previous lemma we have that the two conditions are the same. □

Now let $\psi \in \mathcal{H}$ such that $H\psi = 0$, Then $(u_x v_y - v_x u_y)$ is a factor of ψ for every $(x,y) \in \mathcal{E}$. The polynomial ring these are contained in is a unique factorization domain, so if A and B are factors of ψ then $\psi = AB\gamma$ for some unique γ. Then $\psi = \phi \prod_{(x,y)\in\mathcal{E}} (u_x v_y - u_y v_x)$ and the converse also holds. Thus ψ is unique up to a polynomial of the remaining boundary variables ϕ. □

Now letting T_n be the previously defined increasing and exhaustive sequence of subtrees, we can associate a finite volume state $\psi_n(B)$ of the Hamiltonian defined on T_n which is dependent on the boundary condition $B_n \in M_2^{\otimes|\partial T_n|}$, where $\partial T_n = \{x \in \mathcal{V}_{n+1} \setminus \mathcal{V}_n\}$. We define the infinite-volume ground states of

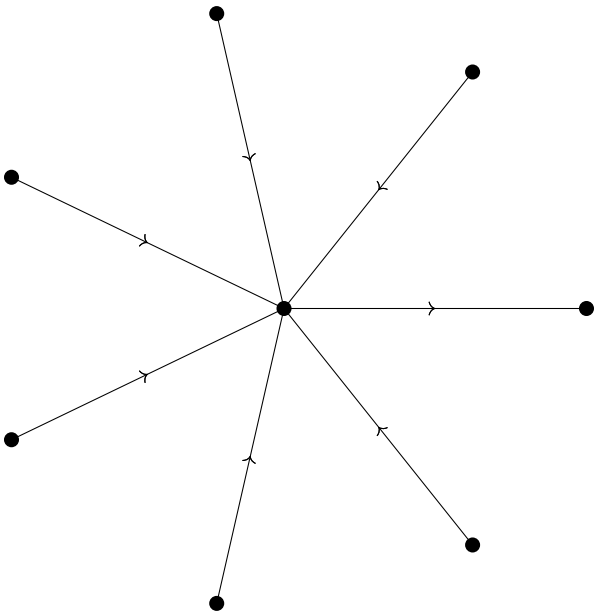


FIGURE 3.1. Vertex of degree $d = 7$ with one outgoing index and six ingoing indices

the model as weak limits of finite volume states where

$$\omega^n_{B_n}(A) = \langle \psi_n(B_n), A\psi_n(B_n) \rangle \tag{3.18}$$

where $A \in \mathcal{A}$ is a quasi-local observable and B_n is a sequence of boundary conditions on T_n. We say the infinite volume state $\omega : \mathcal{A} \to \mathbb{C}$ defined as

$$\omega(A) := \lim_{n\to\infty} \omega^n_{B_n}(A) \tag{3.19}$$

is unique if the limit is independent of the sequence of boundary conditions B_n. One way to compute whether the infinite volume state is independent of the boundary variables taken in the limit is to compute the spectra of the transfer operators, as in the 1-dimensional case.

In order to understand the AKLT model further, we fully describe the transfer operator on a single site of degree d.

3.2. AKLT model transfer operator on a single site

First we describe each site operator on the lattice. From the valence bond solid picture we have that we can view each physical site as composed of d spin-$\frac{1}{2}$ entities living on the edges projected onto a physical spin-$\frac{d}{2}$ space. Note we have a physical spin-$d/2$ representation $\mathcal{P}_{(d/2)}$ of $\mathfrak{su}(2)$ of dimension $d + 1$ which can be uniquely found as a subspace, meaning

$$\mathcal{P}_{(d/2)} \subset \bigotimes_{i=1}^{d} \mathcal{P}_{(1/2)} \tag{3.1}$$

where $\mathcal{P}_{(1/2)}$ is the 2-dimensional spin representation. We have then that the uniqueness of this subspace implies the existence of an SU(2) intertwiner

$$V^* : \mathbb{C}^{d+1} \otimes (\mathbb{C}^2)^{\otimes d-1} \to \mathbb{C}^2 \tag{3.2}$$

which maps the ingoing space and physical space $\mathbb{C}^{d+1} \otimes (\mathbb{C}^2)^{\otimes d-1}$ to the outgoing auxiliary space $\mathbb{C}^2$ and satisfies

$$V(S_i) = (\mathbb{1}_{\text{tp}} \otimes S_i^{d-1} + S_i^{\text{tp}} \otimes \mathbb{1}_{d-1}) \tag{3.3}$$

where S_i^d is the ith SU(2) generator in $\mathcal{P}_{(d/2)}$ and S_i^{tp} is the ith SU(2) generator on $\bigotimes_{i=1}^d \mathcal{P}_{(1/2)}$. We can construct an associated non-normalized transfer operator $\mathbb{F} : M_2(\mathbb{C})^{\otimes d-1} \to M_2(\mathbb{C})$ given by

$$\mathbb{F}(X \otimes Y) = V^*(X \otimes Y)V \tag{3.4}$$

where we normalize so that $\mathbb{F}(\mathbb{1} \otimes \mathbb{1}) = c\mathbb{1}$. We will refer to this unnormalized transfer operator for ease of computation in some places. To compute infinite volume expectations on the tree, we will need to know how $\mathbb{F}$ evaluates when $X = \mathbb{1}$ and Y is arbitrary; in this case we have that the transfer operator breaks down into spin subspaces labeled by j as

$$\mathbb{F}(\mathbb{1} \otimes Y) = \mathbb{F}(Y) = \sum_{j=-\frac{d-1}{2}}^{\frac{d-1}{2}} W_j Y W_j^* \tag{3.5}$$

where

$$S = |\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|, \qquad W_j = SP_j \tag{3.6}$$

where P_j is the projection onto the subspace of spin j; note we drop the dependence on the identity since it will be taken for granted. We choose our boundary indices to be written in the Hilbert-Schmidt basis, that is, as sums of tensor products of Pauli matrices, and we will see that we can provide a formula for each term in this basis. We can compute these weights by choosing a basis of unnormalized symmetric states in the following way.

Let $|w_k\rangle$ be the symmetric state with k ups so that the total spin is $j = k - (d-1)/2$ (for example $|w_1\rangle$ for $d = 3$ would be $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$). On a site of degree d one can write the unnormalized projection as

$$P_j' = |\uparrow\rangle\langle w_{k-1}| + |\downarrow\rangle\langle w_k| \tag{3.7}$$

We then normalize by total number of terms which is the length of w_{k-1} and w_k added together, which gives $\binom{d-1}{k-1} + \binom{d-1}{k} = \binom{d}{k}$ so that

$$P_j = \frac{1}{\sqrt{\binom{d}{k}}}(|\uparrow\rangle\langle w_{k-1}| + |\downarrow\rangle\langle w_k|) \tag{3.8}$$

as this ensures the normalization condition $\tilde{\mathbb{F}}(\mathbb{1}) = \mathbb{1}$; note we differentiate our normalized transfer operator $\tilde{\mathbb{F}}$ from our unnormalized transfer operator $\mathbb{F}$. Thus we have

$$\tilde{\mathbb{F}}(M) = \frac{2}{d+1}\sum_k W_j M W_j^* \tag{3.9}$$

where $M \in M_2(\mathbb{C})^{\otimes d-1}$ and $\tilde{\mathbb{F}} : M_2(\mathbb{C})^{\otimes d-1} \to M_2(\mathbb{C})$. Note that our normalization now insures

$$\tilde{\mathbb{F}}(\mathbb{1}^{\otimes d-1}) = \frac{2}{d+1}\sum_{k=0}^{d-1}\frac{\binom{d-1}{k}}{\binom{d}{k}}\mathbb{1} = \frac{2}{d+1}\sum_{k=0}^{d-1}\frac{d-k}{d}\mathbb{1} = \frac{2}{d+1}\frac{d+1}{2}\mathbb{1} = \mathbb{1} \tag{3.10}$$

In order to know how this operator acts on a general matrix we look at how it acts on a tensor product of Pauli matrices. We have explicitly

$$\mathbb{1} = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Letting

$$M = (\sigma_1)^{\otimes k_1} \otimes (\sigma_2)^{\otimes k_2} \otimes (\sigma_3)^{\otimes k_3} \otimes (\mathbb{1})^{\otimes d-1-k_1-k_2-k_3} \tag{3.11}$$

where this tensor product is taken in any order we have the following identity

Theorem 3.2.1. For our unnormalized transfer operator defined in 3.5, if all the k_i's are even in expression 3.11 we have

$$\mathbb{F}(M) = \frac{1}{k_1+k_2+k_3+1}\frac{\binom{\frac{k_1+k_2+k_3}{2}}{\frac{k_1}{2},\frac{k_2}{2},\frac{k_3}{2}}}{\binom{k_1+k_2+k_3}{k_1,k_2,k_3}}\mathbb{1} \tag{3.12}$$

Proof. We provide an elementary proof by providing some intuition using the spin-flip operator σ_1. First we assume $k := k_1 + k_2 + k_3$ is even. Starting with the case of $M = (\sigma_1)^{\otimes k} \otimes \mathbb{1}^{\otimes(d-1-k)}$ we note that in order for the term $W_j M W_j^*$ will be nonzero for the number of terms in W_j which have the same spin after flipping the first k of them. This implies that the first k of the total string of length d are an equal number of ups and downs. Thus the number of ways to choose such a configuration with

total number of j spin ups is

$$\binom{k}{k/2}\binom{d-k}{j-k/2} \tag{3.13}$$

Noting that the normalization of W_j is $\binom{d}{j}$ and summing over all possible values of j we get that

$$\mathbb{F}(M) = \frac{1}{d+1}\sum_{j\geq k/2}^{d-k/2}\frac{\binom{k}{k/2}\binom{d-k}{j-k/2}}{\binom{d}{j}}\mathbb{1} = \frac{1}{d+1}\sum_{j\geq k/2}^{d-k/2}\frac{\binom{j}{k/2}\binom{d-j}{k/2}}{\binom{d}{k}}\mathbb{1} = \frac{1}{d+1}\frac{\binom{d+1}{k+1}}{\binom{d}{k}}\mathbb{1} = \frac{1}{k+1}\mathbb{1}$$

where we have rewritten the sum on the second line and used Vandermonde's identity to get a closed form for the sum, and divided by two since we have two total outputs $\langle\uparrow|$ and $\langle\downarrow|$. A similar calculation shows that for k odd we have

$$\mathbb{F}(M) = -\frac{1}{k+2}\sigma_1 \tag{3.14}$$

By invariance these numbers will be the same for σ_2 and σ_3 with the same output for odd with a different index.

For the case of

$$M = (\sigma_1)^{\otimes k_1}\otimes(\sigma_2)^{\otimes k_2}\otimes(\sigma_3)^{\otimes k_3}\otimes(\mathbb{1})^{\otimes d-1-k_1-k_2-k_3} \tag{3.15}$$

we make the same counting argument and notice that the number of configurations is

$$\sum_{j\geq(k_1+k_2+k_3)/2}^{d-(k_1+k_2+k_3)/2}\binom{k_1}{k_1/2}\binom{k_2}{k_2/2}\binom{k_3}{k_3/2}\binom{d-k_1-k_2-k_3}{j-k_1/2-k_2/2-k_3/2} \tag{3.16}$$

and so by normalization we have

$$\begin{aligned}\mathbb{F}(M) &= \frac{1}{d+1}\sum_{j\geq(k_1+k_2+k_3)/2}^{d-(k_1+k_2+k_3)/2}\frac{\binom{k_1}{k_1/2}\binom{k_2}{k_2/2}\binom{k_3}{k_3/2}\binom{d-k_1-k_2-k_3}{j-k_1/2-k_2/2-k_3/2}}{\binom{d}{j}}\mathbb{1}\\ &= \frac{1}{d+1}\sum_{j\geq(k_1+k_2+k_3)/2}^{d-(k_1+k_2+k_3)/2}\frac{\binom{\frac{k_1+k_2+k_3}{2}}{k_1/2,k_2/2,k_3/2}\binom{d-j}{k_1/2+k_2/2+k_3/2}\binom{j}{k_1/2,k_2/2,k_3/2}}{\binom{k}{k_1,k_2,k_3}\binom{d}{k_1+k_2+k_3}}\mathbb{1}\\ &= \frac{1}{k_1+k_2+k_3+1}\frac{\binom{\frac{k_1+k_2+k_3}{2}}{k_1/2,k_2/2,k_3/2}}{\binom{k_1+k_2+k_3}{k_1,k_2,k_3}}\mathbb{1}\end{aligned}$$

where again we have used Vandermonde's identity □

Theorem 3.2.2. Letting $k_i' := k_i + k_i \mod 2$, if at most one k_i is such that $k_i \neq k_i'$ then

$$\mathbb{F}(M) = \frac{1}{k_1'+k_2'+k_3'+1}\frac{\binom{\frac{k_1'+k_2'+k_3'}{2}}{\frac{k_1'}{2},\frac{k_2'}{2},\frac{k_3'}{2}}}{\binom{k_1'+k_2'+k_3'}{k_1',k_2',k_3'}}\prod_i(-\sigma_i)^{k_i'-k_i} \tag{3.17}$$

PROOF. Similar as above, using Vandermonde's identity. □

Note that because of normalization the coefficients in for the transfer operator are not dependent on the degree of the site d, only on the number of occupied edges.

We will choose a simple boundary condition to help understand how the concatenations of these operators work on trees, for which the product boundary condition is an invariant subspace of $\mathbb{F}$. Denote the vector $\mathbf{x} = [x_1, x_2, x_3]$ with $||\mathbf{x}|| \leq 1$ and denote $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]$ We will explore the action of the transfer operator on the vector-parametrized boundary condition

$$B(\mathbf{x}) = (\mathbb{1} + \mathbf{x} \cdot \boldsymbol{\sigma})^{\otimes(d-1)} \tag{3.18}$$

We evaluate

$$\begin{aligned}\mathrm{Tr}(\mathbb{F}(B(\mathbf{x}))) &= \sum_{k_i \text{even}} \binom{d-1}{k_1, k_2, k_3, d-1-k} \cdot \frac{\binom{\frac{k_1+k_2+k_3}{2}}{\frac{k_1}{2}, \frac{k_2}{2}, \frac{k_3}{2}}}{\binom{k_1+k_2+k_3+1}{k_1, k_2, k_3}} \cdot x_1^{k_1} x_2^{k_2} x_3^{k_3} \\ &= \sum_{k_i \text{ even}} \frac{1}{k+1} \binom{d-1}{k} \cdot \binom{\frac{k_1+k_2+k_3}{2}}{\frac{k_1}{2}, \frac{k_2}{2}, \frac{k_3}{2}} \cdot x_1^{k_1} x_2^{k_2} x_3^{k_3} = \sum_{k \text{ even}} \frac{1}{k+1} \binom{d-1}{k} \cdot ||\mathbf{x}||^k \\ &= \frac{(1+||\mathbf{x}||)^d - (1-||\mathbf{x}||)^d}{2d \cdot ||\mathbf{x}||}\end{aligned}$$

We will call this function $f_d(\mathbf{x}) = f_d(||\mathbf{x}||)$ for convenience.

Similarly we have

$$\begin{aligned}\mathrm{Tr}(\mathbb{F}(B(\mathbf{x})) \cdot \sigma_i) &= - \sum_{\substack{k_1+k_2+k_3=k \\ k_1 \text{odd}, k_2, k_3 \text{even}}} \binom{d-1}{k_1, k_2, k_3, d-2-k} \cdot \frac{\binom{\frac{k_1+1+k_2+k_3}{2}}{\frac{k_1+1}{2}, \frac{k_2}{2}, \frac{k_3}{2}}}{\binom{k_1+k_2+k_3}{k_1+1, k_2, k_3}} \cdot x_1^{k_1} x_2^{k_2} x_3^{k_3} \\ &= - \sum_{\substack{k_1+k_2+k_3=k \neq 0 \\ k_i \text{even}}} \binom{d-1}{k_1-1, k_2, k_3, d-k} \cdot \frac{\binom{\frac{k_1+k_2+k_3}{2}}{\frac{k_1}{2}, \frac{k_2}{2}, \frac{k_3}{2}}}{\binom{k_1+k_2+k_3}{k_1, k_2, k_3}} \cdot x_1^{k_1-1} x_2^{k_2} x_3^{k_3} \\ &= - \sum_{\substack{k_1+k_2+k_3=k \neq 0 \\ k_i \text{even}}} \frac{k_1}{d} \binom{d}{k_1, k_2, k_3, d-k} \cdot \frac{\binom{\frac{k_1+k_2+k_3}{2}}{\frac{k_1}{2}, \frac{k_2}{2}, \frac{k_3}{2}}}{\binom{k_1+k_2+k_3}{k_1, k_2, k_3}} \cdot x_1^{k_1-1} x_2^{k_2} x_3^{k_3} \\ &= -\frac{1}{d} \frac{\partial}{\partial x_i} f_{d+1}(\mathbf{x}) = -\frac{x_i}{d||\mathbf{x}||} f'_{d+1}(||\mathbf{x}||)\end{aligned}$$

Lastly we note that our trace normalized operator $\tilde{\mathbb{F}}$ evaluates as

$$||\tilde{\mathbb{F}}(B(\mathbf{x})) - \mathbb{1}|| = -F_d(||\mathbf{x}||) := \left| \frac{f'_{d+1}(||\mathbf{x}||)}{df_d(||\mathbf{x}||)} \right| \tag{3.19}$$

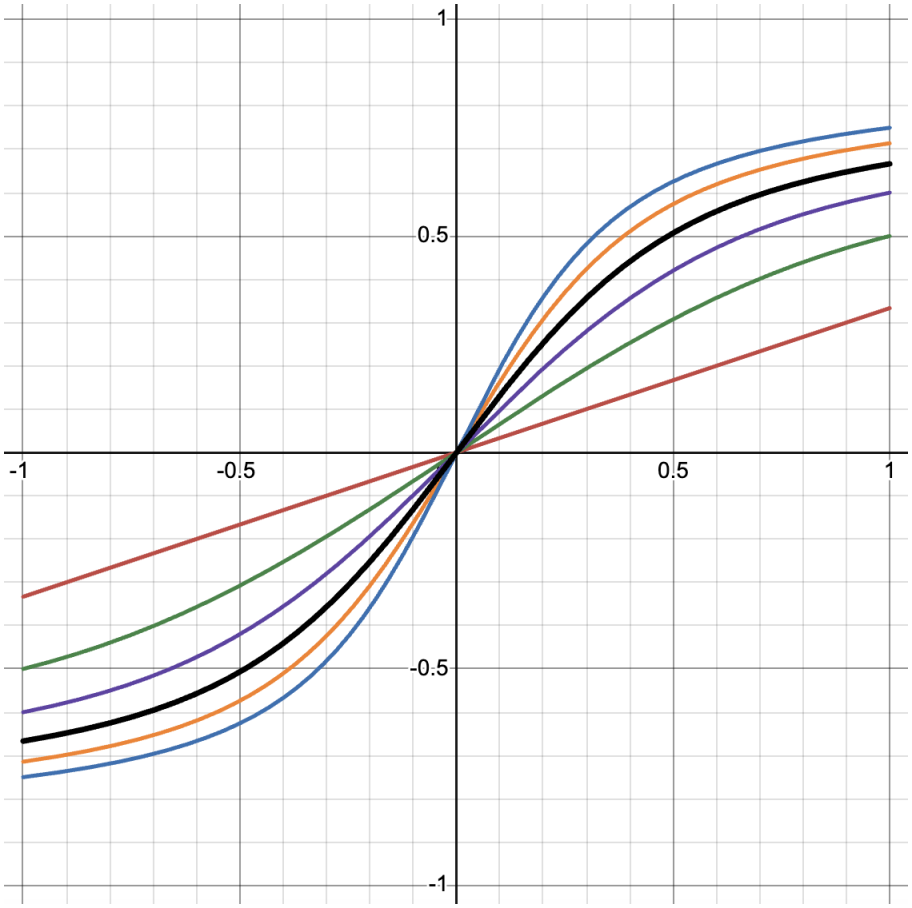


FIGURE 3.2. Plot of $F_d(t)$ for $d = \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}$

$$= \frac{(d+1)\cdot||\mathbf{x}||\cdot\left((1+||\mathbf{x}||)^d+(1-||\mathbf{x}||)^d\right)-\left((1+||\mathbf{x}||)^{d+1}-(1+||\mathbf{x}||)^{d+1}\right)}{(d+1)||\mathbf{x}||\cdot((1+||\mathbf{x}||)^d-(1-||\mathbf{x}||)^d)} \tag{3.20}$$

$$= \frac{1}{d+1}\left(d\cdot\coth(d\cdot\tanh^{-1}(||\mathbf{x}||))-\frac{1}{||\mathbf{x}||}\right) \tag{3.21}$$

3.2.1. The transfer function $F_d(t)$. We analyze the transfer function $F_d(t)$

THEOREM 3.2.3. For $d \geq 2$ we have the following properties of

$$F_d(t) = -\frac{1}{d+1}\left(d\cdot\coth(d\cdot\tanh^{-1}(t))-\frac{1}{t}\right) \tag{3.22}$$

a) $F_d(t)$ is odd and $F_d(0) = 0$

b) $||F_d(t)|| < 1$

c) $F_d(t)$ is analytic

d) $F_d'(t) > 0$

e) $F_d''(t) < 0$ on $(0, 1]$ and $F_d''(t) > 0$ on $[-1, 0)$.

f) $F_d'(0) = \frac{1-d}{3}$

g)

$$-\left(\frac{d-1}{3}\right)\cdot t \leq F_d(t) \leq -\left(\frac{3}{(d-1)t}+1\right)^{-1} \tag{3.23}$$

PROOF. All of these except the right-hand side the last inequality are given in, or follow quickly from those in [10]; the right-hand side of g) follows from the continued fraction expansion for $\coth(t)$ [53]. Note that

$$d\coth(dx) = \frac{1}{x} + \cfrac{d^2x}{3+\cfrac{d^2x^2}{5+\cfrac{d^2x^2}{7+\cfrac{d^2x^2}{\ddots}}}} \tag{3.24}$$

and $F_d(\tanh(x)) \leq F_d(x)$ so

$$\begin{aligned} F_d(x)^{-1} \geq F_d(\tanh(x))^{-1} &= -\left(\frac{1}{d+1}\left(d\coth(dx)-\coth(x)\right)\right)^{-1} \\ &= -(d+1)\left(\cfrac{d^2x}{3+\cfrac{d^2x^2}{5+\cfrac{d^2x^2}{7+\cfrac{d^2x^2}{\ddots}}}} - \cfrac{x}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\cfrac{x^2}{\ddots}}}}\right)^{-1} \\ \geq -\frac{3}{(d-1)x} - \frac{1}{(d-1)}\cfrac{d^2x}{5+\cfrac{d^2x^2}{7+\cfrac{d^2x^2}{\ddots}}} &\geq -\left(\frac{3}{(d-1)x}+1\right) \end{aligned}$$

□

The last inequality allows us to bound iterates of these functions. This already provides us with enough to give a non-uniqueness condition for Cayley trees, which duplicates the result of [10].

3.3. Cayley trees: two proof methods

Our tree in this case is an infinite tree $T^d = \{\mathcal{V}_{T^d}, \mathcal{E}_{T^d}\}$ where for all $x \in \mathcal{V}_{T^d}$ we have $\deg(x) = d$. T^d has a distinguished root $\overline{0}$ and we can define the set of transfer operators on our volumes T^d_n as

$$\tilde{\mathbb{F}}_{n,d} : M_2^{\otimes|\partial T^d_n|} \to M_2^{\otimes|\partial T^d_{n-1}|} \tag{3.1}$$

where $\tilde{\mathbb{F}}_{n,d} = \tilde{\mathbb{F}}_d^{\otimes(d-1)^n}$ where $\tilde{\mathbb{F}}_d : M_2^{\otimes d-1} \to M_2$ is the single site transfer operator analyzed in the previous section. We provide two proofs.

THEOREM 3.3.1. For $d \geq 5$, the AKLT ground state is not unique.

PROOF 1. Note $\tilde{\mathbb{F}}_{n,d} = \tilde{\mathbb{F}}_d^{\otimes(d-1)^n}$ and $\tilde{\mathbb{F}}_d(B(\mathbf{x})) = \mathbb{1} + F_d(||\mathbf{x}||)\cdot\frac{\mathbf{x}}{||\mathbf{x}||}$ if $||\mathbf{x}|| \neq 0$ and $\tilde{\mathbb{F}}_d(B((0))) = \mathbb{1}$. Thus if there exists a value $t_0 \in [0,1]$ with $F_d(t_0) = \pm t_0$ then $\tilde{\mathbb{F}}_d(B(\mathbf{x})) - \mathbb{1} = \pm\mathbf{x}$ for all $\mathbf{x}$ with $||\mathbf{x}|| = t_0$.

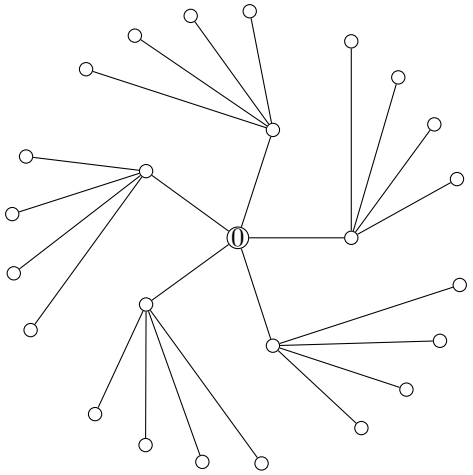


FIGURE 3.3. First two layers of Cayley tree of degree $d = 5$

Defining the limiting boundary conditions $B_n(\mathbf{x}) = (\mathbb{1} + \mathbf{x} \cdot \sigma)^{\otimes \partial T_n}$ we have for some $A \in \mathcal{A}$ with $\text{supp}(A) = \{\overline{0}\}$ we have

$$\begin{aligned}\langle \psi_n(B_n(\mathbf{x})), A\psi_n(B_n(\mathbf{x}))\rangle &= \langle \psi_n(\tilde{\mathbb{F}}^{\circ n}(B_n(\mathbf{x})), A\psi_n(\tilde{\mathbb{F}}^{\circ n}(B_n(\mathbf{x}))\rangle \\ &= \langle \psi_0(B_1(\pm\mathbf{x})), A\psi_0(B_1(\pm\mathbf{x}))\rangle \neq \langle \psi_0(\mathbb{1}), A\psi_0(\mathbb{1})\rangle\end{aligned}$$

since $||x|| = t_0 \neq 0$. Thus we need to know if $F_d(t_0) = \pm t_0$ has a solution on $[0, 1]$. We have $F_d(0) = 0$ and $F_d(t) \leq 0$ on $[0, 1]$ and $|F_d(1)| \leq 1$; thus we will have a solution to $F_d(t) = -t$ if $F_d'(0) = \frac{1-d}{3} < -1$ which occurs exactly when $d \geq 5$. □

PROOF 2. Alternatively, notice that the bound $F_d(x) \geq \left(\frac{3}{(d-1)x} + 1\right)^{-1}$ we have that

$$\bigcirc_{i=1}^{n} F_d(x) \geq \left(\frac{1}{x}\prod_{i=1}^{n}\frac{3}{d_i - 1} + 1 + \sum_{k=1}^{n-1}\prod_{i=1}^{k}\frac{3}{d_i - 1}\right)^{-1} \tag{3.2}$$

Thus for a fixed $x \in (0, 1]$ we have

$$\lim_{n\to\infty} \bigcirc_{i=1}^{n} F_d(x) \geq \lim_{n\to\infty}\left(\frac{1}{x}\prod_{i=1}^{n}\frac{3}{d_i - 1} + 1 + \sum_{k=1}^{n-1}\prod_{i=1}^{k}\frac{3}{d_i - 1}\right)^{-1} = 1 - \frac{3}{d-1} \tag{3.3}$$

which is greater than 0 when $d \geq 5$. Thus $x_d := \lim_{n\to\infty} ||\tilde{\mathbb{F}}_{T_n}(t_0) - \mathbb{1}|| > 0$ so the sequences of boundary conditions $B_n(t_0)$ and $B_n(0)$ lead to different infinite volume expectations. □

These two proofs lead to two different ways to generalize the result. The first proof will generalize to graphs which are not necessarily trees, which we can analyze using a graphical or tensor-network expression for the transfer operator; we explain this in the next section. The second proof generalizes to irregular trees with some nice features, which we analyze in Section 3.5.

