Braided Tensor Categories Describing Anyons in Quantum Lattice Systems: Symmetries, Fermions, and the Fractional Quantum Hall Effect

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Abstract

This dissertation investigates the structure of superselection sectors in quantum lattice systems, with an emphasis on the role of symmetry and fermionic degrees of freedom. First, I consider systems with an on-site unitary action of a compact abelian group G, and show that when the reference representation is a G-invariant product representation, the superselection sectors are classified by the Pontryagin dual of G. Second, I extend a construction by Ogata of a braided monoidal C^*-tensor category to lattice systems with fermionic degrees of freedom by introducing a twisted version of approximate Haag duality. This allows for the construction of a braided monoidal C^*-tensor supercategory that captures anyonic excitations in such systems. Finally, in joint work with Martin Fraas, Sven Bachmann, and Yoshiko Ogata, we analyze the quantization of Hall conductance in infinite lattice systems and show that its denominator is bounded above by the number of isomorphism classes of simple objects in an associated braided tensor category.

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CHAPTER 1

Introduction

Informal Summary of Results. This dissertation explores properties of anyons and superselection sectors of quantum lattice systems. The work is motivated by questions arising in the study of topological phases of matter, especially the fractional quantum Hall effect (FQHE), but many of the results apply more broadly. The overall approach is in the spirit of the Doplicher–Haag–Roberts (DHR) framework, in which sectors are described as representations of a quasi-local observable algebra, and their composition gives rise to a tensor category.

The first main result concerns how requiring an on-site symmetry to be respected affects the classification of superselection sectors. In particular, I considered systems with a unitary on-site action symmetry by a compact abelian group G, and modified the superselection criterion and equivalence relation to require compatibility with this G-action. I showed that in this case, if the reference representation comes from a G-invariant product state, the superselection sectors are classified by the Pontryagin dual of G. This generalizes a result of Naaijkens and Ogata which showed that, when no symmetry is imposed, the sector theory for such a product state is trivial.

The second result extends a construction by Ogata of a braided monoidal C^*-tensor category from a quantum spin system satisfying approximate Haag duality. I generalize this to quantum lattice systems with fermionic degrees of freedom, satisfying approximate twisted Haag duality. The twist in approximate twisted Haag duality, as well as in the modification of the superselection criterion, serves to account for the anticommutation of odd operators with disjoint support. The result is a construction of a braided monoidal C^*-tensor *supercategory*, capturing anyonic excitations in a quantum lattice system with fermionic degrees of freedom.

The third result, joint work with Martin Fraas, Sven Bachmann, and Yoshiko Ogata, concerns the quantization of Hall conductance in certain infinite lattice systems. We showed that the denominator of the Hall conductance is bounded above by the number of inequivalent anyon types - that is, by the number of isomorphism classes of simple objects in an associated braided tensor category. This replaces the role of the ground state degeneracy in the case of the FQHE on a finite torus.

Prior work. The DHR approach, developed by Doplicher, Haag, and Roberts [16, 22], offers a rigorous method for analyzing superselection sectors in relativistic quantum field theory. It introduces a superselection criterion to identify localized excitation sectors and shows that these sectors carry the structure of a braided C^*-tensor category. In this framework, elementary excitations correspond to objects in the category, and their braiding corresponds to a natural braiding structure ϵ. The original DHR papers [16] also introduced the *twisted commutant* and what has come to be known as *twisted Haag duality* to deal with fermionic charges, and addresses models with a unitary action of a compact gauge group.

More recently, the DHR framework has been adapted to quantum lattice systems [37], extending its applicability beyond relativistic settings. A full mathematical formalization of the resulting structure in lattice systems was provided by Ogata [41], who formulated a C^*-algebraic setting for describing anyonic excitations. For an overview of other theoretical approaches to anyons, see the review [33].

Naaijkens and Ogata proved in [4] that for interactions are necessary for nontrivial superselection sectors to occur. They showed that if the reference representation used in the superselection criterion is a product representation, then any representation satisfying the criterion is equivalent to the reference representation.

Various definitions of monoidal supercategories – and their relationships – have been discussed in [1].

The fractional quantum Hall effect (FQHE) has been studied from several perspectives. A microscopic approach for systems with a finite number of electrons was developed by Avron and Seiler [3], and it was shown that this can lead to a rational Hall conductance [26]. A topological field theory description of quantum Hall fluids in the bulk, capturing features such as fractional quantization and anyonic excitations, was developed by Fröhlich and collaborators [18, 19, 20].

In lattice systems, Hastings and Michalakis introduced a setting for the Quantum Hall Effect involving interacting particles with a $U(1)$ symmetry on a finite torus, governed by a gapped local Hamiltonian [24]. Building on this, Bachmann, Bols, De Roeck, and Fraas established rigorous quantization results under the assumption that the Hamiltonian has p locally indistinguishable ground states (along with additional technical conditions) [6, 7]. In particular, they showed that the Hall conductance κ satisfies

$$2\pi\kappa = \frac{q}{p} + O(L^{-\infty})$$

where $q \in \mathbb{Z}$ and L is the linear size of the torus. This implies rational quantization of conductance in the thermodynamic limit, assuming that the ground state in the plane arises as a limit of ground states on large tori. The local indistinguishability condition, known as local topological quantum order (LTQO), was introduced in [13, 34] and is widely believed to hold in standard quantum Hall models, though it has proven challenging to prove. Recent progress in this direction was made in [32].

The connection between Hall conductance and the charge and statistics of excitations has its origins in the seminal works of Laughlin [30, 31] and Arovas, Schrieffer, and Wilczek [1]. Laughlin showed that inserting a 2π flux through the system leads to a localized excitation carrying fractional charge $2\pi\kappa$, while Arovas, Schrieffer, and Wilczek showed that transporting another such excitation around it results in a statistical phase of $e^{i(2\pi)^2\kappa}$ – characteristic of Abelian anyons. In a finite volume setting, this was proved in [8], and this was extended to infinite volume in [25].

Outline. Chapter 2 introduces the background needed for the rest of this dissertation. It covers foundational concepts from category theory, including braided and symmetric monoidal categories, as well as the algebra of quasilocal observables for quantum spin systems, GNS representations, the superselection criterion, and superalgebras.

Chapter 3 analyzes the consequences of modifying the superselection criterion, and the notion of equivalence of representations, by requiring compatibility with an on-site unitary action of a compact abelian symmetry group G.

Chapter 4 extends Ogata's construction [6] from quantum spin systems satisfying approximate Haag duality to quantum lattice systems with fermionic degrees of freedom satisfying approximate twisted Haag duality. In this setting, the resulting structure is a braided C^*-tensor supercategory rather than an ordinary braided C^*-tensor category. The chapter also introduces the definitions of braided C^*-tensor supercategories and approximate twisted Haag duality.

Chapter 5 is about a category associated with the anyons in the Fractional Quantum Hall Effect in certain infinite lattice systems, and proves the result about the quantization of Hall conductance where the number of simple objects up to isomorphisms is finite.

Motivation. In the study of the fractional quantum Hall effect (FQHE), the fraction $\frac{1}{2}$ has never been observed as the coefficient $\frac{q}{p}$ in the quantized Hall conductance $\frac{q}{p} \cdot 2\pi\frac{e^2}{\hbar}$, even though many other fractional values – mostly with odd denominators – have been seen, and some even-denominator fractions appear in special circumstances. Notably, $\frac{1}{2}$ does occur in the bosonic version of the FQHE, suggesting that its absence in fermionic systems is tied to the role of fermions. While several explanations have been given for why the fraction $\frac{1}{2}$ does not occur as the coefficient, this work was motivated by the goal of giving a new explanation based on symmetry and categorical structure.

In particular, Chapter 5 shows that, under certain assumptions, the coefficient $\frac{q}{p}$ can be interpreted categorically: the denominator p corresponds to the number of simple superselection sectors generated (under the monoidal product) by a particular representation. Although this result is currently limited to a bosonic setting, Chapter 4 develops the technical framework needed to extend it to fermionic systems. By then combining this with the symmetry-based analysis of Chapter 3, the goal is to show that in fermionic systems with a $U(1)$ symmetry compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading, the coefficient $\frac{q}{p}$ – when expressed via this categorical construction – cannot equal $\frac{1}{2}$; specifically, when $p = 2$, the corresponding q must be even. This would offer a structural, symmetry-based explanation for the absence of $\frac{1}{2}$ in fermionic FQHE systems.

The results in these three chapters provide a solid foundation for completing such a symmetry-based explanation in future work.

CHAPTER 2

Setup and Background

2.1. braided monoidal categories

A category $\mathcal{C}$ is a collection $\mathrm{ob}(\mathcal{C})$ of "objects" and for each pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$, a collection $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of "morphisms from A to B", and for any $A, B, C \in \mathrm{ob}(\mathcal{C})$, a map ("composition") $\circ_{A,B,C} : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ (or simply $\circ$ rather than $\circ_{A,B,C}$ when there is no ambiguity) satisfying the following conditions:

(1) For all objects $A \in \mathrm{ob}(\mathcal{C})$ there exists a morphism ("the identity morphism on A") $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that for all objects $B \in \mathrm{ob}(\mathcal{C})$ and all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $f \circ_{A,A,B} 1_A = f$, and for all morphisms $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, $1_A \circ_{B,A,A} g = g$. (It can immediately be seen that from the existence of such an 1_A that it is unique.)

(2) The composition is associative, in the following sense: For all objects $A, B, C, D \in \mathrm{ob}(\mathcal{C})$, and all morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, all morphisms $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, and all morphism $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, $(h \circ_{B,C,D} g) \circ_{A,B,D} f = h \circ_{A,C,D} (g \circ_{A,B,C} f)$.

For $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, A is called the domain of f and B is called the codomain of f. The collections $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are called "hom-spaces" or "hom-sets". When we know what the domain and codomain of f and g are, specifying these in the composition is unnecessary, and so, for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $g \circ_{A,B,C} f$ will be written as simply $g \circ f$.

A category $\mathcal{C}$ is called "locally small" if for any two objects of the category, A, B, the collection $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set (rather than a proper class). Let us be concerned only with locally small categories.

For two categories $\mathcal{C}, \mathcal{D}$, a *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a map from the objects of $\mathcal{C}$, $\mathrm{ob}(\mathcal{C})$, to the objects of $\mathcal{D}$, $\mathrm{ob}(\mathcal{D})$; we write $F(A)$ for the image in $\mathrm{ob}(\mathcal{D})$ of an object $A \in \mathrm{ob}(\mathcal{C})$.

- for each pair of objects $A, B \in \text{ob}(\mathcal{C})$, a map from $\text{Hom}_{\mathcal{C}}(A, B)$ to $\text{Hom}_{\mathcal{D}}(F(A), F(B))$ sending $f : A \to B$ to $F(A) : F(A) \to F(B)$ where these maps satisfy the following properties:
  (1) Identity morphisms are preserved : For any $A \in \text{ob}(\mathcal{C})$, $F(1_A) = 1_{F(A)}$
  (2) Composition is preserved: For any $A, B, C \in \text{ob}(\mathcal{C})$, and any $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, C)$, $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* is defined similarly, except that rather than maps from $\text{Hom}_{\mathcal{C}}(A, B)$ to $\text{Hom}_{\mathcal{D}}(F(A), F(B))$, it has maps from $\text{Hom}_{\mathcal{C}}(A, B)$ to $\text{Hom}_{\mathcal{D}}(F(B), F(A))$, and satisfies $F(g \circ f) = F(f) \circ F(g)$ (it "reverses the arrows"/reverses the direction of the morphisms).

If we say "functor" and don't specify "contravariant functor", then we will mean a covariant functor.

Given two categories $\mathcal{C}, \mathcal{D}$, we can define a product category $\mathcal{C} \times \mathcal{D}$ whose objects $\text{ob}(\mathcal{C} \times \mathcal{D}) = \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{D})$, and where for $(A, B), (C, D) \in \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{D})$, $\text{Hom}_{\mathcal{C} \times \mathcal{D}}((A, B), (C, D)) = \text{Hom}_{\mathcal{C}}(A, C) \times \text{Hom}_{\mathcal{D}}(B, D)$, and where, for $f_1 : C \to C', g_1 : D \to D', f_2 : C' \to C'', g_2 : D' \to D''$ we have $(f_2, g_2) \circ_{(C,D),(C',D'),(C'',D'')} (f_1, g_1) = (f_2 \circ_{C,C',C''} f_1, g_2 \circ_{D,D',D''} g_1)$.

A functor whose domain is a product of two categories in this way is called a "bifunctor".

Given two categories $\mathcal{C}, \mathcal{D}$, and two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ consists of, for every object $x \in \text{ob}(\mathcal{C})$, a morphism $\eta_x : F(x) \to G(x)$, such that for every $x, y \in \text{ob}(\mathcal{C})$ and $f : x \to y$, the following diagram commutes:

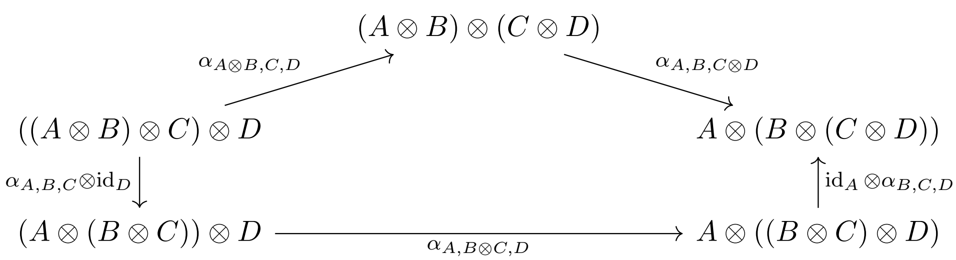
$$\begin{array}{ccc} G(y) & \xleftarrow{G(f)} & G(x) \\ {\scriptstyle \eta_y}\uparrow & & \uparrow{\scriptstyle \eta_x} \\ F(y) & \xleftarrow[F(f)]{} & F(x) \end{array}$$

i.e. $\eta_y \circ F(f) = G(f) \circ \eta_x$.

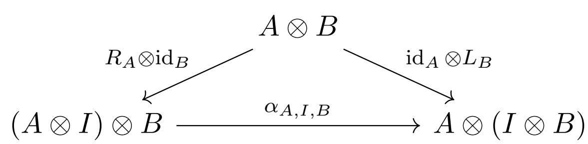
A *natural isomorphism* $\eta : F \Rightarrow G$ is a natural transformation such that each $\eta_x : F(x) \to G(x)$ is an isomorphism. For a natural isomorphism $\eta : F \Rightarrow G$, we have its inverse $\eta^{-1} : G \Rightarrow F$ (with $(\eta^{-1})_x = (\eta_x)^{-1}$) which is also a natural isomorphism.

A monoidal category consists of a category $\mathcal{C}$ along with some extra decoration, where together they satisfy certain properties. Specifically, it is equipped with a distinguished object $I \in \text{ob}(\mathcal{C})$ (the "identity object"), a bifunctor $(- \otimes -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and, for all $A, B, C \in \text{ob}(\mathcal{C})$, natural isomorphisms $\alpha_{A,B,C}$ ("the associator"), L_A (the "left unitor"), and R_A (the "right unitor"), where these natural isomorphisms are $\alpha : ((-\otimes -)\otimes -) \cong (-\otimes(-\otimes -))$, $L : (I \otimes -) \cong \mathrm{Id}_{\mathcal{C}}$, and $R : (-\otimes I) \cong \mathrm{Id}_{\mathcal{C}}$, and which satisfy the following identities:

(1) "The pentagon identity" : For all $A, B, C, D \in \mathrm{ob}(\mathcal{C})$, the following diagram commutes:



(2) "The triangle identity" : For all $A, B \in \mathrm{ob}(\mathcal{C})$ the following diagram commutes:

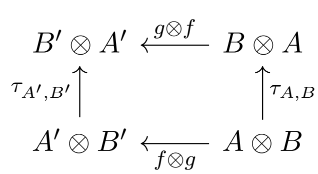


Let Vect denote the category whose objects are vector spaces over the complex numbers and whose morphisms are linear maps between those vector spaces.

Then Vect, equipped with the usual tensor product on vector spaces, $\mathbb{C}$ as the identity object, and $\alpha_{A,B,C}((u \otimes v) \otimes w) := v \otimes (u \otimes w)$ for any vectors u, v, w in the vector spaces A, B, C respectively, is an example of a monoidal category.

A braided monoidal category consists of a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with a natural transformation $\tau : (-\otimes -) \Rightarrow (-\otimes -) \circ (\mathrm{swap})$ where "$(-\otimes -) \circ (\mathrm{swap})$" denotes the bifunctor $(-\otimes -)$ with the order of the two inputs switched, and which satisfies the hexagon identities and the triangle identities. That is to say:

- naturality: for all $A, A', B, B' \in \mathrm{ob}(\mathcal{C})$, and all $f \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, B')$ the following diagram commutes:



- hexagon identities: for all $A, B, C \in \mathcal{C}$, the following diagrams commute:

$$\begin{array}{ccccc}
(A\otimes B)\otimes C & \xrightarrow{\alpha_{A,B,C}} & A\otimes(B\otimes C) & \xrightarrow{\tau_{A,B\otimes C}} & (B\otimes C)\otimes A \\
{\scriptstyle \tau_{A,B}\otimes \mathrm{id}_C}\downarrow & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\
(B\otimes A)\otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B\otimes(A\otimes C) & \xrightarrow[\mathrm{id}_B\otimes\tau_{A,C}]{} & B\otimes(C\otimes A)
\end{array}$$

$$\begin{array}{ccccc}
A\otimes(B\otimes C) & \xrightarrow[\alpha^{-1}_{A,B,C}]{} & (A\otimes B)\otimes C & \xrightarrow[\tau_{A\otimes B,C}]{} & C\otimes(A\otimes B) \\
{\scriptstyle \mathrm{id}_A\otimes\tau_{B,C}}\downarrow & & & & \downarrow{\scriptstyle \alpha^{-1}_{C,A,B}} \\
A\otimes(C\otimes B) & \xrightarrow[\alpha^{-1}_{A,C,B}]{} & (A\otimes C)\otimes B & \xrightarrow[\tau_{A,C}\otimes\mathrm{id}_B]{} & (C\otimes A)\otimes B
\end{array}$$

- the triangle identity: for $A \in \mathrm{ob}(\mathcal{C})$ the following diagram commutes:

$$\begin{array}{ccc}
 & A & \\
{\scriptstyle L_A}\swarrow & & \searrow{\scriptstyle R_A} \\
I\otimes A & \xrightarrow{\tau_{I,A}} & A\otimes I
\end{array}$$

.

There is a standard way to equip category Vect with a braiding to make it into a braided monoidal category: for any $A, B \in \mathrm{ob}(\mathrm{Vect})$, and any $u \in A$ and $v \in B$, define $\tau_{A,B}(u \otimes v) := v \otimes u$, and extend linearly to obtain $\tau_{A,B} : \mathrm{Hom}_{\mathrm{Vect}}(A \otimes B, B \otimes A)$. Because this braiding τ is such that $\tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A\otimes B}$ for all $A, B \in \mathrm{ob}(\mathrm{Vect})$, Vect is what is known as a "symmetric monoidal category".

A category $\mathcal{C}$ is said to be "enriched in Vect" if for each pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$, the hom-space $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is equipped with structure that makes it an object of the category Vect, in such a way that each $\circ_{A,B,C} : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ is bilinear.
There is a more general notion in category theory of a category $\mathcal{C}$ enriched in some given monoidal category $\mathcal{V}$, where instead of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ necessarily being a set, it is instead an object in $\mathcal{V}$, and composition is given by a morphism $\mathrm{Hom}_{\mathcal{C}}(B, C) \otimes \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ in $\mathcal{V}$ rather than a function between sets, and where conditions analogous to the requirements on the composition map in the definition of a category are imposed on this composition morphism. However, in this text we will only consider enrichment in a couple monoidal categories (Vect and SVect) where the objects have underlying sets and have their monoidal products fit nicely with the cartesian product of sets - e.g. bilinear maps from a product of two vector spaces corresponding to linear maps from the tensor product of those vector spaces - and as such we will not need the general theory of enrichment, and

can instead treat enriched categories as ordinary categories whose hom-sets carry extra structure and where the composition is compatible with these extra properties. So, rather than giving the general definition of enrichment in an abitrary monoidal category, we just define enrichment in the particular monoidal categories we are concerned with, Vect and (later) $\underline{\text{SVect}}$.

A monoidal category $(\mathcal{C}, \otimes, I)$ is "enriched in Vect" if, in addition to the category $\mathcal{C}$ being enriched in Vect, the bifunctor $(- \otimes -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ has the property that for any $A, B, C, D \in \text{ob}(\mathcal{C})$, the map $\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(C, D) \to \text{Hom}_{\mathcal{C}}(A \otimes C, B \otimes D)$ is bilinear. (This definition comes from the enriched version of the definition of a monoidal category, where instead of having $(-\otimes-)$ a bifunctor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, instead having $(-\otimes-)$ be a bifunctor from $\mathcal{C} \boxtimes \mathcal{C}$, where $\text{Hom}_{\mathcal{C}\boxtimes\mathcal{C}}((A, B), (C, D)) = \text{Hom}_{\mathcal{C}}(A, C) \otimes \text{Hom}_{\mathcal{C}}(B, D)$, and the bifunctor, being enriched in Vect, is required to be a linear map from $\text{Hom}_{\mathcal{C}}(A, C) \otimes \text{Hom}_{\mathcal{C}}(B, D)$ to $\text{Hom}_{\mathcal{C}}(A \otimes B, C \otimes D)$, therefore, restricting to a bilinear map from $\text{Hom}_{\mathcal{C}}(A, C) \times \text{Hom}_{\mathcal{C}}(B, D)$ to $\text{Hom}_{\mathcal{C}}(A \otimes B, C \otimes D)$.)

2.2. Super vector spaces and Superalgebras

Let $\underline{\text{SVect}}$ be the symmetric monoidal category of "super vector spaces (over $\mathbb{C}$) with even morphisms", defined as:

- $\text{ob}(\underline{\text{SVect}})$ is the set of super vector spaces, i.e. $(\mathbb{Z}/2\mathbb{Z})$-graded vector spaces (over $\mathbb{C}$), where for $A \in \text{ob}(\underline{\text{SVect}})$, A is $A_0 \oplus A_1$ for some $A_0, A_1 \in \text{ob}(\text{Vect})$
- for $A, B \in \text{ob}(\underline{\text{SVect}})$, the morphisms from A to B are the even linear maps from A to B, i.e. $\text{Hom}_{\underline{\text{SVect}}}(A, B) = \text{Hom}_{\text{Vect}}(A_0, B_0) \oplus \text{Hom}_{\text{Vect}}(A_1, B_1)$, so for every $f \in \text{Hom}_{\underline{\text{SVect}}}(A, B)$ there are some $g \in \text{Hom}_{\text{Vect}}(A_0, B_0)$ and $h \in \text{Hom}_{\text{Vect}}(A_1, B_1)$ with $f = g \oplus h$
- the identity morphism for $A = A_0 \oplus A_1$ is $\text{id}_A = \text{id}_{A_0} \oplus \text{id}_{A_1}$
- for $A, B \in \text{ob}(\underline{\text{SVect}})$, $A \otimes B$ is defined as $(A \otimes B) := (A \otimes B)_0 \oplus (A \otimes B)_1$ where $(A \otimes B)_0 := (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$ and $(A \otimes B)_1 := (A_0 \otimes B_1) \oplus (A_1 \otimes B_0)$ where the $\oplus$ and $\otimes$ in the right hand sides of these equations are the $\oplus$ and $\otimes$ of Vect
- for $f = g \oplus h \in \text{Hom}_{\underline{\text{SVect}}}(A, B)$ and $j = k \oplus l \in \text{Hom}_{\underline{\text{SVect}}}(C, D)$, $f \otimes j \in \text{Hom}_{\underline{\text{SVect}}}(A \otimes C, B \otimes D)$ is defined as $((g \otimes k) \oplus (h \otimes l)) \oplus ((g \otimes l) \oplus (h \otimes k))$ with $(g \otimes k) \oplus (h \otimes l) \in \text{Hom}_{\text{Vect}}((A \otimes C)_0, (B \otimes D)_0)$ and $(g \otimes l) \oplus (h \otimes k) \in \text{Hom}_{\text{Vect}}((A \otimes C)_1, (B \otimes D)_1)$
- the identity object is $\mathbb{C} \oplus 0$

- the associator and unitor morphisms are obtained from the associator and unitor morphisms of Vect in a straightforward way
- for $A, B \in \text{ob}(\underline{\text{SVect}})$, the braiding morphism is $\tau_{A,B} \in \text{Hom}_{\underline{\text{SVect}}}(A \otimes B, B \otimes A)$ defined as $(\tau_{A_0,B_0} \oplus (-1) \cdot \tau_{A_1,B_1}) \oplus (\text{swap} \circ (\tau_{A_0,B_1} \oplus \tau_{A_1,B_0}))$ with $(\tau_{A_0,B_0} \oplus (-1) \cdot \tau_{A_1,B_1}) \in \text{Hom}_{\text{Vect}}((A \otimes B)_0, (B \otimes A)_0)$ and $\text{swap} \circ (\tau_{A_0,B_1} \oplus \tau_{A_1,B_0}) \in \text{Hom}_{\text{Vect}}((A \otimes B)_1, (B \otimes A)_1)$, where, for $s, s' = 0, 1$, $\tau_{A_s,B_{s'}} \in \text{Hom}_{\text{Vect}}(A_s \otimes B_{s'}, B_{s'} \otimes A_s)$ are the braiding morphisms from Vect, and where $\text{swap} : (B_1 \otimes A_0) \oplus (B_0 \otimes A_1) \to (B_0 \otimes A_1) \oplus (B_1 \otimes A_0)$ is given by the block matrix $\begin{pmatrix} 0 & \text{id}_{(B_1 \otimes A_0)} \\ \text{id}_{(B_0 \otimes A_1)} & 0 \end{pmatrix}$.

  In particular, if $u \in A$ and $v \in B$ and both are homogeneous, then $\tau_{A,B}(u \otimes v) = (-1)^{|u||v|}(v \otimes u)$ where $|u|, |v|$ are the grades of u, v respectively.

(There is a "forgetful functor" from SVect to Vect which forgets the grading and the braiding structure, but preserves the monoidal structure, and in this sense SVect without its braiding can be seen as a subcategory of Vect without its braiding.)

In a way that will be elaborated on shortly, superalgebras relate to SVect in a way that corresponds directly to how algebras relate to Vect.

Similar to how categories can be enriched in Vect, they can also be enriched in SVect. A category $\mathcal{C}$ is said to be enriched in SVect if each hom space $\text{Hom}_{\mathcal{C}}(A, B)$ is given the structure of a super vector space (a $\mathbb{Z}/2\mathbb{Z}$-graded vector space), so that is is an object of SVect, and the composition is both bilinear and even, in the sense that the extension of the bilinear map $\circ_{A,B,C} : \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(A, C)$ to the linear map $\circ_{A,B,C} : \text{Hom}_{\mathcal{C}}(B, C) \otimes \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(A, C)$, is an even (grade preserving) linear map (and so, a morphism in SVect). A category enriched in SVect is called a "supercategory".

There is also a definition for what it means for something to be a monoidal supercategory, i.e. a monoidal category enriched in SVect, analogous to the definition of a monoidal category enriched in Vect, with both being instances of a more general definition of a monoidal category being enriched in another monoidal category. However, unlike a monoidal category enriched in Vect, a monoidal category enriched in SVect is not in general a monoidal category when considered as an ordinary category rather than as an enriched category. The definition of a (strict) monoidal category enriched in SVect, i.e. of a (strict) monoidal supercategory, is given later in Definition 105.

One example of a category enriched in SVect is the category SVect, whose objects are the same as those of SVect, but where its hom spaces are, rather than the vector spaces of all *even* linear maps from the domain to the codomain, instead, the super vector space of *all* linear maps from the domain to the codomain.

A *superalgebra* is, concretely, a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, meaning a super vector space $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ equipped with an (associative) bilinear multiplication that respects the grading, i.e. for $x \in \mathfrak{A}_i$ and $y \in \mathfrak{A}_j$, $xy \in \mathfrak{A}_{i+j}$.
However, in the same way that super vector spaces are distinguished from mere $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, superalgebras are distinguished from $\mathbb{Z}/2\mathbb{Z}$-graded algebras by the introduction of signs when swapping odd components.
For example, instead of the usual commutator

$$[A, B] := AB - BA,$$

superalgebras use the *supercommutator*

$$[A, B]_{\pm} := AB - (-1)^{|A||B|}BA,$$

(for homogeneous elements $A, B \in \mathfrak{A}$ of degrees $|A|, |B| \in \mathbb{Z}/2\mathbb{Z}$, and extended bilinearly for general elements). When the supercommutator of two elements is zero, we say that the two elements supercommute. Two homogeneous elements supercommute when one of the two is even and they commute, or if both are odd and they anticommute.
Similarly, the tensor product of superalgebras is defined so that elements of the two tensor factors supercommute. Given superalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, their tensor product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is the superalgebra whose underlying super vector space is the tensor product of the super vector spaces underlying the respective superalgebras, and where the multiplication is defined such that

$$(A \otimes B) \cdot (C \otimes D) := (-1)^{|B||C|}((A \cdot C) \otimes (B \cdot D))$$

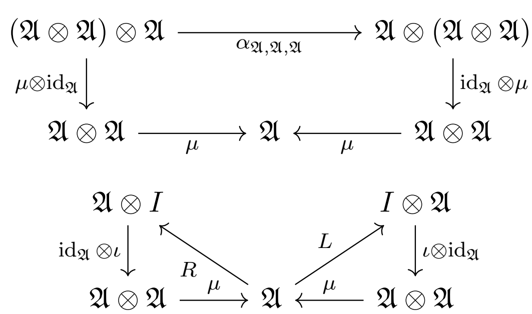
for all homogeneous $A, C \in \mathfrak{A}_1$ and $B, D \in \mathfrak{A}_2$, where $|B|, |C|$ are the grades of B, C respectively, and where the multiplication for general elements is defined by extending this bilinearly. (The ι for the tensor product is defined by the tensor product of the ι maps for the two superalgebras.)

In particular, this means that, for $A \in \mathfrak{A}_1$ and $B \in \mathfrak{A}_2$

$$(1 \otimes B)(A \otimes 1) = (-1)^{|A||B|}(A \otimes B) = (-1)^{|A||B|}(A \otimes 1)(1 \otimes B),$$

i.e., $(1 \otimes B)$ and $(A \otimes 1)$ supercommute.

In terms of category theory, the analogy is this: (unital) algebras (over $\mathbb{C}$) are monoid objects in the monoidal category Vect, while superalgebras are monoid objects in the monoidal category <u>SVect</u>. What it means to say that an algebra (respectively superalgebra) is a monoid object in Vect (respectively <u>SVect</u>) is that an algebra (respectively superalgebra) is an object $\mathfrak{A}$ of Vect (respectively <u>SVect</u>) equipped with morphisms $\mu : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ and $\iota : I \to \mathfrak{A}$ such that the following diagrams commute:



where R, L, α are the unitors and associator for Vect (respectively <u>SVect</u>) and I is the identity object of Vect (respectively <u>SVect</u>), i.e. $\mathbb{C}$. These same diagrams define a monoid object in any monoidal category.

In any braided monoidal category, there is a definition of a kind of product of two monoid objects, and this gives the definition of the tensor product of two algebras or two superalgebras:
Given two monoid objects $(X, \mu_X : X \otimes X \to X, \iota_X : I \to X)$ and $(Y, \mu_Y : Y \otimes Y \to Y, \iota_Y : I \to Y)$ the product of these monoid objects is $X \otimes Y$ equipped with

$$\mu_{X \otimes Y} := (\mu_X \otimes \mu_Y) \circ (\mathrm{id}_X \otimes \tau_{X,Y} \otimes \mathrm{id}_Y)$$

(where the necessary associators are left implicit, and $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ is the braiding) and $\iota_{X \otimes Y} := (\iota_X \otimes \iota_Y) \circ R_I$ (where $R_I = L_I : I \to I \otimes I$ is the unitor isomorphism for I).

This defines the tensor product both for algebras (when the monoid objects are monoid objects in Vect) and for superalgebras (when the monoid objects are monoid objects in SVect). When the category is SVect, the braiding $\tau_{X,Y}$ introduces the sign that appears in the multiplication in the earlier concrete definition.

(One may view a (super)algebra as the space of endomorphisms of an object in a one-object category enriched in Vect (respectively SVect), and (super)algebra endomorphisms as (super)endofunctors. Likewise with monoid objects in any monoidal category. This partially motivates some notation in Chapter 4.)

2.3. Lattice Spin Systems

Let $(\Gamma, d : \Gamma \times \Gamma \to \mathbb{R})$ be a discrete metric space. For the purposes here, (Γ, d) will generally be a Delone set in $\mathbb{R}^2$, especially a lattice. For each $x \in \Gamma$, let $\mathcal{H}_{\{x\}}$ be a finite-dimensional Hilbert space, and let $\mathcal{A}_{\{x\}} := \mathcal{B}(\mathcal{H}_{\{x\}})$, the algebra of (bounded) operators on $\mathcal{H}_{\{x\}}$, considered as a C^*-algebra, where the norm is the operator norm. For any finite subset Λ of Γ, define the C^*-algebra $\mathcal{A}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{A}_{\{x\}}$ and $\mathcal{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{H}_{\{x\}}$. For such finite Λ, $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. For infinite subsets $\Lambda \subseteq \Gamma$, define $\mathcal{A}_\Lambda$ to be direct limit of $\mathcal{A}_{\Lambda'}$ over finite subsets Λ' of Λ.

More specifically, consider $\mathcal{P}_0(\Lambda)$, the set of finite subsets of Λ, with $\mathcal{P}_0(\Lambda)$ considered as an upwards-directed set under the order of set inclusions. For finite subsets $\Lambda_1, \Lambda_2 \in \mathcal{P}_0(\Lambda)$, if $\Lambda_1 \subseteq \Lambda_2$ there is an inclusion $\iota_{\Lambda_1,\Lambda_2} : \mathcal{A}_{\Lambda_1} \hookrightarrow \mathcal{A}_{\Lambda_2}$ defined by $\iota_{\Lambda_1,\Lambda_2}(A) := A \otimes 1_{\mathcal{H}_{\Lambda_2\setminus\Lambda_1}}$. These inclusions are compatible, in that for $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{P}_0(\Lambda)$ with $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3$, $\iota_{\Lambda_2,\Lambda_3} \circ \iota_{\Lambda_1,\Lambda_2} = \iota_{\Lambda_1,\Lambda_3}$. $(\{\mathcal{A}_{\Lambda'} | \Lambda' \in \mathcal{P}_0(\Lambda)\}, \{\iota_{\Lambda_1,\Lambda_2} | \Lambda_1 \subseteq \Lambda_2 \in \mathcal{P}_0(\Lambda)\})$ is then a directed system in the category of C^*-algebras, and it has a direct limit, a C^*-algebra which is to be called $\mathcal{A}_\Lambda$.

The specific construction of this limiting C^* algebra is essentially by taking a union $\bigcup_{\Lambda'\in\mathcal{P}_0(\Lambda)} \mathcal{A}_{\Lambda'}$ where for $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{P}_0(\Lambda)$, $\mathcal{A}_{\Lambda_1}$ is identified with its image under $\iota_{\Lambda_1,\Lambda_2}$ in $\mathcal{A}_{\Lambda_2}$, and then $\mathcal{A}_\Lambda$ is finally obtained by taking the Cauchy completion (with distance given by the norm), of this union. Identify each $\mathcal{A}_{\Lambda'}$ with its image in this $\mathcal{A}_\Lambda$. Define $\mathcal{A}_{\Lambda,loc}$ to be the subset of $\mathcal{A}_\Lambda$ given by the union of the $\mathcal{A}_{\Lambda'}$ for $\Lambda' \in \mathcal{P}_0(\Lambda)$.

Set $\mathcal{A} := \mathcal{A}_\Gamma$ and $\mathcal{A}_{loc} := \mathcal{A}_{\Gamma,loc}$.

Definition 1. A state on a unital C^*-algebra $\mathfrak{A}$ is a linear map $\omega : \mathfrak{A} \to \mathbb{C}$ which is "positive" in that for all $A \in \mathfrak{A}$, $\omega(A^*A) \geq 0$, and such that $\omega(1) = 1$.

The definition of a state on a not-necessarily-unital C^*-algebra is similar, but non-unital C^*-algebras will not be relevant here.

Definition 2. A Gelfand-Naimark-Segal (GNS) representation for state ω on a C^*-algebra $\mathfrak{A}$ is a triple $(\mathcal{H}, \pi, \Omega)$ such that $\mathcal{H}$ is a Hilbert space, $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ a $*$-representation, and $\Omega \in \mathcal{H}$ a unit vector such that $\forall A \in \mathfrak{A}$, $\omega(A) = \langle \Omega | \pi(A) \Omega \rangle$ and such that Ω is "cyclic", meaning that $\{\pi(A)\Omega | A \in \mathfrak{A}\}$ is norm-dense in $\mathcal{H}$.

The GNS construction, for any C^*-algebra $\mathfrak{A}$ and any state ω on $\mathfrak{A}$, constructs a GNS representation of ω. GNS representations of a given state are unique up to unitary equivalence, in that if ω is a state on a C^*-algebra $\mathfrak{A}$ and $(\mathcal{H}, \pi, \Omega)$ and $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\Omega})$ are both GNS representations of ω, there exists a unitary linear map $U : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $\tilde{\pi} = \mathrm{Ad}(U) \circ \pi$ and $\tilde{\Omega} = U\Omega$.

Definition 3. With our $\mathcal{A}$ and given a reference representation $(\mathcal{H}_\pi, \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi))$ of $\mathcal{A}$, another representation $(\mathcal{H}_\rho, \rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho))$ of $\mathcal{A}$ satisfies the superselection criterion with respect to $(\mathcal{H}_\pi, \pi)$ if, for all cones Λ, there exists a unitary $V_{\rho,\Lambda} : \mathcal{H}_\rho \to \mathcal{H}_\pi$ such that $\mathrm{Ad}(V_{\rho,\Lambda}) \circ \rho|_{\mathcal{A}_{\Lambda^c}} = \pi|_{\mathcal{A}_{\Lambda^c}}$.

This definition can be interpreted as saying that, for any cone Λ, there is some unitary $V_{\rho,\Lambda}$ that localizes the ways that ρ differs from the reference representation to within Λ, so that for any observable supported outside of Λ, i.e. for any $A \in \mathcal{A}_{\Lambda^c}$, $\mathrm{Ad}(V_{\rho,\Lambda}) \circ \rho$ agrees with the reference representation π on A.

CHAPTER 3

Symmetry-Compatible Superselection Sectors in Quantum Spin Systems with Compact Abelian Symmetry

3.1. Setup

In this chapter we will modify the setting for lattice spin systems (Section 2.3) by equipping the algebra with a unitary on-site action of a locally compact abelian group.

DEFINITION 4. For any topological group G, a *system of on-site unitary G actions* consists of a a collection of group homomorphisms $((g \in G) \mapsto (U_{\{x\},g} \in \mathcal{U}(\mathcal{H}_{\{x\}}))_{x\in\Gamma}$.

In this chapter we will almost entirely stick with one system of on-site unitary G actions, with one exception where we will deal with two. As such, the phrasing will be with the system and group being fixed. So:

Let G be a locally compact abelian group, and let $\widehat{G}$ denote its Pontryagin dual; that is, the group $\widehat{G} := \mathrm{Hom}(G, U(1))$ of continuous group homomorphisms from G to $U(1)$, equipped with pointwise multiplication.

Recall the set of sites Γ and for each $x \in \Gamma$ the finite-dimensional Hilbert space $\mathcal{H}_{\{x\}}$.

For each $x \in \Gamma$, let $U_{\{x\},\bullet} : G \to \mathcal{U}(\mathcal{H}_{\{x\}}) \subset \mathcal{A}_{\{x\}}$ be a unitary representation of G on $\mathcal{H}_{\{x\}}$—that is, a continuous group homomorphism from G to the unitary elements of $\mathcal{A}_{\{x\}}$. I.e. fix a system of on-site unitary G-actions $(U_{\{x\},\bullet} : G \to \mathcal{U}(\mathcal{H}_{\{x\}}))_{x\in\Gamma}$.

Define the induced automorphism $\alpha_{\{x\},\bullet} : G \to \mathrm{Aut}(\mathcal{A}_{\{x\}})$ by

$$\alpha_{\{x\},g} := \mathrm{Ad}(U_{\{x\},g}).$$

For any finite subset $\Lambda \subset \Gamma$, define

$$U_{\Lambda,g} := \prod_{x\in\Lambda} U_{\{x\},g}, \quad \alpha_{\Lambda,g} := \mathrm{Ad}(U_{\Lambda,g}).$$

Then these are group homomorphisms from G into $\mathcal{U}_\Lambda$ and $\mathrm{Aut}(\mathcal{A}_\Lambda)$, respectively:

$$(g \mapsto U_{\Lambda,g}) \in \mathrm{Hom}(G, \mathcal{U}_\Lambda), \quad (g \mapsto \alpha_{\Lambda,g}) \in \mathrm{Hom}(G, \mathrm{Aut}(\mathcal{A}_\Lambda)).$$

For infinite $\Lambda \subseteq \Gamma$, define the action on $A \in \mathcal{A}_\Lambda$ via the limit

$$\alpha_{\Lambda,g}(A) := \lim_{\substack{\Lambda' \nearrow \Lambda \\ \text{finite}}} \alpha_{\Lambda',g}(A).$$

This limit exists for $A \in \mathcal{A}_{\Lambda,\mathrm{loc}}$, and more generally in $\mathcal{A}_{\mathrm{loc}}$, because the sequence is eventually constant. From this, by density and continuity, the definition extends to all of $\mathcal{A}_\Lambda$, and more generally to all of $\mathcal{A}$.

Definition 5. A state $\omega : \mathcal{A} \to \mathbb{C}$ is called *G-invariant* if

$$\omega \circ \alpha_g = \omega \quad \text{for all } g \in G.$$

The following lemma is well-known:

Lemma 6. *Let ω be a G-invariant state, and let $(\mathcal{H}, \pi, \Omega)$ be a GNS representation of ω. Then there exists a continuous group homomorphism*

$$U^{(\pi)} = (g \mapsto U_g^{(\pi)}) : G \to \mathcal{U}(\mathcal{H})$$

such that for all $g \in G$,

$$\pi \circ \alpha_g = \mathrm{Ad}(U_g^{(\pi)}) \circ \pi, \qquad U_g^{(\pi)}\Omega = \Omega.$$

This is a standard result; see, e.g., [Bratteli & Robinson, Operator Algebras and Quantum Statistical Mechanics I, Corollary 2.3.17 and the start of section 4.3.1].

Definition 7. Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a group action. A *G-covariant representation* of $(\mathcal{A}, \alpha)$ is a pair $((\mathcal{H}_\pi, \pi), U_\bullet^{(\pi)})$ consisting of a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ and a strongly continuous unitary representation $U_\bullet^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi)$ such that for all $g \in G$,

$$\pi \circ \alpha_g = \mathrm{Ad}(U_g^{(\pi)}) \circ \pi.$$

Remark 8. Lemma 6 shows that every GNS representation of a G-invariant state naturally gives rise to a G-covariant representation.

Definition 9. For two G-covariant representations $((\mathcal{H}_{\pi_1}, \pi_1), U_\bullet^{(\pi_1)}), ((\mathcal{H}_{\pi_2}, \pi_2), U_\bullet^{(\pi_2)})$, a continuous linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is a *G-equivariant map* if $\forall A \in \mathcal{A}, T\pi_1(A) = \pi_2(A)T$ and $\forall g \in G, TU_g^{(\pi_1)} = U_g^{(\pi_2)}T$.

We now define a version of the superselection criterion (Definition 3) suitable for G-covariant representations:

Definition 10. With our $\mathcal{A}$ and the on-site action $g \mapsto \alpha_g$, and a G-covariant representation $(\mathcal{H}_\pi, \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U_\bullet^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ to serve as the reference representation , another G-covariant representation $(\mathcal{H}_\rho, \rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho), U_\bullet^{(\rho)} : G \to \mathcal{U}(\mathcal{H}_\rho))$ satisfies the G-equivariant version of the superselection criterion with respect to $(\mathcal{H}_\pi, \pi, U_\bullet^{(\pi)})$ if, for all cones Λ, there exists a unitary $V_{\rho,\Lambda} : \mathcal{H}_\rho \to \mathcal{H}_\pi$ that is a G-equivariant map (i.e. such that for all $g \in G$, $V_{\rho,\Lambda}U_g^{(\rho)} = U_g^{(\pi)}V_{\rho,\Lambda}$) and such that $\mathrm{Ad}(V_{\rho,\Lambda}) \circ \rho|_{\mathcal{A}_{\Lambda^c}} = \pi|_{\mathcal{A}_{\Lambda^c}}$.

3.2. $\widehat{G}$-grading

In this section, we show that when G is compact and abelian, the various G-actions—on finite-volume Hilbert spaces $\mathcal{H}_\Lambda$, algebras $\mathcal{A}_\Lambda$, the global algebra $\mathcal{A}$, and Hilbert spaces $\mathcal{H}_\pi$ associated with G-covariant representations—induce a natural $\widehat{G}$-grading on each of these spaces.

We also construct, for each character $\phi \in \widehat{G}$, a continuous projection operator P_ϕ onto the grade-ϕ component, for each space viewed as a Banach space. When the space is an algebra, the grading is compatible with the multiplication; when it's a Hilbert space, the grade components are orthogonal. These projections arise as the Fourier coefficients of the group action, treating the action as a function from G to the space of bounded linear operators.

The following lemma is well-known and will be useful:

Lemma 11. *For any compact abelian group G, for any $\phi \in \widehat{G}$, $\int_{g\in G} \phi(g)d\mu(g) = \delta_{\phi,\hat{1}}$ where μ is the normalized Haar measure for G and $\hat{1}$ is the identity element of $\widehat{G}$.*

Proof. First, the integral $\int_{g\in G} \phi(g)d\mu(g)$ is well-defined because G is compact and $\phi(g)$ is measurable (it is continuous) and $|\phi(g)| = 1$ for all $g \in G$.

For any $g_2 \in G$,

$$\phi(g_2) \cdot \int_{g\in G} \phi(g)d\mu(g) = \int_{g\in G} \phi(g_2)\phi(g)d\mu(g) = \int_{g_2 g\in g_2 G} \phi(g_2 g)d\mu(g_2 g) = \int_{g\in G} \phi(g)d\mu(g)$$

by the translation-invariance of the Haar measure. Therefore, $(\phi(g_2)-1)\cdot\int_{g\in G}\phi(g)d\mu(g)=0$. So, as $\mathbb{C}$ is a field, either $\phi(g_2)-1=0$ or $\int_{g\in G}\phi(g)d\mu(g)=0$. If ϕ is not the identity element of $\widehat{G}$, then there is some $g_2\in G$ such that $\phi(g_2)\neq\hat{1}(g_2)=1$ and so the first factor isn't zero, and so $\int_{g\in G}\phi(g)d\mu(g)=0$. On the other hand, if ϕ is the identity element of $\widehat{G}$, then $\int_{g\in G}\phi(g)d\mu(g)=\int_{g\in G}1d\mu(g)=1$. So, $\int_{g\in G}\phi(g)d\mu(g)=\delta_{\phi,\hat{1}}$, as claimed. □

First, as a simpler test case that avoids any issues of convergence and interchanging orders of integration, suppose that the group G is finite.

THEOREM 12. *Let $B\in\{\mathcal{H}_\Lambda,\mathcal{A}_\Lambda,\mathcal{H}_\pi,\mathcal{B}(\mathcal{H}_\pi)\}$, G a finite abelian group, and $f:G\to\mathrm{Aut}(B)\subseteq\mathrm{End}(B)$ one of the aforementioned G actions and a group homomorphism, and $\mu(S):=\frac{|S|}{|G|}$ be the normalized Haar measure on G. Then, for $\phi\in\widehat{G}$,*

$$P_\phi:=\int_{g\in G}f(g)\phi(g^{-1})d\mu(g)=\frac{1}{|G|}\sum_{g\in G}f(g)\phi(g^{-1})$$

(a) For all $g\in G$, $f(g)\circ P_\phi=\phi(g)\cdot P_\phi$

(b) P_ϕ is a projection, and for $\phi_1,\phi_2\in\widehat{G}$, $P_{\phi_1}\circ P_{\phi_2}=\delta_{\phi_1,\phi_2}P_{\phi_1}=\delta_{\phi_1,\phi_2}P_{\phi_2}$

(c) $f(g)=\sum_{\phi\in\widehat{G}}\phi(g)P_\phi$. In particular, $id_B=f(1_G)=\sum_{\phi\in\widehat{G}}P_\phi$.

PROOF. First, part (a):

$$\begin{aligned}f(g)\circ P_\phi&=f(g)\circ(\frac{1}{|G|}\sum_{g_2\in G}\phi(g_2^{-1})f(g_2))\\&=\frac{1}{|G|}\sum_{g_2\in G}\phi(g_2^{-1})f(g)\circ f(g_2)\\&=\frac{1}{|G|}\sum_{g_2\in G}\phi(g)\phi((gg_2)^{-1})f(gg_2)\\&=\phi(g)\frac{1}{|G|}\sum_{gg_2\in gG}\phi((gg_2)^{-1})f(gg_2)=\phi(g)P_\phi.\end{aligned}$$

Now for part (b): For $\phi_1, \phi_2 \in \widehat{G}$ (not necessarily distinct),

$$\begin{aligned}
P_{\phi_1} \circ P_{\phi_2} &= (\int_{g_1 \in G} \phi_1(g_1^{-1}) f(g_1) d\mu(g_1)) \circ P_{\phi_2} \\
&= \int_{g_1 \in G} \phi_1(g_1^{-1}) f(g_1) \circ P_{\phi_2} d\mu(g_1) \\
&= \int_{g_1 \in G} \phi_1(g_1^{-1}) \phi_2(g_1) \cdot P_{\phi_2} d\mu(g_1) \\
&= (\int_{g_1 \in G} (\phi_1^{-1}\phi_2)(g_1) d\mu(g_1)) P_{\phi_2} \\
&= \delta_{\phi_1^{-1}\phi_2, \hat{1}} P_{\phi_2} = \delta_{\phi_1, \phi_2} P_{\phi_2}.
\end{aligned}$$

The fifth equality in the above is due to lemma 11, and the third is by part (a).

For part (c):

$$\begin{aligned}
\sum_{\phi \in \widehat{G}} \phi(g) P_\phi &= \sum_{\phi \in \widehat{G}} \phi(g) \frac{1}{|G|} \sum_{g_2 \in G} \phi(g_2^{-1}) f(g_2) \\
&= \frac{1}{|G|} \sum_{\phi \in \widehat{G}} \sum_{g_2 \in G} \phi((g_2 g^{-1})^{-1}) f(g_2 g^{-1}) \circ f(g) \\
&= \sum_{g_2 \in G} (\frac{1}{|G|} \sum_{\phi \in \widehat{G}} \phi((g_2 g^{-1})^{-1})) f(g_2 g^{-1}) \circ f(g) \\
&= (\sum_{g_2 \in G} \delta_{g_2, g} f(g_2 g^{-1})) \circ f(g) \\
&= \mathrm{id}_B \circ f(g) = f(g).
\end{aligned}$$

The fourth equality is by the same reasoning as Lemma 11, except with G and $\widehat{G}$ switched, using the normalized Haar measure on $\widehat{G}$ instead of on G. Note that these measures are only both normalizable when G is finite as it is here. □

For each of these spaces, if $x \in B$ is such that $(f(g))(x) = \phi(g) \cdot x$ for all $g \in G$, then $x = P_\phi(x)$, i.e. is homogeneous of grade ϕ. If for some finite $\Lambda \subset \Gamma$, $v \in \mathcal{H}_\Lambda$ is homogeneous of grade ϕ_1 and $A \in \mathcal{A}_\Lambda$ is homogeneous of grade ϕ_2, then $U_{\Lambda,g} A v = U_{\Lambda,g} A U_{\Lambda,g}^* U_{\Lambda,g} v = \phi_2(g) A \phi_1(g) v = (\phi_2\phi_1)(g) A v$, so Av has grade $\phi_2\phi_1$. The same applies for $v \in \mathcal{H}_\pi$ and $A \in \mathcal{B}(\mathcal{H}_\pi)$ for a G-covariant representation $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U_\bullet^{(\pi)})$.

Proposition 13. *For G just compact abelian rather than finite and abelian, with μ the normalized Haar measure for G, $P_\phi := \int_{g\in G} \phi(g^{-1})f(g)d\mu(g)$ exists. In addition:*

(a) For all $g \in G$, $f(g) \circ P_\phi = \phi(g) \cdot P_\phi$

(b) P_ϕ is a projection, and for $\phi_1, \phi_2 \in \widehat{G}$, $P_{\phi_1} \circ P_{\phi_2} = \delta_{\phi_1,\phi_2} P_{\phi_1} = \delta_{\phi_1,\phi_2} P_{\phi_2}$

Proof. For each $\phi \in \widehat{G}$, $\left\|\phi(g^{-1})f(g)\right\| = 1$, and $\phi(g^{-1})f(g)$ is measurable, so the Bochner integral $P_\phi := \int_{g\in G} \phi(g^{-1})f(g)d\mu(g)$ exists, and

$$\begin{aligned} \|P_\phi\| &= \left\| \int_{g\in G} \phi(g^{-1})f(g)d\mu(g) \right\| \\ &\leq \int_{g\in G} \left\|\phi(g^{-1})f(g)\right\| d\mu(g) = 1. \end{aligned}$$

To show (a):

As composition on the left with $f(g)$ is a continuous linear map from $\mathcal{B}(B)$ to $\mathcal{B}(B)$,

$$\begin{aligned} f(g) \circ P_\phi &= f(g) \circ \int_{g_2\in G} \phi(g_2^{-1})f(g_2)d\mu(g_2) \\ &= \int_{g_2\in G} \phi(g_2^{-1})f(g) \circ f(g_2)d\mu(g_2) \\ &= \int_{g_2\in G} \phi(g)\phi((gg_2)^{-1})f(gg_2)d\mu(g_2) \\ &= \phi(g) \cdot P_\phi. \end{aligned}$$

To show (b):

Composition on the right with P_ϕ is also a continuous linear map from $\mathcal{B}(B)$ to $\mathcal{B}(B)$, and so

$$\begin{aligned} P_{\phi_1} \circ P_{\phi_2} &= \left(\int_{g\in G} \phi_1(g^{-1})f(g)d\mu(g) \right) \circ P_{\phi_2} \\ &= \int_{g\in G} \phi_1(g^{-1})f(g) \circ P_{\phi_2} d\mu(g) \\ &= \int_{g\in G} \phi_1(g^{-1})\phi_2(g)P_{\phi_2} d\mu(g) \\ &= \left(\int_{g\in G} (\phi_1^{-1}\phi_2)(g)d\mu(g) \right) P_{\phi_2} \\ &= \delta_{\phi_1,\phi_2} P_{\phi_2} \end{aligned}$$

where that last equality is by Lemma 11. □

Proposition 14. *For any finite subset Λ of Γ there is a $\widehat{G}$-grading on $\mathcal{A}_\Lambda$ such that for any $\phi \in \widehat{G}$, $A \in \mathcal{A}_\Lambda$ is homogeneous of degree ϕ iff $\forall g \in G,\ \alpha_{\Lambda,g}(A) = \phi(g) \cdot A$.*

Proof. As $\mathcal{A}_\Lambda$ is a matrix algebra over $\mathbb{C}$, it can be made into a finite-dimensional Hilbert space by equipping it with the Hilbert-Schmidt inner product, $\langle A|B\rangle := \mathrm{Tr}(A^*B)$.
For each $g \in G$, and each $A, B \in \mathcal{A}_\Lambda$, as $\alpha_{\Lambda,g} := \mathrm{Ad}(U_{\Lambda,g})$,

$$\begin{aligned}\langle \alpha_{\Lambda,g}(A)|\alpha_{\Lambda,g}(B)\rangle &= \mathrm{Tr}(\alpha_{\Lambda,g}(A)^*\alpha_{\Lambda,g}(B)) \\ &= \mathrm{Tr}(U_{\Lambda,g}A^*BU_{\Lambda,g}^*) = \mathrm{Tr}(A^*B) = \langle A|B\rangle,\end{aligned}$$

So $\alpha_{\Lambda,g}$ acts unitarily with respect to the Hilbert-Schmidt inner product.
Since G is abelian, the automorphisms $\{\alpha_{\Lambda,g}\}_{g\in G}$ form a commuting family of unitaries on the finite-dimensional Hilbert space $\mathcal{A}_\Lambda$. Any such family can be simultaneously diagonalized, so there exists an orthonormal basis of $\mathcal{A}_\Lambda$ consisting of simultaneous eigenvectors for all $\alpha_{\Lambda,g}$.
For each such basis eigenvector A, let $\lambda_{A,g} \in S^1$ be the eigenvalue satisfying $\alpha_{\Lambda,g}(A) = \lambda_{A,g}A$. One verifies that $g \mapsto \lambda_{A,g}$ is a group homomorphism into S^1:

$$\alpha_{\Lambda,g_1g_2}(A) = \alpha_{\Lambda,g_1}(\alpha_{\Lambda,g_2}(A)) = \lambda_{A,g_2}\alpha_{\Lambda,g_1}(A) = \lambda_{A,g_1}\lambda_{A,g_2}A,$$

so $\lambda_{A,g_1g_2} = \lambda_{A,g_1}\lambda_{A,g_2}$.
This homomorphism is continuous as it is the composition of the continuous map $g \mapsto \alpha_{\Lambda,g}(A)$ with the continuous functional $B \mapsto \langle A|B\rangle$ on $\mathcal{A}_\Lambda$. Thus, each eigenvector A is homogeneous of some grade $\phi \in \widehat{G}$, defined by $\phi(g) := \lambda_{A,g}$.
For each $\phi \in \widehat{G}$, let us write $\mathcal{A}_\Lambda^\phi$ for the subspace of $\mathcal{A}_\Lambda$ consisting of all vectors homogeneous of degree ϕ. Since the simultaneous eigenspaces of a family of commuting normal operators form a direct sum decomposition of the space, we have

$$\mathcal{A}_\Lambda = \bigoplus_{\phi\in\widehat{G}_\Lambda} \mathcal{A}_\Lambda^\phi,$$

where $\widehat{G}_\Lambda \subseteq \widehat{G}$ is the (finite) set of characters that appear as eigenvalues in the decomposition of $\mathcal{A}_\Lambda$. Because $\mathcal{A}_\Lambda$ is finite dimensional, there are only finitely many ϕ such that $\mathcal{A}_\Lambda^\phi$ has a non-zero element.

To see that this direct sum decomposition defines a $\widehat{G}$-grading, note that for $A \in \mathcal{A}_\Lambda^\phi$ and $B \in \mathcal{A}_\Lambda^\psi$,

$$\alpha_{\Lambda,g}(AB) = \alpha_{\Lambda,g}(A)\alpha_{\Lambda,g}(B) = \phi(g)A \cdot \psi(g)B = (\phi\psi)(g)AB,$$

so $AB \in \mathcal{A}_\Lambda^{\phi\psi}$.

Lastly, note that since the ϕ-graded subspaces are pairwise orthogonal:

For $A_\phi \in \mathcal{A}_\Lambda^\phi, B_\psi \in \mathcal{A}_\Lambda^\psi$, $\langle A_\phi | B_\psi \rangle = \langle \alpha_{\Lambda,g}(A_\phi) | \alpha_{\Lambda,g}(B_\psi) \rangle = (\phi^{-1}\psi)(g) \cdot \langle A_\phi | B_\psi \rangle$ either $\phi = \psi$ or $\langle A_\phi | B_\psi \rangle = 0$. So, the decomposition is unique and canonical.

Therefore, this defines a $\widehat{G}$-grading on $\mathcal{A}_\Lambda$, and $A \in \mathcal{A}_\Lambda$ is homogeneous of degree ϕ if and only if $\alpha_{\Lambda,g}(A) = \phi(g)A$ for all $g \in G$. □

Remark 15. This $\widehat{G}$-grading on $\mathcal{A}_\Lambda$ for finite regions $\Lambda \subset \Gamma$ extends to a $\widehat{G}$-grading on $\mathcal{A}_{\mathrm{loc}}$ (for finite subsets $\Lambda_1, \Lambda_2 \subset \Gamma$ such that $\Lambda_1 \subseteq \Lambda_2$, the inclusion of $\mathcal{A}_{\Lambda_1}$ into $\mathcal{A}_{\Lambda_2}$ is compatible with the $\widehat{G}$-grading on each, so $\mathcal{A}_{\mathrm{loc}}$ gets such a grading as well).

It at least largely extends to $\mathcal{A}$ as a whole as well (Proposition 13 still applies of course), but there may be issues with convergence (analogous to those if Fourier series) if one wishes to represent $A \in \mathcal{A}$ as $\sum_{\phi \in \widehat{G}} P_\phi(A)$.

Definition 16. An on-site unitary G-action consists consists of such a system of a unitary representation $(g \in G) \mapsto (U_{\{x\},g} \in \mathcal{U}_{\{x\}})$ for each $x \in \Gamma$.

3.2.1. $\widehat{G}$-grading on Hilbert spaces.

To define a $\widehat{G}$-grading on the infinite-dimensional Hilbert spaces associated with G-covariant representations of $(\mathcal{A}, \alpha)$, we begin with standard results from the theory of unitary representations.

By the Peter–Weyl theorem [3], for any compact group G and any continuous unitary representation $U : G \to \mathcal{U}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, there is a decomposition of $\mathcal{H}$ into a direct sum of finite-dimensional irreducible subrepresentations. In particular, when G is abelian, each irreducible subrepresentation is one-dimensional. Thus,

$$\mathcal{H} = \bigoplus_{\phi \in \widehat{G}} \mathcal{H}_\phi, \quad \text{where } \mathcal{H}_\phi := \{v \in \mathcal{H} \mid \forall g \in G,\ U(g)v = \phi(g)v\}.$$

For each $\phi \in \widehat{G}$, let p_ϕ denote the orthogonal projection onto $\mathcal{H}_\phi$. These subspaces are mutually orthogonal, and every vector $v \in \mathcal{H}$ can be written as

$$v = \sum_{\phi \in \widehat{G}} p_\phi v,$$

with unconditional convergence. These projections p_ϕ can also be written in integral form as:

$$p_\phi v = \int_G \phi(g^{-1})\, U_g v \, d\mu(g),$$

where μ is the normalized Haar measure on G.

Now define, for each $\phi \in \widehat{G}$, the projection $P_\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$P_\phi(A) := \int_G \phi(g^{-1})\, \mathrm{Ad}(U_g)(A)\, d\mu(g)\,.$$

LEMMA 17. *Let $\phi_1, \phi_2 \in \widehat{G}$, $A \in \mathcal{B}(\mathcal{H})$ and $v \in \mathcal{H}$. Then:*

$$P_{\phi_1}(A) \cdot p_{\phi_2} v = p_{\phi_1\phi_2}\, A\, p_{\phi_2} v.$$

PROOF.

$$\begin{aligned}
P_{\phi_1}(A) \cdot p_{\phi_2} v &= \int_{g \in G} \phi_1(g^{-1})\, \mathrm{Ad}(U_g)(A) d\mu(g) p_{\phi_2} v \\
&= \int_G \phi_1(g^{-1})\, U_g A U_g^*\, p_{\phi_2} v \, d\mu(g) \\
&= \int_G \phi_1(g^{-1})\, U_g A \underbrace{U_g^* p_{\phi_2} v}_{=\phi_2(g^{-1}) p_{\phi_2} v} d\mu(g) \\
&= \int_G \phi_1(g^{-1}) \phi_2(g^{-1})\, U_g A p_{\phi_2} v \, d\mu(g) \\
&= \int_G (\phi_1\phi_2)(g^{-1})\, U_g A p_{\phi_2} v \, d\mu(g) \\
&= p_{\phi_1\phi_2}\, A\, p_{\phi_2} v.
\end{aligned}$$

□

DEFINITION 18. For $(\mathcal{H}_1, U^{(1)} : G \to \mathcal{U}(\mathcal{H}_1))$ and $(\mathcal{H}_2, U^{(2)} : G \to \mathcal{U}(\mathcal{H}_2))$ two Hilbert spaces equipped with a continuous unitary G-action, a bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be homogeneous of grade $\phi \in \widehat{G}$ if $U_g^{(2)} T (U_g^{(1)})^* = \phi(g) T$ for all $g \in G$.

Note that when $(\mathcal{H}_1, U^{(1)}) = (\mathcal{H}_2, U^{(2)})$ this is equivalent to $P_\phi(T) = T$.

Projecting onto the ϕ-graded components for these maps between distinct Hilbert spaces equipped with unitary G-actions also works the same (mutatis mutandis) as for bounded operators from one such Hilbert space to itself, but we will not need that here.

LEMMA 19. *For $i = 1, 2, 3$ let $(\mathcal{H}_i, U^{(i)} : G \to \mathcal{U}(\mathcal{H}_i))$ be Hilbert spaces equipped with a continuous unitary G-action. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear map which is homogeneous of grade $\phi \in \widehat{G}$, and let $S : \mathcal{H}_2 \to \mathcal{H}_3$ be a bounded linear map which is homogeneous of grade $\psi \in \widehat{G}$.*
Then, $S \circ T : \mathcal{H}_1 \to \mathcal{H}_3$ is homogeneous of grade $\psi\phi$.

PROOF. For all $g \in G$,

$$\begin{aligned} U_g^{(3)} ST \cdot (U_g^{(1)})^* &= (U_g^{(3)} S \cdot (U_g^{(2)})^*)\,(U_g^{(2)} T \cdot (U_g^{(1)})^*) \\ &= (\psi(g)S)(\phi(g)T) \\ &= \psi(g)\phi(g)ST = (\psi\phi)(g)ST. \end{aligned}$$

□

Now specialize to the case where $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ is a G-covariant representation of $(\mathcal{A}, \alpha)$, with $U := U^{(\pi)}$.

LEMMA 20. *Let $A \in \mathcal{B}(\mathcal{H}_\pi)$. If $A \neq 0$, then:*

(1) There exists $\phi \in \widehat{G}$ such that $P_\phi(A) \neq 0$.
(2) For every such ϕ, there exists a non-zero vector $u \in \mathcal{H}_\pi$ and $\phi_1 \in \widehat{G}$ such that:
   - *u is homogeneous of grade ϕ_1 (i.e., $u = p_{\phi_1} u$),*
   - *$Au \neq 0$,*
   - *$P_\phi(A)u \neq 0$.*

PROOF. Let $A \in \mathcal{B}(\mathcal{H}_\pi)$ be non-zero. Then there exists some $v \in \mathcal{H}_\pi$ such that $Av \neq 0$. As $\mathcal{H}_\pi$ decomposes as $\bigoplus_{\phi_1 \in \widehat{G}} (\mathcal{H}_\pi)_{\phi_1}$, we can write:

$$v = \sum_{\phi_1 \in \widehat{G}} p_{\phi_1} v, \quad \text{with unconditional convergence.}$$

Then,

$$Av = \sum_{\phi_1 \in \widehat{G}} Ap_{\phi_1} v.$$

Since $Av \neq 0$, at least one term in the sum is non-zero, say $Ap_{\phi_1} v \neq 0$ for some $\phi_1 \in \widehat{G}$.

Now decompose $Ap_{\phi_1} v$ into its graded components. There exists $\phi_2 \in \widehat{G}$ such that:

$$p_{\phi_2} Ap_{\phi_1} v \neq 0.$$

Set $\phi := \phi_2 \phi_1^{-1}$, so that $\phi_2 = \phi\phi_1$.

By Lemma 17, we have:

$$P_\phi(A) \cdot p_{\phi_1} v = p_{\phi\phi_1} Ap_{\phi_1} v = p_{\phi_2} Ap_{\phi_1} v \neq 0.$$

Therefore, $P_\phi(A) \neq 0$.

