Affine Springer fibers and triply-graded link homology

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To all my friends and family.

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Abstract

We find the Borel-Moore homology of unramified affine Springer fibers for GL_n under the assumption that they are equivariantly formal and relate them to generalized Haiman ideals. For $n = 3$, we give an explicit description of these ideals, compute their Hilbert series, generators and relations, and compare them to generalized (q,t)-Catalan numbers. We also compute the triply-graded Khovanov-Rozansky homology for Coxeter braids on up to 4 strands and compare the result, proving a version of a conjecture of Oblomkov, Rasmussen, and Shende in this case.

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CHAPTER 1

Introduction

In this thesis, we study the relationship between affine Springer fibers and Khovanov-Rozansky homology of links using geometric, algebraic, and combinatorial techniques. In addition to link homology, affine Springer fibers are related to the Langlands program [25], Hilbert schemes of singular curves [28, 29], and to coherent sheaves on the Hilbert scheme of $\mathbb{C}^2$ [18]. Sometimes these relationships are discussed in terms of compactified Jacobians, which are equivalent to affine Springer fibers [26].

Given a matrix $\gamma \in \mathfrak{gl}_n((t))$ (or $\mathfrak{sl}_n((t))$), the affine Springer fiber Sp_γ is a certain ind-subvariety of the affine Grassmanian, see Definition 2.1.3. Its characteristic polynomial $p(\lambda, t) = \det(\gamma - \lambda I)$ defines a singular curve C_γ in $\mathbb{C}^2$, called the spectral curve of γ. As long as C_γ is reduced, Sp_γ depends only on its spectral curve C_γ, in particular on its completion at the origin $(C_\gamma, 0)$. In this paper we will work with γ with distinct eigenvalues so that C_γ will be reduced.

Fact 1.0.1 ([34]). If $(C_\gamma, 0)$ is irreducible, then Sp_γ is a projective variety, but if $(C_\gamma, 0)$ is not irreducible, Sp_γ is an ind-variety with infinitely many irreducible components.

Intersecting the C_γ with a small sphere around the origin (where C_γ is often singular) gives a link L_γ in S^3. Each irreducible component of $(C_\gamma, 0)$ corresponds to a component of the link, and the intersection numbers of irreducible components are the linking numbers of the corresponding link components. Any smooth components of the curve correspond to unknots. The class of links that can be realized from an algebraic curve in this way are called algebraic links.

Oblomkov, Rasmussen, and Shende [31] have conjectured that for all algebraic links, the homology of Sp_γ is closely related to the triply-graded Khovanov-Rozansky homology [22, 23] (also called HHH) of L_γ. This relationship has previously been shown for all torus knots, and for (n, nd)-torus links by Kivinen in [24].

Conjecture 1.0.2 (ORS [31]). *If L_γ is the link associated to γ, we have*

$$\mathrm{gr}_P H_*(\mathrm{Sp}_\gamma) \otimes \mathbb{C}[\mathbf{x}] \cong \mathrm{HHH}^{a=0}(L_\gamma)$$

where gr_P is a certain perverse filtration on $H_*(\mathrm{Sp}_\gamma)$.

Example 1.0.3. For the matrix

$$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix},$$

the characteristic polynomial is given by $p(\lambda, t) = \lambda^2 - t^3$. The associated link to this curve in $\mathbb{C}^2$ is a trefoil. The affine Springer fiber Sp_γ is isomorphic to $\mathbb{P}^1$, and the reduced HHH homology of the trefoil is isomorphic to $H_*(\mathbb{P}^1)$.

In this thesis, we calculate the homology for a large class of affine Springer fibers with γ of the form

$$\gamma = \begin{pmatrix} z_1 t^{d_1} & & 0 \\ & \ddots & \\ 0 & & z_n t^{d_n} \end{pmatrix}, \tag{1.1}$$

with $z_i \in \mathbb{C}^*$ pairwise distinct and $d_i > 0$, under the assumption that Sp_γ is equivariantly formal (see Definition 2.1.6).

The characteristic polynomial of this γ is $\prod_i (\lambda - z_i t^{d_i})$. The corresponding curve C_γ has n smooth irreducible components with pairwise intersection numbers $d_{ij} = \min(d_i, d_j)$. So the corresponding link L_γ is a link of n unknots with pairwise linking numbers d_{ij}. These are part of a class of links called Coxeter links, which can be expressed as the closure of a Coxeter braid.

In Chapter 4, we calculate HHH of Coxeter braids L_γ for $n = 3$ and $n = 4$, and explicitly compare the result to $H_*(\mathrm{Sp}_\gamma)$ for $n = 3$.

1.1. Affine Springer fibers

In Chapter 2, we focus on computing the equivariant Borel-Moore homology $H_*^T(\mathrm{Sp}_\gamma)$ with respect to the natural torus action of $T = (\mathbb{C}^*)^n$ on Sp_γ (explained in Section 2.1). Given the assumption that Sp_γ is equivariantly formal with respect to T, we can recover the ordinary Borel-Moore homology of Sp_γ by quotienting out by the equivariant parameters, see Fact 2.1.7.

In order to calculate $H_*^T(\mathrm{Sp}_\gamma)$, we view it as a module over

$$R = H_T^*(pt) \otimes \mathbb{C}[\Lambda] \cong \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm].$$

Here, the t_i's are our equivariant parameters, and the x_i's parametrize the integer lattice Λ which acts on Sp_γ by translations.

In [10], Goresky, Kottwitz, and MacPherson (GKM) conjectured that Sp_γ is pure for all unramified (i.e. diagonal) γ. The following is a more narrow conjecture that is what we will rely on in this paper.

Conjecture 1.1.1 ([10]). *For γ as above, Sp_γ is equivariantly formal, as defined in Definition 2.1.6.*

Assuming that Sp_γ is equivariantly formal, we can calculate its equivariant Borel-Moore homology.

Theorem 1.1.2. *Consider γ as in (1.1). Define the ideal*

$$\mathcal{J} = \mathcal{J}_\gamma = \bigcap_{i<j} (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq R$$

with $d_{ij} = min(d_i, d_j)$. If Sp_γ is equivariantly formal, then as R-modules,

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J} \quad \text{where} \quad \Delta = \prod_{i<j} (t_i - t_j)^{d_{ij}}.$$

These $\mathcal{J}$ ideals are a generalization of ideals considered by Haiman in his work on the Hilbert schemes of points [9], so we refer to them as generalized Haiman ideals, or just Haiman ideals. Since R is a domain, multiplication by Δ is injective, and

$$H_*^T(\mathrm{Sp}_\gamma) \cong \Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J}$$

as modules over $R = \mathbb{C}[t_1, \dots, t_n, \mathsf{x}_1^\pm, \dots, x_n^\pm]$. It is still useful to keep track of Δ if we want to retain the localization information of $H_*^T(\mathrm{Sp}_\gamma)$, but we can omit Δ when we only care about $H_*^T(\mathrm{Sp}_\gamma)$ as an R-module. Given the assumption that Sp_γ is equivariantly formal, we can recover the ordinary Borel-Moore homology of Sp_γ as well by simply quotienting by the action of t's.

Corollary 1.1.3. *For γ as in (1.1), if Sp_γ is equivariantly formal, then*

$$H_*(\mathrm{Sp}_\gamma) \cong \mathcal{J}/(\mathbf{t})\mathcal{J}.$$

Here $(\boldsymbol{t}) \subseteq H^*_T(pt) \cong \mathbb{C}[\mathbf{t}]$ is the maximal ideal generated by $t_1, \dots, t_n$.

If $n = 3, 4$, it is known that Sp_γ is equivariantly formal, shown in [27] and [7] respectively. It is also known for the equivalued case, when $d_i = d$ for all i, due to GKM [10].

Corollary 1.1.4. *If $n \leq 4$, or if $d_i = d$ for all i, then*

$$\Delta H^T_*(\mathrm{Sp}_\gamma) \cong \mathcal{J} = \bigcap_{i<j} (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

The equivalued case of 1.1.4 was previously shown by Kivinen in [24] using GKM theory as defined in [12]. The proof of Theorem 1.1.2 relies on this result by Kivinen.

1.1.1. The Coulomb branch algebra. In [15] Gorsky, Kivinen and Oblomkov define a graded algebra with some specific properties called the graded Coulomb branch algebra $\mathcal{A}_G = \bigoplus_{d=0}^\infty \mathcal{A}_d$. Here we consider the case $G = \mathrm{GL}_n$. One of the key properties is that for any $\gamma \in \mathfrak{g}$, the direct sum of homologies of affine Springer fibers

$$F_\gamma = \bigoplus_{k=0}^\infty H_*(\mathrm{Sp}_{t^k\gamma})$$

is a graded module over $\mathcal{A}_G$ or, equivalently, that there is a corresponding quasi-coherent sheaf $\mathcal{F}_\gamma$ on Proj $\bigoplus_{d=0}^\infty \mathcal{A}_d$. They conjecture the following.

Conjecture 1.1.5 ([15]). *The module F_γ is finitely generated and the sheaf $\mathcal{F}_\gamma$ is coherent.*

Theorem 1.1.6. *Conjecture 1.1.5 holds for $G = \mathrm{GL}_3$ and γ as in (1.1).*

This result and many of the results in Section 1.2 rely on the specific combinatorics of the ideal $\mathcal{J}$ when $n = 3$, which is covered in detail in Chapter 3.

1.2. Generalized Haiman ideals

For the rest of the introduction we assume that the d_i's are ordered: $d_1 \leq \ldots \leq d_n$. We will consider a similar ideal to $\mathcal{J}$ defined above,

$$J'(d_1,\ldots,d_n) = \bigcap_{i<j}(t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq \mathbb{C}[t_1,\ldots,t_n,x_1,\ldots,x_n].$$

The ideal $\mathcal{J}$ is obtained from $J'(d_1,\ldots,d_n)$ by localization in $(x_1\cdots x_n)$.

In Section 3.1.1, we define two rational functions, $H(d_1,\ldots,d_n)$ and $F(d_1,\ldots,d_n)$. The function $F(d_1,\ldots,d_n)$ is also known as the generalized (q,t)-Catalan number, see [**13**].

Conjecture 1.2.1. *a) The Hilbert series of the ideal $J'(d_1,\ldots,d_n)$ equals $H(d_1,\ldots,d_n)$.*
b) The Hilbert series of the generating set $J'(d_1,\ldots,d_n)/\mathfrak{m}J'(d_1,\ldots,d_n)$ equals $F(d_1,\ldots,d_n)$, where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} = (t_1,\ldots,t_n,x_1,\ldots,x_n)$.

In particular, Conjecture 1.2.1 implies that $F(d_1,\ldots,d_n)$ is a polynomial in q and t with nonnegative coefficients (see [**13**, Conjecture 1.3]) and provides an explicit algebraic interpretation of these coefficients. Similarly, the conjecture implies that $H(d_1,\ldots,d_n)$ is a power series in q and t with nonnegative coefficients. In Theorem 3.1.10, we show that this Conjecture holds for $n = 3$.

Theorem 1.2.2. *Conjecture 1.2.1 holds for $n = 3$.*

If $d_i = d$ for all i, we will say that the ideal J' is equivalued. In [**9**], Haiman shows the following.

Theorem 1.2.3 (Haiman [**9**]). *For any n,*

1. *The ideal $J'(d,\ldots,d)$ is free as a $\mathbb{C}[\mathbf{t}]$-module.*
2. *The ideal $J'(d,\ldots,d)$ is equal to a product,*

$$J'(d,\ldots,d) = J'(1,\ldots,1)^d.$$

It is easy to see that in general

$$J'(d_1,\ldots,d_n)\cdot J'(d'_1,\ldots d'_n) \subseteq J'(d_1+d'_1,\ldots,d_n+d'_n).$$

We conjecture that Theorem 1.2.3 can be generalized to the non-equivalued case, and that the above inclusion is always an equality.

CONJECTURE 1.2.4. *For any n, assume $d_1 \le d_2 \le \cdots \le d_n$. Then,*

1. *The ideal $J'(d_1, \ldots, d_n)$ is free as a $\mathbb{C}[\mathbf{t}]$-module.*
2. *The ideal $J'(d_1, \ldots, d_n)$ can be written as the product*

$$J'(d_1, \ldots, d_n) = J'(1, \ldots, 1)^{d_1} \cdot J'(0, 1, \ldots, 1)^{d_2 - d_1} \cdot \ldots \cdot J'(0, \ldots, 0, 1)^{d_{n-1} - d_{n-2}}.$$

Statement (1) immediately follows in any cases where Sp_γ is known to be equivariantly formal, in particular for $n \le 4$. In Corollary 3.1.5 we show that statement (2) holds in the $n = 3$ case.

THEOREM 1.2.5. *The ideal $J'(d_1, d_2)$ can be written as a product*

$$J'(d_1, d_2) = J'(1, 1)^{d_1} \cdot J'(0, 1)^{d_2 - d_1}$$

1.3. Khovanov-Rozansky homology of Coxeter braids

Khovanov-Rozansky homology, also called HHH, is a triply-graded link homology theory that categorifies the HOMFLY-PT polynomial, which is itself a generalization of the Jones and Alexander polynomials. It is a powerful but often difficult to compute invariant.

In Chapter 4, we use a recursive process of Hogancamp and Elias [8] to compute $\mathrm{HHH}(L_\gamma)$ for closures of Coxeter braids on 3 and 4 strands and show that they are parity (see Definition 1.3.1). Previously, this method has been used by Hogancamp to compute HHH for $T(n, dn)$ torus links in [19], and by Hogancamp and Mellit to compute HHH for all torus links in [20].

Given integers $0 \le d_1 \le d_2 \le \cdots \le d_n$, we define the n-stranded braid

$$\beta(d_1, \ldots, d_n) = \mathrm{FT}_n^{d_1} \mathrm{FT}_{n-1}^{d_2 - d_1} \cdots \mathrm{FT}_2^{d_{n-1} - d_{n-2}}$$
$$= \mathrm{JM}_n^{d_1} \, \mathrm{JM}_{n-1}^{d_2} \cdots \mathrm{JM}_2^{d_{n-1}},$$

where FT_k and JM_k represent a full twist and a Jucys-Murphy element on the first k strands respectively. These braids are part of the family of Coxeter braids defined in [30]. For the sake of comparison to the ideal $J'(d_1, \ldots, d_n)$, we focus on pure Coxeter braids, whose closures have the maximum number of components. We expect that our calculations of HHH can be extended to all Coxeter braids as defined in [30].

The three gradings of $\mathrm{HHH}(\beta)$ are typically denoted Q, T, and A. However, in line with [8], we use the change of variables

$$q = Q^2, \quad t = T^2 Q^{-2}, \quad a = AQ^{-2}.$$

DEFINITION 1.3.1. For any braid β, we say that $\mathrm{HHH}(\beta)$ is parity if it is supported in only even homological (T) degrees. We will also say β itself is parity if $\mathrm{HHH}(\beta)$ is parity.

CONJECTURE 1.3.2. *The Coxeter braid $\beta(d_1, \ldots, d_n)$ is parity for all n and for all $0 \le d_1 \le \cdots \le d_n$.*

If Conjecture 1.3.2 holds, the following theorem of Gorsky and Hogancamp gives a description of the $a = 0$ piece of HHH, using the y-ified Khovanov-Rozansky homology HY as defined in [14].

THEOREM 1.3.3 (Gorsky, Hogancamp [14]). *Assume that $\beta = \mathrm{JM}_n^{d_1} \ldots \mathrm{JM}_2^{d_{n-1}}$ and $\mathrm{HHH}^{a=0}(\beta)$ is parity. Then*

(1) $\mathrm{HY}^{a=0}(\beta) = \mathrm{HHH}^{a=0}(\beta) \otimes \mathbb{C}[t_1, \ldots, t_n]$ *and* $\mathrm{HHH}^{a=0}(\beta) = \mathrm{HY}^{a=0}(\beta)/(y)$

(2) $\mathrm{HY}^{a=0}(\beta) = J'(d_1, \ldots, d_n)$.

In particular, this also implies that the ideal $J'(d_1, \ldots, d_n)$ is free over $\mathbb{C}[t_1, \ldots, t_n]$.

Putting this together with the conjectures and results from Section 1.1, we get the following specialization of the ORS Conjecture (Conjecture 1.0.2).

CONJECTURE 1.3.4. *For γ as in 1.1 and $\mathcal{J} = \mathcal{J}(d_1, \ldots, d_n)$, we have the following isomorphism of $\mathbb{C}[\mathbf{x}^{\pm}, \mathbf{t}]$-modules:*

$$\mathrm{HY}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^{\pm}] \cong \mathcal{J} \cong \Delta H_*^T(\mathrm{Sp}_\gamma)$$

and

$$\mathrm{HHH}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^{\pm}] \cong \mathcal{J}/(\mathbf{y})\mathcal{J} \cong H_*(\mathrm{Sp}_\gamma).$$

This conjecture has been shown in the equivalued case due to work from Hogancamp [19] and Kivinen [24]. In fact we use the computation of $\mathrm{HHH}(\mathrm{FT}_4^k) = \mathrm{HHH}(T(k, 4k))$ from [19] as a base case for our recursion (which corresponds to $d_1 = d_2 = d_3 = d_4 = k$), rather than re-derive it.

In Section 4.1, we show that Coxeter braids on 3 strands are parity, and explicitly computes $\mathrm{HHH}^{a=0}(\beta)$. In Section 4.2, we show that Coxeter braids on 4 strands are also parity. Thus we show the following result.

THEOREM 1.3.5. *Conjecture 1.3.4 holds for $n \leq 4$.*

This description doesn't guarantee that we can write out $\mathrm{HHH}^{a=0}(\beta)$ explicitly, as finding the bigraded Hilbert series for $J'(d_1, \ldots, d_n)$ is difficult in general. However, the recursive method we use to show that a braid is parity in Chapter 4 simultaneously gives a recursive way to compute $\mathrm{HHH}^{a=0}(\beta)$ as a rational function in q and t.

COROLLARY 1.3.6. *For $n \leq 4$, the ideal $J' = J'(d_1, \ldots, d_n)$ is free as a module over $\mathbb{C}[t_1, \ldots, t_n]$, and its bigraded Hilbert series is given by*

$$\frac{1}{(1-t)^4}\mathrm{HHH}^{a=0}(\beta).$$

For $n = 3$, the bigraded Hilbert series for $J'(d_1, d_2, d_3)$ is explicitly computed in general in Section 3.2.2. For $n = 4$ there is no similar explicit calculation but the recursions in Section 4.2 can be used to compute this series explicitly for any particular d_1, d_2, d_3, d_4.

