Splicing Braid Varieties: Geometry, Cluster Algebras, and Cohomology

By

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To Ryan and Troy.

Contents

Abstract

This dissertation investigates the geometry and topology of braid varieties, a class of smooth affine algebraic varieties arising from positive braids, through the novel perspective of splicing maps. These varieties have connections to a wide array of mathematical structures including positroid varieties, cluster algebras, and Legendrian link invariants.

We begin by defining braid varieties both algebraically, via products of braid matrices, and geometrically, as sequences of flags satisfying certain position conditions. These two descriptions are shown to be equivalent, connecting braid varieties to classical and modern geometric objects. Special attention is given to two-stranded braid varieties, which serve as an illustrative and computable model throughout the thesis.

The next step is to view two-stranded braid varieties as special cases of open positroid varieties, providing a concrete realization of these spaces inside Grassmannians. Through this identification, we introduce standard form matrices and describe these varieties in terms of Plücker coordinates, enabling explicit computation and a bridge to cluster algebra structures.

We then construct cluster structures on open positroid varieties associated with braid varieties, making use of triangulations of polygons and the combinatorics of Plücker coordinates. A key insight is the realization of the U_{fan} chart, an explicit cluster chart for two-stranded braid varieties. These structures allow us to link braid varieties to cluster algebras.

The central innovation of the thesis lies in the development of a splicing map, a geometric and algebraic tool inspired by braid composition. We show that the splicing map is a quasi-cluster isomorphism preserving the cluster structure. In particular, we describe how the splicing map reflects the combinatorics of torus link multiplication and how it acts on positroid varieties via transformations of Plücker coordinates.

Finally, we compute the cohomology of braid varieties using a combination of Alexander duality, de Rham theory, and recursive polynomial identities arising from braid matrix factorizations. The cohomology ring is presented explicitly in terms of generators and relations derived from the recursive polynomials defining the varieties, shedding light on their topological invariants.

This work provides a framework for studying braid varieties through tools from cluster algebras, positroid geometry, and cohomological methods. The splicing map construction offers a practical way to build and understand more complex braid varieties from simpler ones, while the cohomology calculations give a clearer

picture of their topological structure. These results suggest a number of potential connections to other areas like representation theory and low-dimensional topology, and open the door for future exploration.

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CHAPTER 1

Introduction

Braid varieties are a class of smooth affine algebraic varieties associated with positive braids. These varieties emerge from the study of braid groups, which have deep connections to various branches of mathematics, from topology and algebraic geometry to representation theory and combinatorics.

Although braid varieties have become an important area of study, their historical development and origins are not as clearly established as those of other algebraic varieties. This is due to the fact that braid matrices, used in the definition of braid varieties, have appeared in a variety of mathematical contexts over the centuries. The earliest notable appearances trace back to L. Euler's work on continuants in the 18^{th} century [**12**], which generalize determinants and describe recursive structures on matrices. These continuants, which are closely linked to continued fractions, can be viewed as precursors to braid matrices. Continued fractions encode recursive relationships and transformations, providing a foundation for the combinatorial representation of braid transformations in braid matrices. In the 19^{th} century, G. Stokes studied solutions to linear differential equations near irregular singularities [**33**]. The Stokes phenomenon describes how the asymptotic solutions change as one crosses Stokes lines near singularities. This phenomenon is related to braid varieties through the study of how configurations interact as you travel along a path. Braid groups, which govern the algebraic structure of braids, can be viewed as geometric generalizations of this phenomenon. This connection between differential equations and braid groups laid the groundwork for developments in the study of wild character varieties, particularly in the work of P. Boalch [**1, 2**], who established explicit links between moduli spaces and braid group actions.

Further connections to braid varieties emerge in representation theory through the work of M. Broué and J. Michel [**3**], who investigated algebraic varieties associated to finite groups of Lie type, called Deligne-Lusztig varieties, using techniques that reflect the structure of braid groups and their associated flag varieties. These ideas are closely tied to the geometry of Bruhat decompositions and double coset representatives, themes that also appear in the structure of braid varieties. P. Deligne [**9**] studied braid group actions in the setting of algebraic and arithmetic geometry, focusing on their role in the structure of fundamental groups and their representations. Similar structures arise in the study of braid varieties, where monodromy and flag-based descriptions play a central role.

In the context of low-dimensional topology and contact geometry, braid varieties are closely related to augmentation varieties of Legendrian links. T. Kálmán [**24**] gave a combinatorial model for the Legendrian contact differential graded algebra (DGA), where the algebraic data of positive braids encode invariants of Legendrian knots. These Legendrian invariants turn out to have deep connections with the geometry of braid varieties, particularly in their interpretation as moduli spaces of augmentations or constructible sheaves. Similarly, A. Mellit's work [**27**] on the curious Lefschetz property in the cohomology of character varieties shows that the topology of such moduli spaces is often governed by recursive algebraic patterns reminiscent of those that define braid varieties.

Braid varieties play a central role in bridging concepts in algebraic geometry, representation theory, low-dimensional topology, and symplectic geometry. They appear not merely as isolated geometric objects but as rich intersections of ideas from many mathematical disciplines. This thesis builds on these perspectives by developing new tools to understand how braid varieties decompose, interact with cluster structures, and how their cohomology rings, particularly in the two-stranded case, can be described explicitly.

Organization. Chapters 2 - 4 are generally preparatory. Chapter 2 focuses on defining braid varieties, discussing their properties, and delivering an explicit, recursive formula for two-stranded braid varieties [Lemma 2.1.6]. The recursive formula proves to be a useful tool for simplifying the process of computing the singular cohomology of two-stranded braid varieties, as compared to the approach in [**26**]. Chapter 3 focuses on positroid varieties, with the main goal of connecting braid varieties to open positroid varieties. Here, we define an explicit map between two-stranded braid varieties and big cells in $\mathrm{Gr}(2, n)$, i.e., the top dimensional positroid variety in $\mathrm{Gr}(2, n)$. Chapter 4 defines cluster algebras and provides explicit constructions for cluster structures on open positroids varieties associated to torus braids including two-stranded braid varieties.

Chapter 5 is the main focus of this paper, where we define the splicing map for maximal dimension open positroid varieties associated with torus links (Theorem 5.1.10) and provide an explicit splicing map for two-stranded braid varieties (Theorem 5.4.1). We demonstrate that the splicing map is a quasi-cluster isomorphism (Theorem 5.2.3) and investigate various properties of the map, including its geometric connection to the decomposition of braid varieties, as well as the quasi-associativity of the splicing map for two-stranded braid varieties (Theorems 5.4.3 and 5.4.5). In Chapter 6, we compute the cohomology of two-stranded braid varieties (Theorem 6.1.3) and describe its ring structure using generators and relations (Theorem 6.2.5). Finally, in Section 6.2.4, we examine the effects of the two-stranded splicing map on the cohomology and its ring structure.

CHAPTER 2

Braid varieties

Consider the positive braid monoid on n strands, Br_n^+, defined as

$$\mathrm{Br}_n^+ = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1, \ldots, n-2,\ \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| > 1\rangle$$

where σ_i is the positive crossing between the i and $i+1$ strand.

Let $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell}$ be a positive braid on n strands. We define the braid variety associated to β in two ways: algebraically in terms of braid matrices or geometrically as a sequence of flags. Refer to [**6, 7, 5, 18**] for more information and context on braid varieties.