Set $u := p_{\phi_1} v$. Then u is homogeneous of grade ϕ_1, $Au \neq 0$, and $P_\phi(A)u \neq 0$.

For the second part: let $\phi \in \widehat{G}$ be such that $P_\phi(A) \neq 0$. Then apply the above argument to $P_\phi(A)$ in place of A to find such a homogeneous vector u.

□

Lemma 21. *If $A \in \mathcal{B}(\mathcal{H}_\pi)$ and there is exactly one $\phi \in \widehat{G}$ such that $P_\phi(A) \neq 0$, then $P_\phi(A) = A$, i.e. A is homogeneous of grade ϕ.*

Proof. For all $\phi_1 \in \widehat{G}$, if there is $v \in \mathcal{H}_\pi$ such that $Ap_{\phi_1} v \neq p_{\phi\phi_1} Ap_{\phi_1} v$, then with $Ap_{\phi_1} v = \sum_{\phi_2 \in \widehat{G}} p_{\phi_2} Ap_{\phi_1} v$ must have $p_{\phi_2} Ap_{\phi_1} v \neq 0$ for some $\phi_2 \neq \phi\phi_1$. Set $\phi' = \phi_2 \phi_1^{-1}$ so that $\phi'\phi_1 = \phi_2$. Then, $P_{\phi'}(A)\phi_1 v = p_{\phi'\phi_1} Ap_{\phi_1} v \neq 0$ (by Lemma 17), and therefore $P_{\phi'}(A) \neq 0$, with $\phi' = \phi_2\phi_1^{-1} \neq (\phi\phi_1)\phi_1^{-1} = \phi$.

But, we are given that for $\phi' \neq \phi$ that $P_{\phi'}(A) = 0$. Therefore, it must be that for all $\phi_1 \in \widehat{G}$ that for all $v \in \mathcal{H}_\pi$ that $Ap_{\phi_1} v = p_{\phi\phi_1} Ap_{\phi_1} v$.

So, $Ap_{\phi_1} = p_{\phi\phi_1} Ap_{\phi_1}$ for all $\phi_1 \in \widehat{G}$.

For any $v \in \mathcal{H}_\pi$, $v = \sum_{\phi_1} p_{\phi_1} v$, (which converges unconditionally, so as, A is bounded,) therefore $Av = \sum_{\phi_1} A\, p_{\phi_1} v$ (also converging unconditionally).

And, as for all $\phi_1 \in \widehat{G}$, $Ap_{\phi_1} = p_{\phi\phi_1} Ap_{\phi_1}$, therefore $Av = \sum_{\phi \in \widehat{G}} p_{\phi\phi_1} Ap_{\phi_1} v$.

By Lemma 17, $P_\phi(A)p_{\phi_1}v = p_{\phi\phi_1}Ap_{\phi_1}v$. So,

$$Av = \sum_{\phi_1 \in \widehat{G}} P_\phi(A)p_{\phi_1}v = P_\phi(A)\sum_{\phi_1 \in \widehat{G}} p_{\phi_1}v = P_\phi(A)v.$$

So, $A = P_\phi(A)$. □

3.3. Refining the $\widehat{G}$-grading

The goal of this section is to conclude, under certain additional conditions, that if a product of two operators with disjoint support is homogeneous of grade $\hat{1} \in \widehat{G}$, then both of the factors are themselves homogeneous (possibly of nontrivial grade).

Definition 22. Let $((g \in G) \mapsto (U_{\{x\},g} \in \mathcal{U}(\mathcal{H}_{\{x\}}))_{x\in\Gamma}$ be a system of on-site unitary G-actions, and let $\Lambda \subset \Gamma$. Then, for each $x \in \Gamma$, define

$$\tilde{U}_{\{x\},(g_1,g_2)} := \begin{cases} U_{\{x\},g_1} & \text{if } x \in \Lambda \\ U_{\{x\},g_2} & \text{if } x \in \Lambda^c. \end{cases}$$

This defines a system $((g_1, g_2) \mapsto \tilde{U}_{\{x\},(g_1,g_2)})_{x\in\Gamma}$ of on-site unitary $G \times G$-actions.
(More generally, this can be extended to any partition of Γ, with one copy of G for each part. Here, we consider only the case of a region and its complement.)
From this, for each finite region $\Lambda_1 \in \mathcal{P}_0(\Gamma)$, define the maps $(g_1, g_2) \mapsto \tilde{U}_{\Lambda_1,(g_1,g_2)}$ and $(g_1, g_2) \mapsto \tilde{\alpha}_{\Lambda_1,(g_1,g_2)}$ as for any system of on-site unitary group actions, and likewise define $(g_1, g_2) \mapsto \tilde{\alpha}_{(g_1,g_2)}$.
Note that for all $g \in G$ and for all finite $\Lambda_1 \in \mathcal{P}_0(\Gamma)$, that $\tilde{U}_{\Lambda_1,(g,g)} = U_{\Lambda_1,g}$. Also note that for all $g \in G$, $\tilde{\alpha}_{(g,g)} = \alpha_g$.

The Pontryagin dual of $G \times G$ is (isomorphic to) the group $\widehat{G} \times \widehat{G}$.
With this system of on-site unitary $G \times G$ actions, an element of $\mathcal{A}_\Lambda$ which has grade $\phi \in \widehat{G}$ with respect to the grading obtained from the system of on-site unitary G actions, has grade $(\phi, \hat{1}) \in \widehat{G} \times \widehat{G}$ with respect to the grading obtained from this system of on-site unitary $G \times G$-actions. Likewise, an element of $\mathcal{A}_{\Lambda^c}$ of grade $\phi \in \widehat{G}$ with respect to the grading from the G action, has grade $(\hat{1}, \phi) \in \widehat{G} \times \widehat{G}$ with respect to the grading from this $G \times G$ action.
This is because the $G \times G$ action acts on $\mathcal{A}_\Lambda$ only by the first copy of G, and acts on $\mathcal{A}_{\Lambda^c}$ only by the second copy of G. So, $\tilde{\alpha}_{(g,1)}|_{\mathcal{A}_\Lambda} = \alpha_g|_{\mathcal{A}_\Lambda}$ and $\tilde{\alpha}_{(1,g)}|_{\mathcal{A}_{\Lambda^c}} = \alpha_g|_{\mathcal{A}_{\Lambda^c}}$.

The purpose of the next few lemmas is largely in order to circumvent issues of convergence that may arise when trying to express operators $A \in \mathcal{B}(\mathcal{H}_\pi)$ as sums of their homogeneous components, either with respect to the $\widehat{G}$-grading or the $(\widehat{G} \times \widehat{G})$-grading. (These issues do not arise when G is finite.)

LEMMA 23. *Let* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ *be a* G*-covariant representation of* $(\mathcal{A}, \alpha)$ *and* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), \tilde{U}^{(\pi)} : G \times G \to \mathcal{U}(\mathcal{H}_\pi))$ *be a* $G \times G$*-covariant representation of* $(\mathcal{A}, \tilde{\alpha})$. *Suppose that* $\forall g \in G, \tilde{U}^{(\pi)}_{(g,g)} = U^{(\pi)}_g$. *Then, for all* $\phi_1, \phi_2 \in \widehat{G}$,

$$p_{\phi_1\phi_2} p_{(\phi_1,\phi_2)} = p_{(\phi_1,\phi_2)} = p_{(\phi_1,\phi_2)} p_{\phi_1\phi_2}\,,$$

where the projections $p_{(\phi_1,\phi_2)}$ *for* $(\phi_1, \phi_2) \in \widehat{G} \times \widehat{G}$ *are the projections onto the* $(\phi_1, \phi_2) \in \widehat{G} \times \widehat{G}$ *grade components of* $\mathcal{H}_\pi$, *defined using* $\tilde{U}^{(\pi)}$ *just as the projections* p_ϕ *are defined using* $U^{(\pi)}$.

PROOF. First to show that $p_{\phi_1\phi_2} p_{(\phi_1,\phi_2)} = p_{(\phi_1,\phi_2)}$:

$$\begin{aligned} p_{\phi_1\phi_2} p_{(\phi_1,\phi_2)} &= \int_{g \in G} (\phi_1\phi_2)(g^{-1}) U^{(\pi)}_g d\mu(g)\, p_{(\phi_1,\phi_2)} \\ &= \int_{g \in G} (\phi_1\phi_2)(g^{-1}) U^{(\pi)}_g p_{(\phi_1,\phi_2)} d\mu(g) \\ &= \int_{g \in G} (\phi_1\phi_2)(g^{-1}) \tilde{U}^{(\pi)}_{(g,g)} p_{(\phi_1,\phi_2)} d\mu(g) \\ &= \int_{g \in G} (\phi_1\phi_2)(g^{-1}) (\phi_1, \phi_2)(g, g) p_{(\phi_1,\phi_2)} d\mu(g) \\ &= \int_{g \in G} (\phi_1\phi_2)(g^{-1}) \phi_1(g)\phi_2(g) p_{(\phi_1,\phi_2)} d\mu(g) \\ &= \int_{g \in G} p_{(\phi_1,\phi_2)} d\mu(g) = p_{(\phi_1,\phi_2)}. \end{aligned}$$

Showing that $p_{(\phi_1,\phi_2)} = p_{(\phi_1,\phi_2)} p_{\phi_1\phi_2}$ is essentially the same, except that instead of using $\tilde{U}^{(\pi)}_{(g_1,g_2)} p_{(\phi_1,\phi_2)} = (\phi_1, \phi_2)(g_1, g_2) p_{(\phi_1,\phi_2)}$ to conclude that $U^{(\pi)}_g p_{(\phi_1,\phi_2)} = (\phi_1, \phi_2)(g, g) p_{(\phi_1,\phi_2)}$, it uses $p_{(\phi_1,\phi_2)} \tilde{U}^{(\pi)}_{(g_1,g_2)} = (\phi_1, \phi_2)(g_1, g_2) p_{(\phi_1,\phi_2)}$ to conclude that $p_{(\phi_1,\phi_2)} U^{(\pi)}_g = (\phi_1, \phi_2)(g, g) p_{(\phi_1,\phi_2)}$. (One concludes that $p_{(\phi_1,\phi_2)} \tilde{U}^{(\pi)}_{(g_1,g_2)} = (\phi_1, \phi_2)(g_1, g_2) p_{(\phi_1,\phi_2)}$ from the fact that $\tilde{U}^{(\pi)}_{(g_1,g_2)} p_{(\phi_1,\phi_2)} = (\phi_1, \phi_2)(g_1, g_2) p_{(\phi_1,\phi_2)}$ and $p_{(\phi_1,\phi_2)} \tilde{U}^{(\pi)}_{(g_1,g_2)} v = p_{(\phi_1,\phi_2)} \tilde{U}^{(\pi)}_{(g_1,g_2)} \sum_{(\phi_1', \phi_2') \in \widehat{G} \times \widehat{G}} p_{(\phi_1', \phi_2')} v$ for all $v \in \mathcal{H}_\pi$.) $\square$

LEMMA 24. *Let* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ *be a* G*-covariant representation of* $(\mathcal{A}, \alpha)$ *and* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), \tilde{U}^{(\pi)} : G \times G \to \mathcal{U}(\mathcal{H}_\pi))$ *be a* $G \times G$*-covariant representation of* $(\mathcal{A}, \tilde{\alpha})$*. Suppose that* $\forall g \in G, \tilde{U}^{(\pi)}_{(g,g)} = U^{(\pi)}_g$*. Let* $\phi_1, \phi_2 \in \widehat{G}$ *and* $A \in \mathcal{B}(\mathcal{H}_\pi)$*.*
Then, if $P_{(\phi_1,\phi_2)}(A) \neq 0$*, then* $P_{\phi_1\phi_2}(A) \neq 0$*.*

PROOF. Suppose $P_{(\phi_1,\phi_2)}(A) \neq 0$. Then, by Lemma 20 there exists a vector $v \in \mathcal{H}_\pi$ which is homogeneous with respect to the $\widehat{G} \times \widehat{G}$ grading, of some grade $(\psi_1, \psi_2) \in \widehat{G} \times \widehat{G}$, and such that $P_{(\phi_1,\phi_2)}(A)v = P_{(\phi_1,\phi_2)}(A)p_{(\psi_1,\psi_2)}v \neq 0$.
By Lemma 17, $P_{(\phi_1,\phi_2)}(A)p_{(\psi_1,\psi_2)}v = p_{(\phi_1,\phi_2)(\psi_1,\psi_2)}Ap_{(\psi_1,\psi_2)}v$.
By Lemma 23, $p_{(\psi_1,\psi_2)}v = p_{\psi_1\psi_2}p_{(\psi_1,\psi_2)}v$, so, (using $v = p_{(\psi_1,\psi_2)}v$) we have $v = p_{\psi_1\psi_2}v$.
Therefore,

$$\begin{aligned}
p_{(\phi_1,\phi_2)(\psi_1,\psi_2)}P_{\phi_1\phi_2}(A)v &= p_{(\phi_1,\phi_2)(\psi_1,\psi_2)}P_{\phi_1\phi_2}(A)p_{\psi_1\psi_2}v \\
&= p_{(\phi_1,\phi_2)(\psi_1,\psi_2)}p_{\phi_1\phi_2\psi_1\psi_2}Ap_{\psi_1\psi_2}v \\
&= p_{(\phi_1\psi_1,\phi_2\psi_2)}p_{\phi_1\phi_2\psi_1\psi_2}Ap_{\psi_1\psi_2}v \\
&= p_{(\phi_1\psi_1,\phi_2\psi_2)}Ap_{(\psi_1,\psi_2)}v \\
&= P_{(\phi_1,\phi_2)}(A)p_{(\psi_1,\psi_2)}v \neq 0.
\end{aligned}$$

(The fourth equality is using Lemma 23 on the left part of the expression and $p_{\psi_1\psi_2}v = v = p_{(\psi_1,\psi_2)}v$ on the right part of the expression.)
So, as $p_{(\phi_1,\phi_2)(\psi_1,\psi_2)}P_{\phi_1\phi_2}(A)v \neq 0$, therefore $P_{\phi_1\phi_2}(A)v \neq 0$, and so $P_{\phi_1\phi_2}(A) \neq 0$. $\square$

LEMMA 25. *Let* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ *be a* G*-covariant representation of* $(\mathcal{A}, \alpha)$ *and* $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), \tilde{U}^{(\pi)} : G \times G \to \mathcal{U}(\mathcal{H}_\pi))$ *be a* $G \times G$*-covariant representation of* $(\mathcal{A}, \tilde{\alpha})$*. Suppose that* $\tilde{U}^{(\pi)}_{(g,g)} = U^{(\pi)}_g$ *for all* $g \in G$*.*
Let $A \in \pi(\mathcal{A}_\Lambda)''$ *and* $B \in \pi(\mathcal{A}_{\Lambda^c})''$*. Then:*
For all $(g_1, g_2) \in G \times G$*,* $\mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(A) = \mathrm{Ad}(U^{(\pi)}_{g_1})$ *and* $\mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(B) = \mathrm{Ad}(U^{(\pi)}_{g_2})$*.*
Furthermore, $P_{(\phi,\hat{1})}(A) = P_\phi(A)$ *and* $P_{(\hat{1},\phi)}(B) = P_\phi(B)$*.*

PROOF. The arguments about $A \in \pi(\mathcal{A}_\Lambda)''$ (involving the first copy of G and $\widehat{G}$) and about $B \in \pi(\mathcal{A}_{\Lambda^c})''$ (involving the second copies) are symmetric, so we give the proof only for A.

Since for all $g \in G$,

$$\mathrm{Ad}(\tilde{U}^{(\pi)}_{(1,g)})(\pi(a)) = \pi(\tilde{\alpha}_{(1,g)}(a)) = \pi(a) \quad \text{for all } a \in \mathcal{A}_\Lambda,$$

it follows that $\tilde{U}^{(\pi)}_{(1,g)} \in \pi(\mathcal{A}_\Lambda)' = (\pi(\mathcal{A}_\Lambda)'')'$. (Here, 1 denotes the identity element of G.)
Thus, for $g \in G$, and $A \in \pi(\mathcal{A}_\Lambda)''$, $\mathrm{Ad}(\tilde{U}^{(\pi)}_{(1,g)})(A) = A$.
So, for $(g_1, g_2) \in G \times G$ and $A \in \pi(\mathcal{A}_\Lambda)''$,

$$\begin{aligned}\mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(A) &= \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_1)} \cdot \tilde{U}^{(\pi)}_{(1,g_1^{-1}g_2)})(A) \\ &= \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_1)})(A) = \mathrm{Ad}(U^{(\pi)}_{g_1})(A)\,.\end{aligned}$$

For the grading projections, we compute, for $A \in \pi(\mathcal{A}_\Lambda)''$ and $\phi \in \widehat{G}$:

$$\begin{aligned}P_{(\phi,\hat{1})}(A) &= \int_{(g_1,g_2)\in G\times G} \overline{(\phi,\hat{1})(g_1,g_2)}\, \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(A)\, d\mu_{G\times G}(g_1,g_2) \\ &= \int_{(g_1,g_2)\in G\times G} \phi(g_1^{-1})\, \mathrm{Ad}(U^{(\pi)}_{g_1})(A)\, d\mu_{G\times G}(g_1,g_2) \\ &= \int_{g_1\in G} \phi(g_1^{-1})\, \mathrm{Ad}(U^{(\pi)}_{g_1})(A)\, d\mu_G(g_1) = P_\phi(A),\end{aligned}$$

since the integral over g_2 contributes nothing (the integrand is constant in g_2).
The argument for $P_{(\hat{1},\phi)}(B) = P_\phi(B)$ is the same, with the roles of Λ and Λ^c reversed. □

LEMMA 26. *Let $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ be a G-covariant representation of $(\mathcal{A}, \alpha)$ and $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), \tilde{U}^{(\pi)} : G \times G \to \mathcal{U}(\mathcal{H}_\pi))$ be a $G \times G$-covariant representation of $(\mathcal{A}, \tilde{\alpha})$.*
Then, for all $\phi \in \widehat{G}$ and all $(\phi_1, \phi_2) \in \widehat{G} \times \widehat{G}$, for all regions $\Lambda_1 \subseteq \Gamma$, for all $X \in \pi(\mathcal{A}_{\Lambda_1})''$, $P_\phi(X), P_{(\phi_1,\phi_2)}(X) \in \pi(\mathcal{A}_{\Lambda_1})''$. (In particular one can apply this to $\Lambda_1 = \Lambda$ or to $\Lambda_1 = \Lambda^c$.)

PROOF. Let $\Lambda_1 \subseteq \Gamma$.
As α and $\tilde{\alpha}$ are on-site and $U^{(\pi)}$ and $\tilde{U}^{(\pi)}$ represent them for π, for all $g \in G$ and all $(g_1, g_2) \in G \times G$, $\mathrm{Ad}(U^{(\pi)}_g)(\pi(\mathcal{A}_{\Lambda_1})) = \pi(\mathcal{A}_{\Lambda_1})$ and $\mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(\pi(\mathcal{A}_{\Lambda_1})) = \pi(\mathcal{A}_{\Lambda_1})$.
For any set S of operators and any unitary U, $(\mathrm{Ad}(U)(S))'' = \mathrm{Ad}(U)(S'')$.
Therefore, for $X \in \pi(\mathcal{A}_{\Lambda_1})''$, $\mathrm{Ad}(U^{(\pi)}_g)(X), \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(X) \in \pi(\mathcal{A}_{\Lambda_1})''$.
What remains then is that the Bochner integrals defining $P_\phi(A) = \int_{g\in G} \phi(g^{-1})\, \mathrm{Ad}(U^{(\pi)}_g)(X) d\mu_G(g)$ and $P_{(\phi_1,\phi_2)}(A) = \int_{(g_1,g_2)\in G\times G} (\phi_1, \phi_2)((g_1, g_2)^{-1})\, \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(X) d\mu_{G\times G}(g_1, g_2)$ of functions with values in $\pi(\mathcal{A}_{\Lambda_1})''$, are still in $\pi(\mathcal{A}_{\Lambda_1})''$.

Since $\pi(\mathcal{A}_{\Lambda_1})''$ is a von Neumann algebra, it is a Banach space, and so is closed under (converging) Bochner integrals of operator-valued functions taking values in it. So these absolutely convergent Bochner integrals in $\pi(\mathcal{A}_{\Lambda_1})''$ converge to values in $\pi(\mathcal{A}_{\Lambda_1})''$.
I.e. $P_\phi(X), P_{(\phi_1,\phi_2)}(X) \in \pi(\mathcal{A}_{\Lambda_1})''$.

□

Lemma 27. *Let $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)} : G \to \mathcal{U}(\mathcal{H}_\pi))$ be a G-covariant representation of $(\mathcal{A}, \alpha)$ and $(\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), \tilde{U}^{(\pi)} : G \times G \to \mathcal{U}(\mathcal{H}_\pi))$ be a $G \times G$-covariant representation of $(\mathcal{A}, \tilde{\alpha})$. Suppose that $\forall g \in G, \tilde{U}^{(\pi)}_{(g,g)} = U^{(\pi)}_g$.*
Suppose also that π has the property that if $X \in \pi(\mathcal{A}_\Lambda)''$ and $Y \in \pi(\mathcal{A}_{\Lambda^c})''$ and X, Y are non-zero, then $XY \neq 0$, where Λ is the region in terms of which the $(G \times G)$-action is defined.
Let $A \in \pi(\mathcal{A}_\Lambda)''$ and $B \in \pi(\mathcal{A}_{\Lambda^c})''$.
Then, if AB is non-zero and is homogeneous of grade $\hat{1} \in \widehat{G}$, then A and B are each homogeneous with respect to the $\widehat{G}$-grading, and their grades are inverses of each-other.

Proof. As $AB \neq 0$, $A \neq 0$ and $B \neq 0$. So, by Lemma 20, as A and B are non-zero, there is at least one $\phi_A \in \widehat{G}$ such that $P_{\phi_A}(A) \neq 0$, and at least one $\phi_B \in \widehat{G}$ such that $P_{\phi_B}(B) \neq 0$.
By Lemma 25, because the $(G\times G)$-action acts only by the first copy of G on $\pi(\mathcal{A}_\Lambda)''$ and only by the second copy of G on $\pi(\mathcal{A}_{\Lambda^c})''$, for such $\phi_A, \phi_B \in \widehat{G}$, $P_{\phi_A}(A) = P_{(\phi,\hat{1})}(A)$ and $P_{\phi_B}(B) = P_{(\hat{1},\phi_B)}(B)$.
For any $X \in \mathcal{B}(\mathcal{H}_\pi)$, $\mathrm{Ad}(U^{(\pi)}_g)(P_{(\phi_1,\phi_2)}(A)) = \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g,g)})(P_{(\phi_1,\phi_2)}(A)) = (\phi_1,\phi_2)(g,g)P_{(\phi_1,\phi_2)}(A) = (\phi_1\phi_2)(g)P_{(\phi_1,\phi_2)}(A)$. So, if an operator has grade (ϕ_1,ϕ_2) with respect to the $(\widehat{G}\times\widehat{G})$-grading, it therefore has grade $\phi_1\phi_2$ with respect to the $\widehat{G}$-grading. So, if $P_{(\phi_1,\phi_2)}(X) \neq 0$, then $P_{\phi_1\phi_2}(X) \neq 0$
For sake of contradiction, suppose that there are some $\phi_A, \phi_B \in \widehat{G}$ such that $\phi_A\phi_B \neq \hat{1}$ and such that $P_{\phi_A}(A) \neq 0$ and $P_{\phi_B}(B) \neq 0$. For $(g_1, g_2) \in G \times G$,

$$\begin{aligned}\mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(AB) &= \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(A)\ \mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(B) \\ &= \mathrm{Ad}(U^{(\pi)}_{g_1})(A)\ \mathrm{Ad}(U^{(\pi)}_{g_2})(B),\end{aligned}$$

so

$$P_{(\phi_A,\phi_B)}(AB) = \int_{(g_1,g_2)\in G\times G} \overline{(\phi_A,\phi_B)(g_1,g_2)}\,\mathrm{Ad}(\tilde{U}^{(\pi)}_{(g_1,g_2)})(AB)d\mu_{G\times G}(g_1,g_2)$$

$$
\begin{aligned}
&= \int_{(g_1,g_2)\in G\times G} \phi_A(g_1^{-1})\phi_B(g_2^{-1})\,\mathrm{Ad}(U_{g_1}^{(\pi)})(A)\ \mathrm{Ad}(U_{g_2}^{(\pi)})(B)d\mu_{G\times G}(g_1,g_2)\\
&= \int_{(g_1,g_2)\in G\times G} \phi_A(g_1^{-1})\,\mathrm{Ad}(U_{g_1}^{(\pi)})(A)\,\phi_B(g_2^{-1})\,\mathrm{Ad}(U_{g_2}^{(\pi)})(B)d\mu_{G\times G}(g_1,g_2)\\
&= \int_{g_1\in G}\int_{g_2\in G} \phi_A(g_1^{-1})\,\mathrm{Ad}(U_{g_1}^{(\pi)})(A)\,\phi_B(g_2^{-1})\,\mathrm{Ad}(U_{g_2}^{(\pi)})(B)d\mu_G(g_2)d\mu_G(g_1)\\
&= \int_{g_1\in G} \phi_A(g_1^{-1})\,\mathrm{Ad}(U_{g_1}^{(\pi)})(A)\int_{g_2\in G} \phi_B(g_2^{-1})\,\mathrm{Ad}(U_{g_2}^{(\pi)})(B)d\mu_G(g_2)d\mu_G(g_1)\\
&= \int_{g_1\in G} \phi_A(g_1^{-1})\,\mathrm{Ad}(U_{g_1}^{(\pi)})(A)d\mu_G(g_1)\int_{g_2\in G} \phi_B(g_2^{-1})\,\mathrm{Ad}(U_{g_2}^{(\pi)})(B)d\mu_G(g_2)\\
&= P_{\phi_A}(A)P_{\phi_B}(B).
\end{aligned}
$$

By Lemma 26, as $A \in \pi(\mathcal{A}_\Lambda)''$ and $B \in \pi(\mathcal{A}_{\Lambda^c})''$, therefore $P_{\phi_A}(A) \in \pi(\mathcal{A}_\Lambda)''$ and $P_{\phi_B}(B) \in \pi(\mathcal{A}_{\Lambda^c})''$.

Because $P_{\phi_A}(A) \in \pi(\mathcal{A}_\Lambda)''$ and $P_{\phi_B}(B) \in \pi(\mathcal{A}_{\Lambda^c})''$ are non-zero, by the property of π that the product of a any pair of non-zero operators from $\pi(\mathcal{A}_\Lambda)'', \pi(\mathcal{A}_{\Lambda^c})''$ respectively is non-zero, their product is non-zero.

Therefore, $P_{(\phi_A,\phi_B)}(AB) = P_{\phi_A}(A)P_{\phi_B}(B) \neq 0$.

By Lemma 24, as $P_{(\phi_A,\phi_B)}(A) \neq 0$, therefore $P_{\phi_A\phi_B}(A) \neq 0$.

But, we said that $\phi_A\phi_B \neq \hat{1}$ and we are given that AB is homogeneous of grade $\hat{1}$. Therefore, we have a contradiction, so the assumption that there are ϕ_A, ϕ_B such that $\phi_A \cdot \phi_B \neq \hat{1}$ and $P_{\phi_A}(A) \neq 0$ and $P_{\phi_B}(B) \neq 0$ must have been false.

Therefore, for all ϕ_A, ϕ_B such that $P_{\phi_A}(A) \neq 0$ and $P_{\phi_B}(B) \neq 0$, we must have that $\phi_A\phi_B = \hat{1}$.

So, for any $\phi_A \in \widehat{G}$, if $P_{\phi_A}(A) \neq 0$, the only $\phi_B \in \widehat{G}$ such that $P_{\phi_B}(B)$ can be non-zero, is $\phi_B = \phi_A^{-1}$, and so B must be homogeneous of grade $\phi_B = \phi_A^{-1}$. By the same reasoning, A must also be homogeneous. So, there is exactly one $\phi_A \in \widehat{G}$ such that $P_{\phi_A}(A) \neq 0$ and exactly one $\phi_B \in \widehat{G}$ such that $P_{\phi_B}(B) \neq 0$, and $\phi_A\phi_B = \hat{1}$.

Therefore, by Lemma 21 we have that $P_{\phi_A}(A) = A$ and $P_{\phi_B}(B) = B$, i.e. that A and B are homogeneous of grades ϕ_A and ϕ_B (with $\phi_A\phi_B = \hat{1}$) as desired. □

3.4. Classification of Superselection sectors with respect to a Product Representation

From Theorem 4.5 of [4] we have

THEOREM 28 (Theorem 4.5 from [4]). *Let $\Lambda \subset \Gamma$ be a cone. Let $\pi_\Lambda : \mathcal{A}_\Lambda \to \mathcal{B}(\mathcal{H}_\Lambda)$ and $\pi_{\Lambda^c} : \mathcal{A}_{\Lambda^c} \to \mathcal{B}(\mathcal{H}_{\Lambda^c})$ be irreducible representations of $\mathcal{A}_\Lambda$ and $\mathcal{A}_{\Lambda^c}$ respectively. Let $\pi_0 := \pi_\Lambda \otimes \pi_{\Lambda^c}$.*
Then, if any irreducible representation $\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\sigma)$ satisfies the superselection criterion (Definition 3) with respect to π_0, then σ is unitarily equivalent to π_0, i.e. there exists a unitary $U : \mathcal{H}_\sigma \to \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^c}$ such that $\mathrm{Ad}(U) \circ \sigma = \pi_0$.

REMARK 29. Let $\Lambda \subset \Gamma$. Let $(\pi_\Lambda, U^{(\pi_\Lambda)})$ and $(\pi_{\Lambda^c}, U^{(\pi_{\Lambda^c})})$ be G-covariant representations of $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ and $(\mathcal{A}_{\Lambda^c}, \alpha_{\Lambda^c})$ respectively. Set $\pi_0 := \pi_\Lambda \otimes \pi_{\Lambda^c}$ and $U^{(\pi_0)} := (g \mapsto U_g^{(\pi_\Lambda)} \otimes U_g^{(\pi_{\Lambda^c})})$. Then $(\pi_0, U^{(\pi_0)})$ is a G-covariant representation of $(\mathcal{A}, \alpha)$. In addition, if one defines the map $\tilde{U}^{(\pi_0)} := ((g_1, g_2) \mapsto U_{g_1}^{(\pi_\Lambda)} \otimes U_{g_2}^{(\pi_{\Lambda^c})}) : G \times G \to \mathcal{U}(\mathcal{H}_{\pi_\Lambda} \otimes \mathcal{H}_{\pi_{\Lambda^c}})$ then $(\pi_0, \tilde{U}^{(\pi_0)})$ is a $G \times G$-covariant representation of $(\mathcal{A}, \tilde{\alpha})$, such that $\forall g \in G, \tilde{U}_{(g,g)}^{(\pi_0)} = U_g^{(\pi_0)}$.

THEOREM 30. *Let $\Lambda \subset \Gamma$ be a cone. Let $(\pi_\Lambda, U_\bullet^{(\pi_\Lambda)})$ and $(\pi_{\Lambda^c}, U_\bullet^{(\pi_{\Lambda^c})})$ be G-covariant representations of $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ and $(\mathcal{A}_{\Lambda^c}, \alpha_{\Lambda^c})$ respectively, with $\pi_\Lambda : \mathcal{A}_\Lambda \to \mathcal{B}(\mathcal{H}_\Lambda)$ and $\pi_{\Lambda^c} : \mathcal{A}_{\Lambda^c} \to \mathcal{B}(\mathcal{H}_{\Lambda^c})$ being irreducible representations. Let $(\pi_0, U_\bullet^{(\pi_0)})$ be the G-covariant representation of $(\mathcal{A}, \alpha)$ obtained as $\pi_0 := \pi_\Lambda \otimes \pi_{\Lambda^c}$ and $U_g^{(\pi_0)} := U_g^{(\pi_\Lambda)} \otimes U_g^{(\pi_{\Lambda^c}}$.*
Let $(\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\sigma), U_\bullet^{(\sigma)})$ be a G-covariant representation of $\mathcal{A}$ which satisfies the G-symmetry respecting version of the superselection criterion with respect to $(\pi_0, U_\bullet^{(\pi_0)})$ and let σ be an irreducible representation.
Then, there exists a unique $\phi \in \widehat{G}$ such that there is a unitary $U : \mathcal{H}_\sigma \to \mathcal{H}_{\pi_0} = \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^c}$ of grade ϕ (in the sense defined in Definition 18) such that $\mathrm{Ad}(U) \circ \sigma = \pi_0$. (There are also no non-homogeneous $U : \mathcal{H}_\sigma \to \mathcal{H}_{\pi_0}$ that satisfy $\mathrm{Ad}(U) \circ \sigma = \pi_0$.)
In this sense, the irreducible G-covariant representations of $(\mathcal{A}, \alpha)$ satisfying the G-equivariant version of the superselection criterion with respect to $(\pi_0, U^{(\pi_0)})$ are classified by $\widehat{G}$ up to G-equivariant unitary equivalence.

PROOF. As $(\sigma, U_\bullet^{(\sigma)})$ satisfies the G-symmetry respecting version of the superselection criterion with respect to $(\pi_0, U_\bullet^{(\pi_0)})$, σ satisfies the superselection criterion (the version not dealing with a G-action, Definition 3) with respect to π_0.
Therefore, by Theorem 28, there exists a unitary $U : \mathcal{H}_\sigma \to \mathcal{H}_{\pi_0}$ such that $\mathrm{Ad}(U) \circ \sigma = \pi_0$.
Because $(\sigma, U_\bullet^{(\sigma)})$ satisfies the the G-equivariant version of the superselection criterion (Definition 10) with respect to $(\pi_0, U_\bullet^{(\pi_0)})$, there exist G-equivariant unitaries $V_{\sigma,\Lambda}, V_{\sigma,\Lambda^c} : \mathcal{H}_\sigma \to \mathcal{H}_{\pi_0}$

such that $\mathrm{Ad}(V_{\sigma,\Lambda}) \circ \sigma|_{\mathcal{A}_{\Lambda^c}} = \pi_0|_{\mathcal{A}_{\Lambda^c}}$ and $\mathrm{Ad}(V_{\sigma,\Lambda^c}) \circ \sigma|_{\mathcal{A}_\Lambda} = \pi_0|_{\mathcal{A}_\Lambda}$. So $\mathrm{Ad}(V^*_{\sigma,\Lambda}) \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} = \sigma|_{\mathcal{A}_{\Lambda^c}}$ and $\mathrm{Ad}(V^*_{\sigma,\Lambda^c}) \circ \pi_0|_{\mathcal{A}_\Lambda} = \sigma|_{\mathcal{A}_\Lambda}$, and therefore $\mathrm{Ad}(UV^*_{\sigma,\Lambda}) \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} = \mathrm{Ad}(U) \circ \sigma|_{\mathcal{A}_{\Lambda^c}} = \pi_0|_{\mathcal{A}_{\Lambda^c}}$ and $\mathrm{Ad}(UV^*_{\sigma,\Lambda^c}) \circ \pi_0|_{\mathcal{A}_\Lambda} = \pi_0|_{\mathcal{A}_\Lambda}$. Now, using $\pi_0 = \pi_\Lambda \otimes \pi_{\Lambda^c}$ so $\pi_0|_{\mathcal{A}_{\Lambda^c}} = 1_{\mathcal{H}_\Lambda} \otimes \pi_{\Lambda^c}$ and $\pi_0|_{\mathcal{A}_\Lambda} = \pi_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}}$ we have $\mathrm{Ad}(UV^*_{\sigma,\Lambda}) \circ (1_{\mathcal{H}_\Lambda} \otimes \pi_{\Lambda^c}) = (1_{\mathcal{H}_\Lambda} \otimes \pi_{\Lambda^c})$ and $\mathrm{Ad}(UV^*_{\sigma,\Lambda^c}) \circ (\pi_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}}) = (\pi_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})$. Therefore, $UV^*_{\sigma,\Lambda} \in (1_{\mathcal{H}_\Lambda} \otimes \pi_{\Lambda^c}(\mathcal{A}_{\Lambda^c}))' = (\mathcal{B}(\mathcal{H}_\Lambda) \otimes 1_{\mathcal{H}_{\Lambda^c}})$. Let $V_\Lambda \in \mathcal{U}(\mathcal{H}_\Lambda)$ be the unitary such that $UV^*_{\sigma,\Lambda} = V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}}$ and let $V_{\Lambda^c} \in \mathcal{U}(\mathcal{H}_{\Lambda^c})$ be the unitary such that $UV_{\sigma,\Lambda^c} = 1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c}$.
Then, $(V^*_\Lambda \otimes V_{\Lambda^c}) = (V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^* \cdot (1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c}) = (UV^*_{\sigma,\Lambda})^*(UV_{\sigma,\Lambda^c}) = V_{\sigma,\Lambda}V^*_{\sigma,\Lambda^c}$.
As $V_{\sigma,\Lambda}V^*_{\sigma,\Lambda^c} : \mathcal{H}_{\pi_0} \to \mathcal{H}_{\pi_0}$ is a composition of two G-equivariant maps, it is also equivariant.
At this point, we wish to apply Lemma 27 to $(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^* \cdot (1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c})$ having grade $\hat{1}$. Apply the refining of the $\widehat{G}$ grading in Section 3.3 where the region $\Lambda \subset \Gamma$ chosen is the cone Λ. As described in Remark 29, for $\tilde{U}^{(\pi_0)}_{(g_1,g_2)} := U^{(\pi_\Lambda)}_{g_1} \otimes U^{(\pi_{\Lambda^c})}_{g_2}$, $(\pi_0, \tilde{U}^{(\pi_0)})$ is a $G\times G$-covariant representation of $(\mathcal{A}, \tilde{\alpha})$ such that $\forall g \in G, \tilde{U}^{(\pi_0)}_{(g,g)} = U^{(\pi_0)}_g$. In addition, as $\pi_0 = \pi_\Lambda \otimes \pi_{\Lambda^c}$, $\pi_0(\mathcal{A}_\Lambda)'' = \pi_\Lambda(\mathcal{A}_\Lambda)'' \otimes 1_{\mathcal{H}_{\Lambda^c}}$ and $\pi_0(\mathcal{A}_{\Lambda^c})'' = 1_{\mathcal{H}_\Lambda} \otimes \pi_{\Lambda^c}(\mathcal{A}_{\Lambda^c})''$, and so for any non-zero $X \in \pi_0(\mathcal{A}_\Lambda)''$ and non-zero $Y \in \pi_0(\mathcal{A}_{\Lambda^c})''$ we have $XY \neq 0$. Therefore, the conditions of Lemma 27 are satisfied, so for $A = (V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^*$ and $B = (1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c})$, and $AB = V_{\sigma,\Lambda}V^*_{\sigma,\Lambda^c}$ being G-equivariant, i.e. having grade $\hat{1} \in \widehat{G}$, we conclude that $A = (V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^*$ and $B = (1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c})$ are each homogeneous with respect to the $\widehat{G}$-grading, with grades inverses of each-other. Say ϕ is the grade of $(1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c})$, so ϕ^{-1} is the grade of $(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^*$.
For all $g \in G$, $\mathrm{Ad}(U^{(\pi_0)}_g)((V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^*) = \phi^{-1}(g)(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^*$, so $\mathrm{Ad}(U^{(\pi_0)}_g)((V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})) = (\phi^{-1}(g)(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})^*)^* = \phi(g)(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})$, so $(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}})$ is homogeneous of grade ϕ as well.
So, with $(V_\Lambda \otimes 1_{\mathcal{H}_{\Lambda^c}}) = (UV^*_{\sigma,\Lambda})$ and $(1_{\mathcal{H}_\Lambda} \otimes V_{\Lambda^c}) = UV_{\sigma,\Lambda^c}$ both homogeneous of grade ϕ, multiplying either by $V_{\sigma,\Lambda}$ or V_{σ,Λ^c} respectively on the right, we get that U is homogeneous of grade ϕ as well (by Lemma 19), because $V_{\sigma,\Lambda}$ and V_{σ,Λ^c} are G-equivariant, i.e. of grade $\hat{1}$, and $\phi\hat{1} = \phi$.
Finally, if for some unitary $U_2 : \mathcal{H}_\sigma \to \mathcal{H}_{\pi_0}$ is satisfies $\mathrm{Ad}(U_2)\circ\sigma = \pi_0$, then, because $\pi_0 = \mathrm{Ad}(U)\circ\sigma$, we have $\mathrm{Ad}(U^*U_2) \circ \sigma = \sigma$, and so $U^*U_2 \in \sigma(\mathcal{A})'$, and therefore because σ is irreducible, $U^*U_2 \in \sigma(\mathcal{A})' = \mathbb{C}1_{\mathcal{H}_\sigma}$ and so U_2 is just U multiplied by a phase factor, and so has the same grade ϕ.
Hence, the irreducible G-covariant representations of $(\mathcal{A}, \alpha)$ satisfying the G-equivariant version of the superselection criterion (relative to $(\pi_0, U^{(\pi_0)}_\bullet)$), are classified by $\widehat{G}$ up to G-equivariant unitary equivalence. □

In particular, if $\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}$ where $\omega_\Lambda : \mathcal{A}_\Lambda \to \mathbb{C}$ and ω_{Λ^c} are pure G-invariant states, then for $(\mathcal{H}_\Lambda, \pi_\Lambda : \mathcal{A}_\Lambda \to \mathcal{B}(\mathcal{H}_\Lambda), \Omega_{\omega_\Lambda})$ and $(\mathcal{H}_{\Lambda^c}, \pi_{\Lambda^c} : \mathcal{A}_{\Lambda^c} \to \mathcal{B}(\mathcal{H}_{\Lambda^c}), \Omega_{\omega_{\Lambda^c}})$ the GNS representations of ω_Λ and ω_{Λ^c} respectively, by Lemma 6, there exist $U^{(\pi_\Lambda)} : G \to \mathcal{U}(\mathcal{H}_\Lambda)$ and $U^{(\pi_{\Lambda^c})} : G \to \mathcal{U}(\mathcal{H}_{\Lambda^c})$ such that $(\pi_\Lambda, U^{(\pi_\Lambda)})$ and $(\pi_{\Lambda^c}, U^{(\pi_{\Lambda^c})})$ are G-covariant representations of $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ and $(\mathcal{A}_{\Lambda^c}, \alpha_{\Lambda^c})$ respectively. And, with $\mathcal{H}_{\pi_0} := \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^c}$, $\pi_0 := \pi_\Lambda \otimes \pi_{\Lambda^c}$, $\Omega_\omega := \Omega_{\omega_\Lambda} \otimes \Omega_{\omega_{\Lambda^c}}$, $(\mathcal{H}_{\pi_0}, \pi_0, \Omega_\omega)$ is a GNS representation of ω, and for $U^{(\pi_0)} := (g \mapsto (U_g^{(\pi_\Lambda)} \otimes U_g^{(\pi_{\Lambda^c})}))$, $(\pi_0, U^{(\pi_0)})$ is a G-covariant representation of $(\mathcal{A}, \alpha)$. So, the conditions of Theorem 30 hold. So, (using the fact that GNS representations are unique up to unitary equivalence) for $(\mathcal{H}_{\pi_0}, \pi_0, \Omega_\omega)$ a GNS representation of such a G-invariant product state $\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}$ and $U^{(\pi_0)}$ the unitary G-action on $\mathcal{H}_{\pi_0}$ that fixes Ω_ω, and makes $(\pi_0, U^{(\pi_0)})$ a G-covariant representation of $(\mathcal{A}, \alpha)$, the irreducible G-covariant representations that satisfy the G-equivariant version of the superselection criterion with respect to $(\pi_0, U^{(\pi_0)})$ are classified by $\widehat{G}$ up to G-equivariant unitary equivalence.

CHAPTER 4

Braided C^*-Tensor Supercategories from Fermionic Lattice Systems with Approximate Twisted Haag Duality

4.1. Introduction

The goal of this chapter is to extend a construction [6] introduced by Ogata for extracting braided C^*-tensor categories from gapped quantum spin systems. This construction, building on the Doplicher–Haag–Roberts (DHR) approach to superselection sectors, applies to bosonic spin systems satisfying approximate Haag duality. Here, we generalize it to the fermionic case by assuming a modified form of duality, which we call approximate twisted Haag duality.

4.1.1. High-Level Overview of Approach. In [6], after picking a reference representation, a subalgebra of the algebra of bounded operators on the Hilbert space for that representation is selected. Given an algebra, the endomorphisms of that algebra as objects and intertwiners between those endomorphisms, forms a strict monoidal category. For the particular algebra chosen, there are endomorphisms corresponding to different representations satisfying the superselection criterion. This sub-collection of endomorphisms is closed under composition, and includes the identity endomorphism, so the full subcategory of the category of endomorphisms of the algebra, with just the objects that correspond to the representations satisfying the superselection criterion, is also a strict monoidal category. In addition to this, a braiding morphisms is constructed and shown to be a braiding morphism, and the category is shown to have subobjects and direct sums, and to be independent of certain choices. Therefore, it is found to be a braided C^*-tensor category. The approach here is similar. However, many things are instead $\mathbb{Z}/2\mathbb{Z}$-graded. In particular, the algebra of bounded operators on the ($\mathbb{Z}/2\mathbb{Z}$-graded) Hilbert space for the representation, is regarded as a superalgebra, as is the subalgebra of it which is selected. The endomorphisms of this subalgebra which correspond to the representations of the algebra satisfying the version of the superselection criterion under consideration, are all grade-preserving. Given a superalgebra, the grade-preserving endomorphisms (as objects), together with intertwiners defined in the graded (or "super") sense, form a monoidal supercategory. A number of key properties must then be verified to ensure the analogous versions of them still hold in this graded setting.

4.1.2. Main Result. The main result of this chapter is the following theorem. For the more precise statement and proof, see Theorem 108.

THEOREM 31. *Let $\mathcal{A}$ be the quasilocal algebra of a quantum lattice system with fermionic degrees of freedom. Let π_0 be an irreducible grade-preserving representation that satisfies the approximate twisted Haag duality, and where $\pi_0(\mathcal{A}_\Lambda)''_{even}$ and $(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{even}$ are properly infinite factors. Then the representations satisfying the superselection criterion with respect to π_0, and which are localized to a chosen fixed cone Λ_0, form a braided strict C^*-tensor supercategory (Definition 107).*

The assumption that $\pi_0(\mathcal{A}_\Lambda)''_{\text{even}}$ and $(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{\text{even}}$ are properly infinite factors is something which I believe should follow under assumptions of π_0 being a GNS representation of a pure gapped ground state for a uniformly bounded finite range even interaction along with some other reasonable assumptions, but I have not yet managed to prove this, and therefore I make the assumption about the factors. The place this assumption is used is in order to show that the category has direct sums and subobjects. The other parts of the result do not depend on this assumption.

4.2. Setup and Assumptions

4.2.1. Two-Dimensional Quantum Lattice Systems. Let Γ be a lattice in $\mathbb{R}^2$. Technically it does not need to be a lattice in the strict sense, only a Delone set: that is, there exist constants $r, R > 0$ such that for all $x, x' \in \Gamma$ with $x \neq x'$, we have $d(x, x') > r$, and such that for all $p \in \mathbb{R}^2$ there exists an $x \in \Gamma$ such that $d(x, p) < R$.

However, we will call it a "lattice" even though we really mean Delone set. The reader may think of Γ as being $\mathbb{Z}^2$ throughout.

For each $x \in \Gamma$ let $\mathcal{H}_{\{x\}} = \mathcal{H}_{\{x\},\text{even}} \oplus \mathcal{H}_{\{x\},\text{odd}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded finite-dimensional (and not 0-dimensional) Hilbert space. We also impose the following condition:

ASSUMPTION 4.2.1 (to guarantee existence of odd local unitaries in large enough regions). *The set*

$$\{x \in \Gamma | \dim(\mathcal{H}_{\{x\},even}) = \dim(\mathcal{H}_{\{x\},odd}) \geq 1\}$$

is also a Delone set.

The reader may wish to suppose that this set is all of Γ. Nothing essential will be lost by doing so.
For $x \in \Gamma$ let $\mathcal{A}_{\{x\}} := \mathcal{B}(\mathcal{H}_{\{x\}})$ which, viewed not only as a C^*-algebra, but also as a superalgebra, where the even (resp. odd) part consists of the grade-preserving (resp. grade-reversing) operators.
Let $U_{\alpha_F,\{x\}} := 1_{\mathcal{H}_{\{x\},\text{even}}} - 1_{\mathcal{H}_{\{x\},\text{odd}}} \in \mathcal{A}_{\{x\},\text{even}}$. For $A = A_0 + A_1 \in \mathcal{A}_{\{x\}}$,

$$\text{Ad}(U_{\alpha_F,\{x\}})(A_0 + A_1) = A_0 - A_1.$$

Here and throughout, $A = A_0 + A_1$ means that A_0 is the even part of A and A_1 is the odd part.
Fix a total order on Γ. The particular order will not matter as long as it is kept consistent.
For finite $\Lambda \subset \Gamma$, $\mathcal{A}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{A}_{\{x\}}$ where this tensor product is taken in the order induced by the total ordering chosen on Γ, and refers to the tensor product for superalgebras. (The tensor product for superalgebras is such that, if A, B, C, D are each homogeneous, then $(A \otimes B) \cdot (C \otimes D) = (-1)^{|B||C|}(AC) \otimes (BD)$ where $|B|, |C|$ are the grades of B, C respectively.)
For finite subsets $\Lambda_1, \Lambda_2 \subset \Gamma$ satisfying $\Lambda_1 \subseteq \Lambda_2$ there is an inclusion $\mathcal{A}_{\Lambda_1} \hookrightarrow \mathcal{A}_{\Lambda_2}$ by taking the tensor product with the identity operator from $\mathcal{A}_{\Lambda_2 \setminus \Lambda_1}$. These form a directed system.
For infinite $\Lambda \subseteq \Gamma$, define $\mathcal{A}_\Lambda := \lim\limits_{\Lambda' \to \Lambda} \mathcal{A}_{\Lambda'}$ where this is the direct limit of the directed system consisting of the C^*-algebras $\mathcal{A}_{\Lambda'}$ for finite $\Lambda' \subset \Lambda$, and the inclusions between them.
In particular $\mathcal{A}_\Gamma$ is defined this way. Set $\mathcal{A} := \mathcal{A}_\Gamma$.
Throughout, for $\Lambda \subseteq \mathbb{R}^2$, Λ will often be identified with $\Lambda \cap \Gamma$ when the context is such that a subset of Γ is required.
Define $\alpha_F \in \text{Aut}(\mathcal{A})$ as, for $A = A_0 + A_1 \in \mathcal{A}$, $\alpha_F(A_0 + A_1) := A_0 - A_1$.
For finite $\Lambda \subset \Gamma$, define $U_{\alpha_F,\Lambda} := \prod_{x\in\Lambda} U_{\alpha_F,\{x\}}$, and define $\alpha_{F,\Lambda} := \text{Ad}(U_{\alpha_F,\Lambda})$. Note that $\alpha_F(A) = \lim\limits_{\Lambda \to \Gamma} \alpha_\Lambda(A)$ where the limit is over finite $\Lambda \subset \Gamma$.
Let $\mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space, and $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be an irreducible grade-preserving $*$-representation of the C^*-algebra $\mathcal{A}$.
Let $U_{\alpha_F} := 1_{\mathcal{H}_{\text{even}}} - 1_{\mathcal{H}_{\text{odd}}}$. $\text{Ad}(U_{\alpha_F}) \circ \pi_0 = \pi_0 \circ \alpha_F$. For $A = A_0 + A_1 \in \mathcal{B}(\mathcal{H})$, $\text{Ad}(U_{\alpha_F})(A_0 + A_1) = A_0 - A_1$.
So, for $\alpha : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathcal{A})$ a group homomorphism which sends the identity element to the identity and the non-identity element to α_F, and for $U_\alpha : \mathbb{Z}/2\mathbb{Z} \to \mathcal{U}(\mathcal{H})$ the group homomorphism which sends the non-identity element of $\mathbb{Z}/2\mathbb{Z}$ to U_{α_F}, (π_0, U_α) is a $(\mathbb{Z}/2\mathbb{Z})$-covariant representation of

$(\mathcal{A}, \alpha)$. Because α is entirely determined by α_F, and likewise U_α is entirely determined by U_{α_F}, we will describe this as (π_0, U_{α_F}) being a $(\mathbb{Z}/2\mathbb{Z})$-covariant representation of $(\mathcal{A}, \alpha_F)$.

DEFINITION 32. Given a representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho)$ and a unitary $U_\rho \in \mathcal{U}(\mathcal{H}_\rho)$, we say that (ρ, U_ρ) is a $(\mathbb{Z}/2\mathbb{Z})$-covariant representation of $(\mathcal{A}, \alpha_F)$ if $\mathrm{Ad}(U_\rho) \circ \rho = \rho \circ \alpha_F$ and $U_\rho^2 = 1$.

LEMMA 33. *For any cone Λ, there exists an odd unitary $B_1 \in \mathcal{A}_{\Lambda,loc}$.*

PROOF. By assumption 4.2.1 there is some radius such that for any ball in $\mathbb{R}^2$ of that radius, there is a site $x \in \Gamma$ in that ball such that $\dim(\mathcal{H}_{\{x\},\text{even}}) = \dim(\mathcal{H}_{\{x\},\text{odd}}) \geq 1$. And, for such a site x, there is an odd unitary in $\mathcal{A}_{\{x\}}$. (One can take an orthonormal basis of $\mathcal{H}_{\{x\},\text{even}}$ and of $\mathcal{H}_{\{x\},\text{odd}}$, and then map each basis element of the former to a different basis element of the latter, and visa versa, and extend linearly.)

For any cone $\Lambda \subset \mathbb{R}^2$, and any positive radius, there is a ball of that radius that is a subset of Λ. Therefore, there exists a site $x \in \Lambda \cap \Gamma$ such that there exists an odd unitary $B_1 \in \mathcal{A}_{\{x\}} \subset \mathcal{A}_{\Lambda,loc}$. □

REMARK 34. Conversely, for any finite subset R of Γ, if there is an odd unitary in $\mathcal{A}_R$, there must be a site $x \in R$ such that $\dim(\mathcal{H}_{\{x\},\text{even}}) = \dim(\mathcal{H}_{\{x\},\text{odd}}) \geq 1$. This is because for such a unitary to exist implies $\dim(\mathcal{H}_{R,\text{even}}) = \dim(\mathcal{H}_{R,\text{odd}})$, and the tensor product of two $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional vector spaces only has the dimensions of the even and odd parts equal if at least one of the two tensor factors has the dimensions of the even and odd parts equal.

4.2.2. Approximate Twisted Haag Duality.

Following Equation 4.7 of [2]:

DEFINITION 35. For any C^* subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$, define the twist of $\mathfrak{A}$ to be $\mathfrak{A}^t := \{A_0 + U_{\alpha_F} A_1 | A_0 + A_1 \in \mathfrak{A}\}$(where subscripts indicate parity).

Define "the twisted commutant" to be the commutant of the twist,

i.e. $\mathfrak{A}^{t\prime} = (\mathfrak{A}^t)'$.

Also define the linear map $\mathrm{tw} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ as $\mathrm{tw}(A_0 + A_1) := A_0 + U_{\alpha_F} A_1$.

Note that $\mathfrak{A}^{tt} = \mathfrak{A}$.

DEFINITION 36. *Twisted locality* is the condition on π that for any two disjoint cones Λ_1, Λ_2, that $\pi(\mathcal{A}_{\Lambda_1}) \subseteq \pi(\mathcal{A}_{\Lambda_2})^{t\prime}$.

Twisted locality is obtained automatically, just as the usual locality, $\pi(\mathcal{A}_{\Lambda_1}) \subseteq \pi(\mathcal{A}_{\Lambda_2})'$ is obtained automatically for spin systems.

Following Definition 4.3 of [2],

DEFINITION 37. A representation π of $\mathcal{A}$ satisfies Twisted Haag duality if, for all cones Λ,

$$\pi(\mathcal{A}_{\Lambda^c})^{t\prime} = \pi(\mathcal{A}_\Lambda)''.$$

Combining that definition with the definition of Approximate Haag duality given in [6],

DEFINITION 38. A irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ satisfies Approximate Twisted Haag duality if: $\forall \varphi \in (0, 2\pi)$, $\forall \varepsilon > 0$ such that $\varphi + 4\varepsilon < 2\pi$, there exists $R_{\varphi,\varepsilon} > 0$ and for $\delta > 0$ exists decreasing functions $f_{\varphi,\varepsilon,\delta} : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} f_{\varphi,\varepsilon,\delta}(t) = 0$ such that

(i) for all cones Λ such that $|\arg \Lambda| = \varphi$, there exists an even graded unitary $U_{\Lambda,\varepsilon} \in \mathcal{U}(\mathcal{H})$ such that

$$\pi_0\left(\mathcal{A}_{\Lambda^c}\right)^{t\,\prime} \subseteq \mathrm{Ad}(U_{\Lambda,\varepsilon})\left(\pi_0\left(\mathcal{A}_{(\Lambda - R_{\varphi,\varepsilon} e_\Lambda)_\varepsilon}\right)''\right)$$

(ii) $\forall \delta > 0, \forall t \geq 0$ there exists an even graded unitary $\tilde{U}_{\Lambda,\varepsilon,\delta,t} \in \pi(\mathcal{A}_{(\Lambda - t)_{\varepsilon+\delta}})''$ such that

$$\left\|U_{\Lambda,\varepsilon} - \tilde{U}_{\Lambda,\varepsilon,\delta,t}\right\| \leq f_{\varphi,\varepsilon,\delta}(t).$$

For cones Λ, let $R_{\Lambda,\varepsilon}$ denote $R_{|\arg \Lambda|,\varepsilon}$. Also, for $t \in \mathbb{R}$, let $\Lambda + t$ denote $\Lambda + te_\Lambda$.

While in [6] often $\pi_0(\mathcal{A}_{\Lambda^c})'$ appears for various cones Λ, in almost all of those cases the analogous result here will have $\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$ in its place. As such, for all cones Λ we define $\mathfrak{A}(\Lambda) := \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$.

DEFINITION 39. For $\alpha \in \mathrm{Aut}(\mathcal{A})$, α is approximately-factorizable if:

(i): for each cone Λ and each $\delta > 0$, there exist automorphism $\beta_\Lambda, \tilde{\beta}_\Lambda \in \mathrm{Aut}(\mathcal{A}_\Lambda)$, $\beta_{\Lambda^c}, \tilde{\beta}_{\Lambda^c} \in \mathrm{Aut}(\mathcal{A}_{\Lambda^c})$ and $\Xi_{\Lambda,\delta}, \tilde{\Xi}_{\Lambda,\delta} \in \mathrm{Aut}(\mathcal{A}_{\Lambda_\delta \cap (\Lambda^c)_\delta})$ and unitaries $v_{\Lambda,\delta}, \tilde{v}_{\Lambda,\delta} \in \mathcal{A}$ such that

$$\alpha = \mathrm{Ad}(v_{\Lambda\delta}) \circ \Xi_{\Lambda,\delta} \circ (\beta_\Lambda \otimes \beta_{\Lambda^c}),$$

$$\alpha^{-1} = \mathrm{Ad}(\tilde{v}_{\Lambda\delta}) \circ \tilde{\Xi}_{\Lambda,\delta} \circ (\tilde{\beta}_\Lambda \otimes \tilde{\beta}_{\Lambda^c}).$$

(ii): For each $\delta, \delta' > 0$ and $\varphi \in (0, 2\pi)$, there exists a decreating function $g_{\varphi,\delta,\delta'}(t)$ with domain $\mathbb{R}_{\geq 0}$ such that $\lim_{t\to\infty} g_{\varphi,\delta,\delta'}(t) = 0$ and such that for any cone Λ such that $|\arg \Lambda| = \varphi$, for all $t \geq 0$, there exist unitaries $v'_{\Lambda,\delta,\delta',t}, \tilde{v}'_{\Lambda,\delta,\delta',t} \in \mathcal{A}_{(\Lambda-t)_{\delta+\delta'}}$ satisfying

$$\left\| v_{\Lambda,\delta} - v'_{\Lambda,\delta,\delta',t} \right\|, \left\| \tilde{v}_{\Lambda,\delta} - \tilde{v}'_{\Lambda,\delta,\delta',t} \right\| \leq g_{\varphi,\delta,\delta'}(t)$$

for unitaries $v_{\Lambda,\delta}, \tilde{v}_{\Lambda,\delta}$ in (i).

Definition 40. For $\alpha \in \mathrm{Aut}(\mathcal{A})$, α is approximately-factorizable in a grade-preserving way if it is approximately-factorizable and the automorphisms $\beta_\Lambda, \tilde{\beta}_\Lambda, \beta_{\Lambda^c}, \tilde{\beta}_{\Lambda^c}, \Xi_{\Lambda,\delta}, \tilde{\Xi}_{\Lambda,\delta}$ and the unitaries $v_{\Lambda,\delta}, \tilde{v}_{\Lambda,\delta}, v'_{\Lambda,\delta,\delta',t}, \tilde{v}'_{\Lambda,\delta,\delta',t}$ can be chosen such that all those automorphisms are grade-preserving and all those unitaries are even.

An automorphism α which is approximately-factorizable in a grade-preserving way will of course itself be grade-preserving.
Following Proposition 1.3 of [6]:

Proposition 41. *Let $(\mathcal{H}, \pi_0)$ be an irreducible grade-preserving representation of $\mathcal{A}$ which satisfies approximate twisted Haag duality. Then for any automorphism $\alpha \in \mathrm{Aut}(\mathcal{A})$ that is approximately-factorizable in a grade-preserving way, $(\mathcal{H}, \pi_0 \circ \alpha)$ also satisfies approximate twisted Haag duality.*

Proof. The proof is very similar to the proof of Proposition 1.3 of [6], and the length of the changes that need to be made is much smaller than the length of the overall proof, so only the changes will be described.
After $\pi_0(\mathcal{A}_{(\Lambda_\delta)^c}) \subseteq \mathrm{Ad}(\pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta})))(\pi_0 \circ \alpha(\mathcal{A}_{\Lambda^c}))$ is obtained, rather than taking the commutant of this, take the twisted commutant, yielding

$$(\mathrm{Ad}(\pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta})))(\pi_0 \circ \alpha(\mathcal{A}_{\Lambda^c})))^{t\prime} \subseteq \pi_0(\mathcal{A}_{(\Lambda_\delta)^c})^{t\prime}.$$

As $\pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta}))$ is even, $\mathrm{Ad}(\pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta})))((\pi_0 \circ \alpha(\mathcal{A}_{\Lambda^c}))^{t\prime}) \subseteq \pi_0(\mathcal{A}_{(\Lambda_\delta)^c})^{t\prime}$, and applying the approximate twisted Haag duality in place of the application of approximate Haag duality, we get $\mathrm{Ad}(\pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta})))((\pi_0 \circ \alpha(\mathcal{A}_{\Lambda^c}))^{t\prime}) \subseteq \mathrm{Ad}(U_{\Lambda_\delta,\delta} \cdot \pi_0 \circ \alpha(\tilde{v}_{\tilde{\Lambda}_{2,\delta},\delta}))((\pi_0 \circ \alpha(\mathcal{A}_{\tilde{\Lambda}_{3\delta}}))'')$ and so

$$(\pi_0 \circ \alpha(\mathcal{A}_{\Lambda^c}))^{t\prime} \subseteq \mathrm{Ad}(\pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta}))^* \cdot U_{\Lambda_\delta,\delta} \cdot \pi_0(\alpha(\tilde{v}_{\tilde{\Lambda}_{2,\delta},\delta})))((\pi_0 \circ \alpha(\mathcal{A}_{\tilde{\Lambda}_{3\delta}}))'').$$

With this, along with other details included in the proof of Proposition 1.3 of [6], defining $U^{(1)}_{\Lambda,\varepsilon} := \pi_0(\alpha(\tilde{v}_{\Lambda_\delta,\delta}))^* \cdot U_{\Lambda_\delta,\delta} \cdot \pi_0(\alpha(\tilde{v}_{\tilde{\Lambda}_{2,\delta},\delta}))$ (and noting that it is even) is seen to show that $(\mathcal{H}, \pi_0 \circ \alpha)$

satisfies part (i) of Definition 38 (approximate twisted Haag duality). The part of the proof in [6] that shows that part (ii) of approximate Haag duality holds for the $(\mathcal{H}, \pi_0 \circ \alpha)$ in that setting, carries over to this setting to show that the $(\mathcal{H}, \pi_0 \circ \alpha)$ of this setting satisfies part (ii) of Definition 38 (approximate twisted Haag duality), with no changes being made. (The reason no changes need to be made here is essentially due to the fact that this part of the proof and statement only makes reference to double commutants, never to single commutants or to two operators commuting). ☐

4.2.3. Superselection Criterion for Lattice Systems with Fermionic Degrees of Freedom. Recall our fixed reference representation of $\mathcal{A}$, $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, which is irreducible and grade-preserving.

To account for the anticommutation of odd operators with disjoint support, we define a variation on the superselection criterion (see Definition 3) as follows:

Definition 42. A $(\mathbb{Z}/2\mathbb{Z})$-covariant representation $(\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}), U_\rho \in \mathcal{U}(\mathcal{H}))$ of $(\mathcal{A}, \alpha_F)$ satisfies the superselection criterion with respect to (π_0, U_{α_F}) if:
for all cones Λ, there exists an even unitary $V_{\rho,\Lambda} \in \mathcal{U}(\mathcal{H})$ such that:
for all $A_0 \in \mathcal{A}_{\Lambda^c,\text{even}}$, $\text{Ad}(V_{\rho,\Lambda}) \circ \rho(A_0) = \pi_0(A_0)$,
and for all $A_1 \in \mathcal{A}_{\Lambda^c,\text{odd}}$, $\text{Ad}(V_{\rho,\Lambda})(U_\rho \rho(A_1)) = U_{\alpha_F} \pi_0(A_1)$.

Let $\mathcal{O}_0$ be the set of grade-preserving representations of $\mathcal{A}$ on $\mathcal{H}$ which satisfy the superselection criterion with respect to π_0.
For $\rho \in \mathcal{O}_0$, and Λ a cone, let $\mathcal{V}_{\rho,\Lambda}$ be the set of even unitaries $V_{\rho,\Lambda}$ which satisfy the two conditions at the end of the above definition.
$\mathcal{V}_{\rho,\Lambda}$ will always be non-empty by virtue of $\rho \in \mathcal{O}_0$.

For $\rho \in \mathcal{O}_0$, define the superselection sector $[\rho]$ to be the set of all $\sigma \in \mathcal{O}_0$ such that there exists an even unitary $U \in \mathcal{U}(\mathcal{H})$ such that $\text{Ad}(U) \circ \sigma = \rho$.
For any cone Λ, define

$$\mathcal{O}_\Lambda := \{(\rho, U_\rho) \in \mathcal{O}_0 |\ \rho|_{\mathcal{A}_{\Lambda^c}} = \pi_0|_{\mathcal{A}_{\Lambda^c}}\} .$$

Definition 43. For $(\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}), U_\rho)$ a representation of $\mathcal{A}$ equipped with an implementation of α_F, define the function $\rho^t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ as $\rho^t(A_0 + A_1) := \rho(A_0) + U_\rho \rho(A_1)$, for all $A_0 \in \mathcal{A}_{\text{even}}$ and $A_1 \in \mathcal{A}_{\text{odd}}$.

Note that ρ^t is not an algebra homomorphism!

Definition 44. For $(\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}), U_\rho)$ and $(\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H}), U_\sigma)$ two representations of $\mathcal{A}$ each equipped with their respective implementations of α_F, define R to be an intertwiner from (ρ, U_ρ) to (σ, U_σ) if $\forall A \in \mathcal{A}$, $R\rho^t(A) = \sigma^t(A)R$ (i.e. $R \cdot (\rho(A_0) + U_\rho \rho(A_1)) = (\sigma(A_0) + U_\sigma \sigma(A_1)) \cdot R$). Let $R : (\rho, U_\rho) \to (\sigma, U_\sigma)$ denote that R is an intertwiner from (ρ, U_ρ) to (σ, U_σ).