REMARK 1.3.7. Note that Conjecture 3.1.8 predicts a closed formula for the Hilbert series as a sum of certain rational functions over 10 standard Young tableaux of size 4. We plan to verify this conjecture in a future work.

We are optimistic that Theorem 1.3.3 can be generalized to arbitrary n using the same recursive process, proving one side of Conjecture 1.3.4. This would confirm that the ideal $J'(d_1, \ldots, d_n)$ is free over y's and give a recursive formula for its bigraded Hilbert series. We would also expect that understanding this ideal and therefore $H_*(\mathrm{Sp}_\gamma)$ better would help to find affine cells for Sp_γ in general.

1.4. The fundamental domain of affine Springer fibers

Finally, in Chapter 5 we discuss the cells of the fundamental domain of Sp_γ, as described by Chen in [6], and relate this to the combinatorics of J' for $n = 3$. We show that there is a bijection between half of the cells in the fundamental domain and the generators of the ideal $J'(d_1, d_2)$. We expect that this bijection indicates a stronger relationship between the cells of Sp_γ and the combinatorics of J' and of generalized (q, t)-Catalan numbers.

CHAPTER 2

Affine Springer Fibers

This chapter is based on work from [32] published in IMRN.

2.1. Background

First, some notation. Let $\mathcal{K} = \mathbb{C}((t))$ be the field of Laurent power series in t, and $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of power series in t. For nonzero $f \in \mathcal{K}$, let $\nu(f)$ denote the order of f, which is the degree of the smallest nonzero term. Throughout, $H_*^T(X)$ will refer to the equivariant Borel-Moore homology of X, and $H_*(X)$ refers to regular Borel-Moore homology. All tensor products are over $\mathbb{C}$ unless otherwise indicated.

Definition 2.1.1. A *lattice* $\Lambda \subseteq \mathcal{K}^n$ is a free $\mathcal{O}$-submodule of $\mathcal{K}^n$ of rank n such that $\Lambda \otimes_{\mathcal{O}} \mathcal{K} = \mathcal{K}^n$. In other words, it is the $\mathcal{O}$-span of a basis of $\mathcal{K}^n$ over $\mathcal{K}$.

Definition 2.1.2. The *affine Grassmanian* $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ of GL_n over $\mathbb{C}$ is an ind-scheme defined as the space of all lattices $\Lambda \subseteq \mathcal{K}^n$.

We will always be working over $\mathbb{C}$, so we will use $\mathrm{Gr}(\mathrm{GL}_n)$ or $\mathrm{Gr}(\mathrm{SL}_n)$ for the affine Grassmanian, or just Gr when the group is clear. We can equivalently define the affine Grassmanian for GL_n as

$$\mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O}),$$

as $\mathrm{GL}_n(\mathcal{K})$ acts transitively on the space of lattices, and the stabilizer of the standard lattice $\mathcal{O}^n$ is precisely $\mathrm{GL}_n(\mathcal{O})$. We will often conflate a matrix $g \in \mathrm{GL}_n(\mathcal{K})$ with its coset representative in $\mathrm{Gr}(\mathrm{GL}_n)$.

We define $\mathrm{Gr}(\mathrm{SL}_n)$ similarly, either as $\mathrm{SL}_n(\mathcal{K})/\mathrm{SL}_n(\mathcal{O})$, or as the space of lattices of SL_n type. We say that a lattice Λ is of SL_n type if it can be written as $\Lambda = g\mathcal{O}^n$ for some $g \in \mathrm{SL}_n(\mathcal{K})$.

Definition 2.1.3 ([21]). The *affine Springer fiber* Sp_γ of an element $\gamma \in \mathfrak{gl}_n(\mathcal{K})$ is a sub ind-scheme of the affine Grassmanian, defined accordingly as the space of lattices $\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}$ such that $\gamma\Lambda \subseteq \Lambda$, or as the space of $g \in \mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O})$ such that $g^{-1}\gamma g \in \mathfrak{gl}_n(\mathcal{O})$.

FACT 2.1.4 ([34]). If γ is regular, semi-simple, and topologically nilpotent, then Sp_γ is finite dimensional, although it can still have infinitely many irreducible components. In our cases, these conditions essentially are that γ is diagonalizable, has distinct eigenvalues λ_i (over $\overline{\mathcal{K}}$), and that $\nu(\lambda_i) > 0$ for all i.

EXAMPLE 2.1.5 ([34]). Consider the matrix

$$\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$$

in $\mathfrak{sl}_2(\mathcal{K})$. Then Sp_γ looks like an infinite chain of $\mathbb{P}^1$'s connected at 0 and ∞. There is a $\mathbb{C}^*$ action that scales each $\mathbb{P}^1$, and a $\mathbb{Z}$ action that translates them.

There is a natural action of the centralizer $C(\gamma)$ on Sp_γ. In the case where γ is diagonal, this gives a torus action of $T = (\mathbb{C}^*)^n$ and lattice action of $\Lambda = \mathbb{Z}^n$ on Sp_γ over GL_n (or $(\mathbb{C}^*)^{n-1}$ and $\mathbb{Z}^{n-1}$ over SL_n) that can be respectively seen as multiplication by the matrices in $C(\gamma)$:

$$\lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and

$$Z = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix}.$$

In general for a T-ind-scheme X, $H_*^T(X)$ is naturally a module over $H_T^*(pt)$ via the cap product.

DEFINITION 2.1.6. We will say that X is *equivariantly formal* if $H_*^T(X)$ is free as a module over $H_T^*(pt)$.

FACT 2.1.7. (GKM [12]) If X is equivariantly formal, then

$$H_*(X) \cong H_*^T(X)/(\mathbf{t})H_*^T(X),$$

as modules over $H_T^*(pt)$. Here $(\mathbf{t}) \subseteq H_T^*(pt) \cong \mathbb{C}[\mathbf{t}]$ is the maximal ideal generated by $t_1, \ldots, t_n$.

We will need the following localization lemma as stated by Brion.

LEMMA 2.1.8 (Brion [5]). *Let X be a T-ind-scheme and $T' \subseteq T$ a subtorus. If $i : X^{T'} \to X$ is the inclusion of T'-fixed points, then the induced map*

$$i_* : H_*^T(X^{T'}) \to H_*^T(X)$$

is an isomorphism after inverting finitely many characters of T that restrict nontrivially to T'. Further, if X is equivariantly formal, then the induced map i_* is injective, and we have that

$$\prod_{\chi, \chi_{T'} \neq 0} \chi H_*^T(X) \subseteq H_*^T(X^{T'})$$

where we take the product over all characters of T that restrict nontrivially to T'.

We will also make frequent use of the Iwasawa decomposition for $\mathrm{Gr}(\mathrm{GL}_n)$, which tells us that all $g \in \mathrm{Gr}(\mathrm{GL}_n)$ can be represented by a product DU of a diagonal matrix

$$D = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix} \tag{2.1}$$

with a unipotent matrix U with 1's on the diagonal and entries χ_{ij} above the diagonal [34]. Further, these χ_{ij}'s are unique up to $\mathcal{O}$, so we can choose them to have all coefficients of nonnegative powers of t be 0, so that each matrix DU represents a unique element $g \in \mathrm{Gr}$.

LEMMA 2.1.9. *The T-fixed points of $\mathrm{Gr}(\mathrm{GL}_n)$ can be uniquely represented by diagonal matrices D as in the Iwasawa decomposition.*

PROOF. Let $\lambda \in T$ and $g = DU$ be as in the Iwasawa decomposition. Since $\lambda^{-1} \in \mathrm{GL}_n(\mathcal{O})$, up to multiplication on the right by $\mathrm{GL}_n(\mathcal{O})$, we get

$$\lambda g = \lambda g \lambda^{-1} = DU',$$

where D is as above, and U' is unipotent with $\frac{\lambda_i}{\lambda_j}\chi_{ij}$ above the diagonal. If g is a fixed point under the action of T, we must have $\frac{\lambda_i}{\lambda_j}\chi_{ij} = \chi_{ij}$ for all i, j and for all $\lambda \in T$. This can only happen if $\chi_{ij} = 0$ for all i, j, so $g = D$ is diagonal as desired. □

Since the T-action on Sp_γ comes from the action on Gr, the T-fixed points of Sp_γ are simply the T-fixed points of Gr that are contained in Sp_γ.

Lemma 2.1.10. *If γ is diagonal and the orders of its eigenvalues are all nonnegative, then*

$$\mathrm{Sp}_\gamma^T = \mathrm{Gr}^T.$$

Proof. For any γ, $\mathrm{Sp}_\gamma^T \subseteq \mathrm{Gr}^T$ as stated above. If $g \in \mathrm{Gr}^T$, then

$$g = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix}$$

by Lemma 2.1.9. As g and γ are diagonal, $g^{-1}\gamma g = \gamma$, and $\gamma \in \mathfrak{gl}_n(\mathcal{O})$, since the eigenvalues of γ are all in $\mathcal{O}$. So $\mathrm{Gr}^T \subseteq \mathrm{Sp}_\gamma^T$. □

In particular, this means that the T-fixed points of Sp_γ are discrete and isomorphic to the integer lattice $\Lambda = \mathbb{Z}^n$, so can view $H_*(\mathrm{Sp}_\gamma)$ as a module over

$$H_T^*(\mathrm{Sp}_\gamma^T) \cong H_T^*(pt) \otimes \mathbb{C}[\Lambda] \cong \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm].$$

Here, the t_i's are our equivariant parameters, and a monomial $x_1^{a_1} \cdots x_n^{a_n}$ corresponds to the fixed point $\mathrm{diag}(t^{a_1}, \dots, t^{a_n})$. The lattice Λ acts on Sp_γ^T and on Sp_γ by translation.

Lemma 2.1.11. *Fix $i < j$. If $T' \subseteq T$ is a codimension 1 subtorus cut out by $t_i = t_j$, then the T'-fixed points of $\mathrm{Gr}(\mathrm{GL}_n)$ are of the form DU, where D is as in (2.1) and*

$$U = \begin{pmatrix} 1 & & \chi_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}$$

with all χ's zero except for χ_{ij}.

Proof. As before, up to equivalence,

$$\lambda g = \lambda g \lambda^{-1} = DU',$$

where U' has $\frac{\lambda_k}{\lambda_l}\chi_{kl}$ above the diagonal. If g is a fixed point for T, since $\lambda_i = \lambda_j$, χ_{ij} can be arbitrary, but $\chi_{kl} = 0$ for all $(k,l) \neq (i,j)$. So the fixed points are as described. □

Corollary 2.1.12. *If T' is a codimension 1 subtorus of T cut out by $t_i = t_j$, then*

$$\mathrm{Gr}(\mathrm{GL}_n)^{T'} \cong \mathrm{Gr}(\mathrm{GL}_2) \times \mathbb{Z}^{n-2}.$$

Proof. Each of the T' fixed points is represented by DU above. Looking at the 2×2 submatrix of DU in rows and columns i, j, we see a copy of $\mathrm{Gr}(\mathrm{GL}_2)$. The rest of the m_i are free integers, and there are $n-2$ of them. □

2.2. Homology of unramified affine Springer fibers

We want to find the equivariant Borel-Moore homology $H_*^T(\mathrm{Sp}_\gamma)$ of the class of affine Springer fibers with

$$\gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} = \begin{pmatrix} z_1 t^{d_1} & & 0 \\ & \ddots & \\ 0 & & z_n t^{d_n} \end{pmatrix}.$$

Here $z_i \in \mathbb{C}^*$ are pairwise distinct and $d_i \geq 0$. We can assume up to a change of basis that $d_1 \leq \cdots \leq d_n$.

It is known that for γ as above, Sp_γ is equivariantly formal (see Definition 2.1.6) over GL_n for $n \leq 4$ (see [27] for $n = 3$ and [7] for $n = 4$), and it is known to be equivariantly formal for all n if $d_i = d$ for all i [11]. But it is not known over GL_n in general. It would be sufficient to know that the homology of Sp_γ is supported in even degrees, and we conjecture that this is the case for all n. We will need to assume that Sp_γ is equivariantly formal in order to calculate its homology.

We consider $H_*^T(\mathrm{Sp}_\gamma)$ as a module over

$$H_T^*(\mathrm{pt}) \otimes \mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[t_1, \ldots, t_n, x_1^\pm, \ldots, x_n^\pm] = R.$$

Theorem 2.2.1. *Consider γ as in (1.1). Define the ideal*

$$\mathcal{J} \subseteq R$$

$$\mathcal{J} = \bigcap_{i<j} (t_i - t_j, x_i - x_j)^{d_{ij}}$$

with $d_{ij} = min(d_i, d_j)$. If Sp_γ is equivariantly formal, then as R-modules,

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J},$$

where $\Delta = \prod_{i<j}(t_i - t_j)^{d_{ij}}$.

Note that multiplication by Δ is injective, so

$$H_*^T(\mathrm{Sp}_\gamma) \cong \Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J}$$

as R-modules. It can be useful to keep track of Δ if we want to retain the localization information of $H_*^T(\mathrm{Sp}_\gamma)$, but we can omit Δ when we only care about $H_*^T(\mathrm{Sp}_\gamma)$ as an R-module.
The rough outline of the proof of Theorem 2.2.1 is as follows:

(1) Take a codimension one subtorus $T' \subseteq T$. The T'-fixed points of Sp_γ are essentially isomorphic to an affine Springer fiber $\mathrm{Sp}_{\tilde{\beta}}$ with $\tilde{\beta} \in \mathfrak{gl}_2$ whose homology is known.
(2) Relate the homology of $\mathrm{Sp}_{\tilde{\beta}}$ to that of Sp_γ using Lemma 2.1.8.
(3) Take enough subtori T' and piece together their homologies to find the homology of Sp_γ.

Step 3 will require the assumption that Sp_γ is equivariantly formal.

Lemma 2.2.2. *If $T' \subseteq T$ is the subtorus cut out by $t_i = t_j$, then up to $\mathbb{Z}^{n-2}$, the T'-fixed points of Sp_γ are isomorphic to an affine Springer fiber over GL_2,*

$$\mathrm{Sp}_\gamma^{T'} \cong \mathrm{Sp}_{\beta_{ij}} \times \mathbb{Z}^{n-2},$$

where

$$\beta_{ij} = \begin{pmatrix} z_i t^{d_i} & 0 \\ 0 & z_j t^{d_j} \end{pmatrix}.$$

Proof. In Lemma 2.1.11 we've already characterized the T'-fixed points of Gr as DU, where U is unipotent with only a single nonzero χ_{ij}. We just need to check which of those fixed points are in Sp_γ. If $g \in \mathrm{Gr}(\mathrm{GL}_n)^{T'}$, then

$$g^{-1}\gamma g = \begin{pmatrix} z_1 t^{d_1} & & \chi_{ij}(z_i t^{d_i} - z_j t^{d_j}) \\ & \ddots & \\ 0 & & z_n t^{d_n} \end{pmatrix}.$$

Again looking at the 2×2 i, j submatrix, we see a matrix identical to $g^{-1}\beta_{ij}g$. So a T'-fixed point g is in Sp_γ if and only if the 2×2 matrix

$$\begin{pmatrix} t^{m_i} & \chi_{ij} \\ 0 & t^{m_j} \end{pmatrix}$$

is in $\mathrm{Sp}_{\beta_{ij}}$, i.e. $\mathrm{Sp}_\gamma \cong \mathrm{Sp}_{\beta_{ij}} \times \mathbb{Z}^{n-2}$. □

LEMMA 2.2.3. *Given $\beta_{ij} \in \mathfrak{gl}_2$ as in Theorem 1, we have $\mathrm{Sp}_{\beta_{ij}} \cong \mathrm{Sp}_{\tilde{\beta}_{ij}}$, where*

$$\tilde{\beta}_{ij} = \begin{pmatrix} z_i t^{d_{ij}} & 0 \\ 0 & z_j t^{d_{ij}} \end{pmatrix}$$

and $d_{ij} = min(d_i, d_j)$.

PROOF. Again using the Iwasawa decomposition, write $g = DU$, where

$$U = \begin{pmatrix} 1 & \chi_{ij} \\ 0 & 1 \end{pmatrix}.$$

Then,

$$g^{-1}\gamma g = \begin{pmatrix} \gamma_i & \chi_{ij}(\gamma_i - \gamma_j) \\ 0 & \gamma_j \end{pmatrix}.$$

By definition, $g \in \mathrm{Sp}_{\beta_{ij}}$ if and only if $\chi_{ij}(\gamma_i - \gamma_j) \in \mathcal{O}$. Since we assume that the z_i are distinct, $\nu(\gamma_i - \gamma_j) = \min(\nu(\gamma_i), \nu(\gamma_j)) = \min(d_i, d_j) = d_{ij}$. So $g \in \mathrm{Sp}_{\beta_{ij}}$ if and only if χ_{ij} has order at least $-d_{ij}$. This is the same as the condition for g to be in $\mathrm{Sp}_{\tilde{\beta}_{ij}}$, since

$$g^{-1}\tilde{\beta}_{ij}g = \begin{pmatrix} z_i t^{d_{ij}} & (z_i - z_j)\chi_{ij} t^{d_{ij}} \\ 0 & z_j t^{d_{ij}} \end{pmatrix}.$$

□

REMARK 2.2.4. The one-dimensional quotient torus T/T' naturally acts on $\mathrm{Sp}_\gamma^{T'}$. On the other hand, T/T' is isomorphic to the one-dimensional torus $(\mathbb{C}^*)^2/\mathbb{C}^*$ which acts on $\mathrm{Sp}_{\beta_{ij}}$ and on $\mathrm{Sp}_{\tilde{\beta}_{ij}}$. The isomorphisms constructed in Lemmas 2.2.2 and 2.2.3 are T/T'-equivariant.

The homology for the affine Springer fiber of matrices like $\tilde{\beta}_{ij}$ with all powers the same is known. It is found using GKM theory by Kivinen in [24].