We begin by defining the braid variety $X(\beta)$ algebraically. For the positive braid β, we assign a complex variable z (see Figure 2.1) and an $n \times n$ matrix $B_i(z)$ at each crossing σ_i. We define the matrix $B_i(z)$ as

$$B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & z & -1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}$$

where the non-trivial 2×2 embedded matrix is at the i and $i+1$ row and column. We define the ***braid matrix*** associated to $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell}$ as

$$B_\beta(z_1, \ldots, z_\ell) = B_{i_1}(z_1) \cdots B_{i_\ell}(z_\ell) \in SL(n, \mathbb{C}[z_1, \ldots, z_\ell]).$$

One can check that braid matrices satisfy the braid relations up to a change of variables, given by

$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_1z_3 - z_2)B_{i+1}(z_1), \qquad \text{for all } i \in [1, n-2] \tag{2.1}$$

$$B_i(z_1)B_j(z_2) = B_j(z_2)B_i(z_1), \qquad \text{for } |i-j| > 1. \tag{2.2}$$

Definition 2.0.1. The ***braid variety*** $X(\beta)$ is defined by

$$X(\beta) := \left\{ (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell : \begin{pmatrix} 0 & \ldots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \ldots & 0 \end{pmatrix} B_\beta(z_1, \ldots, z_\ell) \text{ is upper-triangular.} \right\}$$

From Equations (2.1) and (2.2), we see that under braid moves the resulting braid varieties up to a change of variables are equivalent and conclude that:

Theorem 2.0.2. *Braid varieties are braid invariants.*

Note that there is a surjective homomorphism $\pi : \mathrm{Br}_n^+ \to S_n$, given by $\pi(\sigma_i) = s_i$ where $s_1, \ldots, s_{n-1}$ are simple transpositions of i and $i+1$ in S_n. The Demazure product $\delta : \mathrm{Br}_n^+ \to S_n$ is defined inductively by:

$$\delta(e) = e,\ \delta(\beta\sigma_i) = \begin{cases} \delta(\beta)s_i & \text{if } \delta(\beta)s_i > \delta(\beta) \\ \delta(\beta) & \text{else,} \end{cases}$$

Unlike π, the Demazure product δ is not a morphism of monoids. As an abuse of notation, we will denote by w_0 the longest element in both S_n and Br_n^+.

Now, we define braid varieties geometrically, consider the variety of complete flags

$$\mathrm{Fl}_n = \{0 = \mathcal{F}_0 \subset \mathcal{F}_1 \cdots \subset \mathcal{F}_n = \mathbb{C}^n\},\ \dim \mathcal{F}_i = i.$$

We say that two flags $\mathcal{F}$ and $\mathcal{F}'$ are in position s_i, denoted $\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$, if $\mathcal{F}_j = \mathcal{F}'_j$ for $j \neq i$ and $\mathcal{F}_i \neq \mathcal{F}'_i$.

Definition 2.0.3. The ***braid variety*** $X(\beta)$ is defined as the space of sequences of flags

$$\mathcal{F}^{(0)} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)} \cdots \xrightarrow{s_{i_{\ell-1}}} \mathcal{F}^{(\ell-1)} \xrightarrow{s_{i_\ell}} \mathcal{F}^{(\ell)}$$

such that $\mathcal{F}^{(0)}$ is the standard flag and $\mathcal{F}^{(\ell)}$ is the antistandard flag in $\mathbb{C}^n$:

$$\mathcal{F}_i^{(0)} = \langle e_1, \ldots, e_i \rangle,\ \mathcal{F}_i^{(\ell)} = \langle e_{j-i+1}, \ldots, e_n \rangle.$$

We will often use the abbreviation $\mathcal{F}^{(0)} \overset{\beta}{\dashrightarrow} \mathcal{F}^{(\ell)}$.

Theorem 2.0.4. *Definitions 2.0.1 and 2.0.3 for braid varieties are equivalent.*

Proof. Suppose that $X(\beta)$ is defined as in Definition 2.0.1 for some braid $\beta = \sigma_{i_1} \dots \sigma_{i_\ell}$. For each crossing σ_{i_j} where $j \in 1, \dots, \ell$, we have $\pi(\sigma_{i_j}) = s_{i_j}$. Fix $\mathcal{F}^{(0)}$ as the standard flag. We see that multiplication by $B_{\sigma_{i_1}}(z_1)$ gives $\mathcal{F}^{(0)}_m = \mathcal{F}^{(1)}_m$ for $m \neq i_1$ and $\mathcal{F}^{(0)}_{i_1} \neq \mathcal{F}^{(1)}_{i_1}$, therefore, $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ are in position s_{i_1}, i.e., $\mathcal{F}^{(0)} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)}$. Conversely, if $\mathcal{F}^{(0)} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)}$, then there exists a unique z_1 such that $B_{i_1}(z_1)\mathcal{F}^{(0)} = \mathcal{F}^{(1)}$. By induction, we have that $B_{\sigma_{i_1}}(z_1) \dots B_{\sigma_{i_\ell}}(z_\ell) = B_\beta(z_1, \dots, z_\ell)$ describes the sequence of flags $\mathcal{F}^{(0)} \xrightarrow{s_{i_1}} \mathcal{F}^{(1)} \dots \xrightarrow{s_{i_{\ell-1}}} \mathcal{F}^{(\ell-1)} \xrightarrow{s_{i_\ell}} \mathcal{F}^{(\ell)}$.

Multiplication of the antidiagonal matrix with 1s along the antidiagonal corresponds to a half twist, therefore, we see that $\mathcal{F}^{(\ell)}$ is the antistandard flag in $\mathbb{C}^n$. Therefore, describing Definition 2.0.3. □

As a consequence of Escobar's work on brick manifolds [**10**],

Theorem 2.0.5 (Casals-Gorsky-Gorsky-Le-Shen-Simental[**5**]). *The braid variety $X(\beta)$ is a smooth, irreducible affine algebraic variety of dimension $l(\beta) - l(\omega_0)$, where ω_0 is the lift of the longest word. The variety $X(\beta)$ is nonempty if and only if the Demazure product $\delta(\beta) = w_0$.*

2.1. Two-stranded braid varieties

In this section we focus on the case $n = 2$ and $\beta \in \mathrm{Br}_2^+$. Given that there is only one possible crossing, we then refer to the two-stranded braid with ℓ crossings as σ^ℓ. The braid matrices on two strands are given as a product of the matrices

$$B(z) = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2.1.1. The ***two-stranded braid variety*** $X(\sigma^\ell)$ is defined by the equation

$$X(\sigma^\ell) := \left\{ (z_1, \dots, z_\ell) \in \mathbb{C}^\ell : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B(z_1) \cdots B(z_\ell) \text{ is upper-triangular.} \right\}$$

If β and β' are related by braid moves then $X(\beta) \simeq X(\beta')$, this isomorphism arises from the invariance of braid matrices.

Remark 2.1.2. The modification of -1 in Definition 2.1.1 compared to Definition 2.0.1 is made solely to guarantee that the product of braid matrices lies in $SL(2, \mathbb{C})$, thereby ensuring total positivity and does not alter the braid variety.

EXAMPLE 2.1.3. Let $\beta = \sigma^1 \in \mathrm{Br}_2^+$, the braid matrix is given by

$$B_\beta(z_1) = \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore, the braid variety is defined as

$$\begin{aligned} X(\sigma^1) &= \left\{ z_1 \in \mathbb{C} : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix} \text{ is upper-triangular.} \right\} \\ &= \left\{ z_1 \in \mathbb{C} : \begin{pmatrix} -1 & 0 \\ z_1 & -1 \end{pmatrix} \text{ is upper triangular} \right\} = \{z_1 = 0\}. \end{aligned}$$

More precisely, $X(\sigma^1)$ is a point.