We will often write ρ in place of (ρ, U_ρ), leaving U_ρ implicit.
It can readily be seen that if $R_1 : \rho_1 \to \rho_2$ and $R_2 : \rho_2 \to \rho_3$, then $R_2 R_1 : \rho_1 \to \rho_3$

Proposition 45. *For $\rho \in \mathcal{O}_0$ be an* irreducible *grade-preserving representation of $\mathcal{A}$. Then U_ρ is either U_{α_F} or $-U_{\alpha_F}$.*

Proof of this is at 109. It can be seen that both values are possibilities, because for any $\rho \in \mathcal{O}_0$, it can be seen that $U_{\rho \circ \alpha_F} = -U_\rho$.

Lemma 46. *For $\rho \in \mathcal{O}_0$, then,*
if $U_\rho = U_{\alpha_F}$, then for all cones Λ, for all $V_{\rho,\Lambda} \in \mathcal{V}_{\rho,\Lambda}$, for all $A \in \mathcal{A}_{\Lambda^c}$, have that $\operatorname{Ad}(V_{\rho,\Lambda}) \circ \rho(A) = \pi_0(A)$,
and, if $U_\rho = -U_{\alpha_F}$, then for all cones Λ, for all $V_{\rho,\Lambda} \in \mathcal{V}_{\rho,\Lambda}$, for all $A \in \mathcal{A}_{\Lambda^c}$, have that $\operatorname{Ad}(V_{\rho,\Lambda}) \circ \rho(A) = \pi_0 \circ \alpha_F(A)$

Proof of this is at 110.
(If ρ is irreducible, then by the result before this one, one of the two cases in this result will apply. But, even if ρ is not irreducible, if U_ρ is as described, one can still apply this.)

4.3. Analogous Results for Defining the Category

This section presents statements and definitions that are closely analogous to those in [6], which are used in constructing the braided monoidal C^*-supercategory.

Throughout, assume that π_0, in addition to being irreducible, also satisfies approximate twisted Haag duality.

As defined in equation A.7 of [6], define $\mathcal{C}(\theta, \varphi)$ to be the set of cones for which no ray contained in that cone has an angle in the interval $[\theta-\varphi, \theta+\varphi]$ (where a ray of the form $\vec{x}+\{t(\cos(\beta), \sin(\beta))|t \in [0, \infty)\}$ is said to have angle β). A pair (θ, φ) is said to "label a forbidden direction" if $\theta \in \mathbb{R}$ and $\varphi \in (0, \pi)$, so that the image $[\theta-\varphi, \theta+\varphi]$, under the usual map from $\mathbb{R}$ to S^1 is a proper non-empty subset of S^1. It is called a "forbidden direction" because it is the range of angles that the cones used aren't allowed to include.

The category constructed will be as follows: We fix a forbidden direction (θ, φ) and a cone $\Lambda_0 \in \mathcal{C}(\theta, \varphi)$. The objects of the category will be the representations in

$$\mathcal{O}_{\Lambda_0,*} := \left\{\rho \in \mathcal{O}_0 \;\middle|\; \rho|_{\mathcal{A}_{\Lambda_0^c}} = \pi_0|_{\mathcal{A}_{\Lambda_0^c}} \text{ and } U_\rho = U_{\alpha_F}\right\},$$

i.e. those representations satisfying the superselection criterion (Definition 42), which are localized in Λ_0, and which have the same implementation of the parity operator.

We will define a sub-superalgebra $\mathcal{B}(\theta, \varphi) \subseteq \mathcal{B}(\mathcal{H})$ (Definition 48) using the reference representation π_0. For each $\rho \in \mathcal{O}_{\Lambda_0,*}$ there is a unique corresponding grade preserving endomorphism $T_\rho^{(\theta,\varphi),\Lambda_0,1}$ of $\mathcal{B}(\theta, \varphi)$, given by Definition 57. The hom spaces of the category are defined as the spaces of intertwiners between these corresponding endomorphisms (Definition 74), in a sense that matches with the definition of the hom spaces for the monoidal supercategory of the category of grade-preserving endomorphisms of the superalgebra $\mathcal{B}(\theta, \varphi)$. In fact, our category will be, up to relabeling the objects, a full subcategory of that category. The supermonoidal product of the category is defined on objects as $\rho \otimes \sigma = T_\rho^{(\theta,\varphi),\Lambda_0,1} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0$ and is such that $T_{\rho\otimes\sigma}^{(\theta,\varphi),\Lambda_0,1} = T_\rho^{(\theta,\varphi),\Lambda_0,1} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1}$, and the supermonoidal product on the morphisms (Definition 81) is defined in a way that is equivalent to how the monoidal product for morphisms in the category of grade-preserving endomorphisms of a superalgebra is defined. The braiding morphisms are defined by taking a limit of some products of morphisms that transport where the representations are localized to relative to the reference representation, to different cones that are far from each-other, described in Definition 92. The direct sums and subobjects (so that even projections split) are constructed using isometries obtained using some factors being properly infinite, in Lemma 99 and Lemma 100 respectively.

4.3.1. Superselection sectors and their extensions. Following Lemma 2.2 from [6]:

LEMMA 47. *Let Λ_1, Λ_2 be cones, $\rho, \sigma \in \mathcal{O}_0$, $V_{\rho,\Lambda_1} \in \mathcal{V}_{\rho,\Lambda_1}, V_{\sigma,\Lambda_2} \in \mathcal{V}_{\sigma,\Lambda_2}$, and $R \in \mathcal{B}(\mathcal{H})$ an intertwiner from ρ to σ. Then $V_{\sigma,\Lambda_2} R V^*_{\rho,\Lambda_1} \in \pi(\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c})^{t\prime}$*

PROOF. Let $A \in \mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c}$.

$$V_{\sigma,\Lambda_2} R V^*_{\rho,\Lambda_1} \pi^t(A) = V_{\sigma,\Lambda_2} R \rho^t(A) V^*_{\rho,\Lambda_1} = V_{\sigma,\Lambda_2} \sigma^t(A) R V^*_{\rho,\Lambda_1} = \pi^t(A) V_{\sigma,\Lambda_2} R V^*_{\rho,\Lambda_1}$$

Using that $\mathrm{Ad}(V_{\rho,\Lambda_1}) \circ \rho^t|_{\mathcal{A}_{\Lambda_1^c}} = \pi^t|_{\mathcal{A}_{\Lambda_1^c}}$ and likewise for V_{σ,Λ_2} and that $\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c} = \mathcal{A}_{\Lambda_1^c} \cap \mathcal{A}_{\Lambda_2^c}$. So, for all $\pi^t(A) = \pi(A_0) + U_{\alpha_F}\pi(A_1) \in (\pi(\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c})^t$, $V_{\sigma,\Lambda_2} R V^*_{\rho,\Lambda_1}$ commutes with it, i.e. $V_{\sigma,\Lambda_2} R V^*_{\rho,\Lambda_1} \in \pi(\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c})^{t\prime}$. ☐

Based on the Definition 2.3 of [6]:

DEFINITION 48. For $\theta \in \mathbb{R}, \varphi \in (0, \pi)$, define $\mathcal{B}(\theta, \varphi) := \overline{\bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}}$.
Also define $\mathcal{B}_0(\theta, \varphi) := \bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_\Lambda)''$.

By twisted locality, for all cones Λ, $\pi_0(\mathcal{A}_\Lambda) \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$, so $\pi_0(\mathcal{A}_\Lambda)'' \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$. Therefore $\mathcal{B}_0(\theta, \varphi) \subseteq \mathcal{B}(\theta, \varphi)$ and $\overline{\mathcal{B}_0(\theta, \varphi)}^{\|\cdot\|} \subseteq \mathcal{B}(\theta, \varphi)$. Under the assumption of (full, not approximate) twisted Haag duality, it immediately follows that $\mathcal{B}(\theta, \varphi) = \overline{\bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_\Lambda)''}^{\|\cdot\|} = \overline{\mathcal{B}_0(\theta, \varphi)}^{\|\cdot\|}$. It also follows withough this stronger assumption, as seen in the next lemma.
Entirely following Lemma 2.4 of [6]:

LEMMA 49. *Assuming approximate twisted Haag duality, the unitary $U_{\Lambda,\varepsilon} \in \mathcal{B}(\theta, \varphi)$ (where $U_{\Lambda,\varepsilon}$ is from the definition of approximate twisted Haag duality, Definition 38) and the norm closure of $\mathcal{B}_0(\theta, \varphi)$ is $\mathcal{B}(\theta, \varphi)$.*

The proof is essentially the same as the proof of Lemma 2.4 of [6]:

PROOF. As $\mathcal{B}(\theta, \varphi)$ is the norm closure of $\bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$, to show that the norm closure of $\mathcal{B}_0(\theta, \varphi)$ is $\mathcal{B}(\theta, \varphi)$ it suffices to show that $\forall \Lambda \in \mathcal{C}(\theta, \varphi)$, $\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime} \subseteq \overline{\mathcal{B}_0(\theta, \varphi)}^{\|\cdot\|}$.
For all $\Lambda \in \mathcal{C}(\theta, \varphi)$, pick $\varepsilon > 0$ such that $\Lambda_{4\varepsilon} \in \mathcal{C}(\theta, \varphi)$. For all $t \geq 0$ and $\delta > 0$, $\tilde{U}_{\Lambda,\varepsilon,\delta}(t) \in \pi_0(\mathcal{A}_{(\Lambda-t)_{\varepsilon+\delta}})''$. For δ small enough that $\Lambda_{\varepsilon+\delta} \in \mathcal{C}(\theta, \varphi)$, we then have $\tilde{U}_{\Lambda,\varepsilon,\delta,t} \in \pi_0(\mathcal{A}_{(\Lambda-t)_{\varepsilon+\delta}})'' \subseteq \mathcal{B}_0(\theta, \varphi) \subseteq \mathcal{B}(\theta, \varphi)$, and that $\left\| \tilde{U}_{\Lambda,\varepsilon,\delta,t} - U_{\Lambda,\varepsilon} \right\| \leq f_{|\arg\Lambda|,\varepsilon,\delta}(t)$. Therefore, as $\tilde{U}_{\Lambda,\varepsilon,\delta,t} \to U_{\Lambda,\varepsilon}$ with respect to the norm topology, and $\tilde{U}_{\Lambda,\varepsilon,\delta,t} \in \mathcal{B}_0(\theta, \varphi)$, we have that $U_{\Lambda,\varepsilon}$ is in the norm closure of $\mathcal{B}_0(\theta, \varphi)$.

As $\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime} \subseteq \mathrm{Ad}(U_{\Lambda,\varepsilon})(\pi_0(\mathcal{A}_{(\Lambda-R_{\Lambda,\varepsilon})_\varepsilon})'')$, and $\pi_0(\mathcal{A}_{(\Lambda-R_{\Lambda,\varepsilon})_\varepsilon})'' \subseteq \mathcal{B}_0(\theta,\varphi) \subseteq \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|}$ and $U_{\Lambda,\varepsilon} \in \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|}$, and so $\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime} \subseteq \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|}$. So, we have that for all cones $\Lambda \in \mathcal{C}(\theta,\varphi)$ that $\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime} \subseteq \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|}$. So $\bigcup_{\Lambda\in\mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime} \subseteq \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|}$, so $\mathcal{B}(\theta,\varphi) \subseteq \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|}$. So $\overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|} = \mathcal{B}(\theta,\varphi)$. □

Following Lemma 2.5 of [6]:

LEMMA 50. *For any cone* Λ *and any unitary* $u_\Lambda \in \mathfrak{A}(\Lambda), \varepsilon, \delta > 0$ *with* $|\arg\lambda| + 4\varepsilon < 2\pi$ *and* $t \geq R_{|\arg\Lambda|,\varepsilon}$, *there is a unitary* $\tilde{u}_\Lambda \in \pi_0(\mathcal{A}_{(\Lambda-t)_{\varepsilon+\delta}})''$ *such that* $\|u_\Lambda - \tilde{u}_\Lambda\| \leq 2f_{|\arg\Lambda|,\varepsilon,\delta}(t)$, *namely,* $\tilde{u}_\Lambda := \mathrm{Ad}(\tilde{U}_{\Lambda,\varepsilon,\delta,t}U^*_{\Lambda,\varepsilon})(u_\Lambda)$. *In addition, if* u_Λ *is homogeneous,* $\tilde{u}_\Lambda$ *will have the same parity.*

PROOF. This follows directly from the definition of approximate twisted Haag duality, Definition 38. □

Following Definition 2.6 of [6], except that $R_{\varphi,\varepsilon}$ is for us the $R_{\varphi,\varepsilon}$ that appears in Definition 38 the definition of approximate twisted Haag duality rather than the one in the definition of approximate Haag duality. These definitions are as follows:

DEFINITION 51. We say that a set of cones $\mathcal{S}_1$ is distal from a set of cones $\mathcal{S}_2$ if there are cones $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ and $\varepsilon > 0$ such that $\bigcup\limits_{\Lambda_1\in\mathcal{S}_1} \Lambda_1 \subseteq \tilde{\Lambda}_1$, $\bigcup\limits_{\Lambda_2\in\mathcal{S}_2} \Lambda_2 \subseteq \tilde{\Lambda}_2$, and $(\tilde{\Lambda}_1 - R_{\tilde{\Lambda}_1,\varepsilon})_\varepsilon \subseteq \tilde{\Lambda}_2^c$.
We also say that a cone Λ_1 is distal from a cone Λ_2 if $\{\Lambda_1\}$ is distal from $\{\Lambda_2\}$. If Λ_1 is distal from Λ_2, then the two are disjoint. Let (θ,φ) label a forbidden direction. We of two sets of cones $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{C}(\theta,\varphi)$ that $\mathcal{S}_1$ is distal from $\mathcal{S}_2$ with forbidden direction (θ,φ) if $\mathcal{S}_1$ is distal from $\mathcal{S}_2$ and the cones $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ and $\varepsilon > 0$ above can be chosen such that $(\tilde{\Lambda}_1)_\varepsilon, (\tilde{\Lambda}_2)_\varepsilon \in \mathcal{C}(\theta,\varphi)$ and $\arg((\tilde{\Lambda}_1)_\varepsilon) \cap \arg((\tilde{\Lambda}_2)_\varepsilon) = \emptyset$.
Finally, we say that $\mathcal{S}_1 \perp_{(\theta,\varphi)} \mathcal{S}_2$ if $\mathcal{S}_1$ is distal from $\mathcal{S}_2$ and $\mathcal{S}_2$ is distal from $\mathcal{S}_1$ both with forbidden direction (θ,φ). And, we say that $\Lambda_1 \perp_{(\theta,\varphi)} \Lambda_2$ if $\{\Lambda_1\} \perp_{(\theta,\varphi)} \{\Lambda_2\}$.

Analogy of Lemma 2.8 of [6]:

LEMMA 52. *Let* Λ_1, Λ_2 *be cones such that* Λ_1 *is distal from* Λ_2. *For* $i = 1,2$, *let* $X^{i,t_i} \in \pi(\mathcal{A}_{(\Lambda_i+t_i)^c})^{t\prime}$ *for each* $t_i \geq 0$, *such that* $\sup_{t_i\geq 0}\left\|X^{i,t_i}\right\| < \infty$. *Then*

$$\lim_{t_1,t_2\to\infty}\left\|[X^{1,t_1}, X^{2,t_2}]_\pm\right\| = 0$$

Proof. Since Λ_1 is distal from Λ_2, there exist cones $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ and $\varepsilon > 0$ such that

$$\Lambda_1 \subset \tilde{\Lambda}_1 \subset (\tilde{\Lambda}_1 - R_{\tilde{\Lambda}_1,\varepsilon})_\varepsilon \subset \tilde{\Lambda}_2^c, \quad \Lambda_2 \subset \tilde{\Lambda}_2.$$

For $t_1, t_2 \geq 0$, we have

$$\Lambda_1 + t_1 \subseteq \Lambda_1 \subseteq \tilde{\Lambda}_1 \subseteq (\tilde{\Lambda}_1 - R_{\tilde{\Lambda}_1,\varepsilon})_\varepsilon \subseteq \tilde{\Lambda}_2^c \subseteq \Lambda_2^c \subseteq (\Lambda_2 + t_2)^c.$$

By approximate twisted Haag duality we have

$$\pi(\mathcal{A}_{\Lambda_1+t_1})^{t\prime} \subseteq \mathrm{Ad}(U_{\Lambda_1+t_1,\frac{\varepsilon}{2}})(\pi(\mathcal{A}_{(\Lambda+t_1-R_{\Lambda_1,\frac{\varepsilon}{2}})_\varepsilon})'')$$

and for $t_1 \geq R_{\Lambda_1,\frac{\varepsilon}{2}}$, we have $(\Lambda_1 + t_1 - R_{\Lambda_1,\frac{\varepsilon}{2}})_{\frac{\varepsilon}{2}} \subseteq (\Lambda_1)_{\frac{\varepsilon}{2}} \subseteq (\Lambda_2 + t_2)^c$, and so

$$\pi(\mathcal{A}_{\Lambda_1+t_1})^{t\prime} \subseteq \mathrm{Ad}(U_{\Lambda_1+t_1,\frac{\varepsilon}{2}})(\pi(\mathcal{A}_{(\Lambda_2+t_2)^c})'').$$

Now, as $X^{1,t_1} \in \pi(\mathcal{A}_{\Lambda_1+t_1})^{t\prime}$, we have that $\mathrm{Ad}(U^*_{\Lambda_1+t_1,\frac{\varepsilon}{2}})(X^{1,t_1}) \in \pi(\mathcal{A}_{(\Lambda_2+t_2)^c})''$.
Decomposing into even and odd parts, $\mathrm{Ad}(U_{\Lambda_1+t_1,\frac{\varepsilon}{2}})(X^{1,t_1}) = A_0 + A_1$, we have $A_0 + U_{\alpha_F} A_1 \in (\pi(\mathcal{A}_{(\Lambda_2+t_2)^c})'')^t$. Since $X^{2,t_2} \in \pi(\mathcal{A}_{(\Lambda_2+t_2)^c})^{t\prime} = ((\pi(\mathcal{A}_{(\Lambda_2+t_2)^c})'')^t)'$, we therefore have

$$[X^{2,t_2}, A_0 + U_{\alpha_F} A_1] = 0.$$

Let $\tilde{U}_{t_1} := \tilde{U}_{\Lambda_1+t_1,\frac{\varepsilon}{2},\frac{\varepsilon}{2},t_1/2} \in \pi(\mathcal{A}_{(\Lambda_1+t_1-t_1/2)_\varepsilon})'' \subseteq \pi(\mathcal{A}_{(\Lambda_2+t_2)^c})''$. It is even, so it is also in $(\pi(\mathcal{A}_{(\Lambda_2+t_2)^c})'')^t$, so:

$$[X^{2,t_2}, \tilde{U}_{t_1}] = 0.$$

Now write:

$$\begin{aligned}
[X^{2,t_2}, U_{\Lambda_1+t_1,\frac{\varepsilon}{2}}] &= [X^{2,t_2}, \tilde{U}_{t_1} + (U_{\Lambda_1+t_1,\frac{\varepsilon}{2}} - \tilde{U}_{t_1})] \\
&= [X^{2,t_2}, \tilde{U}_{t_1}] + [X^{2,t_2}, U_{\Lambda_1+t_1,\frac{\varepsilon}{2}} - \tilde{U}_{t_1}] \\
&= [X^{2,t_2}, U_{\Lambda_1+t_1,\frac{\varepsilon}{2}} - \tilde{U}_{t_1}].
\end{aligned}$$

So,

$$\left\| [X^{2,t_2}, U_{\Lambda_1+t_1,\frac{\varepsilon}{2}}] \right\| \leq 2 \left\| X^{2,t_2} \right\| \cdot \left\| U_{\Lambda_1+t_1,\frac{\varepsilon}{2}} - \tilde{U}_{t_1} \right\| \leq 2 \left\| X^{2,t_2} \right\| f_{|\arg \Lambda_1|,\frac{\varepsilon}{2},\frac{\varepsilon}{2}}(\frac{t_1}{2}).$$

Now, for $U := U_{\Lambda_1+t_1,\frac{\varepsilon}{2}}$, estimate the graded commutator:

$$\begin{aligned}
\left\|[X^{1,t_1}, X^{2,t_2}]_\pm\right\| &= \left\|\mathrm{Ad}(U^*)([X^{1,t_1}, X^{2,t_2}]_\pm)\right\| \\
&= \left\|[\mathrm{Ad}(U^*)(X^{1,t_1}), \mathrm{Ad}(U^*)(X^{2,t_2})]_\pm\right\| \\
&= \left\|[\mathrm{Ad}(U^*)(X^{1,t_1}), X^{2,t_2} + U^*[X^{2,t_2}, U]]_\pm\right\| \\
&\le \left\|[\mathrm{Ad}(U^*)(X^{1,t_1}), X^{2,t_2}]_\pm\right\| + \left\|[\mathrm{Ad}(U^*)(X^{1,t_1}), U^*[X^{2,t_2}, U]]_\pm\right\| \\
&\le \left\|[\mathrm{Ad}(U^*)(X^{1,t_1}), X^{2,t_2}]_\pm\right\| + 3\left\|X^{1,t_1}\right\| \cdot \left\|[X^{2,t_2}, U]\right\| \\
&\le \left\|[\mathrm{Ad}(U^*)(X^{1,t_1}), X^{2,t_2}]_\pm\right\| + 3\left\|X^{1,t_1}\right\| \cdot 2\left\|X^{2,t_2}\right\| f_{|\arg\Lambda_1|,\frac{\varepsilon}{2},\frac{\varepsilon}{2}}(\frac{t_1}{2}),
\end{aligned}$$

From $X^{1,t_1} \in \pi(\mathcal{A}_{(\Lambda_1+t_1)^c})^{t\prime}$, it follows that its even and odd parts, $(X^{1,t_1})_0, (X^{1,t_1})_1$ belong to $\pi(\mathcal{A}_{(\Lambda_1+t_1)^c})^{t\prime}$ as well. Therefore, applying the conclusion that

$$[X^{2,t_2}, (\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})(X^{1,t_1}))_0 + U_{\alpha_F} \cdot (\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})(X^{1,t_1}))_1] = 0$$

to each of $(X^{1,t_1})_0, (X^{1,t_1})_1$ individually, we get that $[X^{2,t_2}, \mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_0)] = 0$ and $[X^{2,t_2}, U_{\alpha_F} \cdot \mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_1)] = 0$. These are equivalent to

$$[(X^{2,t_2})_0, (\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_0))] + [(X^{2,t_2})_1, (\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_0))] = 0$$

and

$$U_{\alpha_F}[(X^{2,t_2})_0, (\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_1))] - U_{\alpha_F}\{(X^{2,t_2})_1, (\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_1))\} = 0,$$

which, by both of these sums having the form of an even expression plus an odd expression, and both being zero, is equivalent to $[(X^{2,t_2})_0, \mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_0)]$, $[(X^{2,t_2})_1, \mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_0)]$, $[(X^{2,t_2})_0, \mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_1)]$ and $\{(X^{2,t_2})_1, \mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})((X^{1,t_1})_1)\}$ each being 0, which implies

$$[\mathrm{Ad}(U_{(\Lambda_1+t_1),\frac{\varepsilon}{2}})(X^{1,t_1}), X^{2,t_2}]_\pm = 0$$

Therefore, $\left\|[X^{1,t_1}, X^{2,t_2}]_\pm\right\| \le 6\left\|X^{1,t_1}\right\| \left\|X^{2,t_2}\right\| f_{|\arg\Lambda_1|,\frac{\varepsilon}{2},\frac{\varepsilon}{2}}(\frac{t_1}{2})$,
and so $\lim_{t_1,t_2\to\infty} \left\|[X^{1,t_1}, X^{2,t_2}]_\pm\right\| = 0$.

□

Starting with the usual (counterclockwise) cyclic order on S^1, if we remove the interval associated with the forbidden direction, $(\theta-\varphi,\theta+\varphi)$ then the remaining subset of S^1 inherits a linear order from the cyclic order on S^1.

DEFINITION 53. Let (θ,φ) label a forbidden direction.

For $\Lambda_1,\Lambda_2\in\mathcal{C}(\theta,\varphi)$, define $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$ to mean that all directions for Λ_1 are counterclockwise, with respect to cutting the cyclic order on the space of directions at the forbidden direction (θ,φ) to get a linear order, from all directions for Λ_2. That is, for $i=1,2$, there are $\theta_i\in\mathbb{R},\varphi_i\in(0,\pi)$ such that $(\theta_i-\varphi_i,\theta_i+\varphi_i)$ is the range of angles for Λ_i and such that

$$\theta+\varphi<\theta_1-\varphi_1<\theta_1+\varphi_1<\theta_2-\varphi_2<\theta_2+\varphi_2<2\pi+\theta-\varphi.$$

DEFINITION 54. For a given forbidden direction (θ,φ) and a cone $\Lambda\in\mathcal{C}(\theta,\varphi)$, define

$$\kappa_{\Lambda,\theta,\varphi}:=\left\{K_\Lambda\in\mathcal{C}(\theta,\varphi)|\Lambda \underset{(\theta,\varphi)}{\curvearrowleft} K_\Lambda \text{ and } K_\Lambda \text{ distal from } \Lambda \text{ with forbidden direction } (\theta,\varphi)\right\}.$$

Following lemma 2.12 of [6]

PROPOSITION 55. *Let $\theta\in\mathbb{R},\varphi\in(0,\pi),\rho\in\mathcal{O}_0$, and $V_{\rho,\Lambda}\in\mathcal{V}_{\rho,\Lambda}$. Then, for all cones $\Lambda\in\mathcal{C}(\theta,\varphi)$*

(i) For all cones $K_\Lambda\in\kappa_{\Lambda,\theta,\varphi}$ and all $V_{\rho,K_\Lambda}\in\mathcal{V}_{\rho,K_\Lambda}$, there exists a cone $C_\Lambda\in\mathcal{C}(\theta,\varphi)$ such that $\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})\subseteq\pi_0(\mathcal{A}_{C^c_\Lambda})^{t\prime}\subseteq\mathcal{B}(\theta,\varphi)$, and for all $A\in\mathcal{A}_\Lambda$,

$$\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})\circ\pi_0^t(A)=\mathrm{Ad}(V_{\rho,\Lambda_0})\circ\rho^t(A)$$

(ii) For all cones $K_\Lambda,\tilde{K}_\Lambda\in\kappa_{\Lambda,\theta,\varphi}$ and all $V_{\rho,K_\Lambda}\in\mathcal{V}_{\rho,K_\Lambda}$ and $V_{\rho,\tilde{K}_\Lambda}\in\mathcal{V}_{\rho,\tilde{K}_\Lambda}$,
$\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''}=\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\tilde{K}_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''}$ and
$\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''t}=\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\tilde{K}_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''t}$.

PROOF. To prove part (i), no non-obvious changes need to be made to the proof for part (i) of Lemma 2.12 of [6].

For part (ii),

the reasoning which obtains a cone $\tilde{\Lambda}_1\in\mathcal{C}(\theta,\varphi)$ and $L_1,\tilde{L}_1\geq 0$ such that $\{K_\Lambda+L_1,\tilde{K}_\Lambda+\tilde{L}_1\}$ is distal from Λ with forbidden direction (θ,φ) applies with no changes.

By 47,

$V_{\rho,K_\Lambda+L_1}V^*_{\rho,K_\Lambda}\in\pi_0(\mathcal{A}_{K^c_\Lambda})^{t\prime}$

$V_{\rho,\tilde{K}_\Lambda+\tilde{L}_1}V^*_{\rho,K_\Lambda+L_1} \in \pi_0(\mathcal{A}_{\tilde{\Lambda}_1^c})^{t\prime}$
$V_{\rho,\tilde{K}_\Lambda}V^*_{\rho,\tilde{K}_\Lambda+\tilde{L}_1} \in \pi_0(\mathcal{A}_{\tilde{K}_\Lambda^c})^{t\prime}$.
As $\Lambda \subseteq K_\Lambda^c \cap (\tilde{\Lambda}_1)^c \cap (\tilde{K}_\Lambda)^c$, for $A_0 + U_{\alpha_F}A_1 \in \pi_0(\mathcal{A}_\Lambda)^t \subseteq \pi_0(\mathcal{A}_{K_\Lambda^c})^t \cap \pi_0(\mathcal{A}_{(\tilde{\Lambda}_1)^c})^t \cap \pi_0(\mathcal{A}_{(\tilde{K}_\Lambda)^c})^t$, $V_{\rho,K_\Lambda+L_1}V^*_{\rho,K_\Lambda}$, $V_{\rho,\tilde{K}_\Lambda+\tilde{L}_1}V^*_{\rho,K_\Lambda+L_1}$, and $V_{\rho,\tilde{K}_\Lambda}V^*_{\rho,\tilde{K}_\Lambda+\tilde{L}_1}$ each commute with it. So, as

$$V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*} = (V_{\rho,\tilde{K}_\Lambda}V^*_{\rho,\tilde{K}_\Lambda+\tilde{L}_1})(V_{\rho,\tilde{K}_\Lambda+\tilde{L}_1}V^*_{\rho,K_\Lambda+L_1})(V_{\rho,K_\Lambda+L_1}V^*_{\rho,K_\Lambda}),$$

therefore

$$\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(A_0 + U_{\alpha_F}A_1) = A_0 + U_{\alpha_F}A_1.$$

In particular, for even $A_0 \in (\pi_0(\mathcal{A}_\Lambda)^t)_{\text{even}} = (\pi_0(\mathcal{A}_\Lambda))_{\text{even}}$, we have $\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(A_0) = A_0$, and for odd $A_1 \in (\pi_0(\mathcal{A}_\Lambda)^t)_{\text{odd}}$ we have $\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(U_{\alpha_F}A_1) = U_{\alpha_F}A_1$.
As U_{α_F} commutes with all even operators, and $V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*}$ is even, we have

$$U_{\alpha_F}A_1 = \mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(U_{\alpha_F}A_1) = U_{\alpha_F}\,\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(A_1)$$

and so $\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(A_1) = A_1$ for odd $A_1 \in \pi_0(\mathcal{A}_\Lambda)$.
Therefore, for $A_0 + A_1 \in \pi_0(\mathcal{A}_\Lambda)$, $\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})(A_0 + A_1) = A_0 + A_1$. Therefore, by continuity, the same is true for $A_0 + A_1 \in \pi_0(\mathcal{A}_\Lambda)''$, so $\mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})|_{\pi_0(\mathcal{A}_\Lambda)''} = id|_{\pi_0(\mathcal{A}_\Lambda)''}$.
Therefore,

$$\begin{aligned}\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''} &= \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\tilde{K}_\Lambda}) \circ \mathrm{Ad}(V_{\rho,\tilde{K}_\Lambda}V_{\rho,K_\Lambda^*})|_{\pi_0(\mathcal{A}_\Lambda)''} \\ &= \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\tilde{K}_\Lambda}) \circ id|_{\pi_0(\mathcal{A}_\Lambda)''} \\ &= \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\tilde{K}_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''}.\end{aligned}$$

Similarly, $\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''t} = \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\tilde{K}_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''t}$. □

Following Definition 2.13 in [6]:

Definition 56. For (θ,φ) labeling a forbidden direction, and $\Lambda_0, \Lambda \in \mathcal{C}(\theta,\varphi)$, and $\rho \in \mathcal{O}_0$ and $V_{\rho,\Lambda_0} \in \mathcal{V}_{\rho,\Lambda_0}$, define $T^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}_{\rho,\Lambda} : \pi_0(\mathcal{A}_\Lambda)'' \to \mathcal{B}(\theta,\varphi)$ by
$T^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}_{\rho,\Lambda} := \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})|_{\pi_0(\mathcal{A}_\Lambda)''}$
(By 55, this homomorphism is independent of the choice of $K_\Lambda \in \kappa_{\Lambda,\theta,\varphi}$ and of $V_{\rho,K_\Lambda} \in \mathcal{V}_{\rho,K_\Lambda}$)

Following Lemma 2.14 in [6]:

Definition 57. Let (θ,φ) specify a forbidden direction. Let $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. Let $\rho \in \mathcal{O}_0$. Let $V_{\rho,\Lambda_0} \in \mathcal{V}_{\rho,\Lambda_0}$.
Define $T_{\rho,0}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} : \mathcal{B}_0(\theta,\varphi) \to \mathcal{B}(\theta,\varphi)$ as
$T_{\rho,0}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(x) := T_{\rho,\Lambda}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(x)$ for $x \in \pi_0(\mathcal{A}_\Lambda)''$ with $\Lambda \in \mathcal{C}(\theta,\varphi)$.
Define $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} : \mathcal{B}(\theta,\varphi) \to \mathcal{B}(\theta,\varphi)$ to be the unique (norm-continuous) linear extension of $T_{\rho,0}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$ to the domain of $\mathcal{B}(\theta,\varphi)$.

Proposition 58. $T_{\rho,0}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} : \mathcal{B}_0(\theta,\varphi) \to \mathcal{B}(\theta,\varphi)$ *as defined above is well-defined, and is a* $*$*-homomorphism, as is its extension* $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} : \mathcal{B}(\theta,\varphi) \to \mathcal{B}(\theta,\varphi)$.

The proof of this is no different than the proof of the corresponding part of lemma 2.14 of [6]. The following lemma corresponds to said 2.14 of [6]:

Lemma 59. *Let* (θ,φ) *specify a forbidden direction. Let* $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. *Let* $\rho \in \mathcal{O}_0$. *Let* $V_{\rho,\Lambda_0} \in \mathcal{V}_{\rho,\Lambda_0}$. $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} : \mathcal{B}(\theta,\varphi) \to \mathcal{B}(\theta,\varphi)$ *has the following properties:*

(i): *For all* $\Lambda \in \mathcal{C}(\theta,\varphi)$, $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$ *is ultraweak-continuous on* $\pi_0(\mathcal{A}_\Lambda)''$.
(ii): *For all* $A \in \mathcal{A}$, $\mathrm{tw} \circ T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) = \mathrm{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$.

In addition, it also satisfies

(a): *It is unique in the sense that if* $X_\rho : \mathcal{B}(\theta,\varphi) \to \mathcal{B}(\mathcal{H})$ *is a* $*$*-homomorphism which is, for all* $\Lambda \in \mathcal{C}(\theta,\varphi)$, *ultraweak-continuous on* $\pi_0(\mathcal{A}_\Lambda)''$, *and if for all* $A \in \mathcal{A}$, $\mathrm{tw} \circ X_\rho \circ \pi_0(A) = \mathrm{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$, *then* $X_\rho = T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$.
(b): *For all cones* Λ', *and all* $V_{\rho,\Lambda'} \in \mathcal{V}_{\rho,\Lambda'}$, $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0|_{\mathcal{A}_{(\Lambda')^c}} = \mathrm{Ad}(V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*) \circ \pi_0|_{\mathcal{A}_{(\Lambda')^c}}$. *And, for* $\tilde{\rho} = T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0$, $(\tilde{\rho}, U_{\tilde{\rho}} = U_{\alpha_F})$ *satisfies the superselection criterion wrt* π_0, *and for any* $V_{\rho,\Lambda'} \in \mathcal{V}_{\rho,\Lambda'}$, $V_{\rho,\Lambda'} V_{\rho,\Lambda_0}^* \in \mathcal{V}_{\tilde{\rho},\Lambda'}$.
(d): $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0|_{\mathcal{A}_{(\Lambda_0)^c}} = \pi_0|_{\mathcal{A}_{(\Lambda_0)^c}}$ *and* $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0 \in \mathcal{O}_{\Lambda_0}$.
(e): *For* $\Lambda \in \mathcal{C}(\theta,\varphi)$ *there exists* $C_\Lambda \in \mathcal{C}(\theta,\varphi)$ *such that* $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(\pi_0(\mathcal{A}_\Lambda)'') \subseteq \pi_0(\mathcal{A}_{C_\Lambda^c})^{t\prime} = \mathfrak{A}(C_\Lambda)$.

Proof. For (i), $T_{\rho}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}|_{\pi_0(\mathcal{A}_\Lambda)''} = T_{\rho,\Lambda}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} = \mathrm{Ad}(V_{\rho,\Lambda_0} V_{\rho,K_\Lambda}^*)|_{\pi_0(\mathcal{A}_\Lambda)''}$ which is ultraweak continuous.

To see that it satisfies (ii):

For any $A \in \mathcal{A}_{loc}$, there exists $\Lambda \in \mathcal{C}(\theta,\varphi)$ s.t. $A \in \mathcal{A}_\Lambda$, and so such that

$$\begin{aligned}\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) &= T_{\rho,\Lambda}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_0) + U_{\alpha_F} T_{\rho,\Lambda}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_1) \\ &= \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,K_\Lambda}^*)(\pi_0(A_0)) + U_{\alpha_F} \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,K_\Lambda}^*)(\pi_0(A_1)) \\ &= \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,K_\Lambda}^*)(\pi_0^t(A_0 + A_1)) \\ &= \operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)\end{aligned}$$

where the last equality is using $A \in \mathcal{A}_\Lambda$ and Lemma 55. So for all $A \in \mathcal{A}_{loc}$, $\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) = \operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$. For $A \in \mathcal{A}$ and $(A_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{A}_{loc}$ and $A_n \to A$ with respect to the norm on the C^* algebra, as $\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0$ and $\operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t$ are norm-continuous, $\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) = \lim_n \operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_n) = \lim_n \operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A_n) = \operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$, so for all $A \in \mathcal{A}$, $\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) = \operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$.

To prove (a), the only change needed compared to the proof of the analogous statement in [6] is to show that for $\Lambda \in \mathcal{C}(\theta,\varphi)$ that X_ρ and $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$ coincide on $\pi_0(\mathcal{A}_\Lambda)$. As for $A \in \mathcal{A}$, $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_0) + U_{\alpha_F} T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_1)$ and $X_\rho(A_0) + U_{\alpha_F} X_\rho(A_1)$ are both equal to $\operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$, they are equal. In particular, for even $A_0 \in \mathcal{A}$, $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_0) = X_\rho \circ \pi_0(A_0)$ and for odd $A_1 \in \mathcal{A}$, $U_{\alpha_F} T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_1) = U_{\alpha_F} X_\rho \circ \pi_0(A_1)$ and so $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_1) = X_\rho \circ \pi_0(A_1)$, and so $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_0 + A_1) = X_\rho \circ \pi_0(A_0 + A_1)$, so they coincide on $\pi_0(\mathcal{A})$, and so in particular on $\pi_0(\mathcal{A}_\Lambda)$ for all $\Lambda \in \mathcal{C}(\theta,\varphi)$. The remainder of the argument that they coincide on their domain $\mathcal{B}(\theta,\varphi)$ (i.e. are equal) is unchanged from the argument for this in Lemma 2.14 of [6].

To prove (b):

Let Λ' be a cone, not necessarily in $\mathcal{C}(\theta,\varphi)$. Let $A = A_0 + A_1 \in \mathcal{A}_{(\Lambda')^c}$. By (ii) $\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) = \operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A)$. By $\rho \in \mathcal{O}_0$ there exists $V_{\rho,\Lambda'}$ such that $\operatorname{Ad}(V_{\rho,\Lambda'}) \circ \rho^t|_{\mathcal{A}_{(\Lambda')^c}} = \pi_0^t|_{\mathcal{A}_{(\Lambda')^c}}$ and therefore such that $\rho^t|_{\mathcal{A}_{(\Lambda')^c}} = \operatorname{Ad}(V_{\rho,\Lambda'}^*) \circ \pi_0^t|_{\mathcal{A}_{(\Lambda')^c}}$. So, $\operatorname{Ad}(V_{\rho,\Lambda_0}) \circ \rho^t(A) = \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*) \circ \pi_0^t(A)$. So $\operatorname{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A) = \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*) \circ \pi_0^t(A)$. Because $V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*$ is even, it follows that $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_0) = \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*) \circ \pi_0(A_0)$ and $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0(A_1) = \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*) \circ \pi_0(A_1)$, and so $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0|_{\mathcal{A}_{(\Lambda')^c}} = \operatorname{Ad}(V_{\rho,\Lambda_0} V_{\rho,\Lambda'}^*) \circ \pi_0|_{\mathcal{A}_{(\Lambda')^c}}$. So, for $\tilde{\rho} = T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0$ and $U_{\tilde{\rho}} =$

U_{α_F}, we have that $\tilde{\rho}^t|_{\mathcal{A}_{(\Lambda')^c}} = \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\Lambda'}) \circ \pi_0^t|_{\mathcal{A}_{(\Lambda')^c}}$, and so $\mathrm{Ad}(V_{\rho,\Lambda'}V^*_{\rho,\Lambda_0}) \circ \tilde{\rho}^t|_{\mathcal{A}_{(\Lambda')^c}} = \pi_0^t|_{\mathcal{A}_{(\Lambda')^c}}$, i.e. $V_{\rho,\Lambda'}V^*_{\rho,\Lambda_0} \in \mathcal{V}_{\tilde{\rho},\Lambda'}$. This holds for all cones Λ', and so this shows that $\tilde{\rho} = T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0 \in \mathcal{O}_0$.
The proof for part (d) is no different from the proof in lemma 2.14 of [6] ((d) follows immediately from part (b) when choosing $\Lambda' = \Lambda_0$ and choosing V_{ρ,Λ_0} as the value for $V_{\rho,\Lambda'}$, which can be done due to choosing $\Lambda' = \Lambda_0$).
To prove (e): For $\Lambda \in \mathcal{C}(\theta,\varphi)$, by lemma 55, for $K_\Lambda \in \kappa_{\Lambda,\theta,\varphi}$ and V_{ρ,K_Λ} there exists a cone $C_\Lambda \in \mathcal{C}(\theta,\varphi)$ such that $\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}) \subseteq \pi_0(\mathcal{A}_{C_\Lambda^c})^{t\prime}$.
(I.e. such that $\mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})(\mathfrak{A}(\Lambda)) \subseteq \mathfrak{A}(C_\Lambda)$.)
And, by twisted locality, $\pi_0(\mathcal{A}_\Lambda)'' \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$, therefore

$$\begin{aligned}
T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(\pi_0(\mathcal{A}_\Lambda)'') &= T_{\rho,\Lambda}^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(\pi_0(\mathcal{A}_\Lambda)'') \\
&= \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})(\pi_0(\mathcal{A}_\Lambda)'') \\
&\subseteq \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,K_\Lambda})(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}) \\
&\subseteq \pi_0(\mathcal{A}_{C_\Lambda^c})^{t\prime} = \mathfrak{A}(C_\Lambda).
\end{aligned}$$

□

As an immediate corollary, we have the following, which is also analogous to part (ii) of Lemma 2.14 of [6]:

LEMMA 60. *If $U_\rho = \pm U_{\alpha_F}$,*

(ii) for all $A_0 + A_1 \in \mathcal{A}$,
$T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(\pi_0(A_0) + \pi_0(A_1)) = \mathrm{Ad}(V_{\rho,\Lambda_0})(\rho(A_0) \pm \rho(A_1))$
(So, if $U_\rho = U_{\alpha_F}$, then $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0 = \mathrm{Ad}(V_{\rho,\Lambda_0}) \circ \rho$ and if $U_\rho = -U_{\alpha_F}$ then $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0 = \mathrm{Ad}(V_{\rho,\Lambda_0}) \circ \rho \circ \alpha_F$.) $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} = \mathrm{Ad}(V_{\rho,\Lambda_0}V^*_{\rho,\Lambda}) \circ \pi_0|_{\mathcal{A}_{\Lambda^c}}$

Following Lemma 2.15 of [6]:

LEMMA 61. *(i) Let (θ,φ) label a forbidden direction, $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$, and $(\rho_1, U_{\rho_1}), (\rho_2, U_{\rho_2}) \in \mathcal{O}_0$ such that there is an even unitary $W \in \mathcal{U}(\mathcal{H})$ such that $\rho_2^t = \mathrm{Ad}(W) \circ \rho_1^t$. Then, for any $V_{\rho_i,\Lambda_0} \in \mathcal{V}_{\rho_i,\Lambda_0}$ for $i = 1,2$, $T_{\rho_1}^{(\theta,\varphi),\Lambda_0,V_{\rho_1,\Lambda_0}} = \mathrm{Ad}(V_{\rho_1,\Lambda_0}W^*V^*_{\rho_2,\Lambda_0}) \circ T_{\rho_2}^{(\theta,\varphi),\Lambda_0,V_{\rho_2,\Lambda_0}}$*

(ii) For (θ_i,φ_i) for $i=1,2$ labeling forbidden directions such that $(\theta_1-\varphi_1,\theta_1+\varphi_1)\subseteq(\theta_2-\varphi_2,\theta_2+\varphi_2)$, $\mathcal{B}(\theta_2,\varphi_2)\subseteq\mathcal{B}(\theta_1,\varphi_1)$ and $\mathcal{C}(\theta_2,\varphi_2)\subseteq\mathcal{C}(\theta_1,\varphi_1)$. For any $\Lambda_0\in\mathcal{C}(\theta_2,\varphi_2)$, any $\rho\in\mathcal{O}_0$, and any $V_{\rho,\Lambda_0}\in\mathcal{V}_{\rho,\Lambda_0}$, we have $T_\rho^{(\theta_1,\varphi_1),\Lambda_0,V_{\rho,\Lambda_0}}|_{\mathcal{B}(\theta_2,\varphi_2)}=T_\rho^{(\theta_2,\varphi_2),\Lambda_0,V_{\rho,\Lambda_0}}$.

(iii) Let (θ,φ) label a forbidden direction. Let $\Lambda_0,\Lambda_1\in\mathcal{C}(\theta,\varphi)$, $\rho\in\mathcal{O}_0$, $V_{\rho,\Lambda_0}\in\mathcal{V}_{\rho,\Lambda_0}$ and $V_{\rho,\Lambda_1}\in\mathcal{V}_{\rho,\Lambda_1}$. Then $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}=\mathrm{Ad}(V_{\rho,\Lambda_0}V_{\rho,\Lambda_1}^*)\circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}$.

Proof. For part (i), For any $\tilde\Lambda\in\mathcal{C}(\theta,\varphi)$ and any $V_{\rho_2,\tilde\Lambda}\in\mathcal{V}_{\rho_2,\tilde\Lambda}$,

$$\mathrm{Ad}(V_{\rho_2,\tilde\Lambda}W)\circ\rho_1^t|_{\mathcal{A}_{\tilde\Lambda^c}}=\mathrm{Ad}(V_{\rho_2,\tilde\Lambda})\circ\rho_2^t|_{\mathcal{A}_{\tilde\Lambda^c}}=\pi_0^t|_{\mathcal{A}_{\tilde\Lambda^c}}$$

and this, combined with the fact that $V_{\rho_2,\tilde\Lambda}W$ is even, means that $V_{\rho_2,\tilde\Lambda}W\in\mathcal{V}_{\rho_1,\tilde\Lambda}$. For $\Lambda\in\mathcal{C}(\theta,\varphi)$ apply that in the case of $\tilde\Lambda=K_\Lambda$ for some $K_\Lambda\in\kappa_{\Lambda,\theta,\varphi}$, By Definitions 56 and 57,

$$\begin{aligned}T_{\rho_1}^{(\theta,\varphi),\Lambda_0,V_{\rho_1,\Lambda_0}}|_{\pi_0(\mathcal{A}_\Lambda)''}&=\mathrm{Ad}(V_{\rho_1,\Lambda_0}(V_{\rho_2,K_\Lambda}W)^*)|_{\pi_0(\mathcal{A}_\Lambda)''}\\&=\mathrm{Ad}(V_{\rho_1,\Lambda_0}W^*V_{\rho_2,\Lambda_0}^*)\circ\mathrm{Ad}(V_{\rho_2,\Lambda_0}V_{\rho_2,K_\Lambda}^*)|_{\pi_0(\mathcal{A}_\Lambda)''}\\&=\mathrm{Ad}(V_{\rho_1,\Lambda_0}W^*V_{\rho_2,\Lambda_0}^*)\circ T_{\rho_2}^{(\theta,\varphi),\Lambda_0,V_{\rho_2,\Lambda_0}}|_{\pi_0(\mathcal{A}_\Lambda)''}.\end{aligned}$$

So, as $\mathrm{Ad}(V_{\rho_1,\Lambda_0}W^*V_{\rho_2,\Lambda_0}^*)\circ T_{\rho_2}^{(\theta,\varphi),\Lambda_0,V_{\rho_2,\Lambda_0}}$ is norm-continuous on $\mathcal{B}(\theta,\varphi)$ and ultraweak-continuous on $\pi_0(\mathcal{A}_\Lambda)''$ for each $\Lambda\in\mathcal{C}(\theta,\varphi)$, and as it coincides with $T_{\rho_1}^{(\theta,\varphi),\Lambda_0,V_{\rho_1,\Lambda_0}}$ on each $\pi_0(\mathcal{A}_\Lambda)''$ for $\Lambda\in\mathcal{C}(\theta,\varphi)$, it coincides on all of $\mathcal{B}_0(\theta,\varphi)$ and then by continuity on all of $\mathcal{B}(\theta,\varphi)$.
That is to say, $\mathrm{Ad}(V_{\rho_1,\Lambda_0}W^*V_{\rho_2,\Lambda_0}^*)\circ T_{\rho_2}^{(\theta,\varphi),\Lambda_0,V_{\rho_2,\Lambda_0}}=T_{\rho_1}^{(\theta,\varphi),\Lambda_0,V_{\rho_1,\Lambda_0}}$, as desired.
The proof for part (ii) is essentially unchanged from the proof of part (ii) of lemma 2.15 of [6]; one need only replace "$X_\rho\circ\pi_0=\mathrm{Ad}(\overline{V}_{\rho,\Lambda_0})\circ\rho$" with $\mathrm{tw}\circ X_\rho\circ\pi_0=\mathrm{Ad}(V_{\rho,\Lambda_0})\circ\rho^t$.
For part (iii), For $X_\rho:=\mathrm{Ad}(V_{\rho,\Lambda_0}V_{\rho,\Lambda_1}^*)\circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}:\mathcal{B}(\theta,\varphi)\to\mathcal{B}(\mathcal{H})$, a $*$-homomorphism which is ultraweak-continuous on $\pi_0(\mathcal{A}_\Lambda)''$ for all $\Lambda\in\mathcal{C}(\theta,\varphi)$. And,

$$\begin{aligned}\mathrm{Ad}(V_{\rho,\Lambda_0})\circ\rho^t&=\mathrm{Ad}(V_{\rho,\Lambda_0}V_{\rho,\Lambda_1}^*)\circ\mathrm{Ad}(V_{\rho,\Lambda_1})\circ\rho^t\\&=\mathrm{Ad}(V_{\rho,\Lambda_0}V_{\rho,\Lambda_1}^*)\circ\mathrm{tw}\circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}\circ\pi_0\\&=\mathrm{tw}\circ\mathrm{Ad}(V_{\rho,\Lambda_0}V_{\rho,\Lambda_1}^*)\circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}\circ\pi_0\\&=\mathrm{tw}\circ X_\rho\circ\pi_0.\end{aligned}$$

(The second equality is by Lemma 59(ii), and the third equality is because $\mathrm{tw}\circ\mathrm{Ad}(U)=\mathrm{Ad}(U)\circ\mathrm{tw}$ for even unitaries U.) Therefore, by Lemma 59(a), $X_\rho=T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$, as desired. □

The proof of Lemma 2.16 of [6] goes through essentially without modification (One need only switch out which lemmas are used to the corresponding ones here, as well as the definition of $\mathfrak{A}(\Lambda)$ to the one used here, etc.):

LEMMA 62. *Let (θ,φ) label a forbidden direction. Let $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. Then, for any $\varepsilon > 0$, $\Lambda \in \mathcal{C}(\theta,\varphi)$ such that $\Lambda_0 \subseteq \Lambda$, any $\rho \in \mathcal{O}_0$ and any $V_{\rho,\Lambda_0} \in \mathcal{V}_{\rho,\Lambda_0}$,*

$$T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}(\pi_0(\mathcal{A}_\Lambda)'') \subseteq \mathfrak{A}(\Lambda_\varepsilon) = \pi_0(\mathcal{A}_{(\Lambda_\varepsilon)^c})^{t\prime}.$$

Lemma 2.17 of [6] works in this setting when $\mathfrak{A}(\Lambda) := \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$ rather than $\pi_0(\mathcal{A}_{\Lambda^c})'$ as in [6], with no change to the proof other than referring to approximate twisted Haag duality rather than to approximate Haag duality:

LEMMA 63. *Let (θ,φ) label a forbidden direction. Then, for any $\Lambda_0, \Lambda \in \mathcal{C}(\theta,\varphi)$, and $\rho \in \mathcal{O}_0$, and any V_{ρ,Λ_0}, $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$ is ultraweak-continuous on $\mathfrak{A}(\Lambda) = \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$.*

Lemma 2.18 of [6] states:

LEMMA 64. *Let (θ,φ) label a forbidden direction. Let $\rho \in \mathcal{O}_0$, $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta,\varphi)$, $t \geq 0$, $\varepsilon, \delta > 0$ with $(\Lambda_1)_{\varepsilon+\delta}, (\Lambda_2)_{\varepsilon+\delta} \in \mathcal{C}(\theta,\varphi)$ and $|\arg \Lambda_2| + 4\varepsilon < 2\pi$. Let $V_{\rho,\Lambda_1} \in \mathcal{V}_{\rho,\Lambda_1}$. Recall from definition 38 what $U_{\Lambda_2,\varepsilon}$ and $f_{|\arg\Lambda_2|,\varepsilon,\delta}(t)$ refer to. Suppose $(\Lambda_2 - t)_{\varepsilon+\delta} \subseteq \Lambda_1^c$. Then we have*

$$\left\| T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(U_{\Lambda_2,\varepsilon})U_{\Lambda_2,\varepsilon}^* - 1 \right\| \leq 2 f_{|\arg\Lambda_2|,\varepsilon,\delta}(t)$$

The proof of this lemma given there works in this modified setting without modification.

REMARK 65. (π_0, U_{α_F}) is of course in $\mathcal{O}_0$, and for every cone Λ, $1 \in \mathcal{V}_{\pi_0,\Lambda}$. As such, for each Λ, if we choose 1 for V_{π_0,K_Λ} when constructing $T_{\rho,\Lambda}^{(\theta,\varphi),\Lambda_0,1}$, we can see that each is the identity, and so $T_\rho^{(\theta,\varphi),\Lambda_0,1} = \mathrm{id}_{\mathcal{B}(\theta,\varphi)}$.

4.3.2. Like section 3 (the composition).

DEFINITION 66. Let (θ,φ) label a forbidden direction. Let $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. Let $\{V_{\eta,\Lambda_0} \in \mathcal{V}_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0}$ be a choice of a $V_{\eta,\Lambda_0} \in \mathcal{V}_{\eta,\Lambda_0}$ for each $\eta \in \mathcal{O}_0$. Let $D = ((\theta,\varphi), \Lambda_0, \{V_{\eta,\Lambda_0} \in \mathcal{V}_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$.
For $\rho, \sigma \in \mathcal{O}_0$ define

$$\rho \circ_D \sigma := T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,V_{\sigma,\Lambda_0}} \circ \pi_0 : \mathcal{A} \to \mathcal{B}(\theta,\varphi).$$

For such a D, for each $\eta \in \mathcal{O}_0$ define $T_\eta^D := T_\eta^{(\theta,\varphi),\Lambda_0,V_{\eta,\Lambda_0}}$ where those parameters of (θ,φ), Λ_0 and V_{η,Λ_0} come from D. So, $\rho \circ_D \sigma = T_\rho^D \circ T_\sigma^D \circ \pi_0$.

The proof of Lemma 3.2 of [6] applies here to obtain the following:

LEMMA 67. *Let D, D' be as in Definition 66, and let $\rho, \sigma \in \mathcal{O}_0$. Then there exists an even unitary U such that $\rho \circ_D \sigma = \mathrm{Ad}(U) \circ (\rho \circ_{D'} \sigma)$.*

Closely following Lemma 3.3 of [6]:

LEMMA 68. *Let $D = ((\theta,\varphi),\Lambda_0,\{V_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$ as in Definition 66. Then, for any $\rho,\sigma \in \mathcal{O}_0$, for $\tau = \rho \circ_D \sigma$ and $U_\tau = U_{\alpha_F}$, $(\tau, U_\tau) \in \mathcal{O}_0$, and in particular, $(\tau, U_\tau = U_{\alpha_F}) \in \mathcal{O}_{\Lambda_0,*}$.*

PROOF. Let $D = ((\theta,\varphi),\Lambda_0,\{V_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$, and let $\rho,\sigma \in \mathcal{O}_0$. Let $\tau = \rho \circ_D \sigma$.
For each cone Λ:
Let $D' = ((\theta',\varphi'),\Lambda,\{V_{\eta,\Lambda}\}_{\eta\in\mathcal{O}_0})$ as in Definition 66 (choosing (θ',φ') such that $\Lambda \in \mathcal{C}(\theta',\varphi')$). By Lemma 67 there exists an even unitary $U_{\tau,\Lambda}$ such that $\mathrm{Ad}(U_{\tau,\Lambda}) \circ \tau = \mathrm{Ad}(U_{\tau,\Lambda}) \circ (\rho \circ_D \sigma) = \rho \circ_{D'} \sigma$. By Lemma 59 part (b), for all $A \in \mathcal{A}_{\Lambda^c}$,

$$
\begin{aligned}
\mathrm{Ad}(U_{\tau,\Lambda}) \circ \tau|_{\mathcal{A}_{\Lambda^c}} &= \mathrm{Ad}(U_{\tau,\Lambda}) \circ (\rho \circ_D \sigma)|_{\mathcal{A}_{\Lambda^c}} \\
&= \rho \circ_{D'} \sigma|_{\mathcal{A}_{\Lambda^c}} \\
&= T_\rho^{(\theta',\varphi'),\Lambda,V_{\rho,\Lambda}} \circ T_\sigma^{(\theta',\varphi'),\Lambda,V_{\sigma,\Lambda}} \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} \\
&= T_\rho^{(\theta',\varphi'),\Lambda,V_{\rho,\Lambda}} \circ \mathrm{Ad}(V_{\sigma,\Lambda}V_{\sigma,\Lambda}^*) \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} \\
&= \mathrm{Ad}(V_{\rho,\Lambda}V_{\rho,\Lambda}^*) \circ \mathrm{Ad}(1) \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} \\
&= \pi_0|_{\mathcal{A}_{\Lambda^c}}.
\end{aligned}
$$

For $U_\tau = U_{\alpha_F}$, $\tau^t = \mathrm{tw} \circ \tau$, and as $U_{\tau,\Lambda}$ is even, $\mathrm{Ad}(U_{\tau,\Lambda}) \circ \tau^t = \mathrm{Ad}(U_{\tau,\Lambda}) \circ \mathrm{tw} \circ \tau = \mathrm{tw} \circ \mathrm{Ad}(U_{\tau,\Lambda}) \circ \tau$, and therefore

$$
\begin{aligned}
\mathrm{Ad}(U_{\tau,\Lambda}) \circ \tau^t|_{\mathcal{A}_{\Lambda^c}} &= \mathrm{tw} \circ \mathrm{Ad}(U_{\tau,\Lambda}) \circ \tau|_{\mathcal{A}_{\Lambda^c}} \\
&= \mathrm{tw} \circ \pi_0|_{\mathcal{A}_{\Lambda^c}} = \pi_0^t|_{\mathcal{A}_{\Lambda^c}},
\end{aligned}
$$

so, $V_{\tau,\Lambda} = U_{\tau,\Lambda} \in \mathcal{V}_{\tau,\Lambda}$. So, as for all Λ there is an even unitary $V_{\tau,\Lambda} \in \mathcal{V}_{\tau,\Lambda}$, we have that $(\tau, U_\tau) = ((\rho \circ_D \sigma), U_{\alpha_F}) \in \mathcal{O}_0$.

In particular, for $\Lambda = \Lambda_0$, we can choose $D' = D$ which then gives us $\mathrm{Ad}(1) \circ \tau^t|_{\mathcal{A}_{\Lambda_0^c}} = \pi_0^t|_{\mathcal{A}_{\Lambda_0^c}}$ and so $(\tau, U_\tau) = ((\rho \circ_D \sigma), U_{\alpha_F}) \in \mathcal{O}_{\Lambda_0,*}$. □

Following Lemma 3.4 of [6]:

Lemma 69. *Let* $D = ((\theta, \varphi), \Lambda_0, \{V_{\eta,\Lambda_0}\}_{\eta \in \mathcal{O}_0})$ *as in Definition 66. Let* $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathcal{O}_0$. *Let* $W_\rho : \rho_1 \to \rho_1$ *and* $W_\sigma : \sigma_1 \to \sigma_2$ *be even unitaries (so,* $\mathrm{Ad}(W_\rho) \circ \rho_1^t = \rho_2^t$ *and* $\mathrm{Ad}(W_\sigma) \circ \sigma_1^t = \sigma_2^t$*). Then there is an even unitary* $U \in \mathcal{U}(\mathcal{H})$ *such that*

$$\rho_1 \circ_D \sigma_1 = \mathrm{Ad}(U) \circ (\rho_2 \circ_D \sigma_2).$$

Proof. By Lemma 61(i),

$$T^D_{\rho_1} = T^{(\theta,\varphi),\Lambda_0,V_{\rho_1,\Lambda_0}}_{\rho_1} = \mathrm{Ad}(V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0}) \circ T^{(\theta,\varphi),\Lambda_0,V_{\rho_2,\Lambda_0}}_{\rho_2} = \mathrm{Ad}(V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0}) \circ T^D_{\rho_2}$$

and

$$T^D_{\sigma_1} = T^{(\theta,\varphi),\Lambda_0,V_{\sigma_1,\Lambda_0}}_{\sigma_1} = \mathrm{Ad}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0}) \circ T^{(\theta,\varphi),\Lambda_0,V_{\sigma_2,\Lambda_0}}_{\sigma_2} = \mathrm{Ad}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0}) \circ T^D_{\sigma_2}.$$

As $W_\rho : \rho_1 \to \rho_2$ and $W_\sigma : \sigma_1 \to \sigma_2$, by Lemma 47, $W_\rho, W_\sigma \in \pi_0(\mathcal{A}_{(\Lambda_0 \cup \Lambda_0)^c})^{t\prime} = \mathfrak{A}(\Lambda_0) \subseteq \mathcal{B}(\theta, \varphi)$. So,

$$\begin{aligned}
\rho_1 \circ_D \sigma_1 &= T^D_{\rho_1} \circ T^D_{\sigma_1} \circ \pi_0 \\
&= (\mathrm{Ad}(V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0}) \circ T^D_{\rho_2}) \circ (\mathrm{Ad}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0}) \circ T^D_{\sigma_2}) \circ \pi_0 \\
&= \mathrm{Ad}(V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0} T^D_{\rho_2}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0})) \circ T^D_{\rho_2} \circ T^D_{\sigma_2} \circ \pi_0 \\
&= \mathrm{Ad}(V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0} T^D_{\rho_2}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0})) \circ (\rho_2 \circ_D \sigma_2).
\end{aligned}$$

So, there is an even unitary $U \in \mathcal{U}(\mathcal{H})$ (specifically $U = V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0} T^D_{\rho_2}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0})$) such that $\rho_1 \circ_D \sigma_1 = \mathrm{Ad}(U) \circ (\rho_2 \circ \sigma_2)$, as desired. □

Remark 70. When expressed in the notation for the monoidal product of intertwiners for the T endomorphisms, which will be introduced in the next section, this $V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0} T^D_{\rho_2}(V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0})$ is $(V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0}) \otimes (V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0})$ where $V_{\rho_1,\Lambda_0} W^*_\rho V^*_{\rho_2,\Lambda_0} : T^D_{\rho_2} \to T^D_{\rho_1}$ and $V_{\sigma_1,\Lambda_0} W^*_\sigma V^*_{\sigma_2,\Lambda_0} : T^D_{\sigma_2} \to T^D_{\sigma_1}$.

Lemma 3.5 of [6] applies in this setting as well when $\mathfrak{A}(\Lambda)$ is defined as we have defined it here, with the proof going through without modification other than citing Lemmas 59 and 63 as the analogies to lemmas 2.14 and 2.17 of [6]:

LEMMA 71. *Let (θ, φ) label a forbidden direction. For $i = 1, 2$, let $\Lambda_i \in \mathcal{C}(\theta, \varphi)$, $\rho_i \in \mathcal{O}_0$, and $V_{\rho_i,\Lambda_i} \in \mathcal{V}_{\rho_i,\Lambda_i}$. Then, $T_{\rho_1}^{(\theta,\varphi),\Lambda_1,V_{\rho_1,\Lambda_1}} \circ T_{\rho_2}^{(\theta,\varphi),\Lambda_2,V_{\rho_2,\Lambda_2}}$ is ultraweak-continuous on $\mathfrak{A}(\Lambda)$.*

Following Lemma 3.6 of [6]:

LEMMA 72. *Let $D = ((\theta, \varphi), \Lambda_0 \in \mathcal{C}(\theta, \varphi), \{V_{\eta,\Lambda_0} \in \mathcal{V}_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$ be as in Definition 66. Let $\rho, \sigma \in \mathcal{O}_0$.*
Set $(\gamma, U_\gamma) = (\rho \circ_D \sigma,\, U_{\alpha_F}) \in \mathcal{O}_0$. Then

$$T_\gamma^D = \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ T_\rho^D \circ T_\sigma^D,$$

and $1 \in \mathcal{V}_{\gamma,\Lambda_0}$ and $V_{\gamma,\Lambda_0} \in \mathfrak{A}(\Lambda_0) \subseteq \mathcal{B}(\theta, \varphi)$.

PROOF. This proof is essentially the same as the proof of Lemma 3.6 of [6].
For $X := \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ T_\rho^D \circ T_\sigma^D : \mathcal{B}(\theta, \varphi) \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. By Lemma 71, X is ultraweak-continuous on $\pi_0(\mathcal{A}_\Lambda)''$ for each $\Lambda \in \mathcal{C}(\theta, \varphi)$. Furthermore, $X \circ \pi_0 = (\mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ T_\rho^D \circ T_\sigma^D) \circ \pi_0 = \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ (T_\rho^D \circ T_\sigma^D \circ \pi_0) = \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ \gamma$. So $\mathrm{tw} \circ X \circ \pi_0 = \mathrm{tw} \circ \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ \gamma = \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ \mathrm{tw} \circ \gamma = \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ \gamma^t$ (as $U_\gamma = U_{\alpha_F}$). Therefore, by the uniqueness in Lemma 59(a), $X = T_\gamma^D$, so $T_\gamma^D = \mathrm{Ad}(V_{\gamma,\Lambda_0}) \circ T_\rho^D \circ T_\sigma^D$, as desired.
As shown in Lemma 68, $1 \in \mathcal{V}_{(\gamma,U_\gamma),\Lambda_0}$ and $(\gamma, U_\gamma) \in \mathcal{O}_{\Lambda_0,*}$. So, as $V_{\gamma,\Lambda_0}, 1 \in \mathcal{V}_{(\gamma,U_\gamma),\Lambda_0}$, by Lemma 47, $V_{\gamma,\Lambda_0} = V_{\gamma,\Lambda_0} \cdot 1^* \in \mathfrak{A}(\Lambda_0)$, as desired. □

Following Lemma 3.7 of [6], we have associativity up to an even unitary of this kind of composition. The proof is essentially unchanged from the one in [6].:

LEMMA 73. *Let $((\theta, \varphi), \Lambda_0 \in \mathcal{C}(\theta, \varphi), \{V_{\eta,\Lambda_0} \in \mathcal{V}_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$ as in Definition 66.*
Let $\rho, \sigma, \gamma \in \mathcal{O}_0$. Then, there exists an even unitary $U_{D,\rho,\sigma,\gamma} \in \mathcal{U}(\mathcal{H})$ such that

$$(\rho \circ_D \sigma) \circ_D \gamma = \mathrm{Ad}(U_{D,\rho,\sigma,\gamma}) \circ (\rho \circ_D (\sigma \circ_D \gamma)).$$

PROOF. Let $\tau_1 = \rho \circ_D \sigma$ and $\tau_2 = \sigma \circ_D \gamma$. By Lemma 72,

$$\begin{aligned}(\rho \circ_D \sigma) \circ_D \gamma = \tau_1 \circ_D \gamma = T^D_{\tau_1} \circ T^D_\gamma \circ \pi_0 \\ &= (\mathrm{Ad}(V_{\tau_1,\Lambda_0}) \circ T^D_\rho \circ T_\sigma) \circ T^D_\gamma \circ \pi_0 = \mathrm{Ad}(V_{\tau_1,\Lambda_0}) \circ T^D_\rho \circ (\mathrm{Ad}(V^*_{\tau_2,\Lambda_0}) \circ T^D_{\tau_2} \circ \pi_0) \\ &= \mathrm{Ad}(V_{\tau_1,\Lambda_0} T^D_\rho(V^*_{\tau_2,\Lambda_0})) \circ T^D_\rho \circ T^D_{\tau_2} \circ \pi_0 \\ &= \mathrm{Ad}(V_{\tau_1,\Lambda_0} T^D_\rho(V^*_{\tau_2,\Lambda_0})) \circ (\rho \circ_D \tau_2) = \mathrm{Ad}(V_{\tau_1,\Lambda_0} T^D_\rho(V^*_{\tau_2,\Lambda_0})) \circ (\rho \circ_D (\sigma \circ_D \gamma)).\end{aligned}$$

In the fifth equality, using that $V^*_{\tau_2,\Lambda_0} \in \mathcal{B}(\theta, \varphi)$ by Lemma 72. Note that $V_{\tau_1,\Lambda_0} T^D_\rho(V^*_{\tau_2,\Lambda_0})$ is even. □

4.3.3. Like section 4 (The Intertwiners).

DEFINITION 74. For grade-preserving endomorphisms T_1, T_2 of $\mathcal{B}(\theta, \varphi)$, define (T_1, T_2) to be

$$(T_1, T_2) := \{R \in \mathcal{B}(\mathcal{H}) | \forall x \in \mathcal{B}(\theta, \varphi), R \cdot \mathrm{tw} \circ T_1(x) = \mathrm{tw} \circ T_2(x) \cdot R\},$$

the intertwiners from T_1 to T_2. $R \in (T_1, T_2)$ will also be denoted $R : T_1 \to T_2$.