Theorem 2.2.5 (Kivinen [24]). *If*

$$\gamma = \begin{pmatrix} z_1 t^d & & 0 \\ & \ddots & \\ 0 & & z_n t^d \end{pmatrix},$$

then $\Delta H_*^T(\mathrm{Sp}_\gamma)$ *injects into*

$$H_T^*(pt) \otimes \mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[t_1, \dots, t_n, x_1^{\pm}, \dots, x_n^{\pm}],$$

where $\Delta = \prod_{i<j}(t_i - t_j)^d$. *As a submodule, there is a canonical isomorphism*

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \prod_{i<j}(t_i - t_j, x_i - x_j)^d.$$

Corollary 2.2.6. *We have the following canonical isomorphism of* $\mathbb{C}[t_i - t_j, x_i^{\pm}, x_j^{\pm}]$*-modules:*

$$(t_i - t_j)^{d_{ij}} H_*^{T/T'}(\mathrm{Sp}_{\tilde{\beta}_{ij}}) \cong (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq \mathbb{C}[t_i - t_j, x_i^{\pm}, x_j^{\pm}].$$

Here T/T' *acts on* $\mathrm{Sp}_{\tilde{\beta}_{ij}}$ *as in Remark 2.2.4.*

Now in order to piece together these homologies of $\mathrm{Sp}_{\tilde{\beta}_{ij}}$, we use the following fact:

Lemma 2.2.7 (Algebraic Hartogs' lemma). *If A is an integrally closed Noetherian integral domain, then*

$$A = \bigcap_{p \text{ codimension } 1} A_p,$$

where we take the intersection of A_p over all codimension 1 prime ideals of A inside the fraction field Frac(A). If M is a free A-module, then we also have

$$M = \bigcap_{p \text{ codimension } 1} M_p,$$

where the intersection is taken inside $M \otimes_A \mathit{Frac}(A)$.

Since we assumed that Sp_γ is equivariantly formal, $H_*^T(\mathrm{Sp}_\gamma)$ is free over $H_T^*(pt)$. Now we can prove Theorem 2.2.1.

Proof of Theorem 2.2.1. By Lemma 2.1.8, we have that, up to localization away from $(t_i - t_j)$,

$$H_*^T(\mathrm{Sp}_\gamma) \cong_{loc} H_*^T(\mathrm{Sp}_\gamma^{T'}) \cong H_*^{T/T'}\left(\mathrm{Sp}_\gamma^{T'}\right) \otimes H_{T'}^*(pt) = H_*^{T/T'}(\mathrm{Sp}_{\tilde{\beta}_{ij}} \times \mathbb{Z}^{n-2}) \otimes H_{T'}^*(pt).$$

Here $\cong_{loc}$ indicates an isomorphism after localization. Note that after localization, $\Delta = (t_i - t_j)^{d_{ij}}$ up to an invertible factor, so by Lemma 2.2.3 we get

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong_{loc} (t_i - t_j)^{d_{ij}} H_*^{T/T'}\left(\mathrm{Sp}_{\widetilde{\beta}_{i,j}}\right) \otimes H_{T'}^*(pt) \otimes \mathbb{C}[\mathbb{Z}^{n-2}] = (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq \mathbb{C}[\mathbf{t}, \mathbf{x}^\pm].$$

We have inclusion map

$$\mathrm{Sp}_\gamma^{T'} \xrightarrow{i} \mathrm{Sp}_\gamma$$

and by Lemma 2.1.8

$$\prod_{(k,\ell)\neq(i,j)} (t_k - t_\ell) H_*^T(\mathrm{Sp}_\gamma) \subseteq i_* H_*^T(\mathrm{Sp}_\gamma^{T'}).$$

We also have that

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \subseteq \prod_{(k,\ell)\neq(i,j)} (t_k - t_\ell)(t_i - t_j)^{d_{ij}} H_*^T(\mathrm{Sp}_\gamma).$$

By the above, this is contained in

$$i_*(t_i - t_j)^{d_{ij}} H_*^T(\mathrm{Sp}_\gamma^{T'}) = (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

We conclude that

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \subseteq (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

This holds for all codimension-1 subtori $T' = (t_i - t_j) \subseteq T$ with $i < j$, so we have

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \subseteq \bigcap_{i<j} (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

In fact we have already seen that $(t_i - t_j, x_i - x_j)^{d_{ij}}$ is exactly the localization $H_*^T(\mathrm{Sp}_\gamma)_p$ where $p = (t_i - t_j)$.

So by Lemma 2.2.7 we conclude that

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \bigcap_{i<j} (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

□

This proof required the assumption that Sp_γ is equivariantly formal. If $d_i = d$ for all i, then it is known to be equivariantly formal [11] and we recover the homology result of Theorem 2.2.5 from Kivinen [24].

Conjecture 2.2.8. Sp_γ *is equivariantly formal for all* $d_1, \ldots, d_n$ *and all* n.

CHAPTER 3

Generalized Haiman Ideals

This chapter is based on work from [32] published in IMRN.

3.1. Generators and basis for $n = 3$

When $n = 3$, it is known that Sp_γ is equivariantly formal, so by Theorem 2.2.1, up to denominators (Δ), its equivariant Borel-Moore homology is isomorphic to the ideal $\mathcal{J} \subseteq \mathbb{C}[t_1, t_2, t_3, x_1^\pm, x_2^\pm, x_3^\pm]$ defined as

$$\mathcal{J} = \mathcal{J}(d_1, d_2) = (t_1 - t_2, x_1 - x_2)^{d_1} \cap (t_1 - t_3, x_1 - x_3)^{d_1} \cap (t_2 - t_3, x_2 - x_3)^{d_2}$$

seen as a module over $\mathbb{C}[t_1, t_2, t_3, x_1^\pm, x_2^\pm, x_3^\pm]$. Here we are assuming that $d_1 \leq d_2 \leq d_3$, so that d_1, d_1, d_2 are equal to the pairwise minima d_{ij}.

We consider a similar ideal $J' \subset \mathbb{C}[t_1, t_2, t_3, x_1, x_2, x_3]$

$$J' = J'(d_1, d_2) = (t_1 - t_2, x_1 - x_2)^{d_1} \cap (t_1 - t_3, x_1 - x_3)^{d_1} \cap (t_2 - t_3, x_2 - x_3)^{d_2}$$

It is easy to see that $\mathcal{J} = J' \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^\pm]$, so the generators for $\mathcal{J}$ over $\mathbb{C}[\mathbf{t}, \mathbf{x}^\pm]$ will be the same as generators of J' over the polynomial ring $\mathbb{C}[\mathbf{t}, \mathbf{x}]$. Next we will do a change of variables:

$$a = t_1 - t_2,\ b = x_1 - x_2,\ c = t_3 - t_2,\ d = x_3 - x_2$$

and consider the ideal

$$J = J(d_1, d_2) = (a, b)^{d_1} \cap (c, d)^{d_2} \cap (a - c, b - d)^{d_1}$$

over $R = \mathbb{C}[a, b, c, d]$. Clearly, we get

$$J'(d_1, d_2) = J(d_1, d_2) \otimes_{\mathbb{C}} \mathbb{C}[x_1 + x_2 + x_3, t_1 + t_2 + t_3],$$

so again all three ideals have the same generators up to this change of variables.

We can also consider these as bigraded ideals, where the t_i's (or a and c) have bidegree q, and the x_i's (b and d) have bidegree t.

We will frequently use the polynomial

$$ad - bc = d(a-c) - c(b-d) \in (a,b) \cap (c,d) \cap (a-c, b-d).$$

THEOREM 3.1.1. *The ideal* $J = (a,b)^{d_1} \cap (c,d)^{d_2} \cap (a-c, b-d)^{d_1}$ *over* $R = \mathbb{C}[a,b,c,d]$ *has the following families of generators* $(0 \le j \le d_1)$*:*

(1) $A_{i,j} = a^{d_1-j}c^{d_2-j}(a-c)^i(b-d)^{d_1-j-i}(ad-bc)^j, 1 \le i \le d_1 - j$. *These generators have bidegree* $q^{d_1+d_2-j+i}t^{d_1-i}$*, and there are* $d_1 - j$ *of these for a fixed* j*. They are characterized by* $\deg_t < d_1$.

(2) $B_{i,j} = a^{d_1-j-i}b^i d^{d_2-j}(b-d)^{d_1-j}(ad-bc)^j, 1 \le i \le d_1 - j$. *Such generators have bidegree* $q^{d_1-i}t^{d_1+d_2-j+i}$*, and there are* $d_1 - j$ *of these for a fixed* j*. They are characterized by* $\deg_q < d_1$.

(3) $C_{i,j} = a^{d_1-j}c^i d^{d_2-j-i}(b-d)^{d_1-j}(ad-bc)^j, 1 \le i \le d_2 - j$. *Such generators have bidegree* $q^{d_1+i}t^{d_1+d_2-j-i}$*, and there are* $d_2 - j$ *of these for a fixed* j*. They are characterized by* $\deg_q > d_1, \deg_t \ge d_1$.

(4) $D_j = a^{d_1-j}d^{d_2-j}(b-d)^{d_1-j}(ad-bc)^j$ *has bidegree* $q^{d_1}t^{d_1+d_2-j}$*, there is one such generator for each* j*. They are characterized by* $\deg_q = d_1$.

REMARK 3.1.2. For $j = d_1$, the generators $A_{i,j}$ and $B_{i,j}$ are not defined, while $C_{i,d_1} = c^i d^{d_2-d_1-i}(ad-bc)^{d_1}$ for $1 \le i \le d_2 - d_1$ and $D_{d_1} = d^{d_2-d_1}(ad-bc)^{d_1}$.

In particular, it is easy to see that there is at most one generator in each (q,t)-bidegree, see also Proposition 3.1.12. Also notice that we chose the generators in Theorem 3.1.1 such that the monomial factor in $A_{i,j}$ does not contain b or d, and the monomial factors in $B_{i,j}, C_{i,j}, D_j$ do not contain bc (unless $j = d_1$).

Theorem 3.1.1 follows from Proposition 3.1.3, which we prove in Section 3.2.1.

PROPOSITION 3.1.3. *The ideal* $J(d_1, d_2)$ *has the following basis (over* $\mathbb{C}$*):*

$$m(a,c)A_{i,j}, m(a,b,d)B_{i,j}, m(a,c,d)C_{i,j}, m(a,d)D_k \ (j \le d_1 - 1)$$

where m are arbitrary monomials in the corresponding variables. For $j = d_1$ we have to add all polynomials of the form

$$a^\alpha b^\beta c^\gamma d^\delta (ad-bc)^{d_1},\ \gamma+\delta \geq d_2 - d_1.$$

EXAMPLE 3.1.4. For $d_1 = d_2 = 1$ we get the following 5 generators of $J(1,1)$:

$$A_{1,0} = ac(a-c),\ B_{1,0} = bd(b-d),\ C_{1,0} = ac(b-d),\ D_0 = ad(b-d),\ D_1 = (ad-bc).$$

We can change the variables back to see that the generators of $\mathcal{J}$ over $\mathbb{C}[t_1,t_2,t_3,x_1^\pm,x_2^\pm,x_3^\pm]$ are

$$A_{1,0} = (t_1-t_2)(t_3-t_2)(t_1-t_3),\ B_{1,0} = (x_1-x_2)(x_3-x_2)(x_1-x_3),$$

$$C_{1,0} = (t_1-t_2)(t_3-t_2)(x_1-x_3),\ D_0 = (t_1-t_2)(x_3-x_2)(x_1-x_3),$$

and

$$D_1 = \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ t_1 & t_2 & t_3 \end{pmatrix}$$

COROLLARY 3.1.5. *We have that*

$$J(d_1,d_2) = J(1,1)^{d_1} \cdot J(0,1)^{d_2-d_1},$$

$$J'(d_1,d_2) = J'(1,1)^{d_1} \cdot J'(0,1)^{d_2-d_1}$$

and

$$\mathcal{J}(d_1,d_2) = \mathcal{J}(1,1)^{d_1} \cdot \mathcal{J}(0,1)^{d_2-d_1}.$$

PROOF. We prove the first equation, and the other two equations follow immediately. The containment $J(1,1)^{d_1} \cdot J(0,1)^{d_2-d_1} \subseteq J(d_1,d_2)$ is clear, so it is sufficient to show that any generator of $J(d_1,d_2)$ can be written as a product of d_1 generators of $J(1,1)$ (listed in Example 3.1.4) and d_2-d_1 generators of $J(0,1) = (c,d)$. Indeed:

$$A_{i,j} = a^{d_1-j}c^{d_2-j}(a-c)^i(b-d)^{d_1-j-i}(ad-bc)^j = (ac(a-c))^i \cdot (ac(b-d))^{d_1-j-i} \cdot (ad-bc)^j \cdot c^{d_2-d_1},$$

$$B_{i,j} = a^{d_1-j-i}b^i d^{d_2-j}(b-d)^{d_1-j}(ad-bc)^j = (ad(b-d))^{d_1-j-i}(bd(b-d))^i(ad-bc)^j d^{d_2-d_1},$$

$$C_{i,j} = a^{d_1-j}c^i d^{d_2-j-i}(b-d)^{d_1-j}(ad-bc)^j =$$

$$(ac(b-d))^x(ad(b-d))^{d_1-j-x}(ad-bc)^j c^{i-x} d^{d_2-d_1-i+x},$$

where $x=\max(0,i+d_1-d_2)$. Note that $i\le d_2-j$, so $i+d_1-d_2\le d_1-j$ and hence $x\le d_1-j$. Also, $d_1\le d_2$, so $i+d_1-d_2\le i$ and $x\le i$. Therefore all exponents are indeed nonnegative. Finally,

$$D_j=a^{d_1-j}d^{d_2-j}(b-d)^{d_1-j}(ad-bc)^j=(ad(b-d))^{d_1-j}(ad-bc)^j d^{d_2-d_1}.$$

□

REMARK 3.1.6. Note that by Remark 3.1.2 the polynomials $a^\alpha b^\beta c^\gamma d^\delta (ad-bc)^{d_1}$, $\gamma+\delta\ge d_2-d_1$ are either multiples of C_{i,d_1} or of D_{d_1}.

In [15], Gorsky, Kivinen, and Oblomkov define a graded algebra $\mathcal{A}_G=\bigoplus_{d=0}^{\infty}\mathcal{A}_d$, depending only on a reductive group G, with some specific properties. One of the key properties is that for any $\gamma\in\mathfrak{g}$, the direct sum of homologies of affine Springer fibers

$$F_\gamma=\bigoplus_{k=0}^{\infty}H_*(\mathrm{Sp}_{t^k\gamma})$$

is a graded module over $\mathcal{A}_G$, or equivalently, that there is a corresponding quasi-coherent sheaf $\mathcal{F}_\gamma$ on $\mathrm{Proj}\bigoplus_{d=0}^{\infty}\mathcal{A}_d$. They conjecture that F_γ is finitely generated and that this sheaf is coherent [15, Conjecture 8.1]. In the case where $G=\mathrm{GL}_n$, they show that this graded algebra is generated in degrees 0 and 1, and that $\mathrm{Proj}\bigoplus_{d=0}^{\infty}\mathcal{A}_d=\mathrm{Hilb}^n(\mathbb{C}^*\times\mathbb{C})$. A special case of this conjecture follows from Theorem 3.1.1 and its corollaries.
Indeed, it is proved in [15] that $\mathcal{A}_0$ is the space of symmetric polynomials in $\mathbb{C}[t_1,\dots,t_n,x_1^{\pm},\dots,x_n^{\pm}]$, and $\mathcal{A}_1$ is the space of antisymmetric polynomials.

THEOREM 3.1.7. *In the case of* $G=\mathrm{GL}_3$ *and* γ *as in (1.1), the graded module* F_γ *is finitely generated over* $\mathcal{A}_G$*, and defines a coherent sheaf on* $\mathrm{Hilb}^3(\mathbb{C}^*\times\mathbb{C})$*, i.e. Conjecture 8.1 holds in this case.*

PROOF. By Corollary 3.8.3 in [9], the ideal generated by $\mathcal{A}_1$ is exactly $\mathcal{J}(1,1)$. There is a natural inclusion of ideals

$$\mathcal{J}(d_1,d_2)\cdot\mathcal{J}(1,1)\to\mathcal{J}(d_1+1,d_2+1).$$

It follows from Corollary 3.1.5 that this map is actually surjective as well. Since $H_*(\mathrm{Sp}_{t^k\gamma})$ corresponds to the ideal $\mathcal{J}(d_1+k, d_2+k)$, this shows that the module F_γ is generated in degree 0, and therefore finitely generated by the generators of $\mathcal{J}(d_1, d_2)$. □

3.1.1. Hilbert Series.

Let us introduce two rational functions

$$H(d_1,\dots,d_n) = \sum_T \frac{z_1^{d_n}\cdots z_n^{d_1}}{(1-q)^n(1-t)^n}\prod_{i=2}^n \frac{1}{(1-z_i^{-1})}\prod_{i<j}\omega(z_i/z_j)$$

and

$$F(d_1,\dots,d_n) = \sum_T z_1^{d_n}\cdots z_n^{d_1}\prod_{i=2}^n \frac{1}{(1-z_i^{-1})(1-qtz_{i-1}/z_i)}\prod_{i<j}\omega(z_i/z_j).$$

Here the sums are over standard tableaux T with n boxes, z_i is the (q,t)-content $q^{c-1}t^{r-1}$ of the box labeled by i in row r and column c in T, and $\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}$. By convention, all the factors of the form $(1-1)$ in the above products (either in the numerator or in denominator) should be ignored.

The function $F(d_1,\dots,d_n)$ is also known as generalized (q,t)-Catalan number, see [**13**] for more details and context. Note that the order of the d_i is reversed here compared to [**13**].

Conjecture 3.1.8. *We have that:*

a) The Hilbert series of the ideal $J'(d_1,\dots,d_n)$ equals $H(d_1,\dots,d_n)$.

b) The Hilbert series of the generating set $J'(d_1,\dots,d_n)/\mathfrak{m}J'(d_1,\dots,d_n)$ equals $F(d_1,\dots,d_n)$. Here $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} = (t_1,\dots,t_n,x_1,\dots,x_n)$.

In particular, this conjecture implies that $F(d_1,\dots,d_n)$ is a polynomial in q and t with nonnegative coefficients (see [**13**, Conjecture 1.3]) and provides an explicit algebraic interpretation of these coefficients. Similarly, the conjecture implies that $H(d_1,\dots,d_n)$ is a power series in q and t with nonnnegative coefficients.