EXAMPLE 2.1.4. Let $\beta = \sigma^2 \in \mathrm{Br}_2^+$ with braid matrix

$$B_\beta(z_1, z_2) = \begin{pmatrix} z_1 z_2 - 1 & -z_1 \\ z_2 & -1 \end{pmatrix}$$

then the braid variety associated to β is

$$\begin{aligned} X(\sigma^2) &= \left\{ (z_1, z_2) \in \mathbb{C}^2 : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 z_2 - 1 & -z_1 \\ z_2 & -1 \end{pmatrix} \text{ is upper-triangular.} \right\} \\ &= \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 - 1 = 0\} \cong \{z_1 \in \mathbb{C} : z_1 \neq 0\} \end{aligned}$$

It is important to note that the choice of coordinate z_1 on $X(\sigma^2) = \{z_1 \neq 0\}$ is not unique in this case, we may have also chosen $X(\sigma^2) = \{z_2 \neq 0\}$. However, the choice of $X(\sigma^2) = \{z_1 \neq 0\}$ is helpful when developing an inductive way to describe the braid variety in order to compute its cohomology.

EXAMPLE 2.1.5. Let $\beta = \sigma^3 \in \mathrm{Br}_2^+$, see Figure 2.1, with braid matrix

$$B_\beta(z_1, z_2, z_3) = \begin{pmatrix} z_1 z_2 z_3 - z_3 - z_1 & 1 - z_1 z_2 \\ z_2 z_3 - 1 & -z_2 \end{pmatrix}$$

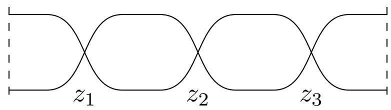


FIGURE 2.1. The braid $\beta = \sigma^3$, where each crossing is assigned a complex variable z, belongs to the positive braid monoid. Therefore, we can omit crossing information since all crossings are positive.

then the braid variety associated to β is

$$X(\sigma^3) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1z_2z_3 - z_3 - z_1 & 1 - z_1z_2 \\ z_2z_3 - 1 & -z_2 \end{pmatrix} \text{ is upper-triangular.} \right\}$$

$$= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1z_2z_3 - z_3 - z_1 = 0\} \cong \{(z_1, z_2) \in \mathbb{C}^2 : z_1z_2 - 1 \neq 0\}$$

There is an inductive relationship between $X(\sigma^\ell)$ and $X(\sigma^{\ell-1})$, we explore this concept further by first establishing general formulas for the braid matrices then extending these results to the polynomials defining the braid varieties. Moreover, with these results we show that the braid variety $X(\sigma^\ell)$ is smooth.

LEMMA 2.1.6 (Hughes[**23**], Chantraine-Ng-Sivek[**8**]). *One can express the braid matrix for $\beta = \sigma^k$ as*

$$B_\beta(z_1, \ldots, z_\ell) = \begin{pmatrix} F_\ell(z_1, \ldots, z_\ell) & -F_{\ell-1}(z_1, \ldots, z_{\ell-1}) \\ F_{\ell-1}(z_2, \ldots, z_\ell) & -F_{\ell-2}(z_2, \ldots, z_{\ell-1}) \end{pmatrix}$$

where

$$F_\ell(z_i, \ldots, z_{i+\ell}) = z_{i+\ell}F_{\ell-1}(z_i, \ldots, z_{i+\ell-1}) - F_{\ell-2}(z_i, \ldots, z_{i+\ell-2}) \tag{2.1}$$

with initial values $F_1(z_i) = z_i$, $F_0 \equiv 1$ and $F_{-1} \equiv 0$.

PROOF. We proceed with induction on ℓ. Clearly,

$$B_{\sigma^1}(z_1) = \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_1(z_1) & -F_0(\emptyset) \\ F_0(\emptyset) & -F_{-1}(\emptyset) \end{pmatrix}$$

Suppose

$$B_{\sigma^\ell}(z_1, \ldots, z_\ell) = \begin{pmatrix} F_\ell(z_1, \ldots, z_\ell) & -F_{\ell-1}(z_1, \ldots, z_{\ell-1}) \\ F_{\ell-1}(z_2, \ldots, z_\ell) & -F_{\ell-2}(z_2, \ldots, z_{\ell-1}) \end{pmatrix}$$

Then

$$\begin{aligned}
B_{\sigma^{\ell+1}}(z_1,\dots,z_\ell,z_{\ell+1}) &= B_{\sigma^\ell}(z_1,\dots,z_\ell)B_\sigma(z_{\ell+1}) \\
&= \begin{pmatrix} F_\ell(z_1,\dots,z_\ell & -F_{\ell-1}(z_1,\dots,z_{\ell-1}) \\ F_{\ell-1}(z_2,\dots,z_\ell) & -F_{\ell-2}(z_2,\dots,z_{\ell-1}) \end{pmatrix} \begin{pmatrix} z_{\ell+1} & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} z_{\ell+1}F_\ell(z_1,\dots,z_\ell) - F_{\ell-1}(z_1,\dots,z_{\ell-1} & -F_{\ell-1}(z_1,\dots,z_{\ell-1}) \\ z_{\ell+1}F_{\ell-1}(z_2,\dots,z_\ell) - F_{\ell-2}(z_2,\dots,z_\ell) & -F_{\ell-1}(z_2,\dots,z_\ell) \end{pmatrix} \\
&= \begin{pmatrix} F_{\ell+1}(z_1,\dots,z_{\ell+1}) & -F_\ell(z_1,\dots,z_\ell) \\ F_\ell(z_2,\dots,z_\ell) & -F_{\ell-1}(z_2,\dots,z_\ell) \end{pmatrix}
\end{aligned}$$

□

THEOREM 2.1.7 (Hughes [**23**]). *The braid variety $X(\sigma^\ell)$ is defined in $\mathbb{C}^\ell$ by the equation $F_\ell(z_1,\dots,z_\ell) = 0$ where F_ℓ is given by the recursion* (2.1)*.*

Moreover, if $F_\ell(z_1,\dots,z_\ell) = 0$, then $F_{\ell-1}(z_1,\dots,z_{\ell-1}) \neq 0$ and $z_\ell = \dfrac{F_{\ell-2}(z_1,\dots,z_{\ell-2})}{F_{\ell-1}(z_1,\dots,z_\ell)}$.

PROOF. By Lemma 2.1.6, we express the braid matrix as

$$B_\beta(z_1,\dots,z_\ell) = \begin{pmatrix} F_\ell(z_1,\dots,z_\ell) & -F_{\ell-1}(z_1,\dots,z_{\ell-1}) \\ F_{\ell-1}(z_2,\dots,z_\ell) & -F_{\ell-2}(z_2,\dots,z_{\ell-1}) \end{pmatrix}$$

Using the definition for a braid variety, we find that

$$\begin{aligned}
X(\sigma^\ell) &= \left\{ (z_1,\dots,z_\ell) \in \mathbb{C}^\ell : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B_\beta(z_1,\dots,z_\ell) \text{ is upper-triangular} \right\} \\
&= \{(z_1,\dots,z_\ell) \in \mathbb{C}^\ell : F_\ell(z_1,\dots,z_\ell) = 0\}
\end{aligned}$$

Given that $F_\ell(z_1,\dots,z_\ell) = 0$ and $F_\ell = z_\ell F_{\ell-1} - F_{\ell-2}$. If $F_{\ell-1} \neq 0$, then we can solve the equation $F_\ell = 0$ for z_ℓ:

$$F_\ell = z_\ell F_{\ell-1} - F_{\ell-2} = 0, \qquad z_\ell = \frac{F_{\ell-2}}{F_{\ell-1}}.$$

Suppose instead that $F_{\ell-1}(z_1, \ldots, z_{\ell-1}) = 0$ and given that $F_\ell(z_1, \ldots, z_\ell) = 0$ by the definition of $X(\sigma^\ell)$, then $F_{\ell-2}(z_1, \ldots, z_{\ell-2}) = 0$. By proceeding with downward induction on ℓ, we conclude that $F_\ell(z_1, \ldots, z_\ell) = 0$ for all ℓ, contradicting $F_0 = 1$. Therefore, $F_{\ell-1}(z_1, \ldots, z_{\ell-1}) \neq 0$.