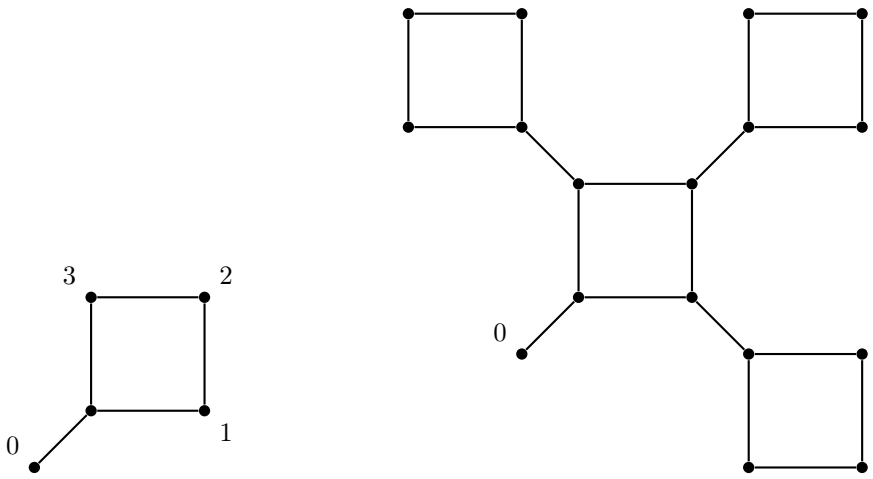


FIGURE 3.4. First and second layers of the quasi-Cayley tree generated by the cell on the left with root 0 and boundary $\{1, 2, 3\}$.

3.4. Tree-like graphs: the graphical picture

When our graph T is easily constructed via tree-like concatenations of finite volume graphs Λ (as the Cayley tree is constructed as a treelike concatenation of the single site graphs) one can leverage the formulae to obtain information on the ground state. We will give a general formula for some of the quantities we have used so far.

Let $\Lambda = (\mathcal{V}_\Lambda, \mathcal{E}_\Lambda)$ be a finite bipartite connected graph and let us define the **root** $\overline{0} \in \mathcal{V}_\Lambda$ with $\deg(\overline{0}) = 1$ and the **boundary** $\partial\Lambda \subset \mathcal{V}_\Lambda$ of Λ. We construct the **tree with cell** Λ with boundary $\partial\Lambda$ in the following way. Let $d = |\partial\Lambda| + 1$ and $\partial\Lambda = \{x_1, ..., x_{d-1}\}$ and let $T_d = (\mathcal{V}_{T_d}, \mathcal{E}_{T_d})$ be the Cayley tree of degree d. To each $x \in \mathcal{V}_{T_d}$ associate a copy Λ^x of Λ. Now to each neighbor $N_x = \{y \in \mathcal{V}_{T_d} | (y, x) \in \mathcal{E}_{T_d}\} = \{y_1^x, ..., y_{d-1}^x\}$ we add an edge $e_{y_k^x} = (x_k, \overline{0}_{y_k^x})$. Thus we get a treelike graph where to each point in our boundary of Λ we have a connection to another copy of Λ attached to the root $\overline{0}$. We can define the transfer operator $\tilde{\mathbb{F}}_\Lambda : M_2^{\otimes\partial\Lambda} \to M_2^{\{\overline{0}\}}$ and define the modified boundary condition

$$B(\mathbf{x}) = \bigotimes_{y \in \partial\Lambda} (\mathbb{1} + (-1)^{\chi_A(y)} \mathbf{x} \cdot \sigma) \tag{3.1}$$

where

$$\chi_A(y) = \begin{cases} 1 & y \in \mathcal{V}_\Lambda^A \\ 0 & y \in \mathcal{V}_\Lambda^B \end{cases} \tag{3.2}$$

where $\mathcal{V}_\Lambda^A$ and $\mathcal{V}_\Lambda^B$ are the bipartition of Λ. In order to give an explicit formula for the values of the transfer operator acting on these boundary conditions we will need to describe a set of labeled loop diagrams in Λ. To each vertex $x_k \in \mathcal{V}_{\partial\Lambda}$ associate a new vertex v_k and edge $e_k = (v_k, x_k)$ and call

$\Lambda' = (\mathcal{V}_\Lambda \cup \bigcup_{i=1}^{d-1}\{v_k\}, \mathcal{E}_\Lambda \cup \bigcup_{i=1}^{d-1}\{e_k\})$ the **augmentation** of Λ. Then we define

$$\mathcal{G} = \{\Gamma \subset \Lambda' \mid \deg x = 0 \bmod 2,\ \forall x \in \mathcal{V}_{\Lambda'}\} \tag{3.3}$$

and

$$\mathcal{G}_k = \{\Gamma \in \mathcal{G} :\ |\mathcal{V}_\Gamma \cap \{v_1, ..., v_{d-1}\}| = k\} \tag{3.4}$$

Defining the weight

$$W(\Gamma) = \prod_{x \in \mathcal{V}_\Gamma} \frac{-1}{\deg(x)+1} \tag{3.5}$$

we have the following.

Theorem 3.4.1. Let Λ be a finite bipartite graph and $\tilde{\mathbb{F}}_\Lambda$ and $B_\Lambda(t)$ as above. Then

$$F_\Lambda(t) = -||\tilde{\mathbb{F}}_\Lambda(B_\Lambda(t)) - \mathbb{1}|| = \frac{p_\Lambda(t)}{q_\Lambda(t)} \tag{3.6}$$

where

$$p_\Lambda(t) = \sum_{k \text{ odd}} t^k \sum_{\Gamma \in \mathcal{G}_k} W(\Gamma) \tag{3.7}$$

and

$$q_\Lambda(t) = \sum_{k \text{ even}} t^k \sum_{\Gamma \in \mathcal{G}_k} W(\Gamma) \tag{3.8}$$

Moreover $F_\Lambda(t) \leq 0$ on $[0,1]$ and $F_\Lambda(0) = 0$ and $F_\Lambda(1) \geq -1$.

Proof. The operator $\mathbb{F}_\Lambda$ is a concatenation of single site operators $\mathbb{F}_x$ for each $x \in C$ where the concatenation of indices is consistent with the edges connecting vertices. Specifically we have

$$\mathbb{F}_x = T^{e'_1,...,e'_k}_{e_1,...,e_j} \tag{3.9}$$

where $j+k = \deg(x)$ and T is the invariant intertwining tensor that finds $M_{d+1} \subset M_2^{\otimes d}$. Thus the entire transfer operator is such that

$$\mathbb{F}_\Lambda = \sum_{e_i, e'_j \in \mathcal{E}_\Lambda} \bigotimes_{\mathcal{V}} \mathbb{F}_x \delta_{e_i, e'_j} \tag{3.10}$$

Thus from the formulae above any nonzero term in the transfer operator corresponds to a weighted sum over a subset $\Gamma \subset C$ such that $\deg(x)$ is even for all indices except any in ∂C, since every nonzero concatenation must have an entering and leaving index. Thus we can write it as $\sum_{\Gamma \in \mathcal{G}} W(\Gamma)$ for some weight function $W(\cdot)$. Moreover this weight function factorizes sitewise; we notice that at each site $x \in \Gamma$

if $\deg(x) = 2k$ then the weight of the $\mathbb{F}_x$ is given from our above formula as $-\frac{1}{2k+1} = -\frac{1}{\deg(x)+1}$ and is zero if $\deg(x)$ is odd. Then every graph Γ can be decomposed into connected components which intersect the boundary and those that do not. The key thing to note is that if $\Gamma = \{\gamma_1, ..., \gamma_n\}$ are the connected components of Γ, then there is at most one component γ_j intersecting $\overline{0}$ and the boundary. Thus we have for $\Gamma \in \mathcal{G}_k$ and k even

$$W(\Gamma) = ||\mathbf{x}||^k \prod_i W(\gamma_i) \tag{3.11}$$

where

$$W(\gamma) = \prod_{x \in \mathcal{V}_\gamma} \frac{-1}{\deg(x)+1} \tag{3.12}$$

Then

$$\mathrm{Tr}(\mathbb{F}_\Lambda(B_\Lambda(\mathbf{x}))) = \sum_{k \text{ even}} ||\mathbf{x}||^k \sum_{\Gamma \in \mathcal{G}_k} W(\Gamma) = q_\Lambda(||\mathbf{x}||) \tag{3.13}$$

is a function of $||\mathbf{x}||$ only, since every configuration in $\mathcal{G}_k$ for k even does not touch $\overline{0}$. For the σ_i parts of the output, we have that there must be a path from the boundary to the root; thus $\Gamma \in \mathcal{G}_k$ for k odd, and we have that

$$\mathrm{Tr}(\mathbb{F}_\Lambda(B_\Lambda(\mathbf{x})) \cdot \sigma_i) = \frac{x_i}{||\mathbf{x}||} \sum_{k \text{ odd}} ||\mathbf{x}||^k \sum_{\Gamma \in \mathcal{G}_k} W(\Gamma) = \frac{x_i}{||\mathbf{x}||} \cdot p_\Lambda(||\mathbf{x}||) \tag{3.14}$$

and so for our trace-normalized operator $\tilde{\mathbb{F}}_\Lambda$ we have

$$||\tilde{\mathbb{F}}_\Lambda(B(\mathbf{x})) - \mathbb{1}|| = -F_\Lambda(||\mathbf{x}||) \tag{3.15}$$

where

$$F_\Lambda(t) = \frac{p_\Lambda(t)}{q_\Lambda(t)} \tag{3.16}$$

with the forms given in the theorem. $F_\Lambda(0) = 0$ then follows since $p_\Lambda(t)$ is odd and $q_\Lambda(0) \neq 0$ since Λ is bipartite; $-1 \leq F_\Lambda(t) \leq 0$ for all $t \in [0, 1]$ follows from the complete positivity of $\mathbb{F}_\Lambda$ and trace normalization of $\tilde{\mathbb{F}}_\Lambda$. □

Note our transfer function for the cell in Figure 3.4 is $F_\Lambda(t) = -\frac{26t}{82+24t^2}$ and $F'_\Lambda(0) = -\frac{13}{41}$ which fails the hypothesis of Theorem 3.4.1.

Corollary 3.4.1. For a rooted quasi-tree T_Λ with generating finite bipartite cell graph Λ, the AKLT Hamiltonian does not have a unique ground state in the infinite volume limit if

$$\frac{p'_\Lambda(0)}{q_\Lambda(0)} = \frac{\sum_{\Gamma\in\mathcal{G}_1} W(\Gamma)}{\sum_{\Gamma\in\mathcal{G}_0} W(\Gamma)} < -1 \tag{3.17}$$

Proof. As $F_\Lambda(t) \leq 0$ on $[0,1]$, $F_\Lambda(0) = 0$, and $F_\Lambda(1) \geq -1$ we have that there is a solution to $F_\Lambda(t) = -t$ if $F'_\Lambda(0) = \frac{p'_\Lambda(0)}{q_\Lambda(0)} = \frac{\sum_{\Gamma\in\mathcal{G}_1} W(\Gamma)}{\sum_{\Gamma\in\mathcal{G}_0} W(\Gamma)} < -1$. Call this solution t_Λ. Now let T^n_Λ be the nth layer of the tree generated from the cell Λ; then for an observble A with $\mathrm{supp}(A) = \{\overline{0}\}$ we have that for the finite volume state ψ_n on T^n_Λ we have

$$\begin{aligned}
&\lim_{n\to\infty} \langle \psi_n(B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1)), A\psi_n(B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1))\rangle = \lim_{n\to\infty} \langle \psi_0(\tilde{\mathbb{F}}_{T^n_\Lambda}(B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1))), A\psi_0(\tilde{\mathbb{F}}_{T^n_\Lambda}(B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1)))\rangle \\
&= \lim_{n\to\infty} \langle \psi_0(\tilde{\mathbb{F}}^{\circ n}_\Lambda(B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1))), A\psi_0(\tilde{\mathbb{F}}^{\circ n}_C(B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1)))\rangle = \langle \psi_0(B_{T^0_\Lambda}(\pm t_\Lambda\cdot\sigma_1))), A\psi_0(B_{T^0_\Lambda}(\pm t_\Lambda\cdot\sigma_1))\rangle \\
&\neq \lim_{n\to\infty} \langle \psi_n(\tilde{\mathbb{F}}_{T^n_\Lambda}(B_{T^n_\Lambda}(\mathbf{0}))), A\psi_n(\tilde{\mathbb{F}}_{T^n_\Lambda}(B_{T^n_\Lambda}(\mathbf{0})))\rangle = \langle \psi_0(B_{T^0_\Lambda}(\mathbf{0}))), A\psi_0(B_{T^0_\Lambda}(\mathbf{0}))\rangle
\end{aligned}$$

Thus the infinite-volume limits given by the sequence $B_{T^n_\Lambda}(t_\Lambda\cdot\sigma_1)$ and $B_{T^n_\Lambda}(\mathbb{1})$ lead to different expectations for local observables, and so the infinite-volume ground state, as a weak limit of finite-volume states, is not unique. □

3.4.1. Examples.

3.4.1.1. *Decorated Cayley trees.* We can define a class of trees by adding $g \geq 0$ sites to an existing Cayley tree. The function

$$F_{d,g}(t) = \frac{p_{\Lambda_g}(t)}{q_{\Lambda_g}(t)} = \frac{1}{3^g} F_d(t) \tag{3.18}$$

and we have that

$$F'_{d,g}(0) = \frac{1}{3^g}\frac{d-1}{3} \tag{3.19}$$

Thus we have the following corollary:

Corollary 3.4.2. Let Γ be a Cayley tree of degree d with decoration number g. If $d > 3^{g+1}+1$ then the AKLT ground state on Γ is ordered, and if $d < 3^{g+1}+1$ it is disordered and unique.

Proof. From of the simplicity of the tree (including the permutation invariance of the inputs) one can carry out the same analysis as in [10] which agrees with the results of [44]. □

The uniqueness result will not be recoverable in general.

3.4.1.2. *Rooted Quasi-Cayley trees.* We can generate a tree out of an underlying graph Λ which is also a tree. This gives a condition that is much easier to verify.

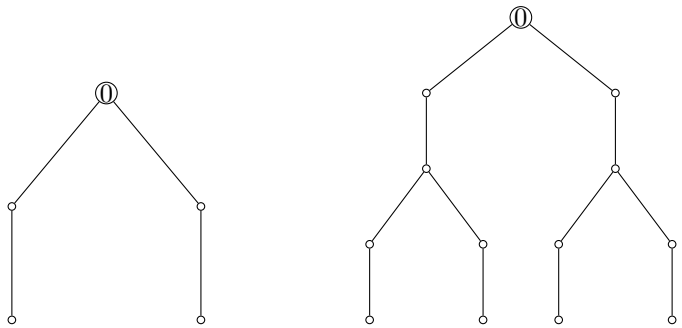


FIGURE 3.5. First two layers of the quasi-Cayley tree generated from the tree on the left; note this is also the degree $d = 2$ Cayley tree with decoration number $g = 1$

THEOREM 3.4.2. Let Γ be a quasi-Cayley tree generated by a cell graph Λ that is also a tree. If

$$\sum_{\gamma \in \mathcal{G}_1} W(\gamma) = \sum_{\gamma \in \mathcal{G}_1} \left(\frac{1}{3}\right)^{|\gamma|} > 1 \tag{3.20}$$

then the AKLT model on this tree does not have a unique ground state.

PROOF. From the above expression for $F_\Lambda(t)$ we have

$$F'_\Lambda(0) = \frac{p'_\Lambda(0)}{q_\Lambda(0)} = \sum_{\gamma \in \mathcal{G}_1} W(\gamma) \tag{3.21}$$

since $q_\Lambda(0)$ is the sum

$$\sum_{\Gamma \in \mathcal{G}_0} W(\Gamma) = 1 \tag{3.22}$$

since $\mathcal{G}_0 = \emptyset$ so $q_\Lambda(0) = 1$ by the convention that $W(\emptyset) = 1$. Thus the condition above ensures that $F'_\Lambda(0) < -1$ and thus has a solution t_Λ where $F_\Lambda(t_\Lambda) = -t_\Lambda$. From a similar argument as Theorem 3.4.1, this implies observables at the origin have different ground state expectations. □

In words, this condition is saying that the sum of all the paths from the root of the cell Λ to any leaves in Λ, weighted by $(-3)^{1-|\gamma|}$ is greater than 1. The proof above is not enough, however, to show that this occurs for every possible choice of boundary condition; also note that this condition depends not only on the tree Λ but also on the root 0.

3.5. Irregular Trees

Following our second proof of the Cayley-tree result using part (g) of Theorem 3.2.3, we can extend the result to irregular trees by starting with a lemma:

LEMMA 3.5.0.1. Let d_i be a sequence of real numbers with $d_i \geq 2$. If the partial products

$$a_n = \prod_{i=1}^{n} \frac{3}{(d_i - 1)} \tag{3.1}$$

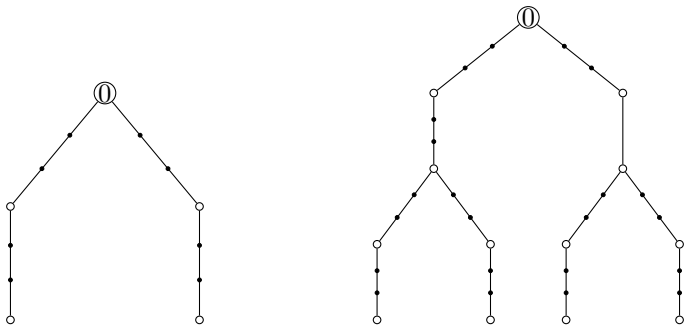


FIGURE 3.6. First two layers of the tree from Figure 2 with decoration number $g = 2$; note this is also the Cayley tree with degree $d = 2$ and decoration number $g = 5$.

is summable, such that $\sum_n a_n < \infty$ then the infinite functional composition $\bigcirc_{i=1}^{\infty} F_{d_i}(x) > 0$ for all $x \in (0, 1]$. Moreover if the sequence of partial products is such that $a_n \to \infty$ then the infinite functional composition $\bigcirc_{i=1}^{\infty} F_{d_i}(x) = 0$ for all $x \in [0, 1]$.

PROOF. Note that from the bound $F_d(x) \geq \left(\frac{3}{(d-1)x} + 1\right)^{-1}$ we have that the iterates

$$\lim_{N\to\infty} \bigcirc_{i=1}^{N} F_{d_i}(x) \geq \left(\frac{1}{x} \cdot \prod_{i=1}^{N} \frac{3}{d_i - 1} + 1 + \sum_{k=1}^{N-1} \prod_{i=1}^{k} \frac{3}{d_i - 1}\right)^{-1} = \lim_{N\to\infty} \left(\frac{1}{x} \cdot a_N + 1 + \sum_{k=1}^{N-1} a_k\right)^{-1}$$
$$= \left(1 + \sum_{k} a_k\right)^{-1} > 0$$

since $a_N \to 0$ and $\sum_k a_k < \infty$.

For the second implication, notice that if the sequence $a_n \to \infty$ then

$$\lim_{n\to\infty} \bigcirc_{i=1}^{n} F_{d_i}(x) \leq \lim_{n\to\infty} x \cdot \prod_{i=1}^{n} \frac{d_i - 1}{3} = \lim_{n\to\infty} x \cdot \frac{1}{a_n} = 0 \tag{3.2}$$

for all $x \in [0, 1]$ □

COROLLARY 3.5.1. Let d_i be a sequence of real numbers with $d_i \geq 2$; if

$$\ln(\mu) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln((d_i - 1)/3) = \lim_{n\to\infty} \frac{1}{n} \ln(a_n) > 0 \tag{3.3}$$

then the infinite function composition

$$\bigcirc_{i=1}^{\infty} F_{d_i}(x) > 0 \tag{3.4}$$

for all $x \in (0, 1]$, and if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln((d_i - 1)/3) \leq 0 \tag{3.5}$$

then

$$\bigcirc_{i=1}^{\infty} F_{d_i}(x) = 0 \tag{3.6}$$

for all $x \in [0,1]$

PROOF. Note that since $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln((d_i-1)/3) = \ln(\mu) > 0$ we have that for any $\epsilon > 0$ we have that there exists an N such that $\frac{1}{n}\sum_{i=1}^{n} \ln((d_i-1)/3) > (1-\epsilon)\ln(\mu) > 0$ for all $n \geq N$; thus we have that $a_n \geq \mu^{(1-\epsilon)n}$; thus

$$\begin{aligned}\lim_{N\to\infty} \bigcirc_{i=1}^{N} F_{d_i}(x) &\geq \left(1 + \sum_{k=1}^{N}\prod_{i=1}^{k}\frac{3}{d_i-1} + \sum_{k=N+1}^{\infty}\prod_{i=1}^{k}\frac{3}{d_i-1}\right)^{-1} \\ &\geq \left(1 + \sum_{k=1}^{N}\prod_{i=1}^{k}\frac{3}{d_i-1} + \frac{1-\mu^{(\epsilon-1)(N+2)}}{1-\mu^{\epsilon-1}}\right)^{-1} > 0\end{aligned}$$

Similarly note that if $\ln(\mu) \leq 0$ then $\frac{1}{a_n} \leq \mu^{(1-\epsilon)n}$ for large enough n, so we have

$$\lim_{N\to\infty} \bigcirc_{i=1}^{N} F_{d_i}(x) \leq \lim_{N\to\infty} \frac{1}{a_N} \cdot x = 0 \tag{3.7}$$

for all $x \in [0,1]$ □

We also provide two examples to show that if

$$\ln(\mu) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln((d_i-1)/3) = 0 \tag{3.8}$$

one can have different macroscopic behavior so $\mu = 1$ is something of a critical value.

THEOREM 3.5.1. There exist two trees Γ_1, Γ_2 such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln((d_i-1)/3) = 0 \tag{3.9}$$

where Γ_1 has a unique disordered ground state, and where Γ_2 has a non-unique ground state.

PROOF. From [10] we know that the Cayley tree of degree 4 has a unique disordered ground state, and since $d_i = 4$ for this tree we get that $\mu = 1$. Let Γ_2 be the tree defined such that $d_{x_i} = d(|x_i - \overline{0}|)$ where $d(n) : \mathbb{N} \to \mathbb{R}$ is given by

$$d(n) = 4 + \lfloor 2\ln_{4/3}(n+7)\rfloor - \lfloor 2\ln_{4/3}(n+6)\rfloor \tag{3.10}$$

Then clearly

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln((d(i)-1)/3) = 0 \tag{3.11}$$

and we have

$$\prod_{i=1}^{n} \frac{3}{d_i - 1} \geq M \cdot \left(\frac{3}{4}\right)^{2\ln_{4/3}(n+6)} = M \cdot \frac{1}{(n+6)^2} \tag{3.12}$$

so the functional composition

$$\bigcirc_{i=1}^{\infty} f_{d_i}(x) \geq (1+M)^{-1} > 0 \tag{3.13}$$

and so by a similar argument as above the ground state on Γ_2 is not unique. □

Note that we do not have a uniform lower bound on the functional composition in the first part of the proof of Corollary 3.5.1. One tantalizing generalization is if we can uniformly lower bound this for trees that are not characterized by a single sequence $\{d_i\}$ of degrees uniformly radiating from the origin. However we have a counterexample to this.

Counterexample 3.5.1. For any $d \geq 5$ define the following tree via the sequence of degrees $\{d_i\}$ as $d_i = 2$ for all $i \leq N$ and $d_i = 5$ for all $i > N$. Then we have $\lim_{n\to\infty} \bigcirc_{i=1}^{n} F_{d_i}(x) = \frac{1}{(-3)^N} \cdot x_5$ where x_5 is the positive solution to $F_5(x_5) = -x_5$ which can be made as small as we like while maintaining that $\mu = \ln(\frac{4}{3})$.

This prevents us from using a simple macroscopic condition (such as all sequences obeying the μ condition) to obtain a bound for trees with arbitrary degrees; however we can still make this work for irregular trees with some added assumptions.

Theorem 3.5.2. Let Γ be a tree with distinguished root $\overline{0}$ such that $\deg(x) = d_i$ for all x such that $|x - \overline{0}| = i$. If

$$\ln(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln((d_i - 1)/3) > 0 \tag{3.14}$$

then the AKLT ground state on this tree is not unique.

Proof. We note that this condition immediately implies that the infinite composition $\bigcirc_{i=1}^{\infty} F_{d_i}(x) > 0$ for $x \in (0, 1]$ by the previous lemma; let N be such that for all $n \geq N$ we have that $\bigcirc_{i=1}^{n} F_{d_i}(x) = -x$ has at least one, solution; let x_n be the smallest solution of $\bigcirc_{i=1}^{n} F_{d_i}(x) = -x$; this sequence is bounded on $(\epsilon, 1]$ for some $\epsilon > 0$ and so has a convergent subsequence $x_{n_k} \to x$ with $x \neq 0$. Let $\mathbf{x}_{n_k} = x_{n_k}\sigma_1$; then the ground state expectations of an observable A with support on $\overline{0}$ of a sequence of boundary conditions

$$\lim_{k\to\infty} \langle \psi_{n_k}(B_{T_{n_k}}(\mathbf{x}_{n_k})), A\psi_{n_k}(B_{T_{n_k}}(\mathbf{x}_{n_k}))\rangle = \lim_{k\to\infty} \langle \psi_0(\tilde{\mathbb{F}}_{T_{n_k}}(B_{T_{n_k}}(\mathbf{x}_{n_k}))), A\psi_0(\tilde{\mathbb{F}}_{T_{n_k}}(B_{T_{n_k}}(\mathbf{x}_{n_k})))\rangle$$
$$= \langle \psi_0(\mathbb{1} + x\sigma_1), A\psi_0(\mathbb{1} + x\sigma_1)\rangle \neq \lim_{k\to\infty} \langle \psi_{n_k}(B_{T_{n_k}}(\mathbf{0})), A\psi_{n_k}(B_{T_{n_k}}(\mathbf{0})\rangle = \langle \psi_0(\mathbb{1}), A\psi_0(\mathbb{1})\rangle$$

□

We can also use the above method to give a result for a general irregular tree with a simple condition.

Theorem 3.5.3. Let Γ be a tree such that there exists a $\mu > 1$ and $C > 0$ such that for every sequence $\{d_{i_k}\}$ such that $|d_{i_k} - \overline{0}| = k$ we have that $\prod_{k=1}^{n} \frac{d_{i_k}-1}{3} \geq C\mu^n$. Then the ground state of this model is not unique.