Example 3.1.9. For $n=2$ we get $J(d_1,d_2) = (x_1-x_2, t_1-t_2)^{d_1}$. We change coordinates to $x_1-x_2 = a, t_1-t_2 = b$ and $\overline{x} = x_1+x_2, \overline{t} = t_1+t_2$, then $J(d_1,d_2)$ has generating set $a^{d_1}, a^{d_1-1}b,\dots,b^{d_1}$, so the Hilbert series for the generating set equals

$$q^{d_1}+q^{d_1-1}t+\dots+t^{d_1} = \frac{q^{d_1}}{1-t/q}+\frac{t^{d_1}}{1-q/t} = F(d_1,d_2).$$

Similarly, $J(d_1, d_2)$ is free over $\mathbb{C}[\overline{x}, \overline{t}]$ with basis $a^\alpha b^\beta, \alpha + \beta \geq d_1$, so by Lemma 3.2.6 below we get the Hilbert series

$$\frac{1}{(1-q)(1-t)} \sum_{\alpha+\beta \geq d_1} q^\alpha t^\beta = \frac{q^{d_1}}{(1-q)^2(1-t)(1-t/q)} + \frac{t^{d_1}}{(1-q)(1-t)^2(1-q/t)}.$$

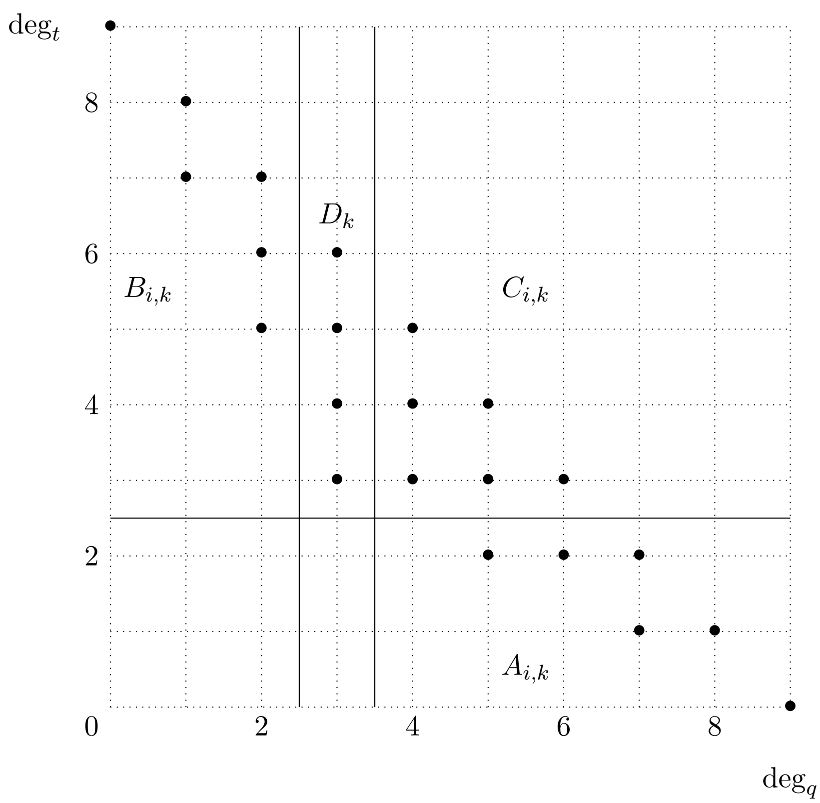
THEOREM 3.1.10. *Conjectures 3.1.8(a) and 3.1.8(b) hold for $n = 3$ for all d_1, d_2, d_3.*

The statement of (a) follows from the Hilbert series calculation in Section 3.2.2, and the proof of (b) will be in Section 3.1.2.

3.1.2. Combinatorics of J.

We've already seen in Example 3.1.4 that $J(1,1)$ has 5 generators, and that in general the generators of J each have a unique (q,t)-bidegree. We can plot the bidegrees of these generators as below.

EXAMPLE 3.1.11. Here is an example where $d_1 = d_2 = 3$:



Each dot represents the q, t degree of a generator of $J(3,3)$. Although the ideal is symmetric in x and t, we break the generators into families in a non-symmetric way.

Every generator lies on a diagonal corresponding to k, which is the degree of $(ad-bc)$ that appears in its definition in Theorem 3.1.1. We see here that the diagonals have 10, 7, 4, and 1 point respectively. Summing all of their degrees gives the generalized q,t-Catalan number

$$F(3,3,3) = [10]_{q,t} + qt[7]_{q,t} + q^2t^2[4]q,t + q^3t^3[1]_{q,t},$$

where $[n+1]_{q,t} := q^n + q^{n-1}t + \cdots + qt^{n-1} + t^n$.

Proposition 3.1.12. *The generators of J are in bijection with the integer lattice points inside the trapezoid bounded by the following inequalities:*

$$2d_1 + d_2 \geq x + y \geq d_1 + d_2,\ \ x + 2y \geq 2d_1 + d_2,\ \ 2x + y \geq 2d_1 + d_2.$$

Proof. It is easy to check that all of the generators of J satisfy these inequalities on their bidegrees $(x,y) = (\deg_q, \deg_t)$. Since the generators all have unique (q,t)-bidegree, it is sufficient to count the number of points in the integer lattice and check that it is the same as the number of generators.

We will count the lattice points going by diagonals, starting with the top diagonal $x+y = 2d_1+d_2$. On this diagonal, $x + 2y \geq 2d_1 + d_2$ and $2x + y \geq 2d_1 + d_2$ are trivially satisfied, since $x, y \geq 0$. So x and y can both range from 0 to $2d_1 + d_2$, and there are $2d_1 + d_2 + 1$ points on this diagonal.

On the next diagonal, $x+y = 2d_1+d_2-1$, we have that $x+2y = 2d_1+d_2-1+y$ and $2x+y = 2d_1+d_2-1+x$ are both at least $2d_1 + d_2$. This implies that $x, y \geq 1$ so we have points

$$(1, 2d_1 + d_2 - 2),\ (2, 2d_1 + d_2 - 1),\ \ldots (2d_1 + d_2 - 2, 1),$$

which amounts to 3 less points than the first diagonal. If we keep going, each diagonal will have 3 less points than the last, until the final diagonal $x + y = d_1 + d_2$, which will have $d_2 - d_1 + 1$ points. If we index the diagonal $x + y = 2d_1 + d_2 - j$ by j, j will range from 0 to d_1. So in total the number of points in the lattice is

$$\sum_{j=0}^{d_1} [2d_1 + d_2 + 1 - 3j].$$

If we count the generators of J as laid out in Theorem 3.1.1, we get

$$A_{i,j} : \sum_{j=0}^{d_1}(d_1 - j), \quad B_{i,j} : \sum_{j=0}^{d_1}(d_1 - j), \quad C_{i,j} : \sum_{j=0}^{d_1}(d_2 - j), \quad D_j : \sum_{j=0}^{d_1} 1,$$

and combining the sums, we get the same count. So we have shown the desired bijection.

□

Proof of Theorem 3.1.10(b). In Example 1.2 in [13], the authors show that

$$F(d_1, d_2, d_3) = [2d_1 + d_2 + 1]_{q,t} + qt[2d_1 + d_2 - 2]_{q,t} + \cdots + q^{d_1}t^{d_1}[d_2 - d_1 + 1]_{q,t},$$

where $[n+1]_{q,t} := q^n + q^{n-1}t + \cdots + qt^{n-1} + t^n$.

We can see from the proof of Proposition 3.1.12 that this exactly matches up with the coordinates of the lattice points, grouped by diagonals, and therefore by Proposition 3.1.12, this is in bijection with the generators of J' and their (q,t) degrees.

Any choice of basis of $J'(d_1, \ldots, d_n)/\mathfrak{m}J'(d_1, \ldots, d_n)$ lifts to a set of generators of J' by Nakayama's lemma. So the Hilbert series of $J'(d_1, \ldots, d_n)/\mathfrak{m}J'(d_1, \ldots, d_n)$ is precisely the degree count of the generators of J', so indeed it is $F(d_1, d_2, d_3)$. □

3.2. Proofs

3.2.1. Proof of Theorem 3.1.1. After doing the change of variables to $\mathbb{C}[a,b,c,d]$, the upshot is that we've reduced the number of variables, and we can use the fact that $J = M \cap (a - c, b - d)^{d_1}$, where $M = (a,b)^{d_1} \cap (c,d)^{d_2}$ is a monomial ideal. In this section we will find a basis for J over $\mathbb{C}$ by characterizing when elements of $(a - c, b - d)^{d_1}$ are in the monomial ideal M, proving Proposition 3.1.3 and by extension Theorem 3.1.1. First, we will need a few key lemmas.

Since M is a monomial ideal, a polynomial f is in M if and only if all monomials m of f that have nonzero coefficients are in M. If m is a monomial, then let $\deg_1(m)$ be the combined (a,b) degree of m, i.e. the sum of its a and b degrees, and similarly let $\deg_2(m)$ be its combined (c,d) degree. Then the monomial m is in M if and only $\deg_1(m) \geq d_1$, and $\deg_2(m) \geq d_2$. Note that these degrees should not be confused with $\deg_q$ and $\deg_t$ defined above.

Consider some $f \in R$ of the form

$$f = \varphi(b - d) + \psi(ad - bc) \in M$$

for some $\varphi, \psi \in R$. Notice that for any $\gamma \in R$, we can modify the coefficients φ, ψ by simultaneously substituting:

$$\varphi \to \varphi + \gamma(ad - bc), \qquad \psi \to \psi - \gamma(b - d) \tag{3.1}$$

without changing f.

LEMMA 3.2.1. *If*

$$f = \varphi(b - d) + \psi(ad - bc) \in M$$

for some $\varphi, \psi \in R$, then up to the relation (3.1), we can assume that $\varphi \in M$.

PROOF. Since M is a monomial ideal, $\varphi(b - d) + \psi(ad - bc) \in M$ if and only if every monomial term of $\varphi(b-d)+\psi(ad-bc)$ is in M. If m is a monomial in the expansion of this expression, either m is in M, or m cancels with some other monomial.

When we expand, all monomials come in pairs from distributing $b - d$ or $ad - bc$. These pairs look like $m\left(\frac{b}{d}\right) - m$ or $m\left(\frac{ad}{bc}\right) - m$, where each term is appropriately divisible so that there are no denominators. If one of these monomials cancels with another monomial, that other monomial also must be part of a pair like the above. For example,

$$\frac{m}{cd}(ad - bc) + \frac{m}{d}(b - d) = m\left(\frac{b}{d}\right)\left(\frac{ad}{bc}\right) - m\left(\frac{b}{d}\right) + m\left(\frac{b}{d}\right) - m = m\left(\frac{b}{d}\right)\left(\frac{ad}{bc}\right) - m.$$

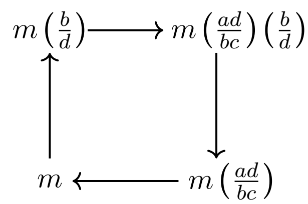
We can continue to follow a chain of cancellation until either we get two terminal monomials that do not cancel with anything, or we eventually reach a monomial that cancels with the starting monomial m, creating a cycle. We can visualize these chains of cancellations by oriented paths in a 2 dimensional lattice. Vertices represent monomials, vertical edges represent a difference of two monomials of the form $m\left(\frac{b}{d}\right)-m$, and horizontal edges represent a difference of the form $m\left(\frac{ad}{bc}\right)-m$. The full path represents the sum of all the pairs of monomials represented by each edge, and the end vertices of the path are the terminal monomials. For example, the above cancellation can be represented by the path:

$$\begin{array}{ccc} m\left(\frac{b}{d}\right) & \longrightarrow & m\left(\frac{ad}{bc}\right)\left(\frac{b}{d}\right) \\ \uparrow & & \\ m & & \end{array}$$

Cycles in these paths correspond exactly to the equivalence

$$\varphi(b-d)+\psi(ad-bc)=(\varphi+\gamma(ad-bc))(b-d)+(\psi-\gamma(b-d))(ad-bc).$$

To see this, let m be a monomial. A cycle looks like this:



This corresponds to the identity

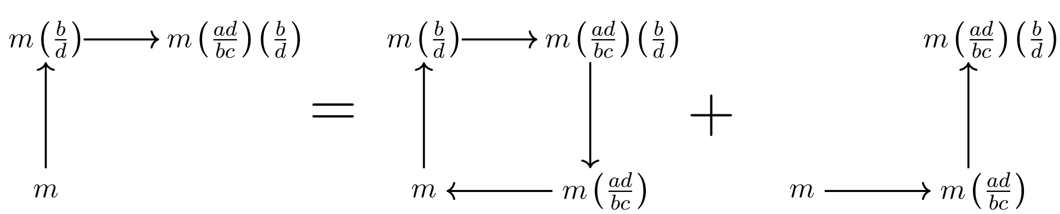
$$\frac{m}{d}(b-d)+\frac{m}{cd}(ad-bc)-\frac{ma}{bc}(b-d)-\frac{m}{bc}(ad-bc)=0.$$

For all of these terms to be monomials, b,c,d must all divide m. Multiplying through by bcd and grouping, we get

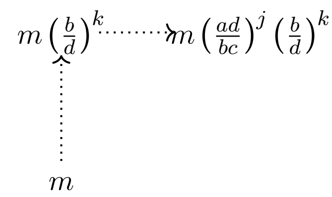
$$-m(ad-bc)(b-d)+m(b-d)(ad-bc)=0.$$

Adding this cycle corresponds to using the above equivalence with $\gamma=m$. Adding any number of these cycles along our path corresponds to modifying the coefficients φ and ψ with relation (3.1) without changing the overall sum f.

We can add and subtract this square loop to any path in order to both eliminate any loops, and to reorder cancellation. For example:



So up to relation (3.1), we can make any path into one with vertical edges first and horizontal edges after, going from m to $m\left(\frac{b}{d}\right)^k$ to $m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j$, with k and j possibly negative or 0.



This reduced path corresponds to $\varphi(b-d)+\psi(ad-bc)$, where

$$\varphi = \frac{m\left(\frac{b}{d}\right)^k - m}{b-d} \text{ and } \psi = \frac{m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j - m\left(\frac{b}{d}\right)^k}{ad-bc}.$$

Notice for any monomial m, $m\left(\frac{ad}{bc}\right)^j \in M$ if and only if $m \in M$, as multiplying by $\frac{a}{b}$ does not change the combined (a,b) degree of a monomial, and the same for $\frac{c}{d}$.

Given a reduced path as above, we know that the terminal monomials m and $m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j$ are in M. But since $m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j \in M$, by the above, $m\left(\frac{b}{d}\right)^k$ is in M. Now we want to show that $\varphi \in M$. Since $m\left(\frac{b}{d}\right)^k \in M$, it follows that $\frac{m}{d^k} \in (c,d)^{d_2}$, since multiplying by b^k does not change the (c,d) degree. Since m is in $(a,b)^{d_1}$, it follows that $\frac{m}{d^k} \in (a,b)^{d_1}$, since it has the same (a,b) degree as m. So $\frac{m}{d^k} \in M$, and therefore indeed

$$\varphi = \frac{m}{d^k}\frac{b^k-d^k}{b-d} \in M.$$

For a general $\varphi(b-d)+\psi(ad-bc)$, we can break the terms into a sum of discrete cancellation chains,

$$\sum_i \left[\varphi_{1,i}(b-d) + \varphi_{2,i}(ad-bc)\right].$$

For each cancellation chain, we have shown that up to equivalence, $\varphi_{1,i} \in M$, and therefore

$$\varphi = \sum_i \varphi_{1,i} \in M.$$

□

LEMMA 3.2.2. *We have that $\varphi(b-d) \in M$ if and only if $\varphi \in M$, and $\varphi(a-c) \in M$ if and only if $\varphi \in M$.*

PROOF. This is essentially what the final argument of the above proof shows. If we expand the expression $\varphi(b-d)$ into monomials, then as in the proof of Lemma 3.2.1, we will get chains of cancellations with two terminal monomials that do not cancel, which looks like the path:

$$\begin{array}{c} m\left(\frac{b}{d}\right)^k \\ \uparrow \\ \vdots \\ m \end{array}$$

This chain corresponds to

$$\varphi = \frac{m\left(\frac{b}{d}\right)^k - m}{b-d},$$

and as before, $m, m\left(\frac{b}{d}\right)^k \in M$ together imply that $\varphi \in M$. Any general $\varphi(b-d)$ can be split into the sum of distinct cancellation chains, and thus $\varphi \in M$.

The same argument applies to $\varphi(a-c)$ since M is symmetric in $(b,d),(a,c)$. □

Next, we characterize how we can best express $f \in (a-c, b-d)^{d_1}$ in order to see when $f \in M$.

Lemma 3.2.3. *Any $f \in (a-c, b-d)$ can be written as*

$$f = \alpha_1(a-c) + \alpha_2(b-d) + \alpha_3(ad-bc),$$

where $\alpha_i \in R$ and α_1 is a polynomial in a and c only.

Proof. If $f \in (a-c, b-d)$, then

$$f = \gamma_1(a-c) + \gamma_2(b-d)$$

for some $\gamma_1, \gamma_2 \in R$. Observe that

$$b(a-c) = a(b-d) + (ad-bc),$$

and

$$d(a-c) = c(b-d) + (ad-bc).$$

So by applying this to any term in γ_1 with a factor of b or d, we can ensure that γ_1 only depends on a and c. □

Lemma 3.2.4. *If $f \in (a-c, b-d)^{d_1} \cap M$, we can write $f = \sum_i f_i$, where*

$$f_i = \sum_{j=0}^{d_1-i} \alpha_{i,j}(a-c)^i(b-d)^{d_1-i-j}(ad-bc)^j$$

and each $f_i \in M$.

Proof. Consider $f \in (a-c, b-d)^{d_1}$. Following Lemma 3.2.3, we can express f as a linear combination of products of $(a-c), (b-d)$, and $(ad-bc)$,

$$f = \sum_{i,j} \alpha_{i,j}(a-c)^i(b-d)^{d_1-i-j}(ad-bc)^j.$$

Further, we can assume that every coefficient of a term with a factor of $(a-c)$ depends only on a and c, because for any term with coefficient α with $i>0$, we can write

$$\alpha(a-c)^i(b-d)^{d_1-i-j}(ad-bc)^j$$

$$= [\alpha(a-c)]\,(a-c)^{i-1}(b-d)^{d_1-i-j}(ad-bc)^j$$

$$= [\alpha_1(a-c)+\alpha_2(b-d)+\alpha_3(ad-bc)]\,(a-c)^{i-1}(b-d)^{d_1-i-j}(ad-bc)^j$$

where α_1 only depends on a and c by Lemma 3.2.3. We can continue this process by induction until all the modified coefficients $\alpha_{i,j}$ only depend on a and c.