$\square$

COROLLARY 2.1.8. *We have* $X(\sigma^\ell) \simeq \{(z_1, \ldots, z_{\ell-1}) \in \mathbb{C}^{\ell-1} : F_{\ell-1}(z_1, \ldots, z_{\ell-1}) \neq 0\}$.

COROLLARY 2.1.9. *The braid variety* $X(\sigma^\ell)$ *is smooth of complex dimension* $\ell - 1$.

PROOF. By Corollary 2.1.8, $X(\sigma^\ell) = \{(z_1, \ldots, z_{\ell-1}) \in \mathbb{C}^{\ell-1} : F_{\ell-1}(z_1, \ldots, z_{\ell-1}) \neq 0\}$. Since $\{(z_1, \ldots, z_{\ell-1}) \in \mathbb{C}^{\ell-1} : F_{\ell-1}(z_1, \ldots, z_{\ell-1}) \neq 0\}$ is an open subset in $\mathbb{C}^{\ell-1}$, then $X(\sigma^\ell)$ is a smooth manifold. $\square$

CHAPTER 3

Positroid varieties

The Grassmannian $\mathrm{Gr}(k, n)$ is a fundamental object in algebraic geometry, representation theory, combinatorics and physics with deep connections to flag varieties and cluster algebras. The ***Grassmannian*** $\mathrm{Gr}(k, n)$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{K}^n$. Alternate notations for the Grassmannian include Gr_k^n, $Gr_k(n)$. For our purposes, we let $\mathbb{K} = \mathbb{C}$, however, the Grassmannian may be defined over a different ring or field, for example it may be defined over $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{F}^q$. The choice of ring or field for the Grassmannian will reveal various properties in its geometrical, topological, and analytic structures.

Given a k-dimensional subspace V we chose a basis and write the basis vectors as rows of the full rank $k \times n$ matrix. We say that two matrices are equivalent, i.e, they represent the same subspace, if they are related by left multiplication by an element in $GL_k(\mathbb{K})$. In other words, points in $\mathrm{Gr}(k, n)$ are described as full rank $k \times n$ matrices up to row operations.

The Grassmannian is a smooth manifold that can be endowed with the structure of a projective smooth algebraic variety using the Plücker embedding. The Plücker embedding maps a k-dimensional subspace given by a matrix V to the set of Plücker coordinates, i.e., determinants of all possible $k \times k$ minors. Under this embedding $\mathrm{Gr}(k, n)$ is realized as a subvariety of projective space $\mathbb{P}^{\binom{n}{k}-1}$.

To explicitly define the map, let $V \in \mathrm{Gr}(k, n)$ and $v_1, \ldots, v_n$ be the columns of V where v_i are k-dimensional vectors. Given an ordered subset $I \in \binom{[n]}{k}$, the *Plücker coordinate* $\Delta_I(V)$ is the minor of $k \times k$ submatrix of V in column set I. We will sometimes consider the exterior algebra $\wedge^\bullet \mathbb{C}^k$, and identify $\Delta_I(V)$ with $v_{i_1} \wedge \cdots \wedge v_{i_k} \in \wedge^k(\mathbb{C}^k) \simeq \mathbb{C}$ for $I = \{i_1, \ldots, i_k\}$.

The row operations have the effect of changing V to AV for an invertible $k \times k$ matrix A. This implies $v_i \mapsto Av_i$ and $\Delta_I \mapsto \det(A)\Delta_I$ for all I. In particular, Δ_I can be considered as projective coordinates on $\mathrm{Gr}(k, n)$, or as affine coordinates on the affine cone $\widehat{\mathrm{Gr}}(k, n)$.

3.1. Open positroid varieties

Positroid varieties are interesting subvarieties of the Grassmannian that can be thought of as juggling patterns. These varieties can be described as intersections of n cyclically shifted Schubert cells and in some cases, as projections of Richardson varieties.

DEFINITION 3.1.1. Let $V = (v_1 \quad v_2 \quad \dots \quad v_n) \in \mathrm{Gr}(k,n)$ where $v_i \in \mathbb{C}^k$, $v_{i+n} = v_i$ and define $f_V : \mathbb{Z} \to \mathbb{Z}$ as

$$f_V(i) = \min\{j \geq i : v_i \in \mathrm{span}(v_{i+1}, \dots, v_j)\}.$$

We say that f_V is a ***bounded affine permutation*** associated to V if it satisfies the following conditions:

(a) $f_V(i+n) = f(i) + n$,

(b) $i \leq f_V(i) \leq i + n$,

(c) $\sum_{i=1}^{n} \big(f_V(i) - i\big) = kn$.

EXAMPLE 3.1.2. Let $V = \begin{pmatrix} 1 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix}$. The bounded affine permutation f_V is given by

i	1	2	3	4	5
$f_V(i)$	3	5	4	6	7

where $v_6 = v_1, v_7 = v_2$

DEFINITION 3.1.3. We define the ***open positroid variety*** as

$$\Pi_f^\circ = \{V \in \mathrm{Gr}(k,n) : f_V = f\}.$$

REMARK 3.1.4. We define a positroid variety Π_f as the Zariski closure of an open positroid variety Π_f°. It is important to note that Π_f is generally not a smooth variety. Therefore, for the purposes of this paper, when we refer to a positroid variety, we mean the open positroid variety Π_f°.

Knutson-Lam-Speyer [**25**] constructed the stratification

$$\mathrm{Gr}(k,n) = \bigsqcup_{f \in \mathbf{B}_{k,n}} \Pi_f^\circ$$

where Π_f° are open positroid varieties indexed by a finite set $\mathbf{B}_{k,n}$ of bounded affine permutations, see [**16**, Section 4.1] for more information. This positroid stratification contains a unique open stratum, the ***top dimensional positroid variety***, defined such that cyclically consecutive Plücker coordinates are non-vanishing, i.e.,

$$\Pi_{k,n}^\circ := \{V \in \mathrm{Gr}(k,n) : \Delta_{1,2,\dots,k}(V), \Delta_{2,3,\dots,k+1}(V), \dots, \Delta_{n,1,2,\dots,k-1}(V) \neq 0\}\,. \tag{3.1}$$

Remark 3.1.5. We sometimes call the unique open stratum the maximal dimension positroid variety or the big cell in the Grassmannian. We note that this variety is defined by the bounded affine permutation $f(i) = i + k$ for all $i \in 1, \dots, n$.

More generally, in [**21**] we define a class of skew shaped positroid varieties associated to skew shaped Young diagrams.

3.2. Positroids as braid varieties

Casals-Gao found an explicit construction [**4**, Section 4] relating $\Pi_{k,n}^\circ$ to the braid variety $X(\beta_{k,n})$ where

$$\beta_{k,n} = (\sigma_1 \dots \sigma_{k-1})^{n-k}(\sigma_1 \dots \sigma_{k-1}) \dots (\sigma_2\sigma_1)\sigma_1 = (\sigma_1 \dots \sigma_{k-1})^{n-k} w_0$$

(see also [**32**]). Here $T(k, n-k) = (\sigma_1 \dots \sigma_{k-1})^{n-k}$ is the $(k, n-k)$ torus braid and $\sigma_1(\sigma_2\sigma_1) \cdots (\sigma_{k-1} \cdots \sigma_1)$ is the specific braid word for the half-twist braid denoted w_0.