REMARK 75. The set of grade-preserving endomorphisms of $\mathcal{B}(\theta, \varphi)$, along with these sets of intertwiners as hom spaces, forms a monoidal supercategory.

LEMMA 76. *If $T_1, T_2 : \mathrm{End}(\mathcal{B}(\theta, \varphi))$ are grade-preserving endomorphisms , and if $R \in (T_1, T_2)$, then its even and odd parts R_0, R_1 are elements of (T_1, T_2) as well.*
In addition, for $R = R_0 + R_1 \in \mathcal{B}(\mathcal{H})$, $R \in (T_1, T_2)$ iff for all $x = x_0 + x_1 \in \mathcal{B}(\theta, \varphi)$, $R_0 \cdot T_1(x_0) = T_2(x_0) \cdot R_0$, $R_1 \cdot T_1(x_0) = T_2(x_0) \cdot R_1$, $R_0 \cdot T_1(x_1) = T_2(x_1) \cdot R_0$ and $R_1 \cdot T_1(x_1) = -T_2(x_1) \cdot R_1$.

PROOF. Let $R_0 + R_1 = R \in (T_1, T_2)$. For even $x_0 \in \mathcal{B}(\theta, \varphi)$ we have $R \cdot T_1(x_0) = R \cdot (\mathrm{tw} \circ T_1(x_0)) = (\mathrm{tw} \circ T_2(x_0)) \cdot R = T_2(x_0) \cdot R$ and so as the even and odd parts of the LHS are equal to the even and odd parts of the RHS respectively, $R_0 \cdot T_1(x_0) = T_2(x_0) \cdot R_0$ and $R_1 \cdot T_1(x_0) = T_2(x_0) \cdot R_1$. For odd $x_1 \in \mathcal{B}(\theta, \varphi)$ we have $R \cdot U_{\alpha_F} \cdot T_1(x_1) = R \cdot (\mathrm{tw} \circ T_1(x_1)) = (\mathrm{tw} \circ T_2(x_1)) \cdot R = U_{\alpha_F} T_2(x_1) \cdot R$. Again the even and odd components of the LHS are equal to the corresponding components of the RHS, so $R_0 \cdot U_{\alpha_F} \cdot T_1(x_1) = U_{\alpha_F} \cdot T_2(x_1) \cdot R_0$, and $R_1 \cdot U_{\alpha_F} \cdot T_1(x_1) = U_{\alpha_F} \cdot T_2(x_1) \cdot R_1$. Equivalently, $R_0 \cdot T_1(x_1) = T_2(x_1) \cdot R_0$ and $R_1 \cdot T_1(x_1) = -T_1(x_1) \cdot R_1$.
Therefore, for general $x = x_0 + x_1 \in \mathcal{B}(\theta, \varphi)$, $R_0 \cdot (\mathrm{tw} \circ T_1(x_0 + x_1)) = R_0 \cdot (T_1(x_0) + U_{\alpha_F} \cdot T_1(x_1)) = R_0 \cdot T_1(x_0) + R_0 \cdot U_{\alpha_F} T_2(x_1)$

$= T_2(x_0) \cdot R_0 + U_{\alpha_F} \cdot T_2(x_1) \cdot R_0 = (\mathrm{tw} \circ T_2(x_0 + x_1)) \cdot R_0$, and likewise $R_1 \cdot (\mathrm{tw} \circ T_1(x_0 + x_1)) = (\mathrm{tw} \circ T_2(x_0 + x_1)) \cdot R_1$.

We already have that for $R = R_0 + R_1 \in (T_1, T_2)$ that for all $x = x_0 + x_1 \in \mathcal{B}(\theta, \varphi)$, $R_0 \cdot T_1(x_0) = T_2(x_0) \cdot R_0$, $R_1 \cdot T_1(x_0) = T_2(x_0) \cdot R_1$, $R_0 \cdot T_1(x_1) = T_2(x_1) \cdot R_0$ and $R_1 \cdot T_1(x_1) = -T_1(x_1) \cdot R_1$.

Now for the reverse direction.

Let $R = R_0 + R_1 \in \mathcal{B}(\mathcal{H})$ be such that for all $x = x_0 + x_1 \in \mathcal{B}(\theta, \varphi)$, those four equalities hold. Then for all $x = x_0 + x_1 \in \mathcal{B}(\theta, \varphi)$,

$$\begin{aligned} R \cdot (\mathrm{tw} \circ T_1(x)) &= R_0 \cdot T_1(x_0) + R_0 \cdot U_{\alpha_F} \cdot T_1(x_1) + R_1 \cdot T_1(x_0) + R_1 \cdot U_{\alpha_F} \cdot T_1(x_1) \\ &= R_0 \cdot T_1(x_0) + U_{\alpha_F} \cdot R_0 \cdot T_1(x_1) + R_1 \cdot T_1(x_0) - U_{\alpha_F} \cdot R_1 \cdot T_1(x_1) \\ &= T_2(x_0) \cdot R_0 + U_{\alpha_F} \cdot T_2(x_1) \cdot R_0 + T_2(x_0) \cdot R_1 + U_{\alpha_F} \cdot T_2(x_1) \cdot R_1 \\ &= T_2(x_0) \cdot (R_0 + R_1) + U_{\alpha_F} \cdot T_2(x_1) \cdot (R_0 + R_1) = (\mathrm{tw} \circ T_2(x)) \cdot R. \end{aligned}$$

I.e. $R \in (T_1, T_2)$. □

REMARK 77. In particular, for $R \in \mathcal{B}(\mathcal{H})_{\mathrm{even}}$, we have $R \in (T_1, T_2)$ iff $\forall x \in \mathcal{B}(\theta, \varphi)$, $R \cdot T_1(x) = T_2(x) \cdot R$.

As an analogy to lemma 4.1 of [6]:

LEMMA 78. *Let (θ, φ) name a forbidden direction. Let $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta, \varphi)$, $\rho \in \mathcal{O}_0$, and $V_{\rho,\Lambda_1} \in \mathcal{V}_{\rho,\Lambda_1}, V_{\rho,\Lambda_2} \in \mathcal{V}_{\rho,\Lambda_2}$. Then $(V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \in (T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}})$.*

PROOF. By lemma 59 part (ii), $\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_i,V_{\rho,\Lambda_i}} \circ \pi_0 = \mathrm{Ad}(V_{\rho,\Lambda_i}) \circ \rho^t$ for $i = 1, 2$.

$\mathrm{Ad}(V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \circ \mathrm{Ad}(V_{\rho,\Lambda_1}) \circ \rho^t = \mathrm{Ad}(V_{\rho,\Lambda_2}) \circ \rho^t$.

For all $A \in \mathcal{A}$, $(V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \cdot (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ \pi_0(A)) \cdot (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1})^* = (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}} \circ \pi_0(A))$, so, for all $A \in \mathcal{A}$, $(V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \cdot (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ \pi_0(A)) = (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}} \circ \pi_0(A)) \cdot (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1})$.

As this holds for all $A \in \mathcal{A}$, in particular, for all $\Lambda \in \mathcal{C}(\theta, \varphi)$, it holds for all $A \in \mathcal{A}_\Lambda$. So, for $\Lambda \in \mathcal{C}(\theta, \varphi)$, we have

$$\forall x \in \pi_0(\mathcal{A}_\Lambda), (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \cdot (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(x)) = (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}}(x)) \cdot (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}).$$

As $T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}}$, tw are each ultraweak-continuous on $\pi_0(\mathcal{A}_\Lambda)''$ (for $\Lambda \in \mathcal{C}(\theta, \varphi)$, by Lemma 59 part (i)), we therefore have that this is also true for $x \in \pi_0(\mathcal{A}_\Lambda)''$.

(Specifically, take $x \mapsto (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \cdot (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(x)) - (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}}(x)) \cdot (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1})$. Because, with domain $\pi_0(\mathcal{A}_\Lambda)''$ it is a continuous function with respect to the ultraweak-topology, so its kernel is closed.) Therefore, it is true for $\mathcal{B}_0(\theta,\varphi) = \bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_\Lambda)''$.

Then, as these functions are also continuous with respect to the norm topology, we also get that it is true for all $x \in \overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|} = \mathcal{B}(\theta,\varphi)$. So,

$$\forall x \in \mathcal{B}(\theta,\varphi), (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \cdot (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(x)) = (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}}(x)) \cdot (V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}),$$

i.e. $(V_{\rho,\Lambda_2} V^*_{\rho,\Lambda_1}) \in (T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\rho^{(\theta,\varphi),\Lambda_2,V_{\rho,\Lambda_2}})$. □

As an analogy to lemma 4.2 of [6]:

LEMMA 79. *Let (θ,φ) name a forbidden direction. Let $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta,\varphi)$, $\rho, \sigma \in \mathcal{O}_0$, and $V_{\rho,\Lambda_1} \in \mathcal{V}_{\rho,\Lambda_1}, V_{\sigma,\Lambda_2} \in \mathcal{V}_{\sigma,\Lambda_2}$. Then $(T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}}) \subseteq \pi_0(\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c})^{t\prime} \subseteq \mathcal{B}(\theta,\varphi)$.*

PROOF. By Lemma 59 part (d), we have $T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ \pi_0|_{\mathcal{A}_{(\Lambda_1)^c}} = \pi_0|_{\mathcal{A}_{(\Lambda_1)^c}}$ and also that $T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \circ \pi_0|_{\mathcal{A}_{(\Lambda_2)^c}} = \pi_0|_{\mathcal{A}_{(\Lambda_2)^c}}$.
Let $R \in (T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}})$.
Let $A \in \mathcal{A}_{(\Lambda_1\cup\Lambda_1)^c} \subseteq \mathcal{A}_{\Lambda_1^c} \cap \mathcal{A}_{\Lambda_2^c}$. Then

$$\begin{aligned} R \cdot (\mathrm{tw} \circ \pi_0(A)) &= R \cdot (\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ \pi_0(A)) \\ &= (\mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \circ \pi_0(A)) \cdot R \\ &= (\mathrm{tw} \circ \pi_0(A)) \cdot R. \end{aligned}$$

(Where the first and last equalities are by the conclusion drawn from Lemma 59 part (d), and the middle inequality is by the definition of $R \in (T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}})$)
So, for $A \in \mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c}$ we have that R commutes with $\mathrm{tw} \circ \pi_0(A)$,
i.e. that $R \in \pi_0(\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c})^{t\prime}$.
So, $(T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}, T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}}) \subseteq \pi_0(\mathcal{A}_{(\Lambda_1\cup\Lambda_2)^c})^{t\prime} \subseteq \mathcal{B}(\theta,\varphi)$, as desired. □

LEMMA 80. *Let (θ,φ) specify a forbidden direction. Let $\Lambda_1, \Lambda_2, \Lambda_2' \in \mathcal{C}(\theta,\varphi)$. Let $\rho, \sigma, \sigma' \in \mathcal{O}_0$. Let $V_{\rho,\Lambda_1}, V_{\sigma,\Lambda_2}, V_{\sigma,\Lambda_2'}$ be from $\mathcal{V}_{\rho,\Lambda_1}, \mathcal{V}_{\sigma,\Lambda_2}, \mathcal{V}_{\sigma,\Lambda_2'}$ respectively.*
Let $S : T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$.
Then, $T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S) : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$

PROOF. First note that, by Lemma 79, $S \in \mathcal{B}(\theta, \varphi)$ and so $T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S)$ is a valid expression. As abbreviations within this proof, let T_ρ denote $T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}$, let T_σ denote $T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}}$ and let $T_{\sigma'}$ denote $T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$.

Now, first let $x_0 \in \mathcal{B}(\theta, \varphi)$ be even. Then

$$\begin{aligned}
T_\rho(S) \cdot (\mathrm{tw} \circ T_\rho \circ T_\sigma(x_0)) &= T_\rho(S) \cdot (T_\rho \circ T_\sigma(x_0)) \\
&= T_\rho(S \cdot T_\sigma(x_0)) \\
&= T_\rho(S \cdot (\mathrm{tw} \circ T_\sigma(x_0))) \\
&= T_\rho((\mathrm{tw} \circ T_{\sigma'}(x_0)) \cdot S) \\
&= (T_\rho \circ T_{\sigma'}(x_0)) \cdot T_\rho(S) \\
&= (\mathrm{tw} \circ T_\rho \circ T_{\sigma'}(x_0)) \cdot T_\rho(S).
\end{aligned}$$

Now, let $x_1 \in \mathcal{B}(\theta, \varphi)$ be odd. Then

$$\begin{aligned}
T_\rho(S) \cdot (\mathrm{tw} \circ T_\rho \circ T_\sigma(x_1)) &= T_\rho(S) \cdot (U_{\alpha_F} \cdot T_\rho \circ T_\sigma(x_1)) \\
&= -T_\rho(S) \cdot (T_\rho \circ T_\sigma(x_1)) \cdot U_{\alpha_F} \\
&= -T_\rho(S \cdot T_\sigma(x_1)) \cdot U_{\alpha_F} \\
&= - - T_\rho(S \cdot U_{\alpha_F} \cdot T_\sigma(x_1) \cdot U_{\alpha_F}) \cdot U_{\alpha_F} \\
&= T_\rho(S \cdot (\mathrm{tw} \circ T_\sigma(x_1)) \cdot U_{\alpha_F}) \cdot U_{\alpha_F} \\
&= T_\rho((\mathrm{tw} \circ T_{\sigma'}(x_1)) \cdot S \cdot U_{\alpha_F}) \cdot U_{\alpha_F} \\
&= T_\rho((U_{\alpha_F} \cdot T_{\sigma'}(x_1)) \cdot S \cdot U_{\alpha_F}) \cdot U_{\alpha_F} \\
&= -T_\rho(T_{\sigma'}(x_1) \cdot U_{\alpha_F} \cdot S \cdot U_{\alpha_F}) \cdot U_{\alpha_F} \\
&= -T_\rho \circ T_{\sigma'}(x_1) \cdot T_\rho(U_{\alpha_F} \cdot S \cdot U_{\alpha_F}) \cdot U_{\alpha_F} \\
&= (\mathrm{tw} \circ T_\rho \circ T_{\sigma'}(x_1)) \cdot \mathrm{Ad}(U_{\alpha_F})(T_\rho(\mathrm{Ad}(U_{\alpha_F})(S))) \\
&= (\mathrm{tw} \circ T_\rho \circ T_{\sigma'}(x_1)) \cdot T_\rho(S).
\end{aligned}$$

So, for an arbitrary $x_0 + x_1 \in \mathcal{B}(\theta, \varphi)$,

$$\begin{aligned} T_\rho(S) \cdot (\mathrm{tw} \circ T_\rho \circ T_\sigma(x_0 + x_1)) &= T_\rho(S) \cdot (\mathrm{tw} \circ T_\rho \circ T_\sigma(x_0)) + T_\rho(S) \cdot (\mathrm{tw} \circ T_\rho \circ T_\sigma(x_1)) \\ &= (\mathrm{tw} \circ T_\rho \circ T_{\sigma'}(x_0)) \cdot T_\rho(S) + (\mathrm{tw} \circ T_\rho \circ T_{\sigma'}(x_1)) \cdot T_\rho(S) \\ &= (\mathrm{tw} \circ T_\rho \circ T_\sigma(x_0 + x_1)) \cdot T_\rho(S). \end{aligned}$$

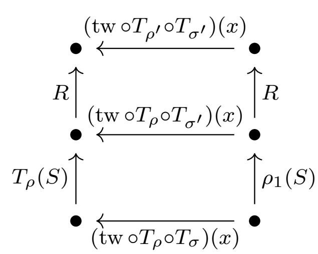
So $T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S) : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$, as desired. □

Definition 81. Let (θ, φ) specify a forbidden direction. Let $\Lambda_1, \Lambda_1', \Lambda_2, \Lambda_2' \in \mathcal{C}(\theta, \varphi)$.
Let $\rho, \rho', \sigma, \sigma' \in \mathcal{O}_0$. Let $V_{\rho,\Lambda_1}, V_{\rho',\Lambda_1'}, V_{\sigma,\Lambda_2}, V_{\sigma,\Lambda_2'}$ be from $\mathcal{V}_{\rho,\Lambda_1}, \mathcal{V}_{\rho',\Lambda_1'}, \mathcal{V}_{\sigma,\Lambda_2}, \mathcal{V}_{\sigma,\Lambda_2'}$ respectively.
Let $R : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \to T_{\rho'}^{(\theta,\varphi),\Lambda_1',V_{\rho',\Lambda_1'}}$ and $S : T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$.
Define $R \otimes S := R \cdot T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S)$

As the analogy to Lemma 4.3 of [6]:

Lemma 82. *Let (θ, φ) specify a forbidden direction. Let $\Lambda_1, \Lambda_1', \Lambda_2, \Lambda_2' \in \mathcal{C}(\theta, \varphi)$. Let $\rho, \rho', \sigma, \sigma' \in \mathcal{O}_0$. Let $V_{\rho,\Lambda_1}, V_{\rho',\Lambda_1'}, V_{\sigma,\Lambda_2}, V_{\sigma,\Lambda_2'}$ be from $\mathcal{V}_{\rho,\Lambda_1}, \mathcal{V}_{\rho',\Lambda_1'}, \mathcal{V}_{\sigma,\Lambda_2}, \mathcal{V}_{\sigma,\Lambda_2'}$ respectively.*
Let $R : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \to T_{\rho'}^{(\theta,\varphi),\Lambda_1',V_{\rho',\Lambda_1'}}$ and $S : T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$.
Then, $R \otimes S : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_{\rho'}^{(\theta,\varphi),\Lambda_1',V_{\rho',\Lambda_1'}} \circ T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$.

Proof. As abbreviations, let $T_\rho = T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}$, $T_{\rho'} = T_{\rho'}^{(\theta,\varphi),\Lambda_1',V_{\rho',\Lambda_1'}}$, $T_\sigma = T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}}$ and $T_{\sigma'} = T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$.
The lower square of the following diagram commutes for all $x \in \mathcal{B}(\theta, \varphi)$ by Lemma 80. The upper square commutes for all $x \in \mathcal{B}(\theta, \varphi)$ by the definition of $R : T_\rho \to T_{\rho'}$, applied in the case of $T_{\sigma'}(x) \in \mathcal{B}(\mathcal{H})$, so that $R \cdot (\mathrm{tw} \circ T_\rho(T_{\sigma'}(x)) = (\mathrm{tw} \circ T_{\rho'}(T_{\sigma'}(x)) \cdot R$. Therefore, for all $x \in \mathcal{B}(\theta, \varphi)$, the entire diagram commutes.



The statement that the rectangle comprised of those two squares commutes, is the statement that $(R \cdot T_\rho(S)) \cdot ((\mathrm{tw} \circ T_\rho \circ T_\sigma)(x)) = ((\mathrm{tw} \circ T_{\rho'} \circ T_{\sigma'})(x)) \cdot (R \cdot T_\rho(S))$, and this holding for all $x \in \mathcal{B}(\theta, \varphi)$

is what it is for $R \cdot T_\rho(S) : T_\rho \circ T_\sigma \to T_{\rho'} \circ T_{\sigma'}$.
So $R \otimes S : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \circ T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_{\rho'}^{(\theta,\varphi),\Lambda_1',V_{\rho',\Lambda_1'}} \circ T_{\sigma'}^{(\theta,\varphi),\Lambda_2',V_{\sigma',\Lambda_2'}}$, as desired. □

The analogy to lemma 4.5 of [6]:

LEMMA 83. *Let (θ, φ) label a forbidden direction. Let $\rho, \rho', \rho'', \sigma, \sigma', \sigma'' \in \mathcal{O}_0$. For $\tau \in \{\rho, \rho', \rho'', \sigma, \sigma', \sigma''\}$ let $\Lambda_\tau \in \mathcal{C}(\theta, \varphi)$ and $V_{\tau,\Lambda_\tau} \in \mathcal{V}_{\tau,\Lambda_\tau}$, and let $T_\tau = T_\tau^{(\theta,\varphi),\Lambda_\tau,V_{\tau,\Lambda_\tau}}$. Let $R : T_\rho \to T_{\rho'}$, $R' : T_{\rho'} \to T_{\rho''}$, $S : T_\sigma \to T_{\sigma'}$ and $S' : T_{\sigma'} \to T_{\sigma''}$.*
Then, $(R' \otimes S')(R \otimes S) = (R'R_0) \otimes (S_0'S) + (R'R_0) \otimes (S_1'S) + (R'R_1) \otimes (S_0'S) - (R'R_1) \otimes (S_1'S)$.
In other words, for S' and R are homogeneous, then

$$(R' \otimes S')(R \otimes S) = (-1)^{|S'||R|}(R'R) \otimes (S'S)$$

(where $|S'|$ and $|R|$ are the grades of S' and R respectively) and if they are not homogeneous, this applies component-wise.

PROOF. This can be seen by Lemma 76 and the definition of the monoidal product $\otimes$ on these intertwiners.
First, suppose R, S' are homogeneous.
By Lemma 76, As $R : T_\rho \to T_{\rho'}$, $T_{\rho'}(S') \cdot R = (-1)^{|S'||R|} R \cdot T_\rho(S')$. So,

$$\begin{aligned}
(R' \otimes S')(R \otimes S) &= (R'T_{\rho'}(S'))(RT_\rho(S)) \\
&= R' \cdot (T_{\rho'}(S) \cdot R) \cdot T_\rho(S) \\
&= (-1)^{|S'||R|} R' \cdot (R \cdot T_\rho(S')) \cdot T_\rho(S) \\
&= (-1)^{|S'||R|} (R'R) T_\rho(S'S) \\
&= (-1)^{|S'||R|} (RR') \otimes (S'S).
\end{aligned}$$

Now we address the case where $R = R_0 + R_1$, and $S' = S_0' + S_1'$ are not homogeneous. By Lemma 76, $R_0, R_1 : T_\rho \to T_{\rho'}$ and $S_0', S_1' : T_{\sigma'} \to T_{\sigma''}$.

$$\begin{aligned}
(R' \otimes S')(R \otimes S) &= (R' \otimes (S_0' + S_1'))((R_0 + R_1) \otimes S) \\
&= ((R' \otimes S_0') + (R' \otimes S_1'))((R_0 \otimes S) + (R_1 \otimes S)) \\
&= (R' \otimes S_0')(R_0 \otimes S) + (R' \otimes S_0')(R_1 \otimes S) + (R' \otimes S_1')(R_0 \otimes S) + (R' \otimes S_1')(R_1 \otimes S)
\end{aligned}$$

$$= (R'R_0) \otimes (S'_0 S) + (R'R_1) \otimes (S'_0 S) + (R'R_0) \otimes (S'_1 S) - (R'R_1) \otimes (S'_1 S),$$

where this is by $\otimes$ being bilinear along with applying the homogeneous case.

□

LEMMA 84. *As in Lemma 82, Let (θ, φ) specify a forbidden direction. Let $\Lambda_1, \Lambda'_1, \Lambda_2, \Lambda'_2 \in \mathcal{C}(\theta, \varphi)$. Let $\rho, \rho', \sigma, \sigma' \in \mathcal{O}_0$. Let $V_{\rho,\Lambda_1}, V_{\rho',\Lambda'_1}, V_{\sigma,\Lambda_2}, V_{\sigma,\Lambda'_2}$ be from $\mathcal{V}_{\rho,\Lambda_1}, \mathcal{V}_{\rho',\Lambda'_1}, \mathcal{V}_{\sigma,\Lambda_2}, \mathcal{V}_{\sigma,\Lambda'_2}$ respectively. Let $R : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \to T_{\rho'}^{(\theta,\varphi),\Lambda'_1,V_{\rho',\Lambda'_1}}$ and $S : T_\sigma^{(\theta,\varphi),\Lambda_2,V_{\sigma,\Lambda_2}} \to T_{\sigma'}^{(\theta,\varphi),\Lambda'_2,V_{\sigma',\Lambda'_2}}$.*
Then for R and S homogeneous,

$$(R \otimes S)^* = (-1)^{|R||S|} \ (R^*) \otimes (S^*),$$

and this extends bilinearly for the non-homogeneous case.

PROOF. First, we will show that this holds for the special case of $R \otimes \mathrm{id}_{\sigma'}$ and $\mathrm{id}_\rho \otimes S$ (where $\mathrm{id}_{\sigma'} = 1 : T_{\sigma'}^{(\theta,\varphi),\Lambda'_2,V_{\sigma',\Lambda'_2}} \to T_{\sigma'}^{(\theta,\varphi),\Lambda'_2,V_{\sigma',\Lambda'_2}}$ and $\mathrm{id}_\rho = 1 : T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}} \to T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}$), and then show that it therefore applies in general.
First,

$$\begin{aligned}(R \otimes \mathrm{id}_{\sigma'})^* &= (R \cdot T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(1))^* \\ &= R^* \\ &= R^* \cdot T_{\rho'}^{(\theta,\varphi),\Lambda'_1,V_{\rho',\Lambda'_1}}(1) \\ &= R^* \otimes 1_{\sigma'} = R^* \otimes (1^*_{\sigma'}).\end{aligned}$$

Second,

$$\begin{aligned}(\mathrm{id}_\rho \otimes S)^* &= (1 \cdot T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S))^* = (T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S))^* \\ &= T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S^*) \\ &= 1 \cdot T_\rho^{(\theta,\varphi),\Lambda_1,V_{\rho,\Lambda_1}}(S^*) \\ &= \mathrm{id}_\rho \otimes (S^*) = (\mathrm{id}^*_\rho) \otimes (S^*).\end{aligned}$$

By Lemma 83, $(R \otimes \mathrm{id}_{\sigma'}) \cdot (\mathrm{id}_\rho \otimes S) = (-1)^{|\mathrm{id}_{\sigma'}||\mathrm{id}_\rho|}(R \cdot \mathrm{id}_\rho) \otimes (\mathrm{id}_{\sigma'} \cdot S) = R \otimes S$.

Therefore,

$$\begin{aligned}(R \otimes S)^* &= ((R \otimes \mathrm{id}_{\sigma'}) \cdot (\mathrm{id}_\rho \otimes S))^* \\ &= (\mathrm{id}_\rho \otimes S)^* \cdot (R \otimes \mathrm{id}_{\sigma'})^* \\ &= (\mathrm{id}_\rho \otimes (S^*)) \cdot ((R^*) \otimes \mathrm{id}_{\sigma'}) \\ &= (-1)^{|S^*||R^*|}((\mathrm{id}_\rho \cdot R^*) \otimes (S^* \cdot \mathrm{id}_{\sigma'})) \\ &= (-1)^{|S||R|} \ (R^*) \otimes (S^*),\end{aligned}$$

(where the fourth equality is again by Lemma 83)

as desired.

The non-homogeneous case is covered by conjugate-linearity of $(-)^*$ and bilinearity of $(- \otimes -)$. □

DEFINITION 85. For $T_1, T_2, T_3, T_4 \in \mathrm{End}(\mathcal{B}(\theta, \varphi))$,

each of the form $T_\rho^{(\theta,\varphi),\Lambda,V_{\rho,\Lambda}}$ for some $\rho \in \mathcal{O}_0$ some $\Lambda \in \mathcal{C}(\theta, \varphi)$ and some $V_{\rho,\Lambda} \in \mathcal{V}_{\rho,\Lambda}$, each for some common choice of forbidden direction (θ, φ) for all four of them,

and $A : T_1 \to T_2$ and $B : T_3 \to T_4$ where $A, B \in \mathcal{B}(\theta, \varphi)$, define

$$[A, B]_{\otimes,\pm} := A \otimes B - (-1)^{|A|\cdot|B|} B \otimes A$$

when A, B are homogeneous, and componentwise if they are not.

(So, $[A_0 + A_1, B_0 + B_1]_{\otimes,\pm} = [A_0, B_0]_{\otimes,\pm} + [A_0, B_1]_{\otimes,\pm} + [A_1, B_0]_{\otimes,\pm} + [A_1, B_1]_{\otimes,\pm}$.)

The analogy to lemma 4.6 of [6]:

LEMMA 86. *Let (θ, φ) label a forbidden direction. Let $\rho, \rho', \sigma, \sigma' \in \mathcal{O}_0$ (not necessarily distinct). For each label $\tau \in \{\rho, \rho', \sigma, \sigma'\}$ let $\Lambda_\tau \in \mathcal{C}(\theta, \varphi)$ and for $t_\tau \geq 0$, $V_{\tau,\Lambda_\tau+t_\tau} \in \mathcal{V}_{\tau,\Lambda_\tau+t_\tau}$. Also set $T_\tau^{t_\tau} = T_\tau^{(\theta,\varphi),\Lambda_\tau+t_\tau,V_{\tau,\Lambda_\tau+t_\tau}}$.*

The labels $\rho, \rho', \sigma, \sigma'$ are used to index associated data (e.g., cones and operators), and these data may differ even when the actual representations are equal.

Suppose that $\{\Lambda_\rho, \Lambda_{\rho'}\} \perp_{(\theta,\varphi)} \{\Lambda_\sigma, \Lambda_{\sigma'}\}$. For $t_\rho, t_{\rho'} \geq 0$, let $R^{t_\rho,t_{\rho'}} \in (T_\rho^{t_\rho}, T_{\rho'}^{t_{\rho'}})$ with $\left\|R^{t_\rho,t_{\rho'}}\right\| \leq 1$ and $R^{t_\rho,t_{\rho'}}$ homogeneous, and for $t_\sigma, t_{\sigma'} \geq 0$, let $S^{t_\sigma,t_{\sigma'}} \in (T_\sigma^{t_\sigma}, T_{\sigma'}^{t_{\sigma'}})$ with $\left\|S^{t_\sigma,t_{\sigma'}}\right\| \leq 1$ and $S^{t_\sigma,t_{\sigma'}}$

homogeneous. Then

$$\lim_{t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty}\left\|R^{t_\rho,t_{\rho'}}\otimes S^{t_\sigma,t_{\sigma'}}-(-1)^{|R^{t_\rho,t_{\rho'}}|\cdot|S^{t_\sigma,t_{\sigma'}}|}S^{t_\sigma,t_{\sigma'}}\otimes R^{t_\rho,t_{\rho'}}\right\|=0,$$

i.e.

$$\lim_{t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty}\left\|[R^{t_\rho,t_{\rho'}},S^{t_\sigma,t_{\sigma'}}]_{\otimes,\pm}\right\|=0.$$

Proof. This proof is following the proof in [6].

First it will demonstrate that $R^{t_\rho,t_{\rho'}}\otimes S^{t_\sigma,t_{\sigma'}}-R^{t_\rho,t_{\rho'}}\cdot S^{t_\sigma,t_{\sigma'}}$ and $S^{t_\sigma,t_{\sigma'}}\otimes R^{t_\rho,t_{\rho'}}-S^{t_\sigma,t_{\sigma'}}\cdot R^{t_\rho,t_{\rho'}}$ both go to 0 as $t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty$ by showing that $T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}})-S^{t_\sigma,t_{\sigma'}}$ and $T_\sigma^{t_\sigma}(R^{t_\rho,t_{\rho'}})-R^{t_\rho,t_{\rho'}}$ both go to 0 as $t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty$. This first part is not much different from the proof of Lemma 4.6 of [6]. The last bit however changes a little due to how the conclusion of Lemma 52 differs from Lemma 2.8 of [6].

By the definition of $\{\Lambda_\rho,\Lambda_{\rho'}\}\perp_{(\theta,\varphi)}\{\Lambda_\sigma,\Lambda_{\sigma'}\}$ there exists $\varepsilon>0$ and $\tilde\Lambda_\rho,\tilde\Lambda_\sigma\in\mathcal{C}(\theta,\varphi)$ such that for $\tau,\upsilon\in\{\rho,\sigma\}$ with $\upsilon\neq\tau$, $(\tilde\Lambda_\tau-R_{\Lambda_\tau,\varepsilon})_\varepsilon\subseteq(\tilde\Lambda_\upsilon)^c$, $(\tilde\Lambda_\tau)_\varepsilon\in\mathcal{C}(\theta,\varphi)$, and $\arg((\tilde\Lambda_\rho)_\varepsilon)\cap\arg((\tilde\Lambda_\sigma)_\varepsilon)=\emptyset$. Choose $\delta>0$ small enough that $(\tilde\Lambda_\rho)_{\varepsilon+\delta},(\tilde\Lambda_\sigma)_{\varepsilon+\delta}\in\mathcal{C}(\theta,\varphi)$ and $\arg((\tilde\Lambda_\rho)_{\varepsilon+\delta})\cap\arg((\tilde\Lambda_\sigma)_{\varepsilon+\delta})=\emptyset$

For any $t\geq 0$, by Lemma A.2 of [6], for sufficiently large (depending on t) $(t_\tau)_{\tau\in\{\rho,\rho',\sigma,\sigma'\}}$, so that for $\tau,\upsilon\in\{\rho,\sigma\}$ with $\upsilon\neq\tau$, $\Lambda_\tau+t_\tau,\Lambda_{\tau'}+t_{\tau'}\subseteq\tilde\Lambda_\tau+t$ and $(\tilde\Lambda_\tau+\frac{t}{2})_{\varepsilon+\delta}\subseteq(\Lambda_\upsilon+t_\upsilon)^c$.

By Lemma 79, we have $S^{t_\sigma,t_{\sigma'}}\in\pi_0(\mathcal{A}_{((\Lambda_\sigma+t_\sigma)\cup(\Lambda_{\sigma'}+t_{\sigma'}))^c})^{t\prime}\subseteq\pi_0(\mathcal{A}_{(\tilde\Lambda_\sigma+t)^c})^{t\prime}$, and likewise that $R^{t_\rho,t_{\rho'}}\in\pi_0(\mathcal{A}_{((\Lambda_\rho+t_\rho)\cup(\Lambda_{\rho'}+t_{\rho'}))^c})^{t\prime}\subseteq\pi_0(\mathcal{A}_{(\tilde\Lambda_\rho+t)^c})^{t\prime}$. Applying approximate twisted Haag duality to this, $\mathrm{Ad}(U^*_{(\tilde\Lambda_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})\subseteq\pi_0(\mathcal{A}_{(\tilde\Lambda_\sigma+t-R_{\tilde\Lambda_\sigma,\varepsilon})_\varepsilon})''$ and $\mathrm{Ad}(U^*_{(\tilde\Lambda_\rho+t),\varepsilon})(R^{t_\rho,t_{\rho'}})\subseteq\pi_0(\mathcal{A}_{(\tilde\Lambda_\rho+t-R_{\tilde\Lambda_\rho,\varepsilon})_\varepsilon})''$. As

$$(\tilde\Lambda_\sigma+(t-R_{\tilde\Lambda_\sigma,\varepsilon}))_\varepsilon\subseteq(\tilde\Lambda_\sigma-R_{\tilde\Lambda_\sigma,\varepsilon})_\varepsilon\subseteq(\tilde\Lambda_\rho)^c\subseteq(\tilde\Lambda_\rho+t)^c\subseteq(\Lambda_\rho+t_\rho)^c$$

and

$$(\tilde\Lambda_\rho+(t-R_{\tilde\Lambda_\rho,\varepsilon}))_\varepsilon\subseteq(\tilde\Lambda_\rho-R_{\tilde\Lambda_\rho,\varepsilon})_\varepsilon\subseteq(\tilde\Lambda_\sigma)^c\subseteq(\tilde\Lambda_\sigma+t)^c\subseteq(\Lambda_\sigma+t_\rho)^c$$

therefore we have $\mathrm{Ad}(U^*_{(\tilde\Lambda_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})\in\pi_0(\mathcal{A}_{(\tilde\Lambda_\sigma+(t-R_{\tilde\Lambda_\sigma,\varepsilon}))_\varepsilon})''\subseteq\pi_0(\mathcal{A}_{(\Lambda_\rho+t_\rho)^c})''$ and similarly $\mathrm{Ad}(U^*_{(\tilde\Lambda_\rho+t),\varepsilon})(R^{t_\rho,t_{\rho'}})\in\pi_0(\mathcal{A}_{(\tilde\Lambda_\rho+t-R_{\tilde\Lambda_\rho,\varepsilon})_\varepsilon})''\subseteq\pi_0(\mathcal{A}_{(\Lambda_\sigma+t_\sigma)^c})''$. Now,

$$\begin{aligned}T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}})&=T_\rho^{t_\rho}(\mathrm{Ad}(U_{(\tilde\Lambda_\sigma+t),\varepsilon})(\mathrm{Ad}(U^*_{(\tilde\Lambda_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})))\\&=\mathrm{Ad}(T_\rho^{t_\rho}(U_{(\tilde\Lambda_\sigma+t),\varepsilon}))\circ T_\rho^{t_\rho}(\mathrm{Ad}(U^*_{(\tilde\Lambda_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})).\end{aligned}$$

Recall Lemma 59 parts (i) and (d). By part (d), $T_\rho^{t_\rho}|_{\pi_0(\mathcal{A}_{(\Lambda_\rho+t_\rho)^c})} = id|_{\pi_0(\mathcal{A}_{(\Lambda_\rho+t_\rho)^c})}$. Because $(\tilde{\Lambda}_\sigma + (t - R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon \subseteq (\Lambda_\rho + t_\rho)^c$, further restricting to $\pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon}) \subseteq \pi_0(\mathcal{A}_{(\Lambda_\rho+t_\rho)^c})$ we have that $T_\rho^{t_\rho}|_{\pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon})} = id|_{\pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon})}$. As $(\tilde{\Lambda}_\sigma + (t - R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon \in \mathcal{C}(\theta,\varphi)$ we have (by part (i) of Lemma 59) that $T_\rho^{t_\rho}$ is ultraweak-continuous on $\pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon})''$, we have that $T_\rho^{t_\rho}|_{\pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon})''} = id|_{\pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon})''}$.
Therefore, as $\mathrm{Ad}(U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}}) \in \pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+(t-R_{\tilde{\Lambda}_\sigma,\varepsilon}))_\varepsilon})''$ we have that

$$T_\rho^{t_\rho}(\mathrm{Ad}(U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})) = \mathrm{Ad}(U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}}).$$

So, from $T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}}) = \mathrm{Ad}(T_\rho^{t_\rho}(U_{(\tilde{\Lambda}_\sigma+t),\varepsilon}))(T_\rho^{t_\rho}(\mathrm{Ad}(U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})))$ we get
$T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}}) = \mathrm{Ad}(T_\rho^{t_\rho}(U_{(\tilde{\Lambda}_\sigma+t),\varepsilon})U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}})$.
Similarly, $T_\sigma^{t_\sigma}(R^{t_\rho,t_{\rho'}}) = \mathrm{Ad}(T_\sigma^{t_\sigma}(U_{(\tilde{\Lambda}_\rho+t),\varepsilon})U^*_{(\tilde{\Lambda}_\rho+t),\varepsilon})(R^{t_\rho,t_{\rho'}})$.
Applying [6, Lemma 2.18]/ Lemma 64 with $\Lambda_\rho + t_\rho$, $\tilde{\Lambda}_\sigma + t$, $\frac{t}{2}$ as the respective values for Λ_1, Λ_2, t from that lemma, we get $\left\|T_\rho^{t_\rho}(U_{\tilde{\Lambda}_\sigma+t,\varepsilon})U^*_{\tilde{\Lambda}_\sigma+t,\varepsilon} - 1\right\| \le 2f_{|\arg\tilde{\Lambda}_\sigma|,\varepsilon,\delta}(\frac{t}{2})$. To apply this lemma we use that $((\tilde{\Lambda}_\sigma + t) - \frac{t}{2})_{\varepsilon+\delta} \subseteq (\Lambda_\rho + t_\rho)^c$ as well as that $(\Lambda_\rho + t_\rho)_{\varepsilon+\delta}, (\tilde{\Lambda}_\sigma + t)_{\varepsilon+\delta} \in \mathcal{C}(\theta,\varphi)$. This is for all $t \ge 0$ and $(t_\tau)_{\tau\in\{\rho,\rho',\sigma,\sigma'\}}$ large enough to satisfy the previously stated conditions, depending on t.
For any $\varepsilon' > 0$, choose $t > 0$ such that $4f_{|\arg\tilde{\Lambda}_\sigma|,\varepsilon,\delta}(\frac{t}{2}) < \varepsilon'$. Then, for this t and corresponding large enough $(t_\tau)_{\tau\in\{\rho,\rho',\sigma,\sigma'\}}$ we have

$$\begin{aligned}
\left\|T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}}) - S^{t_\sigma,t_{\sigma'}}\right\| &= \left\|\mathrm{Ad}(T_\rho^{t_\rho}(U_{(\tilde{\Lambda}_\sigma+t),\varepsilon})U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon})(S^{t_\sigma,t_{\sigma'}}) - S^{t_\sigma,t_{\sigma'}}\right\| \\
&= \left\|[T_\rho^{t_\rho}(U_{(\tilde{\Lambda}_\sigma+t),\varepsilon})U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon}, S^{t_\sigma,t_{\sigma'}}]\right\| \\
&\le 2\left\|S^{t_\sigma,t_{\sigma'}}\right\|\left\|T_\rho^{t_\rho}(U_{(\tilde{\Lambda}_\sigma+t),\varepsilon})U^*_{(\tilde{\Lambda}_\sigma+t),\varepsilon} - 1\right\| \\
&\le 2\cdot 1\cdot 2f_{|\arg\tilde{\Lambda}_\sigma|,\varepsilon,\delta}(\frac{t}{2}) < \varepsilon'.
\end{aligned}$$

Therefore,

$$\lim_{t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty}\left\|T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}}) - S^{t_\sigma,t_{\sigma'}}\right\| = 0.$$

Similarly,

$$\lim_{t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty}\left\|T_\sigma^{t_\sigma}(R^{t_\rho,t_{\rho'}}) - R^{t_\rho,t_{\rho'}}\right\| = 0.$$

Now the proof slightly diverges from the corresponding proof in [6].

Let $s = |R^{t_\rho,t_{\rho'}}||S^{t_\sigma,t_{\sigma'}}|$ where $|R^{t_\rho,t_{\rho'}}|, |S^{t_\sigma,t_{\sigma'}}|$ are the parities of $R^{t_\rho,t_{\rho'}}, S^{t_\sigma,t_{\sigma'}}$ respectively, i.e. $s = 1$ if both are odd and $s = 0$ otherwise.

$$\left\|R^{t_\rho,t_{\rho'}} \otimes -(-1)^s S^{t_\sigma,t_{\sigma'}} \otimes R^{t_\rho,t_{\rho'}}\right\| = \left\|R^{t_\rho,t_{\rho'}} \cdot T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}}) - (-1)^s S^{t_\sigma,t_{\sigma'}} \cdot T_\sigma^{t_\sigma}(R^{t_\rho,t_{\rho'}})\right\|$$
$$\leq \left\|R^{t_\rho,t_{\rho'}} \cdot S^{t_\sigma,t_{\sigma'}} - (-1)^s S^{t_\sigma,t_{\sigma'}} \cdot R^{t_\rho,t_{\rho'}}\right\| + \left\|T_\rho^{t_\rho}(S^{t_\sigma,t_{\sigma'}}) - S^{t_\sigma,t_{\sigma'}}\right\| + \left\|T_\sigma^{t_\sigma}(R^{t_\rho,t_{\rho'}}) - R^{t_\rho,t_{\rho'}}\right\|,$$

and the last two terms both go to 0 in the limit. So,

$$\left\|R^{t_\rho,t_{\rho'}} \otimes S^{t_\sigma,t_{\sigma'}} - (-1)^s S^{t_\sigma,t_{\sigma'}} \otimes R^{t_\rho,t_{\rho'}}\right\| - \left\|[R^{t_\rho,t_{\rho'}}, S^{t_\sigma,t_{\sigma'}}]_\pm\right\|$$

goes to 0. We have that $S^{t_\sigma,t_{\sigma'}} \in \pi_0(\mathcal{A}_{(\tilde{\Lambda}_\sigma+t)^c})^{t\prime}$, and that $R^{t_\rho,t_{\rho'}} \in \pi_0(\mathcal{A}_{(\tilde{\Lambda}_\rho+t)^c})^{t\prime}$. and that $\tilde{\Lambda}_\rho$ is distal from $\tilde{\Lambda}_\sigma$, and so by Lemma 52,

$$\lim_{t\to\infty} \left\|[R^{t_\rho,t_{\rho'}}, S^{t_\sigma,t_{\sigma'}}]_\pm\right\| = 0,$$

where the $(t_\tau)_{\tau\in\{\rho,\rho',\sigma,\sigma'\}}$ are large enough relative to t to satisfy the conditions. And so, as desired,

$$\lim_{t_\rho,t_{\rho'},t_\sigma,t_{\sigma'}\to\infty} \left\|R^{t_\rho,t_{\rho'}} \otimes S^{t_\sigma,t_{\sigma'}} - (-1)^{|R^{t_\rho,t_{\rho'}}|\cdot|S^{t_\sigma,t_{\sigma'}}|} S^{t_\sigma,t_{\sigma'}} \otimes R^{t_\rho,t_{\rho'}}\right\| = 0.$$

□

If $R^{t_\rho,t_{\rho'}} \in (T_\rho^{t_\rho}, T_{\rho'}^{t_{\rho'}})$ and $S^{t_\sigma,t_{\sigma'}} \in (T_\sigma^{t_\sigma}, T_{\sigma'}^{t_{\sigma'}})$ have mixed degree, this can still be applied by splitting them each into their even and odd components.

Definition 87. Let (θ, φ) label a forbidden direction.

For two cones $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta, \varphi)$ we say $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$ to mean the range of angles $\arg \Lambda_2$ is counterclockwise from the range of angles $\arg \Lambda_1$ when the forbidden direction (θ, φ) is taken into account. More precisely, if there are some basepoints $\vec{p}_i \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$ and $\phi_i \in (0, \pi)$ where Λ_i has basepoint $\vec{p}_i$ and range of angles $[\theta_i - \varphi_i, \theta_i + \varphi_i]$ for $i = 1, 2$ and such that

$$\theta + \varphi < \theta_1 - \varphi_1 < \theta_1 + \varphi_1 < \theta_2 - \varphi_2 < \theta_2 + \varphi_2 < 2\pi + \theta - \varphi.$$

The proof of Lemma 4.8 of [6] goes through without modification. Here we restate the lemma:

Lemma 88. *Let (θ, φ) label a forbidden direction. Let $\Lambda_1, \Lambda_2, \Lambda_1', \Lambda_2' \in \mathcal{C}(\theta, \varphi)$. Suppose that $\Lambda_1 \perp_{(\theta,\varphi)} \Lambda_2$, that $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$, and that $\Lambda_1' \perp_{(\theta,\varphi)} \Lambda_2'$, and $\Lambda_2' \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1'$.*

Then, for $i=1,2$ there are $\{\Lambda_i^{(j)}\}_{j=0}^4, \{\Lambda'^{(j)}_i\}_{j=0}^4 \subseteq \mathcal{C}(\theta,\varphi)$ such that

$$\Lambda_i^{(0)} = \Lambda_i, \quad \Lambda'^{(0)}_i = \Lambda'_i, \quad i=1,2$$
$$\{\Lambda_1^{(j)}, \Lambda_1^{(j+1)}\} \perp_{(\theta,\varphi)} \{\Lambda_2^{(j)}, \Lambda_2^{(j+1)}\}, \quad \{\Lambda'^{(j)}_1, \Lambda'^{(j+1)}_1\} \perp_{(\theta,\varphi)} \{\Lambda'^{(j)}_2, \Lambda'^{(j+1)}_2\}, \quad j=0,1,2,3$$
$$\{\Lambda_1^{(4)}, \Lambda'^{(4)}_1\} \perp_{(\theta,\varphi)} \{\Lambda_2^{(4)}, \Lambda'^{(4)}_2\}.$$

So, for each of $\Lambda_1, \Lambda'_1, \Lambda_2, \Lambda'_2$ there is a sequence of 5 cones starting with the one the sequence corresponds to, where for a sequential pair in this sequence, the two of them together are mutually distal (with forbidden direction (θ,φ)) from the corresponding pair from the cone with the other index from $i=1,2$, and where at the end the ones from $i=1$ are together mutually distal from the ones from $i=2$, with forbidden direction (θ,φ).

This is a bit like saying that four cones starting at $\Lambda_1, \Lambda'_1, \Lambda_2, \Lambda'_2$ respectively can be moved around so that the ones that started out at Λ_2, Λ'_2 end up counterclockwise and sufficently separate from the ones that started at Λ_1, Λ'_1 , without the ones that started at Λ_1 and Λ_2 getting too close to each-other, and also without getting too close to where the other had just been, and likewise for the ones starting at Λ'_1 and at Λ'_2.

Lemma 89. *Let I be an upward-directed set. For $t \in I$ and $i=1,2$ let $T_i^t, T_{i'}^t, T_{i''}^t \in \mathrm{End}(\mathcal{B}(\theta,\varphi))$ be endomorphisms of the form required in definition 85. Suppose that $R_i^t : T_i^t \to T_{i'}^t$ and $R_{i'}^t : T_{i'}^t \to T_{i''}^t$ with $R_i^t, R_{i'}^t \in \mathcal{B}(\theta,\varphi)$ as well, that $\lim_{t\in I}[R_1^t, R_2^t]_{\otimes,\pm} = 0$, and that $\lim_{t\in I}[R_{1'}^t, R_{2'}^t]_{\otimes,\pm} = 0$. Also suppose that for some $C>0$ that $\left\|R_i^t\right\|, \left\|R_{i'}^t\right\| \le C$ for all $i=1,2$ and $t \in I$.*
Then, $\lim_{t\in I}[R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm} = 0$, and for all $t\in I$, $\left\|[R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm}\right\| \le 2^7 C^4$

Proof. First suppose that the $R_i^t, R_{i'}^t$ are homogeneous.
By Lemma 83,

$$(R_{1'}^t R_1^t) \otimes (R_{2'}^t R_2^t) = (-1)^{|R_{2'}^t||R_1^t|}(R_{1'}^t \otimes R_{2'}^t)(R_1^t \otimes R_2^t)$$
$$(R_{2'}^t R_2^t) \otimes (R_{1'}^t R_1^t) = (-1)^{|R_{1'}^t||R_2^t|}(R_{2'}^t \otimes R_{1'}^t)(R_2^t \otimes R_1^t).$$

For homogenous A, B of the right type for the expression to be well defined, as $[A,B]_{\otimes,\pm} = A \otimes B - (-1)^{|A||B|} B \otimes A$, therefore also $B \otimes A = (-1)^{|A||B|}(A \otimes B - [A,B]_{\otimes,\pm})$. Applying this,

$$(R_{2'}^t \otimes R_{1'}^t) = (-1)^{|R_{2'}^t||R_{1'}^t|}(R_{1'}^t \otimes R_{2'}^t - [R_{1'}^t, R_{2'}^t]_{\otimes,\pm}) \text{ and}$$
$$(R_2^t \otimes R_1^t) = (-1)^{|R_2^t||R_1^t|}(R_1^t \otimes R_2^t - [R_1^t, R_2^t]_{\otimes,\pm}).$$

So,

$$\begin{aligned}
&(R_{2'}^t R_2^t) \otimes (R_{1'}^t R_1^t) \\
=&(-1)^{|R_{1'}^t||R_2^t| + |R_{2'}^t||R_{1'}^t| + |R_2^t||R_1^t|}(R_{1'}^t \otimes R_{2'}^t - [R_{1'}^t, R_{2'}^t]_{\otimes,\pm})(R_1^t \otimes R_2^t - [R_1^t, R_2^t]_{\otimes,\pm}) \\
=&(-1)^{|R_{1'}^t R_1^t||R_{2'}^t R_2^t| - |R_1^t||R_{2'}^t|}(R_{1'}^t \otimes R_{2'}^t - [R_{1'}^t, R_{2'}^t]_{\otimes,\pm})(R_1^t \otimes R_2^t - [R_1^t, R_2^t]_{\otimes,\pm}) \\
=&(-1)^{|R_{1'}^t R_1^t||R_{2'}^t R_2^t|}(-1)^{|R_1^t||R_{2'}^t|}((R_{1'}^t \otimes R_{2'}^t)(R_1^t \otimes R_2^t) + X^t) \\
=&(-1)^{|R_{1'}^t R_1^t||R_{2'}^t R_2^t|}(((R_{1'}^t R_1^t) \otimes (R_{2'}^t R_2^t) + (-1)^{|R_1^t||R_{2'}^t|} X^t)
\end{aligned}$$

where $X^t = [R_{1'}^t, R_{2'}^t]_{\otimes,\pm} \cdot [R_1^t, R_2^t]_{\otimes,\pm} - [R_{1'}^t, R_{2'}^t]_{\otimes,\pm} \cdot (R_1^t \otimes R_2^t) - (R_{1'}^t \otimes R_{2'}^t) \cdot [R_1^t, R_2^t]_{\otimes,\pm}$. Because the $\|R_i^t\|, \|R_{i'}^t\| \leq C$ and $\lim_{t \in I}[R_1^t, R_2^t]_{\otimes,\pm}$, and $\lim_{t \in I}[R_{1'}^t, R_{2'}^t]_{\otimes,\pm}$ are both 0, $\lim_{t \in I} X^t = 0$. Also, it is easily seen that for all $t \in I$, $\|X^t\| \leq (2 \cdot C^2)^2 + (2 \cdot C^2) \cdot (C^2) + (C^2) \cdot (2C^2) = 8C^4$.
Therefore,

$$\begin{aligned}
[R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm} &= (R_{1'}^t R_1^t) \otimes (R_{2'}^t R_2^t) - (-1)^{|R_{1'}^t R_1^t||R_{2'}^t R_2^t|}(R_{2'}^t R_2^t) \otimes (R_{1'}^t R_1^t) \\
&= (R_{1'}^t R_1^t) \otimes (R_{2'}^t R_2^t) - (R_{1'}^t R_1^t) \otimes (R_{2'}^t R_2^t) + (-1)^s X^t \\
&= 0 + (-1)^s X^t
\end{aligned}$$

where $s = |R_{1'}^t R_1^t||R_{2'}^t R_2^t| + |R_1^t||R_{2'}^t|$. So,

$$\lim_{t \in I}[R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm} = \lim_{t \in I}(-1)^s X^t = 0$$

as desired. And, for all $t \in I$, $\|[R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm}\| = \|X^t\| \leq 8C^4$.
If the $R_i^t, R_{i'}^t$ are not homogeneous, the result can be applied to their even and odd parts, and therefore imply the result for them as well. In this case where they are not homogeneous, to get an upper bound on $\|[R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm}\|$, after splitting it up into the even and odd components of

each of $R_1^t, R_{1'}^t, R_2^t, R_{2'}^t$, there are 2^4 terms which each have norm at most $8C^4$, and so we have an upper bound of, for all $t \in I$, $\left\| [R_{1'}^t R_1^t, R_{2'}^t R_2^t]_{\otimes,\pm} \right\| \leq 2^7 C^4$. $\square$

As an analogy to Lemma 4.9 of [6]:

LEMMA 90. *Let (θ,φ) label a forbidden direction. Let $\Lambda_1, \Lambda_2, \Lambda_1', \Lambda_2' \in \mathcal{C}(\theta,\varphi)$. Suppose that $\Lambda_1 \perp_{(\theta,\varphi)} \Lambda_2$, that $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$, and that $\Lambda_1' \perp_{(\theta,\varphi)} \Lambda_2'$, and $\Lambda_2' \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1'$.*
For $i = 1,2$, let $\rho_i, \rho_i' \in \mathcal{O}_0$, and for each $t_i, t_i' \geq 0$ let $V_{\rho_i,\Lambda_i+t_i} \in \mathcal{V}_{\rho_i,\Lambda_i+t_i}$, $V_{\rho_i',\Lambda_i'+t_i'} \in \mathcal{V}_{\rho_i',\Lambda_i'+t_i'}$, let $T_{\rho_i}^{t_i}$ abbreviate $T_{\rho_i}^{(\theta,\varphi),\Lambda_i+t_i,V_{\rho_i,\Lambda_i+t_i}}$ and $T_{\rho_i'}^{t_i'}$ abbreviate $T_{\rho_i'}^{(\theta,\varphi),\Lambda_i'+t_i',V_{\rho_i',\Lambda_i'+t_i'}}$, and $R_i^{t_i,t_i'} \in (T_{\rho_i}^{t_i}, T_{\rho_i'}^{t_i'})$ with $\left\| R_i^{t_i,t_i'} \right\| \leq 1$.
Then, for $R_i^{t_i,t_i'}$ homogeneous and $s = |R_1^{t_1,t_1'}|\,|R_2^{t_2,t_2'}|$ (i.e. 1 if both are odd and 0 otherwise),

$$\lim_{t_1,t_1',t_2,t_2' \to \infty} \left\| R_1^{t_1,t_1'} \otimes R_2^{t_2,t_2'} - (-1)^s R_2^{t_2,t_2'} \otimes R_1^{t_1,t_1'} \right\| = 0.$$

In the general (not necessarily homogeneous) case, we have

$$\lim_{t_1,t_1',t_2,t_2' \to \infty} \left\| [R_1^{t_1,t_1'}, R_2^{t_2,t_2'}]_{\otimes,\pm} \right\| = 0.$$

PROOF. First, assume that both $R_1^{t_1,t_1'}$ and $R_2^{t_2,t_2'}$ are homogeneous for all t_1, t_1', t_2, t_2'.
By Lemma 4.8 of [6] (Lemma 88) there are $\{\Lambda_i^{(j)}\}_{j=0}^4, \{\Lambda_i'^{(j)}\}_{j=0}^4 \subseteq \mathcal{C}(\theta,\varphi)$ for $i = 1,2$ satisfying

$$\Lambda_i^{(0)} = \Lambda_i, \quad \Lambda_i'^{(0)} = \Lambda_i', \quad i = 1,2$$

$$\{\Lambda_1^{(j)}, \Lambda_1^{(j+1)}\} \perp_{(\theta,\varphi)} \{\Lambda_2^{(j)}, \Lambda_2^{(j+1)}\}, \quad \{\Lambda_1'^{(j)}, \Lambda_1'^{(j+1)}\} \perp_{(\theta,\varphi)} \{\Lambda_2'^{(j)}, \Lambda_2'^{(j+1)}\}, \quad j = 0,1,2,3$$

$$\{\Lambda_1^{(4)}, \Lambda_1'^{(4)}\} \perp_{(\theta,\varphi)} \{\Lambda_2^{(4)}, \Lambda_2'^{(4)}\}.$$

For $i = 1,2$, $j = 0,\ldots,4$ and $t_{i,j}, t_{i,j}' \geq 0$, let $\vec{t} = (t_{i,j}, t_{i,j}' | i = 1,2 \text{ and } j = 0,\ldots,4)$, and let $\vec{t} \to \infty$ mean that $t_{i,j}, t_{i,j}' \to \infty$ for each i,j.
Pick for each i,j, for each $t_{i,j} \geq 0$ a $V_{\rho_i,\Lambda_i^{(j)}+t_{i,j}} \in \mathcal{V}_{\rho_i,\Lambda_i^{(j)}+t_{i,j}}$, and for each $t_{i,j}' \geq 0$ a $V_{\rho_i',\Lambda_i'^{(j)}+t_{i,j}'} \in \mathcal{V}_{\rho_i',\Lambda_i'^{(j)}+t_{i,j}'}$.
For each $\vec{t}$ and for each $i = 1,2$ and $j = 0,\ldots,4$, as abbreviations, set
$T_{\rho_i}^{i,j,\vec{t}} := T_{\rho_i}^{(\theta,\varphi),\Lambda_i^{(j)}+t_{i,j},V_{\rho_i,\Lambda_i^{(j)}+t_{i,j}}}$ and $T_{\rho_i'}^{i',j,\vec{t}} := T_{\rho_i'}^{(\theta,\varphi),\Lambda_i'^{(j)}+t_{i,j}',V_{\rho_i',\Lambda_i'^{(j)}+t_{i,j}'}}$.
For each $\vec{t}$ and each $i = 1,2$ and $j = 0,\ldots,3$ define

$W_i^{(j),\vec{t}} := V_{\rho_i,\Lambda_i^{(j+1)}+t_{i,j+1}} V^*_{\rho_i,\Lambda_i^{(j)}+t_{i,j}}$ and $W'^{(j),\vec{t}}_i := V_{\rho'_i,\Lambda'^{(j+1)}_i+t'_{i,j+1}} V^*_{\rho'_i,\Lambda'^{(j)}_i+t'_{i,j}}$. By Lemma 78, $W_i^{(j),\vec{t}} : T^{i,j,\vec{t}}_{\rho_i} \to T^{i,j+1,\vec{t}}_{\rho_i}$ and $W'^{(j),\vec{t}}_i : T^{i',j,\vec{t}}_{\rho'_i} \to T^{i',j+1,\vec{t}}_{\rho'_i}$.

As the $\{\Lambda_i^{(j)}\}_{j=0}^4, \{\Lambda'^{(j)}_i\}_{j=0}^4 \subseteq \mathcal{C}(\theta,\varphi)$ were chosen according to 88 and $\left\|W_i^{(j),\vec{t}}\right\| = 1 = \left\|W'^{(j),\vec{t}}_i\right\|$, applying lemma 86 we get that, for $j = 0,\dots,3$,

$$\lim_{\vec{t}\to\infty}[W_1^{(j),\vec{t}}, W_2^{(j),\vec{t}}]_{\otimes,\pm} = 0 \quad \text{and} \quad \lim_{\vec{t}\to\infty}[(W'^{(j),\vec{t}}_1)^*, (W'^{(j),\vec{t}}_2)^*]_{\otimes,\pm} = 0\,.$$

For $i = 1, 2$ define

$$\begin{aligned}
W'^{(0\to4),\vec{t}}_i &:= W'^{(3),\vec{t}}_i W'^{(2),\vec{t}}_i W'^{(1),\vec{t}}_i W'^{(0),\vec{t}}_i : T^{i',0,\vec{t}}_{\rho'_i} \to T^{i',4,\vec{t}}_{\rho'_i}\\
W^{(0\to4),\vec{t}}_i &:= W^{(3),\vec{t}}_i W^{(2),\vec{t}}_i W^{(1),\vec{t}}_i W^{(0),\vec{t}}_i : T^{i,0,\vec{t}}_{\rho_i} \to T^{i,4,\vec{t}}_{\rho_i}\\
S_i^{\vec{t}} &:= W'^{(0\to4),\vec{t}}_i R_i^{t_{i,0},t'_{i,0}} (W^{(0\to4),\vec{t}}_i)^* : T^{i,4,\vec{t}}_{\rho_i} \to T^{i',4,\vec{t}}_{\rho'_i}
\end{aligned}$$

where these being intertwiners with the stated domains and codomains follows from composition, along with the adjoint of an intertwiner being an intertwiner in the opposite direction. Note also that as $W'^{(0\to4),\vec{t}}_i$ and $W^{(0\to4),\vec{t}}_i$ are even, the grade of $S_i^{\vec{t}}$ is the same as the grade of $R_i^{t_{i,0},t'_{i,0}}$, and so $s = |R_1^{t_{1,0},t'_{1,0}}| \cdot |R_2^{t_{2,0},t'_{2,0}}| = |S_1^{\vec{t}}| \cdot |S_2^{\vec{t}}|$.

As $\{\Lambda_1^{(4)}, \Lambda'^{(4)}_1\} \perp_{(\theta,\varphi)} \{\Lambda_2^{(4)}, \Lambda'^{(4)}_2\}$, by Lemma 86, $\lim_{\vec{t}\to\infty}[S_1^{\vec{t}}, S_2^{\vec{t}}]_{\otimes,\pm} = 0$.

By iteratively applying Lemma 89, and using the fact that

$$\text{for } j = 0,\dots,3 \quad \lim_{\vec{t}\to\infty}[(W'^{(j),\vec{t}}_1)^*, (W'^{(j),\vec{t}}_2)^*]_{\otimes,\pm} = 0,$$

we get that

$$\lim_{\vec{t}\to\infty}[(W'^{(0\to4),\vec{t}}_1)^*, (W'^{(0\to4),\vec{t}}_2)^*]_{\otimes,\pm} = 0.$$

Similarly, we obtain

$$\lim_{\vec{t}\to\infty}[W_1^{(0\to4),\vec{t}}, W_2^{(0\to4),\vec{t}}]_{\otimes,\pm} = 0$$

from $\lim_{\vec{t}\to\infty}[W_1^{(j),\vec{t}}, W_2^{(j),\vec{t}}]_{\otimes,\pm} = 0$.

As the $W_i^{(0\to4),\vec{t}}, W'^{(0\to4),\vec{t}}_i$ used to define $S_i^{\vec{t}}$ in terms of $R_i^{t_{i,0},t'_{i,0}}$ are unitaries,

$$R_i^{t_{i,0},t'_{i,0}} = (W'^{(0\to4),\vec{t}}_i)^* S_i^{\vec{t}} W_i^{(0\to4),\vec{t}}\,.$$

So, by two more applications of Lemma 89, we get that

$$\lim_{\vec{t}\to\infty}[R_1^{t_{1,0},t'_{1,0}}, R_2^{t_{2,0},t'_{2,0}}]_{\otimes,\pm} = \lim_{\vec{t}\to\infty}[(W'^{(0\to4),\vec{t}}_1)^* S_1^{\vec{t}} W_1^{(0\to4),\vec{t}}, (W'^{(0\to4),\vec{t}}_2)^* S_2^{\vec{t}} W_2^{(0\to4),\vec{t}}]_{\otimes,\pm}$$
$$= 0$$

as desired.