Now we will group terms of f by their combined (b,d) degree. For all $i>0$, since $\alpha_{i,j}$ depends only on a and c, we know that the combined b,d degree of every monomial of

$$\alpha_{i,j}(a-c)^i(b-d)^{d_1-i-j}(ad-bc)^j$$

is $k = \deg_t(m) = d_1 - i$. If any monomial from this term cancels, it must cancel with another term with the same (b,d) degree. So in fact, every monomial with $\deg_t(m) = k$ must come from the sum

$$f_i = \sum_{j=0}^{k} \alpha_{i,j}(a-c)^i(b-d)^{k-j}(ad-bc)^j$$

$$= (a-c)^i \sum_{j=0}^{k} \alpha_{i,j}(b-d)^{k-j}(ad-bc)^j$$

with fixed i. In other words, monomials can cancel within each f_i, but not between them. This implies that for all $i>0$, each $f_i \in M$, since after internal cancellation, each f_i is a sum of monomials in M.

Since $f = \sum_i f_i$ is in M and $f_i \in M$ for all $i>0$,

$$f_0 = \sum_{j=0}^{d_1} \alpha_{i,j} (b-d)^{k-j} (ad-bc)^j$$

must also be in M.

□

By Corollary 3.2.2, we know that

$$f_i = (a-c)^i \sum_{j=0}^{k} \alpha_{i,j} (b-d)^{k-j} (ad-bc)^j \in M$$

implies that

$$\sum_{j=0}^{k} \alpha_{i,j} (b-d)^{k-j} (ad-bc)^j \in M. \tag{3.2}$$

Now we fix i and look at a single f_i, which we will call f to avoid unnecessary indices. Also let $k = d_1 - i$ as before.

Proposition 3.2.5. *If*

$$f = \sum_{j=0}^{k} \alpha_j (b-d)^{k-j} (ad-bc)^j \in M,$$

for some $\alpha_j \in R$, then there exists $\alpha'_j \in R$ such that

$$f = \sum_{j} \alpha'_j (b-d)^{k-j} (ad-bc)^j$$

and each term $\alpha'_j (b-d)^{k-j} (ad-bc)^j$ of the sum is in M.

Proof. We can rewrite f as

$$f = \sum_{j=0}^{k-1} \left[\varphi_j (b-d) + \psi_j (ad-bc)\right] (b-d)^{k-1-j} (ad-bc)^j, \tag{3.3}$$

where initially $\varphi_j = \alpha_j$ for all $0 \le j \le k-1$, $\psi_{k-1} = \alpha_k$, and the rest of the ψ_j's are 0. Essentially we have taken the previous sum for f and added some redundant terms; in particular, ψ_j and φ_{j+1} are coefficients for like terms for $0 \le j \le k-1$. So we have two relations we can use to modify the coefficients of (3.3) without changing the sum:

$$\psi_j \to \psi_j + \gamma, \qquad \varphi_{j+1} \to \varphi_{j+1} - \gamma \tag{3.4}$$

$$\varphi_j \to \varphi_j + \gamma(ad-bc), \qquad \psi_j \to \psi_j - \gamma(b-d). \tag{3.5}$$

The first comes from the redundant coefficients, and the second is the same relation (3.1) used in Lemma 3.2.1.

We will induct on $k = d_1 - i$. If $k = 0$, then $i = d_1$, and our sum (3.2) is just the single term $\alpha_0(a-c)^{d_1}$, which is in M by assumption. Now assume by induction that if

$$\sum_{j=0}^{k-1} \alpha_j (b-d)^{k-1-j}(ad-bc)^j$$

is in M, then we can modify the coefficients using only (3.5) to get all terms $\alpha_j(b-d)^{k-1-j}(ad-bc)^j$ in M. We can apply this to (3.3) with $\alpha_j = \varphi_j(b-d) + \psi_j(ad-bc)$. Using (3.5) on the α_j's actually just amounts to using (3.4) on the φ_j's and ψ_j's, as adding $\gamma(ad-bc)$ to α_j is the same as adding γ to ψ_j, and subtracting $\gamma(b-d)$ from α_j is the same as subtracting γ from φ_j. So the inductive hypothesis implies that up to (3.4), we can get

$$[\varphi_j(b-d) + \psi_j(ad-bc)]\,(b-d)^{k-1-j}(ad-bc)^j$$

in M for all j. By Corollary 3.2.2, this implies that

$$[\varphi_j(b-d) + \psi_j(ad-bc)]\,(ad-bc)^j \in M. \tag{3.6}$$

Notice that multiplying a monomial (polynomial) by $(ad-bc)$ raises its combined (a,b) degree and (c,d) degree each by one. So (3.6) is in $M = (a,b)^{d_1} \cap (c,d)^{d_2}$ if and only if $[\varphi_j(b-d) + \psi_j(ad-bc)] \in N = (a,b)^{d_1-j} \cap (c,d)^{d_2-j}$. Now we apply Lemma 3.2.1 on N to get both $\varphi_j(b-d)$ and $\psi_j(ad-bc)$ in N, and then when multiply by $(ad-bc)^j$, we get that both terms of (3.6) are in M.

So we have shown that if

$$f = \sum_{j=0}^{k-1} [\varphi_j(b-d) + \psi_j(ad-bc)]\,(b-d)^{k-1-j}(ad-bc)^j,$$

up to relations (3.4) and (3.5), we can get all terms of this sum to be in M. Now simply recombine like terms to get

$$f = \sum_j \alpha_j (b-d)^{d_1-i-j}(ad-bc)^j$$

with all terms in M as desired.

□

Now we can lay out a basis for the ideal

$$J(d_1, d_2) = (a, b)^{d_1} \cap (c, d)^{d_2} \cap (a - c, b - d)^{d_1}$$

over $\mathbb{C}$.

Proof of Proposition 3.1.3. We know by Proposition 3.2.5 that $J(d_1, d_2)$ is generated by polynomials of the form

$$\alpha_{i,j}(a - c)^i (b - d)^{d_1 - i - j}(ad - bc)^j,$$

where $\alpha_{i,j}$ only depends on a, c for $i > 0$. Here $0 \le j \le d_1$ and $0 \le i \le d_1 - j$. So as a vector space, $J(d_1, d_2)$ is generated by

$$a^\alpha b^\beta c^\gamma d^\delta (a - c)^i (b - d)^{d_1 - i - j}(ad - bc)^j$$

with the conditions that $\alpha + \beta + j \ge d_1$, $\gamma + \delta + j \ge d_2$, and $\beta = \delta = 0$ if $i > 0$. Among these generators, the only kind of relations remaining are those that come from the fact that $bc = ad - (ad - bc)$. If $i > 0$, then this relation is irrelevant, and we get linearly independent generators of the form

$$m(a, c)A_{i,j}.$$

If $i = 0$ and $\beta, \gamma > 0$, then we can write

$$a^\alpha b^\beta c^\gamma d^\delta = a^{\alpha+1} b^{\beta-1} c^{\gamma-1} d^{\delta+1} + a^\alpha b^{\beta-1} c^{\gamma-1} d^\delta (ad - bc).$$

Continue reducing bc this way until we end up in one of the following situations:

(1) $\gamma = 0$, $\beta \ne 0$, in which case we are left with

$$m(a, b, d)B_{\beta,j}$$

with $0 \le j \le d_1$ and $1 \le \beta \le d_1 - j$.

(2) $\beta = 0$, $\gamma \ne 0$, in which case we are left with

$$m(a, c, d)C_{\gamma,j}$$

with $0 \le j \le d_1$ and $1 \le \gamma \le d_2 - j$.

(3) $\beta = 0$, $\gamma = 0$, in which case we are left with

$$m(a, d)D_j$$

with $0 \le j \le d_1$.

(4) The exponent of $(ad - bc)$ is greater than or equal to d_1, in which case we are left with a linear combination of terms of the form

$$m(a,b,c,d)(ad-bc)^{d_1}.$$

□

Theorem 3.1.1 immediately follows from Proposition 3.1.3.

3.2.2. Hilbert Series Calculation. Let us compute the Hilbert series using the basis in Proposition 3.1.3.

LEMMA 3.2.6. *We have*

$$\sum_{\alpha+\beta\ge s} q^{\alpha}t^{\beta} = \frac{q^s}{(1-q)(1-t/q)} + \frac{t^s}{(1-t)(1-q/t)}.$$

PROOF. We have

$$\sum_{\alpha+\beta\ge s} q^{\alpha}t^{\beta} = \sum_{\beta=0}^{s-1} \frac{q^{s-\beta}t^{\beta}}{(1-q)} + \sum_{\beta=s}^{\infty} \frac{t^{\beta}}{1-q} =$$

$$\frac{q^s - t^s}{(1-q)(1-t/q)} + \frac{t^s}{(1-q)(1-t)}.$$

Now we can use the identity

$$\frac{1}{(1-q)(1-t)} - \frac{1}{(1-q)(1-t/q)} = \frac{1}{(1-t)(1-q/t)}.$$

□

THEOREM 3.2.7. *The bigraded Hilbert series of $J(d_1,d_2)$ is equal to*

$$\frac{q^{2d_1+d_2}}{(1-q)^2(1-t/q)(1-t/q^2)} + \frac{t^{2d_1+d_2}}{(1-t)^2(1-q/t)(1-q/t^2)} +$$

$$\frac{q^{d_1}t^{d_2}(1+t)}{(1-q)(1-t)(1-q/t)(1-t^2/q)} + \frac{q^{d_2}t^{d_1}(1+q)}{(1-t)(1-q)(1-t/q)(1-q^2/t)}.$$

PROOF. We compute the contribution of various basis elements.

1. The contribution of $m(a,c)A_{i,j},\ j \le d_1 - 1$ equals

$$\frac{1}{(1-q)^2} \sum_{j=0}^{d_1-1} \sum_{i=1}^{d_1-j} q^{d_1+d_2-j+i} t^{d_1-i} =$$

$$\frac{1}{(1-q)^2}\sum_{j=0}^{d_1-1}\frac{q^{d_1+d_2-j+1}t^{d_1-1}-q^{2d_1+d_2-2j+1}t^{j-1}}{(1-q/t)}=$$

$$\frac{q^{d_1+d_2+1}t^{d_1-1}-q^{d_2+1}t^{d_1-1}}{(1-q)^2(1-q^{-1})(1-q/t)}-\frac{q^{2d_1+d_2+1}t^{-1}-q^{d_2+1}t^{d_1-1}}{(1-q)^2(1-q/t)(1-t/q^2)}.$$

2. The set $m(a,b,d)B_{i,j}\cup m(a,d)D_j,\ j\le d_1-1$ consists of elements

$$a^\alpha b^\beta d^\gamma(b-d)^{d_1-j}(ad-bc)^j,\ \alpha+\beta\ge d_1-j,\ \gamma\ge d_2-j,$$

so by Lemma 3.2.6 we get the Hilbert series

$$\sum_{j=0}^{d_1-1}\left[\frac{q^{d_1-j}}{(1-q)(1-t/q)}+\frac{t^{d_1-j}}{(1-t)(1-q/t)}\right]\frac{q^jt^{d_1+d_2-j}}{(1-t)}=$$

$$\sum_{j=0}^{d_1-1}\left[\frac{q^{d_1}t^{d_1+d_2-j}}{(1-t)(1-q)(1-t/q)}+\frac{q^jt^{2d_1+d_2-2j}}{(1-t)^2(1-q/t)}\right]=$$

$$\frac{q^{d_1}t^{d_1+d_2}-q^{d_1}t^{d_2}}{(1-t^{-1})(1-t)(1-q)(1-t/q)}+\frac{t^{2d_1+d_2}-q^{d_1}t^{d_2}}{(1-q/t^2)(1-t)^2(1-q/t)}$$

3. Similarly, for $m(a,c,d)C_{i,j}\cup m(a,d)D_j,\ j\le d_1-1$ we get

$$a^\alpha c^\beta d^\gamma(b-d)^{d_1-j}(ad-bc)^j,\ \alpha\ge d_1-j,\ \beta+\gamma\ge d_2-j,$$

so the Hilbert series equals

$$\sum_{j=0}^{d_1-1}\left[\frac{q^{d_2-j}}{(1-q)(1-t/q)}+\frac{t^{d_2-j}}{(1-t)(1-q/t)}\right]\frac{q^{d_1}t^{d_1}}{(1-q)}=$$

$$\frac{q^{d_1+d_2}t^{d_1}-q^{d_2}t^{d_1}}{(1-q^{-1})(1-q)^2(1-t/q)}+\frac{q^{d_1}t^{d_1+d_2}-q^{d_1}t^{d_2}}{(1-t^{-1})(1-q)(1-t)(1-q/t)}.$$

4. We overcount by $m(a,d)D_j,\ j\le d_1-1$ which contribute

$$\frac{1}{(1-q)(1-t)}\sum_{j=0}^{d_1-1}q^{d_1}t^{d_1+d_2-j}=\frac{q^{d_1}t^{d_1+d_2}-q^{d_1}t^{d_2}}{(1-t^{-1})(1-q)(1-t)}$$

5. For $j=d_1$ we have special terms

$$a^\alpha b^\beta c^\gamma d^\delta(ad-bc)^{d_1},\ \gamma+\delta\ge d_2-d_1,$$

which contribute

$$\frac{1}{(1-q)(1-t)}\left[\frac{q^{d_2-d_1}}{(1-q)(1-t/q)}+\frac{t^{d_2-d_1}}{(1-t)(1-q/t)}\right]q^{d_1}t^{d_1}.$$

6. Finally, we collect the coefficients at similar terms:

$$q^{d_1+d_2}t^{d_2}\left[\frac{qt^{-1}}{(1-q)^2(1-q^{-1})(1-q/t)}+\frac{1}{(1-q^{-1})(1-q)^2(1-t/q)}\right]=0;$$

$$q^{d_2}t^{d_1}[-\frac{1}{(1-q)^2(1-q^{-1})(1-q/t)}+\frac{1}{(1-q)^2(1-q/t)(1-t/q^2)}-$$

$$\frac{1}{(1-q^{-1})(1-q)^2(1-t/q)}+\frac{1}{(1-q)(1-t)(1-q)(1-t/q)}]=\frac{q^{d_2}t^{d_1}(1+q)}{(1-t)(1-q)(1-t/q)(1-q^2/t)}.$$

$$q^{2d_1+d_2}\left[-\frac{qt^{-1}}{(1-q)^2(1-q/t)(1-t/q^2)}\right]+\frac{t^{2d_1+d_2}}{(1-q/t^2)(1-t)^2(1-q/t)}=$$

$$\frac{q^{2d_1+d_2}}{(1-q)^2(1-t/q)(1-t/q^2)}+\frac{t^{2d_1+d_2}}{(1-t)^2(1-q/t)(1-q/t^2)};$$

$$q^{d_1}t^{d_1+d_2}[\frac{1}{(1-t^{-1})(1-t)(1-q)(1-t/q)}+\frac{1}{(1-t^{-1})(1-q)(1-t)(1-q/t)}-$$

$$\frac{1}{(1-t^{-1})(1-q)(1-t)}]=0;$$

$$q^{d_1}t^{d_2}[-\frac{t^{-1}}{(1-t^{-1})(1-t)(1-q)(1-t/q)}-\frac{qt^{-2}}{(1-q/t^2)(1-t)^2(1-q/t)}-$$

$$\frac{t^{-1}}{(1-t^{-1})(1-q)(1-t)(1-q/t)}+\frac{t^{-1}}{(1-t^{-1})(1-q)(1-t)}+\frac{1}{(1-q)(1-t)(1-t)(1-q/t)}]=$$

$$\frac{q^{d_1}t^{d_2}(1+t)}{(1-q)(1-t)(1-q/t)(1-t^2/q)}.$$

□

CHAPTER 4

Coxeter Braid Recursions

This chapter is based on work from [32] and [33] for 3 and 4 strands respectively.

Here we compute Khovanov-Rozansky homology of pure Coxeter braids on 3 and 4 strands. For our purposes, we think of Khovanov-Rozansky homology HHH as a certain functor from complexes of Soergel bimodules to triply graded vector spaces over $\mathbb{C}$. To each crossing σ_i in the braid group, we associate a 2-term complex called a Roquier complex. To any braid β, we associate to it a complex of Soergel bimodules by taking the tensor product of Roquier complex for each crossing in the diagram of β from bottom to top. We define HHH(β) to be HHH of the complex associated to its braid diagram. We refer to [16] and references therein for all details.

For some braids, we can use the recursions for triply graded Khovanov-Rozansky homology from [8], as described in [16] to compute HHH. Starting with a braid diagram, we insert certain auxiliary complexes of Soergel bimodules K_n and apply the properties below until all strands are closed up. Figure 4.1 below shows the properties of K_n that we will use in our recursions.

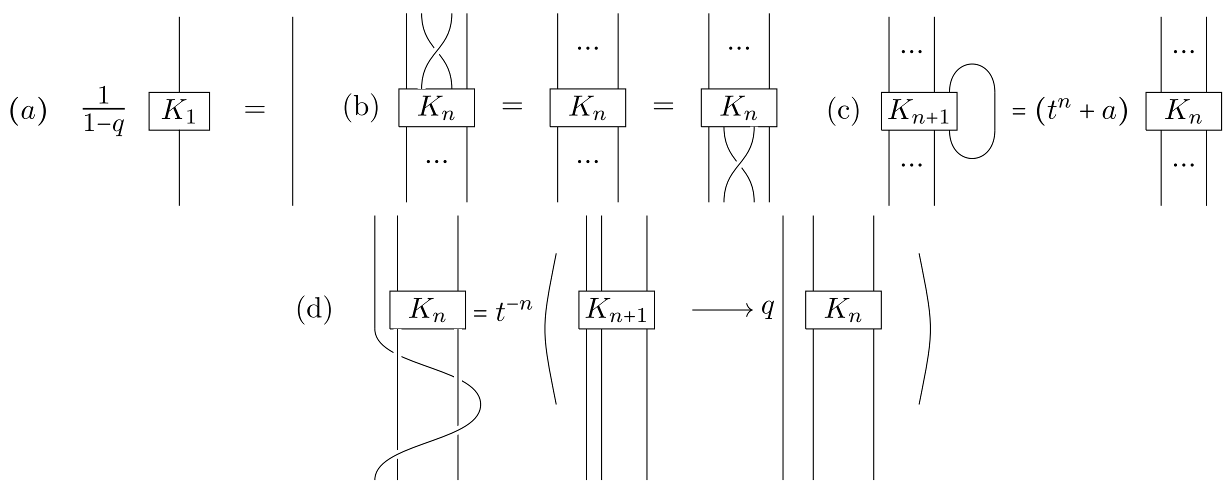


FIGURE 4.1. Recursions for HHH

Essentially, Figure 4.1(a) says that we can insert a K_1 anywhere up to a grading shift. Figure 4.1(b) says that K_n absorbs crossings within the n strands that it spans. Figure 4.1(c) allows us to 'close up' a strand passing through a K_n as long as there are no crossings in the way, and Figure 4.1(d) is the main recursive step that allows K_n to grow if there is a strand wrapping around it. The three gradings of $\mathrm{HHH}(\beta)$ are typically denoted Q, T, and A. However, in line with [8], we use the change of variables

$$q = Q^2, \quad t = T^2 Q^{-2}, \quad a = AQ^{-2}.$$

DEFINITION 4.0.1. For any braid or braid-like diagram β, we say that $\mathrm{HHH}(\beta)$ is parity if it is supported in only even homological (T) degrees, where $T^2 = \frac{t}{q}$.