Define $I(a,i)$ to be the ordered subsets

$$I(a,i) = \{a, a+1, \dots, a+i-1, n-k+i+1, \dots, n\}, \tag{3.1}$$

where $1 \leq i \leq k$ and $a = n-k-j+1$ for $1 \leq j \leq n-k$. Given a matrix $V = (v_1, \dots, v_n)$, we can fill in the bottom row of the braid diagram for $\beta_{k,n}$ by the vectors $v_1, \dots, v_n$. This uniquely determines the subspaces for all other regions as spans $\langle v_i, \dots, v_j \rangle$ for appropriate i, j, see Figures 3.1, 5.1 and 5.2. The conditions $\Delta_{I(a,k)}(V) \neq 0$ are equivalent to the relative position conditions for each crossing of β. The conditions $\Delta_{I(1,i)}(V) \neq 0$ are equivalent to the fact that two flags

$$\mathcal{F}^{(0)} = \{0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \dots \langle v_1, \dots, v_k \rangle\}$$

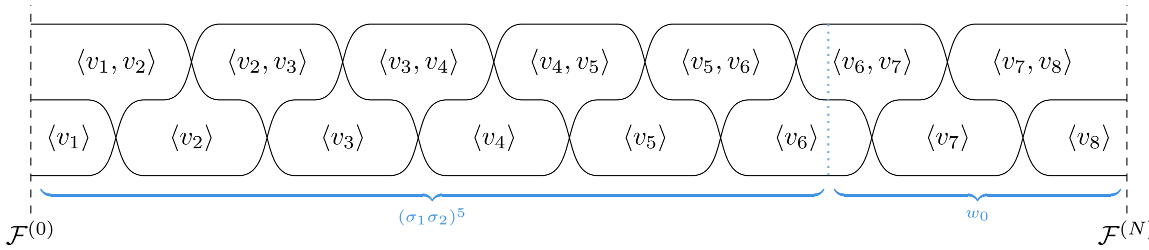


FIGURE 3.1. The braid $\beta_{3,8} = (\sigma_1\sigma_2)^5 w_0$ associated to $\Pi^{\circ,1}_{3,8}$.

and

$$\mathcal{F}^{(N)} = \{0 \subset \langle v_n\rangle \subset \langle v_{n-1}, v_n\rangle \subset \ldots \langle v_{n-k+1}, \ldots, v_n\rangle\}$$

are in position w_0. Therefore there is a unique matrix M such that $M\mathcal{F}^{(0)}$ is the standard flag and $M\mathcal{F}^{(N)}$ is the antistandard flag.

Finally, the flags constructed as above determine the vectors v_i only up to scalars. This can be fixed either by rescaling v_i, or by considering framed flags as in [**4**]. As a result, we obtain the following.

THEOREM 3.2.1 (Casals-Gao[**4**], Shende-Treumann-Williams-Zaslow[**32**]). *Let $\Pi^{\circ,1}_{k,n}$ be the subset of $\Pi^{\circ}_{k,n}$ defined by*

$$\Delta_{b,b+1,\ldots,b+k-1} = \Delta_{I(b,k)} = 1, \quad \text{for } 1 \le b \le n-k.$$

Then $X(\beta_{k,n}) \simeq \Pi^{\circ,1}_{k,n}$.

3.2.1. Two-stranded braid varieties as positroids.

DEFINITION 3.2.2. Let $\Pi^{\circ,1}_{2,n}$ be the subset of the the open positroid variety $\Pi^{\circ}_{2,n}$ such that each $\Delta_{i,i+1} = 1$ for all $1 \le i \le n-1$ and $\Delta_{1,n} \neq 0$.

LEMMA 3.2.3. *Suppose that $v_1, \ldots, v_{\ell+1}$ is a collection of vectors in $\mathbb{C}^2$ such that $v_1 = (1,0)$ and $\det(v_i, v_{i+1}) = 1$. Then there exists a unique collection of parameters $z_1, \ldots, z_\ell$ such that $B(z_1)\cdots B(z_i) = (v_{i+1} \quad -v_i)$ for all i.*

PROOF. Let $v_i = (v_i^1, v_i^2)$, we prove the statement by induction in i. For $i = 1$ we have $v_1 = (1,0)$ and $v_2 = (z,1)$ since $\det(v_i, v_{i+1}) = 1$. For $i > 1$ the vectors v_{i-1}, v_i form a basis of $\mathbb{C}^2$, so we can write $v_{i+1} = \alpha v_{i-1} + \beta v_i$. Now

$$\det(v_i, v_{i+1}) = \alpha\det(v_i, v_{i-1}) + \beta\det(v_i, v_i) = -\alpha\det(v_{i-1}, v_i) + 0 = -\alpha$$

so $\alpha = -1$ and we can denote $z_i = \beta$ and write

$$v_{i+1} = -v_{i-1} + z_i v_i. \tag{3.2}$$

Now

$$\begin{pmatrix} v_{i+1}^1 & -v_i^1 \\ v_{i+1}^2 & -v_i^2 \end{pmatrix} = \begin{pmatrix} v_i^1 & -v_{i-1}^1 \\ v_i^2 & -v_{i-1}^2 \end{pmatrix} \begin{pmatrix} z_i & -1 \\ 1 & 0 \end{pmatrix}$$

and by assumption of induction we have

$$B(z_1)\cdots B(z_{i-1}) = \begin{pmatrix} v_i^1 & -v_{i-1}^1 \\ v_i^2 & -v_{i-1}^2 \end{pmatrix}.$$

□

Remark 3.2.4. Note that $B(z_1)\cdots B(z_i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_{i+1}$.

Lemma 3.2.5. *Let $\Pi^{o}_{2,\ell+1}$ and $\Pi^{o,1}_{2,\ell+1}$ be as described in Equation 3.1 and Definition 3.2.2, then*

a) $\Pi^{\circ,1}_{2,\ell+1}$ is isomorphic to $X(\sigma^\ell)$.

b) $\Pi^{\circ}_{2,\ell+1}$ is isomorphic to $X(\sigma^\ell) \times (\mathbb{C}^*)^\ell$.

Proof. a) We package the vectors v_ℓ in a $2 \times (\ell+1)$ matrix V. Since $\Delta_{1,2} = 1$, we use row operations to ensure that the first column of V is $(1,0)$, so we get

$$V = \begin{pmatrix} 1 & v_2^1 & \cdots & v_{\ell+1}^1 \\ 0 & v_2^2 & \cdots & v_{\ell+1}^2. \end{pmatrix} \in \Pi^{\circ,1}_{2,\ell+1}.$$

By Lemma 3.2.3 we can uniquely find the variables $z_1, \ldots, z_\ell$ such that

$$V = \begin{pmatrix} 1 & F_1(z_1) & \cdots & F_\ell(z_1, \ldots, z_\ell) \\ 0 & F_0 & \cdots & F_{\ell-1}(z_2, \ldots, z_\ell) \end{pmatrix}$$

Note that $\det B_i(z) = 1$, so $\det B_\beta(z_1, \ldots, z_i) = 1$ for any braid β and

$$F_i(z_1, \ldots, z_i) F_i(z_2, \ldots, z_{i+1}) - F_{i+1}(z_1, \ldots, z_{i+1}) F_{i-1}(z_2, \ldots, z_i) = 1, \tag{3.3}$$

so the matrix V indeed satisfies $\Delta_{i,i+1}(V) = 1$. The matrix V belongs to $\Pi^{\circ,1}_{2,\ell}$ if and only if $F_{\ell-1}(z_2, \dots, z_\ell) \neq 0$. In this case, we can use row operations to ensure that $F_\ell(z_1, \dots, z_\ell) = 0$ (we subtract from the first row $F_\ell(z_1, \dots, z_\ell)/F_{\ell-1}(z_2, \dots, z_\ell)$ times the second row).