Proof. For each n let α_k be the index of the leaf corresponding to the minimal sequence

$$\min_{\{i_k\}} \left(\prod_{k=1}^{n} \frac{d_{i_k} - 1}{3} \right) \tag{3.15}$$

use the bound that $f_d(x_1, ..., x_n) \geq f_d(x_\alpha)$ and thus the composition

$$f_{d_{i_0}}(f_{d_{i_1}}(f_{d_{i_2}}(...))) \geq \bigcirc_{k=1}^{n} f_{d_{\alpha_k}}(x) \tag{3.16}$$

and from the above condition we know that from the bounds previously used that

$$\bigcirc_{k=1}^{n} f_{d_{\alpha_k}}(x) \geq \left(\sum_{m=1}^{n} \prod_{k=1}^{m} \frac{3}{d_{a_k} - 1} + 1 \right)^{-1} \geq \left(C \sum_{m=1}^{n} \mu^{-m} + 1 \right)^{-1} \geq \left(\frac{C\mu^{-1}}{1 - \mu^{-1}} + 1 \right)^{-1} > 0 \tag{3.17}$$

Thus the concatenations of the transfer functions are uniformly bounded away from zero so there exists a convergent subsequence that is positive and one that is negative, so the ground state is not unique. □

Note that in this last proof we could not use the geometric mean condition for the minimal sequence, as the functional compositions associated to the set of such sequences is not uniformly bounded below as was shown in Counterexample 1.

One can see the constant 3μ as something of a "connective" constant in that it quantifies the exponential growth of all the self-avoiding paths from the root to the leaves in the tree.

3.6. Bilayer trees

The case of bilayer trees is significantly different, and requires a slightly different analysis; however the same general fixed-point type argument exists.

Bilayer Cayley tree. The bilayer tree is substantially different. This is because we have non-trivial operators which commute with representations of SU(2). Note that in the bilayer tree, if we have the transfer operator at each (bilayer) node in the tree $\tilde{\mathbb{F}} : M_2(\mathbb{C})^{\otimes 2(d-1)} \to M_2(\mathbb{C})^{\otimes 2}$ we look for a pair of density matrices $\rho_0, \rho_1 \in M_2(\mathbb{C})^{\otimes 2}$ such that $T(\rho_i^{\otimes(d-1)}) = \rho_{1-i}$. Note that within $M_2(\mathbb{C})^{\otimes 2}$ we have the Casimir $\sum_i \sigma_i \otimes \sigma_i$ which commutes with the $SU(2)$ representation so we cannot have that $T(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1}$; however T is still trace-preserving. Thus if we define $\boldsymbol{\sigma} = [\sigma_{lk}]$ where $\sigma_{lk} = \sigma_l \otimes \sigma_k$ with $l, k \in \{0, 1, 2, 3\}$; then our boundary conditions B can be written as $\mathbb{1} \otimes \mathbb{1} + \mathbf{x} \cdot \boldsymbol{\sigma}$. We find fixed points of the transfer

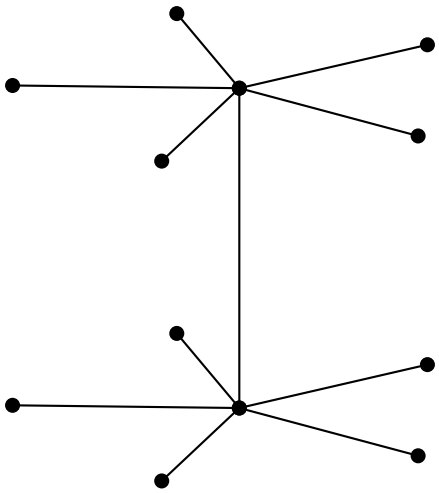


FIGURE 3.7. First layer of bilayer Cayley tree of degree $d = 6$ which has splitting number $g = d - 2 = 4$

operator, the coefficients of which are rational functions of this vector. For simplicity of notation, for a bilayer tree of degree d we call $g - 2$ the **splitting number** of the tree.

THEOREM 3.6.1. The bilayer Cayley tree has a unique ground state for $g = 1, 2$.

PROOF. The case of $g = 1$ is the case of a bilayer AKLT chain which has exponential decay of correlations and a stably-gapped ground state by Theorem 2.5.2. However, by way of an example, we prove this using a fixed point method. Note that the transfer operator can be fully diagonalized so that, starting with a boundary condition

$$B = \mathbb{1} \otimes \mathbb{1} + \sum_i x_{0i} \mathbb{1} \otimes \sigma_i + \sum_i x_{i0} \sigma_i \otimes \mathbb{1} + \sum_{i \neq j} x_{ij} \sigma_i \otimes \sigma_j + \sum_i x_{ii} \sigma_i \otimes \sigma_i = \mathbb{1} \otimes \mathbb{1} + \mathbf{x} \cdot \boldsymbol{\sigma} \tag{3.1}$$

then we have

$$\tilde{\mathbb{F}}(B) = \mathbb{1} \otimes \mathbb{1} + \sum_i f_{i0}(\mathbf{x})(\sigma_i \otimes \mathbb{1}) + \sum_i f_{0i}(\mathbf{x})(\mathbb{1} \otimes \sigma_i) + \sum_{i \neq j} f_{ij}(\mathbf{x})(\sigma_i \otimes \sigma_j) + \sum_i f_{ii}(\mathbf{x})(\sigma_i \otimes \sigma_i) \tag{3.2}$$

where

$$f_{0i}(\mathbf{x}) = \frac{-\frac{1}{3}x_{0i} - \frac{1}{9}x_{i0}}{\left(1 + \frac{\sum_i x_{ii}}{9}\right)}, \qquad f_{i0}(\mathbf{x}) = \frac{-\frac{1}{3}x_{i0} - \frac{1}{9}x_{0i}}{\left(1 + \frac{\sum_i x_{ii}}{9}\right)} \tag{3.3}$$

$$f_{ij}(\mathbf{x}) = \frac{\frac{1}{9}x_{ij}}{\left(1 + \frac{\sum_i x_{ii}}{9}\right)}, \qquad f_{ii}(\mathbf{x}) = \frac{1 + \frac{x_{ii}}{9}}{\left(1 + \frac{\sum_i x_{ii}}{9}\right)} \tag{3.4}$$

we search for a fixed point $f_{kl}(\mathbf{x}) = \pm x_{kl}$ and so we have the set of equations

$$\pm x_{0i}\left(1 + \frac{\sum_i x_{ii}}{9}\right) = -\frac{1}{3}x_{0i} - \frac{1}{9}x_{i0}, \qquad \pm x_{i0}\left(1 + \frac{\sum_i x_{ii}}{9}\right) = -\frac{1}{3}x_{i0} - \frac{1}{9}x_{0i} \tag{3.5}$$

$$\pm x_{ij}\left(1+\frac{\sum_i x_{ii}}{9}\right)=\frac{1}{9}x_{ij},\qquad \pm x_{ii}\left(1+\frac{\sum_i x_{ii}}{9}\right)=1+\frac{x_{ii}}{9} \tag{3.6}$$

which has solution $x_{i0}=x_{0i}=x_{ij}=0$ and

$$x_{ii}=\frac{\left(\frac{1}{9}+\frac{x_{ii}}{9}\right)}{\left(1+\frac{\sum_i x_{ii}}{9}\right)} \tag{3.7}$$

or

$$\left(\frac{1}{9}+\frac{\sum_i x_{ii}}{9}\right)=\left(\sum_i x_{ii}\right)\left(1+\frac{\sum_i x_{ii}}{9}\right) \tag{3.8}$$

which has solution $\left(\sum_i x_{ii}\right)^2+8\sum_i x_{ii}-3=0$ and thus $\sum_i x_{ii}=-4+\sqrt{19}$ which is within the unit sphere. This value, while not being nonzero, is a fixed point of $\tilde{\mathbb{F}}$ that does not break the SU(2) symmetry; moreover, the functions f converge exponentially quickly to this fixed point, so we have a unique ground state.

For $g=2$, like before, we have our bilayer transfer operator $\tilde{E}:M_2(\mathbb{C})^{\otimes 4}\to M_2(\mathbb{C})^{\otimes 2}$ where we can explicitly calculate for $B=\mathbb{1}\otimes\mathbb{1}+\mathbf{x}\cdot\sigma$ that we have

$$\tilde{\mathbb{F}}(B)=\mathbb{1}\otimes\mathbb{1}+\sum_i f_{i0}(\mathbf{x})(\sigma_i\otimes\mathbb{1})+\sum_i f_{0i}(\mathbf{x})(\mathbb{1}\otimes\sigma_i)+\sum_{i\neq j} f_{ij}(\mathbf{x})(\sigma_i\otimes\sigma_j)+\sum_i f_{ii}(\mathbf{x})(\sigma_i\otimes\sigma_i) \tag{3.9}$$

where each f is a rational function of the vector $\mathbf{x}$. We have that we are solving the equation

$$\pm x_{kl}=f_{kl}(\mathbf{x}) \tag{3.10}$$

and so we get that we need to solve

$$\left(-\frac{2}{3}x_{i0}-\frac{2}{9}x_{0i}-\frac{4}{15}x_{ii}x_{i0}-\sum_{j\neq i}\frac{2}{45}x_{j0}y_{ij}-\sum_{j\neq i}\frac{2}{9}x_{i0}x_{ij}\right)$$
$$\pm x_{i0}\left(1+\sum_i\frac{1}{3}x_{i0}^2+\frac{1}{9}x_{i0}x_{0i}+\sum_i\frac{2}{9}x_{ii}+\sum_i\frac{1}{9}x_{ii}^2+\sum_{i,j}\frac{1}{9}x_{ij}^2\right)=0$$

and

$$\left(-\frac{2}{3}x_{0i}-\frac{2}{9}x_{i0}-\frac{4}{15}x_{ii}x_{0i}-\sum_{j\neq i}\frac{2}{45}x_{0j}y_{ji}-\sum_{j\neq i}\frac{2}{9}x_{0i}x_{ji}\right)$$
$$\pm x_{0i}\left(1+\sum_i\frac{1}{3}x_{i0}^2+\frac{1}{9}x_{i0}x_{0i}+\sum_i\frac{2}{9}x_{ii}+\sum_i\frac{1}{9}x_{ii}^2+\sum_{i,j}\frac{1}{9}x_{ij}^2\right)=0$$

and

$$\left(\frac{2}{9}x_{ij}+\frac{2}{9}x_{i0}x_{0j}+\frac{2}{45}x_{i0}x_{j0}+\frac{2}{75}x_{ij}x_{ii}+\frac{2}{75}x_{ij}x_{jj}+\frac{4}{225}x_{ij}x_{kk}+\frac{2}{225}x_{ik}x_{kj}\right)$$

$$\pm x_{ij}\left(1+\sum_i \frac{1}{3}x_{i0}^2+\frac{1}{9}x_{i0}x_{0i}+\sum_i \frac{2}{9}x_{ii}+\sum_i \frac{1}{9}x_{ii}^2+\sum_{i,j}\frac{1}{9}x_{ij}^2\right)=0$$

and

$$\left(\frac{1}{9}+\frac{2}{9}x_{ii}+\frac{2}{9}x_{i0}x_{0i}+\frac{2}{15}x_{i0}^2+\sum_{j\neq i}\frac{2}{45}x_{i0}x_{j0}+\frac{1}{25}x_{ii}^2+\frac{2}{75}x_{ij}x_{ij}+\sum_{i,j}\frac{1}{225}x_{ij}^2+\sum_{i,j}\frac{2}{225}x_{ii}x_{jj}\right)$$
$$\pm x_{ii}\left(1+\sum_i \frac{1}{3}x_{i0}^2+\frac{1}{9}x_{i0}x_{0i}+\sum_i \frac{2}{9}x_{ii}+\sum_i \frac{1}{9}x_{ii}^2+\sum_{i,j}\frac{1}{9}x_{ij}^2\right)=0$$

This set of equations only has the solution $\mathbf{x}=\mathbf{0}$; thus it must be contracting on boundary conditions and so has a unique limit. The same analysis as in [10] can be done to show this ground state is unique. □

Next we show that we have non-uniqueness for splitting number $g=3$

Theorem 3.6.2. The AKLT Hamiltonian defined on the $g=3$ bilayer Cayley tree does not have a unique ground state.

Proof. We start with the ansatz that we have a boundary condition of the form

$$B(\mathbf{x})=\mathbb{1}\otimes\mathbb{1}+\mathbf{x}\cdot\boldsymbol{\sigma} \tag{3.11}$$

where we note that the space of boundary conditions such that $x_1=x_{0i}=x_{i0}$ and $x_2=x_{ij}=x_{kl}$ and $x_3=x_{ii}=x_{jj}$ is a fixed subspace, so we assume $\mathbf{x}$ satisfies these and search for two boundary conditions B_+ and B_- that are solutions to

$$\tilde{\mathbb{F}}(B_\pm(\mathbf{x})^{\otimes 3})=B_\mp(\mathbf{x}) \tag{3.12}$$

which means

$$f_{0i}(\mathbf{x})=f_1(x_1,x_2,x_3)=-x_1 \tag{3.13}$$

$$f_{ij}(\mathbf{x})=f_2(x_1,x_2,x_3)=x_2 \tag{3.14}$$

$$f_{ii}(\mathbf{x})=f_3(x_1,x_2,x_3)=x_3 \tag{3.15}$$

where f_1,f_2,f_3 are rational functions; define $f_0(\mathbf{x})=\mathrm{Tr}(\mathbb{F}(B_\pm(\mathbf{x})^{\otimes 3}))$, so we can rearrange by searching for a root of the following polynomials:

$$f_1(\mathbf{x})+x_1\cdot f_0(\mathbf{x})=-3x_1-\frac{37}{15}x_1^3-\frac{2}{5}x_1x_2^2-\frac{8}{15}x_1x_3-\frac{4}{25}x_1x_2x_3-\frac{14}{75}x_1x_2^2+\frac{8}{75}x_1x_3^2$$
$$+\frac{2}{25}x_1x_3^2+\frac{22}{25}x_1x_2x_3+\frac{3}{25}x_3^3+\frac{4}{75}x_2^3+\frac{8}{25}x_2^2+2x_2^2+2x_3+6x_1^2=0$$

and

$$f_2(\mathbf{x}) - x_2 \cdot f_0(\mathbf{x}) = -\frac{2}{3} + \frac{7}{5}x_1^2 + \frac{14}{75}x_2^2 + \frac{6}{25}x_3 + \frac{1}{15}x_3^2 + \frac{11}{75}x_2x_3$$
$$+\frac{22}{75}x_2x_1^2 + \frac{4}{25}x_3x_1^2 + \frac{4}{25}x_2x_1^2 + \frac{4}{75}x_3x_2^2 + \frac{2}{25}x_2^3 + \frac{1}{25}x_3^3 = 0$$

and

$$f_3(\mathbf{x}) - x_3 \cdot f_0(\mathbf{x}) = -\frac{8}{9} + \frac{142}{75}x_1^2 + \frac{8}{25}x_2^2 + \frac{1}{3}x_3 + \frac{13}{75}x_3^2 + \frac{1}{15}x_3^3 + \frac{16}{75}x_2x_1^2 + \frac{8}{25}x_3x_2^2 = 0 \tag{3.16}$$

From a simple check with a graphing calculator we find there is a pair of solutions $\mathbf{x}_\pm$ such that $\tilde{\mathbb{F}}(B(\mathbf{x}_\pm)^{\otimes 3}) = B(\mathbf{x}_\mp)$ where $\mathbf{x}_+ \approx [0.3020, 0.0466, 0.1754]$ and $\mathbf{x}_- \approx [-0.3020, 0.0466, 0.1754]$. □

Note that while this is the same degree as the single-layer Cayley tree with degree $d_s = 4$ with splitting number $g_s := d_s - 1 = 3$, the bilayer degree $d_b = 5$ Cayley tree has the same splitting number $g_b = d_b - 2 = 3$; however, in the single-layer Cayley tree we have a unique ground state, whereas in the bilayer case we have a degenerate Néel ordered ground state space. [10].

CHAPTER 4

Local Topological Quantum Order and Spectral Gap Stability in the Hexagonal Lattice AKLT Model

Introduction

4.0.1. Main Results. We consider the spin-3/2 AKLT model defined on the hexagonal lattice considered as a graph, denoted $\Gamma = (\mathcal{V}_\Gamma, \mathcal{E}_\Gamma)$. For any vertex $x \in \mathcal{V}_\Gamma$ the onsite Hilbert space is a four-dimensional complex Hilbert space, denoted $\mathcal{H}_x$, and for any edge $e = (x, y) \in \mathcal{E}_\Gamma$ the nearest-neighbor interaction, denoted P_e, is given by the orthogonal projection on the subspace of total spin 3 contained in $\mathcal{H}_x \otimes \mathcal{H}_y$. Then, for any finite subgraph $\Lambda = (\mathcal{V}_\Lambda, \mathcal{E}_\Lambda)$ of Γ, the local Hamiltonian is given by

$$H_\Lambda = \sum_{e \in \mathcal{E}_\Lambda} P_e \in \mathcal{A}_\Lambda$$

where

$$\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda), \quad \mathcal{H}_\Lambda = \bigotimes_{x \in \mathcal{V}_\Lambda} \mathcal{H}_x$$

are the algebra of local observables and Hilbert space of states associated with Λ, respectively. For completeness, we recall that the support of an observable $A \in \mathcal{A}_\Lambda$, denoted $\mathrm{supp}(A)$ is the smallest set $X \subseteq \Lambda$ such that

$$[A, B \otimes \mathbb{1}_X] = 0 \qquad \forall\, B \in \mathcal{A}_{\Lambda \setminus X}\,.$$

The AKLT model is well-known to be frustration-free, and so its ground state space for any finite Λ is given by

$$\mathcal{G}_\Lambda = \ker(H_\Lambda).$$

This model was first introduced in [3], and subsequently studied in [19] where they showed that there exists a unique frustration-free ground state, $\omega : \mathcal{A}_\Gamma \to \mathbb{C}$, of the infinite system, meaning

$$\omega(P_e) = 0 \quad \forall e \in \mathcal{E}_\Gamma,$$

and that ω has exponential decay of correlations. Here, $\mathcal{A}_\Gamma$ is the C^*-algebra of quasi-local observables defined by

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{\mathrm{loc}}}^{\|\cdot\|}, \quad \mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\substack{\Lambda \subset \Gamma \\ |\mathcal{V}_\Lambda| < \infty}} \mathcal{A}_\Lambda\,.$$

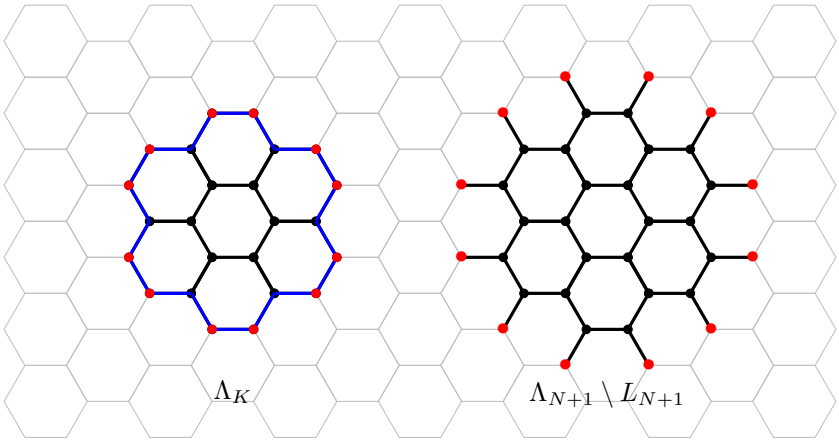


FIGURE 4.1. Illustration of Λ_K and $\Lambda_{N+1} \setminus L_N$. The blue edges comprise L_K

The uniqueness of this state implies that for any sequence of increasing and absorbing finite volumes $\Lambda_n \uparrow \Gamma$, and normalized ground states $\psi_n \in \mathcal{G}_{\Lambda_n}$

$$\lim_{n\to\infty} \langle \psi_n, A\psi_n \rangle = \omega(A)\,. \tag{4.1}$$

The goal of this work is to prove that this ground state satisfies the *local topological quantum order (LTQO)* [6, 32, 37], which captures that any pair of finite volume ground states $\Psi, \Phi \in \mathcal{G}_\Lambda$ are indistinguishable from each other on local observables supported sufficiently far from the boundary of Λ. As a consequence, they are also indistinguishable from the infinite volume ground state.

The precise statement of this indistinguishability will be expressed in terms of a specific sequence of increasing and absorbing finite volumes $\Lambda_N \subseteq \Gamma$. To define these volumes, let $\tilde{\Gamma}$ denote the dual lattice of Γ. For any vertex $\tilde{x}$ in $\tilde{\Gamma}$, let $h_{\tilde{x}}$ denote the finite subgraph of Γ obtained from the union of the hexagon centered at $\tilde{x}$ with all edges $e \in \mathcal{E}_\Gamma$ connected to this hexagon, see Figure 4.1 Then, for a fixed vertex $\tilde{0} \in \tilde{\Gamma}$ define

$$\Lambda_N = \bigcup_{\tilde{x} \in b_{N-1}(\tilde{0})} h_{\tilde{x}} \qquad \forall N \in \mathbb{N}\,, \tag{4.2}$$

where $b_N(\tilde{0}) \subseteq \tilde{\Gamma}$ is the ball of radius N around $\tilde{0}$ with respect to the graph distance in the dual lattice figure here.. In addition, we denote by $\partial\Lambda$ for any finite subgraph $\Lambda \subseteq \Gamma$ as the set of all vertices in Λ that share an edge with some vertex outside of Λ, i.e.

$$\partial\Lambda = \{x \in \mathcal{V}_\Lambda : \exists\, y \in \mathcal{V}_\Gamma \setminus \mathcal{V}_\Lambda \text{ s.t. } (x,y) \in \mathcal{E}_\Gamma\}\,. \tag{4.3}$$

We now state our main result, which says that the finite volume ground states $\psi \in \mathcal{G}_{\Lambda_N}$ are indistinguishable from the the infinite volume ground state on observables $A \in \mathcal{A}_{\Lambda_{K-1}}$ for $K << N$.

Theorem 4.0.1 (Ground State Indistinguishability). *Let $\omega : \mathcal{A}_\Gamma \to \mathbb{C}$ be the unique frustration free ground state of AKLT model on the hexagonal lattice, and Λ_N be as in (4.2). Then, there exists an $\epsilon > 0$ such that for all $N \geq K + 20 \geq 40$*

$$|\langle \Psi, A\Psi \rangle - \omega(A)| \leq 2\|A\| F(N,K) e^{F(N,K)} \tag{4.4}$$

for all $A \in \mathcal{A}_{\Lambda_{K-1}}$ and normalized $\Psi \in \mathcal{G}_{\Lambda_N}$, where $F(N,K) = .9Ke^{-\epsilon(N-K)}$.

Note that if $N \geq K + c\ln(K)$ for any $c \geq \epsilon^{-1}$, the above shows that

$$|\langle \Psi, A\Psi \rangle - \omega(A)| \leq |\partial\Lambda_{K-1}| \|A\| e^{-\epsilon(N-K)} \tag{4.5}$$

where we use that $|\partial\Lambda_{K-1}| = 6(K-1) \geq 1.8e^{.9}K$ for any $K \geq 20$. This will be used to prove that the hexagonal AKLT model satisfies the LTQO property. To state this result, let G_{Λ_N} denote the orthogonal projection onto the ground state space, $\mathcal{G}_{\Lambda_N}$.

Corollary 4.0.2 (LTQO). *Let $\epsilon > 0$ be as in Theorem 4.0.1 and fix any $c \geq \epsilon^{-1}$. Then, for all $K \geq 20$ and $N \geq K + \max\{20, c\ln(K)\}$, one has*

$$\|G_{\Lambda_N} A G_{\Lambda_N} - \omega(A) G_{\Lambda_N}\| \leq 2|\partial\Lambda_{K-1}| \|A\| e^{-\epsilon(N-K)} \qquad \forall\, A \in \mathcal{A}_{\Lambda_{K-1}}\,. \tag{4.6}$$

where the finite volumes are defined as in (4.2).

Since the AKLT model is translation-invariant, the above also holds for any translation of the finite volume sequence from (4.2). We give the proof assuming that Theorem 4.0.1 holds.

Proof. If $A \in \mathcal{A}_{\Lambda_{K-1}}$ is hermitian or anti-hermitian, then by Theorem 4.0.1 and (4.5),

$$\|G_{\Lambda_N} A G_{\Lambda_N} - \omega(A) G_{\Lambda_N}\| = \sup_{\substack{\psi \in \mathcal{G}_{\Lambda_N} \\ \|\psi\|=1}} |\langle \psi, A\psi \rangle - \omega(A)| \leq |\partial\Lambda_{K-1}| \|A\| e^{-\epsilon(N-K)}\,. \tag{4.7}$$

The result for arbitrary $A \in \mathcal{A}_{\Lambda_{k-1}}$ then follows from writing $A = \frac{A+A^*}{2} + \frac{A-A^*}{2}$ and applying (4.7) to each term in the sum. □

The immediate motivation for proving LTQO is that it is an important condition in various spectral gap stability theorems [6]. In particular, it is necessary for any version of such results currently in the literature that may be applied to the AKLT model on the hexagonal lattice. Of course, spectral gap stability only makes sense if the model is, in fact, gapped. There is strong evidence that it is in the case of the AKLT model on the hexagonal lattice. First, Kennedy, Lieb, and Tasaki proved that the ground state ω has exponential decay of correlations [19], a property that, generically, occurs together with a non-vanishing gap in the spectrum above the ground state. Further evidence stems from proofs that

the AKLT model on decorated hexagonal lattices is gapped [2, 28]. Finally, in two almost simultaneous papers, convincing corroboration of a gap was independently obtained by combining analytical methods with sophisticated numerical techniques in [25] and [45].

For completeness, we state a spectral gap stability result for the AKLT model on the hexagonal lattice for a large class of perturbations that would follow from Corollary 4.0.2 given that the model is gapped. Define $\Lambda_{N,\tilde{x}} \subset \Gamma$ as the translation by $\tilde{x} \in \tilde{\Gamma}$ of the set Λ_N defined in (4.2). Suppose $V_{n,\tilde{x}} \in \mathcal{A}_{\Lambda_{n,\tilde{x}}}$, $n \geq 1, \tilde{x} \in \tilde{\Gamma}$ is self-adjoint and define

$$H^V_{\Lambda_N}(s) = H_{\Lambda_N} + s \sum_{\substack{n\geq 1, \tilde{x}\in\tilde{\Gamma} \\ \Lambda_{n,\tilde{x}}\subset\Lambda_N}} V_{n,\tilde{x}}, \quad s \in \mathbb{R}. \tag{4.8}$$

Corollary 4.0.3 (Spectral Gap Stability [37]). *Assume there exist $\gamma > 0$, and $N_0 \geq 1$ such that $\operatorname{spec} H_{\Lambda_N} \cap (0, \gamma) = \emptyset$, for all $N \geq N_0$, and that there are constants $a > 0$, and $\theta \in (0, 1]$ such that*

$$\|V_{n,\tilde{x}}\| \leq e^{-an^\theta}, \text{ for all } n \geq 1, \ \tilde{x} \in \tilde{\Gamma}\,.$$

Then, for all $\gamma_0 \in (0, \gamma)$, there exists $s(\gamma_0) > 0$ and a positive, decreasing sequence $\epsilon_N \to 0$, such that

$$\operatorname{spec} H^V_{\Lambda_N}(s) \subseteq [-\epsilon_N, \epsilon_N] \cup [\gamma_0 + \epsilon_N, \infty) = \emptyset, \text{ for all } N \text{ large enough, and } |s| < s(\gamma_0).$$

As is shown, e.g., in [], the above result implies that the perturbed AKLT model defined by V has a unique gapped ground state ω_s for all $|s| \leq s_0(\gamma_0)$ with gap bounded below by γ_0.