Now we treat the case where the $R_i^{t_i,t'_i}$ are not assumed to be homogeneous. By Lemma 76, we have that $R_{i,\text{even}}^{t_i,t'_i}, R_{i,\text{odd}}^{t_i,t'_i} \in (T_{\rho_i}^{t_i}, T_{\rho'_i}^{t'_i})$, and by Lemma 112(ii), we also have that $\left\|R_{i,\text{even}}^{t_i,t'_i}\right\|, \left\|R_{i,\text{odd}}^{t_i,t'_i}\right\| \leq \left\|R_i^{t_i,t'_i}\right\|$. So, from the condition that $\left\|R_i^{t_i,t'_i}\right\| \leq 1$ we have that $\left\|R_{i,\text{even}}^{t_i,t'_i}\right\|, \left\|R_{i,\text{odd}}^{t_i,t'_i}\right\| \leq 1$. So, the conditions of this lemma apply to the even and odd components of the $R_i^{t_i,t'_i}$, and we have already shown the lemma in the homogeneous case. Therefore, we have

$$\lim_{\vec{t}\to\infty}[R_{1,\alpha}^{t_1,t'_1}, R_{2,\beta}^{t_2,t'_2}]_{\otimes,\pm} = 0, \quad \text{for } \alpha,\beta \in \{\text{even}, \text{odd}\}.$$

So,

$$\lim_{\vec{t}\to\infty}[R_1^{t_1,t'_1}, R_2^{t_2,t'_2}]_{\otimes,\pm} = \lim_{\vec{t}\to\infty}([R_{1,\text{even}}^{t_1,t'_1}, R_{2,\text{even}}^{t_2,t'_2}]_{\otimes,\pm} + [R_{1,\text{even}}^{t_1,t'_1}, R_{2,\text{odd}}^{t_2,t'_2}]_{\otimes,\pm} +$$
$$[R_{1,\text{odd}}^{t_1,t'_1}, R_{2,\text{even}}^{t_2,t'_2}]_{\otimes,\pm} + [R_{1,\text{odd}}^{t_1,t'_1}, R_{2,\text{odd}}^{t_2,t'_2}]_{\otimes,\pm})$$
$$= 0.$$

This completes the proof in the general case. □

The above proof could alternatively have been done using Lemma 83 more directly in place of using Lemma 89 , by using the fact that the $W_i^{(j),\vec{t}}, W'^{(j),\vec{t}}_i$ are all even to avoid the sign terms in the compositions, but this way was easier to phrase.

The conclusion of Lemma 4.10 of [6] applies here as well with no changes, and very little changes in the proof:

Lemma 91. *Let (θ,φ) label a forbidden direction.*

let $\rho = \tau_1 = \tau_{1'}$, $\sigma = \tau_2 = \tau_{2'} \in \mathcal{O}_0$.

For $i = 1,2$ let $t_{i'} = t'_i$ be two notations for the same number. Likewise, let $\Lambda_{i'} = \Lambda'_i$ be two notations for the same cone.

For $i \in \{0,1,1',2,2'\}$ let $\Lambda_i \in \mathcal{C}(\theta,\varphi)$. For $i \in \{1,1',2,2'\}$, for $t_i \geq 0$, let $V_{\tau_i,\Lambda_i+t_i} \in \mathcal{V}_{\tau_i,\Lambda_i+t_i}$. For

$\tau \in \{\rho, \sigma\}$ let $V_{\tau,\Lambda_0} \in \mathcal{V}_{\tau,\Lambda_0}$.

Set $\vec{t} = (t_i)_{i=1,2}$, $\vec{t'} = (t'_i)_{i=1,2}$.

For $i \in \{1,2\}$ set $W^{\vec{t}}_{\tau_i,\Lambda_0,\Lambda_i} := V_{\tau_i,\Lambda_i+t_i} V^*_{\tau_i,\Lambda_0}$ and $W^{\vec{t'}}_{\tau_i,\Lambda_0,\Lambda'_i} := V_{\tau_i,\Lambda'_i+t'_i} V^*_{\tau_i,\Lambda_0}$.

Suppose $\Lambda_2 \perp_{(\theta,\varphi)} \Lambda_1$ and $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$, and that $\Lambda'_2 \perp_{(\theta,\varphi)} \Lambda'_1$ and $\Lambda'_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda'_1$. Then we have

$$\lim_{\vec{t},\vec{t'}\to\infty} \left\| (W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^* (W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}) - (W^{\vec{t'}}_{\sigma\Lambda_0\Lambda'_2} \otimes W^{\vec{t'}}_{\rho\Lambda_0\Lambda'_1})^* (W^{\vec{t'}}_{\rho\Lambda_0\Lambda'_1} \otimes W^{\vec{t'}}_{\sigma\Lambda_0\Lambda'_2}) \right\| = 0.$$

PROOF. For $i = 1,2$ set $W^{\vec{t},\vec{t'}}_{\tau_i,\Lambda_i,\Lambda'_i} := V_{\tau_i,\Lambda'_i+t'_i} V^*_{\tau_i,\Lambda_i+t_i}$.

Set these abbreviations:

For $i = 1,2$ let $T^{\Lambda_0}_{\rho_i} = T^{(\theta,\varphi),\Lambda_0,V_{\tau_i,\Lambda_0}}_{\rho_i}$.

For $i \in \{1,1',2,2'\}$ let $T^{i,t_i}_{\tau_i} = T^{(\theta,\varphi),\Lambda_i+t_i,V_{\tau_i,\Lambda_i+t_i}}_{\tau_i}$.

From Lemma 78, for $i = 1,2$:

$W^{\vec{t}}_{\tau_i,\Lambda_0,\Lambda_i} : T^{\Lambda_0}_{\tau_i} \to T^{i,t_i}_{\tau_i}$, $W^{\vec{t'}}_{\tau_i,\Lambda_0,\Lambda'_i} : T^{\Lambda_0}_{\tau_i} \to T^{i',t'_i}_{\tau_i}$, and $W^{\vec{t},\vec{t'}}_{\tau_i,\Lambda_i,\Lambda'_i} : T^{i,t_i}_{\tau_i} \to T^{i',t'_i}_{\tau_i}$.

As all of these W intertwiners are even, by Lemma 83,

$$\begin{aligned}
&(W^{\vec{t},\vec{t'}}_{\tau_1,\Lambda_1,\Lambda'_1} \otimes W^{\vec{t},\vec{t'}}_{\tau_2,\Lambda_2,\Lambda'_2})(W^{\vec{t}}_{\tau_1,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\tau_2,\Lambda_0,\Lambda_2}) \\
&= (W^{\vec{t},\vec{t'}}_{\tau_1,\Lambda_1,\Lambda'_1} W^{\vec{t}}_{\tau_1,\Lambda_0,\Lambda_1}) \otimes (W^{\vec{t},\vec{t'}}_{\tau_2,\Lambda_2,\Lambda'_2} W^{\vec{t}}_{\tau_2,\Lambda_0,\Lambda_2}) \\
&= W^{\vec{t'}}_{\tau_1,\Lambda_0,\Lambda'_1} \otimes W^{\vec{t'}}_{\tau_2,\Lambda_0,\Lambda'_2}
\end{aligned}$$

and likewise

$$\begin{aligned}
&(W^{\vec{t},\vec{t'}}_{\tau_2,\Lambda_2,\Lambda'_2} \otimes W^{\vec{t},\vec{t'}}_{\tau_1,\Lambda_1,\Lambda'_1})(W^{\vec{t}}_{\tau_2,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\tau_1,\Lambda_0,\Lambda_1}) \\
&= W^{\vec{t'}}_{\tau_2,\Lambda_0,\Lambda'_2} \otimes W^{\vec{t'}}_{\tau_1,\Lambda_0,\Lambda'_1},
\end{aligned}$$

so

(4.1)

$$(W^{\vec{t'}}_{\tau_2,\Lambda_0,\Lambda'_2} \otimes W^{\vec{t'}}_{\tau_1,\Lambda_0,\Lambda'_1})^* (W^{\vec{t'}}_{\tau_1,\Lambda_0,\Lambda'_1} \otimes W^{\vec{t'}}_{\tau_2,\Lambda_0,\Lambda'_2})$$

(4.2)

$$= (W^{\vec{t}}_{\tau_2,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\tau_1,\Lambda_0,\Lambda_1})^* (W^{\vec{t},\vec{t'}}_{\tau_2,\Lambda_2,\Lambda'_2} \otimes W^{\vec{t},\vec{t'}}_{\tau_1,\Lambda_1,\Lambda'_1})^* (W^{\vec{t},\vec{t'}}_{\tau_1,\Lambda_1,\Lambda'_1} \otimes W^{\vec{t},\vec{t'}}_{\tau_2,\Lambda_2,\Lambda'_2})(W^{\vec{t}}_{\tau_1,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\tau_2,\Lambda_0,\Lambda_2}).$$

Applying Lemma 90 with $\rho_i = \rho_i' = \tau_i$ and $R_i^{t_i,t_i'} = W^{\vec{t},\vec{t}'}_{\tau_i,\Lambda_i,\Lambda_i'}$ (for $i = 1,2$) we get $\lim_{\vec{t},\vec{t}'\to\infty}[W^{\vec{t},\vec{t}'}_{\tau_1,\Lambda_1,\Lambda_1'}, W^{\vec{t},\vec{t}'}_{\tau_2,\Lambda_2,\Lambda_2'}]_{\otimes,\pm} = 0$, which, because the $W^{\vec{t},\vec{t}'}_{\tau_i,\Lambda_i,\Lambda_i'}$ are all even, is equivalent to $\lim_{\vec{t},\vec{t}'\to\infty}(W^{\vec{t},\vec{t}'}_{\tau_1,\Lambda_1,\Lambda_1'} \otimes W^{\vec{t},\vec{t}'}_{\tau_2,\Lambda_2,\Lambda_2'} - W^{\vec{t},\vec{t}'}_{\tau_2,\Lambda_2,\Lambda_2'} \otimes W^{\vec{t},\vec{t}'}_{\tau_1,\Lambda_1,\Lambda_1'}) = 0$, which as these are unitaries is equivalent to $\lim_{\vec{t},\vec{t}'\to\infty}((W^{\vec{t},\vec{t}'}_{\tau_2,\Lambda_2,\Lambda_2'} \otimes W^{\vec{t},\vec{t}'}_{\tau_1,\Lambda_1,\Lambda_1'})^*(W^{\vec{t},\vec{t}'}_{\tau_1,\Lambda_1,\Lambda_1'} \otimes W^{\vec{t},\vec{t}'}_{\tau_2,\Lambda_2,\Lambda_2'}) - 1) = 0$ and, substituting this into equation 4.1 above, we get the desired result. □

With this, the definition of the braiding morphism (Definition 4.11 in [6]) works without modification:

DEFINITION 92. Let (θ,φ) label a forbidden direction.
Let $\rho = \tau_1$, $\sigma = \tau_2 \in \mathcal{O}_0$. Let $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. Let $V_{\rho,\Lambda_0} \in \mathcal{V}_{\rho,\Lambda_0}, V_{\sigma,\Lambda_0} \in \mathcal{V}_{\sigma,\Lambda_0}$.
Define $\epsilon_+^{(\Lambda_0)}(\rho,\sigma)$ as follows:
Pick any $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta,\varphi)$ such that $\Lambda_1 \perp_{(\theta,\varphi)} \Lambda_1$ and $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$.
For $i \in \{1,2\}$, for each $t_i \geq 0$, pick any $V_{\tau_i,\Lambda_i+t_i} \in \mathcal{V}_{\tau_i,\Lambda_i+t_i}$.
Set $\vec{t} = (t_i)_{i=1,2}$. For $i \in \{1,2\}$ set $W^{\vec{t}}_{\tau_i,\Lambda_0,\Lambda_i} := V_{\tau_i,\Lambda_i+t_i}V^*_{\tau_i,\Lambda_0}$.
Define

$$\epsilon_+^{(\Lambda_0)}(\rho,\sigma) := \lim_{\vec{t}\to\infty}(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*(W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}).$$

By Lemma 91, this limit exists and is independent of the choices of Λ_1, Λ_2 and of the choices of $V_{\tau_i,\Lambda_i+t_i} \in \mathcal{V}_{\tau_i,\Lambda_i+t_i}$.

Lemma 4.12 from [6] goes through in this setting with essentially no modification to the proof (other than which lemmas are cited to reach the same conclusions, and using Remark 77). It is restated here:

LEMMA 93. *Let (θ,φ) specify a forbidden direction. Let $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. Let $\rho,\sigma \in \mathcal{O}_0$. Let $V_{\rho,\Lambda_0} \in \mathcal{V}_{\rho,\Lambda_0}$ and $V_{\sigma,\Lambda_0} \in \mathcal{V}_{\sigma,\Lambda_0}$. Let T_ρ and T_σ abbreviate $T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}$ and $T_\sigma^{(\theta,\varphi),\Lambda_0,V_{\sigma,\Lambda_0}}$ respectively. Then, $\epsilon_+^{(\Lambda_0)}(\rho,\sigma) \in (T_\rho \circ T_\sigma, T_\sigma \circ T_\rho)$.*

PROOF. Let $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta,\varphi)$ which satisfy $\Lambda_2 \perp_{(\theta,\varphi)} \Lambda_1$ and $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$. Use the notation and abbreviations from Lemma 91. Note that $(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2}) \in (T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0}, T_\rho^{1,t_1} \circ T_\sigma^{2,t_2})$ and that $(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1})^* \in (T_\sigma^{2,t_2} \circ T_\rho^{1,t_1}, T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0})$. For any $A \in \mathcal{A}_{loc}$, for sufficiently large $\vec{t}$,

$A \in \mathcal{A}_{(\Lambda_i+t_i)^c}$ for $i = 1, 2$, and so, by Lemma 59 (d),

$$T_\rho^{1,t_1} \circ T_\sigma^{2,t_2} \circ \pi_0(A) = T_\rho^{1,t_1} \circ \pi_0(A) = \pi_0(A) = T_\sigma^{2,t_2} \circ \pi_0(A) = T_\sigma^{2,t_2} \circ T_\rho^{1,t_1} \circ \pi_0(A).$$

As the W operators are all even, by Remark 77, for all $x \in \mathcal{B}(\theta, \varphi)$ and for all $\vec{t}$,

$$(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\, T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0}(x) = T_\rho^{1,t_1} \circ T_\sigma^{2,t_2}(x)\, (W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2}) \text{ and}$$

$$T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0}(x)\, (W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1})^* = (W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1})^*\, T_\sigma^{2,t_2} \circ T_\rho^{1,t_1}(x).$$

Therefore, for $A \in \mathcal{A}_{loc}$,

$$\begin{aligned}
&(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*(W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2})\, T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0} \circ \pi_0(A) \\
&- \\
&T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0} \circ \pi_0(A)\, (W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*(W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}) \\
&= \\
&(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*\, T_\rho^{1,t_1} \circ T_\sigma^{2,t_2} \circ \pi_0(A)\, (W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}) \\
&- \\
&(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*\, T_\sigma^{2,t_2} \circ T_\rho^{1,t_1} \circ \pi_0(A)\, (W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}),
\end{aligned}$$

and for sufficiently large $\vec{t}$, this is zero. Taking the limit of both sides as $\vec{t} \to \infty$, and applying the definition of $\epsilon_+^{(\Lambda_0)}(\rho, \sigma)$, we get that

$$\epsilon_+^{(\Lambda_0)}(\rho, \sigma)\, T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0} \circ \pi_0(A) - T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0} \circ \pi_0(A)\, \epsilon_+^{(\Lambda_0)}(\rho, \sigma) = 0.$$

Then, as $T_\rho \circ T_\sigma$ and $T_\sigma \circ T_\rho$ are each ultraweak-continuous on $\pi_0(\mathcal{A}_\Lambda)$ for $\Lambda \in \mathcal{C}(\theta, \varphi)$ (by Lemma 71), and are norm continuous, we get that $x \mapsto (\epsilon_+^{(\Lambda_0)}(\rho, \sigma)\, T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0}(x) - T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0}(x)\, \epsilon_+^{(\Lambda_0)}(\rho, \sigma))$ is zero on the norm closure of $\mathcal{B}_0(\theta, \varphi)$, which by Lemma 49 is all of $\mathcal{B}(\theta, \varphi)$.
So, for $x \in \mathcal{B}(\theta, \varphi)$, $\epsilon_+^{(\Lambda_0)}(\rho, \sigma)\, T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0} \circ \pi_0(A) = T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0} \circ \pi_0(A)\, \epsilon_+^{(\Lambda_0)}(\rho, \sigma)$,
i.e. $\epsilon_+^{(\Lambda_0)}(\rho, \sigma) \in (T_\rho^{\Lambda_0} \circ T_\sigma^{\Lambda_0}, T_\sigma^{\Lambda_0} \circ T_\rho^{\Lambda_0})$, as desired. $\square$

Following Lemma 4.13 of [6]:

Lemma 94. *Let (θ,φ) specify a forbidden direction. Let $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$. Let $\rho,\rho',\sigma,\sigma' \in \mathcal{O}_0$. For each $\tau \in \{\rho,\rho',\sigma,\sigma'\}$ let $V_{\tau,\Lambda_0} \in \mathcal{V}_{\tau,\Lambda_0}$. Let $R \in (T_\rho^{(\theta,\varphi),\Lambda_0,V_{\rho,\Lambda_0}}, T_{\rho'}^{(\theta,\varphi),\Lambda_0,V_{\rho',\Lambda_0}})$ and let $S \in (T_\sigma^{(\theta,\varphi),\Lambda_0,V_{\sigma,\Lambda_0}}, T_{\sigma'}^{(\theta,\varphi),\Lambda_0,V_{\sigma',\Lambda_0}})$. For R,S each homogeneous, for $s = |R||S|$,*

$$\epsilon_+^{(\Lambda_0)}(\rho',\sigma')(R\otimes S) = (-1)^s(S\otimes R)\epsilon_+^{(\Lambda_0)}(\rho,\sigma)$$

and otherwise by components.

Proof. Let $\Lambda_1,\Lambda_2 \in \mathcal{C}(\theta,\varphi)$ which satisfy $\Lambda_2 \perp_{(\theta,\varphi)} \Lambda_1$ and $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$. Use the notation and abbreviations from Lemma 91, but note that now that there is ρ',σ' to deal with, and there is no Λ_1' and Λ_2' to deal with, the primes here are indicating something wholly separate. For example, $V_{\rho',\Lambda_1+t_1} \in \mathcal{V}_{\rho',\Lambda_1+t_1}$ may be different from $V_{\rho,\Lambda_1+t_1} \in \mathcal{V}_{\rho,\Lambda_1+t_1}$, and $T_{\rho'}^{1,t_2} = T_{\rho'}^{(\theta,\varphi),\Lambda_1+t_1,V_{\rho',\Lambda_1+t_1}}$.
Set and note that $R^{t_1,t_1} := W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1} R W^{\vec{t}\,*}_{\rho,\Lambda_0,\Lambda_1} \in (T_\rho^{1,t_1}, T_{\rho'}^{1,t_1})$ and $S^{t_2,t_2} := W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} S W^{\vec{t}\,*}_{\sigma,\Lambda_0,\Lambda_2} \in (T_\sigma^{2,t_2}, T_{\sigma'}^{2,t_2})$
As $\Lambda_2 \perp_{(\theta,\varphi)} \Lambda_1$ and $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$, by Lemma 90, $\lim_{\vec{t}\to\infty}[R^{t_1,t_1}, S^{t_2,t_2}]_{\otimes,\pm} = 0$
As $W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1}, W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1}, W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2}, W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2}$ are all even, by Lemma 83,
$(W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})(S\otimes R)(W^{\vec{t}\,*}_{\sigma,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}\,*}_{\rho,\Lambda_0,\Lambda_1}) = (S^{t_2,t_2}\otimes R^{t_1,t_1})$ and $(W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2})(R\otimes S)(W^{\vec{t}\,*}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}\,*}_{\sigma,\Lambda_0,\Lambda_2}) = (R^{t_1,t_1}\otimes S^{t_2,t_2})$.

$$\begin{aligned}
&(S\otimes R)(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1})^*(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\\
&= (W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*(S^{t_2,t_2}\otimes R^{t_1,t_1})(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\\
&= (W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*(-1)^{|R||S|}((R^{t_1,t_1}\otimes S^{t_2,t_2}) - [R^{t_1,t_1}, S^{t_2,t_2}]_{\otimes,\pm})(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\\
&= (-1)^{|R||S|}(((W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*(R^{t_1,t_1}\otimes S^{t_2,t_2})(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\\
&\quad - (W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*[R^{t_1,t_1}, S^{t_2,t_2}]_{\otimes,\pm}(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2}))\\
&= (-1)^{|R||S|}(((W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*(W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2})(R\otimes S)\\
&\quad - (W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*[R^{t_1,t_1}, S^{t_2,t_2}]_{\otimes,\pm}(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})),
\end{aligned}$$

and so

$$\begin{aligned}
(S\otimes R)\epsilon_+^{(\Lambda_0)}(\rho,\sigma) &= \lim_{\vec{t}\to\infty}(S\otimes R)(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1})^*(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\\
&= (-1)^{|R||S|}\lim_{\vec{t}\to\infty}(\,((W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*(W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2})(R\otimes S)
\end{aligned}$$

$$- (W^{\vec{t}}_{\sigma',\Lambda_0,\Lambda_2} \otimes W^{\vec{t}}_{\rho',\Lambda_0,\Lambda_1})^*[R^{t_1,t_1}, S^{t_2,t_2}]_{\otimes,\pm}(W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1} \otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2})\ )$$
$$= (-1)^{|R||S|}\epsilon_+^{(\Lambda_0)}(\rho',\sigma')(R\otimes S).$$

For R, S not homogeneous, $R = R_0 + R_1, S = S_0 + S_1$,

$$(S\otimes R)\epsilon_+^{(\Lambda_0)}(\rho,\sigma) = (S_0\otimes R_0 + S_0\otimes R_1 + S_1\otimes R_0 + S_1\otimes R_1)\epsilon_+^{(\Lambda_0)}(\rho,\sigma)$$
$$= \epsilon_+^{(\Lambda_0)}(\rho',\sigma')(R_0\otimes S_0 + R_1\otimes S_0 + R_0\otimes S_1 - R_1\otimes S_1).$$

□

Following Lemma 4.14 of [6]:

LEMMA 95. *Let (θ,φ) label a forbidden direction. Let $\rho,\sigma\in\mathcal{O}_0$. For $i=0,1$ let $\Lambda_i\in\mathcal{C}(\theta,\varphi)$ and let $D_i = ((\theta,\varphi),\Lambda_i,\{\overline{V}_{\eta,\Lambda_i}\}_{\eta\in\mathcal{O}_0})$ be as in Definition 66. Set $W_{\rho,\Lambda_0,\Lambda_1} := \overline{V}_{\rho,\Lambda_1}\overline{V}^*_{\rho,\Lambda_0}$ and $W_{\sigma,\Lambda_0,\Lambda_1} := \overline{V}_{\sigma,\Lambda_1}\overline{V}^*_{\sigma,\Lambda_0}$. Then,*

$$W_{\rho,\Lambda_0,\Lambda_1}\otimes W_{\sigma,\Lambda_0,\Lambda_1}\in\mathcal{V}_{(\rho\circ_{D_0}\sigma),\Lambda_1},$$

$$1\in\mathcal{V}_{(\rho\circ_{D_0}\sigma),\Lambda_0}$$

PROOF. By Lemma 78, $W_{\rho,\Lambda_0,\Lambda_1} : T_\rho^{D_0}\to T_\rho^{D_1}$ and $W_{\sigma,\Lambda_0,\Lambda_1} : T_\sigma^{D_0}\to T_\sigma^{D_1}$. So, $W_{\rho,\Lambda_0,\Lambda_1}\otimes W_{\sigma,\Lambda_0,\Lambda_1} = W_{\rho,\Lambda_0,\Lambda_1}T_\rho^{D_0}(W_{\sigma,\Lambda_0,\Lambda_1}) : T_\rho^{D_0}\circ T_\sigma^{D_0}\to T_\rho^{D_1}\circ T_\sigma^{D_1}$.

$$\mathrm{Ad}(W_{\rho,\Lambda_0,\Lambda_1}\otimes W_{\sigma,\Lambda_0,\Lambda_1})\circ(\rho\circ_{D_0}\sigma) = \mathrm{Ad}(W_{\rho,\Lambda_0,\Lambda_1}T_\rho^{D_0}(W_{\sigma,\Lambda_0,\Lambda_1}))\circ(T_\rho^{D_0}\circ T_\sigma^{D_0}\circ\pi_0)$$
$$= \mathrm{Ad}(W_{\rho,\Lambda_0,\Lambda_1})\circ T_\rho^{D_0}\circ\mathrm{Ad}(W_{\sigma,\Lambda_0,\Lambda_1})\circ T_\sigma^{D_0}\circ\pi_0$$
$$= T_\rho^{D_1}\circ T_\sigma^{D_1}\circ\pi_0 = \rho\circ_{D_1}\circ\sigma.$$

The third equality is by Lemma 61(iii). By Lemma 68, $((\rho\circ_{D_1}\sigma),U_{\alpha_F})\in\mathcal{O}_{\Lambda_1}$, and therefore $(\rho\circ_{D_1}\sigma)^t|_{\mathcal{A}_{\Lambda_1^c}} = \pi_0^t|_{\mathcal{A}_{\Lambda_1^c}}$. So, for $U_{(\rho\circ_{D_0}\sigma)} = U_{\alpha_F}$ and $U_{(\rho\circ_{D_1}\sigma)} = U_{\alpha_F}$,

$$\mathrm{Ad}(W_{\rho,\Lambda_0,\Lambda_1}\otimes W_{\sigma,\Lambda_0,\Lambda_1})\circ(\rho\circ_{D_0}\sigma)^t|_{\mathcal{A}_{\Lambda_1^c}} = \mathrm{Ad}(W_{\rho,\Lambda_0,\Lambda_1}\otimes W_{\sigma,\Lambda_0,\Lambda_1})\circ\mathrm{tw}\circ(\rho\circ_{D_0}\sigma)|_{\mathcal{A}_{\Lambda_1^c}}$$
$$= \mathrm{tw}\circ\mathrm{Ad}(W_{\rho,\Lambda_0,\Lambda_1}\otimes W_{\sigma,\Lambda_0,\Lambda_1})\circ(\rho\circ_{D_0}\sigma)|_{\mathcal{A}_{\Lambda_1^c}}$$
$$= \mathrm{tw}\circ(\rho\circ_{D_1}\circ\sigma)|_{\mathcal{A}_{\Lambda_1^c}} = (\rho\circ_{D_1}\circ\sigma)^t|_{\mathcal{A}_{\Lambda_1^c}}$$
$$= \pi_0^t|_{\mathcal{A}_{\Lambda_1^c}}.$$

Therefore (also using the fact that $W_{\rho,\Lambda_0,\Lambda_1} \otimes W_{\sigma,\Lambda_0,\Lambda_1}$ is even) $W_{\rho,\Lambda_0,\Lambda_1} \otimes W_{\sigma,\Lambda_0,\Lambda_1} \in \mathcal{V}_{(\rho\circ_{D_0}\sigma),\Lambda_1}$. Also by Lemma 68, $((\rho \circ_{D_0} \sigma), U_{\alpha_F}) \in \mathcal{O}_{\Lambda_0,*}$, so $1 \in \mathcal{V}_{(\rho\circ_{D_0}\sigma),\Lambda_0}$.

□

REMARK 96. We will apply this to the definition 92 for $\epsilon_+^{(\Lambda_0)}((\rho \circ_D \sigma), \cdot)$ and $\epsilon_+^{(\Lambda_0)}(\cdot, (\rho \circ_D \sigma))$. Recall $W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_1} := V_{(\rho\circ_D\sigma),\Lambda_1+t_1} V^*_{(\rho\circ_D\sigma),\Lambda_0}$ for any choice of $V_{(\rho\circ_D\sigma),\Lambda_1+t_1} \in \mathcal{V}_{(\rho\circ_D\sigma),\Lambda_1+t_1}$ and for $V_{(\rho\circ_D\sigma),\Lambda_0} \in \mathcal{V}_{(\rho\circ_D\sigma),\Lambda_0}$.
Now apply the above lemma in the case of Λ_1 and Λ_0 in the above lemma being $\Lambda_i + t_i$ and Λ_0 respectively of definition 92. Then, As $1 \in \mathcal{V}_{(\rho\circ_D\sigma),\Lambda_0}$ we can choose $V_{(\rho\circ_D\sigma),\Lambda_0} = 1$. And, as $W_{\rho,\Lambda_0,\Lambda_i+t_i} \otimes W_{\sigma,\Lambda_0,\Lambda_i+t_i} \in \mathcal{V}_{(\rho\circ_{D_0}\sigma),\Lambda_i+t_i}$.
Then $W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_i+t_i} := (W_{\rho,\Lambda_0,\Lambda_i+t_i} \otimes W_{\sigma,\Lambda_0,\Lambda_i+t_i}) \cdot (1)^* = W_{\rho,\Lambda_0,\Lambda_i+t_i} \otimes W_{\sigma,\Lambda_0,\Lambda_i+t_i}$

Following Lemma 4.16 of [6]:

LEMMA 97 (Hexagon Identities). *Let $D = ((\theta, \varphi), \Lambda_0, \{\overline{V}_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$ be as in Definition 66. For any $\rho, \sigma, \tau \in \mathcal{O}_0$,*

$$\epsilon_+^{(\Lambda_0)}((\rho \circ_D \sigma), \tau) = (\epsilon_+^{(\Lambda_0)}(\rho, \tau) \otimes 1_{T^D_\sigma})(1_{T^D_\rho} \otimes \epsilon_+^{(\Lambda_0)}(\sigma, \tau)),$$

$$\epsilon_+^{(\Lambda_0)}(\rho, (\sigma \circ_D \tau)) = (1_{T^D_\sigma} \otimes \epsilon_+^{(\Lambda_0)}(\rho, \tau))(\epsilon_+^{(\Lambda_0)}(\rho, \sigma) \otimes 1_{T^D_\tau}).$$

PROOF. This proof is nearly the same as the proof of Lemma 4.16 in [6], as, because all the operators involved are even, almost nothing needs to change in the proof (other than citing analogous lemmas/definitions in place of the lemmas/definitions cited).
In applying Definition 92, take $\Lambda_1, \Lambda_2 \in \mathcal{C}(\theta, \varphi)$ satisfying $\Lambda_2 \underset{(\theta,\varphi)}{\curvearrowleft} \Lambda_1$, and $\Lambda_2 \perp$ such that there exist $\tilde{\Lambda}_1, \tilde{\Lambda}_2 \in \mathcal{C}(\theta, \varphi)$ such that

$$(\Lambda_1)_\varepsilon, \Lambda_0 \subset \tilde{\Lambda}_1, \quad \Lambda_2 \subset \tilde{\Lambda}_2$$

$$\arg(\tilde{\Lambda}_1)_{2\varepsilon+\delta+\varepsilon_1+\varepsilon_2} \cap \arg(\tilde{\Lambda}_2)_{2\varepsilon+\delta+\varepsilon_1+\varepsilon_2} = \emptyset,$$

$$(\tilde{\Lambda}_1 - R_{|\arg\tilde{\Lambda}_1|,\varepsilon})_\varepsilon \subset \tilde{\Lambda}_2^c, \quad (\tilde{\Lambda}_2 - R_{|\arg\tilde{\Lambda}_2|,\varepsilon})_\varepsilon \subset \tilde{\Lambda}_1^c,$$

$$(\tilde{\Lambda}_1)_{2(\varepsilon+\varepsilon_1+\varepsilon_2+\delta)}, (\tilde{\Lambda}_2)_{2(\varepsilon+\varepsilon_1+\varepsilon_2+\delta)} \in \mathcal{C}(\theta, \varphi),$$

for some $\delta, \varepsilon > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ small enough. So, in particular, $\Lambda_2 \perp_{(\theta,\varphi)} \Lambda_1$. These conditions are essentially the condition that $\{(\Lambda_1)_\varepsilon, \Lambda_0\} \perp_{(\theta,\varphi)} \{\Lambda_2\}$, along with the extra conditions that the ε in that $(\Lambda_1)_\varepsilon$ be the one used in Definition 51 for the disjointness part, and that $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ can be widened by a bit more than the required ε (instead by $2(\varepsilon+\varepsilon_1+\delta+\varepsilon_2)$) while still remaining in $\mathcal{C}(\theta,\varphi)$ and with disjoint ranges of angles.

For $i=1,2$ and $t_i \geq 0$, pick $D_{\Lambda_i+t_i} = ((\theta,\varphi),\Lambda_i+t_i,\{V_{\eta,\Lambda_i+t_i}\}_{\eta\in\mathcal{O}_0})$ as in Definition 66. Let $\vec{t}=(t_1,t_2)$. For $i,j=1,2$ and $\vec{t}\geq 0$, and for $\eta\in\{\rho,\sigma,\tau\}$ define $W^{\vec{t}}_{\eta,\Lambda_i,\Lambda_j} := V_{\eta,\Lambda_j+t_j}V^*_{\eta,\Lambda_i+t_i} : T^{D_{\Lambda_i+t_i}}_\eta \to T^{D_{\Lambda_j+t_j}}_\eta$ and $W^{\vec{t}}_{\eta,\Lambda_0,\Lambda_i} := V_{\eta,\Lambda_i+t_i}V^*_{\eta,\Lambda_0} : T^D_\eta \to T^{D_{\Lambda_i+t_i}}_\eta$.

By the preceding remark, set

$$W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_1} := W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1}\otimes W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} = W^{\vec{t}}_{\rho,\Lambda_0,\Lambda_1}T^D_\rho(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})$$

$$W^{\vec{t}}_{(\sigma\circ_D\tau),\Lambda_0,\Lambda_2} := W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_2}\otimes W^{\vec{t}}_{\tau,\Lambda_0,\Lambda_2} = W_{\sigma,\Lambda_0,\Lambda_2}T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_0,\Lambda_2})$$

$W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_1} : T^{(\theta,\varphi),\Lambda_0,1}_{(\rho\circ_D\sigma)} \to T^{(\theta,\varphi),\Lambda_1+t_1,W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_1}}_{(\rho\circ_D\sigma)}$. By Lemma 72, $T^D_{(\rho\circ_D\sigma)} = \mathrm{Ad}(V_{(\rho\circ\sigma),\Lambda_0})\circ T^D_\rho\circ T^D_\sigma$. As $1\in\mathcal{V}_{(\rho\circ_D\sigma),\Lambda_0}$, By Lemma 61(iii),

$$\begin{aligned}T^{(\theta,\varphi),\Lambda_0,1}_{(\rho\circ_D\sigma)} &= \mathrm{Ad}(1\cdot V^*_{(\rho\circ_D\sigma),\Lambda_0})\circ T^{(\theta,\varphi),\Lambda_0,V_{(\rho\circ_D\sigma),\Lambda_0}}_{(\rho\circ_D\sigma)} = \mathrm{Ad}(V^*_{(\rho\circ_D\sigma),\Lambda_0})\circ T^D_{(\rho\circ_D\sigma)}\\ &= \mathrm{Ad}(V^*_{(\rho\circ_D\sigma),\Lambda_0})\circ\mathrm{Ad}(V_{(\rho\circ\sigma),\Lambda_0})\circ T^D_\rho\circ T^D_\sigma = T^D_\rho\circ T^D_\sigma.\end{aligned}$$

So, $W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_1} : T^D_\rho\circ T^D_\sigma \to T^{(\theta,\varphi),\Lambda_1+t_1,W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_1}}_{(\rho\circ_D\sigma)}$.

Likewise, $W^{\vec{t}}_{(\sigma\circ_D\tau),\Lambda_0,\Lambda_2} : T^D_\sigma\circ T^D_\tau \to T^{(\theta,\varphi),\Lambda_2+t_2,W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_0,\Lambda_2}}_{(\sigma\circ_D\tau)}$.

We need to show that

$$\lim_{\vec{t}\to\infty}\left\| T^{D^{t_2}_{\Lambda_2}}_\tau\circ T^{D_{\Lambda_1+t_1}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - T^{D_{\Lambda_1+t_1}}_\rho\circ T^{D^{t_2}_{\Lambda_2}}_\tau(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| = 0.$$

To this end, we will show that

$$\lim_{\vec{t}\to\infty}\left\| T^{D_{\Lambda_2+t_2}}_\tau(T^{D_{\Lambda_1+t_1}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})) - T^{D_{\Lambda_1+t_1}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| = 0$$

and that

$$\lim_{\vec{t}\to\infty}\left\| T^{D_{\Lambda_2+t_2}}_\tau(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}\right\| = 0,$$

implying that $\lim_{\vec{t}\to\infty}\left\| T^{D_{\Lambda_1+t_1}}_\rho(T^{D_{\Lambda_2+t_2}}_\tau(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})) - T^{D_{\Lambda_1+t_1}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| = 0$.

For each $\varepsilon_3>0$, choose $s\geq R_{(|\arg\tilde{\Lambda}_1|+2(\varepsilon+\delta+\varepsilon_1)),\frac{1}{2}\varepsilon_2}, R_{|\arg\tilde{\Lambda}_1|,\varepsilon}$ such that $2f_{(|\arg\tilde{\Lambda}_1|+2(\varepsilon+\delta+\varepsilon_1)),\frac{1}{2}\varepsilon_2,\frac{1}{2}\varepsilon_2}(s) < \varepsilon_3$ and $2f_{|\arg\tilde{\Lambda}_1|,\varepsilon,\delta}(s) < \varepsilon_3$.

As $W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} \in \pi_0(\mathcal{A}_{((\Lambda_1+t_1)\cup\Lambda_0)^c})^{t\prime} \subseteq \mathfrak{A}(\tilde{\Lambda}_1)$, by Lemma 50, there exists an even unitary operator $\tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} \in \pi_0(\mathcal{A}_{(\tilde{\Lambda}_1-s)_{\varepsilon+\delta}})''$ such that $\left\|W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} - \tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}\right\| \leq 2f_{|\arg\tilde{\Lambda}_1|,\varepsilon,\delta}(s) < \varepsilon_3$.
By Lemma 62, $T^{D_{\Lambda_1+t_1}}_{\rho}(\tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) \in \mathfrak{A}((\tilde{\Lambda}_1 - s)_{\varepsilon+\delta+\varepsilon_1})$. Applying Lemma 50 again, we get that there is an even unitary $X^{\vec{t},s}_{\sigma,\Lambda_0,\Lambda_1} \in \pi_0(\mathcal{A}_{((\tilde{\Lambda}_1-s)_{\varepsilon+\delta+\varepsilon_1}-s)_{\varepsilon_2}})'' = \pi_0(\mathcal{A}_{(\tilde{\Lambda}_1-2s)_{\varepsilon+\delta+\varepsilon_1+\varepsilon_2}})''$ such that

$$\left\|X^{\vec{t},s}_{\sigma,\Lambda_0,\Lambda_1} - T^{D_{\Lambda_1+t_1}}_{\rho}(\tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| \leq 2f_{(|\arg\tilde{\Lambda}_1|+2(\varepsilon+\delta+\varepsilon_1)),\frac{1}{2}\varepsilon_2,\frac{1}{2}\varepsilon_2}(s) < \varepsilon_3.$$

So, $\left\|X^{\vec{t},s}_{\sigma,\Lambda_0,\Lambda_1} - T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| < 2\varepsilon_3$. For sufficiently large t_2, $(\tilde{\Lambda}_1 - 2s)_{\varepsilon+\delta+\varepsilon_1+\varepsilon_2} \subseteq (\Lambda_2 + t_2)^c$, so $T^{D_{\Lambda_2+t_2}}_{\tau}(X^{\vec{t},s}_{\sigma,\Lambda_0,\Lambda_1}) = X^{\vec{t},s}_{\sigma,\Lambda_0,\Lambda_1}$. Therefore,

$$\lim_{\vec{t}\to\infty}\left\|T^{D_{\Lambda_2+t_2}}_{\tau} \circ T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| < 4\varepsilon_3$$

and this holds for all $\varepsilon_3 > 0$, so

$$\lim_{\vec{t}\to\infty}\left\|T^{D_{\Lambda_2+t_2}}_{\tau} \circ T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| = 0.$$

Similarly, for sufficiently large t_2, $(\tilde{\Lambda}_1 - s)_{\varepsilon+\delta} \subseteq (\Lambda_2 + t_2)^c$ so
$\tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} \in \pi_0(\mathcal{A}_{(\tilde{\Lambda}_1-s)_{\varepsilon+\delta}})'' \subseteq \pi_0(\mathcal{A}_{(\Lambda_2+t_2)^c})''$ so

$$\begin{aligned}\left\|T^{D_{\Lambda_2+t_2}}_{\tau}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}\right\| &= \left\|T^{D_{\Lambda_2+t_2}}_{\tau}(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_1} - \tilde{W}^{\vec{t}}_{\sigma\Lambda_0\Lambda_1}) + T^{D_{\Lambda_2+t_2}}_{\tau}(\tilde{W}^{\vec{t}}_{\sigma\Lambda_0\Lambda_1}) - W^{\vec{t}}_{\sigma\Lambda_0\Lambda_1}\right\| \\ &= \left\|T^{D_{\Lambda_2+t_2}}_{\tau}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} - \tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - (W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1} - \tilde{W}^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| \\ &\leq 2\varepsilon_3,\end{aligned}$$

so

$$\lim_{\vec{t}\to\infty}\left\|T^{D_{\Lambda_1+t_1}}_{\rho} \circ T^{D_{\Lambda_2+t_2}}_{\tau}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| = 0.$$

Combining these, we get the desired

$$\lim_{\vec{t}\to\infty}\left\|T^{D^{t_2}_{\Lambda_2}}_{\tau} \circ T^{D_{\Lambda_1+t_1}}_{\rho}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1}) - T^{D_{\Lambda_1+t_1}}_{\rho} \circ T^{D^{t_2}_{\Lambda_2}}_{\tau}(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\| = 0.$$

In the next bit, to save space, let $W^{\vec{t}}_{\eta,\Lambda_i}$ denote $W^{\vec{t}}_{\eta,\Lambda_0,\Lambda_i}$ and let $T^{\Lambda_i,\vec{t}}_{\eta}$ denote $T^{D_{\Lambda_i+t_i}}_{\eta}$.

By Definition 92:

$$
\begin{aligned}
&\epsilon_+^{(\Lambda_0)}((\rho\circ_D\sigma),\tau)\\
&=\lim_{\vec{t}\to\infty}((W^{\vec{t}}_{\tau,\Lambda_2}\otimes W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_1})^*(W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_1}\otimes W^{\vec{t}}_{\tau,\Lambda_2}))\\
&=\lim_{\vec{t}\to\infty}T^D_\tau((W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_1})^*)W^{\vec{t}*}_{\tau,\Lambda_2}W^{\vec{t}}_{(\rho\circ_D\sigma),\Lambda_1}T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}T^D_\tau(T^D_\rho(W^{\vec{t}*}_{\sigma,\Lambda_1})W^{\vec{t}*}_{\rho,\Lambda_1})W^{\vec{t}*}_{\tau,\Lambda_2}W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho(W^{\vec{t}}_{\sigma,\Lambda_1})T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}W^{\vec{t}*}_{\tau,\Lambda_2}T^{\Lambda_2,\vec{t}}_\tau(W^{\vec{t}*}_{\rho,\Lambda_1}T^{\Lambda_1,\vec{t}}_\rho(W^{\vec{t}*}_{\sigma,\Lambda_1}))T^{\Lambda_1,\vec{t}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_1})W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}W^{\vec{t}*}_{\tau,\Lambda_2}T^{\Lambda_2,\vec{t}}_\tau(W^{\vec{t}*}_{\rho,\Lambda_1})T^{\Lambda_2,\vec{t}}_\tau\circ T^{\Lambda_1,\vec{t}}_\rho(W^{\vec{t}*}_{\sigma,\Lambda_1})T^{\Lambda_1,\vec{t}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_1})W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}W^{\vec{t}*}_{\tau,\Lambda_2}T^{\Lambda_2,\vec{t}}_\tau(W^{\vec{t}*}_{\rho,\Lambda_1})T^{\Lambda_1,\vec{t}}_\rho\circ T^{\Lambda_2,\vec{t}}_\tau(W^{\vec{t}*}_{\sigma,\Lambda_1})T^{\Lambda_1,\vec{t}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_1})W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}W^{\vec{t}*}_{\tau,\Lambda_2}T^{\Lambda_2,\vec{t}}_\tau(W^{\vec{t}*}_{\rho,\Lambda_1})T^{\Lambda_1,\vec{t}}_\rho(T^{\Lambda_2,\vec{t}}_\tau(W^{\vec{t}*}_{\sigma,\Lambda_1})W^{\vec{t}}_{\sigma,\Lambda_1})W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}T^D_\tau(W^{\vec{t}*}_{\rho,\Lambda_1})W^{\vec{t}*}_{\tau,\Lambda_2}W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho(W^{\vec{t}}_{\tau,\Lambda_2}T^D_\tau(W^{\vec{t}*}_{\sigma,\Lambda_1})W^{\vec{t}*}_{\tau,\Lambda_2}W^{\vec{t}}_{\sigma,\Lambda_1})T^D_\rho\circ T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2})\\
&=\lim_{\vec{t}\to\infty}(W^{\vec{t}}_{\tau\Lambda_2}T^D_\tau(W^{\vec{t}}_{\rho,\Lambda_1}))^*W^{\vec{t}}_{\rho,\Lambda_1}T^D_\rho(W^{\vec{t}}_{\tau,\Lambda_2})T^D_\rho(T^D_\tau(W^{\vec{t}*}_{\sigma,\Lambda_1})W^{\vec{t}*}_{\tau,\Lambda_2}W^{\vec{t}}_{\sigma,\Lambda_1}T^D_\sigma(W^{\vec{t}}_{\tau,\Lambda_2}))\\
&=\lim_{\vec{t}\to\infty}((W^{\vec{t}}_{\tau,\Lambda_2}\otimes W^{\vec{t}}_{\rho,\Lambda_1})^*(W^{\vec{t}}_{\rho,\Lambda_1}\otimes W^{\vec{t}}_{\tau,\Lambda_2})\ \cdot\ T^D_\rho((W^{\vec{t}}_{\tau,\Lambda_2}\otimes W^{\vec{t}}_{\sigma,\Lambda_1})^*(W^{\vec{t}}_{\sigma,\Lambda_1}\otimes W^{\vec{t}}_{\tau,\Lambda_2})))\\
&=\epsilon_+^{(\Lambda_0)}(\rho,\tau)\cdot T^D_\rho(\epsilon_+^{(\Lambda_0)}(\sigma,\tau))\\
&=(\epsilon_+^{(\Lambda_0)}(\rho,\tau)\otimes 1_{T^D_\sigma})(1_{T^D_\rho}\otimes\epsilon_+^{(\Lambda_0)}(\sigma,\tau)).
\end{aligned}
$$

Throughout the above chain of equalities, Definition 74 in the case that the input is even so that the twist doesn't do anything ($R\cdot \mathrm{tw}\circ T_1(x_0)=\mathrm{tw}\circ T_1(x_0)\cdot R \iff R\cdot T_1(x_0)=T_1(x_0)\cdot R$ for x_0 even and T_1,T_2 grade preserving) is applied. The 6th equality uses

$$\lim_{\vec{t}\to\infty}\left\|T^{D_{\Lambda_2+t_2}}_\tau\circ T^{D_{\Lambda_1+t_1}}_\rho(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})-T^{D_{\Lambda_1+t_1}}_\rho\circ T^{D_{\Lambda_2+t_2}}_\tau(W^{\vec{t}}_{\sigma,\Lambda_0,\Lambda_1})\right\|=0.$$

The proof for the other relation, $\epsilon_+^{(\Lambda_0)}(\rho,(\sigma\circ_D\tau))=(1_{T^D_\sigma}\otimes\epsilon_+^{(\Lambda_0)}(\rho,\tau))(\epsilon_+^{(\Lambda_0)}(\rho,\sigma)\otimes 1_{T^D_\tau})$, is entirely analogous.

One first picks $\Lambda_1, \Lambda_2, \tilde{\Lambda}_1, \tilde{\Lambda}_2$ in analogous way (this time where $\Lambda_1 \subseteq \tilde{\Lambda}_1$ and $(\Lambda_2)_\varepsilon, \Lambda_0 \subseteq \tilde{\Lambda}_2$) and with this choice (and corresponding choices of $D_{\Lambda_i+t_i}$) shows that

$$\lim_{\vec{t}\to\infty} \left\| T_\rho^{D_{\Lambda_1+t_1}} \circ T_\sigma^{D_{\Lambda_2+t_2}}(W^{\vec{t}}_{\tau,\Lambda_0,\Lambda_2}) - T_\sigma^{D_{\Lambda_2+t_2}} \circ T_\rho^{D_{\Lambda_1+t_1}}(W^{\vec{t}}_{\tau,\Lambda_0,\Lambda_2}) \right\| = 0,$$

and then does the last part analogously as well. □

4.3.4. Like section 5 (Direct Sums, Subobjects, and putting the category together). The statement of Lemma 5.6 of [6] holds here without modification, and the proof is very similar:

LEMMA 98. *Let (θ,φ) label a forbidden direction, and $\Lambda_0 \in \mathcal{C}(\theta,\varphi)$.*
For $\rho, \sigma \in \mathcal{O}_{\Lambda_0}$,

$$(\sigma,\rho) = (T_\rho^{(\theta,\varphi),\Lambda_0,1}, T_\sigma^{(\theta,\varphi),\Lambda_0,1}) \subseteq \mathcal{B}(\theta,\varphi).$$

PROOF. By Lemma 59 (ii), for all $A \in \mathcal{A}$, $\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0(A) = \mathrm{Ad}(1) \circ \rho^t(A) = \rho^t(A)$ and likewise $\mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0(A) = \sigma^t(A)$. So, for any $R \in (\rho,\sigma)$,

$$R \cdot \mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,1}(\pi_0(A)) = R \cdot \rho^t(A) = \sigma^t(A) \cdot R = \mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1}(\pi_0(A)) \cdot R.$$

So, the linear map $A \mapsto (R \cdot \mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,1}(A) - \mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1}(A) \cdot R) : \mathcal{B}(\theta,\varphi) \to \mathcal{B}(\mathcal{H})$ is 0 on $\pi_0(\mathcal{A})$ and therefore on $\pi_0(\mathcal{A}_\Lambda)$. By Lemma 59 part (i) it is ultraweak-continuous on $\pi_0(\mathcal{A}_\Lambda)''$ for $\Lambda \in \mathcal{C}(\theta,\varphi)$, and therefore this map sends all of $\mathcal{B}_0(\theta,\varphi)$ to 0, and, by norm continuity, as $\overline{\mathcal{B}_0(\theta,\varphi)}^{\|\cdot\|} = \mathcal{B}(\theta,\varphi)$, it sends all of $\mathcal{B}(\theta,\varphi)$ to 0 as well. So, for all $A \in \mathcal{B}(\theta,\varphi)$, $R \cdot \mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,1}(A) = \mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1}(A) \cdot R$, i.e. $R \in (T_\rho^{(\theta,\varphi),\Lambda_0,1}, T_\sigma^{(\theta,\varphi),\Lambda_0,1})$.
So,

$$(\sigma,\rho) \subseteq (T_\rho^{(\theta,\varphi),\Lambda_0,1}, T_\sigma^{(\theta,\varphi),\Lambda_0,1}) \subseteq \mathcal{B}(\theta,\varphi)$$

where the second inclusion is by Lemma 79.
Conversely, for any $R : (T_\rho^{(\theta,\varphi),\Lambda_0,1}, T_\sigma^{(\theta,\varphi),\Lambda_0,1})$, $R \cdot \mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0(A) = \mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0(A) \cdot R$. And, again by Lemma 59 (ii), $\mathrm{tw} \circ T_\rho^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0(A) = \rho^t(A)$ and $\mathrm{tw} \circ T_\sigma^{(\theta,\varphi),\Lambda_0,1} \circ \pi_0(A) = \sigma^t(A)$, so $R \cdot \rho^t(A) = \sigma^t(A) \cdot R$, which is the definition (Definition 44) of $R \in (\rho,\sigma)$. So, $(T_\rho^{(\theta,\varphi),\Lambda_0,1}, T_\sigma^{(\theta,\varphi),\Lambda_0,1}) \subseteq (\rho,\sigma)$. So,

$$(\sigma,\rho) = (T_\rho^{(\theta,\varphi),\Lambda_0,1}, T_\sigma^{(\theta,\varphi),\Lambda_0,1}) \subseteq \mathcal{B}(\theta,\varphi),$$

as desired. $\square$

Following Lemma 5.7 of [6]:

LEMMA 99 (Existence of Direct Sums). *Assume that for all cones Λ that the von Neumann algebra $\pi_0(\mathcal{A}_\Lambda)''_{even}$ is properly infinite.*
Let $\rho, \sigma \in \mathcal{O}_{\Lambda_0}$. Then there exists $(\tau, U_\tau) \in \mathcal{O}_{\Lambda_0}$ and even isometries $u \in (\rho, \tau), v \in (\sigma, \tau)$ such that $uu^* + vv^* = 1$.
So, in this sense, there exists a $\tau \in \mathcal{O}_{\Lambda_0}$ such that $\tau = \rho \oplus \sigma$.
In addition, if $U_\sigma = U_\rho = U_{\alpha_F}$ if $\rho, \sigma \in \mathcal{O}_{\Lambda_0,*}$ then $U_\tau = U_{\alpha_F}$, i.e. if $\sigma, \rho \in \mathcal{O}_{\Lambda_0,*}$, then $\tau \in \mathcal{O}_{\Lambda_0,*}$.

PROOF. As for all cones Λ, $\pi_0(\mathcal{A}_\Lambda)''_{\text{even}}$ is properly infinite, by Proposition 1.36 of chapter V of [7], there exist isometries $u_\Lambda, v_\Lambda \in \pi_0(\mathcal{A}_\Lambda)''_{\text{even}}$ such that $p_\Lambda = u_\Lambda u_\Lambda^*$ and $q_\Lambda = v_\Lambda v_\Lambda^*$ are each orthogonal projections and $p_\Lambda + q_\Lambda = 1$. Choose such a pair of isometries for each cone Λ, $((u_\Lambda, v_\Lambda))_{\Lambda \in \{\text{cones}\}}$.
Within this proof, for each pair of cones Λ_2, Λ_1, for any $A, B \in \mathcal{B}(\mathcal{H})$ define

$$\langle A : B \rangle_{\Lambda_2,\Lambda_1} := u_{\Lambda_2} A u_{\Lambda_1}^* + v_{\Lambda_2} B v_{\Lambda_1}^*.$$

Note that

$$\langle A : B \rangle_{\Lambda_3,\Lambda_2} \langle C : D \rangle_{\Lambda_2,\Lambda_1} = \langle AC : BD \rangle_{\Lambda_3,\Lambda_1}$$

$$(\langle A : B \rangle_{\Lambda_2,\Lambda_1})^* = \langle A^* : B^* \rangle_{\Lambda_1,\Lambda_2}$$

$$\langle A : B \rangle_{\Lambda_2,\Lambda_1} + \langle C : D \rangle_{\Lambda_2,\Lambda_1} = \langle A + C : B + D \rangle_{\Lambda_2,\Lambda_1}$$

$$\langle A : B \rangle_{\Lambda_2,\Lambda_1} u_{\Lambda_1} = u_{\Lambda_2} A$$

$$\langle A : B \rangle_{\Lambda_2,\Lambda_1} v_{\Lambda_1} = v_{\Lambda_2} B.$$

Also note that, as $u_\Lambda, v_\Lambda \in \pi_0(\mathcal{A}_\Lambda)''_{\text{even}} \subseteq \pi_0(\mathcal{A}_\Lambda)''$ that if $X \in \pi_0(\mathcal{A}_\Lambda)'$ then $\langle X : X \rangle_{\Lambda,\Lambda} = u_\Lambda X u_\Lambda^* + v_\Lambda X v_\Lambda^* = X(u_\Lambda u_\Lambda^* + v_\Lambda v_\Lambda^*) = X$. Because each of the pairs (u_Λ, v_Λ) have $u_\Lambda, v_\Lambda \in \pi_0(\mathcal{A}_\Lambda)''_{\text{even}}$, these pairings are grade preserving in the sense that the even and odd parts of $\langle (A_0 + A_1) : (B_0 + B_1) \rangle_{\Lambda_2,\Lambda_1}$ are $\langle A_0 : B_0 \rangle_{\Lambda_2,\Lambda_1}$ and $\langle A_1 : B_1 \rangle_{\Lambda_2,\Lambda_1}$ respectively.
Define $\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\tau(A) := \langle \rho(A) : \sigma(A) \rangle_{\Lambda_0,\Lambda_0}.$$

Note that it is grade preserving as ρ and σ are, and as the pairing is. Also note that it is a representation by

$$\begin{aligned}\tau(A)\tau(B) &= \langle\rho(A):\sigma(A)\rangle_{\Lambda_0,\Lambda_0}\langle\rho(B):\sigma(B)\rangle_{\Lambda_0,\Lambda_0}\\ &= \langle\rho(A)\rho(B):\sigma(A)\sigma(B)\rangle_{\Lambda_0,\Lambda_0}\\ &= \langle\rho(AB):\sigma(AB)\rangle_{\Lambda_0,\Lambda_0} = \tau(AB),\end{aligned}$$

and by the analogous reasoning for $\tau(A)+\tau(B)=\tau(A+B)$ and $\tau(A)^*=\tau(A^*)$.

Define also $U_\tau := \langle U_\rho : U_\sigma\rangle_{\Lambda_0,\Lambda_0}$. Then

$$\begin{aligned}\tau^t(A_0+A_1) &= \tau(A_0)+U_\tau\tau(A_1)\\ &= \langle\rho(A_0):\sigma(A_0)\rangle_{\Lambda_0,\Lambda_0} + \langle U_\rho : U_\sigma\rangle_{\Lambda_0,\Lambda_0}\langle\rho(A_1):\sigma(A_1)\rangle_{\Lambda_0,\Lambda_0}\\ &= \langle\rho(A_0):\sigma(A_0)\rangle_{\Lambda_0,\Lambda_0} + \langle U_\rho\rho(A_1) : U_\sigma\sigma(A_1)\rangle_{\Lambda_0,\Lambda_0}\\ &= \langle\rho^t(A_0+A_1):\sigma^t(A_0+A_1)\rangle_{\Lambda_0,\Lambda_0}.\end{aligned}$$

Note that $U_\tau^2 = \langle U_\rho^2 : U_\sigma^2\rangle_{\Lambda_0,\Lambda_0} = \langle 1:1\rangle_{\Lambda_0,\Lambda_0} = 1$ and that U_τ is unitary.

Also note that, as u_Λ, v_Λ are even, that $\langle U_{\alpha_F} : U_{\alpha_F}\rangle_{\Lambda_0,\Lambda_0} = U_{\alpha_F}\cdot(u_{\Lambda_0}u_{\Lambda_0}^* + v_{\Lambda_0}v_{\Lambda_0}^*) = U_{\alpha_F}$, so if $U_\rho = U_{\alpha_F} = U_\sigma$, then $U_\tau = U_{\alpha_F}$ as well.

As $\rho,\sigma\in\mathcal{O}_{\Lambda_0}$, $1\in\mathcal{V}_{\rho,\Lambda_0}$ and $1\in\mathcal{V}_{\sigma,\Lambda_0}$. For every cone Λ pick a $V_{\rho,\Lambda}\in\mathcal{V}_{\rho,\Lambda}$ and a $V_{\sigma,\Lambda}\in\mathcal{V}_{\sigma,\Lambda}$, and in particular for Λ_0 choose $V_{\rho,\Lambda_0}=1$ and $V_{\sigma,\Lambda_0}=1$. Now, for each cone Λ define $W_\Lambda := \langle V_{\rho,\Lambda} : V_{\sigma,\Lambda}\rangle_{\Lambda,\Lambda_0}$. As the $V_{\rho,\Lambda}$ and $V_{\sigma,\Lambda}$ are even and the pairing is grade preserving, these W_Λ are also even. To see that W_Λ is a unitary, note that $W_\Lambda W_\Lambda^* = \langle V_{\rho,\Lambda} : V_{\sigma,\Lambda}\rangle_{\Lambda,\Lambda_0}\langle V_{\rho,\Lambda}^* : V_{\sigma,\Lambda}^*\rangle_{\Lambda_0,\Lambda} = \langle 1:1\rangle_{\Lambda,\Lambda} = 1$ and likewise $W_\Lambda^*W_\Lambda = \langle 1:1\rangle_{\Lambda_0,\Lambda_0} = 1$.

For all $A\in\mathcal{A}_{\Lambda^c}$,

$$\begin{aligned}\mathrm{Ad}(W_\Lambda)\circ\tau^t(A) &= \mathrm{Ad}(\langle V_{\rho,\Lambda}:V_{\sigma,\Lambda}\rangle_{\Lambda,\Lambda_0})(\langle\rho^t(A):\sigma^t(A)\rangle_{\Lambda_0,\Lambda_0})\\ &= \langle\mathrm{Ad}(V_{\rho,\Lambda})\circ\rho^t(A) : \mathrm{Ad}(V_{\sigma,\Lambda})\circ\sigma^t(A)\rangle_{\Lambda,\Lambda}\\ &= \langle\pi_0^t(A):\pi_0^t(A)\rangle_{\Lambda,\Lambda}\\ &= \pi_0^t(A).\end{aligned}$$

(That last equality is by $\pi_0^t(A) \in \pi_0(\mathcal{A}_{\Lambda^c})^t \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime\prime} \subseteq \pi_0(\mathcal{A}_\Lambda)'$ from twisted locality $\pi_0(\mathcal{A}_\Lambda)'' \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$.)

Therefore, as for all cones Λ, W_Λ is an even unitary such that $\mathrm{Ad}(W_\Lambda) \circ \tau^t|_{\mathcal{A}_{\Lambda^c}} = \pi_0|_{\mathcal{A}_{\Lambda^c}}$, so $W_\Lambda \in \mathcal{V}_{\tau,\Lambda}$, and so $\tau \in \mathcal{O}_0$. In particular, $W_{\Lambda_0} = 1 \in \mathcal{V}_{\tau,\Lambda_0}$, so $\tau \in \mathcal{O}_{\Lambda_0}$.

Note that, as for all $A \in \mathcal{A}$, $\tau(A)u_{\Lambda_0} = \langle \rho(A) : \sigma(A) \rangle_{\Lambda_0,\Lambda_0} u_{\Lambda_0} = u_{\Lambda_0}\rho(A)$ and likewise $\tau(A)v_{\Lambda_0} = v_{\Lambda_0}\sigma(A)$, we therefore have $u = u_{\Lambda_0} \in (\rho, \tau)$ and $v = v_{\Lambda_0} \in (\sigma, \tau)$ as the even isometries that were promised. □

Following Lemma 5.12 of [6], with a small change to handle the parity:

LEMMA 100. *Let $\mathcal{N}, \mathcal{M}, \mathcal{R}$ be von Neumann algebras acting on a separable $(\mathbb{Z}/2\mathbb{Z})$-graded Hilbert space $\mathcal{H} = \mathcal{H}_{even} \oplus \mathcal{H}_{odd}$, such that $\mathcal{N}, \mathcal{M} \subseteq \mathcal{R}$ and $\mathcal{N} \subseteq \mathcal{M}'$. Suppose that $\mathcal{M}$ is an infinite factor and $\mathcal{R}$ is a factor. Let w be an* even *unitary on $\mathcal{H}$, and $u \in \mathcal{N}$ be an even isometry such that $w^*uu^*w \in \mathcal{N}$. Suppose in addition that all elements of $\mathcal{R}$ are even (i.e. they preserve the $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathcal{H}$). Then there is an* even *unitary W on $\mathcal{H}$ such that*

$$\forall x \in R', \quad \mathrm{Ad}(Wu^*)(x) = \mathrm{Ad}(u^*w)(x).$$

PROOF. The proof of this is the same as the proof of Lemma 5.12 in [6], except that, due to the assumptions that u and w are even, and the assumption that everything in $\mathcal{R}$ is even, and therefore the $a \in \mathcal{R}$ obtained in said proof is even, therefore, the obtained $W := u^*wa^*u$ is even. □

Following Lemma 5.8 of [6]:

LEMMA 101 (Existence of Subobjects). *Suppose that for all cones Λ that both $\pi_0(\mathcal{A}_\Lambda)''_{even}$ and $(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{even}$ are infinite factors.*

Then, for any $\sigma \in \mathcal{O}_{\Lambda_0}$ and a non-zero even projection $p : \sigma \to \sigma$ such that $pU_\sigma = U_\sigma p$, there exists a $(\tau, U_\tau) \in \mathcal{O}_{\Lambda_0}$ and an isometry $v : \tau \to \sigma$ such that $vv^* = p$.

In particular, if $U_\sigma = U_{\alpha_F}$ (i.e. if $\sigma \in \mathcal{O}_{\Lambda_0,*}$, not just in $\mathcal{O}_{\Lambda_0}$), then $pU_\sigma = U_\sigma p$ is satisfied automatically and $U_\tau = U_{\alpha_F}$ (i.e. $\tau \in \mathcal{O}_{\Lambda_0,*}$ as well).

Proof. To obtain a representation (τ, U_τ) satisfying the superselection criterion and an even isometry $v : \tau \to \sigma$ such that $vv^* = p$, we will produce a family of even isometries Y_Λ for each cone Λ, such that:

- $Y_\Lambda Y_\Lambda^* = p$,
- $Y_{\Lambda_1}^* Y_{\Lambda_2}$ is unitary for all cones Λ_1, Λ_2,
- $\mathrm{Ad}(Y_\Lambda^*) \circ \sigma^t|_{\mathcal{A}_{\Lambda^c}} = \pi_0^t|_{\mathcal{A}_{\Lambda^c}}$.

Given such a family $(Y_\Lambda)_{\Lambda \in \{\text{cones}\}}$, we define a new representation $\tau := \mathrm{Ad}(Y_{\Lambda_0}^*) \circ \sigma$ and set $U_\tau := \mathrm{Ad}(Y_{\Lambda_0}^*)(U_\sigma)$. We will see that Y_{Λ_0} serves as the desired isometry $v : \tau \to \sigma$. First we will show that this τ (equipped with this U_τ) is in $\mathcal{O}_{\Lambda_0}$, and that Y_{Λ_0} indeed serves as the desired isometry $v : \tau \to \sigma$. Afterwards, we will show that such a family $(Y_\Lambda)_\Lambda$ exists.

To see that τ is indeed a representation, note first that it is of course linear and that $\tau(A)^* = \tau(A^*)$. What remains to check is that it is compatible with products:

$$
\begin{aligned}
\tau(A)\tau(B) &= Y_{\Lambda_0}^* \sigma(A) Y_{\Lambda_0} Y_{\Lambda_0}^* \sigma(B) Y_{\Lambda_0} \\
&= Y_{\Lambda_0}^* \sigma(A) p \sigma(B) Y_{\Lambda_0} \\
&= Y_{\Lambda_0}^* \sigma(A) \sigma(B) p Y_{\Lambda_0} \\
&= Y_{\Lambda_0}^* \sigma(AB) Y_{\Lambda_0} = \tau(AB),
\end{aligned}
$$

where the $p\sigma(B) = \sigma(B)p$ used in the third equality is because $p : \sigma \to \sigma$ and $[U_\sigma, p] = 0$, and where the $pY_{\Lambda_0} = Y_{\Lambda_0}$ used in the fourth equality is due to Y_{Λ_0} being an isometry such that $Y_{\Lambda_0} Y_{\Lambda_0}^* = p$.

To see that U_τ has the desired properties for the superselection criterion, note $U_\tau^* = U_\tau$ and that $U_\tau U_\tau^* = Y_{\Lambda_0}^* U_\sigma Y_{\Lambda_0} Y_{\Lambda_0}^* U_\sigma^* Y_{\Lambda_0} = Y_{\Lambda_0}^* U_\sigma p U_\sigma^* Y_{\Lambda_0} = Y_{\Lambda_0}^* p U_\sigma U_\sigma^* Y_{\Lambda_0} = 1$, by $[p, U_\sigma] = 0$.