The braid variety $X(\sigma^\ell)$ is cut out by the equation $\{(z_1, \dots, z_\ell) \in \mathbb{C}^\ell : F_\ell(z_1, \dots, z_\ell) = 0\}$, so we get a map from $\Pi^{\circ,1}_{2,\ell}$ to $X(\sigma^\ell)$. To construct the inverse, observe that $F_\ell(z_1, \dots, z_\ell) = 0$ and (3.3) implies that

$$F_{\ell-1}(z_1, \dots, z_{\ell-1})F_{\ell-1}(z_1, \dots, z_\ell) = 1,$$

so $F_{\ell-1}(z_2, \dots, z_\ell) \neq 0$. Therefore $\Pi^{\circ,1}_{2,\ell} \simeq X(\sigma^\ell)$.

b) Similarly to the above, we can use row operations to ensure any matrix in $\Pi^{\circ}_{2,\ell+1}$ has the first column $(1, 0)$. Now we define a map $\Pi^{\circ,1}_{2,\ell+1} \times (\mathbb{C}^*)^\ell \to \Pi^{\circ}_{2,\ell+1}$ by rescaling all other columns:

$$\varphi : [(v_1, v_2, \dots, v_{\ell+1}), (\lambda_1, \dots, \lambda_\ell)] \mapsto (v_1, \lambda_1 v_2, \dots, \lambda_\ell v_{\ell+1}).$$

The inverse map is clear, since we get

$$\det(\lambda_{i-1}v_i, \lambda_i v_{i+1}) = \lambda_{i-1}\lambda_i,$$

and the scalars λ_i can recovered from the minors $\Delta_{i,i+1}$ for the image of φ.

$\square$

Example 3.2.6. We have

$$B(z_1)B(z_2) = \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z_1 z_2 - 1 & -z_1 \\ z_2 & -1 \end{pmatrix}$$

$$B(z_1)B(z_2)B(z_3) = \begin{pmatrix} z_1 z_2 - 1 & -z_1 \\ z_2 & -1 \end{pmatrix} \begin{pmatrix} z_3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z_1 z_2 z_3 - z_1 - z_3 & 1 - z_1 z_2 \\ z_2 z_3 - 1 & -z_2 \end{pmatrix}.$$

This means that $X(\sigma^3)$ is associated to a point in $\Pi^{\circ,1}_{2,4}$ by packaging v_i in the matrix

$$(v_1 \quad v_2 \quad v_3 \quad v_4) = \begin{pmatrix} 1 & z_1 & z_1 z_2 - 1 & z_1 z_2 z_3 - z_1 - z_3 \\ 0 & 1 & z_2 & z_2 z_3 - 1. \end{pmatrix}$$

3.2.2. Standard form for two-stranded positroids. Throughout this section, we have studied various maps between braid varieties and positroid varieties. To work with such maps, it is useful to fix a specific isomorphism between $X(\sigma^\ell)$ and $\Pi^{\circ,1}_{2,\ell+1}$ which is given by lemmas below.

LEMMA 3.2.7. *Let $M = (v_1 \quad v_2 \quad \dots \quad v_n) \in \Pi^\circ_{2,n}$. There is a unique matrix $A \in GL(2,\mathbb{C})$ such that*

$$AM = \begin{pmatrix} 1 & * & \dots & 0 \\ 0 & 1 & \dots & * \end{pmatrix} = V$$

where $\det A = \Delta_{12}^{-1}(M)$ *and* $\Delta_{ij}(V) = \Delta_{ij}(M) \cdot \det A = \dfrac{\Delta_{ij}(M)}{\Delta_{12}(M)}$.

PROOF. If $M = (v_1 \quad v_2 \quad \dots \quad v_n)$, then acting on the left with the matrix $S = (v_1 \quad v_n)^{-1}$, we obtain

$$S \cdot M = \frac{1}{\Delta_{1n}(M)} \begin{pmatrix} v_n^2 & -v_n^1 \\ -v_1^2 & v_1^1 \end{pmatrix} \begin{pmatrix} v_1^1 & v_2^1 & \dots & v_n^1 \\ v_1^2 & v_2^2 & \dots & v_n^2 \end{pmatrix} = \begin{pmatrix} 1 & * & \dots & 0 \\ 0 & \alpha & \dots & 1 \end{pmatrix}$$

where $\alpha = \det(S)\Delta_{12}(M) = \dfrac{\Delta_{12}(M)}{\Delta_{1n}(M)}$. Now, if we act on the left by $T = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, we obtain

$$T \cdot (S \cdot M) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & * & \dots & 0 \\ 0 & \alpha & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & * & \dots & 0 \\ 0 & 1 & \dots & \alpha^{-1} \end{pmatrix}$$

Let $A = T \cdot S$, then $\det A = (\det T)(\det S) = \left(\dfrac{\Delta_{1n}(M)}{\Delta_{12}(M)}\right)\left(\dfrac{1}{\Delta_{1n}(M)}\right) = \Delta_{12}^{-1}(M)$. □

LEMMA 3.2.8. *Given the standard form matrix*

$$V = \begin{pmatrix} 1 & * & \dots & 0 \\ 0 & 1 & \dots & * \end{pmatrix} \tag{3.4}$$

where $\Delta_{i,i+1} \neq 0$, $\Delta_{1n} \neq 0$, we may rescale the vectors $(v_3, \dots, v_n)$ to $(v_3', \dots, v_n') = (\lambda_3 v_3, \dots, \lambda_n v_n)$ such that $\Delta'_{i,i+1} = 1$. Furthermore, such λ_i are unique.

PROOF. Let

$$v_3' = \frac{v_3}{\Delta_{23}},\ v_4' = \frac{v_4 \cdot \Delta_{23}}{\Delta_{34}},\ \dots,\ v_n' = v_n \prod_{l=2}^{n-1} \Delta_{l,l+1}^{(-1)^{n-l}}$$

Note that with the above rescaling Δ'_{1n} remains nonzero, whereas for $\Delta'_{i,i+1}$ the rescaling gives the desired result:

$$\begin{aligned}\Delta'_{i,i+1} &= \det\begin{pmatrix} v'_i & v'_{i+1}\end{pmatrix} = \det\begin{pmatrix} v_i \prod_{l=2}^{i-1} \Delta_{l,l+1}^{(-1)^{i-l}} & v_{i+1} \prod_{l=2}^{i} \Delta_{l,l+1}^{(-1)^{i+1-l}}\end{pmatrix} \\ &= \Delta_{i,i+1}\Delta_{i,i+1}^{(-1)} = 1.\end{aligned}$$

□

Corollary 3.2.9. *Given a matrix $M \in \Pi^{\circ}_{2,n}$, we can use Lemmas 3.2.7 and 3.2.8 to change M to the matrix*

$$V' = \begin{pmatrix} 1 & * & \dots & 0 \\ 0 & 1 & \dots & * \end{pmatrix}$$

such that $V' \in \Pi^{\circ,1}_{2,n}$. Furthermore, if $M \in \Pi^{\circ,1}_{2,n}$ then $\Delta_{ij}(V') = \Delta_{ij}(M)$.

Proof. We only need to prove the last equation. If $M \in \Pi^{\circ,1}_{2,n}$ with each $\Delta_{i,i+1} = 1$, using Lemma 3.2.7 there exists a unique $V' = AM$, and $\Delta_{ij}(V') = \Delta_{ij}(M)/\Delta_{12}(M) = \Delta_{ij}(M)$. In particular, $\Delta_{i,i+1}(V') = 1$ for all i and we do not require the use of Lemma 3.2.8 to rescale the vectors. □

CHAPTER 4

Cluster algebras

Cluster algebras are a class of commutative rings which were formally introduced by Sergey Fomin and Andrei Zelevinsky in the early 2000s. These algebras were initially motivated by the study of total positivity and Lie algebras. Since their inception, they have emerged as central objects in various branches of mathematics and mathematical physics including representation theory, algebraic geometry, and combinatorics. Cluster algebras are defined by a seed Σ consisting of a quiver, or exchange matrix, and cluster variables, which are a finite collection of algebraically independent elements of the algebra. This seed along with a concept of mutation generates a subring of a field $\mathcal{F}$. For more details on cluster algebras, see [**?**].