We also note that both the LTQO property and spectral gap stability can also be shown for the AKLT model on any d-decorated hexagonal lattice, $d \in \mathbb{N}$, by the arguments provided in this paper. This result, which will be written precisely in Section 4.5, improves the result of [27], which proved these properties for decoration parameters $d \geq 5$.

4.1. Description of the Ground States

The general description of the ground state expectations appears in Section 2.5 2.8, etc.

An immediate consequence of (2.8) and Theorem 2.5.1, is that the expectation of an observable $A \in \mathcal{A}_\Lambda$ in any ground state $\Psi(f) \in \mathcal{G}_\Lambda$ is given by

$$\langle \Psi(f), A\Psi(f)\rangle = \int d\Omega^\Lambda \prod_{(x,y)\in\mathcal{E}_\Lambda} |u_x v_y - v_x u_y|^2 |f|^2 A(\mathbf{\Omega}) = \int d\rho_\Lambda |f|^2 A(\mathbf{\Omega}) \tag{4.1}$$

where, if Ω_x denotes the spherical variable from (2.2) associated to any $x \in \mathcal{V}_G$, one inserts the identity $|u_x v_y - u_x v_y|^2 = \frac{1}{2}(1 - \Omega_x \cdot \Omega_y)$ for every $(x, y) \in \mathcal{E}_\Lambda$ and defines

$$d\rho_\Lambda = \rho_\Lambda d\Omega^\Lambda, \qquad \rho_\Lambda = 2^{-|\mathcal{E}_\Lambda|} \prod_{(x,y)\in\mathcal{E}_\Lambda} (1 - \Omega_x \cdot \Omega_y)\,. \tag{4.2}$$

Recalling (4.2), in the case that $A \in \mathcal{A}_{\Lambda_K}$ and $\Psi(f) \in \mathcal{G}_{\Lambda_N}$ with $N > K + 20 > 40$, Theorem 4.0.1 follows from showing that (4.1) is well-approximated by $\omega(A)$. To accomplish this, we follow [27] and introduce two additional maps, called the *bulk-boundary map* and *bulk state*. We now turn to defining these quantities.

4.1.1. The Bulk-Boundary Map and Bulk State for the Hexagonal Model. Turning back to the hexagonal AKLT model, it was shown in [27] that for any $\Psi(f) \in \mathcal{G}_{\Lambda_N}$ and $A \in \mathcal{A}_{\Lambda_K}$ with $K < N$, (4.1) implies the ground state expectation can be written in terms of $|f|^2$ and the bulk-boundary map

$$\begin{aligned} \omega_N(A; \Omega^{\partial\Lambda_N}) &:= Z_N(A; \Omega^{\partial\Lambda_N}) / Z_N(\Omega^{\partial\Lambda_N}) \\ Z_N(A; \Omega^{\partial\Lambda_N}) &:= \int d\Omega^{\Lambda_N \setminus \partial\Lambda_N} \rho_{\Lambda_N} A(\Omega), \quad Z_N(\Omega^{\partial\Lambda_N}) = Z_N(\mathbb{1}; \Omega^{\partial\Lambda_N}) \end{aligned} \tag{4.3}$$

It is easy to verify that $\omega_N(A; \Omega^{\partial\Lambda_N})$ is a polynomial in the boundary variables $\Omega^{\partial\Lambda_N} = (\Omega_x : x \in \partial\Lambda_N)$ for any $A \in \mathcal{A}_{\Lambda_K}$, and one finds the following expression for any ground state expectation.

Lemma 4.1.0.1 ([27]). Consider the sequence of finite volumes from (4.2), and fix $N > K \geq 1$. Let $A \in \mathcal{A}_{\Lambda_K}$ and $\Psi(f) \in \mathcal{G}_{\Lambda_N}$ where Λ_N is as in (4.2). Then

$$\langle \Psi(f), A\Psi(f) \rangle = \int d\rho_{\Lambda_N} |f|^2 \omega_N(A; \Omega^{\partial\Lambda_N}) \,. \tag{4.4}$$

We give an outline of the proof and point the interested reader to [27] for more details. Let

$$\mathring{\Lambda}_N = \Lambda_N \setminus \partial\Lambda_N \tag{4.5}$$

denote the graph induced from removing all boundary vertices from Λ_N. Lemma 4.1.0.1 is a consequence of recognizing that since $\Lambda_K \subseteq \mathring{\Lambda}_N$,

$$\langle \Psi(f), A\Phi(f) \rangle = \int d\Omega^{\Lambda_N} |f|^2 \rho_{\Lambda_N} A(\Omega) = \int d\Omega^{\partial\Lambda_N} |f|^2 \int d\Omega^{\mathring{\Lambda}_N} \rho_{\Lambda_N} \frac{\int d\Omega^{\mathring{\Lambda}_N} \rho_{\Lambda_N} A(\Omega)}{\int d\Omega^{\mathring{\Lambda}_N} \rho_{\Lambda_N}}$$

and verifying that $Z_N(\Omega^{\partial\Lambda_N}) = \int d\Omega^{\mathring{\Lambda}_N} \rho_{\Lambda_N} > 0$ for all values of the boundary variables. The latter follows since, for any fixed choice of the boundary variables $\Omega^{\partial\Lambda_N} = (\Omega_x : x \in \partial\Lambda_N)$,

$$Z_N(\Omega^{\partial\Lambda_N}) = \|\Psi(g_{\partial\Omega})\|^2 \quad \text{where} \quad g_{\partial\Omega} = \prod_{\substack{(x,y) \in \mathcal{E}_{\Lambda_N} \\ x \in \partial\Lambda_N}} (u_x v_y - u_y v_x) \in \mathcal{H}^{\text{gss}}_{\partial\mathring{\Lambda}_N} \,.$$

The definition for the *bulk state* $\omega_N^{\text{bulk}} : \mathcal{A}_{\Lambda_N} \to \mathbb{C}$ comes from averaging $Z_N(A; \Omega^{\partial\Lambda_N})$ over the boundary variables. Since all ground states $\Psi(f) \in \mathcal{G}_{\Lambda_N}$ only differ by the choice of boundary polynomial, this suggests that $\omega_N^{\text{bulk}}(A)$ should produce a good approximation of $\langle \Psi(f), A\Psi(f) \rangle$ for any normalized $\Psi(f) \in$

$\mathcal{G}_{\Lambda_N}$ and observable $A \in \mathcal{A}_{\Lambda_K}$ with $K << N$. Specifically, for any $A \in \mathcal{A}_{\Lambda_N}$, define

$$\omega_N^{\text{bulk}}(A) := Z_N^{\text{bulk}}(A)/Z^{\text{bulk}} \tag{4.6}$$
$$Z_N^{\text{bulk}}(A) := \int d\rho_{\Lambda_N} A(\Omega) = \int d\Omega^{\partial\Lambda_N} Z_N(A;\Omega^{\partial\Lambda_N}), \quad Z_N^{\text{bulk}} = Z_N^{\text{bulk}}(\mathbb{1})\,.$$

While it is not a priori clear whether ω_N^{bulk} is a state on $\mathcal{A}_{\Lambda_N}$, it is a ground state of H_{Λ_K} for any $K < N$. This can be seen from considering Theorem 2.5.1 with $G = \Lambda = \Lambda_N$, which gives the ground state space for AKLT model obtained from replacing $\mathcal{H}^{(3/2)}$ at each $x \in \partial\Lambda_N$ with $\mathcal{H}^{(1/2)}$. In this case, ω_N^{bulk} is the unique ground state of H_G, which by frustration freeness is also a ground state of H_{Λ_K} for any $K < N$. Thus, by (4.1), it follows that

$$\lim_{N\to\infty} \omega_N^{\text{bulk}}(A) = \omega(A) \qquad \forall\, A \in \mathcal{A}_\Gamma^{\text{loc}}\,.$$

The strategy for proving Theorem 4.0.1 is as follows. For any normalized $\Psi(f) \in \mathcal{G}_{\Lambda_N}$ and fixed observable $A \in \mathcal{A}_{\Lambda_K}$, we can bound

$$\begin{aligned}
|\langle\Psi(f), A\Psi(f)\rangle - \omega(A)| &\leq \left|\langle\Psi(f), A\Psi(f)\rangle - \omega_N^{\text{bulk}}(A)\right| + \left|\omega_N^{\text{bulk}}(A) - \omega(A)\right| \\
&= \left|\int d\rho_{\Lambda_N} |f|^2 [\omega_N(A;\Omega^{\partial\Lambda_N}) - \omega_N^{\text{bulk}}(A)]\right| + \left|\omega_N^{\text{bulk}}(A) - \omega(A)\right| \\
&\leq \sup_{\Omega^{\partial\Lambda_N}} \left|\omega_N(A;\Omega^{\partial\Lambda_N}) - \omega_N^{\text{bulk}}(A)\right| + \lim_{M\to\infty} \left|\omega_N^{\text{bulk}}(A) - \omega_M^{\text{bulk}}(A)\right| \qquad (4.7)
\end{aligned}$$

where we used that $d\rho_{\Lambda_N}|f|^2$ is a probability measure and the supremum is taken over all possible values of the boundary variables. Thus, to prove Theorem 4.0.1, one needs to determine appropriate upper bounds on the two quantities from (4.7) in N. This will be achieved by first showing that both $Z_N(A;\Omega^{\partial\Lambda_N})$ and $Z_N^{\text{bulk}}(A)$ can be written in terms of a hard core polymer representation, and then that the logarithm of each polymer representation has a convergent cluster expansion.

We make two additional comments about this approach. First, any sequence of ground states could have been used in place of ω_N^{bulk} in (4.7). The benefit of using this particular choice comes from the set of polymers associated with this state, which will be introduced in the next section and commented on further there. Second, when we apply the cluster expansion in Section 4.4, it is convenient for computations that the bulk state and bulk-boundary map are defined on the same volume in the first term of (4.7). This is why we insert and remove $\omega_N^{\text{bulk}}(A)$ and apply the triangle inequality rather than simply insert $\omega(A) = \lim_M \omega_M^{\text{bulk}}(A)$.

We now turn to discussing the polymer representations, which was first given for the bulk state in [19] and then modified in [27] to include the bulk-boundary map.

4.1.2. Hard Core Polymer Representations. To simplify some combinatorics in Section 4.3, the desired hard core polymer representation of $Z_N(A;\Omega^{\partial\Lambda_N})$ and $Z_N^{\text{bulk}}(A)$ will be given for observables $A \in \mathcal{A}_{\mathring{\Lambda}_K}$ with $K < N$. The values of quantities are independent of K even though their polymer representations are depend on a suitable value of K.

We first set some notation. First, as above, any subgraph $G = (\mathcal{V}, \mathcal{E}) \subseteq \Gamma$, define the degree of a vertex $x \in \mathcal{V}$ in G, by

$$\deg_G(x) = |\{y \in \mathcal{V} \mid (x,y) \in \mathcal{E}\}|.$$

Second, for any choice of parameters $0 \leq K < N+2$ define

$$\Lambda_{N,K} = \bigcup_{K+2 \leq \tilde{d}(\tilde{x},\tilde{0}) \leq N} h_{\tilde{x}} \tag{4.8}$$

where $\tilde{0}, \tilde{x}$ are as in (4.2), and $\tilde{d}$ is the graph distance in the dual lattice $\tilde{\Gamma}$, This volume is such that the vertex sets of $\Lambda_{N,K}$ and $\mathring{\Lambda}_K$ satisfy $\mathcal{V}_{\Lambda_{N,K}} \cap \mathcal{V}_{\mathring{\Lambda}_K} = \partial\mathring{\Lambda}_K$. As the symbol $A(\boldsymbol{\Omega})$ for any $A \in \mathcal{A}_{\mathring{\Lambda}_K}$ is not a function of Ω_x for any $x \in \mathcal{V}_{\Lambda_{N,K}} \setminus \partial\mathring{\Lambda}_K$, the desired polymer representations will be obtained from using (2.2) to integrate (4.3) and (4.6) over this set of vertices. We first define the polymer sets and weight functions, and then state the representations.

4.1.2.1. *Polymers and Weights.* The desired sets of polymers consists of loops or walks in the hexagonal lattice. That is, they will be subsets of

$$\mathcal{P}_\Gamma = \left\{\gamma = (\mathcal{V}_\gamma, \mathcal{E}_\gamma) \subseteq \Gamma : \gamma \text{ connected } \wedge\ \deg_\gamma(x) \leq 2\ \forall x \in \mathcal{V}_\gamma\right\}. \tag{4.9}$$

The global sets of loops and walks are then defined, respectively, by

$$\mathcal{L}_\Gamma = \{\gamma \in \mathcal{P}_\Gamma : \deg_\gamma(x) = 2\ \forall\, x \in \mathcal{V}_\gamma\}, \quad \mathcal{W}_\Gamma = \mathcal{P}_\Gamma \setminus \mathcal{L}_\Gamma\,.$$

As each $\gamma \in \mathcal{W}_\Gamma$ is connected, it has precisely two vertices $\text{ep}(\gamma) := \{x,y\} \in \mathcal{V}_\gamma$, called the *endpoints*, such that $\deg_\gamma(x) = \deg_\gamma(y) = 1$.

To define the sets of polymers that will be used for the representations of (4.3) and (4.6), for any $N > K \geq 0$ let

$$\mathcal{L}_{N,K} = \{\gamma \in \mathcal{L}_\Gamma : \gamma \subseteq \Lambda_{N,K}\}$$

$$\mathcal{W}_{N,K} = \{\gamma \in \mathcal{W}_\Gamma : \gamma \subseteq \Lambda_{N,K} \wedge \text{ep}(\gamma) \subseteq \partial\Lambda_{N,K}\}$$

$$\mathcal{W}_{N,K}^{\text{bulk}} = \{\gamma \in \mathcal{W}_\Gamma : \gamma \subseteq \Lambda_{N,K} \wedge \text{ep}(\gamma) \subseteq \partial\mathring{\Lambda}_K\}$$

where we use the convention that if $K = 0$, then $\partial\mathring{\Lambda}_K = \emptyset$ and so $\mathcal{W}_{N,K}^{\text{bulk}} = \emptyset$, and

$$\partial\Lambda_{N,K} = \partial\Lambda_N \cup \partial\mathring{\Lambda}_K = \partial\Lambda_N\,.$$

The polymer sets for $Z_N(A;\Omega^{\partial\Lambda_N})$ and $Z_N^{\text{bulk}}(A)$ are then, respectively,

$$\mathcal{P}_{N,K} = \mathcal{L}_{N,K} \cup \mathcal{W}_{N,K}, \quad \mathcal{P}_{N,K}^{\text{bulk}} = \mathcal{L}_{N,K} \cup \mathcal{W}_{N,K}^{\text{bulk}}. \tag{4.10}$$

The main benefit of using the bulk state in (4.7) is that the endpoints of any walk $\gamma \in \mathcal{W}_{N,K}^{\text{bulk}}$ only lie on the *inner boundary*, $\partial\mathring{\Lambda}_K$, of $\Lambda_{N,K}$ and do not involve vertices on the *outer boundary*, $\partial\Lambda_N$.
We define two polymers $\gamma, \gamma' \in \mathcal{P}_\Gamma$ to be *disjoint* if $\mathcal{V}_\gamma \cap \mathcal{V}_{\gamma'} = \emptyset$. For brevity, we write

$$\gamma|\gamma' \quad \text{if} \quad \mathcal{V}_\gamma \cap \mathcal{V}_{\gamma'} = \emptyset, \quad \text{and} \quad \gamma \nmid \gamma' \quad \text{otherwise.} \tag{4.11}$$

A subset of polymers $\{\gamma_1, \dots, \gamma_n\} \subseteq \mathcal{P}_\Gamma$ is called *hard core* if $\gamma_i|\gamma_j$ for all $i \neq j$.
To define the weights associated to each type of polymer, it follows by a direct computation that for any two neighboring edges $(x,y),(x,z) \in \mathcal{E}_\Gamma$,

$$\int d\Omega_x (\Omega_y \cdot \Omega_x)(\Omega_x \cdot \Omega_z) = \frac{1}{3}\Omega_y \cdot \Omega_z, \quad \int d\Omega_x (\Omega_x \cdot \Omega_x) = 1. \tag{4.12}$$

Let $|\gamma| := |\mathcal{E}_\gamma|$ denote the length of any $\gamma \in \mathcal{P}_\Gamma$. The weight function $W(\gamma)$ for any $\gamma \in \mathcal{P}_\Gamma$ is then defined as follows. For $\gamma \in \mathcal{L}_\Gamma$, $|\gamma|$ is even and so by (4.12)

$$W(\gamma) := \int d\Omega^{\mathcal{V}_\gamma} \prod_{(x,y)\in\mathcal{E}_\gamma} (-\Omega_x \cdot \Omega_y) = \left(\frac{1}{3}\right)^{|\gamma|-1} \tag{4.13}$$

For any $\gamma \in \mathcal{W}_\Gamma$, it is straightforward to verify that $|\mathcal{V}_\gamma| = |\gamma| + 1$ and so if $\text{ep}(\gamma) = \{v,w\}$, then by (4.12)

$$W(\gamma) := \int d\Omega^{\mathcal{V}_\gamma \setminus \text{ep}(\gamma)} \prod_{(x,y)\in\mathcal{E}_\gamma} (-\Omega_x \cdot \Omega_y) = \left(-\frac{1}{3}\right)^{|\gamma|-1} \partial\gamma(\Omega), \quad \partial\gamma(\Omega) = -\Omega_v \cdot \Omega_w. \tag{4.14}$$

4.1.2.2. *Hard-Core Polymer Representation of $Z_N(A;\partial\Omega)$.* We can now state the polymer representation of $Z_N(A;\partial\Omega)$, which reduces (4.3) to an integral over the measure associated with $\Omega^{\mathring{\Lambda}_K}$.

Lemma 4.1.0.2. Let $0 \leq K < N$. Then for any $A \in \mathcal{A}_{\mathring{\Lambda}_K}$,

$$Z_N(A;\partial\Omega) = \int d\rho_{\mathring{\Lambda}_K} A(\boldsymbol{\Omega}) \Phi_{N,K}(\boldsymbol{\Omega}) \tag{4.15}$$

where with respect to the weight functions (4.13)-(4.14)

$$\Phi_{N,K}(\boldsymbol{\Omega}) := 2^{-|\mathcal{E}_{\Lambda_{N,K}}|} \sum_{\{\gamma_1,\dots,\gamma_n\}\subseteq\mathcal{P}_{N,K}}^{\text{h.c.}} W(\gamma_1)\cdots W(\gamma_n). \tag{4.16}$$

and the notation 'h.c.' indicates that the sum is taken over all hard core subsets of $\mathcal{P}_{N,K}$.

The proof of this result is given in [27] and follows the approach used in [19]. Here, we provide a shorter version of the proof from [27].

PROOF. Recalling (4.2)-(4.3), one easily finds

$$Z_N(A;\Omega^{\partial\Lambda_N}) = 2^{-|\mathcal{E}_{\Lambda_{N,K}}|}\int d\Omega^{\mathring{\Lambda}_N}\prod_{(x,y)\in\mathcal{E}_{\Lambda_{N,K}}}(1-\Omega_x\cdot\Omega_y)\rho_{\mathring{\Lambda}_K}A(\mathbf{\Omega})$$

$$= 2^{-|\mathcal{E}_{\Lambda_{N,K}}|}\sum_{E\subseteq\mathcal{E}_{\Lambda_{N,K}}}\int d\Omega^{\mathring{\Lambda}_N}\prod_{(x,y)\in E}(-\Omega_x\cdot\Omega_y)\rho_{\mathring{\Lambda}_K}A(\mathbf{\Omega})\,. \tag{4.17}$$

As the measure $d\Omega_x$ is invariant under $\Omega_x \mapsto -\Omega_x$ it follows that any Lebesgue integrable function $f : S^2 \to \mathbb{R}$ satisfies

$$\int d\Omega_x f(-\Omega_x) = \int d\Omega_x f(\Omega_x) \qquad \forall x \in \Gamma\,.$$

Recall that $A(\mathbf{\Omega})$ is a polynomial in the variables Ω_x, $x \in \mathring{\Lambda}_K$, and suppose that $E \subseteq \mathcal{E}_{\Lambda_{N,K}}$ is nonempty and that $\deg_{G_E}(v) \in \{1,3\}$ for some $v \in V_E \cap \mathring{\Lambda}_{N,K}$. Since $f(\Omega_v) = \prod_{(x,v)\in E}\Omega_x\cdot\Omega_v$ is an odd function, by the above identity, integrating over $d\Omega_v$ produces

$$\int d\Omega^{\mathring{\Lambda}_N}\prod_{(x,y)\in E}(-\Omega_x\cdot\Omega_y)\rho_{\mathring{\Lambda}_K}A(\mathbf{\Omega}) = 0\,. \tag{4.18}$$

As $\deg_{G_E}(x) > 0$ for any vertex x in an edge-induced graph, and the geometry of $\Lambda_{N,K}$ is such that $\deg_{\Lambda_{N,K}}(x) = 1$ for all $x \in \partial\Lambda_{N,K}$, (4.18) implies that the sum in (4.17) can be reduced to all subsets E such that

$$\deg_{G_E}(x) = 2 \quad \text{if} \quad x \in V_E \setminus \partial\Lambda_{N,K} \quad \text{and} \quad \deg_{G_E}(x) = 1 \quad \text{if} \quad x \in V_E \cap \partial\Lambda_{N,K}\,. \tag{4.19}$$

Thus, the connected components of G_E comprise a set of hard core polymers $\{\gamma_1,\dots,\gamma_n\} \subseteq \mathcal{P}_{N,K}$. Moreover, given any hard core set of polymers $\{\gamma_1,\dots,\gamma_n\} \subseteq \mathcal{P}_{N,K}$, the union of their respective edge sets $E = \cup_i \mathcal{E}_{\gamma_i}$ produces a set which satisfies (4.19). Therefore,

$$Z_N(A;\Omega^{\partial\Lambda_N}) = 2^{-|\mathcal{E}_{\Lambda_{N,K}}|}\sum_{\{\gamma_1,\dots,\gamma_n\}\subseteq\mathcal{P}_{N,K}}^{h.c.}\int d\Omega^{\mathring{\Lambda}_N}\prod_{i=1}^{n}\prod_{(x,y)\in\mathcal{E}_{\gamma_i}}(-\Omega_x\cdot\Omega_y)\rho_{\mathring{\Lambda}_K}A(\mathbf{\Omega})\,. \tag{4.20}$$

Since $\mathcal{V}_{\mathring{\Lambda}_N} = \mathcal{V}_{\mathring{\Lambda}_{N,K}} \sqcup \mathcal{V}_{\mathring{\Lambda}_K}$, the result follows from using (4.13)-(4.14) and $\int d\Omega_x = 1$ to integrate (4.20) over the variables Ω_x, $x \in \mathcal{V}_{\mathring{\Lambda}_{N,K}}$. □

4.1.2.3. *Hard-Core Polymer Representation of* Z_N^{bulk}. The hard core polymer representation for $Z_N^{\text{bulk}}(A)$, stated below, is almost identical to that $Z_N(A;\Omega^{\partial\Lambda_N})$ with the only difference being that $\mathcal{P}_{N,K}$ is replaced with $\mathcal{P}_{N,K}^{\text{bulk}}$.

LEMMA 4.1.0.3. Let $0 \le K < N$, then for any $A \in \mathcal{A}_{\mathring{\Lambda}_K}$,

$$Z_N^{\text{bulk}}(A) = \int d\rho_{\mathring{\Lambda}_K}A(\mathbf{\Omega})\Phi_{N,K}^{\text{bulk}}(\mathbf{\Omega}) \tag{4.21}$$

where with respect to the weight functions (4.13)-(4.14),

$$\Phi^{\text{bulk}}_{N,K}(\mathbf{\Omega}) := 2^{-|\mathcal{E}_{\Lambda_{N,K}}|} \sum_{\{\gamma_1,\dots,\gamma_n\}\subseteq\mathcal{P}^{\text{bulk}}_{N,K}}^{\text{h.c.}} W(\gamma_1)\cdots W(\gamma_n) \tag{4.22}$$

and the notation indicates the sum is take over hard core subsets of $\mathcal{P}^{\text{bulk}}_{N,K}$.

Proof. The proof of this result is almost identical to that of Lemma 4.1.0.2. The one difference being

$$Z^{\text{bulk}}_N(A) = 2^{-|\Lambda_{N,K}|} \sum_{E\subseteq\mathcal{E}_{\Lambda_{N,K}}} \int d\Omega^{\Lambda_N} \prod_{(x,y)\in E} (-\Omega_x\cdot\Omega_y)\rho_{\mathring{\Lambda}_K} A(\mathbf{\Omega})\,,$$

i.e. the integral is take over $d\Omega^{\Lambda_N}$ instead of $d\Omega^{\mathring{\Lambda}_N}$. Since $\deg_{\Lambda_{N,K}}(x) = 1$ for all $x\in\partial\Lambda_N$, (4.18) implies that the above sum can be reduced to all edge sets E for which $G_E\subseteq\Lambda_{N,K}\setminus\partial\Lambda_N$ and G_E satisfies (4.19). All such edge-induced graphs are in one-to-one correspondence with graphs whose connected components are comprised of hard core sets of polymers $\{\gamma_1,\dots,\gamma_n\}\subseteq\mathcal{P}^{\text{bulk}}_{N,K}$. □

4.1.3. A Useful Comparison. It was shown in Section 4.0.1 that hexagonal AKLT model satisfies LTQO so long as Theorem 4.0.1 holds. LTQO is a key ingredient for proving stability of the ground state gap, that is, the uniform gap of a quantum spin model stays open under sufficiently small perturbations of the interaction. In the proof of the stability results from [], it is important that the LTQO bound makes explicit the dependence on (i) the operator norm of the observable and (ii) the support of the observable. Given (4.3) and (4.6), it is clear that (4.7) can be easily bounded in terms of $\|A(\mathbf{\Omega})\|_{L^\infty}$ and that, by (2.8),

$$\|A\| \le \|A(\mathbf{\Omega})\|_{L^\infty}\,.$$

However, these norms are not equivalent on $\mathcal{A}^{\text{loc}}_\Gamma$. As explained in [27], given any sequence of sites $x_n\in\Gamma$, there is a canonical choice for $A(\mathbf{\Omega})$ so that $A_n = \bigotimes_{i=1}^n(\partial_{u_{x_i}}u_{x_i}/4)\in\mathcal{A}^{\text{loc}}_\Gamma$ satisfies

$$\|A_n\| = 1, \quad \|A_n(\mathbf{\Omega})\|_{L^\infty} = \left(\frac{5}{4}\right)^n \quad \forall\, n\,.$$

The next result, first proved in [27], produces a bound on the two terms from (4.7) that depends on $\|A\|$ and positions us to use cluster expansion techniques to prove how quickly these two quantities converge and how this convergence depends on $\text{supp}(A)$. The main tool for this result is to use the embedding $\mathcal{H}_{\mathring{\Lambda}_K}\subseteq\mathcal{K}_{\mathring{\Lambda}_K}$ where

$$\mathcal{K}_{\mathring{\Lambda}_K} = \bigotimes_{x\in\partial\mathring{\Lambda}_K} L^2(S^2,d\Omega_x)\otimes\bigotimes_{x\in\mathcal{V}_{\Lambda_{K-1}}}\mathcal{H}_x = \bigotimes_{x\in\partial\mathring{\Lambda}_K} L^2(S^2,d\Omega_x)\otimes\mathcal{H}_{\Lambda_{K-1}}\,.$$

Here, note that $\mathcal{V}_{\mathring{\Lambda}_K} = \mathcal{V}_{\Lambda_{K-1}}\sqcup\partial\mathring{\Lambda}_K$. The advantage of considering this larger space is that we can take square roots of certain quantities without leaving the Hilbert space, and, moreover, for any $A\in\mathcal{A}_{\Lambda_{K-1}}$,

the operator norm is unchanged, i.e.

$$\|A\|_{\mathcal{H}_{\Lambda_{K-1}}} = \|A\|_{\mathcal{K}_{\mathring{\Lambda}_K}}$$

where the subscript denotes the Hilbert space in which the norm is taken.