For all $A = A_0 + A_1 \in \mathcal{A}$, $\tau(A_0) + U_\tau \tau(A_1) = \mathrm{Ad}(Y_{\Lambda_0}^*)(\sigma^t(A))$, by:

$$
\begin{aligned}
\tau(A_0) + U_\tau \tau(A_1) &= \mathrm{Ad}(Y_{\Lambda_0}^*)(\sigma(A_0)) + \mathrm{Ad}(Y_{\Lambda_0}^*)(U_\sigma)\, \mathrm{Ad}(Y_{\Lambda_0}^*)(\sigma(A_1))) \\
&= Y_{\Lambda_0}^* \sigma(A_0) Y_{\Lambda_0} + Y_{\Lambda_0}^* U_\sigma Y_{\Lambda_0} Y_{\Lambda_0}^* \sigma(A_1) Y_{\Lambda_0} \\
&= Y_{\Lambda_0}^* \sigma(A_0) Y_{\Lambda_0} + Y_{\Lambda_0}^* U_\sigma p \sigma(A_1) Y_{\Lambda_0} \\
&= Y_{\Lambda_0}^* \sigma(A_0) Y_{\Lambda_0} + Y_{\Lambda_0}^* p U_\sigma \sigma(A_1) Y_{\Lambda_0} \\
&= Y_{\Lambda_0}^* \sigma(A_0) Y_{\Lambda_0} + Y_{\Lambda_0}^* U_\sigma \sigma(A_1) Y_{\Lambda_0} = Y_{\Lambda_0}^* (\sigma(A_0) + U_\sigma \sigma(A_1)) Y_{\Lambda_0}.
\end{aligned}
$$

(The fourth equality is by the assumption that $U_\sigma p = pU_\sigma$.)

For all cones Λ, set $X_\Lambda := Y_\Lambda^* Y_{\Lambda_0}$. By the second property of the family $(Y_\Lambda)_\Lambda$, each X_Λ is a unitary.

Now we show that $X_\Lambda \in \mathcal{V}_{\tau,\Lambda}$. For all cones Λ and all $A = A_0 + A_1 \in \mathcal{A}_{\Lambda^c}$,

$$\begin{aligned}\mathrm{Ad}(X_\Lambda)(\tau(A_0) + U_\tau \tau(A_1)) &= \mathrm{Ad}(X_\Lambda)(Y_{\Lambda_0}^*(\sigma(A_0) + U_\sigma \sigma(A_1))Y_{\Lambda_0}) \\ &= \mathrm{Ad}(Y_\Lambda^* Y_{\Lambda_0} Y_{\Lambda_0}^*)(\sigma^t(A)) = Y_\Lambda^* p \sigma^t(A) p Y_\Lambda \\ &= \mathrm{Ad}(Y_\Lambda^*)(\sigma^t(A)) = \pi_0^t(A),\end{aligned}$$

where the last equality is by the third property of the family $(Y_\Lambda)_\Lambda$ and $A \in \mathcal{A}_{\Lambda^c}$.

So, (τ, U_τ) satisfies the superselection criterion, and so $\tau \in \mathcal{O}_0$. In particular, $X_{\Lambda_0} = Y_{\Lambda_0}^* Y_{\Lambda_0} = 1$, so $1 \in \mathcal{V}_{\tau,\Lambda_0}$, so $\tau \in \mathcal{O}_{\Lambda_0}$ (equipped with U_τ).

Finally, for all $A \in \mathcal{A}$,

$$\begin{aligned}Y_{\Lambda_0} \cdot \tau^t(A) &= Y_{\Lambda_0} \cdot (Y_{\Lambda_0}^* \cdot \sigma^t(A) \cdot Y_{\Lambda_0}) \\ &= (Y_{\Lambda_0} Y_{\Lambda_0}^*) \sigma^t(A) \cdot Y_{\Lambda_0} \\ &= p \cdot \sigma^t(A) \cdot Y_{\Lambda_0} \\ &= \sigma^t(A) \cdot p \cdot Y_{\Lambda_0} \\ &= \sigma^t(A) \cdot Y_{\Lambda_0},\end{aligned}$$

so $Y_{\Lambda_0} : \tau \to \sigma$, and so $v = Y_{\Lambda_0}$ is the promised isometry $v : \tau \to \sigma$ such that $vv^* = p$.

Now, what remains is to show there exists such a family of isometries $(Y_\Lambda)_\Lambda$.

For each cone Λ pick a $V_{\sigma,\Lambda} \in \mathcal{V}_{\sigma,\Lambda}$, and, as $\sigma \in \mathcal{O}_{\Lambda_0}$, for $\Lambda = \Lambda_0$ pick $V_{\sigma,\Lambda_0} = 1$. Note that, for each cone Λ,

$$\forall A \in \mathcal{A}_{\Lambda^c}, \quad p \cdot (\mathrm{Ad}(V_{\sigma,\Lambda}^*) \circ \pi_0^t(A)) = p \cdot \sigma^t(A) = \sigma^t(A) \cdot p = (\mathrm{Ad}(V_{\sigma,\Lambda}^*) \circ \pi_0^t(A)) \cdot p$$

Applying $\mathrm{Ad}(V_{\sigma,\Lambda})$ to both sides of $p \cdot (\mathrm{Ad}(V_{\sigma,\Lambda}^*) \circ \pi_0^t(A)) = (\mathrm{Ad}(V_{\sigma,\Lambda}^*) \circ \pi_0^t(A)) \cdot p$ we obtain $\mathrm{Ad}(V_{\sigma,\Lambda})(p) \cdot \pi_0^t(A) = \pi_0^t(A) \cdot \mathrm{Ad}(V_{\sigma,\Lambda})(p)$, for all $A \in \mathcal{A}_{\Lambda^c}$. So, for all cones Λ,

$$p_\Lambda := \mathrm{Ad}(V_{\sigma,\Lambda})(p) \in \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime} = \mathfrak{A}(\Lambda).$$

Let $\delta > 0$ be the number given in Lemma 5.10 of [6]. For each $\varphi \in (0, 2\pi), \varepsilon > 0$ such that $\varphi + 8\varepsilon < 2\pi$, fix some $t_{\varphi,\varepsilon} \geq R_{\varphi,\varepsilon}$ such that $f_{\varphi,\varepsilon,\varepsilon}(t_{\varphi,\varepsilon}) < \delta$ (Recall the definition of approximate twisted Haag duality, Definition 38 .) For each cone Λ fix some $\varepsilon_\Lambda \in (0, \frac{1}{16}\min(|\arg\Lambda|, |\arg\Lambda^c|))$.
Set $t'_\Lambda := t_{(|\arg\Lambda| - 8\varepsilon_\Lambda), \varepsilon_\Lambda}$.
Set $\Gamma_\Lambda := (\Lambda + t'_\Lambda)_{-4\varepsilon_\Lambda}$.
Note $|\arg\Gamma_\Lambda| = |\arg\Lambda| - 8\varepsilon_\Lambda$, so $t'_\Lambda = t_{(|\arg\Lambda|-8\varepsilon_\Lambda),\varepsilon_\Lambda} = t_{|\arg\Gamma_\Lambda|,\varepsilon_\Lambda} \geq R_{|\arg\Gamma_\Lambda|,\varepsilon_\Lambda}$, so $t'_\Lambda - R_{\Gamma_\Lambda,\varepsilon_\Lambda} \geq 0$.
Choose a cone D_Λ such that $D_\Lambda \subseteq (\Gamma_\Lambda - R_{\Gamma_\Lambda,\varepsilon_\Lambda})_{\varepsilon_\Lambda} \cap \Gamma_\Lambda^c$. Note that

$$D_\Lambda \subseteq (\Gamma_\Lambda - R_{\Gamma_\Lambda,\varepsilon_\Lambda})_{\varepsilon_\Lambda} \subseteq (\Lambda + t'_\Lambda - R_{\Gamma_\Lambda,\varepsilon_\Lambda})_{-3\varepsilon_\Lambda} \subseteq \Lambda_{-3\varepsilon_\Lambda}.$$

By the approximate twisted Haag duality, there is a unitary $U_{\Gamma_\Lambda,\varepsilon_\Lambda} \in \mathcal{U}(\mathcal{H})$ such that

$$\pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime} \subseteq \mathrm{Ad}(U_{\Gamma_\Lambda,\varepsilon_\Lambda})(\pi_0(\mathcal{A}_{(\Gamma_\Lambda - R_{\Gamma_\Lambda,\varepsilon_\Lambda})_{\varepsilon_\Lambda}})''),$$

and another unitary

$$\hat{U}_\Lambda := \tilde{U}_{\Gamma_\Lambda,\varepsilon_\Lambda,\varepsilon_\Lambda,t'_\Lambda} \in \pi_0(\mathcal{A}_{(\Gamma_\Lambda - t'_\Lambda)_{2\varepsilon_\Lambda}})'' = \pi_0(\mathcal{A}_{((\Lambda+t'_\Lambda)_{-4\varepsilon_\Lambda} - t'_\Lambda)_{2\varepsilon_\Lambda}})'' = \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''$$

such that

$$\left\| U_{\Gamma_\Lambda,\varepsilon_\Lambda} - \tilde{U}_{\Gamma_\Lambda,\varepsilon_\Lambda,\varepsilon_\Lambda,t'_\Lambda} \right\| \leq f_{|\arg\Gamma_\Lambda|,\varepsilon_\Lambda,\varepsilon_\Lambda}(t'_\Lambda) = f_{|\arg\Gamma_\Lambda|,\varepsilon_\Lambda,\varepsilon_\Lambda}(t_{|\arg\Gamma_\Lambda|,\varepsilon_\Lambda}) < \delta.$$

(So, $\left\| \hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda} - 1 \right\| < \delta$.)
As

$$\mathrm{Ad}(U^*_{\Gamma_\Lambda,\varepsilon_\Lambda})(\pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime}) \subseteq \pi_0(\mathcal{A}_{(\Gamma_\Lambda - R_{\Gamma_\Lambda,\varepsilon_\Lambda})_{\varepsilon_\Lambda}})'' \subseteq \pi_0(\mathcal{A}_{\Lambda_{-3\varepsilon_\Lambda}})'' \subseteq \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''$$

and as $\hat{U}_\Lambda \in \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''$, we have $\mathrm{Ad}(\hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda})(\pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime}) \subseteq \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''$.
As $D_\Lambda \subseteq \Gamma_\Lambda^c$, $\pi_0(\mathcal{A}_{D_\Lambda})_{\text{even}} \subseteq \pi_0(\mathcal{A}_{\Gamma_\Lambda^c})_{\text{even}} = \pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^t_{\text{even}} \subseteq \pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime\prime}$, so by taking commutants, $\pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime} \subseteq (\pi_0(\mathcal{A}_{D_\Lambda})_{\text{even}})' = (\pi_0(\mathcal{A}_{D_\Lambda})''_{\text{even}})'$.
$\mathcal{H}$ is separable because it is a GNS Hilbert space of a state on $\mathcal{A}$. Apply Lemma 5.10 of [6] to

- $\mathcal{H}$
- the infinite factors $\mathcal{N} = \pi_0(\mathcal{A}_{D_\Lambda})''_{\text{even}} \subseteq \mathcal{M} = \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''_{\text{even}}$
- the projection $p_{\Gamma_\Lambda} \in \pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime} \subseteq (\pi_0(\mathcal{A}_{D_\Lambda})''_{\text{even}})' = \mathcal{N}'$
- the unitary $u = w_\Lambda := \hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda}$.

(Note that it applies here because we have that $\|w_\Lambda - 1\| < \delta$, and that as $p_{\Gamma_\Lambda} \in \pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime}$ that $\mathrm{Ad}(\hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda})(p_{\Gamma_\Lambda}) \in \mathrm{Ad}(\hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda})(\pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime}) \subseteq \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''$, and so as p_{Γ_Λ} is even, $\mathrm{Ad}(\hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda})(p_{\Gamma_\Lambda}) \in \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''_{\text{even}} = \mathcal{M}$.)

Therefore, there exists an isometry $u_\Lambda \in \mathcal{M} = \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''_{\text{even}}$ such that $u_\Lambda u_\Lambda^* = \mathrm{Ad}(w_\Lambda)(p_{\Gamma_\Lambda})$ for $w_\Lambda := \hat{U}_\Lambda U^*_{\Gamma_\Lambda,\varepsilon_\Lambda}$.

Choose a cone $C_\Lambda \subset (\Lambda_{-2\varepsilon_\Lambda})^c \cap \Lambda$.

Apply Lemma 100 in the case of $\mathcal{N} = (\pi_0(\mathcal{A}_{(\Lambda_{-2\varepsilon_\Lambda})^c})^{t\prime})_{\text{even}}$, $\mathcal{M} = \pi_0(\mathcal{A}_{C_\Lambda})''_{\text{even}}$, $\mathcal{R} = (\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{\text{even}}$, where the unitary is w_Λ and the isometry is u_Λ, noting that

$$u_\Lambda \in \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''_{\text{even}} \subseteq \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})'' \subseteq \pi_0(\mathcal{A}_{(\Lambda_{-2\varepsilon_\Lambda})^c})^{t\prime}$$

and is even so therefore $u_\Lambda \in (\pi_0(\mathcal{A}_{(\Lambda_{-2\varepsilon_\Lambda})^c})^{t\prime})_{\text{even}} = \mathcal{N}$ and that $w_\Lambda^* u_\Lambda u_\Lambda^* w_\Lambda = p_{\Gamma_\Lambda} \in \pi_0(\mathcal{A}_{\Gamma_\Lambda^c})^{t\prime} \subseteq \pi_0(\mathcal{A}_{(\Lambda_{-2\varepsilon_\Lambda})^c})^{t\prime}$ and is even therefore $w_\Lambda^* u_\Lambda u_\Lambda^* w_\Lambda = p_{\Gamma_\Lambda} \in (\pi_0(\mathcal{A}_{(\Lambda_{-2\varepsilon_\Lambda})^c})^{t\prime})_{\text{even}}$. Then, by said lemma, there exists an even unitary $W_\Lambda \in \mathcal{U}(\mathcal{H})$ such that for all $x \in \mathcal{R}' = \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime\prime}$,

$$\mathrm{Ad}(W_\Lambda u_\Lambda^*)(x) = \mathrm{Ad}(u_\Lambda^* w_\Lambda)(x).$$

We now have enough to define our family of isometries. For each cone Λ define

$$Y_\Lambda := V^*_{\sigma,\Gamma_\Lambda} w_\Lambda^* u_\Lambda W_\Lambda.$$

Now we show that they satisfy the properties needed. As each of these Y_Λ is a product of some even unitaries and an even isometry, each Y_Λ is an even isometry.

For any $A \in \mathcal{A}_{\Lambda^c}$,

$$\begin{aligned}\mathrm{Ad}(W_\Lambda^* u_\Lambda^* w_\Lambda V_{\sigma,\Gamma_\Lambda}) \circ \sigma^t(A) &= \mathrm{Ad}(W_\Lambda^*) \circ \mathrm{Ad}(u_\Lambda^* w_\Lambda) \circ \pi_0^t(A) = \mathrm{Ad}(W_\Lambda^*) \circ \mathrm{Ad}(W_\Lambda u_\Lambda^*) \circ \pi_0^t(A) \\ &= u_\Lambda^* \pi_0^t(A) u_\Lambda = u_\Lambda^* u_\Lambda \pi_0^t(A) = \pi_0^t(A).\end{aligned}$$

The first equality is from $V_{\sigma,\Gamma_\Lambda} \in \mathcal{V}_{\sigma,\Gamma_\Lambda}$ and $\Lambda^c \subseteq \Gamma_\Lambda^c$. The second equality is by $\pi_0^t(A) \in \pi_0(\mathcal{A}_{\Lambda^c})^t \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime\prime} \subseteq ((\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{\text{even}})'$ and applying the property that W_Λ was obtained as satisfying. The fourth equality is as $u_\Lambda \in \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})''_{\text{even}} \subseteq \pi_0(\mathcal{A}_{\Lambda_{-2\varepsilon_\Lambda}})'' \subseteq \pi_0(\mathcal{A}_{(\Lambda_{-2\varepsilon_\Lambda})^c})^{t\prime} \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$.

So, we have that for each cone Λ, $\mathrm{Ad}(Y_\Lambda^*) \circ \sigma^t|_{\mathcal{A}_{\Lambda^c}} = \pi_0^t|_{\mathcal{A}_{\Lambda^c}}$.

$Y_\Lambda Y_\Lambda^* = V_{\sigma,\Gamma_\Lambda}^* w_\Lambda^* u_\Lambda u_\Lambda^* w_\Lambda V_{\sigma,\Gamma_\Lambda} = p$. For any two cones Λ_1, Λ_2, $Y_{\Lambda_1}^* Y_{\Lambda_2}$ is a unitary, by

$$\begin{aligned}
(Y_{\Lambda_1}^* Y_{\Lambda_2})(Y_{\Lambda_1}^* Y_{\Lambda_2})^* &= Y_{\Lambda_1}^* Y_{\Lambda_2} Y_{\Lambda_2}^* Y_{\Lambda_1} = Y_{\Lambda_1}^* p Y_{\Lambda_1} \\
&= W_{\Lambda_1}^* u_{\Lambda_1}^* w_{\Lambda_1} V_{\sigma,\Gamma_{\Lambda_1}} p V_{\sigma,\Gamma_{\Lambda_1}}^* w_{\Lambda_1}^* u_{\Lambda_1} W_{\Lambda_1} \\
&= W_{\Lambda_1}^* u_{\Lambda_1}^* \operatorname{Ad}(w_{\Lambda_1})(p_{\Gamma_{\Lambda_1}}) u_{\Lambda_1} W_{\Lambda_1} \\
&= W_{\Lambda_1}^* u_{\Lambda_1}^* (u_{\Lambda_1} u_{\Lambda_1}^*) u_{\Lambda_1} W_{\Lambda_1} \\
&= W_{\Lambda_1}^* (u_{\Lambda_1}^* u_{\Lambda_1})(u_{\Lambda_1}^* u_{\Lambda_1}) W_{\Lambda_1} = 1,
\end{aligned}$$

and identical reasoning shows $(Y_{\Lambda_1}^* Y_{\Lambda_2})^*(Y_{\Lambda_1}^* Y_{\Lambda_2}) = (Y_{\Lambda_2}^* Y_{\Lambda_1})(Y_{\Lambda_2}^* Y_{\Lambda_1})^* = 1$.
Therefore, the family of isometries $(Y_\Lambda)_\Lambda$ satisfies the required properties, and so we have that for $\tau := \operatorname{Ad}(Y_{\Lambda_0}^*) \circ \sigma$ and $U_\tau := \operatorname{Ad}(Y_{\Lambda_0}^*)(U_\sigma)$, that $(\tau, U_\tau) \in \mathcal{O}_{\Lambda_0}$ and that Y_{Λ_0} serves as the desired isometry $v : \tau \to \sigma$.

□

4.3.5. Putting it together. Now we will build up to the definition we chose for a braided strict C^*-tensor supercategory so that we can properly state the main result. There are various choices for how to define the $\mathbb{Z}/2\mathbb{Z}$-graded ("super-") versions of things (see [1]), and we have made a choice suiting the results.
Following [1],

DEFINITION 102. A *supercategory* is a category $\mathcal{C}$ enriched in SVect, the symmetric monoidal category whose objects are super vector spaces (over $\mathbb{C}$) and whose morphisms are even linear maps. That is to say, for $A, B \in \text{ob}(\mathcal{C})$, $\text{Hom}_\mathcal{C}(A, B)$ is a super vector space, and for $A, B, C \in \text{ob}(\mathcal{C})$ the composition map $\circ : \text{Hom}_\mathcal{C}(B, C) \times \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{C}(A, C)$ is bilinear and even.
A *superfunctor* is a SVect-enriched functor $F : \mathcal{C} \to \mathcal{D}$ between two supercategories, i.e. a functor such that for $A, B \in \text{ob}(\mathcal{C})$ the function $F(A, B) : \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{D}(F(A), F(B))$ is an even linear map.

Also following [1],

DEFINITION 103. A *strict monoidal supercategory* is a supercategory $\mathcal{C}$ equipped with an identity object $I \in \text{ob}(\mathcal{C})$ and a function $(- \otimes -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which sends $(A, B) \in \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{C})$ to

$(A \otimes B) \in \mathrm{ob}(\mathcal{C})$ and sends $(f,g) \in \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(C,D)$ to $f \otimes g \in \mathrm{hom}_{\mathcal{C}}(A \otimes C, B \otimes D)$ such that:

(a) $(\mathrm{ob}(\mathcal{C}), I, \otimes)$ is a monoid, in that for any $A, B \in \mathrm{ob}(\mathcal{C})$, $A \otimes B \in \mathrm{ob}(\mathcal{C})$, and for any $A, B, C \in \mathrm{ob}(\mathcal{C})$,

$$(A \otimes B) \otimes C = A \otimes (B \otimes C), \qquad I \otimes A = A = A \otimes I.$$

(b) For any $A, B, C, D \in \mathrm{ob}(\mathcal{C})$,

$$(- \otimes -) : \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(C,D) \to \mathrm{Hom}_{\mathcal{C}}((A \otimes C),(B \otimes D))$$

is a bilinear map, and is even in the sense that the corresponding linear map

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \otimes \mathrm{Hom}_{\mathcal{C}}(C,D) \to \mathrm{Hom}_{\mathcal{C}}((A \otimes C),(B \otimes D))$$

is even.

(c) The *super interchange law* holds : For morphisms $f : A \to C$, $g : B \to D$, $h : C \to E$, and $j : D \to F$ with f and j homogeneous,

$$(h \otimes j) \circ (f \otimes g) = (-1)^{|j||f|}(h \circ f) \otimes (j \circ g)$$

and this extends bilinearly to cover general morphisms.

(d) Associativity on morphisms :

For $f : A \to A', g : B \to B', h : C \to C'$, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$

(e) For $f : A \to B$, $f \otimes \mathrm{id}_I = f = \mathrm{id}_I \otimes f$.

(f) For $A, B \in \mathrm{ob}(\mathcal{C})$, $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B}$

Remark 104. The function $(- \otimes -)$ in the above definition has the same datatype as a bifunctor, but is not actually a bifunctor from the category $\mathcal{C} \times \mathcal{C}$ (because it satisfies the super interchange law rather than the interchange law). If one defines the supercategory $\mathcal{C} \boxtimes \mathcal{C}$, the function $(- \otimes -)$ can be described as a superfunctor with domain $\mathcal{C} \boxtimes \mathcal{C}$.

Definition 105. A *braided strict monoidal supercategory* is a strict monoidal supercategory $(\mathcal{C}, \otimes, I)$ equipped with, for each pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$, an even isomorphism

$$\tau_{A,B} : A \otimes B \to B \otimes A$$

called the braiding morphism, such that:

(a) For all morphisms $f : A \to A'$ and $g : B \to B'$, with f and g homogeneous,

$$\tau_{A',B'} \circ (f \otimes g) = (-1)^{|f||g|}(g \otimes f) \circ \tau_{A,B},$$

and this extends bilinearly to general morphisms.

(b) The hexagon identities hold:

$$\tau_{A,B\otimes C} = (\mathrm{id}_B \otimes \tau_{A,C}) \circ (\tau_{A,B} \otimes \mathrm{id}_C),$$

$$\tau_{A\otimes B,C} = (\tau_{A,C} \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \tau_{B,C}).$$

(Here the associators of the hexagon identities have been omitted, leaving these as commutative triangles rather than hexagons, because, as $(\mathcal{C}, \otimes, I)$ is a *strict* supermonoidal category, the associators are each the identity morphism.)

While the above definitions are given in their strict forms and with a more concrete presentation, they arise naturally from a more conceptual setting: the monoidal product can be viewed as a bifunctor that is a superfunctor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$, and the braiding τ arises as a supernatural isomorphism between this functor and its composition with the symmetry functor on $\mathcal{C} \boxtimes \mathcal{C}$. These structures ensure that the coherence conditions and sign rules follow naturally from the categorical formalism, even though we work here with a "compiled down" version suited for explicit computations. (See [1] Definitions 1.1 and 1.4, for this approach.)

Following Definition 2.1.1 of [5] for the definition of a C^*-category,

Definition 106. A category $\mathcal{C}$ is called a C^*-category if

(a) For $A, B \in \mathrm{ob}(\mathcal{C})$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a Banach space, and for all $A, B, C \in \mathrm{ob}(\mathcal{C})$, the composition map

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

is bilinear, and for $f : A \to B$ and $g : B \to C$, $\|g \circ f\| \leq \|g\| \cdot \|f\|$

(b) It is equipped with an anti-linear contravariant functor $(-)^* : \mathcal{C} \to \mathcal{C}$ (which is grade preserving if $\mathcal{C}$ is enriched in SVect) such that for any $A, B \in \mathrm{ob}(\mathcal{C})$ and any $f : A \to B$,

  (a) $f^{**} = f$

  (b) $\|f^* f\| = \|f\|^2$ (in particular, $\mathrm{End}_{\mathcal{C}}(A)$ is a C^*-algebra)

  (c) $f^* f \in \mathrm{End}_{\mathcal{C}}(A)$ is positive as an element of the C^*-algebra that is $\mathrm{End}_{\mathcal{C}}(A)$.

Combining the definition of a monoidal C^*-category in Definition 2.1.1 of [5], with the above definition of a braided strict monoidal category,

Definition 107. A *braided strict C^*-tensor supercategory* (or, as another name, a *braided strict monoidal C^*-supercategory*) is a supercategory $\mathcal{C}$ equipped with an identity object $I \in \mathrm{ob}(\mathcal{C})$, a function with the datatype of a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a braiding τ, and an anti-linear, grade-preserving, contravariant functor $(-)^* : \mathcal{C} \to \mathcal{C}$ such that $(\mathcal{C}, \otimes, I, \tau)$ is a braided strict monoidal supercategory, $(\mathcal{C}, *)$ is a C^*-category, and the following conditions hold:

(a) The identity object I is simple : $\mathrm{Hom}_{\mathcal{C}}(I, I) = \mathbb{C} \cdot \mathrm{id}_I$

(b) $\mathcal{C}$ has direct sums : For any two objects $A, B \in \mathrm{ob}(\mathcal{C})$, there exists an object $A \oplus B$ and even isometries $\iota_A : A \to A \oplus B, \iota_B : B \to A \oplus B$, such that $\iota_A^* \iota_A = \mathrm{id}_A$, $\iota_B^* \iota_B = \mathrm{id}_B$, and $\iota_A \iota_A^* + \iota_B \iota_B^* = \mathrm{id}_{A \oplus B}$

(c) $\mathcal{C}$ has subobjects: for any object $A \in \mathrm{ob}(\mathcal{C})$, and any even projection $p : A \to A$, there exists an object $B \in \mathrm{ob}(\mathcal{C})$ and an even isometry $\iota_p : B \to A$ such that $\iota_p \iota_p^* = p : A \to A$

(d) The category is small, i.e. the class $\mathrm{ob}(\mathcal{C})$ is a set

(e) The involution $(-)^*$ and $(- \otimes -)$ are compatible, in that for any homogeneous morphisms f, g, $(f \otimes g)^* = (-1)^{|f||g|} f^* \otimes g^*$ (non-homogeneous morphisms addressed by bilinearity of $(- \otimes -)$ and conjugate-linearity of $(-)^*$).

Theorem 108. *Let $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be an irreducible representation equipped with a unitary $U_{\alpha_F} \in \mathcal{U}(\mathcal{H})$ that implements the automorphism α_F of $\mathcal{A}$, i.e. such that $\mathrm{Ad}(U_{\alpha_F}) \circ \pi_0 = \pi_0 \circ \alpha_F$. Suppose that π_0 satisfies approximate twisted Haag duality (Definition 38).*

Suppose also that for all cones Λ, that the von Neumann algebras $\pi_0(\mathcal{A}_\Lambda)''_{even}$ and $(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{even}$ are properly infinite factors.

Let $(\theta,\varphi)\in\mathbb{R}\times(0,\pi)$ *define a forbidden direction and let* $\Lambda_0\in\mathcal{C}(\theta,\varphi)$*. Define:*

$$\mathcal{O}_{\Lambda_0}:=\left\{(\rho,U_\rho)\in\mathcal{O}_0\,\middle|\,\rho|_{\mathcal{A}_{\Lambda_0^c}}=\pi_0|_{\mathcal{A}_{\Lambda_0^c}}\right\},\quad \mathcal{O}_{\Lambda_0,*}:=\{(\rho,U_\rho)\in\mathcal{O}_{\Lambda_0}\,|\,U_\rho=U_{\alpha_F}\}\,.$$

Then, there exists a braided strict C^**-tensor supercategory (Definition 107)* $\mathcal{M}$ *given by the data:*

- *objects :* $\mathcal{O}_{\Lambda_0,*}$
- *morphisms : for any* $(\rho,U_\rho),(\sigma,U_\sigma)\in\mathcal{O}_{\Lambda_0,*}$,

$$\mathrm{Hom}_{\mathcal{M}}((\rho,U_\rho),(\sigma,U_\sigma)):=\left\{R\in\mathcal{B}(\mathcal{H})\,\middle|\,\forall x\in\mathcal{A},\ R\rho^t(x)=\sigma^t(x)R\right\},$$

  as in Definition 44
- *monoidal product : defined on objects as in Definition 66, picking* $D=((\theta,\varphi),\Lambda_0,\{V_{\eta,\Lambda_0}\in\mathcal{V}_{\eta,\Lambda_0}\}_{\eta\in\mathcal{O}_0})$ *such that for* $(\eta,U_\eta)\in\mathcal{O}_{\Lambda_0,*}$ $V_{\eta,\Lambda_0}=1$*, and* $\rho\otimes\sigma:=\rho\circ_D\sigma:=T^D_\rho\circ T^D_\sigma\circ\pi_0$ *and* $U_{\rho\otimes\sigma}:=U_{\alpha_F}$*, and defined on morphisms as The (super)monoidal product on morphisms is defined by Definition 81(Using Lemma 98), so for* $R\in\mathrm{Hom}_{\mathcal{M}}((\rho,U_{\alpha_F}),(\rho',U_{\alpha_F}))$ *and* $S\in\mathrm{Hom}_{\mathcal{M}}((\sigma,U_{\alpha_F}),(\sigma',U_{\alpha_F}))$*,* $R\otimes S:=R\cdot T^D_\rho(S)$
- *identity object :* (π_0,U_{α_F})
- *braiding : morphisms* $\epsilon_+^{(\Lambda_0)}(\rho,\sigma)$ *defined in Definitions 92*
- *the involution endofunctor* $(-)^*$ *: the adjoint operation of* $\mathcal{B}(\mathcal{H})$

Following Proof of Theorem 5.1 of [6]

Proof. As for $(\rho,U_\rho)\in\mathcal{O}_{\Lambda_0,*}$, $U_\rho=U_{\alpha_F}$, we can omit the U_ρ here when for $(\rho,U_\rho)\in\mathcal{O}_{\Lambda_0,*}$. For any $\rho,\sigma\in\mathcal{O}_{\Lambda_0,*}$, $\mathrm{Hom}_{\mathcal{M}}(\rho,\sigma)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. It is a closed subspace by virtue of being the intersection of the kernels of a family of continuous linear maps. Therefore, as it is a closed linear subspace of the Banach space $\mathcal{B}(\mathcal{H})$, it is a Banach space, and inherits the operator norm. By Lemma 76 the even and odd parts of each element of this subspace is also an element of this subspace, so this sub-vector-space is in fact a sub-super-vector-space.

For any $\rho,\sigma,\gamma\in\mathcal{O}_{\Lambda_0,*}$, the map

$$\mathrm{Hom}_{\mathcal{M}}(\sigma,\gamma,)\times\mathrm{Hom}_{\mathcal{M}}(\rho,\sigma)\ni(R,S)\mapsto RS\in\mathrm{Hom}_{\mathcal{M}}(\rho,\gamma)$$

is of course bilinear, and is even in the sense that the induced linear map

$$\mathrm{Hom}_{\mathcal{M}}(\sigma,\gamma)\otimes \mathrm{Hom}_{\mathcal{M}}(\rho,\sigma)\to \mathrm{Hom}_{\mathcal{M}}(\rho,\gamma)$$

is even. So, the category is indeed a supercategory.

As the norm on it is the operator norm, $\|RS\| \leq \|R\|\,\|S\|$.

For any $R \in \mathrm{Hom}_{\mathcal{M}}(\rho,\sigma,)$, $R^* \in \mathrm{Hom}_{\mathcal{M}}(\sigma,\rho)$, and $R^{**}=R$.

For $R \in \mathrm{Hom}_{\mathcal{M}}(\rho,\sigma)$ and $S \in \mathrm{Hom}_{\mathcal{M}}(\rho,\sigma)$ $(RS)^* = S^*R^*$, so $(-)^*$ is a contravariant functor.

$\|R^*R\| = \|R\|^2$ (because this $(-)^*$ is the $(-)^*$ of $\mathcal{B}(\mathcal{H})$, which the norm also comes from).

$\mathrm{Hom}_{\mathcal{M}}(\rho,\rho) = \rho(\mathcal{A})^{t\prime}$ is a C^*-algebra, and R^*R is a positive element of it (it is an element of $\mathrm{Hom}_{\mathcal{M}}(\rho,\rho)$, and it is a positive element of $\mathcal{B}(\mathcal{H})$, so it is a positive element of $\mathrm{Hom}_{\mathcal{M}}(\rho,\rho)$).

$(-)^*$ is anti-linear because it is on the Hermitian adjoint on $\mathcal{B}(\mathcal{H})$ which is anti-linear.

$(-)^*$ preserves the grade of homogeneous elements because $(U_{\alpha_F}xU_{\alpha_F})^* = (U_{\alpha_F}x^*U_{\alpha_F})$, so for $\mathrm{Ad}(U_{\alpha_F})(x) = \pm x$, $(\mathrm{Ad}(U_{\alpha_F})(x^*)) = (\mathrm{Ad}(U_{\alpha_F})(x))^* = (\pm x)^*$.

Therefore, $(\mathcal{M},(-)^*)$ is a C^*-(super)category.

Now to see that $(\mathcal{M},\otimes,(\pi_0,U_{\alpha_F}))$ forms a monoidal supercategory.

To do that, first let us see that $(\mathcal{M},\pi_0,\otimes)$ is a monoid.

First, by Lemma 68, for $\rho,\sigma \in \mathcal{O}_{\Lambda_0,*} \subseteq \mathcal{O}_{\Lambda_0}$, $\rho\otimes\sigma := \rho\circ_D\sigma \in \mathcal{O}_{\Lambda_0,*}$, by Λ_0 being the cone specified in D, so $\mathcal{O}_{\Lambda_0,*}$ is closed under $\otimes$.

Second, in Lemma 73 it is seen that for $\rho,\sigma,\gamma \in \mathcal{O}_{\Lambda_0}$, that $(\rho\circ_D\sigma)\circ_D\gamma = \mathrm{Ad}(U)\circ\rho\circ_D(\sigma\circ_D\gamma)$ for some unitary U, and as $\rho,\sigma,\gamma,\rho\circ_D\sigma,\sigma\circ_D\gamma \in \mathcal{O}_{\Lambda_0,*}$, said unitary, $U = V_{(\rho\circ_D\sigma),\Lambda_0}T^D_\rho(V^*_{(\sigma\circ_D\gamma),\Lambda_0})$, is seen to be 1 due to $V_{\eta,\Lambda_0}=1$ for each $\eta\in\mathcal{O}_{\Lambda_0,*}$ for our choice of D, and so $(\rho\circ_D\sigma)\circ_D\gamma = \rho\circ_D(\sigma\circ_D\gamma)$.

So, our $\otimes$ is associative on the nose for objects.

Thirdly, to see that π_0 is an identity element for this monoid, note that $T^D_{\pi_0} = \mathrm{id}_{\mathcal{B}(\theta,\varphi)}$ (by Definition 56, $T^{(\theta,\varphi),\Lambda_0,1}_{\pi_0,\Lambda} = \mathrm{Ad}(1\cdot 1^*)|_{\pi_0(\mathcal{A}_\Lambda)''}$, for any $\Lambda\in\mathcal{C}(\theta,\varphi)$, and so $T^D_{\pi_0} = \mathrm{id}_{\mathcal{B}(\theta,\varphi)}$).

Recall Definition 81: For $\rho,\rho',\sigma,\sigma' \in \mathcal{O}_{\Lambda_0,*}$, $R\in\mathrm{Hom}_{\mathcal{M}}(\rho,\rho') = (T^D_\rho,T^D_{\rho'})$ and $S:\mathrm{Hom}_{\mathcal{M}}(\sigma,\sigma') = (T^D_\sigma,T^D_{\sigma'})$,

$$R\otimes S := R\cdot T^D_\rho(S).$$

Then, as $R\otimes S : (T^D_\rho\circ T^D_\sigma, T^D_{\rho'}\circ T^D_{\sigma'}) = (\rho\otimes\sigma,\rho'\otimes\sigma')$ we have $(-\otimes-)$ defined on morphisms.

Note that this is bilinear, and that it is even in the same sense that composition is even.

By Lemma 82, $R\otimes S : T^D_\rho\circ T^D_\sigma \to T^D_{\rho'}\circ T^D_{\sigma'}$.

The super interchange law holds by Lemma 83.

To see that $(- \otimes -)$ is associative on morphisms, let $R : \rho \to \rho', S : \sigma \to \sigma', G : \gamma \to \gamma'$ for $\rho, \rho', \sigma, \sigma', \gamma, \gamma' \in \mathcal{O}_{\Lambda_0,*}$. Then,

$$\begin{aligned} R \otimes (S \otimes G) &= R \cdot T_\rho^D(S \otimes G) \\ &= R \cdot T_\rho^D(S \cdot T_\sigma^D(G)) \\ &= R \cdot T_\rho^D(S) \cdot T_\rho^D \circ T_\sigma^D(G) \\ &= (R \cdot T_\rho^D(S)) \cdot T_{\rho\otimes\sigma}(G) \\ &= (R \otimes S) \otimes G, \end{aligned}$$

where the third equality is by T_ρ^D being an algebra endomorphism, and the fourth is by Lemma 72 and the choice of D having $V_{\rho\otimes\sigma,\Lambda_0} = 1$. So, it is associative on morphisms as well as on objects.

For $R : \rho \to \rho'$ and $\mathrm{id}_{\pi_0} = 1 : \pi_0 \to \pi_0$, $\mathrm{id}_{\pi_0} \otimes R = 1 \cdot T_{\pi_0}^D(R) = R$ and $R \otimes \mathrm{id}_{\pi_0} = R \cdot T_\rho^D(1) = R$.

Of course, for $1 = \mathrm{id}_\rho : \rho \to \rho$ and $1 = \mathrm{id}_\sigma : \sigma \to \sigma$, $\mathrm{id}_\rho \otimes \mathrm{id}_\sigma = 1 \cdot T_\rho^D(1) = 1 = \mathrm{id}_{\rho\otimes\sigma}$.

Therefore, $(\mathcal{M}, \otimes, \pi_0)$ is a strict monoidal supercategory (Definition 103).

Now to see that the morphisms $\epsilon_+^{(\Lambda_0)}(\rho, \sigma)$ for $\rho, \sigma \in \mathcal{O}_{\Lambda_0,*}$ (Definition 92) make $(\mathcal{M}, \otimes, \pi_0, \epsilon_+^{(\Lambda_0)})$ into a braided strict monoidal supercategory. Lemma 94 shows that it satisfies

$$\epsilon_+^{(\Lambda_0)}(\rho', \sigma') \cdot (R \otimes S) = (-1)^{|R||S|}(S \otimes R) \cdot \epsilon_+^{(\Lambda_0)}(\rho, \sigma)$$

for $R : \rho \to \rho'$ and $S : \sigma \to \sigma'$. And, by Lemma 97 it satisfies the hexagon identities. Also, as $\epsilon_+^{(\Lambda_0)}(\rho, \sigma) : \rho \otimes \sigma \to \sigma \otimes \rho$ is an even unitary, it is an even isomorphism.

So, $(\mathcal{M}, \otimes, \pi_0, \epsilon_+^{(\Lambda_0)})$ is a braided strict monoidal supercategory.

So, we have that $(\mathcal{M}, (-)^*)$ is a C^*-(super)category, and that $(\mathcal{M}, \otimes, \pi_0, \epsilon_+^{(\Lambda_0)})$ is a braided strict monoidal supercategory, so there are only five things left to check.

First, let us see that the identity object π_0 is simple, i.e. $\mathrm{End}_{\mathcal{M}}(\pi_0) = \mathbb{C}\,\mathrm{id}_{\pi_0}$. For $R \in \mathcal{B}(\mathcal{H})$, $R : \pi_0 \to \pi_0$ iff, for all $x \in \mathcal{A}$, $R \cdot \pi_0^t(x) = \pi_0^t(x) \cdot R$, i.e. iff $R \in \pi_0(\mathcal{A})^{t\prime}$. As π_0 is irreducible, $\pi_0(\mathcal{A})' = \mathbb{C} \cdot 1$. By Lemma 111, as $\mathrm{Ad}(U_{\alpha_F})(\pi_0(\mathcal{A})) = \pi_0(\mathcal{A})$, we have $\pi_0(\mathcal{A})^{t\prime} = (\pi_0(\mathcal{A})')^t$. Therefore, $\pi_0(\mathcal{A})^{t\prime} = (\pi_0(\mathcal{A})')^t = (\mathbb{C}1)^t = \mathbb{C}1$. Therefore, $\mathrm{End}_{\mathcal{M}}(\pi_0) = \mathbb{C}\,\mathrm{id}_{\pi_0}$, i.e. the identity object π_0 is simple.

Second, using the assumption that for every cone Λ, $\pi_0(\mathcal{A}_\Lambda)''_{\text{even}}$ is a properly infinite factor, and therefore a properly infinite von Neumann algebra, by Lemma 99, $\mathcal{M}$ has direct sums, in that for any $(\rho, U_\rho = U_{\alpha_F}), (\sigma, U_\sigma = U_{\alpha_F}) \in \mathcal{O}_{\Lambda_0,*}$, there is an object $(\rho \oplus \sigma, U_{\rho\oplus\sigma} = U_{\alpha_F}) \in \mathcal{O}_{\Lambda_0,*}$ and even isometries $\iota_\rho : \rho \to \rho \oplus \sigma$, $\iota_\sigma : \sigma \to \rho \oplus \sigma$, such that $\iota_\rho^*\iota_\rho = \text{id}_\rho = 1$, $\iota_\sigma^*\iota_\sigma = \text{id}_\sigma = 1$, and $\iota_\rho\iota_\rho^* + \iota_\sigma\iota_\sigma^* = \text{id}_{\rho\oplus\sigma} = 1$.

Third, using the assumption that for every cone Λ, $\pi_0(\mathcal{A}_\Lambda)''_{\text{even}}$ and $(\pi_0(\mathcal{A}_{\Lambda^c})^{t\prime})_{\text{even}}$ are both properly infinite factors, by Lemma 101, for any $\sigma = (\sigma, U_\sigma) \in \mathcal{O}_{\Lambda_0}$ and any projection $p : \sigma \to \sigma$ which satisfies $U_\sigma p = pU_\sigma$, there exists a $(\tau, U_\tau) \in \mathcal{O}_{\Lambda_0}$ and an even isometry $\iota_p : \tau \to \sigma$ such that $\iota_p\iota_p^* = p$. And, because for $(\sigma, U_\sigma) \in \mathcal{O}_{\Lambda_0,*}$ we have that $U_\sigma = U_{\alpha_F}$, for any even projection $p : \sigma \to \sigma$ we have that $U_\sigma p = U_{\alpha_F} p = pU_{\alpha_F} = pU_\sigma$, so the lemma applies, and additionally gives us that $U_\tau = U_{\alpha_F}$, so $(\tau, U_\tau) \in \mathcal{O}_{\Lambda_0,*}$.

Fourth, as each $(\rho, U_\rho = U_{\alpha_F}) \in \mathcal{O}_{\Lambda_0,*}$ is a pair of a representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a unitary $U_\rho \in \mathcal{U}(\mathcal{H})$, $\text{ob}(\mathcal{M}) = \mathcal{O}_{\Lambda_0,*} \subset ((\mathcal{A} \to \mathcal{B}(\mathcal{H})) \times \mathcal{U}(\mathcal{H}))$, and is therefore a set, so the category is small.

Finally, for homogeneous $R : \rho \to \rho'$ and $S : \sigma \to \sigma'$, $(R \otimes S)^* = (-1)^{|R||S|} R^* \otimes S^*$ by Lemma 84.

Therefore, $(\mathcal{M}, \otimes, \pi_0, \epsilon_+^{(\Lambda_0)}, (-)^*)$ is a braided strict C^*-tensor supercategory, as desired. □

4.4. Some delayed proofs

Proposition 109. *For $\rho \in \mathcal{O}_0$ be an* irreducible *grade-preserving representation of $\mathcal{A}$. Then U_ρ is either U_{α_F} or $-U_{\alpha_F}$.*

Proof. Let Λ be a cone. By Lemma 33, for any cone there is an odd local unitary in that cone, so let $B_1 \in \mathcal{A}_{\Lambda^c}$, and be an odd unitary. Let $A_0 \in \mathcal{A}_{\Lambda^c}$, and be even. As $\rho \in \mathcal{O}_0$, there exists $V_{\rho,\Lambda} \in \mathcal{V}_{\rho,\Lambda}$. Then, $\text{Ad}(V_{\rho,\Lambda})(U_\rho\rho(A_0B_1)) = U_{\alpha_F}\pi_0(A_0B_1) = U_{\alpha_F}\pi_0(A_0)\pi_0(B_1)$

$= \pi_0(A_0)U_{\alpha_F}\pi_0(B_1) = \text{Ad}(V_{\rho,\Lambda})(\rho(A_0)) \cdot \text{Ad}(V_{\rho,\Lambda})(U_\rho\rho(B_1)) = \text{Ad}(V_{\rho,\Lambda})(\rho(A_0)U_\rho\rho(B_1))$.

So, $\text{Ad}(V_{\rho,\Lambda})(U_\rho\rho(A_0)\rho(B_1)) = \text{Ad}(V_{\rho,\Lambda})(\rho(A_0)U_\rho\rho(B_1))$,

and so $U_\rho\rho(A_0)\rho(B_1) = \rho(A_0)U_\rho\rho(B_1)$.

As B_1 is invertible, multiplying on the right by $\rho(B_1^{-1})$ we get $U_\rho\rho(A_0) = \rho(A_0)U_\rho$.

Therefore, we have that for all even $A_0 \in \mathcal{A}_{\Lambda^c}$, that U_ρ commutes with $\rho(A_0)$.

And, this holds for any cone Λ.

Therefore, for any even $A_0 \in \mathcal{A}_{loc}$, there will be some cone Λ such that $A_0 \in \mathcal{A}_{\Lambda^c}$. and so we have that U_ρ commutes with $\rho(A_0)$.

So, $U_\rho \in \rho(\mathcal{A}_{loc,\text{even}})'$.

$\mathcal{A}_{loc}$ is dense in $\mathcal{A}$, and $\mathcal{A}_{loc,\text{even}}$ is dense in $\mathcal{A}_{\text{even}}$.

Because ρ is continuous, $(A \in \mathcal{A}) \mapsto [U_\rho, \rho(A)]$ is therefore continuous, and its image on $\mathcal{A}_{loc,\text{even}}$ is 0, and therefore as $\mathcal{A}_{loc,\text{even}}$ is dense in $\mathcal{A}_{\text{even}}$, its image on $\mathcal{A}_{\text{even}}$ is also 0, and so $U_\rho \in \rho(\mathcal{A}_{\text{even}})'$.

Because ρ is irreducible, $\rho(\mathcal{A})$ is dense in $\mathcal{B}(\mathcal{H})$, i.e. $\rho(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$.

Because ρ is grade-preserving, $\rho(\mathcal{A})_{\text{even}} = \rho(\mathcal{A}_{\text{even}})$.

As any limit (with respect to the weak topology) of even operators is even,

$(\rho(\mathcal{A})_{\text{even}})'' = (\rho(\mathcal{A})'')_{\text{even}} = \mathcal{B}(\mathcal{H})_{\text{even}}$.

So, $\rho(\mathcal{A}_{\text{even}})'' = \mathcal{B}(\mathcal{H})_{\text{even}}$.

So, $\rho(\mathcal{A}_{\text{even}})' = \rho(\mathcal{A}_{\text{even}})''' = (\rho(\mathcal{A}_{\text{even}})'')' = (\mathcal{B}(\mathcal{H})_{\text{even}})' = \mathbb{C}I_{\mathcal{H}_{\text{even}}} \oplus \mathbb{C}I_{\mathcal{H}_{\text{odd}}}$.

So, $U_\rho \in \mathbb{C}I_{\mathcal{H}_{\text{even}}} \oplus \mathbb{C}I_{\mathcal{H}_{\text{odd}}}$.

As $U_\rho^2 = 1$, therefore U_ρ is of the form $\pm I_{\mathcal{H}_{\text{even}}} + \pm I_{\mathcal{H}_{\text{odd}}}$.

I.e. $U_\rho \in \{1, -1, U_{\alpha_F}, -U_{\alpha_F}\}$.

Now to rule out the first two of these.

For any cone Λ and any odd $A_1 \in \mathcal{A}_{\Lambda^c}$, and any odd invertible $B_1 \in \mathcal{A}_{\Lambda^c}$

$$\begin{aligned}\operatorname{Ad}(V_{\rho,\Lambda})(\rho(A_1B_1)) &= \pi_0(A_1B_1) = \pi_0(A_1)\pi_0(B_1)\\ &= -U_{\alpha_F}\pi_0(A_1)U_{\alpha_F}\pi_0(B_1) = -\operatorname{Ad}(V_{\rho,\Lambda})(U_\rho\rho(A_1)U_\rho\rho(B_1))\end{aligned}$$

so

$$\rho(A_1)\rho(B_1) = -U_\rho\rho(A_1)U_\rho\rho(B_1)\,.$$

As B_1 is invertible, we then get $\rho(A_1) = -U_\rho\rho(A_1)U_\rho$.

This cannot be true if U_ρ is ± 1, and so it must be either U_{α_F} or $-U_{\alpha_F}$, as desired. □

It can be seen that both values are possibilities, because for any $\rho \in \mathcal{O}_0$, it can be seen that $U_{\rho\circ\alpha_F} = -U_\rho$.

LEMMA 110. *Let $\rho \in \mathcal{O}_0$, and suppose $U_\rho = \pm U_{\alpha_F}$. Then, for all cones Λ, all $V_{\rho,\Lambda} \in \mathcal{V}_{\rho,\Lambda}$, and all $A \in \mathcal{A}_{\Lambda^c}$, we have:*

(a) *If $U_\rho = U_{\alpha_F}$, then $\operatorname{Ad}(V_{\rho,\Lambda}) \circ \rho(A) = \pi_0(A)$.*

(b) *If* $U_\rho = -U_{\alpha_F}$, *then* $\operatorname{Ad}(V_{\rho,\Lambda}) \circ \rho(A) = \pi_0 \circ \alpha_F(A)$.

Proof. Let Λ be a cone, $V_{\rho,\Lambda} \in \mathcal{V}_{\rho,\Lambda}$, and $A \in \mathcal{A}_{\Lambda^c}$, with $A = A_0 + A_1$.

$$\begin{aligned}\operatorname{Ad}(V_{\rho,\Lambda})(\rho(A_0) + \rho(A_1)) &= \operatorname{Ad}(V_{\rho,\Lambda})(\rho(A_0) + U_\rho U_\rho \rho(A_1)) \\ &= \pi_0(A_0) + \operatorname{Ad}(V_{\rho,\Lambda})(U_\rho) \operatorname{Ad}(V_{\rho,\Lambda})(U_\rho \rho(A_1)) \\ &= \pi_0(A_0) + U_\rho \cdot (U_{\alpha_F} \pi_0(A_1)) \\ &= \pi_0(A_0) + (U_\rho U_{\alpha_F}) \pi_0(A_1)\end{aligned}$$

The third equality uses that $U_\rho = \pm U_{\alpha_F} \in (\mathcal{B}(\mathcal{H})_{\text{even}})'$ and that $V_{\rho,\Lambda}$ is even, to conclude that $\operatorname{Ad}(V_{\rho,\Lambda})(U_\rho) = U_\rho$. The last expression in this chain of equations is equal to $\pi_0(A_0 + A_1) = \pi_0(A)$ if $U_\rho = U_{\alpha_F}$, and is $\pi_0(A_0 - A_1) = \pi_0 \circ \alpha_F(A)$ if $U_\rho = -U_{\alpha_F}$. Therefore, the we have the desired conclusion, that if $U_\rho = U_{\alpha_F}$ that $\operatorname{Ad}(V_{\rho,\Lambda}) \circ \rho(A) = \pi_0(A)$, and if $U_\rho = -U_{\alpha_F}$ then $\operatorname{Ad}(V_{\rho,\Lambda}) \circ \rho(A) = \pi_0 \circ \alpha_F(A)$. □

Lemma 111. *Let* $\mathfrak{A}$ *be a* C^* *subalgebra of* $\mathcal{B}(\mathcal{H})$*. The following are equivalent:*

(1) $\operatorname{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$

(2) *For* $X = X_0 + X_1 \in \mathcal{B}(\mathcal{H})$, $X \in \mathfrak{A}$ *if and only if* $X_0, X_1 \in \mathfrak{A}$

(3) $\operatorname{Ad}(U_{\alpha_F})(\mathfrak{A}^t) = \mathfrak{A}^t$

In addition, they imply:

(a) $\operatorname{Ad}(U_{\alpha_F})(\mathfrak{A}') = \mathfrak{A}'$

(b) $\mathfrak{A}^{t\prime} = \mathfrak{A}^{\prime t}$

Proof. For (1) $\implies$ (2):
Suppose $\operatorname{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$.
For $X = X_0 + X_1$, if $X \in \mathfrak{A}$, then $\operatorname{Ad}(U_{\alpha_F})(X_0 + X_1) = X_0 - X_1 \in \mathfrak{A}$, so by $\mathfrak{A}$ being a linear subspace of $\mathcal{B}(\mathcal{H})$, $\frac{1}{2}(X + \operatorname{Ad}(U_{\alpha_F})(X)) = \frac{1}{2}((X_0 + X_1) + (X_0 - X_1)) = X_0 \in \mathfrak{A}$ and $\frac{1}{2}(X - \operatorname{Ad}(U_{\alpha_F})(X)) = \frac{1}{2}((X_0 + X_1) - (X_0 - X_1)) = X_1 \in \mathfrak{A}$. Conversely, if $X_0, X_1 \in \mathfrak{A}$, then of course $X = X_0 + X_1 \in \mathfrak{A}$.
For (2) $\implies$ (1):
Suppose that for all $X = X_0 + X_1 \in \mathcal{B}(\mathcal{H})$ that $X \in \mathfrak{A}$ iff $X_0, X_1 \in \mathfrak{A}$. Then, for $X = X_0 + X_1 \in \mathfrak{A}$, we have that $X_0, X_1 \in \mathfrak{A}$, and therefore, as $\mathfrak{A}$ is a vector space, $X_0 - X_1 \in \mathfrak{A}$. So, $X_0 + X_1 \in \mathfrak{A}$ iff

$X_0 - X_1 \in \mathfrak{A}$. I.e. $X \in \mathfrak{A}$ iff $\mathrm{Ad}(U_{\alpha_F})(X) \in \mathfrak{A}$. So, $X \in \mathfrak{A}$ iff $X \in \mathrm{Ad}(U^*_{\alpha_F})(\mathfrak{A}) = \mathrm{Ad}(U_{\alpha_F})(\mathfrak{A})$. i.e. $\mathfrak{A} = \mathrm{Ad}(U_{\alpha_F})(\mathfrak{A})$.

For (1) $\implies$ (3):

Suppose $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$. For $X = X_0 + X_1 \in \mathcal{B}(\mathcal{H})$, $X_0 + U_{\alpha_F}X_1 \in \mathfrak{A}^t$ iff $X_0 + X_1 \in \mathfrak{A}$. $X_0 + U_{\alpha_F}X_1 \in \mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^t)$ iff $\mathrm{Ad}(U^*_{\alpha_F})(X_0 + U_{\alpha_F}X_1) = \mathrm{Ad}(U_{\alpha_F})(X_0 + U_{\alpha_F}X_1) \in \mathfrak{A}^t$. Of course, $\mathrm{Ad}(U_{\alpha_F})(X_0 + U_{\alpha_F}X_1) = X_0 - U_{\alpha_F}X_1$. So $X_0 + U_{\alpha_F}X_1 \in \mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^t)$ iff $X_0 - U_{\alpha_F}X_1 \in \mathfrak{A}^t$ iff $X_0 - X_1 \in \mathfrak{A}$. And, as $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$, $X_0 - X_1 \in \mathfrak{A}$ iff $X_0 + X_1 \in \mathfrak{A}$. So, by $X_0 + U_{\alpha_F}X_1 \in \mathfrak{A}^t$ iff $X_0 + X_1 \in \mathfrak{A}$, $X_0 + U_{\alpha_F}X_1 \in \mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^t)$ iff $X_0 + U_{\alpha_F}X_1 \in \mathfrak{A}^t$, i.e. $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^t) = \mathfrak{A}^t$.

That (3) $\implies$ (1) follows from applying (1) $\implies$ (3) to the case of $\mathfrak{A}^t$ in place of $\mathfrak{A}$, to get that $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^t) = \mathfrak{A}^t$ implies that $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^{tt}) = \mathfrak{A}^{tt}$, and applying the fact that $\mathfrak{A}^{tt} = \mathfrak{A}$.

For (1) $\implies$ (a): Suppose $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$. Then $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}') = (\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}))' = \mathfrak{A}'$.

For (1) $\implies$ (b):

Suppose $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}) = \mathfrak{A}$. By (1) $\implies$ (a) and (1) $\implies$ (3), we have that $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}'^t) = \mathfrak{A}'^t$ and that $\mathrm{Ad}(U_{\alpha_F})(\mathfrak{A}^{t\prime}) = \mathfrak{A}^{t\prime}$.

$$\begin{aligned}
X = X_0 + X_1 \in \mathfrak{A}'^t &\iff X_0, X_1 \in \mathfrak{A}'^t \\
&\iff X_0, U_{\alpha_F}X_1 \in \mathfrak{A}' \\
&\iff \forall Y = Y_0 + Y_1 \in \mathfrak{A}, [X_0, Y] = 0 = [U_{\alpha_F}X_1, Y] \\
&\iff \forall Y = Y_0 + Y_1 \in \mathfrak{A}, (\\
&\quad [X_0, Y_0] = [X_0, Y_1] = [U_{\alpha_F}X_1, Y_0] = [U_{\alpha_F}X_1, Y_1] = 0) \\
&\iff \forall Y = Y_0 + Y_1 \in \mathfrak{A}, (\\
&\quad [X_0, Y_0] = [X_0, Y_1] = [X_1, Y_0] = \{X_1, Y_1\} = 0) \\
&\iff \forall Y = Y_0 + Y_1 \in \mathfrak{A}, (\\
&\quad [X_0, Y_0] = [X_0, U_{\alpha_F}Y_1] = [X_1, Y_0] = [X_1, U_{\alpha_F}Y_1] = 0)
\end{aligned}$$

$$\iff \forall Y = Y_0 + Y_1 \in \mathfrak{A}, ($$

$$[X_0, Y_0 + U_{\alpha_F} Y_1] = 0 = [X_1, Y_0 + U_{\alpha_F} Y_1])$$

$$\iff \forall Y = Y_0 + U_{\alpha_F} Y_1 \in \mathfrak{A}^t, ($$

$$[X_0, Y_0 + U_{\alpha_F} Y_1] = 0 = [X_1, Y_0 + U_{\alpha_F} Y_1])$$

$$\iff X_0, X_1 \in \mathfrak{A}^{t\prime}$$

$$\iff X_0 + X_1 \in \mathfrak{A}^{t\prime}.$$

So, $\mathfrak{A}'^t = \mathfrak{A}^{t\prime}$, as desired. □

4.5. U_{α_F} is not in $\mathcal{B}(\theta, \varphi)$

LEMMA 112. *(i) For any homogeneous $A \in \mathcal{B}(\mathcal{H})$, $\|A\| = \max(\|A|_{\mathcal{H}_{even}}\|, \|A|_{\mathcal{H}_{odd}}\|)$ and there is a sequence $(v_i)_{i\in\mathbb{N}}$ of vectors in $\mathcal{H}$ which are either all in $\mathcal{H}_{even}$ or are all in $\mathcal{H}_{odd}$, and which are all unit vectors, such that $\lim_{i\to\infty} \|Av_i\| = \|A\|$.*

(ii) For any $B = B_0 + B_1 \in \mathcal{B}(\mathcal{H})$, $\|B\| \geq \max(\|B_0\|, \|B_1\|)$.

PROOF. For part (i):

Let $A \in \mathcal{B}(\mathcal{H})$ be homogeneous (either even or odd). Let $A|_{\mathcal{H}_{\text{even}}} : \mathcal{H}_{\text{even}} \to \mathcal{H}$ and $A|_{\mathcal{H}_{\text{odd}}} : \mathcal{H}_{\text{odd}} \to \mathcal{H}$ be the restrictions of A to the domains of $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$. $\|A|_{\mathcal{H}_{\text{even}}}\| = \sup_{v_0 \in \mathcal{H}_{\text{even}}, \|v_0\|=1} \|Av_0\|$ and $\|A|_{\mathcal{H}_{\text{odd}}}\| = \sup_{v_1 \in \mathcal{H}_{\text{odd}}, \|v_1\|=1} \|Av_1\|$. For $v = v_0 + v_1 \in \mathcal{H}$ with $\|v\| = 1$, $\|v_0\|^2 + \|v_1\|^2 = 1$, and $\|A(v_0 + v_1)\|^2 = \|Av_0\|^2 + \|Av_1\|^2$ (because A is homogeneous, and therefore Av_0 and Av_1 have opposite grades, and are therefore orthogonal). So. $\|A(v_0 + v_1)\|^2 = \|Av_0\|^2 + \|Av_1\|^2 \leq \|A|_{\mathcal{H}_{\text{even}}}\|^2 \|v_0\|^2 + \|A|_{\mathcal{H}_{\text{odd}}}\|^2 \|v_1\|^2$. Therefore, $\|A\| = \max(\|A|_{\mathcal{H}_{\text{even}}}\|, \|A|_{\mathcal{H}_{\text{odd}}}\|)$, and whichever of the two is larger, a choice of a sequence of unit vectors $(v_i)_{i\in\mathbb{N}}$ in $\mathcal{H}$ such that $\|Av_i\|$ tends to $\|A\|$, can be chosen such that vectors in the sequence are either all even or are all odd.

For part (ii):

Now, let $B = B_0 + B_1 \in \mathcal{B}(\mathcal{H})$. Consider whichever of $\|B_0\|, \|B_1\|$ is larger, or if they are equal pick one of them arbitrarily. Let $s \in \{0, 1\}$ be the subscript associated with whichever one is chosen. Then, as in part (i) of this lemma, let $(v_i)_{i\in\mathbb{N}}$ be a sequence of homogeneous unit vectors in $\mathcal{H}$ such that $\|B_s v_i\| \to \|B_s\|$. Then, $\|(B_0 + B_1)v_i\|^2 = \|B_s v_i\|^2 + \|B_{(1-s)} v_i\|^2$ (again because $B_s v_i$ and

$B_{(1-s)}v_i$ are orthogonal). $\|(B_0 + B_1)v_i\| \leq \|B_0 + B_1\| \|v_i\| = \|B_0 + B_1\|$. So,

$$\begin{aligned}\|B_0 + B_1\|^2 &\geq \|(B_0 + B_1)v_i\|^2 \\ &= \|B_s v_i\|^2 + \left\|B_{(1-s)}v_i\right\|^2 \\ &\geq \|B_s v_i\|^2 ,\end{aligned}$$

and $\|B_s v_i\| \to \|B_s\| = \max(\|B_0\|, \|B_1\|)$ as $i \to \infty$, so $\|B_0 + B_1\| \geq \max(\|B_0\|, \|B_1\|)$, as desired.

□

LEMMA 113. *Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a grade-preserving representation of $\mathcal{A}$. Let $U_{\alpha_F} = 1_{\mathcal{H}_{even}} - 1_{\mathcal{H}_{odd}} \in \mathcal{U}(\mathcal{H})$ (as usual).*
Then, for any cone Λ and any $A \in \pi(\mathcal{A}_\Lambda)''$, $\|U_{\alpha_F} - A\| \geq 1$.

PROOF. Let Λ be a cone and let $A = A_0 + A_1 \in \pi(\mathcal{A}_\Lambda)''$. As U_{α_F} is even, we have that $A - U_{\alpha_F} = (A - U_{\alpha_F})_0 + (A - U_{\alpha_F})_1 = (A_0 - U_{\alpha_F}) + A_1$. So, by Lemma 112,

$$\|A - U_{\alpha_F}\| \geq \max(\|(A - U_{\alpha_F})_0\|, \|(A - U_{\alpha_F})_1\|) \geq \|A_0 - U_{\alpha_F}\| .$$

Therefore, it will suffice to show that $\|A_0 - U_{\alpha_F}\| \geq 1$.
$A_0 \in \pi(\mathcal{A}_\Lambda)''_{\text{even}}$. Because for every $B \in \mathcal{B}(\mathcal{H})$, $\|[A_0 - U_{\alpha_F}, B]\| \leq 2\|A_0 - U_{\alpha_F}\| \|B\|$, for $B \neq 0$, $\|A_0 - U_{\alpha_F}\| \geq \frac{\|[A_0 - U_{\alpha_F}, B]\|}{2\|B\|}$. So, it suffices to find some $B \in \mathcal{B}(\mathcal{H})$ such that $\|[A_0 - U_{\alpha_F}, B]\| \geq 2\|B\|$.
Pick B to be any non-zero odd $B \in \pi(\mathcal{A}_{\Lambda^c})_{\text{odd}}$. (There exists such a B in any cone by Lemma 33.) By the triangle inequality, $\|[A_0 - U_{\alpha_F}, B]\| \geq \|[U_{\alpha_F}, B]\| - \|[A_0, B]\|$. And, because B is odd, we have $\|[U_{\alpha_F}, B]\| = \|U_{\alpha_F}B - BU_{\alpha_F}\| = \|2U_{\alpha_F}B\| = 2\|B\|$. So, as B is odd and non-zero,

$$\begin{aligned}\|A_0 - U_{\alpha_F}\| &\geq \frac{\|[A_0 - U_{\alpha_F}, B]\|}{2\|B\|} \\ &\geq \frac{\|[U_{\alpha_F}, B]\| - \|[A_0, B]\|}{2\|B\|} \\ &= \frac{2\|B\| - \|[A_0, B]\|}{2\|B\|} = 1 - \frac{\|[A_0, B]\|}{2\|B\|}.\end{aligned}$$

By twisted locality, $\pi(\mathcal{A}_\Lambda)'' \subseteq \pi(\mathcal{A}_{\Lambda^c})^{t\prime}$. Now, as $B \in \pi(\mathcal{A}_{\Lambda^c})_{\text{odd}}$, therefore $U_{\alpha_F}B \in \pi(\mathcal{A}_{\Lambda^c})^t$, and as $A_0 \in \pi(\mathcal{A}_\Lambda)'' \subseteq \pi_0(\mathcal{A}_{\Lambda^c})^{t\prime}$, $[A_0, U_{\alpha_F}B] = 0$. As A_0 is even, then $[A_0, U_{\alpha_F}B] = U_{\alpha_F}[A_0, B]$, so $[A_0, B] = 0$.