Note that $\mathrm{HHH}(\beta)$ is parity if and only if its graded Hilbert series is rational in terms of (q, a, t).

REMARK 4.0.2 ([16]). If both diagrams on the right-hand side of Figure 4.1(d) are parity, then so is the left-hand side, and we can replace the map with addition on the level of rational functions (or a direct sum on the level of complexes). This implies that if we use Figure 4.1(d) repeatedly to break down a braid β into complexes that are known to be parity, then every complex along the way, including β itself, is parity.

We generally restrict our attention to positive braids, as a negative braids are almost never parity. To make the calculation simpler, we translate the braid diagrams into equations where multiplication represents vertical stacking from bottom to top (or equivalently a tensor product of Soergel bimodules). So for example, 4.1(d) translates to

$$K_n \cdot \mathrm{JM}_{n+1} = t^{-n} K_{n+1} + q t^{-n} K_n.$$

We will regularly use the Jucys-Murphy braids

$$\mathrm{JM}_n = \sigma_{n-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{n-1}$$

and the full twist on n strands

$$\mathrm{FT}_n = \mathrm{JM}_2 \, \mathrm{JM}_3 \cdots \mathrm{JM}_n \, .$$

All Jucys-Murphy braids JM_n and full twists FT_n are assumed to be on the first n strands unless otherwise indicated. We will also make regular use of the fact that σ_i commutes with FT_n for all $1 \le i \le n$, σ_i commutes with JM_n for $1 < i < n$, and that JM_m and JM_n commute with each other for any m, n. In other words, the full twist commutes with all crossings on its strands, the Jucys-Murphy braid commutes with internal crossings, and all Jucys-Murphy's (on the initial strands) commute with each other.

Since K_n absorbs crossings by Figure 4.1(b), for $n \leq m$, we have that

$$K_n \mathrm{FT}_m = K_n \, \mathrm{JM}_{n+1} \cdots \mathrm{JM}_m \, .$$

We will also use the conjugation invariance of HHH:

$$\mathrm{HHH}(\alpha\beta) = \mathrm{HHH}(\beta\alpha) \tag{4.1}$$

for any braids α and β. We will denote the conjugate braids by $\alpha\beta \sim \beta\alpha$. We will additionally use the result from Hogancamp [19] that $\mathrm{HHH}(\mathrm{FT}_n^k)$ is parity for all $n, k \geq 0$.

Finally, note that Figure 4.1(a), (b), and (d) are local, i.e. they hold on the level of homotopy equivalences of chain complexes. Braid conjugation and the closing up step (c) are non-local, and only hold on the level of HHH.

4.1. Computations on 3 strands

In this section we show that the link L_γ associated to γ as in (1.1) is parity for $n = 3$, and compute its Khovanov-Rozansky homology. It is the closure of the braid

$$\beta_{d_1,d_2} = (\mathrm{FT}_2)^{d_2-d_1} (\mathrm{FT}_3)^{d_1} = \mathrm{JM}_2^{d_2} \, \mathrm{JM}_3^{d_1} \, .$$

THEOREM 4.1.1. *For all* $d_2 \geq d_1$ $\mathrm{HHH}^{a=0}(\beta_{d_1,d_2})$ *is parity.*

PROOF. We can resolve the first FT_2 by the process:

$$\mathrm{FT}_2 = K_1 \cdot \mathrm{FT}_2 = K_1 \cdot \mathrm{JM}_2 = t^{-1} K_2 + q t^{-1} K_1.$$

Since K_2 absorbs any crossings on the first two strands, this leaves us with

$$\beta_{d_1,d_2} = t^{-1} K_2 \cdot (\mathrm{FT}_3)^{d_1} + q t^{-1} \beta_{d_1,d_2-1}.$$

Now we resolve $\alpha_{d_1} = K_2 \cdot (\mathrm{FT}_3)^{d_1}$ by first writing FT_3 as the product $\mathrm{JM}_2 \cdot \mathrm{JM}_3$. The JM_2 gets absorbed by K_2, so we get

$$K_2 \cdot (\mathrm{FT}_3)^{d_1} = K_2 \cdot \mathrm{JM}_3 \cdot (\mathrm{FT}_3)^{d_1 - 1} = t^{-2} K_3 + q t^{-2} K_2 (\mathrm{FT}_3)^{d_1 - 1},$$

where the K_3 has absorbed the rest of the FT_3's. When we close up K_3, we get t^3, and $\alpha_0 = K_2$ closes up to $\frac{t}{1-q}$. So we get a simple recursion:

$$\mathrm{HHH}^{a=0}(\alpha_{d_1}) = t + q t^{-2} \mathrm{HHH}^{a=0}(\alpha_{d_1 - 1}),$$

with $\mathrm{HHH}^{a=0}(\alpha_0) = \frac{t}{1-q}$. We can write this in a closed form as

$$\mathrm{HHH}^{a=0}(\alpha_{d_1}) = t \left[\frac{1 - (qt^{-2})^{d_1}}{1 - qt^{-2}} + \frac{(qt^{-2})^{d_1}}{q - 1} \right].$$

So overall we have the recursive relation

$$\mathrm{HHH}^{a=0}(\beta_{d_1, d_2}) = \left[\frac{1 - (qt^{-2})^{d_1}}{1 - qt^{-2}} + \frac{(qt^{-2})^{d_1}}{q - 1} \right] + q t^{-1} \mathrm{HHH}^{a=0}(\beta_{d_1, d_2 - 1}), \tag{4.2}$$

with $\beta_{d_1, 0} = (FT_3)^{d_1} = T(3d_1, 3)$, which is parity (with known homology) by [8]. □

To compare this to $H_*(\mathrm{Sp}_\gamma)$, it can be checked by direct computation that the Hilbert series $H(d_1, d_2)$ of Sp_γ satisfies essentially the same recursion (4.2). But we can also apply a theorem of Gorsky and Hogancamp (Proposition 5.5 in [14]). Here HY is the y-ified Khovanov Rozansky homology defined in [14].

Theorem 4.1.2 ([14]). *Assume that* $\beta = JM_1^{d_n} \dots JM_n^{d_1}$, $d_n \geq d_{n-1} \geq \dots \geq d_1$, *and* $\mathrm{HHH}^{a=0}(\beta)$ *is parity. Then:*

(1) $\mathrm{HY}^{a=0}(\beta) = \mathrm{HHH}^{a=0}(\beta) \otimes \mathbb{C}[y_1, \dots, y_n]$ *and* $\mathrm{HHH}^{a=0}(\beta) = \mathrm{HY}^{a=0}(\beta)/(y)$

(2) $I(d_1, \dots, d_n) \subseteq \mathrm{HY}^{a=0}(\beta) \subseteq J'(d_1, \dots, d_n)$, *where* I *is the product*

$$I(d_1, \dots, d_n) = J'(1, \dots, 1)^{d_1} \cdot J'(0, 1, \dots, 1)^{d_2 - d_1} \cdot \dots \cdot J'(0, \dots, 0, 1)^{d_{n-1} - d_{n-2}}.$$

In our case β can be expressed exactly as above, and Theorem 4.1.1 along with Corollary 3.1.5 implies that for $n = 3$:

$$\mathrm{HY}^{a=0}(L_\gamma) = J'(d_1, d_2).$$

Note the analogy between the relationship of HY to HHH in statement (1) of Theorem 1.3.3, and the relationship of $H_*^T(\mathrm{Sp}_\gamma)$ to $H_*(\mathrm{Sp}_\gamma)$ due to Fact 2.1.7. The result is the following weaker version of Conjecture 1.0.2.

THEOREM 4.1.3. *For $n = 3$ and γ as in 1.1,*

$$\mathrm{HY}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, \mathbf{x}^{\pm}] = \Delta H_*^T(\mathrm{Sp}_\gamma)$$

and

$$\mathrm{HHH}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, \mathbf{x}^{\pm}] = H_*(\mathrm{Sp}_\gamma).$$

In order to show Theorem 4.1.3, we used Corollary 3.1.5, but one could instead show that the link splitting map defined in [14] is canonical for β as above, analogous to Proposition 6.11 in [14]. This means that Theorem 4.1.3 can be generalized to higher n solely by showing that β is parity, such as with this same recursive method.

4.2. Computations on 4 strands

Here we compute Khovanov-Rozansky homology of Coxeter braids on 4 strands. Let

$$\beta = \beta(d_1, d_2, d_3, d_4) = (\mathrm{FT}_2)^{d_3 - d_2} \cdot (\mathrm{FT}_3)^{d_2 - d_1} \cdot (\mathrm{FT}_4)^{d_1}.$$

DEFINITION 4.2.1. Let

- $A(n, m, l) = K_1 \mathrm{FT}_2^l \mathrm{FT}_3^m \mathrm{FT}_4^n$,
- $B(n, m) = K_2 \mathrm{FT}_3^m \mathrm{FT}_4^n$,
- $C(n) = K_3 \mathrm{FT}_4^n$.

In this notation,

$$K_1 \beta(d_1, d_2, d_3, d_4) = A(d_1, d_2 - d_1, d_3 - d_2).$$

Assuming that all braids are parity, we can write Figure 4.1(d) in the form

$$K_n \, \mathrm{JM}_{n+1} = t^{-n} K_{n+1} + q t^{-n} K_n.$$

By Figure 4.1(b), $K_n \, \mathrm{JM}_{n+1} = K_n \mathrm{FT}_{n+1}$, so we immediately get the following recursions:

$$A(n,m,l) = t^{-1}B(n,m) + qt^{-1}A(n,m,l-1) \tag{4.3}$$

$$B(n,m) = t^{-2}C(n) + qt^{-2}B(n,m-1) \tag{4.4}$$

$$C(n) = t^{-3}K_4 + qt^{-3}C(n-1) \tag{4.5}$$

These recursions let us proceed by induction, as every application either decreases n, m, or l or shifts us from A to B or B to C. Also note that they don't involve any closing up or conjugation, so they are all local. The majority of the work lies in the 'base cases', when one or more of l, m, n are 0.

4.2.1. Recursion for $C(n)$. Here we calculate HHH recursively for the braids $C(n)$ and for a twisted version $\sigma_3^{2k}C(n)$ that we will need in Section 4.2.2.

LEMMA 4.2.2. *The braid $C(n)$ is parity for all $n \geq 0$, and*

$$\mathrm{HHH}(C(n)) = t^{-3}\frac{1-(qt^{-3})^n}{1-qt^{-3}}\mathrm{HHH}(K_4) + \frac{(qt^{-3})^n(t^2+a)(t+a)(1+a)^2}{(1-q)}.$$

PROOF. The braid $C(0)$ has K_3 on the first 3 strands and an empty fourth strand. We can close up the K_3 using Figure 4.1(c) and introduce a K_1 to close up the last strand, so

$$\mathrm{HHH}(C(0)) = (t^2+a)(t+a)(1+a)^2/(1-q).$$

We also have that

$$\mathrm{HHH}(K_4) = (t^3+a)(t^2+a)(t+a)(1+a).$$

So by induction, both terms on the right hand side of recursion (4.5) are parity. Therefore $C(n)$ is parity, and can be computed recursively. □

LEMMA 4.2.3. *The braid $\sigma_3^{2k}C(n)$ is parity for all $k,n \geq 0$, and*

$$\mathrm{HHH}(\sigma_3^{2k}C(n)) = t^{-3}\frac{1-(qt^{-3})^n}{1-qt^{-3}}\mathrm{HHH}(K_4) + (qt^{-3})^n(t^2+a)(t+a)(1-q)\mathrm{HHH}(\mathrm{FT}_2^k). \tag{4.6}$$

PROOF. For $\sigma_3^{2k}C(0)$, we can close up (using Figure 4.1(c)) the first two strands running through K_3, and then we are left with exactly k full twists on the remaining two strands along with a K_1. So

$$\mathrm{HHH}(\sigma_3^{2k}C(0)) = (t^2+a)(t+a)(1-q)\mathrm{HHH}(\mathrm{FT}_2^k).$$

For $\sigma_3^{2k}C(n)$, we apply recursion (4.5) locally to get

$$\sigma_3^{2k}C(n) = t^{-3}\sigma_3^{2k}K_4 + qt^{-3}\sigma_3^{2k}C(n-1).$$

The σ_3 crossings get absorbed by K_4 (using Figure 4.1(b)), so by induction on n, we conclude that $\sigma_3^{2k}C(n)$ is parity with the desired formula for HHH. □

4.2.2. Recursions for $B(n,m)$. Here we calculate HHH recursively for the braids $B(n,m)$ and for the twisted braids $\sigma_2^2B(n,m)$ that we will need in Section 4.2.3.

LEMMA 4.2.4. *The braid $\sigma_3^{2k}B(0,0)$ is parity for all $k \geq 0$, and*

$$\mathrm{HHH}(\sigma_3^{2k}B(0,0)) = (t+a)(1+a)\mathrm{HHH}(\mathrm{FT}_2^k).$$

PROOF. Recall that $B(0,0)$ consists only of a K_2 on the left two strands, which does not overlap with σ_3^{2k}. We close up the two strands running through K_2 (Figure 4.1(c)) and are left with k full twists on two strands. □

Now we consider the case $B(n,0)$.

LEMMA 4.2.5. *The braid $\sigma_3^{2k}B(n,0)$ is parity for all $n,k \geq 0$, and*

$$\mathrm{HHH}\left[\sigma_3^{2k}B(n,0)\right] = t^{-2}\mathrm{HHH}\left[\sigma_3^{2k}C(n)\right] + qt^{-2}\mathrm{HHH}\left[\sigma_3^{2k+2}B(n-1,1)\right]$$

$$= t^{-2}\mathrm{HHH}\left[\sigma_3^{2k}C(n)\right] + qt^{-4}\mathrm{HHH}\left[\sigma_3^{2k+2}C(n-1)\right] + q^2t^{-4}\mathrm{HHH}\left[\sigma_3^{2k+2}B(n-1,0)\right].$$

In particular we have that $B(n,0)$ is parity for all $n \geq 0$.

PROOF. We have that locally,

$$B(n,0) = K_2\,\mathrm{JM}_3\,\mathrm{JM}_4\,\mathrm{FT}_4^{n-1}$$

$$= t^{-2}K_3\,\mathrm{JM}_4\,\mathrm{FT}_4^{n-1} + qt^{-2}K_2\,\mathrm{JM}_4\,\mathrm{FT}_4^{n-1}$$

$$= t^{-2}C(n) + qt^{-2}K_2\,\mathrm{JM}_4\,\mathrm{FT}_4^{n-1}. \tag{4.7}$$

Here we used Figure 4.1(d) for $K_2\,\mathrm{JM}_3$, and the identity $K_3\,\mathrm{JM}_4 = K_3\mathrm{FT}_4$ which follows from Figure 4.1(b). Now note that $\mathrm{JM}_4 = \sigma_3\,\mathrm{JM}_3\,\sigma_3$, so

$$\sigma_3^{2k}K_2\,\mathrm{JM}_4\,\mathrm{FT}_4^{n-1} = \sigma_3^{2k}K_2\sigma_3\,\mathrm{JM}_3\,\sigma_3\mathrm{FT}_4^{n-1} = \sigma_3^{2k}\sigma_3K_2\,\mathrm{JM}_3\,\mathrm{FT}_4^{n-1}\sigma_3,$$

where we use the fact that σ_3 commutes with both FT_4 and K_2. By conjugation invariance (4.1) we have that

$$\mathrm{HHH}(\sigma_3^{2k}\sigma_3K_2\,\mathrm{JM}_3\,\mathrm{FT}_4^{n-1}\sigma_3) \simeq \mathrm{HHH}(\sigma_3^{2k+2}K_2\,\mathrm{JM}_3\,\mathrm{FT}_4^{n-1}) = \mathrm{HHH}(\sigma_3^{2k+2}B(n-1,1)).$$

Finally, a local application of recursion (4.4) tells us that

$$\sigma_3^{2k+2}B(n-1,1) = t^{-2}\left[\sigma_3^{2k+2}C(n-1)\right] + qt^{-2}\left[\sigma_3^{2k+2}B(n-1,0)\right].$$

We know that $\sigma_3^{2k+2}C(n-1)$ is parity by Lemma 4.2.3, and we know that $\sigma_3^{2k+2n}B(0,0)$ is parity by Lemma 4.2.9. So by induction on n, we can conclude that $\sigma_3^{2k}B(n,0)$ is parity. ☐

COROLLARY 4.2.6. *The braid $B(n,m)$ is parity for all $n,m \geq 0$.*

PROOF. We know that $B(n,0)$ is parity by Lemma 4.2.5 and that $C(n)$ is parity by Lemma 4.2.2. So we use recursion (4.4) and induction on m to conclude that $B(n,m)$ is parity. ☐

Now we show that a twisted version $\sigma_2^2B(n,m)$ is also parity. We start with the step that requires the most caution to keep all braids parity.

LEMMA 4.2.7. *If the right hand side is parity, then*

$$\mathrm{HHH}\left[\sigma_2^2\sigma_3^{2k-2}B(n,0)\right] = t^{-2}\mathrm{HHH}\left[\sigma_3^{2k-2}C(n)\right] + qt^{-4}\mathrm{HHH}\left[\widetilde{\mathrm{JM}}_3\sigma_3^{2k-2}C(n-1)\right] + q^2t^{-4}\mathrm{HHH}\left[\sigma_2^2\sigma_3^{2k}B(n-1,0)\right], \tag{4.8}$$

where $\widetilde{\mathrm{JM}}_3 = \sigma_3\sigma_2^2\sigma_3$ indicates the Jucys-Murphy element on the final 3 strands rather than the first 3 strands.