To simplify notation, we suppress the dependence of the functions $\Phi_{N,K}$, $\Phi^{\text{bulk}}_{N,K}$, Z_N and Z^{bulk}_N on $\mathbf{\Omega}$ for the remainder of this work.

Lemma 4.1.0.4. Fix $N > K \geq 0$. Suppose that $\Phi^{\text{bulk}}_{N,K} > 0$ for any value of $(\Omega_x : x \in \partial\mathring{\Lambda}_K)$. Then, for any $M \geq N$ and $A \in \mathcal{A}_{\Lambda_{K-1}}$,

$$\left|\omega^{\text{bulk}}_N(A) - \omega^{\text{bulk}}_M(A)\right| \leq ||A|| H\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}, \frac{\Phi^{\text{bulk}}_{M,K}}{Z^{\text{bulk}}_M}\right) \tag{4.23}$$

$$\sup_{\Omega^{\partial\Lambda_N}} \left|\omega_N(A;\Omega^{\partial\Lambda_N}) - \omega^{\text{bulk}}_N(A)\right| \leq ||A|| \sup_{\Omega^{\partial\Lambda_N}} H\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}, \frac{\Phi_{N,K}}{Z_N}\right) \tag{4.24}$$

where $H(f,g) = D_\infty(f||g)e^{D_\infty(f||g)}$, $f, g \in L^1(d\rho_{\mathring{\Lambda}_K})$, is defined in terms of the (classical) ∞-Renyi divergence

$$D_\infty(f||g) := \|\log(f) - \log(g)\|_{L^\infty(d\rho_{\mathring{\Lambda}_K})}\,. \tag{4.25}$$

After verifying that a convergence criterion holds, the cluster expansion gives an explicit formula for these logarithms which will be inserted into (4.25) and manipulated to prove Theorem 4.0.1 in Section 4.4. That $\Phi^{\text{bulk}}_{N,K} > 0$ holds will be an immediate consequence of the cluster expansion, so Lemma 4.1.0.4 will apply to our polymer representations. To be explicit, the Renyi-divergences are given by

$$D_\infty\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}\middle\|\frac{\Phi^{\text{bulk}}_{M,K}}{Z^{\text{bulk}}_M}\right) = \sup_{\Omega_x : x\in\partial\mathring{\Lambda}_K}\left|\log\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}\right) - \log\left(\frac{\Phi^{\text{bulk}}_{M,K}}{Z^{\text{bulk}}_M}\right)\right| \tag{4.26}$$

$$\sup_{\Omega^{\partial\Lambda_N}} D_\infty\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}\middle\|\frac{\Phi_{N,K}}{Z_N}\right) = \sup_{\Omega_x : x\in\partial\Lambda_{N,K}}\left|\log\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}\right) - \log\left(\frac{\Phi_{N,K}}{Z_N}\right)\right| \tag{4.27}$$

After giving the proof of Lemma 4.1.0.4, we explain the cluster expansion convergence criterion of [22] in Section 4.2, and then verify it for the polymer representations of $\Phi^{\text{bulk}}_{N,K}$ and $\Phi_{N,K}$ in Section 4.3.

Proof of Lemma 4.1.0.4. Define $\psi_1 = \text{sign}(h)|h|^{1/2}F$ and $\psi_2 = |h|^{1/2}F$ with

$$h = \frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N} - \frac{\Phi^{\text{bulk}}_{M,K}}{Z^{\text{bulk}}_M}, \qquad F = \prod_{(i,j)\in\Lambda_K}(u_i v_j - u_j v_i)$$

then $\psi_1, \psi_2 \in \mathcal{K}_{\mathring{\Lambda}_K}$ since by Lemma 4.1.0.3 (see also (4.10)-(4.14)), $\Phi^{\text{bulk}}_{N,K}$ is a bounded real-valued continuous function of the angles θ_x, ϕ_x defining $(\Omega_x : x \in \partial\mathring{\Lambda}_K)$. Moreover, $Z^{\text{bulk}}_N \neq 0$ since ω^{bulk}_N is a

ground state on $\mathcal{A}_{\mathring{\Lambda}_K}$. Since $\rho_{\Lambda_K} = \overline{F}F$, one has by (4.6) and Lemma 4.1.0.3

$$\left|\omega_N^{\text{bulk}}(A) - \omega_M^{\text{bulk}}(A)\right| = \left| \int d\Omega^{\Lambda_K} \rho_{\Lambda_K} \left[\frac{\Phi_{N,K}^{\text{bulk}}}{Z_N^{\text{bulk}}} - \frac{\Phi_{M,K}^{\text{bulk}}}{Z_M^{\text{bulk}}} \right] A(\Omega) \right| = \left| \langle \psi_1, (\mathbb{1} \otimes A)\psi_2 \rangle_{\mathcal{K}_{\mathring{\Lambda}_K}} \right|$$

$$\leq ||A|| \cdot ||\psi_1||_{\mathcal{K}_{\mathring{\Lambda}_K}} ||\psi_2||_{\mathcal{K}_{\mathring{\Lambda}_K}} = ||A|| \left\| \frac{\Phi_{N,K}^{\text{bulk}}}{Z_N^{\text{bulk}}} - \frac{\Phi_{M,K}^{\text{bulk}}}{Z_M^{\text{bulk}}} \right\|_{L^1(d\rho_{\Lambda_K})}$$

where the last equality follows from $||\psi_1||_{\mathcal{K}_{\mathring{\Lambda}_K}} = ||\psi_2||_{\mathcal{K}_{\mathring{\Lambda}_K}}$. Thus, (4.23) follows from applying Hölder's inequality and the bound $|e^x - 1| \leq |x|e^{|x|}$ to obtain

$$||f - g||_{L^1(\mu)} \leq \left\| \frac{f}{g} - 1 \right\|_{L^\infty(\mu)} \|g\|_{L^1(\mu)} \leq D_\infty(f||g) e^{D_\infty(f||g)} .$$

where, since $\Phi_{N,K}^{\text{bulk}} > 0$,

$$\|\Phi_{N,K}^{\text{bulk}}\|_{L^1(d\rho_{\mathring{\Lambda}_K})} = \int d\rho_{\mathring{\Lambda}_K} \Phi_{N,K}^{\text{bulk}}(\Omega) = Z_N^{\text{bulk}} .$$

For (4.24), recall that $Z_N(\Omega^{\partial\Lambda_N}) \neq 0$ for any choice of boundary variables, see the discussion following Theorem 4.1.0.1. Thus, for any fixed choice of the boundary variables, (4.3), (4.6) and Lemmas 4.1.0.2-4.1.0.3 give

$$\left|\omega_N(A; \Omega^{\partial\Lambda_N}) - \omega_N^{\text{bulk}}(A)\right| = \left| \int d\Omega^{\Lambda_K} \rho_{\Lambda_K} \left[\frac{\Phi_{N,K}^{\text{bulk}}}{Z_N^{\text{bulk}}} - \frac{\Phi_{N,K}}{Z_N} \right] A(\Omega) \right| .$$

The proof then runs analogously to that of (4.23) after recognizing that $\Phi_{N,K}$ is also bounded, real-value continuous function of the angles defining Ω_x, $x \in \partial\mathring{\Lambda}_K$. □

4.2. The Kotecký-Preiss Condition for Convergence of a Cluster Expansion

Given (4.7) and Lemma 4.1.0.4, the proof of Theorem 4.0.1 will be a consequence of estimating the difference of the logarithms in (4.26)-(4.27). In the setting of polymer representations, this logarithm takes an explicit form of a cluster expansion so long as a summability condition is satisfied. The cluster expansion convergence criterion we will use is due to Kotecký and Preiss (KP) [22]. In this section, we introduce this criterion in the context of our polymer sets along with the formula for the cluster expansion. We also state and prove a couple of consequences that we will use in our proofs verifying the KP criterion and Theorem 4.0.1. The cluster expansion and the KP criterion will be stated with respect to an arbitrary nonempty, finite set of polymers $\mathcal{P} \subseteq \mathcal{P}_\Gamma$ where $\mathcal{P}_\Gamma$ is as in (4.9). To begin, we first introduce some notation.

A nonempty set $C = \{\gamma_1, \ldots, \gamma_n\} \subset \mathcal{P}$ is called a *cluster* if the graph obtain from the union $\cup_i \gamma_i := (\cup_i \mathcal{V}_{\gamma_i}, \cup_i \mathcal{E}_{\gamma_i})$ is connected, and the set of all clusters in $\mathcal{P}$ is denoted by $\mathcal{C}(\mathcal{P})$. For any $\gamma \in \mathcal{P}$ and $C \in \mathcal{C}(\mathcal{P})$, we write $\gamma \nmid C$, if $C \cup \{\gamma\}$ is also a cluster or, equivalently, there exists $\gamma' \in C$ such that $\gamma' \nmid \gamma$, see (4.11). Note that $\gamma' \nmid \gamma$ if and only if $\mathcal{V}_\gamma \cap \mathcal{V}_{\gamma'} \neq \emptyset$.

Recalling the weight function, W from (4.13)-(4.14), the *partition function* associated to $\mathcal{P}$ is given by

$$\Xi(\mathcal{P}, W) = \sum_{\vec{\gamma}=\{\gamma_1,\dots,\gamma_n\}\subseteq\mathcal{P}}^{\text{h.c.}} W(\vec{\gamma}), \qquad W(\vec{\gamma}) = \prod_{i=1}^{n} W(\gamma_i) \tag{4.1}$$

where, as before, "h.c" means that $\vec{\gamma}$ is a hard core set of polymer. Note that, with respect to this notation, given any $N > K \geq 1$,

$$\Phi_{N,K} = 2^{-|\mathcal{E}_{\Lambda_{N,K}}|}\Xi(\mathcal{P}_{N,K}, W), \qquad Z_N = 2^{-|\mathcal{E}_{\Lambda_N}|}\Xi(\mathcal{P}_{N,0}, W)$$

and similarly, for $\Phi^{\text{bulk}}_{N,K}$ and Z^{bulk}_N.

A cluster expansion criterion is a condition that, when satisfied, guarantees that the formal sum for $\log \Xi(\mathcal{P}, W)$ given below, called the *cluster expansion*, is absolutely convergent and so the following is a well-defined equality of functions:

$$\log \Xi(\mathcal{P}, W) = \sum_{C\subseteq\mathcal{C}(\mathcal{P})} W^T(C), \qquad W^T(C) = \sum_{n\geq 1} \sum_{\substack{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n:\\ C=\{\gamma_1,\dots,\gamma_n\}}} \varphi(\gamma_1,\dots,\gamma_n)\prod_{i=1}^{n} W(\gamma_i)\,. \tag{4.2}$$

Here, φ is the Ursell function defined by

$$\varphi(\gamma_1) = 1, \qquad \varphi(\gamma_1,\dots,\gamma_n) = \frac{1}{n!}\sum_{\substack{G=(V_n,E):\\ G \text{ connected}}} \prod_{(i,j)\in E} \zeta(\gamma_i,\gamma_j), \quad n \geq 2$$

where $V_n = \{1,\dots,n\}$, and $\zeta(\gamma,\gamma') = -1$ if $\gamma \nmid \gamma'$ and $\zeta(\gamma,\gamma') = 0$ otherwise. We now state the KP cluster expansion convergence criterion, and point the interested reader to [22] for its proof. (See also [11, Chapter 5] for a derivation of the formula in (4.2) and proof of its absolute convergence.)

Theorem 4.2.1 (KP Criterion for Cluster Expansion Convergence [22]). *Let $\mathcal{P} \subseteq \mathcal{P}_\Gamma$ be any nonempty finite set, and suppose there exists $a : \mathcal{P} \to [0,\infty)$ and $b : \mathcal{P} \to [0,\infty)$ such that*

$$\sum_{\gamma'\in\mathcal{P}:\,\gamma'\nmid\gamma} e^{a(\gamma')+b(\gamma')}|W(\gamma')| \leq a(\gamma), \qquad \forall \gamma \in \mathcal{P}. \tag{4.3}$$

Then the sum in (4.2) is absolutely convergent and nonzero Moreover, for any $\gamma \in \mathcal{P}$,

$$\sum_{C\subset\mathcal{C}(\mathcal{P}),C\nmid\gamma} |W^T(C)|e^{b(C)} \leq a(\gamma) \quad \textit{where} \quad b(C) = \sum_{\gamma\in C} b(\gamma)\,. \tag{4.4}$$

The first step for proving Theorem 4.0.1 will be to show that (4.3) holds in the case that $\mathcal{P} = \mathcal{P}_{N,K}$ or $\mathcal{P} = \mathcal{P}^{\text{bulk}}_{N,K}$ for sufficient $N > K \geq 0$. Since $\mathcal{P}^{\text{bulk}}_{N,K} \subseteq \mathcal{P}_{N,K}$, the convergence criterion for either set follows from proving (4.3) holds when $\mathcal{P} = \mathcal{P}_{N,K}$. The fact that (4.3) holds when $K = 0$ was proved in [19]. As such, we focus on the case that $K > 0$ in Section 4.3. Moreover, note that since both φ and W are

real-valued, proving the cluster expansion convergence yields $\log \Phi^{\text{bulk}}_{N,K} \in \mathbb{R}$, from which the condition $\Phi^{\text{bulk}}_{N,K} > 0$ from Lemma 4.1.0.4 follows.

For any $\mathcal{P} \subseteq \mathcal{P}_\Gamma$, the convergence of the cluster expansion would also hold if $b(\gamma) = 0$ for all $\gamma \in \mathcal{P}$. However, the following corollary will be particularly useful for proving Theorem 4.0.1 in the case that a nonzero $b : \mathcal{P} \to [0, \infty)$ can be established.

Corollary 4.2.2 ([22]). *Let $\mathcal{P} \subseteq \mathcal{P}_\Gamma$ be any nonempty, finite set, and suppose that $a, b : \mathcal{P} \to [0, \infty)$ are such that Theorem 4.2.1 holds. Then, for any nonempty $A \subseteq \mathcal{P}$,*

$$\sum_{\substack{C \in \mathcal{C}(\mathcal{P}): \\ C \cap A \neq \emptyset,\, C \nmid \gamma}} |W^T(C)| \leq e^{-b(\gamma, A)} a(\gamma) \tag{4.5}$$

where $b(\gamma, A) = \inf\{b(C) : C \in \mathcal{C}(\mathcal{P}),\, C \nmid \gamma,\, C \cap A \neq \emptyset\}$.

Proof. This bound is a slight modification of an inequality proved in [22]. Since b is nonnegative, by Theorem 4.2.1 it follows that

$$\sum_{\substack{C \in \mathcal{C}(\mathcal{P}): \\ C \cap A \neq \emptyset,\, C \nmid \gamma}} |W^T(C)| e^{b(\gamma, A)} \leq \sum_{C \subset \mathcal{C}(\mathcal{P}),\, C \nmid \gamma} |W^T(C)| e^{b(C)} \leq a(\gamma)$$

as desired. □

The remainder of this section is focused on proving a sufficient condition that guarantees a nonzero function $b : \mathcal{P} \to [0, \infty)$ exists satisfying the conditions of Theorem 4.2.1.

Lemma 4.2.0.1. Suppose that $\mathcal{P}_i \subseteq \mathcal{P}_\Gamma$, $i \in \mathcal{I}$, is any collection of nonempty, finite sets of polymers, and there exists $a : \mathcal{P}_\Gamma \to [0, \infty)$ and $0 < C < 1$ such that for any $i \in I$ and $\gamma \in \mathcal{P}_i$,

$$\sum_{\gamma' \in \mathcal{P}_i : \gamma' \nmid \gamma} e^{a(\gamma')} |W(\gamma')| < C a(\gamma). \tag{4.6}$$

Then there exists $\epsilon > 0$ such that (4.3) holds for any $\mathcal{P}_i$, $i \in I$, with the functions a as above and $b : \mathcal{P}_\Gamma \to [0, \infty)$ defined by $b(\gamma) = \epsilon |\gamma|$.

4.2.1. On the existence of a nonzero decay function $d : \mathcal{P} \to [0, \infty)$.

4.2.1.1. *Making the Partition Function Analytic.* If we complexify our weight function to be a function of a complex variable z we can have

$$W_z(\gamma) = W(\gamma) = \left(\frac{1}{3}\right)^{|\gamma|-1} z^{|\gamma|}$$

then we can make arguments about convergence. It is known (see Friedli and Velenik [11]) that if

$$\sup_D |W_z(\gamma)| \leq W(\gamma)$$

and a certain convergence criterion (i.e. Kotecký-Preiss criterion), then the associated cluster expansion to $W(\gamma)$ converges in the disc D then the series for W_z converges in D and is analytic.
We provide a proof here.

Lemma 4.2.0.2. Assume that $z \mapsto W_z(\gamma)$ is analytic on D, for each $\gamma \in \Gamma$, and that there exists a real weight $\overline{W}(\gamma) \geq 0$ such that

$$\sup_{z \in D} |W_z(\gamma)| \leq \overline{W}(\gamma), \qquad \forall \gamma \in \Gamma,$$

and such that there exists an $a : \Gamma \to \mathbb{R}_{>0}$ such that for each $\gamma_* \in \Gamma$ we have

$$\sum_{\gamma} |\overline{W}(\gamma)| e^{a(\gamma)} |\zeta(\gamma, \gamma_*)| \leq a(\gamma_*)$$

Then

$$\sum_{k \geq 1} \sum_{\gamma_1} ... \sum_{\gamma_k} |\varphi(\gamma_1, ..., \gamma_k)| \prod_{i=1}^{k} |W_z(\gamma_i)| < \infty$$

where φ is the Ursell function, and

$$\Xi(P, W; z) = \exp\left(\sum_{k \geq 1} \sum_{\gamma_1} ... \sum_{\gamma_k} \varphi(\gamma_1, ..., \gamma_k) \prod_{i=1}^{k} W_z(\gamma_i) \right)$$

and we have that $z \mapsto \log \Xi(P, W; z)$ is analytic on D

Proof. Let $\log \Xi(P, W; z) = \sum_{n \geq 1} f_n(z)$ where we have

$$f_n(x) := \sum_{\gamma_1} ... \sum_{\gamma_n} \varphi(\gamma_1, ..., \gamma_n) \prod_{i=1}^{n} W_z(\gamma_i))$$

Now since $|P| < \infty$ we have that f_n is a sum containing only a finite number of terms and is analytic in D. Then if $\sum_n f_n$ is uniformly convergent on compact sets $K \subset D$ then it represents an analytic function on D from Weierstrass' theorem. Then we have

$$\begin{aligned} \sup_{z \in K} \left| \sum_{n \geq 1} f_n(z) - \sum_{n=1}^{N} f_n(z) \right| &\leq \sup_{z \in K} \sum_{n > N} |f_n(z)| \\ &\leq \sum_{n > N} \sup_{z \in K} |f_n(z)| \\ &\leq \sum_{n > N} \sum_{\gamma_1} ... \sum_{\gamma_n} |\varphi(\gamma_1, ..., \gamma_n)| \prod_{i=1}^{n} \overline{W}(\gamma_i) \end{aligned} \tag{4.7}$$

Then we have

$$\sum_{k \geq 1} \sum_{\gamma_1} ... \sum_{\gamma_k} |\varphi(\gamma_1, ..., \gamma_k)| \prod_{i=1}^{k} |W_z(\gamma_i)| < \infty$$

and

$$\Xi(P, W; z) = \exp\left(\sum_{k \geq 1} \sum_{\gamma_1} ... \sum_{\gamma_k} \varphi(\gamma_1, ..., \gamma_k) \prod_{i=1}^{k} W_z(\gamma_i) \right)$$

from the assumption of convergence on compact sets. Then

$$\sum_{n>N}\sum_{\gamma_1}...\sum_{\gamma_n}|\varphi(\gamma_1,...,\gamma_n)|\prod_{i=1}^{n}\overline{W}(\gamma_i)\to 0$$

as $N\to\infty$. Then from the convergence criterion we assumed, from Theorem 5.9 in [11] we get that

$$1+\sum_{k\geq 2}k\sum_{\gamma_2}...\sum_{\gamma_k}|\varphi(\gamma_1,...,\gamma_k)|\prod_{j=2}^{k}|W_z(\gamma_j)|\leq e^{a(\gamma_1)}$$

and so we have convergence in the limit. □

Note that if we replace W with W_z in the result in [19] we have that

$$\log\left(\Xi(P,W;z)\right)=f_{N,K}(z)=\sum_i a_{i,N}^K z^i$$

Then in the limit as $N\to\infty$ this is an analytic function with radius of convergence $r>1$. In particular this means $\sum_i a_{i,N}^K$ is an absolutely convergent series which converges faster than any polynomial power of the index, meaning $\sum_i i^k a_{i,N}^K$ converges for all constant finite k.
More importantly we have that

$$\lim_{N\to\infty}\log\left(\Xi(P,W;z)\right)=\lim_{N\to\infty}f_{N,K}(z)=\sum_i a_i^K z^i$$

converges to an analytic function. Note that this is true because it is the uniform limit of analytic functions which converges on compact subsets.

4.2.2. Radius of convergence $r>1$. The proof of the convergence of $\log\left(\Xi(\mathcal{P}_{N,K},W;z)\right)$ comes from [19] and is by enumeration and calculation of cases of loops. As these numbers are a rough bound, we can easily see (by choosing a radius of convergence within the error rate given by their significant figures) that substituting $W_z(\gamma)$ as defined above we get a convergent expansion and thus an analytic function on a disc of radius $r>1$. This implies all derivatives exist and converge as well.

Lemma 4.2.0.3. The partition function $\log\left(\Xi(\mathcal{P}_{N,K},W;z)\right)$ has radius of convergence $r>1$

Proof. Let $\varepsilon>0$ and let $\tilde{\mathcal{W}}(\gamma)=\left(\frac{1}{3-\varepsilon}\right)^{|\gamma|-1}$; to get convergence we require that

$$\sum_{\gamma'\nmid\gamma}\tilde{W}(\gamma')e^{\tilde{a}(\gamma')}<\tilde{a}(\gamma)$$

and so we have from the above

$$\sum_{\gamma'\nmid\gamma}W(\gamma')\left(\frac{1}{1-\varepsilon/3}\right)^{|\gamma|-1}e^{\tilde{a}(\gamma')}\leq\tilde{a}(\gamma)$$

Now letting $\eta = \frac{1}{1-\varepsilon/3} > 1$ we adjust our Kotecký-Preiss function from [19] so that

$$\tilde{a}(\gamma) = a(\gamma) - |\gamma| \ln \eta$$

so that we have

$$\sum_{\gamma' \nmid \gamma} W(\gamma') e^{a(\gamma')} \leq a(\gamma) - |\gamma| \ln \eta$$

and for small enough ε we retain positivity of the righthand side and for very small ε this gives the same numbers as in the proof of Theorem 4.3.1 for the case of $\epsilon = 0$. Thus we have that the radius of convergence is at least $r \geq \eta > 1$. Thus we have a $d(\gamma) = |\gamma| \ln(|\eta|) > 0$ for nonempty γ, and so the convergence criterion holds for some $\epsilon = \ln(|\eta|) > 0$ □

4.3. Convergence of the Cluster Expansion

In this section, we verify the Kotecký-Preiss criterion for the polymer set $\mathcal{P}_{N,K}$ from (4.10) and weight function W from (4.13)-(4.14) for any N and K sufficiently large. To this end, define $a : \mathbb{N}_{\geq 3} \to [0, \infty)$ by

$$a(l) = \begin{cases} 0.52, & l = 3 \\ 0.56, & l = 4 \\ 0.66, & l = 5 \\ 0.70, & l = 6 \\ 0.15l, & l \geq 7 \end{cases} \tag{4.1}$$

We note that the domain of a reflects the fact that $|\gamma| \geq 3$ for any $\gamma \in \mathcal{P}_{N,K}$.

Theorem 4.3.1 (Cluster expansion convergence for $\mathcal{P}_{N,K}$). *Let $\mathcal{P}_{N,K}$ and W be defined as in (4.10), and (4.13)-(4.14). There exists $\epsilon > 0$ such that if $N - K > 20$ and $K > 20$ or $K = 0$, then for any $\gamma \in \mathcal{P}_{N,K}$*

$$\sum_{\substack{\gamma' \in \mathcal{P}_{N,K} \\ \gamma \nmid \gamma'}} |W(\gamma')| e^{a(|\gamma'|) + \epsilon |\gamma'|} \leq a(|\gamma|)\,. \tag{4.2}$$

Moreover, the above also holds after replacing $\mathcal{P}_{N,K}$ with $\mathcal{P}_{N,K}^{\text{bulk}}$ everywhere since $\mathcal{P}_{N,K}^{\text{bulk}} \subseteq \mathcal{P}_{N,K}$.

The function a above and approach used to prove Theorem 4.3.1 are the same as those used to verify the KP criterion for $\mathcal{P}_{N,0}$ to establish the uniqueness of the frustration-free ground state ω in [19]. We work with $\mathcal{P}_{N,K}$, though, in order to establish the dependence of (4.6) on $|\Lambda_K|$. Since $\mathcal{P}_{N,K} \not\subseteq \mathcal{P}_{N,0}$, the result from [19] cannot be directly invoked to verify (4.2), and certain arguments used in that work need to be modified to work for the more general case.