So, $\|A_0 - U_{\alpha_F}\| \geq 1 - \frac{\|[A_0,B]\|}{2\|B\|} = 1$, and so, as $\|A - U_{\alpha_F}\| \geq \|A_0 - U_{\alpha_F}\|$, we get $\|A - U_{\alpha_F}\| \geq 1$, as desired. □

LEMMA 114. *For any choice of forbidden direction (θ, φ), $U_{\alpha_F} \notin \mathcal{B}(\theta, \varphi)$.*

PROOF. Let (θ, φ) name a forbidden direction. If U_{α_F} were in $\mathcal{B}(\theta, \varphi) = \overline{\bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_\Lambda)''}^{\|\cdot\|}$, there would be a sequence $(A_n)_{n\in\mathbb{N}}$ where each $A_n \in \bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_\Lambda)''$, and such that it satisfies $\lim_{n\to\infty} \|A_n - U_{\alpha_F}\| = 0$. For each $A_n \in \bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi_0(\mathcal{A}_\Lambda)''$, there exists a cone $\Lambda_n \in \mathcal{C}(\theta, \varphi)$ such that $A_n \in \pi_0(\mathcal{A}_{\Lambda_n})''$. By Lemma 113, as $A_n \in \pi_0(\mathcal{A}_{\Lambda_n})''$, $\|A_n - U_{\alpha_F}\| \geq 1$. Therefore, $\|A_n - U_{\alpha_F}\|$ does not converge to 0. So, as no sequence $(A_n)_{n\in\mathbb{N}}$ where each $A_n \in \bigcup_{\Lambda \in \mathcal{C}(\theta,\varphi)} \pi(\mathcal{A}_\Lambda)''$ has $\|A_n - U_{\alpha_F}\|$ converge to 0, therefore $U_{\alpha_F} \notin \mathcal{B}(\theta, \varphi)$, as desired.

□

4.6. On combining with the constructions in the symmetry chapter

We now briefly mention how the construction in this chapter and those in Chapter 3 combine when the compact abelian group G has the $\mathbb{Z}/2\mathbb{Z}$ corresponding to action by α_F, as a subgroup.

Let G be a compact abelian group equipped with an injective inclusion of $\mathbb{Z}/2\mathbb{Z}$ into G. Let $F \in G$ be the image in G of the non-identity element of $\mathbb{Z}/2\mathbb{Z}$. Through Pontryagin duality, this inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow G$ induces a surjection $\widehat{G} \twoheadrightarrow \widehat{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$.

If, in addition to the on-site unitary $\mathbb{Z}/2\mathbb{Z}$ action $\{U_{\alpha_F,\{x\}}\}_{x\in\Gamma}$, there is an a system of on-site unitary G-actions (Definition 4) $((g \in G) \mapsto (U_{\{x\},g} \in \mathcal{U}(\mathcal{H}_{\{x\}})))_{x\in\Gamma}$ such that $U_{\{x\},F} = U_{\alpha_F,\{x\}}$ for all $x \in \Gamma$, then the constructions and definitions in this chapter and in Chapter 3, are compatible.

Because G is abelian, $U_{\{x\},g}$ commutes with $U_{\{x\},F} = U_{\alpha_F,\{x\}}$, and so $U_{\{x\},g}$ is even. Any G-covariant representation $(\rho, U^{(\rho)})$ then has $(\rho, U_\rho = U^{(\rho)}_F)$ as a $\mathbb{Z}/2\mathbb{Z}$-covariant representation. And, again by G being abelian, $U^{(\rho)}_g$ commutes with $U^{(\rho)}_F = U_\rho$, so $U^{(\rho)}_g$ is even in that sense (and if $U_\rho = U_{\alpha_F}$ then it is also even in terms of commuting with U_{α_F}). For two G-covariant representations $(\rho, U^{(\rho)}), (\sigma, U^{(\sigma)})$, if $U_\rho = U_{\alpha_F}$ and $U_\sigma = U_{\alpha_F}$, then any G-covariant map from $(\rho, U^{(\rho)})$ to $(\sigma, U^{(\sigma)})$ will also be even.

The definitions Definition 10 and Definition 42 can then be combined straightforwardly as:

Definition 115. With our $\mathcal{A}$ and the on-site action $g \mapsto \alpha_g$, and a G-covariant representation $(\mathcal{H}_\pi, \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi), U^{(\pi)}_\bullet : G \to \mathcal{U}(\mathcal{H}_\pi))$ to serve as the reference representation , another G-covariant representation $(\mathcal{H}_\rho, \rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho), U^{(\rho)}_\bullet : G \to \mathcal{U}(\mathcal{H}_\rho))$ satisfies the G-equivariant version of the superselection criterion for systems with fermionic degrees of freedom with respect to $(\mathcal{H}_\pi, \pi, U^{(\pi)})$ if for all cones Λ, there exists a unitary $V_{\rho,\Lambda} : \mathcal{H}_\rho \to \mathcal{H}_\pi$ that is a G-equivariant map (i.e. such that for all $g \in G$, $V_{\rho,\Lambda} U^{(\rho)}_g = U^{(\pi)}_g V_{\rho,\Lambda}$) and such that:
for all $A_0 \in \mathcal{A}_{\Lambda^c,\text{even}}$, $\text{Ad}(V_{\rho,\Lambda}) \circ \rho(A_0) = \pi_0(A_0)$,
and for all $A_1 \in \mathcal{A}_{\Lambda^c,\text{odd}}$, $\text{Ad}(V_{\rho,\Lambda})(U^{(\rho)}_F \rho(A_1)) = U^{(\pi)}_F \pi_0(A_1)$.

In the case that $\mathcal{H}_\rho = \mathcal{H}_\pi$ and $U^{(\rho)}_F = U^{(\pi)}_F = U_{\alpha_F}$, anything representation satisfying this definition will satisfy both Definition 10 and Definition 42 (when the former is interpreted to apply in this setting). That all the results in this chapter continues to go through is immediate (under the assumptions of π_0 irreducible, $U^{(\pi_0)}_F = U_{\alpha_F}$, and π_0 satisfying approximate twisted Haag duality) as going from satisfying Definition 42 to satisfying this definition, only adds additional assumptions. That everything in Chapter 3 still works out is also straightforward. The $\widehat{G}$-gradings on algebras and vectors spaces are then refinements of the grading into even and odd parts, in accordance with the surjection $\widehat{G} \twoheadrightarrow \widehat{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$.

CHAPTER 5

Tensor category describing anyons in the quantum Hall effect and quantization of conductance

Tensor category describing anyons in the quantum Hall effect and quantization of conductance

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Dedicated to the memory of Professor Huzihiro Araki

Abstract. In this study, we examine the quantization of Hall conductance in an infinite plane geometry. We consider a microscopic charge-conserving system with a pure, gapped infinite-volume ground state. While Hall conductance is well-defined in this scenario, existing proofs of its quantization have relied on assumptions of either weak interactions, or properties of finite volume ground state spaces, or invertibility. Here, we assume that the conditions necessary to construct the braided C^*-tensor category (aka braided monoidal C^*-category) which describes anyonic excitations are satisfied, and we demonstrate that the Hall conductance is rational if the tensor category is finite.

5.1. Introduction

For an effectively two-dimensional system, such as a metal plate or a single graphene layer, the applied electric field and the induced current are two-component vectors. According to Ohm's law, for small fields, the current is proportional to the applied field. The matrix that relates them is called the conductance matrix. In an insulator, the current can only flow in the direction transversal to the applied field. The corresponding conductance matrix is antisymmetric, and Ohm's law takes the form

$$\vec{J} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \vec{V} \tag{5.1}$$

where we call the off-diagonal conductance κ the Hall conductance.

The quantum Hall effect refers to the behaviour of κ at low temperatures. As observed by Kitzling [29] and Tsui, Störmer and Gossard [45], whenever the material is insulating, i.e., the conductance matrix is as in (5.1), the Hall conductance is a fractional multiple of a universal constant.[1] The effect is called integer quantum Hall effect if the Hall conductance is a whole number and fractional quantum Hall effect if the Hall conductance is a non-integer rational number.

The integer quantum Hall effect is well modelled by non-interacting electrons in disordered media. The fact that κ is integer-valued in this case is now reasonably well understood, and it is beyond the scope of this article to review the extensive body of literature on this topic. Let us mention that integer quantization remains true in the case of weak interactions [21] and under the additional assumption that the ground state is invertible [25]. As a consequence, electron-electron interactions must be included to obtain a non-integer Hall conductance, which introduces significant analytical challenges. Consequently, the fractional quantum Hall effect is mathematically much less understood. A microscopic framework for a finite number of interacting electrons was already developed by Avron and Seiler in [3], resulting in a possibly rational Hall conductance [26]. A topological field theory of quantum Hall fluids in the bulk, which yields fractional quantization and anyonic excitations, was developed in the early 90's by Fröhlich and collaborators, [18, 20] and again later [19]. An interacting microscopic framework with a well-defined thermodynamic limit was only provided twenty years later in the work of Hastings and Michalakis [24].

[1] We will use units in which this constant is equal to $(2\pi)^{-1}$, and consequently, $2\pi\kappa$ is a rational number.

The setting of Hastings and Michalakis and of subsequent works [6,7,35] involves a gapped Hamiltonian for interacting particles with a $U(1)$ symmetry on a finite torus of linear size L. Assuming that the Hamiltonian has p locally indistinguishable ground states (along with some further technical assumptions), it is proved that

$$2\pi\kappa = \frac{q}{p} + O(L^{-\infty}),$$

i.e. there exists $q \in \mathbb{Z}$ such that $|2\pi\kappa - q/p|$ vanishes faster than any inverse power of L as $L \to \infty$. This implies, see [6], quantization of conductance in the plane, provided we assume that the ground state in the plane is a limit of ground states of embedded tori. Since they are locally indistinguishable it does not matter in the limit which torus ground states are used. This plausible assumption, often referred to as LTQO for Local Topological Quantum Order and introduced in [13, 34], is likely satisfied in all standard quantum Hall models (in fact, it was forseen already in [46]) but it is currently difficult to prove, see however [32] for recent progress in this direction.

In this article, we will show that Hall conductance is quantized in the infinite plane geometry without assuming LTQO. We want to avoid this assumption not due to the lack of proof – we will anyway have to assume analytical properties we can't prove in any concrete model – but because not having it leads to an intriguing intellectual puzzle: What replaces the ground states degeneracy on the finite torus in the denominator p of the quantum Hall conductance fraction? We will show here that p is upper bounded by the rank of the braided C^*-tensor category associated with the ground state [41], which describes the anyonic excitations in the system. A parallel approach was taken in [25, 43], where the infinite volume assumption is the invertibility of the state.

The connection between rational Hall conductance and the properties of low-energy excitations was first described in the works of Laughlin [30, 31], and Arovas, Schrieffer, Wilczek [1]. Laughlin demonstrated that insertion of a 2π flux produces an excitation with a fractional charge $2\pi\kappa$ at the point of insertion. Arovas, Schrieffer, and Wilczek then showed that if a second excitation is adiabatically moved around the first, it acquires phase $e^{i(2\pi)^2\kappa}$. This means that the excitation is an Abelian anyon. In a finite volume setting that is very close to the present one, this was proved in [8], and was extended to the infinite volume in [25]. The connection exemplifies the interplay between macroscopic properties of a system, such as Hall conductance, and its microscopic properties, like the statistics of elementary excitations.

In this work, we use the theory developed by Doplicher, Haag and Roberts [16, 22] for relativistic quantum field theories, recently adapted to lattice systems [37], to describe anyon excitations. See the review [33] for other approaches to describing anyons. The DHR approach uses a superselection criterion to define excitation sectors, and proceeds to show that there is a natural braided C^*-tensor category structure associated with these sectors. In particular, physical elementary excitations correspond to objects in this category, and the physical braiding of two excitations corresponds to the braiding structure ϵ in the category. A complete mathematical setting in the context of quantum lattice systems was first described by Ogata [41], and we will use this particular framework here.
As mentioned above, the way how to construct Abelian anyons in fractional quantum Hall effect was introduced in [8] and later expanded on and used to prove quantization for invertible systems in [25]. Neither of these works construct the anyons as objects of a braided C^*-tensor category. Firstly no exact framework existed at that time, and secondly (speaking for authors of [8]) it seemed at the time that technical details associated with the precise construction might obscure the relatively simple idea behind the construction. We now feel that this has changed and that there is a need for uniform setting and precise definitions. The main technical part of this work, see Section 5.5, is the construction of some objects in the braided C^*-tensor category $\mathcal{M}$ associated with the ground state. Echoing [1] and [18], the braiding properties of these objects will be connected to the Hall conductance. In Section 5.6, we then prove that under the assumption that there is finite number of superselection sectors, Hall conductance κ is indeed a rational number.

5.2. Setting and results

We follow the setting and notation of [41], which expands on the usual framework of 2-dimensional lattice spin systems. We consider a lattice $\mathbb{Z}^2$ and to each point $x \in \mathbb{Z}^2$ we associate an algebra $\mathcal{A}_{\{x\}}$ isomorphic to the algebra of $d \times d$ matrices for some fixed $d > 1$. For a finite subset Z of $\mathbb{Z}^2$ we define $\mathcal{A}_Z = \otimes_{x \in Z} \mathcal{A}_{\{x\}}$. For $Z_1 \subset Z_2$, the algebra $\mathcal{A}_{Z_1}$ is canonically embedded in $\mathcal{A}_{Z_2}$ by tensoring operators in $\mathcal{A}_{Z_1}$ with the identity. For infinite $Z \subset \mathbb{Z}^2$, the algebra $\mathcal{A}_Z$ is defined as an inductive limit of algebras associated with finite subsets of Z. We denote $\mathcal{A} = \mathcal{A}_{\mathbb{Z}^2}$. For each $Z \subset \mathbb{Z}^2$, we fix the conditional expectation $\mathbb{E}_Z : \mathcal{A} \to \mathcal{A}_Z$ onto $\mathcal{A}_Z$ preserving the trace. The algebra of local observables is denoted by $\mathcal{A}_{\text{loc}}$.

We will use notation, definitions and some results about interactions and dynamics that are summarized in Appendix 5.A. While most of what we use should be standard for an expert in the field, the notion of an anchored interaction which was introduced in [9] might be an exception.
We consider an interaction $h \in \mathcal{J}$, here $\mathcal{J}$ is a class of interactions that are sufficiently local and uniformly bounded (see the appendix for the exact definition), and assume that it has a finite range, i.e. there exists $r > 0$ such that $\mathrm{diam}(S) > r$ implies $h_S = 0$. We denote $\{\tau_t^h : t \in \mathbb{R}\}$ the dynamics, namely the one parameter group of automorphisms, generated by h.

ASSUMPTION 5.2.1. *The dynamics τ^h has a unique gapped ground state ω.*

Precisely, this means that there is a unique state ω satisfying

$$\frac{\omega(A^*[h,A])}{\omega(A^*A)} \geq g > 0 \tag{5.2}$$

for all local A such that $\omega(A) = 0$. It is then automatically a ground state, i.e., $\omega(A^*[h,A]) \geq 0$ for all local observables A, and is pure [44]. We denote the GNS representation of ω by $(\mathcal{H}, \pi, \Omega)$.
Note that we do not assume that ω is the unique state satisfying the condition $\omega(A^*[h,A]) \geq 0$, namely there may in general be other such 'algebraic ground states'.

5.2.1. Braided C^*-tensor category associated with π. In this section we recall, to the extent that we will need in this work, the construction of braided C^*-tensor category described in [41]. It requires the approximate Haag duality. We do not present the full definition here and refer reader to [41, Definition 1.1] (c.f. 38).

ASSUMPTION 5.2.2. *The GNS representation $(\mathcal{H}, \pi, \Omega)$ of ω satisfies the approximate Haag duality.*

We denote $\boldsymbol{e}_\beta := (\cos\beta, \sin\beta)$ and set

$$\Lambda_{\boldsymbol{a},\theta,\varphi} := \{\boldsymbol{a} + t\boldsymbol{e}_\beta \mid t > 0,\ \beta \in (\theta - \varphi, \theta + \varphi)\} \tag{5.3}$$

for $\theta \in \mathbb{R}$, $\boldsymbol{a} \in \mathbb{R}^2$, and $\varphi \in (0, \pi)$. We call a subset of this shape a cone and use the same notation for the subset $\Lambda_{\boldsymbol{a},\theta,\varphi} \cap \mathbb{Z}^2$ of the lattice. It is important that the empty set and $\mathbb{R}^2$ are not cones. Strict Haag duality is the statement that $\pi(\mathcal{A}_{\Lambda^c})' = \pi(\mathcal{A}_\Lambda)''$ for all cones Λ while the approximate version allows for 'tails' on the outside of the cones.

We now define superselection sectors with respect to the GNS representation $(\mathcal{H}, \pi, \Omega)$ of the gapped ground state ω, see Assumption 5.2.1. We note that the representation is irreducible because the ground state ω is pure.

Recalling Definition 3:

DEFINITION 116. We say that a representation σ of $\mathcal{A}$ on $\mathcal{H}$ satisfies the *superselection criterion* with respect to π if

$$\sigma|_{\mathcal{A}_{\Lambda^c}} \simeq \pi|_{\mathcal{A}_{\Lambda^c}},$$

for any cone Λ. Here, $\simeq$ denotes unitary equivalence.

We denote by $\mathcal{O}_0$ all representations of $\mathcal{A}$ on $\mathcal{H}$ that satisfy the superselection criterion. Equivalence of representations splits $\mathcal{O}_0$ into equivalence classes, which are called superselection sectors.

THEOREM ([41], Theorems 5.2 & 6.1). *Given Assumptions 5.2.1, 5.2.2, the superselection sectors form a braided C^*-tensor category.*

We call this category $\mathcal{M}$ and refer to [41] for precise definitions. We will recall the construction in Section 5.3. For the moment we only note that objects in the category are representations satisfying the superselection criteria, and morphisms are their intertwiners. The braiding, $\epsilon(\rho, \sigma)$ of objects ρ, σ, encodes the exchange statistics of the anyons corresponding to ρ, σ. We will also introduce a braiding statistics $\theta(\rho, \sigma)$ which will be the phase obtained by moving σ counterclockwise around ρ, see (5.6).

5.2.2. Charge conservation. We consider an on-site $U(1)$ symmetry generated by an interaction $q \in \mathcal{J}$ such that operators $q_{\{x\}} \in \mathcal{A}_{\{x\}}$ have integer spectrum for all $x \in \mathbb{Z}^2$, and $q_S = 0$ if S is not a singleton. The operator $q_{\{x\}}$ encodes physical charge at site x, and for any finite region Z we denote

$$Q_Z := \sum_{x \in Z} q_{\{x\}},$$

and refer to it as the *charge* in the set Z. By assumption, $\mathrm{Spec}(Q_Z) \subset \mathbb{Z}$. For any (finite or not) subset Z, let δ^q_Z be the derivation associated with $q|_Z$, the restriction of q to Z — see Appendix 5.A.6 for the notion of restriction of an interaction — and let α^Z be the corresponding family of automorphisms. Note that $\alpha^Z_{2\pi} = \mathrm{id}$, justifying the name $U(1)$ symmetry. We denote $\delta^q = \delta^q_{\mathbb{Z}^2}$, and $\alpha = \alpha^{\mathbb{Z}^2}$.

We assume that our system is $U(1)$ invariant in the following sense.

ASSUMPTION 5.2.3. *For any finite $S, Z \subset \mathbb{Z}^2$ such that $S \subset Z$,*

$$[h_S, Q_Z] = 0. \tag{5.4}$$

We immediately note that in conjunction with Assumption 5.2.1, this implies the $U(1)$-invariance of the state, namely $\omega \circ \alpha_\phi = \omega$ for all $\phi \in \mathbb{R}$.

Assumptions 5.2.1, 5.2.3 allow to construct a self-adjoint operator $J \in \mathcal{A}$ whose expectation value

$$\kappa := \omega(J) \tag{5.5}$$

is the Hall conductance of the system [5, 24]. We provide details of this construction in Section 5.4. An alternative construction of an observable corresponding to Hall conductance is given in [43] using the framework of higher Berry curvature [2].

5.2.3. Results. The first theorem that we will prove makes an explicit connection between the braided C^*-tensor category, specifically the braiding statistics $\theta(\rho, \rho)$ briefly introduced at the end of Section 5.2.1 and defined in (5.6) below, and the Hall conductance κ. As discussed in the introduction, versions of this theorem are in [8, 25].

THEOREM 117 (Existence of Anyons). *Given Assumptions 5.2.1 – 5.2.3, there exists a simple object $\rho \in \mathcal{M}$ such that*

$$\theta(\rho, \rho) = e^{-i(2\pi)^2\kappa}.$$

The second theorem that we prove addresses quantization of the Hall conductance.

THEOREM 118 (Quantization of Hall conductance). *Suppose Assumptions 5.2.1 – 5.2.3 hold, and assume that there is a finite number p', of equivalence class of simple objects in $\mathcal{M}$. Then there exists an integer $p \leq p'$ such that*

$$2\pi\kappa \in \mathbb{Z}/p.$$

5.2.4. Outline. In the following section we provide details about construction of the braided C^*-tensor category. In Section 5.4 we define the Hall conductance. In Section 5.5 we prove Theorem 117, and in Section 5.6 we prove Theorem 118. Finally, Appendix A contains all we need about interactions and their associated objects, and Appendix B has some technical parts related to the definition of the braiding statistics θ on the braided tensor category.

5.3. Construction of braided C^*-tensor category

The idea to use superselection sectors to describe anyon ground state excitations was first described in the context of algebraic quantum field theory in [14]. It was recently adapted to quantum spin systems [15, 36, 37, 41]. A representation σ that is quasi-equivalent to π, without any restriction, corresponds to local excitations of the ground state. A representation σ that satisfies the superselection criteria but is not quasi-equivalent to π corresponds to anyon excitation: We often visualize them as excitations created by an endomorphism acting along a string going from the point of the excitations to infinity, which is, in particular, localized inside a cone. This is the case in some exactly solvable models [27, 28], see [36, 37].

In order to construct the braided C^*-tensor category, we shall now make various choices but the resulting category is independent of these choices [41]. Let $\mathcal{C}$ be the set of all cones (5.3) such that $[\theta-\varphi, \theta+\varphi] \cap [\frac{3\pi}{2}-\frac{\pi}{4}, \frac{3\pi}{2}+\frac{\pi}{4}] = \emptyset \mod 2\pi$. This makes a choice for what is called the forbidden direction, see Figure 5.1. Let

$$\mathcal{B} := \overline{\cup_{\Lambda\in\mathcal{C}} \pi\left(\mathcal{A}_\Lambda\right)''},$$

where the overline indicates the norm closure. For each cone Λ and $\sigma \in \mathcal{O}_0$, we set

$$\mathcal{V}_{\sigma,\Lambda} := \{V_{\sigma,\Lambda} \in \mathcal{U}(\mathcal{H}) \mid \mathrm{Ad}(V_{\sigma,\Lambda}) \circ \sigma|_{\mathcal{A}_{\Lambda^c}} = \pi|_{\mathcal{A}_{\Lambda^c}}\},$$

which is a nonempty set by the very definition of $\mathcal{O}_0$. We also denote by $\mathcal{O}_\Lambda$ the set of all $\sigma \in \mathcal{O}_0$ with $\mathbb{I} \in \mathcal{V}_{\sigma,\Lambda}$. $\mathcal{O}_\Lambda$ represents anyonic excitations supported in Λ.

We fix a cone $\Lambda_0 := \Lambda_{\mathbf{0},\frac{\pi}{2},\frac{5\pi}{8}} \in \mathcal{C}$, the objects in the category $\mathcal{M}$ are the elements of $\mathcal{O}_{\Lambda_0}$. In order to introduce a tensor product of objects (and later braiding), we first pull $\sigma \in \mathcal{O}_{\Lambda_0}$ to a map on the algebra $\mathcal{B}$. There exists a unique *-homomorphism T_σ of $\mathcal{B}$ such that

$$T_\sigma \circ \pi = \sigma$$

and T_σ is weakly continuous on $\pi\left(\mathcal{A}_\Lambda\right)''$, for every $\Lambda \in \mathcal{C}$.

For two objects $\sigma_1, \sigma_2 \in \mathcal{O}_{\Lambda_0}$, their tensor product is defined as

$$\sigma_1 \otimes \sigma_2 := T_{\sigma_1} \circ T_{\sigma_2} \circ \pi.$$

The morphisms of $\mathcal{M}$ are the intertwiners

$$\mathrm{Hom}(\sigma_1, \sigma_2) := \{V \in B(\mathcal{H}) \mid V\sigma_1 = \sigma_2 V\}.$$

To define braiding we fix the following two cones $\Lambda_2 := \Lambda_{\mathbf{0},\pi,\frac{\pi}{8}}, \Lambda_1 := \Lambda_{\mathbf{0},\frac{\pi}{2},\frac{\pi}{8}}$. For $\rho \in \mathcal{O}_{\Lambda_1}$ and $\sigma \in \mathcal{O}_{\Lambda_0}$ the braiding $\epsilon(\rho, \sigma)$, of ρ, σ is defined as the norm limit

$$\epsilon(\rho, \sigma) := \lim_{s \to \infty} V^*_{\sigma,\Lambda_2(s)} T_\rho \left(V_{\sigma,\Lambda_2(s)}\right).$$

Here and later, we use a notation $\Lambda_{\boldsymbol{a},\theta,\varphi}(s) = \Lambda_{\boldsymbol{a},\theta,\varphi} + s\boldsymbol{e}_\theta$. The braiding is independent of the choice of unitaries $V_{\sigma,\Lambda_2(s)} \in \mathcal{V}_{\sigma,\Lambda_2(s)}$, and it intertwines $\rho \otimes \sigma$ with $\sigma \otimes \rho$, i.e. $\epsilon(\rho, \sigma) \in \mathrm{Hom}(\rho \otimes \sigma, \sigma \otimes \rho)$. If $\rho = \pi$, then $T_\rho = \mathrm{id}$ and hence $\epsilon(\pi, \sigma) = 1$ for all σ.

Remark 119. As shown in Lemma 147, this definition is a special case of Definition 4.11 in [6], c.f. Definition 92.

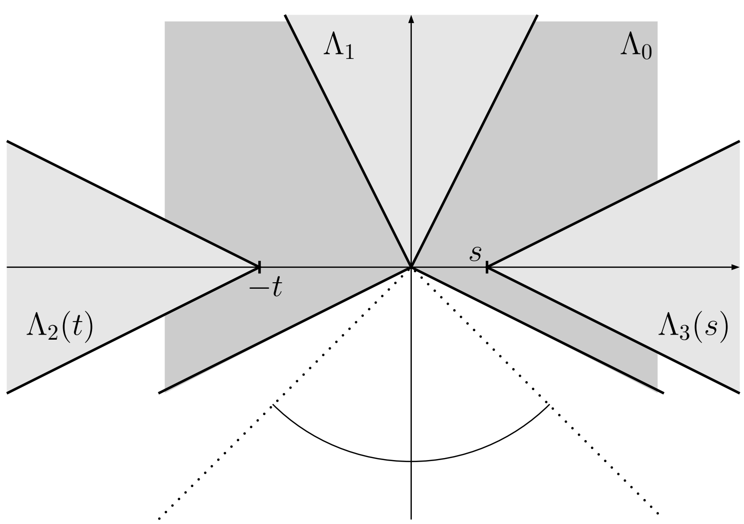

Figure 5.1. The various cones used in the construction of the category $\mathcal{M}$. Forbidden directions are represented by the arc in the lower half plane.

We will also need the braiding statistics, $\theta(\rho,\sigma)$, associated with winding of the anyon σ around ρ. We fix another cone $\Lambda_3=\Lambda_{\mathbf{0},0,\frac{\pi}{8}}$, and for $\rho\in\mathcal{O}_{\Lambda_1}$ and $\sigma\in\mathcal{O}_{\Lambda_0}$, we define

$$\theta(\rho,\sigma)=\lim_{t\to\infty}\epsilon(\rho,\operatorname{Ad}V_{\sigma,\Lambda_3(t)}\circ\sigma). \tag{5.6}$$

The limit is well defined, see (i) of the next lemma. In this article we will only encounter Abelian anyons in which case the braiding statistics is proportional to identity: This is reflected in the assumptions and statements of the following lemma. While a 'braiding statistics' or 'statistical phase' has been defined in many different ways in the mathematical literature and expresses the same phenomenology (among the close analogs, see Section 2 in [**17**], Section 2 in [**42**] or Section 8.5 in [**23**]), the authors are not aware of Definition (5.6) having appeared before.

LEMMA 120. *Suppose that Assumptions 5.2.1, 5.2.2 hold. Let* $\rho,\sigma\in\mathcal{O}_{\Lambda_1}$. *Suppose that* σ *is of the form* $\sigma=\pi\circ\tilde{\sigma}$ *for some* $\tilde{\sigma}\in\operatorname{Aut}(\mathcal{A})$ *such that* $\tilde{\sigma}|_{\mathcal{A}_{\Lambda_1^c}}=\operatorname{id}_{\mathcal{A}_{\Lambda_1^c}}$. *Then*

(i) $\theta(\rho,\sigma)$ *is well defined, and independent of the choice of* $V_{\sigma,\Lambda_3(t)}$,

(ii) $\theta(\rho,\sigma)\in\operatorname{Hom}(\rho,\rho)$.

Suppose in addition that $\rho=\pi\circ\tilde{\rho}$ *for some automorphism* $\tilde{\rho}$. *Then*

(iii) $\theta(\rho,\sigma)=e^{i\theta}\operatorname{id}$, *for some* $\theta\in\mathbb{R}$,

(iv) For $\rho'=\operatorname{Ad}_V\circ\rho\in\mathcal{O}_{\Lambda_1}$ *and* $\sigma'=\operatorname{Ad}_W\circ\sigma\in\mathcal{O}_{\Lambda_0}$

$$\theta(\rho',\sigma')=\theta(\rho,\sigma),$$

(v) $\theta(\rho_1\otimes\rho_2,\sigma)=\theta(\rho_1,\sigma)\theta(\rho_2,\sigma)$.

PROOF. The proof of (i) is quite technical and similar to the proofs of existence of $\epsilon(\rho,\sigma)$ in [**41**]. We postpone it to Appendix 5.B. Since similar techniques are required for the proof of part (ii), we similarly postpone it, see Lemma 145.

Since ρ is irreducible by the additional assumption, the point (ii) implies that $\theta(\rho,\sigma)$ is proportional to identity. Because T_ρ is a unital $*$-endomorphism, $\theta(\rho,\sigma)$ is a unitary as the norm limit of a family of unitaries. It follows that $\theta(\rho,\sigma)$ is a phase. This proves (iii).

Manifestly, $\theta(\rho, \sigma') = \theta(\rho, \sigma)$. So to prove (iv), it remains to compute $\theta(\rho', \sigma)$. To this end, let $\sigma'_s = \mathrm{Ad}(V_{\sigma,\Lambda_3(s)}) \circ \sigma$. Pick $V_{\sigma'_s,\Lambda_2(t)} = V_{\sigma,\Lambda_2(t)} V^*_{\sigma,\Lambda_3(s)}$.

$$\begin{aligned}\theta(\rho', \sigma) &= \lim_{s\to\infty} \epsilon(\mathrm{Ad}(V) \circ \rho, \mathrm{Ad}(V_{\sigma,\Lambda_3(s)}) \circ \sigma) \\ &= \lim_{s\to\infty} \lim_{t\to\infty} [[V^*_{\sigma'_s,\Lambda_2(t)}, V]] \,\mathrm{Ad}(V)(\epsilon(\rho, \sigma'_s)) \\ &= \lim_{s\to\infty} \lim_{t\to\infty} [[V_{\sigma,\Lambda_3(s)} V^*_{\sigma,\Lambda_2(t)}, V]] \,\mathrm{Ad}(V)(\epsilon(\rho, \sigma'_s)).\end{aligned}$$

where we used Lemma 146 in the second equality and denote $[[U_1, U_2]] = U_1 U_2 U_1^* U_2^*$ for the commutator of unitaries. For $A \in \mathcal{A}_{\Lambda_1^c}$, we have that

$$\pi(A) = \rho'(A) = \mathrm{Ad}(V)(\rho(A)) = \mathrm{Ad}(V)(\pi(A)),$$

namely $V \in \pi(\mathcal{A}_{\Lambda_1^c})'$. Hence $\lim_{s\to\infty} \lim_{t\to\infty}[[V_{\sigma,\Lambda_3(s)} V^*_{\sigma,\Lambda_2(t)}, V]] = 1$ by Lemma 139 and we get

$$\theta(\rho', \sigma) = \mathrm{Ad}(V)(\theta(\rho, \sigma)).$$

With this, Part (iv) follows from (iii).

To prove (v), we recall [41] that for $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{O}_{\Lambda_0}$,

$$\epsilon\left(\sigma_1 \otimes \sigma_2, \sigma_3\right) = \epsilon(\sigma_1, \sigma_3) T_{\sigma_1}\left(\epsilon(\sigma_2, \sigma_3)\right).$$

From the definition of θ, we then get

$$\theta(\rho_1 \otimes \rho_2, \sigma) = \theta(\rho_1, \sigma) T_{\rho_1}\left(\theta(\rho_2, \sigma)\right),$$

and claim then follows again from (iii). □

5.4. Definition of Hall conductance

There are various, equivalent, formulas for Hall conductance. These formulas fall into two classes, the first class expresses Hall conductance as the adiabatic curvature of a certain ground state bundle. The second class expresses Hall conductance as a charge pumped upon insertion of a 2π flux. The formulas can be proved from the Kubo formula [11], so the starting point is a matter of taste. We decided to start with a formula from the first class because it is most naturally formulated in the infinite volume limit. However, in the process of proving our main theorems we will need a formula from the second class which we will establish as a lemma below.

To define Hall conductance, we use a partition of space in four quadrants,

$$A := \{(x,y) \in \mathbb{Z}^2 \mid 0 \le x, 0 \le y\},$$

$$B := \{(x,y) \in \mathbb{Z}^2 \mid x \le -1, 0 \le y\},$$

$$C := \{(x,y) \in \mathbb{Z}^2 \mid x \le -1, y \le -1, \},$$

$$D := \{(x,y) \in \mathbb{Z}^2 \mid 0 \le x, y \le -1\},$$

see Figure 5.2. We will use the notation $Z_1Z_2 = Z_1 \cup Z_2$ for any two sets Z_1, Z_2. For example, AB is the upper half plane. In addition, for a set Z, we set $Z_N := Z \cap [-N,N]^{\times 2}$.

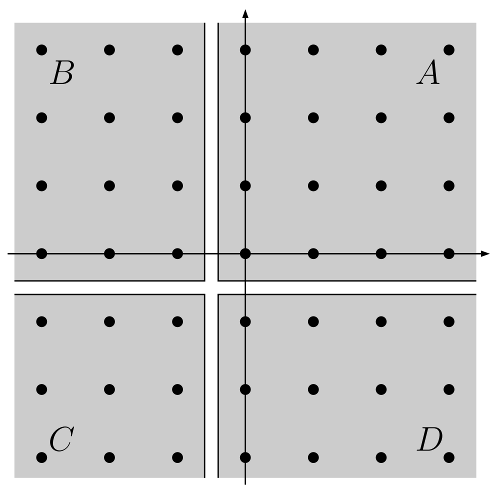

FIGURE 5.2. The four quadrants used to define the Hall conductance

For a region $Z \subset \mathbb{Z}^2$, we define

$$k^Z = -\int dt W(t) \tau_t^h(\delta_Z^q(h)), \tag{5.7}$$

with $W(t)$ a super-polynomially decaying function such that $i\sqrt{2\pi}\hat{W}(k) = 1/k$ for $|k| \ge g$. Here, we use the specific definition of an interaction given in (5.28), and so k^Z is a bonafide interaction. Next lemma gives basic properties of this interaction. We recall that the Hamiltonian has finite range r, and let $\partial Z := \{x \in \mathbb{Z}_2 : \mathrm{dist}(x,Z) \le r, \mathrm{dist}(x,Z^c) \le r\}$.

LEMMA 121. *Suppose that Assumptions 5.2.1 and 5.2.3 hold. Then for any $Z \subset \mathbb{Z}^2$,* *(i)* k^Z *is anchored in* ∂Z*,*

(ii) $\delta^q(k^Z_S) = 0$,

(iii) $k^Z = -k^{Z^c}$.

Proof. The statement (i) follows from Lemma 133. Note the non-trivial definition of time evolution of an interaction given in (5.28). Charge conservation, Assumption 5.2.3, implies (ii,iii).

□

A consequence of (i) and Lemma 132 is that $i[k^{AB}, k^{AD}]$ is summable. With this, we can define the Hall conductance via (5.5) with

$$J = \sum_S i[k^{AB}, k^{AD}]_S. \tag{5.8}$$

It is in no way apparent that expectation value of J is adiabatic curvature of some ground state bundle, we refer reader to [5] for the details about this bundle.

As announced in the first paragraph, we will need to connect this definition to a different formula that we will use later. We are going to do this in the remaining part of this section.

For a region Z we define an interaction $\overline{q}^Z = q|_Z - k^Z$, and denote β^Z the associated family of automorphisms. β^Z_ϕ corresponds to threading flux ϕ through the boundary of Z, see [8]. The function W in (5.7) is chosen so that the state ω is invariant, namely

$$\omega \circ \delta^{\overline{q}^Z} = 0, \qquad \omega \circ \beta^Z_\phi = \omega \tag{5.9}$$

for all regions Z, see [10]. For finite Z, the operator $K_Z = \sum_S k^Z_S$ is well-defined and the invariance above can be phrased as

$$\omega([\overline{Q}_Z, A]) = 0 \tag{5.10}$$

for all $A \in \mathcal{A}$, where $\overline{Q}_Z := Q_Z - K_Z$.

We claim that the interaction $i[\overline{q}^{AB}, k^{AD}]$ is also summable and that

$$\omega(\sum_S i[\overline{q}^{AB}, k^{AD}]_S) = 0.$$

To establish this, we start by recalling that k^{AD} is anchored in $\partial(AD)$, see Lemma 121(i). We now split the sum to two parts. First of all, if $S \subset AB$, we have that

$$[\overline{q}^{AB}, k^{AD}]_S = \sum_{S_1 \cup S_2 = S} [\overline{q}^{AB}_{S_1}, k^{AD}_{S_2}] = - \sum_{S_1 \cup S_2 = S} [k^{AB}_{S_1}, k^{AD}_{S_2}],$$

by Lemma 121(ii). By Lemma 132, the sum $\sum_{S_1 \cup S_2 \subset AB} [k^{AB}_{S_1}, k^{AD}_{S_2}]$ is absolutely convergent, in particular, we can write it as

$$- \sum_{S_2 \subset AB} \sum_{S_1 \subset AB} [k^{AB}_{S_1}, k^{AD}_{S_2}] = \sum_{S_2 \subset AB} \sum_{S_1 \subset AB} [\overline{q}^{AB}_{S_1}, k^{AD}_{S_2}],$$

where we used that for all $S_2 \subset AB$, $\sum_{S_1 \subset AB} [q_{S_1}, k^{AD}_{S_2}] = \delta^q(k^{AD}_{S_2}) = 0$. It might be worth noting that the double sum on the RHS is not absolutely convergent anymore. Second of all, we consider those sets S that intersect $(AB)^c$. In fact, the anchoring of k^{AD} implies that we are considering only those that intersect both $\partial(AD)$ and $(AB)^c$. On the one hand, the sum

$$\sum_{S_1, S_2 : (S_1 \cup S_2) \cap \partial(AD) \cap (AB)^c \neq \emptyset} [k^{AB}_{S_1}, k^{AD}_{S_2}]$$

is absolutely convergent by Lemma 132. On the other hand, since the interaction q is strictly on site,

$$\sum_{S : S \cap \partial(AD) \cap (AB)^c \neq \emptyset} [q^{AB}, k^{AD}]_S = \sum_{S_1, S_2 : S_2 \cap \partial(AD) \cap (AB)^c \neq \emptyset} [q^{AB}_{S_1}, k^{AD}_{S_2}]$$

and the sum on the RHS is absolutely convergent. Altogether, we have now established that the commutator is summable. The above argument also yields that if the two sums are put added to each other, we obtain a convergent sum,

$$\sum_S i[\overline{q}^{AB}, k^{AD}]_S = \sum_{S_2} \delta^{\overline{q}^{AB}}(k^{AD}_{S_2}). \tag{5.11}$$

It follows that the expectation vanishes since $\omega(\delta^{\overline{q}^{AB}}(k^{AD}_{S_2})) = 0$ for every S_2. By the same argument, the equality also holds with AB and AD exchanged. So we established two new expressions for Hall conductance,

$$\omega(\sum_S [k^{AB}, k^{AD}]_S) = \omega(\sum_S [q^{AB}, k^{AD}]_S) = \omega(\sum_S [k^{AB}, q^{AD}]_S).$$

Adding the last two, and subtracting the first and a zero $\sum_S [q^{AB}, q^{AD}]_S$ we then get

$$\omega(J) = \omega(-i \sum_S [\overline{q}^{AB}, \overline{q}^{AD}]_S), \tag{5.12}$$

with J defined in (5.8). To avoid any confusion, we remark that the expectation on the RHS looks formally zero by (5.9). However, the double sum $\sum_{S_1,S_2} [\overline{q}^{AB}_{S_1}, \overline{q}^{AD}_{S_2}]$ is not convergent so (5.9) is not applicable.

So far, the Hall conductance has been connected to adiabatic curvature. We now show that the definition above can also be related to charge transport. We start with a formal calculation (which, to be clear, is wrong!). By differentiating under the integral,

$$(\beta^{AD}_{2\pi})^{-1}(\overline{q}^{AB}) - \overline{q}^{AB} = -\int_0^{2\pi} (\beta^{AD}_\phi)^{-1} \delta_{\overline{q}^{AD}}(\overline{q}^{AB}) d\phi,$$

however LHS and RHS are not equal as interactions based on our definitions (5.26,5.28) of manipulating interactions. Continuing with formal calculations (which ignore that sums are not absolutely convergent), we conclude that

$$\sum_S \Big((\beta^{AD}_{2\pi})^{-1}(\overline{q}^{AB}) - \overline{q}^{AB} \Big)_S = \int_0^{2\pi} (\beta^{AD}_\phi)^{-1} \Big(\sum_S i[\overline{q}^{AB}, \overline{q}^{AD}]_S \Big) d\phi,$$

and the expectation of the RHS is $-2\pi\kappa$ by (5.12). This way, we obtained a formal connection between change of charge under the action of $\beta^{AD}_{2\pi}$ and Hall conductance.

It likely won't be any surprise to the reader that to make the calculation correctly we need to regularize the expression. There are many ways how to do that, our regularization resembles [8] (see also Lemma 125). To this end, for $r > 0$, we decompose

$$(\beta^{AD}_{2\pi})^{-1} = \gamma^{1,r}\gamma^{0,r}, \tag{5.13}$$

where γ are automorphisms such that

(i) $\gamma^{0,r}$ (resp. $\gamma^{1,r}$) is generated by TDI $g_{0,r}$ (resp. $g_{1,r}$) anchored in $\partial(AD) \cap \{x_2 \leq r\}$ (resp. $\partial(AD) \cap \{x_2 \geq r\}$), moreover $(g_{1,r})_S = 0$ unless $S \subset AB$,

(ii) there exists function $f \in \mathcal{F}$ and a constant C such that $\|g_{j,r}\|_f \leq C$ holds for $j = 0, 1$ and all $r \geq 0$,

(iii) the TDIs are charge conserving, i.e. $[(g_{j,r})_S, Q_S] = 0$ for $j = 0, 1$, $r \geq 0$ and all finite S.

For conceptual clarity, the existence of this decomposition is assumed here, with a choice of $\gamma^{0,r}$ being given explicitly when the lemma will be used in the proof of Theorem 117.

Lemma 122. *Let*

$$J_0 = \int_0^{2\pi} (\beta_\phi^{AD})^{-1}\Big(i\sum_S [\overline{q}^{AB}, \overline{q}^{AD}]_S\Big) d\phi.$$

Then

$$\omega(J_0) = -2\pi\omega(J),$$

and

$$\lim_{r\to\infty}\lim_{N\to\infty} \gamma^{0,r}(\overline{Q}_{(AB)_N}) - \overline{Q}_{(AB)_N} = J_0,$$

where the limits are in the uniform topology of the C^*-algebra.

Proof. Since $[\overline{q}^{AB}, \overline{q}^{AD}]$ is summable, the equality $\omega(J_0) = -2\pi\omega(J)$ follows immediately from (5.12) and the invariance of ω under the action of β_s^{AD}.

It remains to prove the last statement. We split the limit into two parts using $\overline{Q}_{(AB)_N} = Q_{(AB)_N} - K_{(AB)_N}$. We first consider the charge contribution. We fix $r > 0$, and we are going to show that the limit of $\gamma^{0,r}(Q_{(AB)_N}) - Q_{(AB)_N}$ as $N \to \infty$ exists. For $M > N$,

$$Q_{(AB)_M} - Q_{(AB)_N} = \sum_{x\in(AB)_M\setminus(AB)_N} q_x.$$

Using Lemma 134 we have for $|x| \gg r$,

$$\|\gamma^{0,r}(q_x) - q_x\| \leq f(|x|/2),$$

and

$$\|(\gamma^{0,r}(Q_{(AB)_M}) - Q_{(AB)_M}) - (\gamma^{0,r}(Q_{(AB)_N}) - Q_{(AB)_N})\| \leq \sum_{|x|\geq N} f(|x|/2).$$

Since $f(|x|/2)$ is summable, the sum is going to zero as $N \to \infty$. Hence the sequence is Cauchy and therefore it has a limit.

The decomposition (5.13) yields that

$$\gamma^{0,r}(Q_{(AB)_N}) - Q_{(AB)_N} = (\gamma^{1,r})^{-1}((\beta_{2\pi}^{AD})^{-1}(Q_{(AB)_N}) - Q_{(AB)_N}) \\ + (\gamma^{1,r})^{-1}(Q_{(AB)_N}) - Q_{(AB)_N}.$$

Since $Q_{(AB)_N}$ is a bonafide element of the algebra, we can differentiate under the integral sign to get

$$(\beta_{2\pi}^{AD})^{-1}(Q_{(AB)_N}) - Q_{(AB)_N} = -\int_0^{2\pi} (\beta_\phi^{AD})^{-1}\delta_{\overline{q}^{AD}}(Q_{(AB)_N})d\phi.$$

Now

$$\begin{aligned}\delta_{\overline{q}^{AD}}(Q_{(AB)_N}) &= -\sum_S i[q^{(AB)_N}, \overline{q}^{AD}]_S \\ &= -\sum_S i[q^{AB}, \overline{q}^{AD}]_S + \sum_S i[q^{(AB)_N^c}, \overline{q}^{AD}]_S,\end{aligned}$$

where the convergence of these sums was established in the paragraphs preceding the lemma. Hence,

$$\gamma^{0,r}(Q_{(AB)_N}) - Q_{(AB)_N} = (\gamma^{1,r})^{-1}\Big(\int_0^{2\pi} (\beta_\phi^{AD})^{-1}\big(\sum_S i[q^{AB}, \overline{q}^{AD}]_S\big)d\phi\Big) + J_N,$$

where

$$J_N = -\int_0^{2\pi} (\beta_s^{AD})^{-1}\sum_S i[q^{(AB)_N^c}, \overline{q}^{AD}]_S ds + (\gamma^{1,r})^{-1}(Q_{(AB)_N}) - Q_{(AB)_N}.$$

The automorphism $(\gamma^{1,r})^{-1}$ is generated by a TDI, let's call it g, that is charge conserving and supported in AB. Then we can write the last term as

$$(\gamma^{1,r})^{-1}(Q_{(AB)_N}) - Q_{(AB)_N} = \sum_{S: S\cap(AB)_N^c \neq \emptyset} \int_0^1 \tau_s^g(i[(g_s)_S, Q_{(AB)_N}])ds,$$

which gives a decomposition $J_N = \sum_S (j_N)_S$ with j_N anchored in $(AB)_N^c$. We established above that J_N has a limit, and since it is anchored on the complement of a square that eventually covers all of $\mathbb{Z}^2$, the limit is a multiple of the identity. But J_N is traceless for all N and hence the limit is zero. In conclusion, we obtained

$$\lim_{N\to\infty} \gamma^{0,r}(Q_{(AB)_N}) - Q_{(AB)_N} = (\gamma^{1,r})^{-1}\Big(\int_0^{2\pi} (\beta_\phi^{AD})^{-1}\big(\sum_S i[q^{AB}, \overline{q}^{AD}]_S\big)d\phi\Big).$$

As $(\gamma^{1,r})^{-1}(A) \to A$ for all $A \in \mathcal{A}$ we get

$$\lim_{r\to\infty}\lim_{N\to\infty} \gamma^{0,r}(Q_{(AB)_N}) - Q_{(AB)_N} = \int_0^{2\pi} (\beta_\phi^{AD})^{-1}\big(\sum_S i[q^{AB}, \overline{q}^{AD}]_S\big)d\phi. \tag{5.14}$$

Regarding the second part associated with $K_{(AB)_N}$, Lemma 134 gives that the contribution of $(k^{(AB)_N})_S$ to $\gamma^{0,r}(K_{(AB)_N}) - K_{(AB)_N}$ decays with the distance of S from the origin. This means that we can directly take the limit to get

$$\lim_{N\to\infty}(\gamma^{0,r}(K_{(AB)_N}) - K_{(AB)_N}) = \sum_S \gamma^{0,r}(k_S^{AB}) - k_S^{AB}.$$

Using the same lemma, we have that the sum on the RHS is uniformly convergent in r (we assumed that TDIs $g^{0,r}$ are uniformly bounded) so we can also take the limit in r to get

$$\lim_{r\to\infty}\lim_{N\to\infty}(\gamma^{0,r}(K_{(AB)_N}) - K_{(AB)_N}) = \sum_S (\beta_{2\pi}^{AD})^{-1}(k_S^{AB}) - k_S^{AB}.$$

Finally, we can now differentiate term by term under the integral to get

$$\begin{aligned}\lim_{r\to\infty}\lim_{N\to\infty}(\gamma^{0,r}(K_{(AB)_N}) - K_{(AB)_N}) &= -\sum_S \int_0^{2\pi} (\beta_\phi^{AD})^{-1} i[\overline{q}^{AD}, k_S^{AB}]d\phi \\ &= -\int_0^{2\pi}(\beta_\phi^{AD})^{-1}\big(i\sum_S [\overline{q}^{AD}, k^{AB}]_S\big)d\phi, \end{aligned} \tag{5.15}$$

where in the second line we used that $[\overline{q}^{AD}, k^{AB}]_S$ is absolutely summable so even though the lines are not equal for each S, they have the same sum. (Recall (5.11).)

Adding now (5.14,5.15) back together finishes the proof. □

We end this section by proving that J_0 commutes with the ground state. We recall an elementary lemma.

LEMMA 123. *Let $M \in \mathcal{A}$ be such that $\omega([M, A]) = 0$ for all $A \in \mathcal{A}$. Then*

$$\omega(MA) = \omega(M)\omega(A)$$

for any $A \in \mathcal{A}$.

PROOF. In the GNS representation $(\mathcal{H}, \pi, \Omega)$ of ω, we let P be the orthogonal projection onto the space spanned by Ω. Since ω is pure, π is irreducible and so $\pi(\mathcal{A})'' = (\mathbb{C}\cdot 1)' = B(\mathcal{H})$, namely $\pi(\mathcal{A})$ is weakly dense in $B(\mathcal{H})$. For $B \in B(\mathcal{H})$, let $(A_\alpha)_\alpha$ be a net in $\mathcal{A}$ converging weakly to B.

Then

$$\begin{aligned}\mathrm{Tr}([P,\pi(M)]B) &= \langle\Omega|\pi(M)B\Omega\rangle - \langle\Omega|B\pi(M)\Omega\rangle \\ &= \lim_{\alpha\to\infty}\langle\Omega|\pi(MA_\alpha)\Omega\rangle - \langle\Omega|\pi(A_\alpha M)\Omega\rangle \\ &= \lim_{\alpha\to\infty}\omega([M,A_\alpha]) = 0\end{aligned}$$

by the assumption. Since this is true for any $B \in B(\mathcal{H})$, we conclude that $[P,\pi(M)] = 0$, and hence

$$\omega(MA) = \mathrm{Tr}(P\pi(M)\pi(A)) = \mathrm{Tr}(P\pi(M)P\pi(A)) = \omega(M)\omega(A)$$

because P is a one-dimensional projection. □

For any finite Z, the conditions of the lemma are satisfied for $\overline{Q}_Z$ by (5.10) and we get

$$\omega(\overline{Q}_Z A) = \omega(\overline{Q}_Z)\omega(A) = \omega(Q_Z)\omega(A). \tag{5.16}$$

To get the second equality we noted that because Z is finite,

$$K_Z = \int W(t)\tau_t^h(\delta_h(Q_Z))dt.$$

Since ω is a ground state of τ_t^h, it is in particular invariant and $\omega \circ \delta_h = 0$ so that $\omega(K_Z) = 0$ by the formula above. It then follows from the definition of $\overline{Q}_Z$ that $\omega(\overline{Q}_Z) = \omega(Q_Z)$.
We note for later purposes that

$$\pi(\overline{Q}_Z)\Omega = \omega(Q_Z)\Omega. \tag{5.17}$$

This follows immediately by applying both sides of the identity $\pi(\overline{Q}_Z)P = P\pi(\overline{Q}_Z)$ to Ω:

$$\pi(\overline{Q}_Z)\Omega = \langle\Omega|\pi(\overline{Q}_Z)\Omega\rangle\Omega.$$

Lemma 124. *Suppose that Assumptions 5.2.1 and 5.2.3 hold. Then*

$$\omega(J_0 A) = \omega(J_0)\omega(A) \tag{5.18}$$

holds for all $A \in \mathcal{A}$.

PROOF. By (5.27), we have

$$\omega\Big([\sum_S[\overline{q}^{AB},\overline{q}^{AD}]_S,A]\Big)=\omega(\delta_{\overline{q}^{AB}}\delta_{\overline{q}^{AD}}(A)-\delta_{\overline{q}^{AD}}\delta_{\overline{q}^{AB}}(A)),$$

for any local $A\in\mathcal{A}$ and the RHS is equal to zero by (5.9). Using the second part of (5.9) we then get

$$\omega([J_0,A])=0,$$

for all $A\in\mathcal{A}$ and the statement follows from Lemma 123. □

5.5. Construction of an object in $\mathcal{M}$ associated with the $U(1)$ symmetry

Having defined the current observable J_0, we now turn to the explicit construction of the representation $\rho\in\mathcal{M}$ whose existence was announced in Theorem 117 and prove that it has statistical properties stated therein.

First of all, we note that while the TDI $\overline{q}^Z$ that generates β^Z_ϕ is anchored in Z, the automorphism $\beta^Z_{2\pi}$ has a trivial action far away from ∂Z because q_x have integer spectrum and k^Z is anchored in ∂Z. Concretely, $\beta^Z_{2\pi}$ can be obtained from a TDI that is anchored in ∂Z:

LEMMA 125. *Fix $Z\subset\mathbb{Z}^2$. There exists a TDI, $\tilde{k}^Z$, anchored in ∂Z such that*

$$\beta^Z_{2\pi}=\tau^{\tilde{k}^Z}_{2\pi}$$

PROOF. Recall that α^Z_ϕ is the family of automorphisms associated with the charge $q|_Z$. Since $\alpha^Z_{2\pi}=\mathrm{id}$, we have

$$\beta^Z_{2\pi}=\beta^Z_{2\pi}\circ(\alpha^Z_{2\pi})^{-1}=\mathrm{id}+\int_0^{2\pi}\partial_\phi\left(\beta^Z_\phi\circ(\alpha^Z_\phi)^{-1}\right)d\phi.$$

Computing the derivative, we have

$$\partial_\phi\left(\beta^Z_\phi\circ(\alpha^Z_\phi)^{-1}\right)=\beta^Z_\phi\circ\Big(\delta_{\overline{q}^Z}-\delta_{q^Z}\Big)\circ(\alpha^Z_\phi)^{-1}.$$

Using $\overline{q}^Z-q^Z=-k^Z$ we get

$$\partial_\phi\left(\beta^Z_\phi\circ(\alpha^Z_\phi)^{-1}\right)=\left(\beta^Z_\phi\circ(\alpha^Z_\phi)^{-1}\right)\circ\alpha^Z_\phi\circ\delta_{-k^Z}\circ(\alpha^Z_\phi)^{-1},$$

and the lemma holds with TDI $\tilde{k}^Z(\phi)=-\alpha^Z_\phi(k^Z)$. The TDI is anchored in ∂Z by Lemma 121 and Lemma 133. □

When an arbitrary TDI h is acted upon by the $U(1)$ automorphism, and only in the case, we will make an exception to (5.28)and define

$$(\alpha_\phi(h))_S := \alpha_\phi(h_S).$$

This is more convenient and α manifestly respects anchoring because it acts on-site.

To a cone $\Lambda = \Lambda_{\boldsymbol{a},\theta,\varphi}$ we associate the half space $Z_\Lambda := \{x \in \mathbb{R}^2 : f_\theta \cdot x \geq 0\}$, where f_θ is the unit vector obtained by rotating e_θ clockwise by 90 degrees, see Figure 5.3. Then by Lemma 125, $\beta_{2\pi}^{Z_\Lambda}$ is generated by TDI $\tilde{k}^{Z_\Lambda}$. Let ρ_s^Λ be the family of automorphisms generated by TDI $\tilde{k}^{Z_\Lambda}|_\Lambda$. Finally we put $\rho^\Lambda := \rho_{2\pi}^\Lambda$.

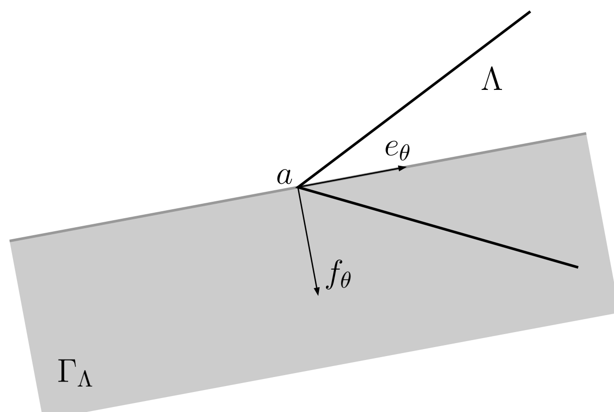

FIGURE 5.3. The half plane Z_Λ associated with the cone Λ

Since $\tilde{k}^{Z_\Lambda}$ is anchored on ∂Z_Λ, the automorphism ρ^Λ is generated by a TDI anchored on the axis of the cone Λ, which will be useful when we do perturbation theory. It is also possible to express ρ^Λ using the interaction $\overline{q}$. Since restrictions commute with the on-site automorphism $\alpha_s^{Z_\Lambda}$, we have that $\tilde{k}_s^{Z_\Lambda}|_\Lambda = -\alpha_s^{\Lambda\cap Z_\Lambda}(k^{Z_\Lambda}|_\Lambda)$, and we see that

$$\partial_s(\rho_s^\Lambda \circ \alpha_s^{\Lambda\cap Z_\Lambda}) = (\rho_s^\Lambda \circ \alpha_s^{\Lambda\cap Z_\Lambda}) \circ (-\delta_{k^{Z_\Lambda}|_\Lambda} + \delta_q^{\Lambda\cap Z_\Lambda}) = (\rho_s^\Lambda \circ \alpha_s^{\Lambda\cap Z_\Lambda}) \circ \delta_{\overline{q}^{Z_\Lambda}|_\Lambda}.$$

Since $\overline{q}$ is a constant interaction, this implies that $\rho_s^\Lambda = \exp\left(s\delta_{\overline{q}^{Z_\Lambda}|_\Lambda}\right) \circ (\alpha_s^{\Lambda\cap Z_\Lambda})^{-1}$, and in particular

$$\rho^\Lambda = \exp\left(2\pi\delta_{\overline{q}^{Z_\Lambda}|_\Lambda}\right). \tag{5.19}$$

This expression will be more useful for algebraic manipulations.

The next lemma states that, for any Λ, the representation $\pi \circ \rho^{\Lambda}$ satisfies the super-selection criteria and that in addition, all these automorphisms belong to the same super-selection sector.

Lemma 126. *For all cones $\Lambda, \Lambda' \subset \Lambda_0$,*

(i) $\pi \circ \rho^{\Lambda} \in \mathcal{O}_{\Lambda_0}$,
(ii) $\pi \circ \rho^{\Lambda} \simeq \pi \circ \rho^{\Lambda'}$.

Moreover there exists unitaries $V_{r,t} \in \mathcal{A}$ such that

$$\beta_{2\pi}^{AB} \circ \mathrm{Ad}[V_{r,t}] = \rho^{\Lambda_2(r)} \circ (\rho^{\Lambda_3(t)})^{-1}.$$

Proof. We start with the last part of the lemma. Recalling the definiton of the cones, Figure 5.1, we see that $Z_{\Lambda_3(t)} = CD$ for all t and so $\rho^{\Lambda_3(t)} = \exp\left(2\pi\delta_{\overline{q}^{CD}|_{\Lambda_3(t)}}\right)$ by (5.19). Using Lemma 121(iii), we have

$$\overline{q}^{CD}|_{\Lambda_3(t)} = q|_{\Lambda_3(t)} - \overline{q}^{AB}|_{\Lambda_3(t)},$$

and we conclude that

$$(\rho^{\Lambda_3(t)})^{-1} = \exp\left(2\pi\delta_{\overline{q}^{AB}|_{\Lambda_3(t)}}\right).$$

Moreover, $\rho^{\Lambda_2(r)} = \exp\left(2\pi\delta_{\overline{q}^{AB}|_{\Lambda_2(r)}}\right)$ and since the cones $\Lambda_2(r), \Lambda_3(t)$ are disjoint, we get

$$\rho^{\Lambda_2(r)} \circ (\rho^{\Lambda_3(t)})^{-1} = \exp\left(2\pi\delta_{\overline{q}^{AB}|_{\Lambda_2(r)\Lambda_3(t)}}\right).$$

Equivalently, this is an automorphism generated by the TDI $\tilde{k}^{AB}|_{\Lambda_2(r)\Lambda_3(t)}$. On the other hand, $\beta_{2\pi}^{AB}$ is generated by TDI $\tilde{k}^{AB}$. The claim then follows from Lemma 136, used for $X = \partial(AB)$ and $Z = \Lambda_2(r)\Lambda_3(t)$, and Lemma 135.

We now turn to the claim (ii). By the same reasoning that we used for the specific cones above, we get that for any non-overlapping cones Λ, Λ' there exists a region Z and a unitary $V_{\Lambda,\Lambda'}$ such that

$$\beta_{2\pi}^{Z} \circ \mathrm{Ad}[V_{\Lambda,\Lambda'}] = \rho^{\Lambda} \circ (\rho^{\Lambda'})^{-1},$$

see Figure 5.4.

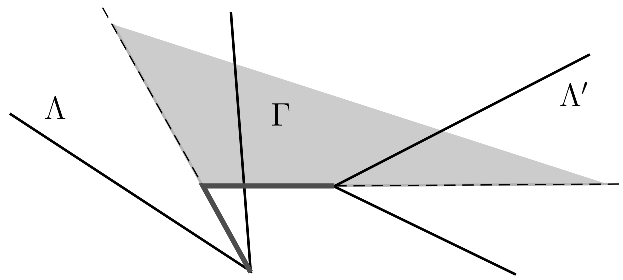

FIGURE 5.4. The region Z corresponding to the disjoint cones Λ, Λ'. The unitary $V_{\Lambda,\Lambda'}$ is almost localized along the thicker grey line.

The invariance (5.9) implies that β_ϕ^Z is unitarily implementable in the GNS representation, namely there are unitaries v_ϕ^Z such that $v_\phi^Z\Omega = \Omega$ and

$$\pi \circ \beta_\phi^Z = \mathrm{Ad}\left[v_\phi^Z\right] \circ \pi. \tag{5.20}$$

It follows that

$$\pi \circ \rho^\Lambda \circ (\rho^{\Lambda'})^{-1} = \mathrm{Ad}\left[v_\phi^Z\right] \circ \pi \circ \mathrm{Ad}[V_{\Lambda,\Lambda'}] = \mathrm{Ad}\left[v_\phi^Z \pi(V_{\Lambda,\Lambda'})\right] \circ \pi$$

which is (ii). If Λ, Λ' overlap, we find a cone Λ'', possibly ignoring the forbidden direction, that does not overlap with either. Then by the above, $\pi \circ \rho^\Lambda \simeq \pi \circ \rho^{\Lambda''} \simeq \pi \circ \rho^{\Lambda'}$, concluding the proof. Finally, (i) holds by construction. □

We note that the proof provides an explicit intertwiner $V_{\rho^{\Lambda_3(t)},\Lambda_2(r)}$:

LEMMA 127. *The unitary* $V_{\rho^{\Lambda_3(t)},\Lambda_2(r)} := v_{2\pi}^{AB}\pi(V_{r,t}) \in \mathcal{U}(\mathcal{H})$ *is such that*

$$\left(\pi \circ \rho^{\Lambda_2(r)}(A)\right) V_{\rho^{\Lambda_3(t)},\Lambda_2(r)} = V_{\rho^{\Lambda_3(t)},\Lambda_2(r)} \left(\pi \circ \rho^{\Lambda_3(t)}(A)\right), \quad A \in \mathcal{A}.$$

In fact, more can be said. Indeed, the proof of Lemma 135 gives

$$V_{r,t} = \mathrm{T}\epsilon i \int_0^{2\pi} G_s ds, \quad G_s = \tau_s^{\tilde{k}^{AB}|_{\Lambda_2(r)\Lambda_3(t)}} \Bigg(\sum_{S: S\cap(\Lambda_2(r)\Lambda_3(t))^c \neq \emptyset} \tilde{k}_S^{AB} \Bigg). \tag{5.21}$$

Since $\delta_{\tilde{k}^{AB}|_{\Lambda_2(r)\Lambda_3(t)}}$ acts trivially on the terms that are completely supported on the complement of $\Lambda_2(r)\Lambda_3(t)$, we have that

$$G_s = \sum_{S: S\subset(\Lambda_2(r)\Lambda_3(t))^c} \tilde{k}_S^{AB} + \tilde{G}_s$$

where $\tilde{G}_s$ is an observable that is almost localized at the apexes, $a_2(r), a_3(t)$, of $\Lambda_2(r)$ and $\Lambda_3(t)$. Specifically, there exists $f \in \mathcal{F}$ and a constant C, both independent of r, t, such that $\|\tilde{G}_s\|_f \le C$ and $\|\tilde{G}_s(S)\| \le f(\mathrm{dist}(S, \{a_2(r), a_3(t)\})$.

Lemma 128. *Let $c_N = \epsilon 2\pi i \omega(Q_{AB_N})$. Then*

$$v_{2\pi}^{AB} = \operatorname*{s-lim}_{N\to\infty} \overline{c}_N \pi\left(\epsilon 2\pi i \overline{Q}_{AB_N}\right)$$

Proof. First we note that (5.17) implies that $\pi(\epsilon 2\pi i \overline{Q}_{AB_N})\Omega = c_N\Omega$. Then for any $A \in \mathcal{A}$,

$$\overline{c}_N \pi\left(\epsilon 2\pi i \overline{Q}_{AB_N}\right)\pi(A)\Omega = \pi\left(\mathrm{Ad}\left[\epsilon 2\pi i \overline{Q}_{AB_N}\right](A)\right)\Omega$$

while

$$v_{2\pi}^{AB}\pi(A)\Omega = \pi\left(\beta_{2\pi}^{AB}(A)\right)\Omega$$

by (5.20). Hence

$$\overline{c}_N \pi\left(\epsilon 2\pi i \overline{Q}_{AB_N}\right)\pi(A)\Omega \quad - \quad v_{2\pi}^{AB}\pi(A)\Omega \quad = \quad \pi\left(\mathrm{Ad}\left[\epsilon 2\pi i \overline{Q}_{AB_N}\right](A) - \beta_{2\pi}^{AB}(A)\right)\Omega$$

and the claim follows from the strong limit

$$\beta_{2\pi}^{AB} = \operatorname*{s-lim}_{N\to\infty} \mathrm{Ad}\left[\epsilon 2\pi i \overline{Q}_{AB_N}\right]$$

and the cyclicity of Ω with respect to $\pi(\mathcal{A})$. □

Remark 129. Note from (5.20) that $v_{2\pi}^{AB}$ can be arbitrarily well approximated by elements in $\cup_{\Lambda\in\mathcal{C}}\pi(\mathcal{A}_{\Lambda^c})'$. In particular, by the approximate Haag duality, $v_{2\pi}^{AB}$ belongs to $\mathcal{B}$. More precisely, for each $\varepsilon > 0$, we may choose a cone $\Lambda_\varepsilon \in \mathcal{C}$ and a unitary $U_\varepsilon \in \mathcal{A}$ such that $\pi(U_\varepsilon)v_{2\pi}^{AB} \in \pi\left(\mathcal{A}_{\Lambda_\varepsilon^c}\right)'$ and $\|U_\varepsilon - \mathbb{I}\| < \varepsilon$. Note also that we may further choose a sequence of unitaries $U_\varepsilon^N \in \mathcal{A}$ such that $\pi\left(U_\varepsilon^N \epsilon 2\pi i \overline{Q}_{AB_N}\right) \in \pi\left(\mathcal{A}_{\Lambda_\varepsilon^c}\right)'$ and $\left\|U_\varepsilon^N - U_\varepsilon\right\| \to 0$, $N \to \infty$. Hence we obtain

$$\pi(U_\varepsilon)v_{2\pi}^{AB} = \operatorname*{s-lim}_{N\to\infty} \overline{c}_N \pi(U_\varepsilon^N)\pi\left(\epsilon 2\pi i \overline{Q}_{AB_N}\right) \tag{5.22}$$

in $\pi\left(\mathcal{A}_{\Lambda_\varepsilon^c}\right)'$.