PROOF. Since the equation (4.7) holds locally, we can multiply both sides on the left by $\sigma_2^2\sigma_3^{2k-2}$. For the first term in (4.7), if we include the additional crossings, note that

$$t^{-2}\left[\sigma_2^2\sigma_3^{2k-2}C(n)\right] \sim t^{-2}\left[\sigma_3^{2k-2}K_3\mathrm{FT}_4^n\sigma_2^2\right] = t^{-2}\left[\sigma_3^{2k-2}K_3\sigma_2^2\mathrm{FT}_4^n\right] = t^{-2}\left[\sigma_3^{2k-2}C(n)\right].$$

Here we conjugate σ_2^2 and pass it through FT_4^n until it is absorbed by K_3. Therefore

$$t^{-2}\mathrm{HHH}\left[\sigma_2^2\sigma_3^{2k-2}C(n)\right] = t^{-2}\mathrm{HHH}\left[\sigma_3^{2k-2}C(n)\right].$$

For the second term in (4.7), we can again write $\mathrm{JM}_4 = \sigma_3\,\mathrm{JM}_3\,\sigma_3$, so

$$\sigma_2^2\sigma_3^{2k-2}K_2\,\mathrm{JM}_4\,\mathrm{FT}_4^{n-1} = \sigma_2^2\sigma_3^{2k-2}K_2\sigma_3\,\mathrm{JM}_3\,\sigma_3\mathrm{FT}_4^{n-1} = \sigma_2^2\sigma_3^{2k-1}K_2\,\mathrm{JM}_3\,\sigma_3\mathrm{FT}_4^{n-1}.$$

Applying recursion (4.4) gives

$$t^{-2}\left[\sigma_2^2\sigma_3^{2k-1}K_3\sigma_3\mathrm{FT}_4^{n-1}\right] + qt^{-2}\left[\sigma_2^2\sigma_3^{2k-1}K_2\sigma_3\mathrm{FT}_4^{n-1}\right]. \tag{4.9}$$

Now we slide σ_3 through FT_4 and conjugate by it in the first term in (4.9), and slide it past K_2 in the second term in (4.9), which gives

$$t^{-2}\left[\sigma_3\sigma_2^2\sigma_3^{2k-1}K_3\mathrm{FT}_4^{n-1}\right] + qt^{-2}\left[\sigma_2^2\sigma_3^{2k}K_2\mathrm{FT}_4^{n-1}\right]$$

$$= t^{-2}\left[\widetilde{\mathrm{JM}}_3\sigma_3^{2k-2}C(n-1)\right] + qt^{-2}\left[\sigma_2^2\sigma_3^{2k}B(n-1,0)\right].$$

□

So in order to fully resolve $\sigma_2^2\sigma_3^{2k-2}B(n,0)$, we need still to resolve the cases $\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(n)$ and $\sigma_2^2\sigma_3^{2k}B(0,0)$.

LEMMA 4.2.8. *The braid $\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(n)$ is parity.*

PROOF. Consider $n = 0$ first. We close up the first strand (apply Figure 4.1(c)), so

$$\mathrm{HHH}(\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(0)) = \mathrm{HHH}(\widetilde{\mathrm{JM}}_3\sigma_3^{2k}K_3) = (t^2 + a)\mathrm{HHH}(\mathrm{JM}_3\,\sigma_2^{2k}K_2).$$

Note that the first two braids here are on 4 strands, while the last is on 3 strands after the first strand was closed up. Now conjugate by JM_3 and apply Figure 4.1(d) to get

$$\mathrm{JM}_3\,\sigma_2^{2k}K_2 \sim \sigma_2^{2k}K_2\,\mathrm{JM}_3 = t^{-2}\left[\sigma_2^{2k}K_3\right] + qt^{-2}\left[\sigma_2^{2k}K_2\right].$$

The K_3 in the first term absorbs the extra crossings (Figure 4.1(b)), and we can close up the first strand of the second term (Figure 4.1(d)). So

$$t^{-2}\left[\sigma_2^{2k}K_3\right] + qt^{-2}\left[\sigma_2^{2k}K_2\right] = t^{-2}\left[K_3\right] + qt^{-2}(t+a)\left[K_1\mathrm{FT}_2^k\right],$$

where the final term is now on two strands. So we conclude that $\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(0)$ is parity, and

$$\mathrm{HHH}(\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(0)) = t^{-2}(t^2+a)\left[(t^2+a)(t+a)(1+a) + q(t+a)(1-q)\mathrm{HHH}(\mathrm{FT}_2^k)\right].$$

To resolve $\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(n)$, again we apply recursion (4.5). All extra crossings will be absorbed by K_4, so by induction on n and the above base case, $\widetilde{\mathrm{JM}}_3\sigma_3^{2k}C(n)$ is parity. □

LEMMA 4.2.9. *The braid* $\sigma_2^2\sigma_3^{2k}B(0,0)$ *is parity, and*

$$\mathrm{HHH}(\sigma_2^2\sigma_3^{2k}B(0,0)) = (t+a)t^{-1}\left[(t+a)(1-q) + q(1+a)\right]\mathrm{HHH}(\mathrm{FT}_2^k).$$

PROOF. By definition, $\sigma_2^2\sigma_3^{2k}B(0,0) = \sigma_2^2\sigma_3^{2k}K_2$. First, we close up one strand passing through K_2 on the left to obtain

$$(t+a)\sigma_1^2\sigma_2^{2k}K_1 \sim (t+a)\sigma_2^{2k}K_1\sigma_1^2 = (t+a)\sigma_2^{2k}K_1\,\mathrm{JM}_2\,.$$

on three strands. Then we use Figure 4.1(d) to get

$$\sigma_2^{2k}K_1\,\mathrm{JM}_2 = t^{-1}[\sigma_2^{2k}K_2] + qt^{-1}[\sigma_2^{2k}K_1].$$

For both terms, we can close up the first strand and have k full twists remaining on the last two strands, getting

$$t^{-1}\left[(t+a)(1-q)\mathrm{FT}_2^k + q(1+a)\mathrm{FT}_2^k\right].$$

□

LEMMA 4.2.10. *The braid* $\sigma_2^2B(n,m)$ *is parity.*

PROOF. We can see that $\sigma_2^2\sigma_3^{2k}B(n,0)$ is parity by Lemma 4.2.7, as all terms on the right hand side of (4.8) are parity by Lemma 4.2.8 and Lemma 4.2.7.

It follows that $\sigma_2^2 B(n,m)$ is parity from induction on recursion (4.4) and the above, as once again the extra crossings are absorbed by K_3. □

4.2.3. Recursions for $A(n,m,l)$. Now we know how to fully resolve all of the B's and C's, so the only thing left is A.

LEMMA 4.2.11. *If the right hand side is parity, then*

$$A(n,m,0) = t^{-1}B(n,m) + qt^{-1}\left[\sigma_2^2 A(n,m-1,1)\right]$$

$$= t^{-1}B(n,m) + qt^{-2}\left[\sigma_2^2 B(n,m-1)\right] + q^2t^{-3}B(n,m-1) + q^3t^{-3}A(n,m-1,0).$$

PROOF. First, use Figure 4.1(d) to get that

$$A(n,m,0) = K_1\mathrm{FT}_3^m\mathrm{FT}_4^n = K_1\,\mathrm{JM}_2\,\mathrm{JM}_3\,\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n$$

$$= t^{-1}K_2\,\mathrm{JM}_3\,\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n + qt^{-1}K_1\,\mathrm{JM}_3\,\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n. \tag{4.10}$$

Since $K_2\,\mathrm{JM}_3 = K_2\mathrm{FT}_3$, the first term in (4.10) is just $t^{-1}B(n,m)$. For the second term in (4.10), note that $\mathrm{JM}_3 = \sigma_2\,\mathrm{JM}_2\,\sigma_2$ and that σ_2 commutes with FT_3, FT_4, and K_1. Sliding one σ_2 to the top and conjugating by it and moving the other σ_2 down past K_1, we get that

$$K_1\,\mathrm{JM}_3\,\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n = K_1\sigma_2\,\mathrm{JM}_2\,\sigma_2\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n = \sigma_2K_1\,\mathrm{JM}_2\,\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n\sigma_2$$

$$\sim \sigma_2^2K_1\,\mathrm{JM}_2\,\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n = \sigma_2^2A(m,n-1,1).$$

Overall we have that

$$\mathrm{HHH}[A(n,m,0)] = t^{-1}\mathrm{HHH}(B(n,m)) + qt^{-1}\mathrm{HHH}\left[\sigma_2^2A(n,m-1,1)\right].$$

If we apply recursion (4.3) again, we get that

$$\sigma_2^2A(n,m-1,1) = t^{-1}\left[\sigma_2^2B(n,m-1)\right] + qt^{-1}\sigma_2^2A(n,m-1,0)$$

$$= t^{-1}\left[\sigma_2^2B(n,m-1)\right] + qt^{-1}A(n,m-1,1),$$

noticing that

$$\sigma_2^2 A(n, m-1, 0) = \sigma_2^2 K_1 \mathrm{FT}_3^{m-1}\mathrm{FT}_4^n = (\sigma_1\sigma_2)K_1\sigma_1^2\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n(\sigma_1\sigma_2)^{-1}$$

$$\sim K_1\sigma_1^2\mathrm{FT}_3^{m-1}\mathrm{FT}_4^n = A(n, m-1, 1).$$

Here we used that $\sigma_2 = (\sigma_1\sigma_2)\sigma_1(\sigma_1\sigma_2)^{-1}$ and the conjugating braid $(\sigma_1\sigma_2)$ commutes with K_1, FT_3 and FT_4. Finally, we apply recursion (4.3) one more time to $A(n, m-1, 1)$ to get the desired result. □

We have already shown that most of the terms on the right hand side of the equation in Lemma 4.2.11 are parity. For the final base case, we refer to the work of Hogancamp in [19], noting that the torus link T(4,4n) is the closure of the braid FT_4^n.

Lemma 4.2.12. *(Hogancamp [19]) The braid $A(n,0,0) = (1-q)\mathrm{FT}_4^n$ is parity and $\mathrm{HHH}(A(n,0,0))$ can be computed recursively.*

Corollary 4.2.13. *The braid $A(n, m, 0)$ is parity for all $n, m \geq 0$.*

Proof. This follows by induction from Lemma 4.2.11, as we have shown that every B term on the right hand side is parity (Lemma 4.2.6 and Lemma 4.2.10) and that the base case is parity (Lemma 4.2.12). □

Theorem 4.2.14. *Assume that $0 \leq d_1 \leq d_2 \leq d_3$. Then the braid*

$$\beta = \beta(d_1, d_2, d_3, d_4) = (\mathrm{FT}_2)^{d_3-d_2} \cdot (\mathrm{FT}_3)^{d_2-d_1} \cdot (\mathrm{FT}_4)^{d_1}$$

is parity, and $\mathrm{HHH}(\beta)$ can be computed using the recursive process laid out above.

Proof. We write $\mathrm{HHH}(\beta) = \frac{1}{1-q}\mathrm{HHH}(A(d_1, d_2-d_1, d_3-d_2))$, so it is sufficient to prove that $A(n, m, l)$ is parity for all $n, m, l \geq 0$. Apply recursion (4.3) repeatedly, reducing to terms of the form $B(n, m)$ and $A(n, m, 0)$. These are parity by Corollary 4.2.6 and Corollary 4.2.13 respectively, and we can continue following the recursions laid out above to compute $\mathrm{HHH}(\beta)$. □

CHAPTER 5

Generalized (q,t)-Catalan numbers and the fundamental domain for $n = 3$

This chapter is based on joint work with Eugene Gorsky from [32].

In [6] Zongbin Chen introduced a notion of the fundamental domain for an unramified affine Springer fiber, which captures the behavior of cells in Sp_γ under translations by the lattice Λ. More precisely, for $n \leq 4$ (and conjecturally in general) Sp_γ admits a cell decomposition with cells parametrized by the lattice Λ. There is one torus fixed point in each cell. In general, the dimension of a cell is a complicated piecewise-linear function on Λ which stabilizes outside of the fundamental domain $\mathcal{P}$. The cells corresponding to points in Λ outside $\mathcal{P}$ can be obtained by translation of cells corresponding to points in $\mathcal{P}$.

At the same time, by Theorem 1.1.2 the (non-equivariant) homology of Sp_γ as a module over Λ is captured by $\mathcal{J}/(y)\mathcal{J}$ as a module over $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, so the cells in $\mathcal{P}$ should correspond to the generators of $\mathcal{J}$, and (following Conjecture 1.2.1) to the generalized (q,t)-Catalan numbers $F(d_n, \ldots, d_1)$.

In this appendix we explore the definition and some general properties of $\mathcal{P}$ and establish its precise relation with the generalized (q,t)-Catalan numbers for $n = 3$. We hope to generalize this to higher n in future work.

5.0.1. The Fundamental Domain. We define the action of S_n on $\mathbb{R}^n$ by $\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Note that for the basis vectors $\mathbf{e}_i$ we have $\sigma e_i = e_{\sigma^{-1}(i)}$.

We start with the matrix $\gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ as in (1.1), where γ_i are pairwise distinct monomials with order d_i. We will always assume that $d_1 \leq \ldots \leq d_n$. Define $d_{ij} = \min(d_i, d_j)$ for $i \neq j$, noting that d_{ij} is the order of $\gamma_i - \gamma_j$.

DEFINITION 5.0.1. (Compare with [6, Proposition 2.8]) We define the polytope $\mathcal{P}(d_1, \ldots, d_n)$ as the convex hull of the points $p_\sigma = \sigma(b_{1,\sigma}, \ldots, b_{n,\sigma})$ where:

$$b_{i,\sigma} = \sum_{j<i} d_{\sigma^{-1}(i),\sigma^{-1}(j)}, \ \sigma \in S_n.$$

EXAMPLE 5.0.2. For $n = 2$ we get two points $p_e = (0, d_1)$ and $p_{(1\ 2)} = (d_1, 0)$, and $\mathcal{P}$ is the segment connecting them.

EXAMPLE 5.0.3. For $n = 3$ we get 6 points shown in the following table

σ	$(b_{1,\sigma}, b_{2,\sigma}, b_{3,\sigma})$	p_σ
e	$(0, d_1, d_1 + d_2)$	$(0, d_1, d_1 + d_2)$
$(1\ 2)$	$(0, d_1, d_1 + d_2)$	$(d_1, 0, d_1 + d_2)$
$(1\ 3)$	$(0, d_2, 2d_1)$	$(2d_1, d_2, 0)$
$(2\ 3)$	$(0, d_1, d_1 + d_2)$	$(0, d_1 + d_2, d_1)$
$(1\ 2\ 3)$	$(0, d_1, d_1 + d_2)$	$(d_1, d_1 + d_2, 0)$
$(1\ 3\ 2)$	$(0, d_2, 2d_1)$	$(2d_1, 0, d_2)$

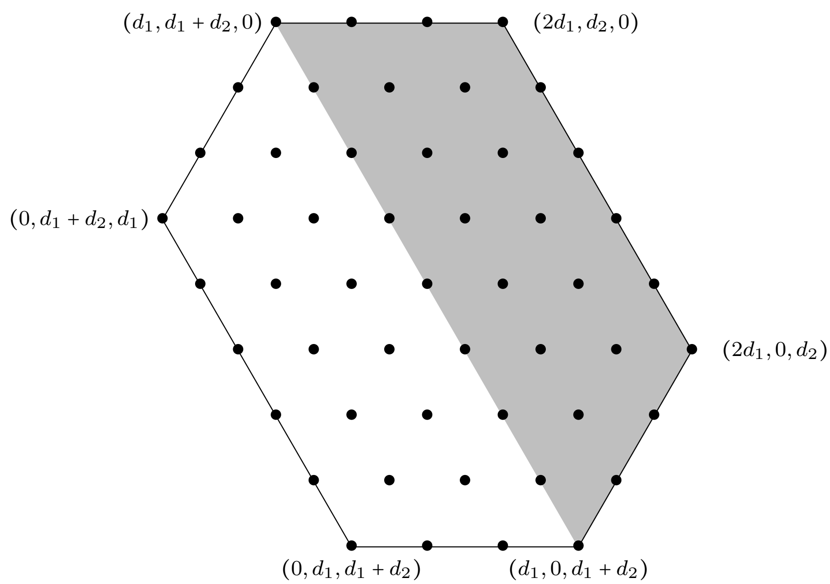


FIGURE 5.1. Fundamental domain for $(d_1, d_2) = (3, 5)$

EXAMPLE 5.0.4. Suppose that $d_i = d$ for all i. Then $b_{i,\sigma} = d(i - 1)$,

$$p_\sigma = (d(\sigma(1) - 1), \ldots, d(\sigma(n) - 1)),$$

and $\mathcal{P}$ is the standard $(n-1)$-dimensional permutahedron dilated d times.

We now establish some general properties of $\mathcal{P}$.

Lemma 5.0.5. *The polytope $\mathcal{P}(d_1,\ldots,d_n)$ is contained in the hyperplane $\sum x_i = \sum_{i<j} d_{ij}$.*

Proof. For any σ we have $\sum_i b_{i,\sigma} = \sum_{j<i} d_{\sigma^{-1}(i),\sigma^{-1}(j)} = \sum_{i<j} d_{ij}$. □

Proposition 5.0.6. *Let $\mathbf{e}_i$ denote the i-th basis vector. Then $\mathcal{P}(d_1,\ldots,d_n)$ is the Minkowski sum of $\binom{n}{2}$ segments connecting $d_{ij}\mathbf{e}_i$ and $d_{ij}\mathbf{e}_j$.*

Proof. Let $\mathcal{P}'(d_1,\ldots,d_n)$ denote the above Minkowski sum. We can write

$$p_\sigma = \sigma\left(\sum_{i<j} d_{\sigma^{-1}(i),\sigma^{-1}(j)}\mathbf{e}_i\right) = \sum_{i<j} d_{\sigma^{-1}(i),\sigma^{-1}(j)}\mathbf{e}_{\sigma^{-1}(i)} = \sum_{\sigma(i)<\sigma(j)} d_{ij}\mathbf{e}_i.$$

Given a permutation σ and $i<j$, we can choose one end of the segment connecting $d_{ij}\mathbf{e}_i$ and $d_{ij}\mathbf{e}_j$ as follows: if $\sigma(i)<\sigma(j)$, we choose $\mathbf{e}_i$, otherwise we choose $\mathbf{e}_j$. Clearly, the sum of these points equals p_σ, so $p_\sigma \in \mathcal{P}'(d_1,\ldots,d_n)$. Therefore $\mathcal{P}(d_1,\ldots,d_n) \subseteq \mathcal{P}'(d_1,\ldots,d_n)$.
On the other hand, $\mathcal{P}'(d_1,\ldots,d_n)$ is a zonotope with edges parallel to the edges of the standard permutahedron $\mathcal{P}(1,\ldots,1)$. It follows e.g. from [2, Section 9] that the vertices of $\mathcal{P}'(d_1,\ldots,d_n)$ are in bijection with the vertices of $\mathcal{P}(1,\ldots,1)$, and are given by p_σ. So $\mathcal{P}'(d_1,\ldots,d_n) \subset \mathcal{P}(d_1,\ldots,d_n)$.
□

Example 5.0.7. For $n=3$ we get three segments $[(d_1,0,0),(0,d_1,0)]$, $[(d_1,0,0),(0,0,d_1)]$ and $[(0,d_2,0),(0,0,d_2)]$.