A cluster algebra $\mathcal{A}$ is defined by a skew-symmetric integer matrix $\widetilde{B}$ of size $(n+m)\times n$ called the ***extended exchange matrix***, the top $n \times n$ part B is a skew shaped integer matrix where

$$\widetilde{B}_{ij} = \begin{cases} a & \text{if there are } a \text{ arrows from vertex } i \text{ to vertex } j; \\ -a & \text{if there are } a \text{ arrows from vertex } j \text{ to vertex } i; \\ 0 & \text{otherwise} \end{cases}$$

Alternatively, we can consider the ***ice quiver*** Q associated to $\widetilde{B}$ as a finite directed graph with no 1- or 2-cycles such that the number of vertices $|Q_0| = n+m$. We can specify that the vertices are either frozen $x_i \in Q_0^f$, or mutable $x_i \in Q_0 \setminus Q_0^f$. Note that $|Q_0^f| = m$.

Let $\mathcal{F}$ be a field with transcendence degree $n+m$ over $\mathbb{C}$, i.e., $\mathcal{F} \cong \mathbb{C}(\overline{x})$ where $\overline{x} = (x_1, \ldots, x_{n+m})$ is the transcendence basis for $\mathcal{F}$ and are defined as ***cluster variables***. We say that $\Sigma = (Q, \overline{x})$ is a ***seed*** of $\mathcal{A}$.

Definition 4.0.1. For each mutable vertex x_k, we define the ***mutation*** of a seed Σ as $\mu_k(\Sigma) = (\mu_k(Q), \overline{x}')$ where $\overline{x}' = (x'_1, \ldots, x'_{n+m})$ is given by

$$x'_k x_k = \left(\prod_{\widetilde{B}_{ki} \geq 0} x_i^{\widetilde{B}_{ki}} + \prod_{\widetilde{B}_{kj} \leq 0} x_j^{-\widetilde{B}_{kj}} \right), \qquad x'_i = x_i \quad \text{if } i \neq k. \tag{4.1}$$

When performing a mutation, we modify the quiver Q according to the following rules to obtain $Q' = \mu_k(Q)$:

(1) If there is a path of the vertices $i \to k \to j$, then we add an arrow from i to j.

(2) Any arrows incident to k change orientation.

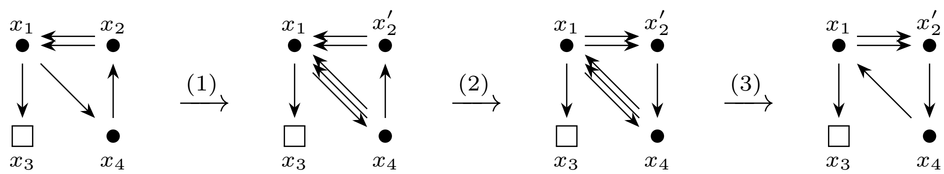
(3) Remove a maximal disjoint collection of 2-cycles produced in Steps (1) and (2).

We say that two seeds Σ, Σ' are ***mutation equivalent*** in the cluster algebra if they are related by a finite sequence of mutations $\underline{\mu}$, and note that μ_k is an involution.

The cluster algebra $\mathcal{A} \subset \mathcal{F}$ is generated by all cluster variables in all seeds under mutation.

The geometric interpretation of a cluster algebra is a cluster variety. Define the ***cluster variety*** $X = Spec(\mathcal{A})$ as an affine algebraic variety given by a collection of open charts $U \simeq (\mathbb{C}^*)^{n+m}$ where each chart U is parametrized by cluster coordinates $x_1, \ldots, x_{n+m}$ which are invertible on U and extend to regular functions on X. If the coordinate extends to a non-vanishing regular function on X then we call it frozen, otherwise we call the coordinate mutable. Given the condition of mutation as described above, the ring of functions on X is generated by all cluster variables in all charts.

Example 4.0.2. We demonstrate the process of mutation by mutating the following quiver at x_2, following the procedure detailed in Definition 4.0.1. Here, the variables x_1, x_3, x_4 remain unchanged, whereas $x_2' = \dfrac{x_4 + x_1^2}{x_2}$.



4.1. Quasi-cluster homomorphisms

It is possible for a cluster algebra $\mathcal{A}$ to be defined by two non-mutation equivalent seeds Σ, Σ', i.e., $\mathcal{A}(\Sigma) \cong \mathcal{A} \cong \mathcal{A}(\Sigma')$ yet $\Sigma \neq \underline{\mu}(\Sigma')$ for all $\underline{\mu}$. In general, cluster structures for a commutative algebra $\mathcal{A}$ is not unique.

Example 4.1.1. Let $\Sigma = (Q, \overline{x})$ be the seed described by

$$x_a \to x_1 \to x_b$$

where the blue vertices are frozen. The associated cluster algebra is defined as

$$\mathcal{A}(\Sigma) = \mathbb{C}[x_1, x_1', x_a^{\pm 1}, x_b^{\pm 1}]/(x_1 x_1' = x_a + x_b).$$

Now, consider the seed $\Sigma' = (Q', \overline{y})$ described by

$$y_a \to y_1 \qquad y_b$$

where the blue vertices are frozen. The associated cluster algebra is defined as

$$\mathcal{A}(\Sigma') = \mathbb{C}[y_1, y_1', y_a^{\pm 1}, y_b^{\pm 1}]/(y_1 y_1' = y_a + 1).$$

We can define an isomorphism between the two cluster algebras given by $y_1 \mapsto x_1 x_b^{-1}, y_a \mapsto x_a x_b^{-1}$, $y_b \mapsto x_b$ and $y_1' \mapsto x_1'$; however, the two seeds Σ, Σ' are not mutation equivalent.

This particular property is defined as a quasi-equivalence between cluster algebras. We use the notion of ***exchange ratios***, given a mutable vertex x_k in a seed Σ the exchange ratio $\widehat{y_k}$ is defined:

$$\widehat{y_i} = \frac{\prod_{\widetilde{B}_{ki} \geq 0} x_i^{\widetilde{B}_{ki}}}{\prod_{\widetilde{B}_{kj} \leq 0} x_j^{-\widetilde{B}_{kj}}}. \tag{4.1}$$

Definition 4.1.2. (Fraser[**14**], Fraser–Sherman-Bennett[**15**]) Let $A(\Sigma), A(\Sigma')$ be cluster algebras of rank $n+m$, each with m frozen variables. Let $\overline{x} = \{x_1, \ldots, x_{n+m}\}$ be the cluster variables of Σ, and $\overline{x}' = \{x_1', \ldots x_{n+m}'\}$ be the cluster variables of Σ'. A ***quasi-cluster isomorphism*** is an algebra isomorphism $f : A(\Sigma) \to A(\Sigma')$ satisfying the following conditions:

(1) For each frozen variable $x_j \in \overline{x}$, $f(x_j)$ is a Laurent monomial in the frozen variables of $\overline{x}'$.

(2) For each mutable variable $x_i \in \overline{x}$, $f(x_i)$ coincides with x_i', up to multiplication by a Laurent monomial in the frozen variables of $\overline{x}'$.

(3) The exchange ratios are preserved, i.e., for each mutable variable x_i of Σ, $f(\widehat{y_i}) = \widehat{y}_i'$.

By the main result of [**14**], it is sufficient to check the conditions of quasi-equivalence in one cluster, and they will automatically hold in every other cluster.