We are now ready to prove our first main result, namely that the braiding statistics $\theta(\rho,\rho)$ is nothing but the exponential of the Hall conductance, which in turn was defined via (5.5,5.8) as the expectation value of the adiabatic curvature.

Proof of Theorem 117. We claim that $\rho := \pi\circ\rho^{\Lambda_1}$ has the stated properties. By Lemma 126, $\rho\in\mathcal{M}$, and it is simple because π is irreducible. So it remains to prove the braiding relation $\theta(\rho,\rho) = \epsilon 2\pi i\omega(J_0)$.

Using item (iv) in Lemma 120 and Lemma 126, we have for any $r\geq 0$, $\theta(\rho,\rho) = \theta(\pi\circ\rho^{\Lambda_1(r)},\rho)$. In particular,

$$\theta(\rho,\rho) = \lim_{r\to\infty}\theta\left(\pi\circ\rho^{\Lambda_1(r)},\rho\right).$$

We now pick $V_{\rho,\Lambda_3(t)}$ such that $\mathrm{Ad}\, V_{\rho,\Lambda_3(t)}\circ\rho = \pi\circ\rho^{\Lambda_3(t)}$ to get

$$\theta\left(\pi\circ\rho^{\Lambda_1(r)},\rho\right) = \lim_{t\to\infty}\epsilon(\pi\circ\rho^{\Lambda_1(r)},\pi\circ\rho^{\Lambda_3(t)}).$$

By Lemma 127,

$$\epsilon(\pi\circ\rho^{\Lambda_1(r)},\pi\circ\rho^{\Lambda_3(t)}) = \lim_{s\to\infty}\pi(V_{s,t}^*)(v_{2\pi}^{AB})^* T_{\rho^{\Lambda_1(r)}}(v_{2\pi}^{AB}\pi(V_{s,t})).$$

Now

$$T_{\rho^{\Lambda_1(r)}}(v_{2\pi}^{AB}\pi(V_{s,t})) = T_{\rho^{\Lambda_1(r)}}(\pi\beta_{2\pi}^{AB}(V_{s,t})v_{2\pi}^{AB}) = \pi(\rho^{\Lambda_1(r)}\beta_{2\pi}^{AB}(V_{s,t}))T_{\rho^{\Lambda_1(r)}}(v_{2\pi}^{AB})$$

and the explicit expression (5.21) implies that $\lim_{r\to\infty}\rho^{\Lambda_1(r)}\beta_{2\pi}^{AB}(V_{s,t}) = \beta_{2\pi}^{AB}(V_{s,t})$ uniformly in s,t. Hence,

$$\theta(\rho,\rho) = \lim_{r\to\infty}(v_{2\pi}^{AB})^* T_{\rho^{\Lambda_1(r)}}(v_{2\pi}^{AB}) = \lim_{r\to\infty}\left\langle\Omega\middle|(v_{2\pi}^{AB})^* T_{\rho^{\Lambda_1(r)}}(v_{2\pi}^{AB})\Omega\right\rangle,$$

where we used that $\theta(\rho,\rho)$ is a scalar in the last equality. With this, the weak continuity of $T_{\rho^{\Lambda_1(r)}}$ on each $\pi\left(\mathcal{A}_{\Lambda^c}\right)'$, $\Lambda\in\mathcal{C}$, Remark 129 and Lemma 128, we get

$$\theta(\rho,\rho) = \lim_{r\to\infty}\lim_{N\to\infty}\omega(\epsilon{-2\pi i\overline{Q}_{(AB)_N}}\rho^{\Lambda_1(r)}(\epsilon{2\pi i\overline{Q}_{(AB)_N}})).$$

Now we consider the automorphism $\theta_r := (\beta_{2\pi}^{AD})^{-1}\circ\rho^{\Lambda_1(r)}$ of $\mathcal{A}$, which is such that

$$\lim_{r\to\infty}\|(\beta_{2\pi}^{Z_{\Lambda_1}})^{-1}\circ\rho^{\Lambda_1(r)}(A) - (\beta_{2\pi}^{Z_{\Lambda_1}})^{-1}(A)\| = 0 \tag{5.23}$$

holds for all all $A \in \mathcal{A}$. Using (5.16), we have that

$$\theta(\rho,\rho) = \lim_{r\to\infty}\lim_{N\to\infty} \omega(\epsilon-2\pi i\overline{Q}_{(AB)_N}\theta_r(\epsilon 2\pi i\overline{Q}_{(AB)_N})). \tag{5.24}$$

Let now $v_{r,N}(\phi) = \epsilon-i\phi\overline{Q}_{(AB)_N}\theta_r(\epsilon i\phi\overline{Q}_{(AB)_N})$ and $v(\phi) = \lim_{r\to\infty}\lim_{N\to\infty} v_{r,N}(\phi)$ uniformly in ϕ, which are such that $v_{r,N}(0) = 1$ and therefore $v(0) = 1$. Note that $\gamma^{0,r} = \theta_r$ is a possible choice in the decomposition (5.13), so by Lemma 122 we have that

$$\lim_{r\to\infty}\lim_{N\to\infty}(\theta_r(\overline{Q}_{(AB)_N}) - \overline{Q}_{(AB)_N}) = J_0, \tag{5.25}$$

and since

$$\partial_\phi(\epsilon-i\phi\overline{Q}_{(AB)_N}\theta_r(\epsilon i\phi\overline{Q}_{(AB)_N})) = i\epsilon-i\phi\overline{Q}_{(AB)_N}(\theta_r(\overline{Q}_{(AB)_N}) - \overline{Q}_{(AB)_N})\theta_r(\epsilon i\phi\overline{Q}_{(AB)_N}).$$

we conclude from (5.25) that

$$\partial_\phi v_\phi = i\beta^{AB}_{-\phi}\left(J_0\right)v_\phi.$$

the proof of Lemma 124 now implies that $\partial_\phi\omega(v_\phi) = i\omega(J_0)\omega(v_\phi)$ and therefore

$$\theta(\rho,\rho) = \omega(v_{2\pi}) = e^{2\pi i\omega(J_0)},$$

which is the equation that we aimed to prove. ☐

5.6. Quantization of Hall conductance

We finally prove Theorem 118. We start with a Lemma that is a corollary of Lemma 120.

Lemma 130. *Suppose that $\rho \simeq \pi$, then $\theta(\rho,\sigma) = 1$ for any $\sigma \in \mathcal{O}_{\Lambda_0}$.*

Proof. By point (iv) in Lemma 120 we have

$$\theta(\rho,\sigma) = \theta(\pi,\sigma).$$

The right hand side is manifestly equal to the unity. ☐

Now we are ready for the proof of the theorem.

Proof of Theorem 118. Consider the object ρ constructed in Theorem 117. For any $n \in \mathbb{N}$, $\rho^{\otimes n}$ is an irreducible object in $\mathcal{M}$. By the assumption that there are finite number p' of simple objects, there exists $p \leq p'$ such that $\rho^{\otimes p} \simeq \pi$. By the previous lemma we then have

$$\theta(\rho^{\otimes p}, \rho) = 1.$$

On the other hand by Theorem 117 and Lemma 120(v),

$$\theta(\rho^{\otimes p}, \rho) = \theta(\rho, \rho)^p = e^{2\pi i \omega(J_0) p}.$$

The two equations and the definition (5.5) then imply the stated quantization of the Hall conductance κ. □

5.A Manipulating interactions

We use one of the standard setups to manipulate interactions. We follow [9]. We consider the l^∞-norm on $\mathbb{Z}^2$. For $x \in \mathbb{Z}^2$ and $r > 0$, $B_r(x)$ indicates the ball in $\mathbb{Z}^2$ centered at x with radius r with respect to this norm. The diameter of a subset $S \subset \mathbb{Z}^2$ with respect to this norm is denoted by $\mathrm{diam}(S)$.

5.A.1 Interactions. Let $\mathcal{F}$ be a class of strictly positive, non-decreasing functions $f : \mathbb{N}^+ \to \mathbb{R}^+$ that decay faster than any power, i.e. $\lim_{r\to\infty} f(r) r^p = 0$ for all $p > 0$. An interaction $h : S \Subset \mathbb{Z}^2 \to h_S \in \mathcal{A}_S$ is a map associating a finite subset of $\mathbb{Z}^2$ with an operator in $\mathcal{A}_S$. We will only consider interactions for which

$$\|h\|_f = \sup_{x \in \mathbb{Z}_2} \sum_{S \ni x} \frac{\|h_S\|}{f(1 + \mathrm{diam}(S))}$$

is finite for some $f \in \mathcal{F}$, we will call the set of all such interactions by $\mathcal{J}$. We will denote interactions with lower case letters and use upper case letter for their quadratures. For a set X we put

$$H_X = \sum_{S : S \cap X \neq \emptyset} h_S,$$

whenever the sum converges in the norm topology. The sum, in particular, converges for X finite, provided $h \in \mathcal{J}$. If the sum exists for $X = \mathbb{Z}^2$, we call the interaction summable. A derivation δ^h_X, associated with h and a region X, acts as

$$\delta^h_X(A) = \sum_{S : S \cap X \neq \emptyset} i[h_S, A], \quad A \in \mathcal{A}_{\mathrm{loc}}.$$

For $X = \mathbb{Z}^2$, we omit X and write δ^h. If δ^h is inner, we will call the interaction h an inner interaction. Summable interactions are inner, but there are inner interactions that are not summable. If H_X exists then $\delta^h_X(A) = i[H_X, A]$.

5.A.2 Anchored interactions. We say that an interaction h is anchored in a set $X \subset \mathbb{Z}^2$ if $h \in \mathcal{J}$ and $S \cap X = \emptyset$ implies that $h_S = 0$. If an interaction is anchored in a finite region X then $H_X = H_{\mathbb{Z}^2}$ exists.

Anchoring can be readily connected to more standard forms of locality.

LEMMA 131. *Suppose that h is anchored in X and $A \in \mathcal{A}_Y$. Then*

$$\|\delta_h(A)\| \leq 2f(\operatorname{dist}(X,Y))|Y|\|h\|_f\|A\|,$$

holds for any $f \in \mathcal{F}$.

PROOF. We have

$$\delta_h(A) = \sum_{\substack{S: S\cap X \neq \emptyset, \\ S \cap Y \neq \emptyset}} i[h_S, A].$$

So we get

$$\begin{aligned}
\|\delta_h(A)\| &\leq 2\sum_{y\in Y} \sum_{\substack{S: y\in S, \\ S\cap X\neq\emptyset}} \|h_S\|\|A\| \\
&\leq 2\sum_{y\in Y} \sum_{\substack{S: y\in S, \\ S\cap X\neq\emptyset}} \frac{\|h_S\|}{f(1+\operatorname{diam}(S))} f(1+\operatorname{diam}(S))\|A\| \\
&\leq 2|Y|\|h\|_f f(1+\operatorname{dist}(X,Y))\|A\|,
\end{aligned}$$

which is what we were supposed to prove. We note that the RHS might be infinite in which case the inequality is trivial. □

5.A.3 Commutators. For interactions h, h' we define their commutator $[h, h']$ as

$$[h,h']_S = \sum_{\substack{S_1,S_2: S_1\cup S_2 = S, \\ S_1\cap S_2\neq\emptyset}} [h_{S_1}, h'_{S_2}]. \tag{5.26}$$

If $h, h' \in \mathcal{J}$ then $[h, h'] \in \mathcal{J}$, and

$$-i\delta_{[h,h']} = \delta_h \delta_{h'} - \delta_{h'} \delta_h. \tag{5.27}$$

Furthermore, if h is anchored in X then $[h, h']$ is anchored in X.
It would be convenient if $[h, h']$ was anchored in the intersection of the anchors of h, h'. Alas, this is not the case. As a partial substitute, we will use the following criteria to decide if a commutator is inner.

LEMMA 132. *Let h (resp. h') be interactions anchored in X (resp. X'). Assume that for all $f \in \mathcal{F}$,*

$$\sum_{x \in X} \sum_{x' \in X'} f(|x - x'|) < \infty.$$

Then $[h, h']$ is summable.

Note that the assumption above is to be understood as a constraint on the sets and not on the family $\mathcal{F}$. It is satisfied in particular whenever X, X' are two non-parallel strips of finite width.

PROOF. By the definition (5.26) of the commutator of interactions, it suffices to show that

$$\mathcal{S} = \sum_{S \cap X \neq \emptyset} \sum_{\substack{S' \cap X' \neq \emptyset \\ S \cap S' \neq \emptyset}} \|h_S\| \|h'_{S'}\|$$

is convergent. Let $\tilde{f}$ be such that $\|h\|_{\tilde{f}} < \infty$, $\|h'\|_{\tilde{f}} < \infty$ and let

$$g(r) := \max_{r_1 + r_2 = r+1} \tilde{f}(r_1) \tilde{f}(r_2).$$

Then $g \in \mathcal{F}$ and we write

$$\mathcal{S} \leq \sum_{\substack{x \in X \\ x \in X'}} \sum_{\substack{S : x \in S, x' \in S' \\ S \cap S' \neq \emptyset}} \frac{\|h_S\|}{\tilde{f}(1 + \mathrm{diam}(S))} \frac{\|h'_{S'}\|}{\tilde{f}(1 + \mathrm{diam}(S'))} g(2 + \mathrm{diam}(S) + \mathrm{diam}(S')).$$

By the geometry of S, S' in the above sum, $\mathrm{diam}(S) + \mathrm{diam}(S') \geq |x - x'|$ so we get

$$\mathcal{S} \leq \|h\|_{\tilde{f}} \|h'\|_{\tilde{f}} \sum_{x \in X} \sum_{x \in X'} g(|x - x'|),$$

which is finite by assumption. $\square$

5.A.4 Time-evolution. For $A \in \mathcal{A}$ and a site $x \in \mathbb{Z}^2$, we define a decomposition of A

$$A = \sum_{n=0}^{\infty} A_{x,n},$$

where

$$A_{x,n} := \mathbb{E}_{B_n(x)}[A] - \mathbb{E}_{B_{n-1}(x)}[A],$$

for $n \geq 1$ and $A_{x,0} := \mathbb{E}_{B_0(x)}[A]$. For an automorphism β, and an interaction h anchored in X, we define the time evolved interaction as

$$\beta(h)_{B_k(x)} := \sum_{S: x \in S \cap X} \frac{1}{|S \cap X|} \beta(h_S)_{x,k}, \tag{5.28}$$

for $x \in X$ and $k \geq 0$. We define $\beta(h)_S = 0$ for any other S.

For an interaction h, we denote τ_s^h the group of automorphisms generated by δ^h. We will repeatedly use

Lemma 133. *Suppose that $h \in \mathcal{J}$ and that h' is anchored in X. Then $\tau_s^h(h')$ is anchored in X for all $s \geq 0$.*

The proof is in [9, Lemma 5.2.].

5.A.5 Quadratures. For a series of interactions h_j, we define

$$\left(\sum_j h_j \right)_S := \sum_j (h_j)_S,$$

provided the sum exists in norm sense. Likewise, provided that the integral on the RHS exists in the Bochner sense, we put

$$\left(\int h_t w(t) dt \right)_S = \int (h_t)_S w(t) dt,$$

for a family of interactions h_t and weight function $w(t)$.

5.A.6 Restrictions. For an interaction h and a region Z we define

$$(h|_Z)_S = \begin{cases} 0 & S \cap Z^c \neq \emptyset \\ h_S & S \subset Z. \end{cases}$$

The automorphisms, $\tau_s^{h|_Z}$, associated with $h|_Z$ act strictly in Z.

5.A.7 Time dependent interactions.

Time dependent interaction (TDI) is a map $s \in I \subset \mathbb{R} \to h_s \in \mathcal{J}$, with I an interval of $\mathbb{R}$. Furthermore we require that

(i) the map $s \in I \to (h_s)_S$ is continuous for all $S \subset \mathbb{Z}^2$,

(ii) there exist $f \in \mathcal{F}$ such that

$$\sup_{s\in I} \|h_s\|_f < \infty.$$

Operations on interactions extend point-wise to TDIs. The role of TDI's is to generate time evolution, to a TDI h we associate a family of automorphisms τ_s^h that satisfies the equation

$$\partial_s \tau_s^h(A) = \tau_s^h(\delta_{h_s}(A)), \quad A \in \mathcal{A}_{\text{loc}}.$$

Anchored interactions generate an automorphism that acts trivially far away from the anchoring region. This is quantified by the following lemma.

LEMMA 134. *Let h be a TDI anchored in X, τ_s^h the associated automorphism, and $A \in \mathcal{A}_Y$,*

$$\|\tau_1^h(A) - A\| \leq 2|Y| \sup_{s\in[0,1]} \|h_s\|_f f(\text{dist}(X,Y))\|A\|,$$

holds for any $f \in \mathcal{F}$.

PROOF. By differentiating under integral we get

$$\tau_1^h(A) - A = \int_0^1 \tau_s^h(\delta_{h_s}(A))ds,$$

and the statement follows from Lemma 131. □

5.A.8 Perturbation theory.

LEMMA 135. *Let h, h' be two TDIs, and τ_s^h, $\tau_s^{h'}$ the associated automorphisms. Suppose that $h - h'$ is inner, i.e. there exists a family $D_s \in \mathcal{A}$ such that*

$$\delta_{h_s}(A) - \delta_{h'_s}(A) = i[D_s, A], \tag{5.29}$$

holds for all $A \in \mathcal{A}$. Then there exists a unitary $V_s \in \mathcal{A}$ such that

$$V_s \tau_h^s(A) = \tau_{h'}^s(A) V_s.$$

holds for all $A \in \mathcal{A}$.

Proof. We have

$$\partial_s(\tau_h^s \circ (\tau_{h'}^s)^{-1}(A)) = \tau_h^s(\delta_{h_s} - \delta_{h'_s})(\tau_{h'}^s)^{-1}(A) = \tau_h^s \circ (\tau_{h'}^s)^{-1}[i\tau_{h'}^s(D_s), A],$$

by the assumption. In other words, the family of automorphisms $\tau_h^s \circ (\tau_{h'}^s)^{-1}$ is generated by the family of self-adjoint elements $\tau_{h'}^s(D_s) \in \mathcal{A}$. It follows immediately that $\tau_h^s \circ (\tau_{h'}^s)^{-1} = \mathrm{Ad}[V_s^*]$ where V_s is the time-ordered exponential of $\tau_{h'}^t(D_t)$. □

The lemma will be mainly used in the context of localizing interactions.

Lemma 136. *Let h be a TDI anchored in a region X. Suppose that $Z \subset \mathbb{Z}^2$ is a region such that there exists constants C_1, C_2 so that*

$$\mathrm{dist}(Z^c, X \cap B_{0,n}^c) \geq C_1 + C_2 n$$

holds for all integers n with $B_{0,n} := B_0(n)$. Then the TDI $h - h|_Z$ is inner.

Proof. Since, by definition,

$$(h - h|_Z)_S = \begin{cases} 0 & S \subset Z, \\ h_S & \text{otherwise,} \end{cases}$$

we get

$$\sum_S \|(h - h|_Z)_S\| = \sum_{S \cap Z^c \neq \emptyset} \|h_S\|.$$

Since h is anchored in X we can add a condition $S \cap X \neq \emptyset$ to the last sum. Then we bound it as

$$\sum_{\substack{S \cap Z^c \neq \emptyset \\ S \cap X \neq \emptyset}} \|h_S\| \leq \sum_{n=0}^{\infty} \sum_{x \in B_n \cap X} \sum_{\substack{S \ni x \\ S \cap Z^c \neq \emptyset \\ S \cap X \cap B_{n-1}^c \neq \emptyset}} \|h_S\|.$$

Any set S in the last sum is such that includes points in both Z^c and $X \cap B^c_{0,n-1}$. The diameter of such set is bigger than $C_1 + C_2(n-1)$ by assumption. If f is such that $\|h\|_f < \infty$, then

$$\begin{aligned}\sum_S \|(h - h|_Z)_S\| &\leq \sum_{n=0}^{\infty} \sum_{x \in B_n \cap X} \sum_{\substack{S \ni x \\ S \cap Z^c \neq \emptyset \\ S \cap B^c_{n-1} \neq \emptyset}} \frac{\|h_S\|}{f(1+\mathrm{diam(S)})} f(1 + C_1 + C_2(n-1)) \\ &\leq \|h\|_f \sum_{n=0}^{\infty} (2n+1)^2 f(1 + C_1 + C_2(n-1)),\end{aligned}$$

and the series is convergent since f decays faster than any inverse power. □

5.B Braiding statistics associated with winding

The goal of this appendix is to finish the proof of Lemma 120. Throughout the appendix we assume that assumptions 5.2.1, 5.2.2 hold.

A technical tool that we will use is a Lemma that follows from approximate Haag duality, see [41, Lemma 2.5]. We will use the notation $(\Lambda_{\boldsymbol{a},\theta,\varphi})_\epsilon := \Lambda_{\boldsymbol{a},\theta,\varphi+\epsilon}$.

LEMMA 137. *Let $\varepsilon > 0$, and $\delta > 0$, and let Λ be a cone such that $|\arg \Lambda| + 4\varepsilon < 2\pi$. Let $A \in \pi(\mathcal{A}_{\Lambda^c})'$. Then, under the assumption of approximate Haag duality, for all $r > R_{|\arg \Lambda|,\varepsilon}$ there exists $A'_r \in \pi(\mathcal{A}_{(\Lambda(-r))_{\varepsilon+\delta}})''$ such that $\|A - A'_r\| \leq 2 f_{|\arg \Lambda|,\varepsilon,\delta}(r) \, \|A\|$. Here $f.(r)$ is a decreasing function that vanishes in the limit $r \to \infty$.*

Specifically, there exists a unitary, $\tilde{U}_r$, depending on $\Lambda, \varepsilon, \delta$, such that $A'_r = \mathrm{Ad}(\tilde{U}_r)A$ satisfies these conditions.

LEMMA 138. *Let $\sigma \in \mathcal{O}_{\Lambda_0}$. Then*

$$\lim_{s,t\to\infty} \left\| [V_{\sigma,\Lambda_3(s)} V^*_{\sigma,\Lambda_2(t)}, A] \right\| = 0$$

holds for all $A \in \pi(\mathcal{A})$.

PROOF. First, suppose that $A \in \pi(\mathcal{A}_{loc})$. Then, there is some finite set on which A is supported. So, as $\bigcup_{n\in\mathbb{N}} \Lambda_1(-n) = \mathbb{R}^2$, there exists some $n \in \mathbb{N}$ such that A is supported in $\Lambda_1(-n)$, i.e., $A \in \pi(\mathcal{A}_{\Lambda_1(-n)}) \subseteq \pi(\mathcal{A}_{\Lambda_1(-n)})''$. For $s, t > n\tan(\frac{\pi}{8})$, $\Lambda_1(-n) \subset (\Lambda_3(s) \cup \Lambda_2(t))^c$, and so $\pi(\mathcal{A}_{\Lambda_1(-n)})'' \subset (\pi(\mathcal{A}_{(\Lambda_3(s)\cup\Lambda_2(t))^c})')'$, and therefore $A \in \pi(\mathcal{A}_{\Lambda_1(-n)})'' \subset (\pi(\mathcal{A}_{(\Lambda_3(s)\cup\Lambda_2(t))^c})')'$. By [41, Lemma 2.2], $V_{\sigma,\Lambda_3(s)} V^*_{\sigma,\Lambda_2(t)} \in \pi(\mathcal{A}_{(\Lambda_3(s)\cup\Lambda_2(t))^c})'$, and so $[A, V_{\sigma,\Lambda_3(s)} V^*_{\sigma,\Lambda_2(t)}] = 0$.

We conclude that for all $A \in \pi(\mathcal{A}_{\text{loc}})$, $\lim_{s,t\to\infty} \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A]\right\| = 0$. For $A \in \pi(\mathcal{A})$, the statement follows by density of $\pi(\mathcal{A}_{loc})$ in $\pi(\mathcal{A})$. □

LEMMA 139. *Let $\sigma \in \mathcal{O}_{\Lambda_0}$. Then*

$$\lim_{s,t\to\infty} \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A]\right\| = 0$$

holds for all $A \in \pi(\mathcal{A}_{\Lambda_1^c})'$.

PROOF. Let $A \in \pi(\mathcal{A}_{\Lambda_1^c})'$. Pick $\varepsilon > 0$ and $\delta > 0$ such that $|\arg\Lambda_1| + 4\varepsilon < 2\pi$. For concreteness, pick $\varepsilon = \delta = 10^{-3}$. Then, by Lemma 137, for all $r > R_{|\arg\Lambda_1|,\varepsilon}$ there exists $A'_r \in \pi(\mathcal{A}_{(\Lambda_1(-r))_{\varepsilon+\delta}})''$ such that $\|A - A'_r\| \le 2f_{|\arg\Lambda_1|,\varepsilon,\delta}(r)$. If $(\Lambda_1(-r))_{\varepsilon+\delta} \subset (\Lambda_3(s)\cup\Lambda_2(t))^c$, then $A'_r \in \pi(\mathcal{A}_{(\Lambda_1(-r))_{\varepsilon+\delta}})'' \subseteq (\pi(\mathcal{A}_{(\Lambda_3(s)\cup\Lambda_2(t))^c})')'$. By [41, Lemma 2.2], $V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)} \in \pi(\mathcal{A}_{(\Lambda_3(s)\cup\Lambda_2(t))^c})'$. Therefore, A'_r commutes with $V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}$. So, whenever $(\Lambda_1(-r))_{\varepsilon+\delta} \subset (\Lambda_3(s)\cup\Lambda_2(t))^c$,

$$\begin{aligned}
\left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A]\right\| &= \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A'_r + (A - A'_r)]\right\| \\
&= \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A'_r] + [V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, (A - A'_r)]\right\| \\
&\le \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A'_r]\right\| + \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, (A - A'_r)]\right\| \\
&\le 0 + 2\left\|V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}\right\| \|A - A'_r\| \\
&\le 4f_{|\arg\Lambda_1|,\varepsilon,\delta}(r).
\end{aligned}$$

Therefore, pick $r = \max(R_{|\arg\Lambda_1|}, \cot(\frac{\pi}{8} + \varepsilon + \delta)\cdot\min(t,s) - 1)$, so that for sufficiently large s, t, $(\Lambda_1(-r))_{\varepsilon+\delta} \subset (\Lambda_3(s)\cup\Lambda_2(t))^c$ and also so that $r \to \infty$ as $\min(s,t) \to \infty$, so that this upper bound of $4f_{|\arg\Lambda_1|,\varepsilon,\delta}(r)$ on $\left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A]\right\|$ goes to 0 as $t, s \to \infty$. So, $\lim_{s\to\infty}\lim_{t\to\infty} \left\|[V_{\sigma,\Lambda_3(s)}V^*_{\sigma,\Lambda_2(t)}, A]\right\| = 0$. □

LEMMA 140. *Let $\sigma \in \mathcal{O}_{\Lambda_0}$. For $i = a, b$ let Λ_i be a cone, and $V_{\sigma,\Lambda_i} \in \mathcal{V}_{\sigma,\Lambda_i}$. Then, $V^*_{\sigma,\Lambda_b}V_{\sigma,\Lambda_a} \in \sigma(\mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c})'$.*

PROOF. This proof is very similar to that of [41, Lemma 2.2]. Let $A \in \mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c} = \mathcal{A}_{\Lambda_a^c} \cap \mathcal{A}_{\Lambda_b^c}$. Then

$$\mathrm{Ad}(V^*_{\sigma,\Lambda_b}V_{\sigma,\Lambda_a}) \circ \sigma(A) = \mathrm{Ad}(V^*_{\sigma,\Lambda_b}) \circ \pi(A) = \sigma(A).$$

So, for all $A \in \mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c}$, $[V^*_{\sigma,\Lambda_b}V_{\sigma,\Lambda_a}, \sigma(A)] = 0$, i.e. $V^*_{\sigma,\Lambda_b}V_{\sigma,\Lambda_a} \in \sigma(\mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c})'$. □

Lemma 141. *Let Λ be a cone such that $\Lambda \subseteq \Lambda_0$, and such that Λ is disjoint from Λ_3. Let $\tilde{\sigma} \in \mathrm{Aut}(\mathcal{A})$ and assume that $\tilde{\sigma}|_{\mathcal{A}_{\Lambda^c}} = \mathrm{id}_{\mathcal{A}_{\Lambda^c}}$. Let $\sigma = \pi \circ \tilde{\sigma} \in \mathcal{O}_\Lambda$. Then for all $s_1, s_2 \geq s \geq 0$ and $V_{\sigma,\Lambda_3(s_1)} \in \mathcal{V}_{\sigma,\Lambda_3(s_1)}$ and $V_{\sigma,\Lambda_3(s_2)} \in \mathcal{V}_{\sigma,\Lambda_3(s_2)}$, $V^*_{\sigma,\Lambda_3(s_2)} V_{\sigma,\Lambda_3(s_1)} \in \pi(\mathcal{A}_{\Lambda_3(s)^c})'$.*

Proof. By Lemma 140, $V^*_{\sigma,\Lambda_3(s_2)} V_{\sigma,\Lambda_3(s_1)} \in \sigma(\mathcal{A}_{(\Lambda_3(s_1)\cup\Lambda_3(s_2))^c})' \subseteq \sigma(\mathcal{A}_{\Lambda_3(s)^c})'$. As $\tilde{\sigma}$ is an automorphism which is the identity when restricted to $\mathcal{A}_{\Lambda^c}$, it is also an automorphism when restricted to $\mathcal{A}_\Lambda$. And, as $\Lambda_3(s)^c \supseteq \Lambda$, $\tilde{\sigma}$ is also an automorphism when restricted to $\mathcal{A}_{\Lambda_3(s)^c}$. As such, $\sigma(\mathcal{A}_{\Lambda_3(s)^c}) = \pi(\tilde{\sigma}(\mathcal{A}_{\Lambda_3(s)^c})) = \pi(\mathcal{A}_{\Lambda_3(s)^c})$. So, $V^*_{\sigma,\Lambda_3(s_2)} V_{\sigma,\Lambda_3(s_1)} \in \sigma(\mathcal{A}_{\Lambda_3(s)^c})' = \pi(\mathcal{A}_{\Lambda_3(s)^c})'$. □

Lemma 142. *Let $\Lambda_a, \Lambda_b \in \mathcal{C}$ be disjoint subsets of Λ_0, and let $\sigma_a \in \mathcal{O}_{\Lambda_a}$ and $\sigma_b \in \mathcal{O}_{\Lambda_b}$. Let $R \in \mathrm{Hom}(\sigma_a \otimes \sigma_b, \sigma_b \otimes \sigma_a)$. Then $R \in \pi(\mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c})'$.*

Proof. This proof is essentially the same as that of [41, Lemma 4.2]. For $A \in \mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c}$, $\sigma_a \otimes \sigma_b(A) = \pi(A) = \sigma_b \otimes \sigma_a(A)$. As for all $A \in \mathcal{A}$, $R \cdot (\sigma_a \otimes \sigma_b)(A) = (\sigma_b \otimes \sigma_a)(A) \cdot R$, in particular, for all $A \in \mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c}$, $R \cdot \pi(A) = R \cdot (\sigma_a \otimes \sigma_b)(A) = (\sigma_b \otimes \sigma_a)(A) \cdot R = \pi(A) \cdot R$. So, $R \in \pi(\mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c})'$. □

In particular, $\epsilon(\sigma_a, \sigma_b) \in \pi(\mathcal{A}_{(\Lambda_a\cup\Lambda_b)^c})'$.

Lemma 143. *Suppose $\rho \in \mathcal{O}_{\Lambda_1}$ and $\sigma \in \mathcal{O}_{\Lambda_0}$, and $V \in \mathcal{U}(\mathcal{H})$ such that $\sigma' = \mathrm{Ad}(V) \circ \sigma \in \mathcal{O}_{\Lambda_0}$ as well. Then, $\epsilon(\rho, \sigma') = \mathrm{Ad}(V)(\epsilon(\rho,\sigma)) \cdot V \cdot T_\rho(V^*)$.*

Proof. Picking $V_{\sigma',\Lambda_2(t)} = V_{\sigma,\Lambda_2(t)} V^*$,

$$\begin{aligned}
\epsilon(\rho,\sigma') &= \lim_{t\to\infty} V^*_{\sigma',\Lambda_2(t)} T_\rho(V_{\sigma',\Lambda_2(t)}) \\
&= \lim_{t\to\infty} V V^*_{\sigma,\Lambda_2(t)} T_\rho(V_{\sigma,\Lambda_2(t)}) T_\rho(V^*) \\
&= \lim_{t\to\infty} \mathrm{Ad}(V)(V^*_{\sigma,\Lambda_2(t)} T_\rho(V_{\sigma,\Lambda_2(t)})) \cdot V \cdot T_\rho(V^*) \\
&= \mathrm{Ad}(V)(\epsilon(\rho,\sigma)) \cdot V \cdot T_\rho(V^*).
\end{aligned}$$

That $V_{\sigma,\Lambda_2(t)}, V \in \mathcal{B}$ follows from $V_\sigma, \Lambda_2(t) \in \pi(\mathcal{A}_{(\Lambda_2(t)\cup\Lambda_0)^c})' = \pi(\mathcal{A}_{\Lambda_0^c})'$ and $V \in \pi(\mathcal{A}_{(\Lambda_0\cup\Lambda_0)^c})' = \pi(\mathcal{A}_{\Lambda_0^c})'$ and $\pi(\mathcal{A}_{\Lambda_0^c})' \subseteq \mathcal{B}$, so the second equation splitting $T_\rho(V_{\sigma',\Lambda_2(t)})$ into $T_\rho(V_{\sigma,\Lambda_2(t)}) T_\rho(V^*)$ is legitimate. □

Lemma 144. *Let $\Lambda \subset \Lambda_0$ be disjoint from $(\Lambda_3(s_\Lambda))_{2\cdot 10^{-3}}$ for some s_Λ. Let $\rho \in \mathcal{O}_{\Lambda_1}$ and $\sigma \in \mathcal{O}_\Lambda$. Let σ be of the form $\sigma = \pi \circ \tilde{\sigma}$ for some $\tilde{\sigma} \in \mathrm{Aut}(\mathcal{A})$ such that $\tilde{\sigma}|_{\mathcal{A}_{\Lambda^c}} = \mathrm{id}_{\mathcal{A}_{\Lambda^c}}$. For $s > 0$, let*

$V_{\sigma,\Lambda_3(s)} \in \mathcal{V}_{\sigma,\Lambda_3(s)}$. *Let* $\sigma_{\Lambda_3(s)} := \mathrm{Ad}(V_{\sigma,\Lambda_3(s)}) \circ \sigma$. *Then,* $\lim_{s\to\infty} \epsilon(\rho, \sigma_{\Lambda_3(s)})$ *exists, and is independent of the choice of* $V_{\sigma,\Lambda_3(s)} \in \mathcal{V}_{\sigma,\Lambda_3(s)}$.

Proof. We start by showing that the limit exists. This will be done by showing that the sequence is Cauchy. By Lemma 143 , $\epsilon(\rho, \sigma_{\Lambda_3(s)}) = V_{\sigma,\Lambda_3(s)}\epsilon(\rho,\sigma)T_\rho(V^*_{\sigma,\Lambda_3(s)})$. Using that $T_\rho(V^*_{\sigma,\Lambda_3(s)})$ is unitary, for $s_1, s_2 > s > 0$,

$$\begin{aligned}
&\|\epsilon(\rho, \sigma_{\Lambda_3(s_2)}) - \epsilon(\rho, \sigma_{\Lambda_3(s_1)})\| \\
&\qquad = \left\| V_{\sigma,\Lambda_3(s_2)}\epsilon(\rho,\sigma)T_\rho(V^*_{\sigma,\Lambda_3(s_2)}) - V_{\sigma,\Lambda_3(s_1)}\epsilon(\rho,\sigma)T_\rho(V^*_{\sigma,\Lambda_3(s_1)}) \right\| \\
&\qquad = \left\| V^*_{\sigma,\Lambda_3(s_1)}V_{\sigma,\Lambda_3(s_2)}\epsilon(\rho,\sigma)T_\rho(V^*_{\sigma,\Lambda_3(s_2)})T_\rho(V_{\sigma,\Lambda_3(s_1)}) - \epsilon(\rho,\sigma) \right\| \\
&\qquad = \left\| V^*_{\sigma,\Lambda_3(s_1)}V_{\sigma,\Lambda_3(s_2)}\epsilon(\rho,\sigma)T_\rho(V^*_{\sigma,\Lambda_3(s_2)}V_{\sigma,\Lambda_3(s_1)}) - \epsilon(\rho,\sigma) \right\|.
\end{aligned}$$

Let $V_{\sigma,s_2,s_1} = V^*_{\sigma,\Lambda_3(s_2)}V_{\sigma,\Lambda_3(s_1)}$ so the above becomes

$$\left\|\epsilon(\rho, \sigma_{\Lambda_3(s_2)}) - \epsilon(\rho, \sigma_{\Lambda_3(s_1)})\right\| = \left\| V^*_{\sigma,s_2,s_1}\epsilon(\rho,\sigma)T_\rho(V_{\sigma,s_2,s_1}) - \epsilon(\rho,\sigma)\right\|.$$

By Lemma 141, $V_{\sigma,s_2,s_1} \in \pi(\mathcal{A}_{\Lambda_3(s)^c})'$. By Lemma 142, $\epsilon(\rho,\sigma) \in \pi(\mathcal{A}_{(\Lambda_1\cup\Lambda)^c})'$. By Lemma 137, for $s > 2R_{|\arg\Lambda_3|,\varepsilon}$, setting $V_{\sigma,s_2,s_1,s} = \mathrm{Ad}(\tilde{U}_r)(V_{\sigma,s_2,s_1})$, we obtain $V_{\sigma,s_2,s_1,s} \in \pi(\mathcal{A}_{(\Lambda_3(s-\frac{s}{2}))_{\varepsilon+\delta}})''$, and $\|V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s}\| \leq 2f_{|\arg\Lambda_3|,\varepsilon,\delta}(\frac{s}{2})$. For concreteness, pick $\varepsilon = \delta = 10^{-3}$. Then, for $s > \max(2s_\Lambda, 2R_{|\arg\Lambda_3|,10^{-3}})$, $(\Lambda_3(\frac{s}{2}))_{2\cdot 10^{-3}} \subseteq (\Lambda_3(s_\Lambda))_{2\cdot10^{-3}}$ and is therefore disjoint from Λ, and $(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}} \subseteq (\Lambda_3)_{2\cdot10^{-3}}$ and is therefore disjoint from Λ_1, and so $(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}}$ is disjoint from $\Lambda_1\cup\Lambda$. Therefore $[V^*_{\sigma,s_2,s_1,s}, \epsilon(\rho,\sigma)] = 0$, and we decompose

$$V^*_{\sigma,s_2,s_1}\epsilon(\rho,\sigma) = \epsilon(\rho,\sigma)V^*_{\sigma,s_2,s_1,s} + (V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s})^*\,\epsilon(\rho,\sigma).$$

As $(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}} \in \mathcal{C}$, T_ρ is weak-continuous on $\pi(\mathcal{A}_{(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}}})''$, and as $(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}} \subseteq \Lambda_1^c$, T_ρ is the identity on $\pi(\mathcal{A}_{(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}}})$, so together we get that it is also the identity on $\pi(\mathcal{A}_{(\Lambda_3(\frac{s}{2}))_{2\cdot10^{-3}}})''$, and so $T_\rho(V_{\sigma,s_2,s_1,s}) = V_{\sigma,s_2,s_1,s}$. We get,

$$\begin{aligned}
T_\rho(V_{\sigma,s_2,s_1}) &= T_\rho(V_{\sigma,s_2,s_1,s} + (V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s})) \\
&= V_{\sigma,s_2,s_1,s} + T_\rho(V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s}).
\end{aligned}$$

Therefore, using again that $[V^*_{\sigma,s_2,s_1,s}, \epsilon(\rho,\sigma)] = 0$,

$$V^*_{\sigma,s_2,s_1}\epsilon(\rho,\sigma)T_\rho(V_{\sigma,s_2,s_1}) = \epsilon(\rho,\sigma) + (\epsilon(\rho,\sigma)V^*_{\sigma,s_2,s_1,s})T_\rho(V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s}) \\ + (V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s})^*\,\epsilon(\rho,\sigma)T_\rho(V_{\sigma,s_2,s_1}).$$

So

$$\begin{aligned} \left\|V^*_{\sigma,s_2,s_1}\epsilon(\rho,\sigma)T_\rho(V_{\sigma,s_2,s_1}) - \epsilon(\rho,\sigma)\right\| &\le \left\|\epsilon(\rho,\sigma)V^*_{\sigma,s_2,s_1,s}\right\| \|T_\rho(V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s})\| \\ &\quad + \|(V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s})^*\| \, \|\epsilon(\rho,\sigma)T_\rho(V_{\sigma,s_2,s_1})\| \\ &\le 2\,\|\epsilon(\rho,\sigma)\|\,\|V_{\sigma,s_2,s_1} - V_{\sigma,s_2,s_1,s}\| \\ &\le 2\cdot 2 f_{|\arg\Lambda_3|,10^{-3},10^{-3}}(\frac{s}{2}), \end{aligned}$$

which goes to 0 as $s \to \infty$. Therefore, the sequence $(\epsilon(\rho,\sigma_{\Lambda_3(s)}))_{s\in\mathbb{N}}$ is Cauchy, and the sequence converges, i.e.,

$$\theta(\rho,\sigma) = \lim_{s\to\infty}\epsilon(\rho,\sigma_{\Lambda_3(s)})$$

exists.

Inspecting the proof, we showed that for any choice of $V_{\sigma,\Lambda_3(s)}$, we have

$$\|\theta(\rho,\sigma) - \epsilon(\rho,\sigma_{\Lambda_3(s)})\| \le 4f_{|\arg\Lambda_3|,10^{-3},10^{-3}}(\frac{s}{2}).$$

We will use this to show that the limit is independent of the choice of $V_{\sigma,\Lambda_3(s)} \in \mathcal{V}_{\sigma,\Lambda_3(s)}$.

Let $V_{\sigma,\Lambda_3(s)}, V'_{\sigma,\Lambda_3(s)}$, where for each s, $V_{\sigma,\Lambda_3(s)}, V'_{\sigma,\Lambda_3(s)} \in \mathcal{V}_{\sigma,\Lambda_3(s)}$, be two choices. Now consider a third choice, a sequence $V''_{\sigma,\Lambda_3(s)}$ which for $s < s'$ has $V''_{\sigma,\Lambda_3(s)} = V'_{\sigma,\Lambda_3(s)}$, but for $s \ge s'$ has $V''_{\sigma,\Lambda_3(s)} = V_{\sigma,\Lambda_3(s)}$. By the above bound the limit point of the sequence, which is $\theta(\rho,\sigma)$, has a distance bounded by $4f_{|\arg\Lambda_3|,10^{-3},10^{-3}}(\frac{s}{2})$ from the limit point of the sequence corresponding to the choice $V'_{\sigma,\Lambda_3(s)}$.

So, the limit exists and is independent of the choice of $V_{\sigma,\Lambda_3(s)}$, as desired. □

LEMMA 145. *Let $\rho \in \mathcal{O}_{\Lambda_1}$ and $\sigma \in \mathcal{O}_{\Lambda_0}$. Suppose that σ be of the form $\sigma = \pi \circ \tilde\sigma$ for some $\tilde\sigma \in \mathrm{Aut}(\mathcal{A})$ such that $\tilde\sigma|_{\mathcal{A}_{\Lambda^c}} = \mathrm{id}_{\mathcal{A}_{\Lambda^c}}$. Then, $\theta(\rho,\sigma) \in \mathrm{Hom}(\rho,\rho)$.*

PROOF. The task is to show that for all $A \in \mathcal{A}$, $\theta(\rho,\sigma)\rho(A) = \rho(A)\theta(\rho,\sigma)$. In fact, by density, it is enough to show it for $A \in \mathcal{A}_{\mathrm{loc}}$. Let $\sigma_{\Lambda_3(t)} := \mathrm{Ad}(V_{\sigma,\Lambda_3(t)}) \circ \sigma$. For all t, $\epsilon(\rho,\sigma_{\Lambda_3(t)}) \in$

$\mathrm{Hom}(\rho\otimes\sigma_{\Lambda_3(t)}, \sigma_{\Lambda_3(t)}\otimes\rho)$. Pick $r \in \mathbb{N}$ such that $A \in \mathcal{A}_{\Lambda_1(-r)}$. For $t > \cot(\frac{\pi}{2}-\frac{\pi}{8})r$, $\Lambda_1(-r) \subset \Lambda_3(t)^c$, and so $T_{\sigma_{\Lambda_3(t)}}|_{\pi(\mathcal{A}_{\Lambda_1(-r)})} = \mathrm{id}$. Therefore, $\rho \otimes \sigma_{\Lambda_3(t)}(A) = T_\rho \circ T_{\sigma_{\Lambda_3(t)}} \circ \pi(A) = T_\rho \circ \pi(A) = \rho(A)$, and we get

$$\begin{aligned}\epsilon(\rho, \sigma_{\Lambda_3(t)}) \cdot \rho(A) &= \epsilon(\rho, \sigma_{\Lambda_3(t)}) \cdot (\rho \otimes \sigma_{\Lambda_3(t)})(A)\\ &= (\sigma_{\Lambda_3(t)} \otimes \rho)(A) \cdot \epsilon(\rho, \sigma_{\Lambda_3(t)})\\ &= (T_{\sigma_{\Lambda_3(t)}} \circ T_\rho \circ \pi(A)) \cdot \epsilon(\rho, \sigma_{\Lambda_3(t)})\\ &= T_{\sigma_{\Lambda_3(t)}}(\rho(A)) \cdot \epsilon(\rho, \sigma_{\Lambda_3(t)}).\end{aligned}$$

And so,

$$\begin{aligned}\theta(\rho, \sigma) \cdot \rho(A) &= \lim_{t\to\infty} \epsilon(\rho, \sigma_{\Lambda_3(t)}) \cdot \rho(A)\\ &= \lim_{t\to\infty} T_{\sigma_{\Lambda_3(t)}}(\rho(A)) \cdot \epsilon(\rho, \sigma_{\Lambda_3(t)}).\end{aligned}$$

As for all t, $\epsilon(\rho, \sigma_{\Lambda_3(t)})$ is a unitary, and $\theta(\rho, \sigma) = \lim_{t\to\infty} \epsilon(\rho, \sigma_{\Lambda_3(t)})$, we conclude that $\theta(\rho, \sigma) \cdot \rho(A) \cdot \theta(\rho, \sigma)^* = \lim_{t\to\infty} T_{\sigma_{\Lambda_3(t)}}(\rho(A))$, and in particular that the limit on the right hand side exists. Now to conclude the proof we need to show that this limit is equal to $\rho(A)$. We will use many cones below, and we summarize their position in Figure 5.5. As $A \in \mathcal{A}_{\Lambda_1(-r)}$, $\rho(A) = T_\rho \circ \pi(A) = \mathrm{Ad}(V^*_{\rho,K_{\Lambda_1(-r)}})(\pi(A))$, where $K_{\Lambda_1(-r)}$ can be chosen to be any cone in $\mathcal{C}$ which is

(i) distal from $\Lambda_1(-r)$ with forbidden direction that of $\mathcal{C}$ (for the definition of distal see [41], we will only use that such a cone exists) and

(ii) clockwise between $\Lambda_1(-r)$ and the forbidden direction.

For $C_r = \Lambda_1(-r) \vee \Lambda_1 \vee K_{\Lambda_1(-r)} = \Lambda_1(-r) \vee K_{\Lambda_1(-r)}$ (the smallest cone including both $\Lambda_1(-r)$ and $K_{\Lambda_1(-r)}$), $\pi(\mathcal{A}_{\Lambda_1(-r)}) \subseteq \pi(\mathcal{A}_{C_r}) \subseteq \pi(\mathcal{A}_{C_r^c})'$, we have $V^*_{\rho,K_{\Lambda_1(-r)}} \in \pi(\mathcal{A}_{(\Lambda_1 \cup K_{\Lambda_1(-r)})^c})' \subseteq \pi(\mathcal{A}_{C_r^c})'$, and so $\rho(A) = T_{\rho,\Lambda_1(-r)}(\pi(A)) = \mathrm{Ad}(V^*_{\rho,K_{\Lambda_1(-r)}})(\pi(A)) \in \pi(\mathcal{A}_{C_r^c})'$. Now we want to use approximate Haag duality to find elements of $\mathcal{B}$ which approximate this and on which $T_{\sigma_{\Lambda_3(t)}}$ acts as the identity $T_{\sigma_{\Lambda_3(t)}}|_{\pi(\mathcal{A}_{\Lambda_3(t)^c})} = \mathrm{id}_{\pi(\mathcal{A}_{\Lambda_3(t)^c})}$. So, we want to pick $K_{\Lambda_1(-r)}$ so that we can find expanded versions of the corresponding C_r, to get arbitrarily good (as $t \to \infty$) approximations to $\rho(A)$ there, and where these expanded versions of C_r are both elements of $\mathcal{C}$ and subsets of $\Lambda_3(t)^c$. So, we want to pick an interval of directions which is a little bit clockwise of the interval of directions for $\Lambda_1(-r)$, and a basepoint, so that even after moving it back and widening it a little, it will still be disjoint from $\Lambda_1(-r) = \Lambda_{-re_{\frac{\pi}{2}},\frac{16\pi}{32},\frac{4\pi}{32}}$. Choose the interval of directions for it to be $(\frac{9\pi}{32} - \frac{\pi}{32}, \frac{9\pi}{32} + \frac{\pi}{32})$. Then, for the basepoint, start with the basepoint of $\Lambda_1(-r)$ (where the only intersection would be the common basepoint), and move it forwards from there by enough to make $K_{\Lambda_1(-r)}$ distal from $\Lambda_1(-r)$. Specifically, let $\vec{x}_r = (-r)\boldsymbol{e}_{\frac{\pi}{2}} + (R_{2\frac{\pi}{32},\varepsilon}+2)\boldsymbol{e}_{\frac{9\pi}{32}}$, for $\varepsilon = \frac{\pi}{64}$, and let $K_{\Lambda_1(-r)} = \Lambda_{\vec{x}_r,\frac{9\pi}{32},\frac{\pi}{32}}$. To check that $K_{\Lambda_1(-r)}$ is distal from $\Lambda_1(-r)$, pick $\varepsilon = \frac{\pi}{64}$ and see that as $\frac{\pi}{64} < (\frac{\pi}{2} - \frac{\pi}{8}) - (\frac{9\pi}{32} + \frac{\pi}{32})$, that the range of directions for $(\Lambda_{\vec{x}_r,\frac{9\pi}{32},\frac{\pi}{32}})_\varepsilon$ and $\Lambda_1(-r) = \Lambda_{-re_{\frac{\pi}{2}},\frac{\pi}{2},\frac{\pi}{8}}$ are disjoint, and so $(\Lambda_{\vec{x}_r,\frac{9\pi}{32},\frac{\pi}{32}} - R_{2\frac{\pi}{32},\varepsilon}\vec{e}\boldsymbol{e}_{\frac{9\pi}{32}})_\varepsilon = \Lambda_{(-r\boldsymbol{e}_{\frac{\pi}{2}}+2\boldsymbol{e}_{\frac{9\pi}{32}}),\frac{9\pi}{32},\frac{\pi}{32}+\varepsilon}$ is disjoint from $\Lambda_1(-r)$. From this, and that $(K_{\Lambda_1(-r)})_\varepsilon$ and $(\Lambda_1(-r))_\varepsilon$ are disjoint element of $\mathcal{C}$, we have that $K_{\Lambda_1(-r)}$ is distal from $\Lambda_1(-r)$ with forbidden direction $(\frac{3\pi}{2} - \frac{\pi}{4}, \frac{3\pi}{2} + \frac{\pi}{4})$. It is also clockwise from $\Lambda_1(-r)$ with respect to the forbidden direction. Therefore it is a valid choice for $K_{\Lambda_1(-r)}$.

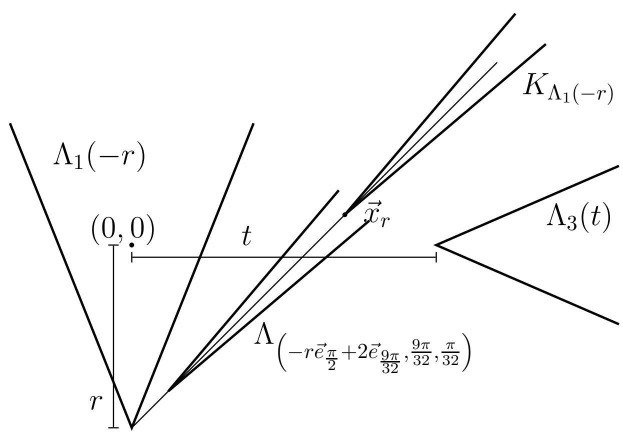

FIGURE 5.5. The cones used in the proof of Lemma 145

With this choice of $K_{\Lambda_1(-r)}$, $C_r = \Lambda_1(-r) \vee K_{\Lambda_1(-r)} = \Lambda_{-r\boldsymbol{e}_{\frac{\pi}{2}},\frac{7\pi}{16},\frac{3\pi}{16}}$.
As $\rho(A) \in \pi(\mathcal{A}_{C_r^c})'$, by Lemma 137 , for $X_{t_2} = \mathrm{Ad}(\tilde{U}_{t_2})(\rho(A))$ we have that, for $t_2 > R_{|\arg C_r|,\varepsilon}$, $X_{t_2} \in \pi(\mathcal{A}_{(C_r)_{\varepsilon+\delta}-t_2\boldsymbol{e}_{C_r}})''$ and $\|X_{t_2} - \rho(A)\| < 2\,\|\rho(A)\|\, f_{|\arg C_r|,\varepsilon,\delta}(t_2)$. For $\varepsilon+\delta < \frac{\pi}{2}$, $((C_r)_{\varepsilon+\delta} - t_2\boldsymbol{e}_{C_r}) \in \mathcal{C}$, and so $T_{\sigma_{\Lambda_3(t)}}$ is strongly continuous on $\pi(\mathcal{A}_{((C_r)_{\varepsilon+\delta}-t_2\boldsymbol{e}_{C_r})})''$. To have $((C_r)_{\varepsilon+\delta} - t_2\boldsymbol{e}_{C_r}) \subset \Lambda_3(t)^c$, we need $\varepsilon + \delta < \frac{\pi}{8}$, and $t > \cot(\frac{\pi}{4} - (\varepsilon + \delta)) \cdot (r + t_2 \cdot (\sin(\frac{7\pi}{16})) - t_2 \cdot \cos(\frac{7\pi}{16})$ (this condition is obtained from the base point of $\Lambda_3(t)$ being to the right of the line which extends the right edge of the cone $((C_r)_{\varepsilon+\delta} - t_2\boldsymbol{e}_{C_r}))$. So, it suffices that $\varepsilon + \delta < \frac{\pi}{8}$ and $t \geq \cot(\frac{\pi}{8}) \cdot (r + t_2)$. So, we can set $t_2 = t\tan(\frac{\pi}{8}) - r$. Now having $((C_r)_{\varepsilon+\delta} - t_2\boldsymbol{e}_{C_r}) \subset \Lambda_3(t)^c$, we have that $T_{\sigma_{\Lambda_3(t)}}$ is the identity

on $\pi(\mathcal{A}_{((C_r)_{\varepsilon+\delta}-t_2\boldsymbol{e}_{C_r})})$, and so by the weak continuity is the identity on $\pi(\mathcal{A}_{((C_r)_{\varepsilon+\delta}-t_2\boldsymbol{e}_{C_r})})''$, and so $T_{\sigma_{\Lambda_3(t)}}(X_{t_2}) = X_{t_2}$. Because both $\rho(A)$ and X_{t_2} are elements of $\mathcal{B}$, we have

$$\begin{aligned} T_{\sigma_{\Lambda_3(t)}}(\rho(A)) &= T_{\sigma_{\Lambda_3(t)}}(X_{t_2} + (\rho(A) - X_{t_2})) \\ &= \rho(A) - (\rho(A) - X_{t_2}) + T_{\sigma_{\Lambda_3(t)}}(\rho(A) - X_{t_2}). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| T_{\sigma_{\Lambda_3(t)}}(\rho(A)) - \rho(A) \right\| &\leq \|(\rho(A) - X_{t_2}\| + \left\| T_{\sigma_{\Lambda_3(t)}}(\rho(A) - X_{t_2}) \right\| \\ &\leq 4 \|\rho(A)\| f_{|\arg C_r|,\varepsilon,\delta}(t_2), \end{aligned}$$

which goes to 0 as t, and therefore t_2, goes to infinity. ☐

LEMMA 146. *Suppose $\rho \in \mathcal{O}_{\Lambda_1}$, $\sigma \in \mathcal{O}_{\Lambda_0}$, and $V \in \mathcal{U}(\mathcal{H})$ is such that $\rho' = \mathrm{Ad}(V) \circ \rho \in \mathcal{O}_{\Lambda_1}$. Then, $\epsilon(\rho', \sigma) = \lim_{t\to\infty} [[V^*_{\sigma,\Lambda_2(t)}, V]] \cdot \mathrm{Ad}(V)(\epsilon(\rho, \sigma))$.*

PROOF.

$$\begin{aligned} \epsilon(\rho', \sigma) &= \lim_{t\to\infty} V^*_{\sigma,\Lambda_2(t)} T_{\rho'}(V_{\sigma,\Lambda_2(t)}) \\ &= \lim_{t\to\infty} V^*_{\sigma,\Lambda_2(t)} \mathrm{Ad}(V) \circ T_\rho(V_{\sigma,\Lambda_2(t)}) \\ &= \lim_{t\to\infty} V^*_{\sigma,\Lambda_2(t)} V T_\rho(V_{\sigma,\Lambda_2(t)}) V^* \\ &= \lim_{t\to\infty} (V^*_{\sigma,\Lambda_2(t)} V V_{\sigma,\Lambda_2(t)} V^*) V (V^*_{\sigma,\Lambda_2(t)} T_\rho(V_{\sigma,\Lambda_2(t)})) V^* \\ &= \lim_{t\to\infty} [[V^*_{\sigma,\Lambda_2(t)}, V]] \, \mathrm{Ad}(V)(V^*_{\sigma,\Lambda_2(t)} T_\rho(V_{\sigma,\Lambda_2(t)})) \\ &= \lim_{t\to\infty} [[V^*_{\sigma,\Lambda_2(t)}, V]] \cdot \mathrm{Ad}(V)(\epsilon(\rho, \sigma)) \end{aligned}$$

☐

LEMMA 147. *Let $\rho \in \mathcal{O}_{\Lambda_1}$ and $\sigma \in \mathcal{O}_{\Lambda_0}$ such that definition 5.3 applies. Then, the resulting morphism $\epsilon(\rho, \sigma)$ coincides with that defined in Definition 4.11 of [6] for $(\theta, \varphi) = (\frac{3\pi}{2}, \frac{\pi}{4})$.*

PROOF. First, let us recall Definition 4.11 of [6], in the case that $(\theta, \varphi) = (\frac{3\pi}{2}, \frac{\pi}{4})$:
To avoid ambiguity with the cones $\Lambda_0 := \Lambda_{\mathbf{0},\frac{\pi}{2},\frac{5\pi}{8}}$, $\Lambda_1 := \Lambda_{\mathbf{0},\frac{\pi}{2},\frac{\pi}{8}}$, and $\Lambda_2 := \Lambda_{\mathbf{0},\pi,\frac{\pi}{8}}$, we add add primes to the variable names in the definition. Let $\Lambda_0' \in \mathcal{C}$ (we take $\Lambda_0' = \Lambda_0$) and $\rho, \sigma \in \mathcal{O}_0$, and pick

any two cones $\Lambda_1', \Lambda_2' \in \mathcal{C}$ such that Λ_2' is counterclockwise from Λ_1' (in the sense applicable for cones in $\mathcal{C}$) and such that Λ_1' is distal from Λ_2' with forbidden direction $(\frac{3\pi}{2}, \frac{\pi}{4})$, and vice versa. Then pick a $V_{\rho,\Lambda_0'} \in \mathcal{V}_{\rho,\Lambda_0'}$, a $V_{\sigma,\Lambda_0'} \in \mathcal{V}_{\sigma,\Lambda_0'}$, and for all $t_1, t_2 \geq 0$ a $V_{\rho,\Lambda_1'(t_1)} \in \mathcal{V}_{\rho,\Lambda_1'(t_1)}$ and $V_{\sigma,\Lambda_2'(t_2)} \in \mathcal{V}_{\sigma,\Lambda_2'(t_2)}$, and set $W^{\vec{t}}_{\rho\Lambda_0'\Lambda_1'} = V_{\rho,\Lambda_1'(t_1)} V^*_{\rho,\Lambda_0}$ and $W^{\vec{t}}_{\sigma\Lambda_0'\Lambda_2'} = V_{\sigma,\Lambda_2'(t_2)} V^*_{\sigma,\Lambda_0'}$. Then define:

$$\epsilon_+^{(\Lambda_0')}(\rho,\sigma) := \lim_{\vec{t}\to\infty} (W^{\vec{t}}_{\sigma\Lambda_0'\Lambda_2'} \otimes W^{\vec{t}}_{\rho\Lambda_0'\Lambda_1'})^* (W^{\vec{t}}_{\rho\Lambda_0'\Lambda_1'} \otimes W^{\vec{t}}_{\sigma\Lambda_0'\Lambda_2'}),$$

where $\vec{t} = (t_1, t_2)$ and $\lim\limits_{\vec{t}\to\infty}$ means $\lim\limits_{t_1\to\infty, t_2\to\infty}$.

Now we will show that for $\rho \in \mathcal{O}_{\Lambda_1}$ and $\sigma \in \mathcal{O}_{\Lambda_0}$, that this reduces to Definition 5.3.

Observe that $\Lambda_2 := \Lambda_{\mathbf{0},\pi,\frac{\pi}{8}}$ and $\Lambda_1 := \Lambda_{\mathbf{0},\frac{\pi}{2},\frac{\pi}{8}}$ are such that Λ_2 is counterclockwise from Λ_1. Moreover, for sufficiently large s, in addition to $\Lambda_2(s)$ and Λ_1 satisfying the condition that $\Lambda_2(s)$ be counterclockwise from Λ_1, the pair also satisfies the condition in [6, Definition 4.11] that the two are distal from each other with forbidden direction $(\frac{3\pi}{2}, \frac{\pi}{4})$, so $\Lambda_2(s)$ and Λ_1 can be the Λ_2' and Λ_1' of [6, Definition 4.11]. So, we set $\Lambda_1' = \Lambda_1$ and $\Lambda_2' = \Lambda_2$ (the requirement of the s in order for $\Lambda_2(s)$ and Λ_1 to be distal will not matter because of the limits that are taken in both definitions, so we take $\Lambda_2' = \Lambda_2$ rather than setting $\Lambda_2' = \Lambda_2(s_0)$ for some sufficiently large s_0, in order to simplify notation).

Because $\rho \in \mathcal{O}_{\Lambda_1} \subset \mathcal{O}_{\Lambda_0}$ (as $\Lambda_1 \subset \Lambda_0$) and $\sigma \in \mathcal{O}_{\Lambda_0}$ we can choose the unitaries $V_{\rho,\Lambda_0} = 1 \in \mathcal{V}_{\rho,\Lambda_0}$ and $V_{\sigma,\Lambda_0} = 1 \in \mathcal{V}_{\sigma,\Lambda_0}$, so that $W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} = V_{\rho,\Lambda_1(t_1)}$ and $W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} = V_{\sigma,\Lambda_2(t_2)}$.

$$(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^* (W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}) = T_\sigma((W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*)(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2})^* (W^{\vec{t}}_{\rho\Lambda_0\Lambda_1}) T_\rho(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2})$$

By Lemma 4.1 of [6], $W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \in (T_\rho, T_\rho^{(\frac{3\pi}{2},\frac{\pi}{4}),\Lambda_1+t_1,V_{\rho,\Lambda_1+t_1}})$ and $W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} \in (T_\sigma, T_\sigma^{(\frac{3\pi}{2},\frac{\pi}{4}),\Lambda_2+t_2,V_{\sigma,\Lambda_2+t_2}})$. Therefore, $T_\sigma((W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*)(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2})^* = (W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2})^* T_\sigma^{(\frac{3\pi}{2},\frac{\pi}{4}),\Lambda_2+t_2,V_{\sigma,\Lambda_2+t_2}}((W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*)$. So, we have

$$\begin{aligned}(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2} &\otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^* (W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} \otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}) \\ &= (W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2})^* T_\sigma^{(\frac{3\pi}{2},\frac{\pi}{4}),\Lambda_2+t_2,V_{\sigma,\Lambda_2+t_2}}\left((W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*\right)(W^{\vec{t}}_{\rho\Lambda_0\Lambda_1}) T_\rho(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}).\end{aligned}$$

Because $\rho \in \mathcal{O}_{\Lambda_1}$, $1 \in \mathcal{V}_{\rho,\Lambda_1}$, so by [41, Lemma 2.2], $W^{\vec{t}}_{\rho\Lambda_0\Lambda_1} = V_{\rho,\Lambda_1+t_1} 1^* \in \pi(\mathcal{A}_{(\Lambda_1\cup\Lambda_1+t_1)^c})' = \pi(\mathcal{A}_{\Lambda_1^c})'$. As $\Lambda_2 + t_2$ is counterclockwise from Λ_1, by [6, Lemma 2.19], we have that

$$\lim_{t_2\to\infty} \left\| T_\sigma^{(\frac{3\pi}{2},\frac{\pi}{4}),\Lambda_2+t_2,V_{\sigma,\Lambda_2+t_2}}|_{\pi(\mathcal{A}_{\Lambda_1^c})'} - \mathrm{id}_{\pi(\mathcal{A}_{\Lambda_1^c})'} \right\| = 0$$

and therefore $\lim_{t_2\to\infty} T_\sigma^{(\frac{3\pi}{2},\frac{\pi}{4}),\Lambda_2+t_2,V_{\sigma,\Lambda_2+t_2}}(V^*_{\rho,\Lambda_1+t_1})(V_{\rho,\Lambda_1+t_1}) = 1$ where this convergence is uniform in t_1. Therefore,

$$\lim_{t_2\to\infty}(W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}\otimes W^{\vec{t}}_{\rho\Lambda_0\Lambda_1})^*(W^{\vec{t}}_{\rho\Lambda_0\Lambda_1}\otimes W^{\vec{t}}_{\sigma\Lambda_0\Lambda_2}) = \lim_{t_2\to\infty}(V_{\sigma,\Lambda_2(t_2)})^* T_\rho(V_{\sigma,\Lambda_2(t_2)}),$$

which is our definition of $\epsilon(\rho,\sigma)$.

Therefore under the conditions of definition 5.3, the morphism $\epsilon(\rho,\sigma)$ applies coincides with [6, Definition 4.11], as desired.

□

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