The key observation for proving Theorem 4.0.1, and hence LTQO, is that the function a and constant ϵ above are uniform in N and K. Furthermore, as is shown in Lemma 4.2.0.1, it is sufficient to prove Theorem 4.3.1 for $\epsilon = 0$ so long as one establishes (4.2) with $a(|\gamma|)$ replaced by $Ca(|\gamma|)$ where $0 < C < 1$ is independent of N and K. To this end, we begin by first inserting the bound

$$|W(\gamma)| \leq 3^{-|\gamma|+1}$$

and then proving that there exists $0 < C < 1$ (independent of N and K) so that for any $\gamma \in \mathcal{P}_{N,K}$,

$$\sum_{\substack{\gamma' \in \mathcal{P}_{N,K} \\ \gamma \nmid \gamma'}} w(|\gamma'|) = \sum_{l=3}^{\infty} w(l) |\mathcal{P}_{N,K}^{\gamma,l}| \leq Ca(|\gamma|) \tag{4.3}$$

where for simplicity we introduce

$$w(l) := \frac{e^{a(l)}}{3^{l-1}}, \qquad \mathcal{P}_{N,K}^{\gamma,l} := \{\gamma' \in \mathcal{P}_{N,K} : \gamma \nmid \gamma',\ |\gamma'| = l\}. \tag{4.4}$$

Hence, one only needs to produce sufficiently tight upper bound on $|\mathcal{P}_{N,K}^{\gamma,l}|$ to verify Theorem 4.3.1. To establish (4.3), the set $\mathcal{P}_{N,K}^{\gamma,l}$ is partitioned into loops and walks, and then the sum is decomposed by the value of l. To make this precise, for any $\gamma \in \mathcal{P}_{N,K}$ and $l \geq 3$, let

$$\mathcal{W}_{N,K}^{\gamma,l} = \{\gamma' \in \mathcal{W}_{N,K} : \gamma \nmid \gamma',\ |\gamma'| = l\} \tag{4.5}$$

$$\mathcal{L}_{N,K}^{\gamma,l} = \{\gamma' \in \mathcal{L}_{N,K} : \gamma \nmid \gamma',\ |\gamma'| = l\} \tag{4.6}$$

denote the set of all walks (resp. loops) of length l that intersect γ. For convenience later on, we also denote the set of all walks (resp. loops) of length l by

$$\mathcal{W}_{N,K}^{l} = \{\gamma \in \mathcal{W}_{N,K} : |\gamma| = l\}, \quad \mathcal{L}_{N,K}^{l} = \{\gamma \in \mathcal{L}_{N,K} : |\gamma| = l\}. \tag{4.7}$$

Given these definitions, the left hand side of (4.3) can be written as

$$\begin{aligned} \sum_{\substack{\gamma' \in \mathcal{P}_{N,K} \\ \gamma \nmid \gamma'}} w(|\gamma'|) = &\sum_{\substack{3 \leq l \leq 20 \\ l \text{ even}}} w(l) |\mathcal{W}_{N,K}^{\gamma,l}| + \sum_{\substack{3 \leq l \leq 20 \\ l \text{ odd}}} w(l) |\mathcal{W}_{N,K}^{\gamma,l}| + \sum_{l=6,10} w(l) |\mathcal{L}_{N,K}^{\gamma,l}| \\ &+ \sum_{12 \leq l \leq 28} w(l) |\mathcal{L}_{N,K}^{\gamma,l}| + \sum_{l \geq 30} w(l) |\mathcal{L}_{N,K}^{\gamma,l}| + \sum_{l \geq 21} w(l) |\mathcal{W}_{N,K}^{\gamma,l}| \end{aligned} \tag{4.8}$$

where the choice of the six specific sums above is motivated by grouping together sets of polymer whose cardinalities will be bound using similar arguments.

4.3.1. Cardinality bounds on polymer sets. For a fixed $\gamma \in \mathcal{P}_{N,K}$, the main challenge for establishing (4.2) comes from bounding the contributions in (4.8) from polymers $\gamma \nmid \gamma'$ of small length,

specifically,

$$\gamma' \in \bigcup_{3\leq l\leq 20} \mathcal{W}^{\gamma,l}_{N,K} \cup \bigcup_{6\leq l\leq 28} \mathcal{L}^{\gamma,l}_{N,K}\,. \tag{4.9}$$

The assumptions on N and K in Theorem 4.3.1 are made to simplify bounding $|\mathcal{W}^{\gamma,l}_{N,K}|$ and $|\mathcal{L}^{\gamma,l}_{N,K}|$ for such l and otherwise play no role in the proof of the main result. We first, however, give generic bounds on the cardinalities of these sets that, while not tight enough to bound all six sums in (4.8), will be sufficient to bound the final two sums. For brevity, we fix N and K such that $N-K>20$ and $K>20$, and suppress these labels from the sets of polymers, e.g. all those appearing in (4.5)-(4.7). All bounds produced in this section are uniform in such a choice.

The generic upper bounds produced on $|\mathcal{W}^{\gamma,l}|$ and $|\mathcal{L}^{\gamma,l}|$ will depend on the type and length of the polymer $\gamma\in\mathcal{P}$. To this end, for any $l,l'\in\mathbb{N}_{\geq 3}$, define

$$M_{\ell\ell}(l,l') = \sup_{\gamma\in\mathcal{L}^l} |\mathcal{L}^{\gamma,l'}| \qquad M_{\ell w}(l,l') = \sup_{\gamma\in\mathcal{L}^l} |\mathcal{W}^{\gamma,l'}| \tag{4.10}$$

$$M_{w\ell}(l,l') = \sup_{\gamma\in\mathcal{W}^l} |\mathcal{L}^{\gamma,l'}| \qquad M_{ww}(l,l') = \sup_{\gamma\in\mathcal{W}^l} |\mathcal{W}^{\gamma,l'}| \tag{4.11}$$

In words, $M_{\ell w}(l,l')$ is the maximum number of walks of length l' that intersect a fixed loop γ of length l, and similarly for the other three functions.

The bounds on (4.10)-(4.11) in Lemma 4.3.0.1 are based on the observation that the geometry of the hexagonal lattice and the definition of $\Lambda_{N,K}$ guarantee that any pair of polymers $\gamma,\gamma'\in\mathcal{P}_{N,K}$ must share an edge e if $\gamma\nmid\gamma'$. To state these bounds, first partition the set of edges of $\Lambda_{N,K}$ by

$$\mathcal{E}^b = \{e\in\mathcal{E}_{N,K} : e \text{ is a boundary edge}\}, \quad \mathcal{E}^i = \mathcal{E}_{N,K}\setminus\mathcal{E}^b$$

where $e=\{v,w\}\in\mathcal{E}_{N,K}$ is a *boundary edge* if either $\deg_{\Lambda_{N,K}}(v)=1$ or $\deg_{\Lambda_{N,K}}(w)=1$, and an *interior edge* otherwise. Then, the following three functions capture the maximum number of polymers of a given type and length that can contain a fixed edge:

$$N^i_w(l) = \sup_{e\in\mathcal{E}^i} |\{\gamma\in\mathcal{W}^l : e\in\gamma\}| \tag{4.12}$$

$$N^b_w(l) = \sup_{e\in\mathcal{E}^b} |\{\gamma\in\mathcal{W}^l : e\in\gamma\}| \tag{4.13}$$

$$N_\ell(l) = \sup_{e\in\mathcal{E}_{N,K}} |\{\gamma\in\mathcal{L}^l :\ e\in\gamma\}| = \sup_{e\in\mathcal{E}^i} |\{\gamma\in\mathcal{L}^l :\ e\in\gamma\}| \tag{4.14}$$

where the second equality for N_ℓ follows since a loop cannot contain a boundary edge.

Lemma 4.3.0.1 ([19]). For any $l\geq 3$ the functions from (4.10)-(4.11) satisfy

$$M_{\ell\ell}(l,l') \leq lN_\ell(l') \tag{4.15}$$

$$M_{w\ell}(l,l') \leq (l-2)N_\ell(l') \tag{4.16}$$

$$M_{\ell w}(l,l') \leq lN_w^i(l') \tag{4.17}$$

$$M_{ww}(l,l') \leq 2N_w^b(l') + (l-2)N_w^i(l') \tag{4.18}$$

Moreover, $N_\ell(l) \leq 2^{l-3}$ for $l \geq 6$, $N_w^b(l) \leq 2^{l-5}$ for $l \geq 11$ and $N_w^i(l) \leq (2l+95)2^{l-10}$ for $l \geq 21$.

We note that, trivially, $N_\ell(l) \leq N_\ell^\Gamma(l)$ where $N_\ell^\Gamma(l)$ is the maximum number of loops of length l in the entire hexagonal lattice, Γ, that intersect a fixed edge, i.e.

$$N_\ell^\Gamma(l) = \sup_{e\in\mathcal{E}_\Gamma} |\{\gamma \in \mathcal{L}_\Gamma : |\gamma| = l\}|.$$

The exact values of $N_\ell^\Gamma(l)$ for $l \leq 28$ were calculated with the aid of a computer, and can be found in Table 4.4.

PROOF. Fix any $\gamma \in \mathcal{W}^l$, and suppose that $\gamma' \in \mathcal{L}^{\gamma,l'}$. Since γ' is a loop, there must be an interior edge $e \in \gamma \cap \gamma'$. Therefore,

$$|\mathcal{L}^{\gamma,l'}| \leq (l-2)N_\ell(l')$$

as γ has exactly $l-2$ interior edges. This implies (4.16). In the case that $\gamma' \in \mathcal{W}^{\gamma,l'}$, it must be that γ' contains either a boundary edge of γ or an interior edge of γ. Since γ has exactly two boundary edges,

$$|\mathcal{W}^{\gamma,l'}| \leq 2N_w^b(l') + (l-2)N_w^i(l')$$

from which (4.18) follows. The bounds on $M_{\ell\ell}(l,l')$ and $M_{\ell w}(l,l')$ follow from the analogous arguments given that every edge of $\gamma \in \mathcal{L}^l$ is an interior edge.

For the bound on $N_\ell(l)$, fix an edge $e = \{v, v'\}$ that must be contained in a loop of size l. Then lay $l-3$ additional edges connected to e through v' to create a walk of length $l-2$ with endpoints v, w. There are at most two choices for laying each of these successive edges, and so there are at most 2^{l-3} such paths in $\Lambda_{N,K}$. If this walk can be closed to a loop, then there are exactly two edges between v and w. The result follows since there is a unique shortest path between any two vertices of the hexagonal lattice that are two edges apart.

To bound $N_w^b(l)$ and $N_w^i(l)$, notice that any walk $\gamma \in \mathcal{W}^l$ containing an edge $e = \{v, v'\} \in \mathcal{E}_{N,K}$ can be viewed as the concatenation of e with two disjoint walks $\gamma_v, \gamma_{v'} \in \mathcal{P}_\Gamma$ that begin at the vertices v and v', respectively, and end at the boundary, $\partial\Lambda_{N,K}$. Here, we use the convention that one of these walks is the empty graph if e is a boundary edge. Thus, $l = |\gamma_v| + |\gamma_{v'}| + 1$. Now, for any $v \in \mathcal{V}_{N,K}$ and edge $e \in \mathcal{E}_{N,K}$ containing v, let $n(v,e,l)$ be the number of walks $\gamma \in \mathcal{P}_\Gamma$ of length l that begin at v, do not

contain e, and end at some vertex $w \in \partial\Lambda_{N,K}$. Then, set

$$N(l) = \sup_{\substack{v \in \mathcal{V}_{N,K} \\ e=\{v,v'\} \in \mathcal{E}_{N,K}}} n(v, e, l).$$

Considering the case that $e \in \mathcal{E}^b$ and $e \in \mathcal{E}^i$ separately, the above observations imply

$$N_w^b(l) \leq N(l-1), \quad N_w^i(l) \leq \sum_{l'=1}^{l-2} N(l')N(l-l'-1) \tag{4.19}$$

where, in the case that $e = \{v, v'\} \in \mathcal{E}^i$, one uses that $\min\{|\gamma_v|, |\gamma_{v'}|\} \geq 1$.

The value of $N(l)$ can be bounded from above with the aid of a computer when, e.g. $l \leq 10$, see Table 4.6. In particular, $N(10) \leq 64$. For $l \geq 11$, the crude bound $N(l) \leq 2^l$ can be improved by noting that any such walk must be the concatenation of a walk of length $l-10$ starting at v and avoiding a fixed edge $e = \{v, v'\}$, and a walk of length ten ending at a boundary edge. Thus, for $l \geq 10$

$$N(l) \leq 2^{l-10} N(10) \leq 2^{l-4}\,. \tag{4.20}$$

Inserting this into (4.19) produces the desired upper bound on N_w^b. For the bound on $N_1^i(l)$, note that $l - l' - 1 \geq 11$ when $l' \leq 10$ and $l \geq 22$, and so (4.20) implies

$$\begin{aligned}\sum_{l'=1}^{l-2} N(l')N(l-l'-1) &= 2\sum_{l'=1}^{10} N(l')N(l-l'-1) + \sum_{l'=11}^{l-12} N(l')N(l-l'-1) \\ &\leq 2\sum_{l'=1}^{10} N(l')2^{l-l'-5} + (l-22)2^{l-9} \leq (2l+95)2^{l-10}\end{aligned}$$

where the last bound is obtained by inserting the values from Table 4.6. In the case that $l = 21$, the same bound follows from applying the above argument to the expansion

$$\sum_{l'=1}^{19} N(l')N(20-l') = 2\sum_{l'=1}^{9} N(l')N(l-l'-1) + N(10)^2\,.$$

□

The above results are sufficient for proving Theorem 4.3.1 when the length of the fixed polymer $|\gamma| = 3$. In the case that $|\gamma| > 3$, the bounds from Lemma 4.3.0.1 can once again be used to bound the last four sums from (4.8). However, additional bounds are needed for the first two sums in (4.8), specifically for the terms associated to $4 \leq l' \leq 20$, which is the focus of the remainder of this section. These bounds will be partitioned depending on the length of the fixed polymer γ.

We begin by considering $4 \leq |\gamma| \leq 6$ and $4 \leq l' \leq 20$. Since $K \geq 20$, in this case, it must be that either (1) $\mathcal{W}^{\gamma,l'}$ is empty (which occurs if γ is a loop that is sufficiently far away from the boundary of $\Lambda_{N,K}$), (2) that the endpoints of any $\gamma' \in \mathcal{W}^{\gamma,l'}$ always belong to the outer boundary, i.e. $\partial\Lambda_N$, or (3) that

the endpoints of any $\gamma' \in \mathcal{W}^{\gamma,l'}$ always belong to the inner boundary, i.e. $\partial\mathring{\Lambda}_K$. Hence, for any polymer $\gamma \in \mathcal{P}$ with $4 \leq |\gamma| \leq 6$ and any $4 \leq l' \leq 20$, we can partition

$$\mathcal{W}^{\gamma,l'} = \mathcal{W}^{\gamma,l',\mathrm{out}} \cup \mathcal{W}^{\gamma,l',\mathrm{in}}$$

where

$$\mathcal{W}^{\gamma,l',\mathrm{out}} = \{\gamma' \in \mathcal{W}^{\gamma,l'} : \mathrm{ep}(\gamma') \subseteq \partial\Lambda_N\}, \qquad \mathcal{W}^{\gamma,l',\mathrm{in}} = \{\gamma' \in \mathcal{W}^{\gamma,l'} : \mathrm{ep}(\gamma') \subseteq \partial\mathring{\Lambda}_K\}\,.$$

Depending on if γ is sufficiently close to the inner or outer boundary, either $|\mathcal{W}^{\gamma,l',\mathrm{out}}| = 0$ for all $4 \leq l' \leq 20$, or $|\mathcal{W}^{\gamma,l',\mathrm{in}}| = 0$ for all $4 \leq l' \leq 20$ (or both). As such, we define

$$\mathcal{W}^{l,\mathrm{out}} = \{\gamma \in \mathcal{W}^{l} : |\mathcal{W}^{\gamma,l',\mathrm{out}}| \neq 0 \text{ for some } 4 \leq l' \leq 20\}, \qquad 4 \leq l \leq 6$$
$$\mathcal{L}^{6,\mathrm{out}} = \{\gamma \in \mathcal{L}^{6} : |\mathcal{W}^{\gamma,l',\mathrm{out}}| \neq 0 \text{ for some } 4 \leq l' \leq 20\},$$

and similarly for $\mathcal{W}^{l,\mathrm{in}}$ and $\mathcal{L}^{6,\mathrm{in}}$. Now, for any $4 \leq l' \leq 20$ and $\# \in \{\mathrm{in}, \mathrm{out}\}$, let

$$S^{\#}_{ww}(l,l') = \sup_{\gamma \in \mathcal{W}^{l,\#}} \sum_{j=3}^{l'} |\mathcal{W}^{\gamma,j,\#}| - \sup_{\gamma \in \mathcal{W}^{l,\#}} \sum_{j=3}^{l'-1} |\mathcal{W}^{\gamma,j,\#}|, \qquad 4 \leq l \leq 6 \tag{4.21}$$

$$S^{\#}_{\ell w}(6,l') = \sup_{\gamma \in \mathcal{L}^{6,\#}} \sum_{j=3}^{l'} |\mathcal{W}^{\gamma,j,\#}| - \sup_{\gamma \in \mathcal{L}^{6,\#}} \sum_{j=3}^{l'-1} |\mathcal{W}^{\gamma,j,\#}| \tag{4.22}$$

and for convenience set

$$S^{\#}_{ww}(l,3) = \sup_{\gamma \in \mathcal{W}^{l,\#}} |\mathcal{W}^{\gamma,3,\#}|, \qquad S^{\#}_{\ell w}(6,3) = \sup_{\gamma \in \mathcal{L}^{6,\#}} |\mathcal{W}^{\gamma,3,\#}|\,. \tag{4.23}$$

Since there are only a finite number of distinct sets $\mathcal{W}^{\gamma,j,\#}$ up to translations in $\Lambda_{N,K}$ and these distinct sets are independent of the choice of N and K, the values of $S^{\#}_{ww}(l,l')$ and $S^{\#}_{\ell w}(6,l')$ can be calculated with the aid of a computer for any l, l' as above. Moreover, the following result holds, which is a slight generalization of a bound from [19] and follows the argument given therein.

Lemma 4.3.0.2 ([19]). For any $\# \in \{\ell, w\}$, let

$$S_{\#w}(l,l') = \max\{S^{\mathrm{out}}_{\#w}(l,l'), S^{\mathrm{in}}_{\#w}(l,l')\}. \tag{4.24}$$

Then, one has that

$$\sup_{\gamma \in \mathcal{W}^{l}} \sum_{l'=3}^{20} |\mathcal{W}^{\gamma,l'}| w(l') \leq \sum_{l'=3}^{20} S_{ww}(l,l') w(l'), \qquad 4 \leq l \leq 6 \tag{4.25}$$

$$\sup_{\gamma \in \mathcal{L}^{6}} \sum_{l'=3}^{20} |\mathcal{W}^{\gamma,l'}| w(l') \leq \sum_{l'=3}^{20} S_{\ell w}(6,l') w(l')\,. \tag{4.26}$$

The precise values of (4.24) were calculated with the aid of a computer for all possible choices of l, l', and are listed in Table 4.3. For notational convenience in this table, we also set

$$S_{ww}(3,l) = M_{ww}(3,l)\,.$$

PROOF. The proofs for (4.25) and (4.26) are analogous, so we give the argument in for (4.25). To begin, note that

$$\sup_{\gamma\in\mathcal{W}^l}\sum_{l'=3}^{20}|\mathcal{W}^{\gamma,l'}|w(l') = \max\left\{\sup_{\gamma\in\mathcal{W}^{l,\mathrm{out}}}\sum_{l'=3}^{20}|\mathcal{W}^{\gamma,l',\mathrm{out}}|w(l'),\ \sup_{\gamma\in\mathcal{W}^{l,\mathrm{in}}}\sum_{l'=3}^{20}|\mathcal{W}^{\gamma,l',\mathrm{in}}|w(l')\right\}. \tag{4.27}$$

Fixing $\# \in \{\mathrm{in}, \mathrm{out}\}$, it follows from (4.24) that

$$\begin{aligned}\sum_{l'=3}^{20} S_{ww}(l,l')w(l') &\geq \sum_{l'=3}^{20} S^{\#}_{ww}(l,l')w(l') \\ &= \sum_{l'=3}^{19}\left((w(l')-w(l'+1))\sup_{\gamma\in\mathcal{W}^{l,\#}}\sum_{j=3}^{l'}|\mathcal{W}^{\gamma,j,\#}|\right) + w(20)\sup_{\gamma\in\mathcal{W}^{l,\#}}\sum_{j=3}^{20}|\mathcal{W}^{\gamma,j,\#}|\end{aligned}$$

Since $w(l)$ is a monotone decreasing function and $w(20) \geq 0$, for any fixed $\gamma \in \mathcal{W}^{l,\#}$ the above implies that

$$\begin{aligned}\sum_{l'=3}^{20} S^{\#}_{ww}(l,l')w(l') &\geq \sum_{l'=3}^{19}\left((w(l')-w(l'+1))\sum_{j=3}^{l'}|\mathcal{W}^{\gamma,j,\#}|\right) + w(20)\sum_{j=3}^{20}|\mathcal{W}^{\gamma,j,\#}| \\ &= \sum_{l'=3}^{20}|\mathcal{W}^{\gamma,j,\#}|w(l')\end{aligned}$$

from which (4.25) follows from (4.24) and (4.27). □

We now turn to considering cardinality bounds on polymer sets for the first two sums in (4.8) in the case that the fixed polymer $\gamma \in \mathcal{P}_{N,K}$ has length $|\gamma| > 6$.

In both cases, we are uniformly bounding the number of walks γ' of length $l' \leq 21$ that intersect a polymer of fixed type. Since $N - K > 20$, the distance between any two vertices $x \in \partial\Lambda_N$ and $y \in \partial\Lambda_K$ necessarily satisfy $d(x,y) \geq 40$. Thus, the endpoints of γ' must both belong to either the inner or outer boundary, i.e.

$$\mathrm{ep}(\gamma') \subseteq \partial\Lambda_K, \quad \text{or} \quad \mathrm{ep}(\gamma') \subseteq \partial\Lambda_N\,. \tag{4.28}$$

Recall that both the inner and outer boundary of $\Lambda_{N,K}$ can be partition into six corners and six sides. As such, the constraint $K \geq 20$ implies that $\mathrm{ep}(\gamma')$ must (1) belong to the same side of the boundary if l' is even, or (2) belong to two neighboring sides encompassing a single corner if l' is odd. As such,

the endpoints of γ' can be uniquely labeled as the left endpoint, $v_L(\gamma')$, and right endpoint, $v_R(\gamma')$, which will be used to produce the desired bounds on $M_{ww}(l,l')$ and $M_{\ell w}(l,l')$. We begin with bounding $M_{ww}(l,l')$, and recall that a hexagon is called a *boundary hexagon* if it contains a vertex $v \in \partial\Lambda_{N,K}$. For a walk γ with both end points in the same connected component of the boundary as in (4.28), considered as a piecewise differentiable curve in $\mathbb{C}\setminus\{0\}$, we denote by $v_R(\gamma)$ that endpoint such that the winding angle from it to the other endpoint is positive. Explicity, for $\gamma : [0,1] \to \mathbb{C}$ be a differentiable parametrization of the walk with $\gamma(0) = v_R(\gamma)$, then

$$\frac{1}{2\pi i}\int_0^1 \frac{\lambda_j(s)}{\gamma(s)}ds > 0.$$

It will be useful to consider sub graphs $B_{N,K} \subset \Gamma$, $0 \le K \le N$which corresponds to rings in the dual lattice. In analogy with the definition of Λ_N in (4.2), we define

$$B_{N,K} = \bigcup_{\hat{x}\in\hat{\Gamma}, K\le|\hat{x}|\le N} h_{\hat{x}}$$

where $h_{\hat{x}}$ is the hexagon centered at $\hat{x}$.

Lemma 4.3.0.3. Fix $l > 6$. Then,

$$M_{ww}(l,l') \le \begin{cases} l/2+1, & l' = 4,5 \\ l/2+2, & l' = 6 \\ (l+l'-1)R(l'), & 7 \le l' \le 21 \end{cases} \tag{4.29}$$

where

$$R(l') = \sup_{v\in\partial\Lambda_{N,K}} |\{\lambda_j \in \mathcal{W}^{l'} : v_R(\lambda_j) = v\}|. \tag{4.30}$$

Proof. Let λ_j be a walk or a loop of length ℓ', contained in $\Lambda_{N,K}$, with both K and $N-K$ large enough (TBD). Define

$$X(\lambda_j) = \left|\left\{v \in \partial\Lambda_{N,K} : \exists\gamma \in \mathcal{W}^l \text{ s.t. } v = v_R(\gamma) \wedge \gamma \nmid \lambda_j\right\}\right|. \tag{4.31}$$

We claim the following bound

$$X(\lambda_j) \le \frac{l'}{2} + l - 2 \tag{4.32}$$

for ℓ in the range $7 \le \ell \le \ell_0$. The constraints we need on N, K, and $N-K$ depend on ℓ_0.

We decompose $\Lambda_{N,K}$ intro three subregions: an inner corridor $C_i = \Lambda_{K+\delta,K}$, an outer corridor $C_o = \Lambda_{N,N-\delta}$, and an interior area given by $B := \Lambda_{N-\delta-1,K+\delta+1}$. The parameter δ is chosen such the walks

γ are contained in one of the corridors, say $C_\sharp$. Hence, the intersections of γ with λ_j, must occur in the corridors. To proceed, we first decompose λ_j in disjoint sections: $\lambda_j = \eta_1 \vee \tilde{\eta}_1 \vee \eta_2 \vee \tilde{\eta}_2 \vee \cdots \vee \tilde{\eta}_{m-1} \vee \eta_m$, each part η_i is contained in a corridor, while the parts $\tilde{\eta}_i$ are contained in B. If λ_j is a loop, η_1 or $\tilde{\eta}_1$ may be empty.

Next, we coarsen the decomposition λ_j by defining $\eta_i \sim \eta_{i+1}$ if $|\tilde{\eta}_i| < \ell_0$, and consider the disjoint decomposition $\lambda_j = \lambda_1 \vee \tilde{\lambda}_1 \vee \lambda_2 \vee \tilde{\lambda}_2 \vee \cdots \vee \tilde{\lambda}_{p-1} \vee \lambda_p$, where each $\lambda_j = \eta_i \vee \tilde{\eta}_i \vee \eta_{i+1} \vee \tilde{\eta}_{i+1} \vee \cdots \vee \tilde{\eta}_{i+q-1} \vee \eta_q$, for some i and q depending on j, such that the $\tilde{\eta}_i$ appearing in λ_p all satisfy $|\tilde{\eta}_i| < \ell_0$, and $|\tilde{\lambda}_j| \geq \ell_0$. Due to the choice of parameters each piece λ_j intersects exactly one of the corridors.

Now, if $\gamma \nmid \lambda_j$, it has to intersect λ_j for some j. It follows that

$$X(\lambda_j) \leq \sum_j X(\lambda_j) \tag{4.33}$$

Consider the following three distinct possibilities for λ_j. Case I and II cover the situation in which there are walks $\gamma_1, \gamma_2 \in \mathcal{W}^\ell$ for which $\gamma_1 \cup \gamma_2 \cup \lambda_j$ is not simply connected in $\Lambda_{N,K}$. In Case I we assume λ_j intersects the inner corridor C_i, while Case II is when λ_j intersects the outer corridor C_o. Case III is the situation where for all $\gamma_1, \gamma_2 \in \mathcal{W}^\ell$, $\gamma_1 \cup \gamma_2 \cup \lambda_j$ is simply connected in $\Lambda_{N,K}$.

Cases I and II: If there are walks $\gamma_1, \gamma_2 \in \mathcal{W}^\ell$ for which $\gamma_1 \cup \gamma_2 \cup \lambda_j$ is not simply connected in $\Lambda_{N,K}$, we will estimate $X(\lambda_j)$ by the trivial bound equal to the number of vertices in the inner and outer boundary of $\Lambda_{N,K}$.