Remark 5.0.8. Quite surprisingly, a similar polytope appeared in a recent work of Alishahi, Gorsky, and Liu on Heegaard Floer homology [1].

5.0.2. Generalized (q,t)-Catalans for $n=3$. Given $d_1 \le d_2$, we can consider the Young diagram $\lambda_{d_1,d_2} = (d_1+d_2, d_1)$. We will draw Young diagrams in French notation, with the corner at $(0,0)$, see Figure 5.2. We also consider the line

$$\ell_{d_1,d_2} = \{x + d_2 y = d_1 + 2d_2 + \varepsilon\}$$

where ε is a small positive number. The following lemma shows that λ_{d_1,d_2} is a triangular partition in the sense of [3].

LEMMA 5.0.9. *The diagram λ_{d_1,d_2} is the largest Young diagram below the line ℓ_{d_1,d_2}. If a diagram μ is strictly below ℓ_{d_1,d_2}, then $\mu \subset \lambda_{d_1,d_2}$.*

PROOF. Let us describe all integer points (x,y) satisfying $x + d_2 y < d_1 + 2d_2 + \varepsilon$. For $y = 1$ we get $x < d_1 + d_2 + \varepsilon$, so $x \le d_1 + d_2$. For $y = 2$ we get $x < d_1 + \varepsilon$, so $x \le d_1$. For $y \ge 3$ we get $x < d_1 + (2-y)d_2 + \varepsilon < 0$, so there are no integer points (here we used $d_1 \le d_2$). The result follows. □

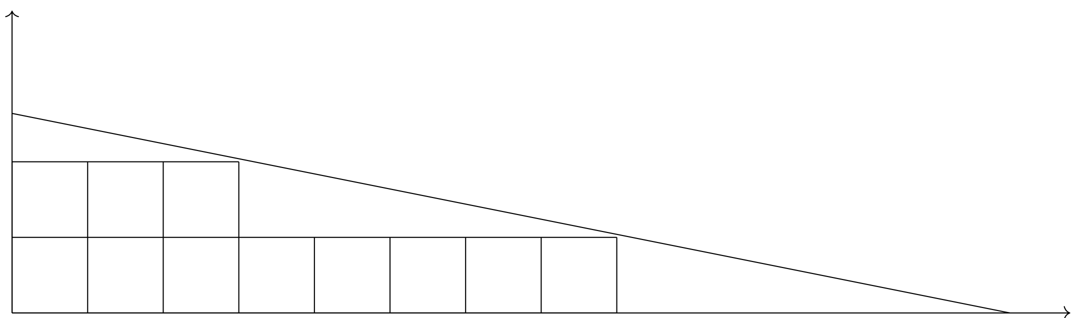


FIGURE 5.2. The line ℓ_{d_1,d_2} and the diagram λ_{d_1,d_2} for $(d_1,d_2) = (3,5)$.

Following [3], we define two statistics on subdiagrams of λ_{d_1,d_2}.

DEFINITION 5.0.10. Given $\mu \subset \lambda_{d_1,d_2}$, we define $\text{area}(\mu) = |\lambda_{d_1,d_2}| - |\mu|$ and

$$\text{dinv}(\mu) = \left\{ \square \in \mu : \frac{a(\square)}{\ell(\square)+1} \le d_2 < \frac{a(\square)+1}{\ell(\square)} \right\}$$

Here $a(\square)$ and $\ell(\square)$ are respectively the arm and the leg of a box $\square$ in μ.

Note that d_2 in the definition of dinv is negative reciprocal to the slope of the line ℓ_{d_1,d_2}.

THEOREM 5.0.11. *The map $\phi : \mu \mapsto (\text{area}(\mu), \text{dinv}(\mu))$ yields a bijection between the subdiagrams $\mu \subset \lambda_{d_1,d_2}$ and the integer points in the trapezoid*

$$\{d_1 + d_2 \le x + y \le 2d_1 + d_2,\ 2x + y \le 2d_1 + d_2,\ x + 2y \le 2d_1 + d_2\}. \tag{5.1}$$

As a consequence, we get the generalized (q,t)-Catalan number

$$\sum_{\mu \subset \lambda_{d_1,d_2}} q^{\text{area}(\mu)} t^{\text{dinv}(\mu)} = F(d_1,d_2). \tag{5.2}$$

Equation (5.2) is a special case of the main result of [4], but we give a more direct proof here generalizing [17, Theorem 4.1].

PROOF. Let us write $\mu = (a+b, a)$, then $0 \le a \le d_1$ and $0 \le b \le d_1 + d_2 - a$. We have the following cases (see Figure 5.3):

(a) If $a + b \le d_2$ then $\mathrm{dinv}(\mu) = a + b$, so

$$\phi(\mu) = (2d_1 + d_2 - 2a - b, a + b) = (x, y).$$

We have $a = 2d_1 + d_2 - x - y, b = 2y + x - 2d_1 - d_2$, so the inequalities $0 \le a, a \le d_1, 0 \le b$ and $a + b \le d_2$ respectively translate to the inequalities $x + y \le 2d_1 + d_2, x + y \ge d_1 + d_2, 2y + x \ge 2d_1 + d_2$ and $y \le d_2$.

(b) If $a + b > d_2, b \le d_2$ then $\mathrm{dinv}(\mu) = 2a + 2b - d_2$, so

$$\phi(\mu) = (2d_1 + d_2 - 2a - b, 2a + 2b - d_2) = (x, y).$$

We have $y = d_2 \mod 2$, and

$$a = \frac{4d_1 + d_2 - 2x - y}{2}, \; b = x + y - 2d_1$$

The inequalities $0 \le a, a \le d_1, 0 \le b, b \le d_2$ and $a + b > d_2$ respectively translate to the inequalities

$$2x + y \le 4d_1 + d_2, \; 2x + y \ge 2d_1 + d_2, \; x + y \ge 2d_1, \; x + y \le 2d_1 + d_2, \; y > d_2.$$

The second, fourth and fifth inequalities define a triangle $\mathbf{T}$ with vertices $(0, 2d_1 + d_2), (d_1, d_2)$ and $(2d_1, d_2)$ with the bottom side removed. The other two inequalities are satisfied on this triangle. In other words, in this case the image of ϕ is the set of all integer points in the triangle $\mathbf{T}$ satisfying $y = d_2 \mod 2$.

(c) If $b > d_2$ then $\mathrm{dinv}(\mu) = 2a + d_2 + 1$, so

$$\phi(\mu) = (2d_1 + d_2 - 2a - b, 2a + d_2 + 1) = (x, y).$$

We have $y = d_2 + 1 \mod 2$, and

$$a = \frac{y - d_2 - 1}{2}, \; b = 2d_1 + 2d_2 + 1 - x - y.$$

The inequalities $0 \le a, a \le d_1, b > d_2$ and $a + b \le d_1 + d_2$ respectively translate to the inequalities

$$y \ge d_2 + 1, \; y \le 2d_1 + d_2 + 1, \; x + y < 2d_1 + d_2 + 1, \; 2x + y \ge 2d_1 + d_2 + 1.$$

Similarly to (b), the image of ϕ in this case is the set of all integer points in the triangle $\mathbf{T}$ satisfying $y = d_2 + 1 \mod 2$. ☐

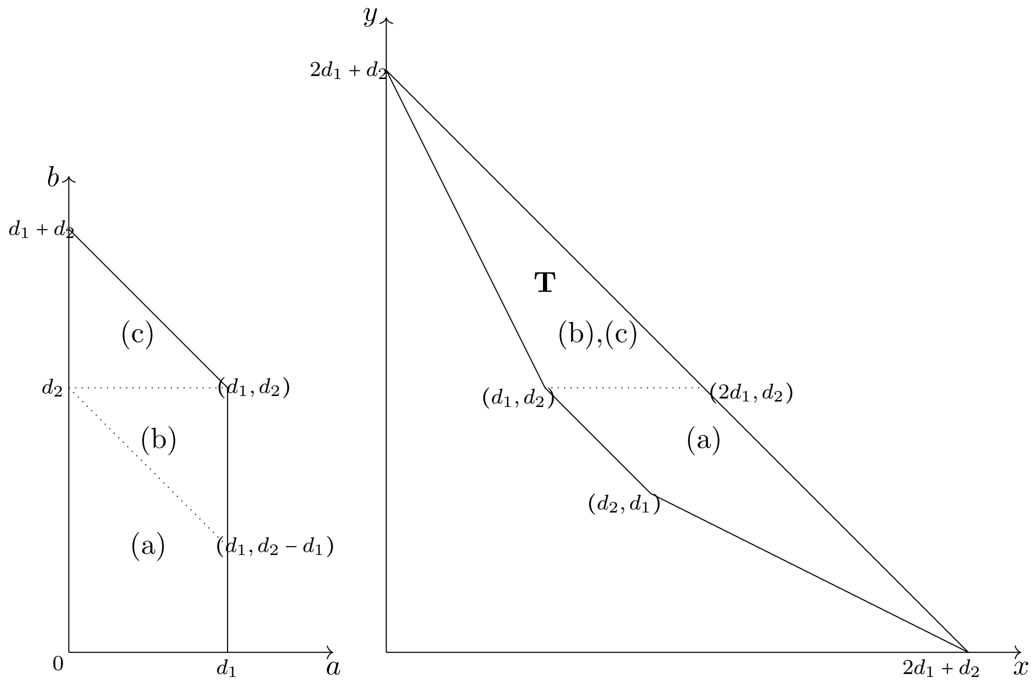


FIGURE 5.3. The bijection ϕ

Finally, we compare the above combinatorial results with the fundamental domain. Observe that for $n = 3$ the fundamental domain $\mathcal{P}(d_1, d_2)$ is a hexagon with an axis of symmetry, which cuts it into two equal halves (see Figure 5.1).

THEOREM 5.0.12. *The integer points in a half of $\mathcal{P}(d_1, d_2)$ (including boundary) are in bijection with the generators of the ideal $\mathcal{J}(d_1, d_2)$.*

PROOF. We construct the desired bijection in several steps:

1) The integer points in a half of $\mathcal{P}(d_1, d_2)$ are in bijection with the subdiagrams of λ_{d_1,d_2}. Indeed, we can write such points as $(d_1, 0, d_1 + d_2) + a(1, 0, -1) + b(0, 1, -1)$. It is easy to see by comparing Figures 5.1 and 5.3 that $0 \le a \le d_1$ and $0 \le b \le d_1 + d_2 - a$, and hence (a, b) define a subdiagram $\mu = (a + b, a)$.

2) By Theorem 5.0.11, we have a bijection ϕ between the subdiagrams of λ_{d_1,d_2} and the points in the trapezoid (5.1).

3) By Proposition 3.1.12, there is a bijection between the generators of $\mathcal{J}(d_1, d_2)$ and the points in the trapezoid (5.1). □

We expect that the bijection in Theorem 5.0.12 is far more than a combinatorial coincidence. In particular, by tracing through the bijections we see that dinv defines a piecewise linear function on the fundamental domain $\mathcal{P}(d_1, d_2)$, and we expect this function to be closely related to the dimension of cells in an appropriately chosen cell decomposition of Sp_γ. The corresponding (equivariant) homology classes of the cells would then correspond to some elements of $\mathcal{J}(d_1, d_2)$, and we expect that these would indeed generate the ideal. We plan to study these questions and generalize them to $n > 3$ in future work.

Bibliography

[1] A. Alishahi, E. Gorsky, B. Liu. Splitting maps in link Floer homology and integer points in permutahedra. arXiv:2307.07741

[2] M. Beck, S. Robins. Computing the continuous discretely. Undergrad. Texts Math. Springer, New York, 2015, xx+285 pp.

[3] F. Bergeron, M. Mazin. Combinatorics of triangular partitions. Enumer. Comb. Appl. 3 (2023), no. 1, Paper No. S2R1, 20 pp.

[4] J. Blasiak, M. Haiman, J. Morse, A. Pun, G. H. Seelinger. A shuffle theorem for paths under any line Forum Math. Pi 11 (2023), Paper No. e5, 38 pp.

[5] Brion, Michel. Poincaré duality and equivariant (co)homology. Michigan Math. J. 48 (2000), 77–92.

[6] Z. Chen. On the fundamental domain of affine Springer fibers. Math. Z. 286 (2017), no. 3–4, 1323–1356.

[7] Chen, Zongbin. Pureté des fibres de Springer affines pour GL_4. (French) [Purity of affine Springer fibers for GL_4] Bull. Soc. Math. France 142 (2014), no. 2, 193–222.

[8] Elias, Ben, and Matthew Hogancamp. "On the computation of torus link homology." Compositio Mathematica 155.1 (2019): 164-205.

[9] Haiman, Mark. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. Journal of the American Mathematical Society 14.4 (2001): 941-1006.

[10] Goresky, Mark; Kottwitz, Robert; Macpherson, Robert. Homology of affine Springer fibers in the unramified case. Duke Math. J.121(2004), no.3, 509–561.

[11] Goresky, Mark; Kottwitz, Robert; MacPherson, Robert. Purity of equivalued affine Springer fibers. Represent. Theory 10 (2006), 130–146.

[12] Goresky, Mark; Kottwitz, Robert; MacPherson, Robert. Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math. 131 (1998), no.1, 25–83.

[13] Gorsky, Eugene; Hawkes, Graham Schilling, Anne ; Rainbolt, Julianne. Generalized q, t-Catalan numbers. Algebr. Comb. 3 (2020), no. 4, 855–886.

[14] Gorsky, Eugene, and Matthew Hogancamp. Hilbert schemes and y–ification of Khovanov–Rozansky homology. Geometry & Topology 26.2 (2022): 587-678.

[15] Gorsky, Eugene, Oscar Kivinen, and Alexei Oblomkov. The affine Springer fiber-sheaf correspondence. arXiv preprint arXiv:2204.00303 (2022).

[16] E. Gorsky, O. Kivinen. J. Simental. Algebra and geometry of link homology: lecture notes from the IHES 2021 Summer School. Bull. Lond. Math. Soc. 55 (2023), no. 2, 537–591.

[17] E. Gorsky, M. Mazin. Compactified Jacobians and q, t-Catalan numbers, II. J. Algebraic Combin. 39 (2014), no. 1, 153–186.

[18] Gorsky, Eugene, Andrei Neguţ, and Jacob Rasmussen. "Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology." Advances in Mathematics 378 (2021): 107542.

[19] M. Hogancamp. Khovanov-Rozansky homology and higher Catalan sequences. 2017. arXiv: 1704.01562.

[20] Hogancamp, Matthew, and Anton Mellit. "Torus link homology." arXiv preprint arXiv:1909.00418 (2019).

[21] D. Kazhdan, G. Lusztig. Fixed point varieties on affine flag manifolds. Israel J. Math. 62 (1988), no. 2, 129–168.

[22] M. Khovanov. Triply–graded link homology and Hochschild homology of Soergel bimodules Internat. J. Math. 18 (2007), no. 8, 869–885.

[23] M. Khovanov, L. Rozansky. Matrix factorizations and link homology.II Geom. Topol. 12 (2008), no. 3, 1387–1425.

[24] Kivinen, Oscar Unramified affine Springer fibers and isospectral Hilbert schemes. Selecta Math. (N.S.) 26 (2020), no. 4, Paper No. 61, 42 pp.

[25] Laumon, Gérard. Aspects géométriques du Lemme Fondamental de Langlands-Shelstad.[Geometric aspects of the Fundamental Lemma of Langlands-Shelstad] International Congress of Mathematicians. Vol. II, 401–419. European Mathematical Society (EMS), Zürich, 2006.

[26] Laumon, Gérard Fibres de Springer et jacobiennes compactifiées.[Springer fibers and compactified Jacobians]. Algebraic geometry and number theory, 515–563. Progr. Math., 253 Birkhäuser Boston, Inc., Boston, MA, 2006

[27] Lucarelli, Vincent Gilbert. Affine pavings for affine Springer fibers for split elements in $PGL(3)$. Thesis (Ph.D.)–The University of Chicago. 2004.

[28] Migliorini, Luca; Shende, Vivek. A support theorem for Hilbert schemes of planar curves. Eur. Math. Soc. (JEMS)15(2013), no.6, 2353–2367.

[29] Maulik, Davesh; Yun, Zhiwei. Macdonald formula for curves with planar singularities. J. Reine Angew. Math.694(2014), 27–48.

[30] Oblomkov, Alexei, and Lev Rozansky. "HOMFLYPT homology of Coxeter links." arXiv preprint arXiv:1706.00124 (2017).

[31] Oblomkov, Alexei; Rasmussen, Jacob; Shende, Vivek. The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link. With an appendix by Eugene Gorsky. Geom. Topol. 22 (2018), no. 2, 645–691.

[32] Turner, Joshua P. "Affine Springer Fibers and Generalized Haiman Ideals (with an Appendix by Eugene Gorsky and Joshua P. Turner)." International Mathematics Research Notices 2024.16 (2024): 11878-11909.

[33] Turner, Joshua P. "Triply Graded Link Homology for Coxeter Braids on 4 Strands." arXiv preprint arXiv:2410.03068 (2024).

[34] Yun, Zhiwei Lectures on Springer theories and orbital integrals. Geometry of moduli spaces and representation theory, 155–215, IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017.