For Case I this gives:

$$X(\lambda_j) \leq \frac{1}{2}|\partial\Lambda_K| = 9 + 6(K-2) \leq \frac{1}{2}|\lambda_j|. \tag{4.34}$$

For Case II we start from the trivial estimate

$$X(\lambda_j) \leq \frac{1}{2}|\partial\Lambda_N| = 9 + 6(N-2).$$

In order to obtain an upper bound in terms of $|\lambda_j|$, we use Lemma 4.3.0.4 and note that $|\lambda_j| \geq 2|\partial C_o \cap \Lambda_{N-\delta+1,N-\delta}| = 18 + 12(N-\delta-2)$. Hence

$$X(\lambda_j) \leq \frac{1}{2}|\lambda_j| + 6\delta. \tag{4.35}$$

Case III: Let $\gamma_1, \gamma_2 \in \mathcal{W}^\ell$ be two walks that intersect λ_j. By assumption, $\gamma_1 \cup \gamma_2 \cup \lambda_j$ is simply connected in $\Lambda_{N,K}$. It follows that there is a walk $\mu \in \mathcal{W}_{N,K}$ with end points w_1 and w_2 in the inner and outer component of $\partial\Lambda_{N,K}$, respectively, with $\mu \cap (\gamma_1 \cup \gamma_2 \cup \lambda_j) = \emptyset$. Clearly, we have $\gamma_1 \cup \gamma_2 \cup \lambda_j \subset \Lambda_{N,K} \setminus \mu$. Let $\tilde{d}$ be the graph distance on $\Lambda_{N,K} \setminus \mu$.

Let $v_1 = v_R(\gamma_1)$ and $v_2 = v_r(\gamma_2)$ and v_1' and v_2' intersection vertices of these walks with λ_j. The triangle inequality gives

$$\tilde{d}(v_1, v_2) \leq \tilde{d}(v_1, v_1') + \tilde{d}(v_1', v_2') + \tilde{d}(v_2', v_2). \tag{4.36}$$

Since $\gamma_1 \nmid \lambda_j$, v_1' can be chosen such that $\tilde{d}(v_1, v_1') \leq \ell - 1$. Similarly, v_2' can be chosen such that $\tilde{d}(v_2, v_2') \leq \ell - 1$. Furthermore, γ_1 and γ_2 intersect λ_j in at least one edge. It suffices to count that edge only once in the estimate for $\tilde{d}(v_1', v_2')$. Therefore, we have

$$\tilde{d}(v_1, v_2) \leq |\lambda_j| + 2\ell - 4. \tag{4.37}$$

Note that $v_R(\gamma_1)$ and $v_2 = v_R(\gamma_2)$ are contained in the same component of the $\partial\Lambda_{N,K}$. Lemma 4.3.0.4 provides a geodesic walk connecting $v_R(\gamma_1)$ and $v_2 = v_R(\gamma_2)$. The number $X(\lambda_j)$ can be bounded in terms of the length of this geodesic walk as follows:

$$X(\lambda_j) \leq \frac{1}{2}\tilde{d}(v_1, v_2). \tag{4.38}$$

Combining inequalities (4.36), (4.37), and (4.38) provides the following upper bound for $X(\lambda_j)$:

$$X(\lambda_j) \leq \frac{1}{2}|\lambda_j| + \ell - 2.$$

This concludes Case III.

The optimal value of delta appears to be $\ell/4$.

For the estimate (4.34) we have

$$X(\lambda_j) \leq \frac{1}{2}|\lambda_j| + \ell - 2.$$

and noting the choice of $\ell_0 = 6\delta$, we have

$$|\lambda_j| = \sum_{j=1}^{p} |\lambda_j| + \sum_{j=1}^{p-1} |\tilde{\lambda}_j| \geq \sum_{j=1}^{p} |\lambda_j| + (p-1)\ell_0.$$

With these results, returning to the inequality (4.33), we find

$$X(\lambda_j) \leq \frac{1}{2}|\lambda_j| + \frac{3}{2}\ell.$$

□

Lemma 4.3.0.4. Two adjacent sides of a fundamental parallelogram in the hexagonal lattice form a geodesic between its opposite vertices.

Proof. By the brick wall isometric isomorphism. □

Finally we need to bound the same but with $\gamma \in \mathcal{L}^l$ for $l > 6$. Thus we have the following

Lemma 4.3.0.5. If $|\gamma| > 6$ then

$$M_{\ell w}(|\gamma|, 4) \leq \left(\frac{|\gamma|}{4} + \frac{1}{2}\right)$$

$$M_{\ell w}(|\gamma|, 5) \leq \left(\frac{|\gamma|}{4} + \frac{1}{2}\right)$$

$$M_{\ell w}(|\gamma|, 6) \leq \left(\frac{|\gamma|}{2} + \frac{3}{2}\right)$$

and

$$M_{\ell w}(|\gamma|, l) \leq \left(\frac{|\gamma|}{4} + l - \frac{5}{2}\right) R(l)$$

when $7 \leq l \leq 20$

Proof. As before we count the number of $\gamma' \in \mathcal{W}^4$ which intersect a given loop of length $|\gamma|$, and find the worst case scenario is the loop γ which hugs the boundary as in Figure 3, since the walks of length 4 which intersect it must hug the boundary as well. Thus the total number of such walks is bounded by $\frac{|\gamma|}{4} + \frac{1}{2}$. For γ' of odd length this same bound works since these walks must span a corner. For $\gamma' \in \mathcal{W}^6$ we have similarly the bound $\frac{|\gamma|}{4} + \frac{3}{2}$. For $\gamma' \in \mathcal{W}^{22>l>6}$ we use the same function $R(l)$, and the worst case scenario is again given in Figure 3; since boundary bonds are at least 2 bonds away from their neighbors we have that the worst case for the number of these bonds is given by $\frac{1}{2}(l-2) + \frac{1}{4}(|\gamma| - 6) + \frac{1}{2}(l-2) = \frac{|\gamma|}{4} + l - \frac{5}{2}$, and thus multiplying by the bound $R(l)$ we get the desired bound. □

Figure 4.2. The shape of the worst case walk, graphic from [19]

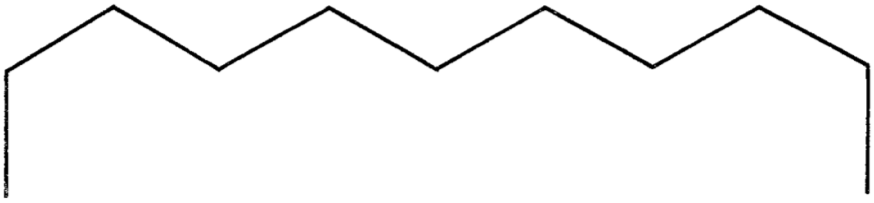
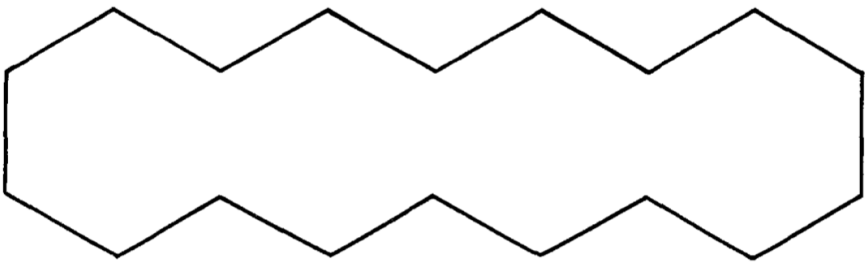


Figure 4.3. The shape of the worst case loop, graphic from [19]



4.3.2. Proof of Theorem 4.3.1. Assume that $N-K>20$ and $K>20$. The case of $K=0$ was proved in [19, Appendix], and is a mild variation of the argument given here for $K>20$. As mentioned above, we prove that (4.3) holds and invoke Lemma 4.2.0.1 to obtain $\epsilon>0$. To this end, we verify (4.3) case by case for $|\gamma|=3,4,5,6$ and $|\gamma|>6$, distinguishing the cases of when γ is a walk or a loop.

Case 1: $\gamma\in\mathcal{W}^3$. This follows identically the proof from [19], which we include for completeness. There are precisely six walks γ of length six in $\Lambda_{N,K}$, one at each of the exterior corners of the volume. The argument is the same in all cases as they are all isomorphic up a rotation of $\Lambda_{N,K}$. In this case, the number of loops of length $l<10$ in $\Lambda_{N,K}$ that intersect γ, i.e. $M_{w\ell}(3,l)$, and walks of length $l\leq 20$ in $\Lambda_{N,K}$ that intersect γ, i.e. $M_{ww}(3,l)$, can be explicitly counted with the aid of a computer. The resulting values are given in the first column of Table 4.1. Note that there are no loops with length $l<6$ or $l=8$ in the hexagonal lattice. Thus, substituting the values from this table, applying Lemma 4.3.0.1, and utilizing Table 4.4 yields

$$\begin{aligned}\sup_{\gamma\in\mathcal{W}^3}\frac{1}{a(3)}\sum_{\substack{\gamma'\in\mathcal{P}_{N,K}:\\ \gamma\nmid\gamma'}}w(|\gamma'|)\leq&\frac{1}{a(3)}\Bigg(\sum_{3\leq l\leq 20}w(l)M_{ww}(3,l)+\sum_{l=6,10}w(l)M_{w\ell}(3,l)\\&+\sum_{l\geq 21}w(l)\left(2^{l-4}+(2l+95)2^{l-10}\right)\\&+\sum_{6\leq l\leq 14}w(2l)N_\ell^\Gamma(2l)+\sum_{l\geq 15}w(2l)2^{2l-3}\Bigg)\\\leq&0.87601\end{aligned}$$

This establishes (4.3) for $\gamma\in\mathcal{W}^3$.

Case 2-5: $\gamma\in\mathcal{W}^4,\mathcal{W}^5,\mathcal{W}^6,\mathcal{L}^6$. These cases all follow the same argument, which is again the analogous version of the proof from [19]. We provide the details for $\gamma\in\mathcal{W}^4$, and give the resulting values produced in all other cases. By Lemma 4.3.0.1-4.3.0.2 and Table 4.3

$$\begin{aligned}\sup_{\gamma\in\mathcal{W}^4}\frac{1}{a(4)}\sum_{\substack{\gamma'\in\mathcal{P}_{N,K}:\\ \gamma\nmid\gamma'}}w(|\gamma'|)\leq&\frac{1}{a(4)}\Bigg(\sum_{3\leq l\leq 20}w(l)S_{ww}(4,l)+\sum_{l=6,10}w(l)M_{w\ell}(4,l)\\&+\sum_{l\geq 21}w(l)\left(2^{l-4}+2(2l+95)2^{l-10}\right)\\&+\sum_{6\leq l\leq 14}2w(2l)N_\ell^\Gamma(2l)+\sum_{l\geq 15}w(2l)2^{2l-2}\Bigg)\\\leq&0.87757\end{aligned}$$

The results for $\gamma \in \mathcal{W}^5, \mathcal{W}^6, \mathcal{L}^6$ use the same argument up to appropriate modifications of the sums. This results in the following values:

$$\sup_{\gamma\in\mathcal{W}^5} \frac{1}{a(5)} \sum_{\substack{\gamma'\in\mathcal{P}_{N,K}:\\ \gamma\nmid\gamma'}} w(|\gamma'|) \leq 0.89544 \tag{4.39}$$

$$\sup_{\gamma\in\mathcal{W}^6} \frac{1}{a(6)} \sum_{\substack{\gamma'\in\mathcal{P}_{N,K}:\\ \gamma\nmid\gamma'}} w(|\gamma'|) \leq 0.95785 \tag{4.40}$$

$$\sup_{\gamma\in\mathcal{L}^6} \frac{1}{a(6)} \sum_{\substack{\gamma'\in\mathcal{P}_{N,K}:\\ \gamma\nmid\gamma'}} w(|\gamma'|) \leq 0.85329 \tag{4.41}$$

Case 6: $\gamma \in \mathcal{W}^l$ with $l > 6$. We again apply Lemma 4.3.0.1 to bound $|M_{w\ell}(l,l')| \leq l 2^{l'-3}$ for any $l' \geq 6$, and moreover,

$$|M_{ww}(l,l')| \leq 2^{l'-4} + (l-2)(2l'+95)2^{l'-10} \leq l(2l'+95)2^{l'-10} \qquad \forall l' \geq 21\,.$$

Thus we have

$$\begin{aligned}\sup_{\gamma\in\mathcal{W}^l} \sum_{\gamma'\nmid\gamma} w(|\gamma'|) \leq & \sum_{\substack{3\leq l'\leq 20:\\ l' \text{ even}}} M_{ww}(|\gamma|,l')w(l') + \sum_{\substack{3\leq l'\leq 20:\\ l' \text{ odd}}} M_{ww}(|\gamma|,l')w(l') \\ & + l\sum_{l'=3}^{\infty} 2^{2l'-3} w(2l') + l\sum_{l'=21}^{\infty} (2l'+95)2^{l'-10} w(l')\end{aligned} \tag{4.42}$$

Case 7: $\gamma \in \mathcal{L}^l$ with $l > 6$. This proceeds analogously to Case 6. Hence, by Lemma 4.3.0.1 we have

$$\begin{aligned}\sup_{\gamma\in\mathcal{L}^l} \sum_{\gamma'\nmid\gamma} w(|\gamma'|) \leq & \sum_{\substack{3\leq l'\leq 20:\\ l' \text{ even}}} M_{w\ell}(|\gamma|,l')w(l') + \sum_{\substack{3\leq l'\leq 20:\\ l' \text{ odd}}} M_{w\ell}(|\gamma|,l')w(l') \\ & + l\sum_{l'=3}^{\infty} 2^{2l'-3} w(2l') + l\sum_{l'=21}^{\infty} (2l'+95)2^{l'-10} w(l')\end{aligned} \tag{4.43}$$

Hence, for any $\gamma \in \mathcal{P}_{N,K}$, we have that

$$\sum_{\substack{\gamma'\in\mathcal{P}_{N,K}\\ \gamma\nmid\gamma'}} w(|\gamma'|) \leq Ca(|\gamma|)$$

Since C is uniform in the choice of N and K, Theorem 4.3.1 holds by Lemma 4.2.0.1.

□

Let us define the following sets:

$$\tilde{\mathcal{W}}^l := \{\gamma \in \mathcal{W}^{\text{bulk}} | \text{ep}(\gamma) \subset \partial\Lambda_N,\ |\gamma| = l\}$$

$$\mathring{\mathcal{W}}^l := \mathcal{W}^l \setminus (\mathcal{W}^{\text{bulk},l} \cup \tilde{\mathcal{W}}^l)$$

To prove Theorem 4.3.1 we must verify this criterion for γ, γ' in the set

$$\mathcal{P}_{N,K} = \bigcup_l \mathcal{W}^l \cup \tilde{\mathcal{W}}^l \cup \mathring{\mathcal{W}}^l \cup \mathcal{L}^l \tag{4.44}$$

Explicitly we verify the uniform upper bound on (4.3.1) for the following summands:

$$\gamma \in \mathcal{W}^l,\ \gamma' \in \mathcal{W}^{l'}$$

$$\gamma \in \mathcal{W}^l,\ \gamma' \in \mathcal{L}^{l'}$$

$$\gamma \in \mathcal{L}^l,\ \gamma' \in \mathcal{W}^{l'}$$

for $l, l' \geq 4$ (there are no walks in $S_{N,K}$ of length 3) and for

$$\gamma \in \mathcal{W}^l,\ \gamma' \in P_{l'}$$

for $l' > 20$, where P_l is the set of all polymers of length l which can begin and end on the boundary.

Note that the first three we will prove do not cover the 12 other pairings implied by (4.44) for $\mathcal{P}_{N,K}$. However this last case will correspond to a very small term in our sum which will cover the cases where $\gamma, \gamma' \in \mathring{\mathcal{W}}^l$ by assuming N and K large enough; the assumption that $N - K > 20$ and $K > 20$ forces $\mathring{\mathcal{W}}^l$ to be empty for $l < 21$, and so for all cases γ' are in $\mathring{\mathcal{W}}$ are covered by our final case; the upper bounds used in the case $\gamma \in \mathcal{W}^{l>6}$ apply to $\gamma \in \mathring{\mathcal{W}}$ as well so these are covered. This also forces the case of $\gamma \in \tilde{\mathcal{W}}^l$, $\gamma' \in \mathcal{W}^{l'}$ and $\gamma \in \mathcal{W}^l$, $\gamma' \in \tilde{\mathcal{W}}^{l'}$ to be accounted for in this final case as well. The remaining cases are proved in [19].

However for these remaining four cases we can make some further reductions. We see that for $\gamma' \in \mathcal{W}^l$ for odd l includes only walks which straddle one or more corners; choosing $K > 20$ ensures that all walks which straddle more than one corner are included in the case of $\gamma' \in P_{>20}$. Similarly $K > 20$ ensures that $\gamma' \in \mathcal{W}^l$ which straddle more than one edge are included in $\gamma' \in P_{>20}$. Thus we restrict ourselves to the subset of walks $\mathcal{W}^l_o \subseteq \mathcal{W}^l$ which straddle at most one corner.

Note further that all $\gamma \in \mathcal{W}^l_o$ for l even have an isomorphic walk of the same length in $\tilde{\mathcal{W}}^l$, so the numbers used in [19] for $\gamma \in \mathcal{W}^4, \mathcal{W}^6$ are all valid for $\gamma' \in \mathcal{W}^l_o$ with l even.

Case 1: $\gamma \in \mathcal{W}_o^4$**.** Up to translation there is only one such walk on any given edge of $\partial\Lambda_K$. In order to bound our sum for any walk $\gamma \in \mathcal{W}_o^4$ we have the following upper bound:

(4.45)
$$\sum_{\gamma' \nmid \gamma} w(\gamma') \leq \sum_{3<l'<21 \text{ even}} M_{ww}(|\gamma|, l')w(l') + \sum_{3<l'<21 \text{ odd}} Q_o(l')w(l') + \sum_{l'=6,10} M_{\ell w}(|\gamma|, l')w(l')$$
$$+ \sum_{10<l'<30 \text{ even}} (|\gamma|-2)N(l')w(l') + \sum_{l'>28 \text{ even}} (|\gamma|-2)2^{l'-3}w(l') + \sum_{l'>20} \left(2P_b(l') + (|\gamma|-2)P_i(l')\right) w(l')$$

And we find that substituting in our values from the Tables 1, 2, 3 and 4 we have

$$\sup_{\gamma \in \mathcal{W}_o^4} \frac{1}{a(\gamma)} \sum_{\gamma' \nmid \gamma} w(\gamma') \leq 0.4473$$

The values for each case are given in Table 5.

Case 2,3: $\gamma \in \mathcal{W}_o^5, \mathcal{W}_o^6$**.** There are only 6 walks of length 5, a unique one at each corner; we tabulate the number of walks of length $l \leq 20$ intersecting it in Table 3 and bound this number for $l > 20$ using our lemmata, as in Case 1. What results are the values in Table 5:

$$\sup_{\gamma \in \mathcal{W}_o^5} \frac{1}{a(\gamma)} \sum_{\gamma' \nmid \gamma} w(\gamma') \leq 0.4034$$

Similarly for $\mathcal{W}_o^6$ we have

$$\sup_{\gamma \in \mathcal{W}_o^6} \frac{1}{a(\gamma)} \sum_{\gamma' \nmid \gamma} w(\gamma') \leq 0.5164$$

Case 4: $\gamma \in \mathcal{L}^6$**.** The case of $\mathcal{L}^6$ is special in that we upper bound the last three pieces of the sum using the analogous upper bounds for loops, without the assumption of endpoints to get

(4.46)
$$\sum_{\gamma' \nmid \gamma} w(\gamma') \leq \sum_{3<l'<21 \text{ even}} M_{w\ell}(|\gamma|, l')w(l') + \sum_{3<l'<21 \text{ odd}} Q_o(l')w(l') + \sum_{l'=6,10} M_{\ell\ell}(|\gamma|, l')w(l')$$
$$+ \sum_{30>l'>10 \text{ even}} |\gamma| N(l')w(l') + \sum_{l'>28 \text{ even}} |\gamma| 2^{l'-3} w(l') + \sum_{l'>20} |\gamma| P_i(l')w(l')$$

Thus for $\mathcal{L}^6$ we have

$$\sup_{\gamma \in \mathcal{L}^6} \frac{1}{a(\gamma)} \sum_{\gamma' \nmid \gamma} w(\gamma') \leq 0.479$$

This form of the bound is also used for $\mathcal{L}^{>6}$.

Case 5: $\gamma \in \mathcal{W}_o^{>6}$**.** In this case we bound the number of loops of length l intersecting γ by $N(l)|\gamma|$, the number of walks of length $l > 20$ we bound above by $|\gamma|(2l+97)2^{l-10}$ for simplicity. Lastly we bound

the number of walks intersecting γ with $l < 20$ by $M_{ww}(|\gamma|, l)$. Thus we have

$$\begin{aligned}\sum_{\gamma' \nmid \gamma} w(\gamma') \leq & \sum_{l' \geq 4, \text{ even}}^{20} M_{ww}(|\gamma|, l')w(l') + \sum_{l' \geq 5, \text{ odd}}^{19} M_{ww}(|\gamma|, l')w(l') \\ & + \sum_{l' \geq 6, \text{ even}}^{\infty} |\gamma| N(l')w(l') + \sum_{l' \geq 21}^{\infty} |\gamma|(2l' + 97)2^{l'-10}w(l')\end{aligned} \tag{4.47}$$

and so from the bound in Lemma 4.3.0.3 we have

$$\begin{aligned}\sum_{\gamma' \nmid \gamma} w(\gamma') \leq & \left(\frac{|\gamma|}{2} + 1\right) w(4) + \left(\frac{|\gamma|}{2} + 2\right) w(6) + \sum_{8 \leq l' \leq 20} \left(\frac{|\gamma|}{2} + l' - 2\right) R(l')w(l') + \sum_{5 \leq l' \leq 19 \text{ odd}} \left(\frac{|\gamma|}{2} + 1\right) w(l') \\ & + \sum_{28 \geq l' \geq 6, \text{ even}} |\gamma| N(l')w(l') + \sum_{l' \geq 30, \text{ even}}^{\infty} |\gamma| 2^{l'-3} w(l') + \sum_{l' \geq 21}^{\infty} |\gamma|(2l' + 97)2^{l'-10}w(l')\end{aligned} \tag{4.48}$$

Thus substituting in the numbers from the other tables we get the values in Table 5 and find

$$\sup_{\gamma \in \mathcal{W}_o^{>6}} \frac{1}{a(\gamma)} \sum_{\gamma' \nmid \gamma} w(\gamma') \leq 0.6503$$

Case 6: $\gamma \in \mathcal{L}^{>6}$**.** This proceeds as in Case 4 and 5. We bound the number of loops of length l intersecting $\gamma \in \mathcal{L}^{>6}$ by $|\gamma| N(l)$ as before; we bound the number of walks of length $l > 20$ intersecting γ by $|\gamma| P_i(l)$; we bound the number of walks of length $3 < l < 21$ by $M_{\ell w}(|\gamma|, l)$. Thus we have

$$\begin{aligned}\sum_{\gamma' \nmid \gamma} w(\gamma') \leq & \sum_{l' \geq 4, \text{ even}}^{20} M_{w\ell}(|\gamma|, l')w(l') + \sum_{l' \geq 5, \text{ odd}}^{19} M_{w\ell}(|\gamma|, l')w(l') \\ & + \sum_{l' \geq 6, \text{ even}}^{\infty} |\gamma| N(l')w(l') + \sum_{l' \geq 21}^{\infty} |\gamma|(2l' + 97)2^{l'-10}w(l')\end{aligned} \tag{4.49}$$

From the bounds given in Lemma 4.3.0.5 we have

$$\begin{aligned}\sum_{\gamma' \nmid \gamma} w(\gamma') \leq & \left(\frac{|\gamma|}{4} + \frac{1}{2}\right) w(4) + \left(\frac{|\gamma|}{4} + \frac{3}{2}\right) w(6) + \sum_{8 \leq l' \leq 20} \left(\frac{|\gamma|}{4} + l' - \frac{5}{2}\right) R(l')w(l') + \sum_{5 \leq l' \leq 19 \text{ odd}} \left(\frac{|\gamma|}{4} + \frac{1}{2}\right) w(l') \\ & + \sum_{28 \geq l' \geq 6, \text{ even}} |\gamma| N(l')w(l') + \sum_{l' \geq 30, \text{ even}}^{\infty} |\gamma| 2^{l'-3} w(l') + \sum_{l' \geq 21}^{\infty} |\gamma|(2l' + 97)2^{l'-10}w(l')\end{aligned} \tag{4.50}$$

Plugging in these bounds we obtain the numbers in Table 5 to get

$$\sup_{\gamma \in \mathcal{L}^{>6}} \frac{1}{a(\gamma)} \sum_{\gamma' \nmid \gamma} w(\gamma') \leq 0.4301$$

This final case combined with Theorem 3.1.1 proves the statement of Theorem 4.3.1 for $\epsilon = 0$. By the result of the appendix we have convergence for some $\epsilon > 0$ and so the theorem is proved.

□

As a general note one can compare our Table 5 with the Table IV in [19] for which many of the values coincide.

4.4. Proof of Theorem 4.0.1

We now give the proof of the main result.

Proof of Theorem 4.0.1. Fix any N, K such that $N > K + 20 > 40$. Since $\mathcal{P}^{\text{bulk}}_{N,K} \subseteq \mathcal{P}_{N,K}$, it follows by Theorem 4.3.1 that Theorem 4.2.1 holds for both $\mathcal{P}_{N,K}$ and $\mathcal{P}^{\text{bulk}}_{N,K}$. As a consequence, $\log(\Phi^{\text{bulk}}_{N,K}) = \log(Z(\mathcal{P}^{\text{bulk}}_{N,K}; W))$ has a convergent cluster expansion. This cluster expansion is real-valued (as the weight function $W(\gamma)$ and the Ursell function are both real-valued) and so $\Phi^{\text{bulk}}_{N,K} > 0$. Therefore, Lemma 4.1.0.4 holds, which when combined with (4.7) yields

$$|\langle \Psi(f), A\Psi(f)\rangle - \omega(A)| \leq \|A\| \left(\sup_{\Omega^{\partial\Lambda_N}} H\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}, \frac{\Phi_{N,K}}{Z_N} \right) + \limsup_M H\left(\frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N}, \frac{\Phi^{\text{bulk}}_{M,K}}{Z^{\text{bulk}}_M} \right) \right) \tag{4.1}$$

for any normalized ground state $\Psi(f) \in \mathcal{G}_{\Lambda_N}$ and observable $A \in \mathcal{A}_{\Lambda_{K-1}}$. Here, recall that ω is the unique infinite volume ground state, and $H(f,g) = D_\infty(f||g)e^{D_\infty(f||g)}$ where

$$D_\infty(f||g) = \|\log(f) - \log(g)\|_{L^\infty(d\rho_{\mathring{\Lambda}_K})}$$

is the classical Renyi-entropy. We proceed by using the cluster expansion for $\Phi^{\text{bulk}}_{N,K}$ and $\Phi_{N,K}$ with (4.26)-(4.27) to appropriately bound D_∞ for the two cases indicated in (4.1).
Since $Z^{\#}_N = \Phi^{\#}_{N,0}$ for $\# \in \{\cdot, \text{bulk}\}$, and Theorem 4.3.1 holds for $K = 0$, by Lemma 4.1.0.2-4.1.0.3,

$$\log\left(\frac{\Phi^{\#}_{N,K}}{Z^{\#}_N} \right) = \log\left(2^{|\mathcal{E}_{\mathring{\Lambda}_K}|} \right) + \sum_{C \in \mathcal{C}(\mathcal{P}^{\#}_{N,K})} W^T(C) - \sum_{C' \in \mathcal{C}(\mathcal{P}^{\#}_{N,0})} W^T(C') \,.$$

Let $\mathcal{D}^{\text{bulk}}_{M,N,K} = \mathcal{C}(\mathcal{P}^{\text{bulk}}_{M,K}) \setminus \mathcal{C}(\mathcal{P}^{\text{bulk}}_{N,K})$. Since $\mathcal{P}^{\text{bulk}}_{N,K} \subseteq \mathcal{P}^{\text{bulk}}_{M,K}$ for all $K \geq 0$ and $M \geq N$, inserting the above into (4.26) gives

$$\begin{aligned} D_\infty\left(\frac{\Phi^{\text{bulk}}_{M,K}}{Z^{\text{bulk}}_M} \middle\| \frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N} \right) &= \sup_{\Omega^{\partial\mathring{\Lambda}_K}} \left| \sum_{C \in \mathcal{D}^{\text{bulk}}_{M,N,K}} W^T(C) - \sum_{C \in \mathcal{D}^{\text{bulk}}_{M,N,0}} W^T(C) \right| \\ &\leq \sup_{\Omega^{\partial\mathring{\Lambda}_K}} \sum_{C \in \mathcal{D}^{\text{bulk}}_{M,N,K} \triangle \mathcal{D}^{\text{bulk}}_{M,N,0}} \left| W^T(C) \right| \end{aligned} \tag{4.2}$$

Similarly, define $\mathcal{D}_{N,K} = \mathcal{C}(\mathcal{P}_{N,K}) \setminus \mathcal{C}(\mathcal{P}^{\text{bulk}}_{N,K})$. Then, since $\mathcal{P}^{\text{bulk}}_{N,K} \subseteq \mathcal{P}_K$, by (4.27),

$$\sup_{\Omega^{\partial\Lambda_N}} D_\infty\left(\frac{\Phi_{N,K}}{Z_N} \middle\| \frac{\Phi^{\text{bulk}}_{N,K}}{Z^{\text{bulk}}_N} \right) \leq \sup_{\Omega^{\partial\Lambda_{N,K}}} \sum_{C \in \mathcal{D}_{N,K} \triangle \mathcal{D}_{N,0}} \left| W^T(C) \right| \tag{4.3}$$

The result will be a consequence of first determining the above symmetric differences and then appropriately applying Lemma 4.2.2.

Recall the definitions of the polymer sets from (4.10) and consider the symmetric difference from (4.2). Any cluster $C \in \mathcal{D}^{\text{bulk}}_{M,N,K}$ contains either a loop from $\mathcal{L}_{M,K}$ or walk from $\mathcal{W}^{\text{bulk}}_{M,K}$ that intersects $\Lambda_M \setminus \Lambda_N$, as otherwise the cluster belongs to $\mathcal{C}(\mathcal{P}^{\text{bulk}}_{N,K})$. Similarly, any cluster $C' \in \mathcal{D}^{\text{bulk}}_{M,N,0}$ contains a loop from $\mathcal{L}_{M,0}$ that intersects $\Lambda_M \setminus \Lambda_N$. Moreover, if every polymer in C' is supported on $\Lambda_{M,K}$, then C' also belongs to $\mathcal{D}^{\text{bulk}}_{M,N,K}$. From this, one can then easily deduce that

$$\mathcal{D}^{\text{bulk}}_{M,N,K} \triangle \mathcal{D}^{\text{bulk}}_{M,N,0} = \left\{ C \in \mathcal{C}(\mathcal{P}^{\text{bulk}}_{M,K}) \cup \mathcal{C}(\mathcal{P}^{\text{bulk}}_{M,0}) : \exists \gamma, \gamma' \in C \text{ s.t. } \gamma \nmid \mathring{\Lambda}_K,\ \gamma' \nmid (\Lambda_M \setminus \Lambda_N) \right\}. \tag{4.4}$$

Note that the polymers γ, γ' need not be distinct, and as any $C \in \Delta^{\text{bulk}}_{M,N,K}$ is a cluster, meaning $\cup_{\gamma \in C} \gamma$ is a connected graph, the set conditions imply that there is some $\gamma \in C$ that intersects $\partial \mathring{\Lambda}_N$.

To calculate the symmetric difference in (4.3), note that any cluster $C \in \mathcal{D}_{N,K}$ contains a walk γ that belongs to $\Lambda_{N,K}$ and has at least one endpoint on $\partial \Lambda_N$. Similarly, any cluster $C' \in \mathcal{D}_{N,0}$ contains a walk that belongs to $\Lambda_N = \Lambda_{N,K}$ and has both endpoints on $\partial \Lambda_N$. If all polymers from C' are supported on $\Lambda_{N,K}$, then $C' \in \mathcal{D}_{N,K}$. Thus, the symmetric differences is the set of clusters that intersect both $\partial \Lambda_N$ and $\mathring{\Lambda}_K$:

$$\mathcal{D}_{N,K} \triangle \mathcal{D}_{N,0} = \{ C \in \mathcal{C}(\mathcal{P}_{N,K}) \cup \mathcal{C}(\mathcal{P}_{N,0}) : \exists \gamma, \gamma' \in C \text{ s.t. } \gamma \nmid \partial \Lambda_N,\ \gamma' \nmid \mathring{\Lambda}_K \}. \tag{4.5}$$

We now turn to considering Lemma 4.2.2, which will be applied four times, once for each set of polymers used in the definitions of (4.4)-(4.5). To this end, for each $K' = K, 0$, define

$$A^{\text{bulk}}_{M,N,K'} = \{ \gamma \in \mathcal{P}^{\text{bulk}}_{M,K'} : \gamma \cap \partial \Lambda_N \neq \emptyset \}, \quad A_{N,K'} = \{ \gamma \in \mathcal{P}_{N,K'} : \gamma \cap \partial \Lambda_N \neq \emptyset \}, \tag{4.6}$$

and let $\gamma^{(K)}$ denote the shortest loop that encases $\mathring{\Lambda}_{K+1}$ (this is also the shortest loop that contains every vertex $x \in \partial \mathring{\Lambda}_{K+1}$). Note that $\gamma^{(K)}$ is common to all polymer sets we are considering:

$$\gamma^{(K)} \in \mathcal{P}^{\text{bulk}}_{M,K} \cap \mathcal{P}^{\text{bulk}}_{M,0} \cap \mathcal{P}_{N,K} \cap \mathcal{P}_{N,0}.$$

Let $d(C) = \sum_{\gamma \in C} \epsilon |\gamma|$ for any cluster C where $\epsilon > 0$ is as in Theorem 4.3.1. Then, for any of the four sets A from (4.6), which is a subset of the polymer set $\mathcal{P}$, the conditions on A imply

$$d(\gamma^{(K)}, A) := \inf\{ d(C) : C \in \mathcal{C}(\mathcal{P}),\, C \nmid \gamma^{(K)},\, C \cap A \neq \emptyset \} \geq 2\epsilon(N - K) - \epsilon.$$

Here, we use $C \nmid \gamma$ means there exists $\gamma' \in C$ such that $\gamma' \nmid \gamma$ and that $\text{dist}(\partial \Lambda_N, \gamma^{(K)}) = 2(N - K) - 1$ with respect to the graph distance.

Therefore, applying Lemma 4.2.2 twice shows,

$$\sum_{C\in\mathcal{D}_{N,K}\triangle\mathcal{D}_{N,0}}|W^T(C)|\leq\sum_{\substack{C\in\mathcal{C}(\mathcal{P}_{N,K}):\\ C\cap A_{N,K}\neq\emptyset,\\ C\nmid\gamma^{(K)}}}|W^T(C)|+\sum_{\substack{C\in\mathcal{C}(\mathcal{P}_{N,0}):\\ C\cap A_{N,0}\neq\emptyset,\\ C\nmid\gamma^{(K)}}}|W^T(C)|\leq 2a(\gamma^{(K)})e^{-2\epsilon(N-K)+\epsilon}$$

where the final expression is independent of the boundary variables $(\Omega_x : x \in \partial\Lambda_{N,K})$. Since $a(\gamma^{(K)}) = .15|\gamma^{(K)}| = .9(2K+1)$, it immediately follows from (4.3) that

$$\sup_{\Omega^{\partial\Lambda_N}} D_\infty\left(\frac{\Phi_{N,K}}{Z_N}\middle\|\frac{\Phi_{N,K}^{\text{bulk}}}{Z_N^{\text{bulk}}}\right)\leq 1.8(2K+1)e^{-2\epsilon(N-K)+\epsilon}:=F(N,K)$$

Analogously, again applying Lemma 4.2.2 twice and arguing as above shows

$$D_\infty\left(\frac{\Phi_{M,K}^{\text{bulk}}}{Z_M^{\text{bulk}}}\middle\|\frac{\Phi_{N,K}^{\text{bulk}}}{Z_N^{\text{bulk}}}\right)\leq F(N,K)$$

Therefore, by (4.1) and the definition of $H(f,g)$), one obtains the desired result

$$|\langle\Psi(f),A\Psi(f)\rangle-\omega(A)|\leq 2F(N,K)e^{F(N,K)}\,.$$

□

4.5. Decorated Cases

We have not directly showed LTQO and ground state indistinguishability for the decorated models for $0 < d < 5$. The cases of the decorated models can be addressed by modifying the cluster expansion with an added decay coefficient (and a slight abuse of notation) $d(\gamma) = d\ln(4.8)$ where d is the decoration number. This cluster expansion clearly converges by direct comparison, and the same bounds hold as in the Kotecký-Preiss case, with the added and greatly increased decay function. Thus one gets LTQO and ground state indistinguishability for this model via the arguments used here and those of [27].

```
import matplotlib.pyplot as plt
import time
import numpy as np
rr=3
\#significant digits
def starts2(n):
\#generates starting points of paths
    s=[]
    for i in range(0,n+1):
        s.append([[-.5-i*1.5,round(3**.5/2*(1-i),rr)],[-1-i*1.5,round
            (3**.5/2*(2-i),rr)]])
    return(s)
def ends2(n,o):
\#generates ending points, second input generates inner corner if o==0,
    otherwise outer corner
    e=[]
    for i in range(0,n+1):
        if(o==0):
            e.append([[(.5+i*1.5),(round(3**.5/2*(1-i),rr))],[(1+i*1.5),(
                round(3**.5/2*(2-i),rr))]])
        elif(o==1):
            e.append([[.5,round(2*(i+1.5)*3**.5/2,rr)],[-.5,round(2*(i+1.5)
                *3**.5/2,rr)]])
    return(e)
def hits(a,b):
\#returns true if two paths a,b intersect at a vertex
    for i in a:
        if i in b:
            return(True)
    return(False)
def lasts(a):
```

```
\#returns last entry of each element in a list of lists
    b=[]
    for i in a:
        b.append(i[-1])
    return(b)
def firsts(a):
\#returns first entry of each element in a list of lists
    b=[]
    for i in a:
        b.append(i[0])
    return(b)
def saw(n,sp,ep,inter,o):
\#generates self-avoiding walks of length n, that start at sp, end at ep,
    intersects the path inter, inner corner o==0 or outer corner o==1
    pths=[]
    for a in sp:
        p=a
        for z in range(0,2**(n-1)):
            d=0
            p=p[:2]
            v=bin(z)[2:].zfill(n-1)
            for e,j in enumerate(v):
                r1=round(p[-1][0]-p[-2][0],rr)
                r2=round((p[-1][1]-p[-2][1]),rr)
                p0=[round(p[-1][0]+r1/2-3**.5/2*r2,rr),round(p
                    [-1][1]+3**.5/2*r1+r2/2,rr)]
                p1=[round(p[-1][0]+r1/2+3**.5/2*r2,rr),round(p
                    [-1][1]-3**.5/2*r1+r2/2,rr)]
                if(j=='0' and p0 in p):
                    d=1
                    break
                elif(j=='0' and p0 not in p):
```

```
                    if(p[-1] in firsts(ep) and e!=len(v)-1):
                        d=1
                        break
                    else:
                        p.append(p0)
                elif(j=='1' and p1 in p):
                    d=1
                    break
                elif(j=='1' and p1 not in p):
                    if(p[-1] in firsts(ep) and e!=len(v)-1):
                        d=1
                        break
                    else:
                        p.append(p1)
            if (p[-1] in firsts(ep)):
                if(hits(p,inter)):
                    if(d!=1):
                        if(not(hits(p[1:-1],firsts(ep)+firsts(sp)))):
                            if(p[::-1] not in pths):
                                pths.append(p)

    return(pths)
plt.scatter(*np.array(firsts(ends2(u,0))).T)
\#plot start points
plt.scatter(*np.array(firsts(starts2(u))).T)
\#plot end points
```

$M_{ab}^N(l,l')$	$(a,l)=(w,3)$	$(a,l)=(w,4)$	$(a,l)=(w,5)$	$(a,l)=(w,6)$	$(a,l)=(\ell,6)$
$(b,l')=(w,3)$	1	2	2	3	2
$(w,4)$	2	2	2	3	2
$(w,5)$	2	1	1	1	1
$(w,6)$	2	2	2	3	2
$(w,7)$	6	2	2	2	2
$(w,8)$	8	9	8	12	10
$(w,9)$	14	7	7	7	7
$(w,10)$	18	22	18	27	24
$(w,11)$	38	20	20	20	20
$(w,12)$	52	70	56	78	70
$(w,13)$	106	62	61	62	62
$(w,14)$	150	224	164	225	221
$(w,15)$	296	193	186	193	193
$(w,16)$	428	655	494	644	641
$(w,17)$	868	606	568	606	606
$(w,18)$	1284	2084	1516	1940	2066
$(w,19)$	2530	1930	1760	1930	1930
$(w,20)$	3818	6504	4692	5793	6578
$(\ell,6)$	1	2	2	3	7
$(\ell,10)$	3	7	6	10	30

TABLE 4.1. Exact values of $M_{ab}^N(l,l')$ for various $3 \le l \le 6$ and $4 \le l' \le 20$

$M_{ab}^K(l,l')$	$(a,l)=(w,4)$	$(a,l)=(w,5)$	$(a,l)=(w,6)$	$(a,l)=(\ell,6)$
$(w,4)$	3	2	4	2
$(w,5)$	0	1	0	1
$(w,6)$	4	2	5	3
$(w,7)$	0	2	0	2
$(w,8)$	15	8	19	13
$(w,9)$	1	8	1	8
$(w,10)$	38	18	48	37
$(w,11)$	3	24	3	24
$(w,12)$	111	54	143	113
$(w,13)$	15	78	14	79
$(w,14)$	329	158	431	360
$(w,15)$	48	246	44	254
$(w,16)$	964	466	1279	1095
$(w,17)$	182	782	166	824
$(w,18)$	2929	1426	3930	3459
$(w,19)$	608	2480	565	2698
$(w,20)$	8859	4366	12020	10783
$(\ell,6)$	2	3	3	7
$(\ell,10)$	7	11	10	30

TABLE 4.2. Exact values of $M_{ab}^K(l,l')$ for various $4 \le l \le 6$ and $4 \le l' \le 20$

	$(a,l)=(w,3)$	$(a,l)=(w,4)$	$(a,l)=(w,5)$	$(a,l)=(w,6)$	$(a,l)=(\ell,6)$
$M_{aw}(l,3)$	1	2	2	3	2
$S_{aw}(l,4)$	2	2	2	3	2
$S_{aw}(l,5)$	2	1	1	1	1
$S_{aw}(l,6)$	2	2	2	3	2
$S_{aw}(l,7)$	6	2	2	2	2
$S_{aw}(l,8)$	8	9	8	12	10
$S_{aw}(l,9)$	14	7	7	7	7
$S_{aw}(l,10)$	18	22	18	27	24
$S_{aw}(l,11)$	38	20	20	20	20
$S_{aw}(l,12)$	52	70	56	78	70
$S_{aw}(l,13)$	106	62	61	62	62
$S_{aw}(l,14)$	150	224	164	225	221
$S_{aw}(l,15)$	296	193	186	193	193
$S_{aw}(l,16)$	428	655	494	644	641
$S_{aw}(l,17)$	868	606	568	606	606
$S_{aw}(l,18)$	1284	2084	1516	1940	2066
$S_{aw}(l,19)$	2530	1930	1760	1930	1930
$S_{aw}(l,20)$	3818	6504	4692	5793	6578
$M_{a\ell}(l,6)$	1	2	2	3	7
$M_{a\ell}(l,10)$	3	7	6	10	30

TABLE 4.3. The tables for $M^N_{ab}(l,l')$ and $M^K_{ab}(l,l')$ should be replaced with one table with these labels for the rows and columns. Note: the numerical values in the table need to be updated for correctness. Moreover, the values in the first column are really the numbers for $M_{ww}(3,l')$ since $S_{ww}(l,l')$ is really only defined for $l \geq 4$. Perhaps the first column should be made into its own table?

$N^\Gamma_\ell(l)$ for $l<30$	
l	$N^\Gamma_\ell(l)$
6	2
8	0
10	10
12	8
14	56
16	96
18	390
20	920
22	3168
24	8592
26	28002
28	81368

TABLE 4.4. $N^\Gamma_\ell(l)$ for $l<30$

Number of walks straddling a corner of odd length $l<20$	
l	$Q_o(l)$
5	1
7	2
9	7
11	20
13	62
15	193
17	606
19	1930

TABLE 4.5. $Q_o(l)$ for odd $l<20$

$N(l)$ for $l < 11$	
l	$N(l)$
1	1
2	2
3	2
4	4
5	6
6	8
7	16
8	24
9	40
10	64

TABLE 4.6. Maximum number of walks $N(l)$ from a fixed site to the boundary of length l

$\gamma'\backslash\gamma$	$\mathcal{W}^4$	$\mathcal{W}^5$	$\mathcal{W}^6$	$\mathcal{L}^6$	$\mathcal{W}^{>6}$	$\mathcal{L}^{>6}$
$\mathcal{W}^4$	0.2316	0.1965	0.2779	0.1853	0.2779	0.1297
$\mathcal{W}^5$	0.0427	0.0362	0.0341	0.0341	0.0227	0.0159
$\mathcal{W}^6$	0.0296	0.0251	0.0355	0.0237	0.0434	0.0221
$\mathcal{W}^7$	0.014	0.0119	0.0112	0.0112	0.0075	0.0052
$\mathcal{W}^8$	0.0244	0.0184	0.0282	0.0217	0.0549	0.0324
$\mathcal{W}^9$	0.0073	0.0062	0.0059	0.0059	0.0039	0.0027
$\mathcal{W}^{10}$	0.0089	0.0062	0.0098	0.0078	0.0224	0.0137
$\mathcal{W}^{10<l<21}$	0.0136	0.0099	0.0123	0.0109	0.025	0.016
$\mathcal{W}^{>20}$	0.0386	0.0462	0.0562	0.076	0.0591	0.0591
$\mathcal{L}^6$	0.0296	0.0377	0.0355	0.0829	0.1105	0.1105
$\mathcal{L}^{10}$	0.0028	0.0038	0.0033	0.0098	0.0152	0.0152
$\mathcal{L}^{>10}$	0.0041	0.0052	0.0066	0.0098	0.0077	0.0077
Total	0.4473	0.4034	0.5164	0.479	0.6503	0.4301

TABLE 4.7. Each column must sum to < 1 for the Kotecký-Preiss condition to be satisfied

$R(l)$ for even l with $2 < l < 22$	
l	$R(l)$
4	1
6	1
8	4
10	9
12	26
14	75
16	215
18	649
20	1943

TABLE 4.8. Number of walks hitting a fixed bond in the boundary and ending to the left of the fixed bond

CHAPTER 5

Conclusion and Outlook

5.1. Conclusion

The three foregoing sets of results are a way of exploring the macroscopic behavior of the ground states of the AKLT model on graphs of varying dimension.

In **Section 2**, we saw that in quasi-one-dimension, as in the case of the one-dimensional chain, all AKLT models constructed in a crystalline way (repeating a bipartite graph linearly such that the resulting concatenation is bipartite) have a stably-gapped ground state with exponential decay of correlations. The model, by virtue of being antiferromagnetic, would be expected to have some kind of antiferromagnetic (i.e. Néel) long-range ordering if the ground states have such an order; the predicted lack of antiferromagnetic ordering in one dimension [31] thus extends to these quantum models where the underlying classical model violates the assumption that we have bounded infinite-volume dynamics; this is because the points of divergence are in regions of exponentially low probability, and also this divergence occurs on a set of measure zero in the configuration space. Thus, while we have slightly extended the set of one-dimensional phases for which we have a stably-gapped unique ground state, the result agrees totally with existing predictions [3], [31], [9].

In **Section 3** we saw that a sufficient growth condition on the number of paths in the graphs (both for irregular trees and for tree-like graphs) led to a long-range order parameter in the form of Néel order and thus ground state degeneracy in these models. This finding is consistent with work on trees [10], [3], [44] that implies the splitting number has a crucial; we establish that the geometric mean of the splittings having average $\mu > 3$ for all sequences of points radiating from the origin creates a long-ranged ordered model.

In **Section 4** we found that the AKLT model on the hexagonal lattice has local topological quantum order and thus the spectral gap above the ground state is stable under sufficiently local perturbations. This is consistent with the expected behavior conjectured in [19].

5.2. Outlook: Refined Conjectures for General Lattices

Finally, we give conjectured behavior for adding layers to bilayer trees.

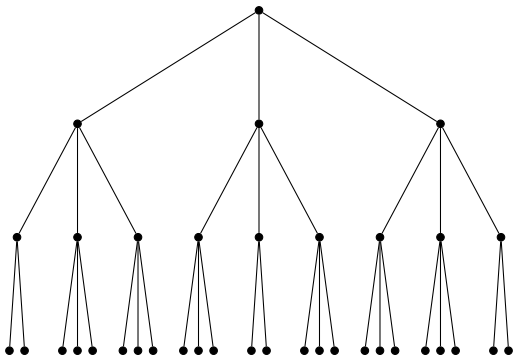


FIGURE 5.1. Levels $2, 3, 4$ of the Lichtenberg figure for the square lattice. Note the missing edges in Layer 4 distinguish it from a degree 4 Cayley tree

CONJECTURE 5.2.1. For the bilayer Cayley tree case, the ground state of the AKLT model is not unique for all $g \geq 3$ where g is the splitting number. For the n-layer Cayley tree case with $n \geq 3$, the ground state is not unique for $g \geq 2$.

We also conjecture some behavior for general graphs which agrees, though is more explicit than the prediction of [3]; the reasoning behind this conjecture is by analogy with the above result involving the connective constant. We construct an associated tree with similar behavior to our lattice.

DEFINITION 5.2.1. Let $\overline{0} \in \mathcal{V}_\Gamma$ be a distinguished origin point of a lattice Γ; define the tree $\mathcal{T}_\Gamma$ of Γ in the following way: $\mathcal{T}_0 = \{\overline{0}\}$, $\mathcal{T}_i = \{(x, y) \, : \, x \in \mathcal{T}_{i-1}, y \in \Gamma, \gamma(\overline{0}, x) \text{ is self avoiding}\}$ where $\gamma(\overline{0}, x)$ is the unique path defined by $\gamma = (\overline{0}, x_1, ..., x_i, y)$ where $x_k \in \mathcal{T}_k$. We call this tree the **Lichtenberg figure** of the graph.

We also give the following definition:

DEFINITION 5.2.2. [29] Let c_n be the number of self-avoiding walks of length n starting at any point in a regular lattice Γ. Then $\lim_{n\to\infty} c_n^{1/n} = \mu_\Gamma$ exists and is called the **connective constant** of Γ.

We conjecture that when $\mu > 3$, the associated Lichtenberg figure obeys the conditions of Theorem 3.5.3, and so will have long-range order. The conditions of the previous theorem will be connected to the growth rate of possible space-filling walks in the graph, which should grow exponentially as the connective constant. Moreover, we conjecture this behavior extends to the lattice itself when the dimension $d > 1$. Thus we state the following:

CONJECTURE 5.2.2. For a general regular bipartite lattice Γ of dimension $d > 1$, if the connective constant $\mu_\Gamma > 3$, then both the Lichtenberg figure associated to Γ and Γ itself have the property that the AKLT ground state is not unique and is antiferromagnetically ordered on these graphs in the infinite-volume limit.

FIGURE 5.2. Lichtenberg figure in perspex block showing fern-like pattern of ionized paths through the material produced by corona discharge from a 2 million volt Van de Graaf electric generator, 1956; courtesy of National Museums of Scotland [1]

In general, the usefulness of tree models is that they allow one to create a one-parameter dynamical system as we did with our transfer function. In inherently higher-dimensional lattices, it becomes harder to get useful bounds on these transfer functions; however, lower bounds may exist in terms of these modified trees; very garish bounds can be used [19], [3] to give asymptotic behavior, but detailed analysis of where critical points lie is generically very hard.

The heuristic reasoning behind Conjecture 5.2.2 is that the difference between the weight function $F_{\Gamma_n}(t)$ for Γ_n and the weight function $F_{\mathcal{T}_{\Gamma_n}}(t)$ for the Lichtenberg figure $\mathcal{T}_{\Gamma_n}$ for Γ_n is expected to be such that

$$F_{\Gamma_n}(t) \sim F_{T_{\Gamma_n}}\left(\frac{\kappa_n}{\mu^n} \cdot t\right) \tag{5.1}$$

where $\kappa_n \geq \epsilon > 0$ for all n. As $n \to \infty$ we get this implies $F_\Gamma(t) > 0$ when $\mu > 3$. This direct comparison is difficult to make in general, but it is expected that the noise parameter $\frac{\kappa_n}{\mu^n}$ has uniformly lower bounded numerator κ_n since the number of loops intersecting a path grows at the same rate as the number of self-avoiding walks [24], so it cannot decay faster. In general, the number of such walks also has a universal polynomial corrective term which is still unproven but suspected [24].

Observing Figure 5.3 we see the original conjecture of AKLT about the macroscopic behavior of these models. The dotted line in the diagram corresponds to the SU(2) AKLT model which is the subject of this thesis; there is an SU(n) generalization of these results, however the transfer function used in 3.5.3 is more difficult to construct; however a simple argument from [8] can be used to show that in the SU(n)

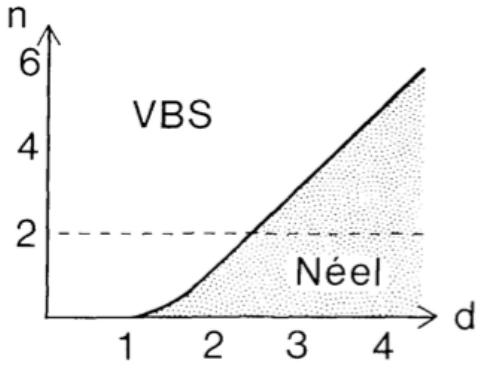


FIGURE 5.3. Conjectured phase diagram for SU(n) models [3] based on dimension d

chain the correlation length is $\frac{1}{\ln(n+1)}$ and so one would suspect that we would have an ordering on any graph with $\mu > n+1$ by similar reasoning.

Although the AKLT conjecture is in terms of the dimension, the dimensionality and connectivity of the lattice are both quantified by the connective constant, so we suspect the conjecture holds. Note the region $d \leq 1$ is verified in this thesis and likely has an easy extension to the SU(n) models.

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