4.2. Cluster structures on open positroids

In 2003, Scott [**29**] established that the homogeneous coordinate ring of $\mathrm{Gr}(k,n)$ denoted $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ has a cluster structure using Postnikov arrangements. In this paper, will we use a different construction using the rectangles seed $\Sigma_{k,n}$ which generates the cluster structure for the Plücker ring $R_{k,n}$ isomorphic to $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ as detailed in [**13**, Section 6.7]. The cluster structure in the Plücker ring $R_{k,n}$ is generated from the mutations on the rectangular seed $\Sigma_{k,n}$. Unlike [**29**], we always assume that the frozen variables are invertible, in fact, we are considering a cluster structure on $\mathbb{C}[\widehat{\Pi}^{\circ}_{k,n}]$.

We first construct the quiver $Q_{k,n}$ where vertices are labeled by rectangles r contained in the $k \times (n-k)$ rectangle R along with the empty rectangle $\varnothing$. The frozen vertices are defined as rectangles of size $k \times j$ for $1 \leq j \leq n-k$, size $i \times (n-k)$ for $1 \leq i \leq k$, and $\emptyset$. The arrows connect from the $i \times j$ rectangle to the $i \times (j+1)$ rectangle, the $(i+1) \times j$ rectangle, and the $(i-1) \times (j-1)$ rectangle with the conditions that the rectangle has nonzero dimension, fits inside of R and does not connect two frozens. There is also an arrow from the $\varnothing$ rectangle to the 1×1 rectangle, see Figure 4.1.

Each rectangle r contained in the $k \times (n-k)$ rectangle R corresponds to a k-element subset of $[n]$ representing a Plücker coordinate. This correspondence is determined by positioning r in R such that the upper left corner coincides with the upper left corner of R. There exists a path from the upper right corner to the lower left corners of R which traces out the smaller rectangle r, with steps from 1 to n, where the map from r to $I(r)$ is given by the vertical steps of the path, see Figure 4.2. Define

$$\tilde{x}^{k,n} = \{\Delta_{I(r)} : r \text{ rectangle contained in } k \times (n-k) \text{ rectangle}\}$$

We may now define the rectangles seed $\Sigma_{k,n} = (\tilde{x}^{k,n}, \widetilde{B}(Q_{k,n}))$.

We can summarize (and slightly rephrase) the above constructions as follows. We define ordered subsets

$$I(a,i) = \{a, a+1, \ldots, a+i-1, n-k+i+1, \ldots, n\}, \tag{4.1}$$

where $a = n-k-j+1$.

Theorem 4.2.1 (Scott [**29**]). *The cluster variables in the initial seed are given by the minors* $\Delta_{I(a,i)}$ *for* $1 \leq a \leq n-k$ *and* $1 \leq i \leq k$*, and an additional frozen variable* $\Delta_{n-k+1,\ldots,n}$*. Furthermore:*

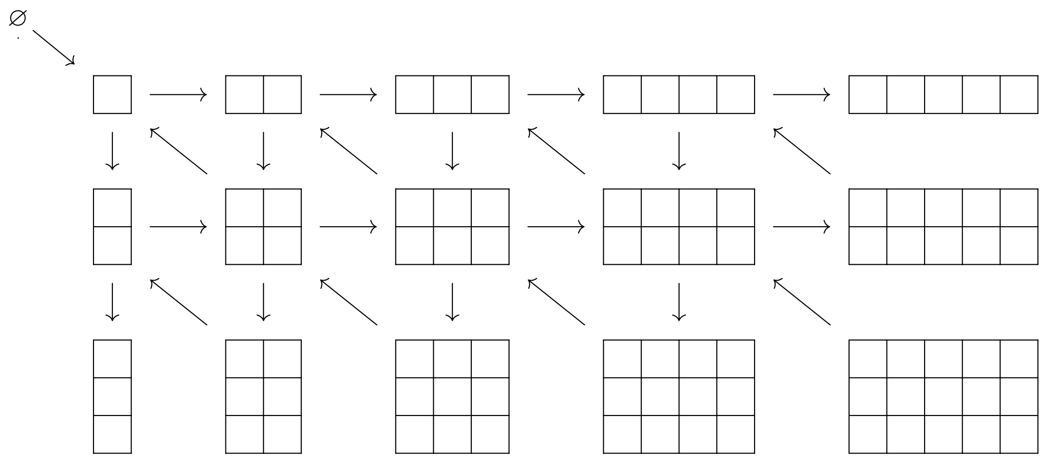


FIGURE 4.1. The quiver $Q_{3,8}$. Vertices are labeled by rectangles contained in a 3×5 rectangle. The grid is arranged such that the rectangles width increases from left to right and the heights increase from top to bottom.

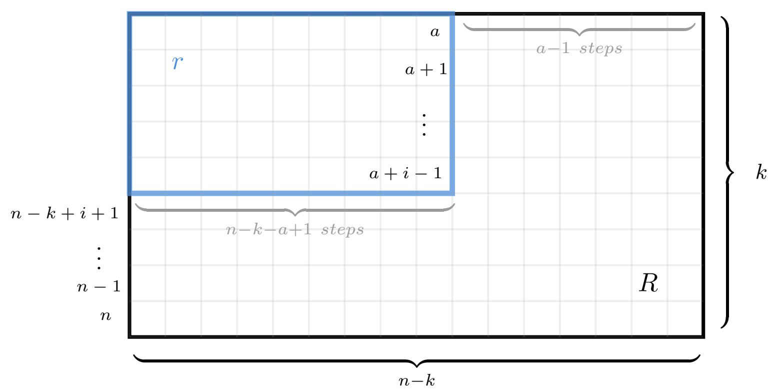


FIGURE 4.2. The Plücker coordinate $\Delta_{I(r)}$ corresponding to a rectangle r is given by the vertical steps in the path from the upper right corner to the lower left corner of the rectangle R of size $k \times (n-k)$ that cuts out the rectangle r positioned in the upper left corner of R.

1) The variables $\Delta_{I(a,i)}$ are frozen for $a = 1$ and $i = k$, and mutable otherwise.

2) The quiver $Q_{k,n}$ consists of the following arrows:

$$\begin{array}{ccc} \Delta_{I(a,i)} & \longrightarrow & \Delta_{I(a-1,i)} \\ \downarrow & \nwarrow & \downarrow \\ \Delta_{I(a,i+1)} & \longrightarrow & \Delta_{I(a-1,i+1)} \end{array} \tag{4.2}$$

3) There is an additional arrow $\Delta_{n-k+1,\ldots,n} \to \Delta_{I(n-k,1)}$.

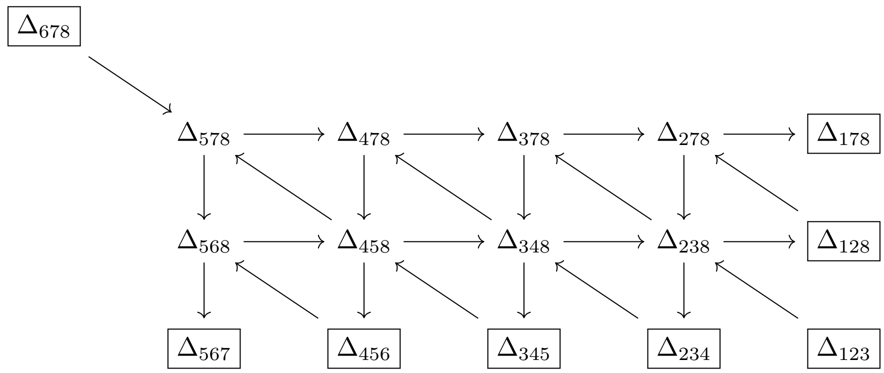


FIGURE 4.3. Cluster variables in $\mathrm{Gr}(3,8)$ corresponding to the rectangles seed.

See Figure 4.3 for the $\widehat{\Pi}^\circ_{3,8}$ example and Figure 4.4 for the general case.

Below we will also use the cluster structure on $\Pi^\circ_{k,n}$. The corresponding quiver can be obtained from $Q_{k,n}$ by deleting the vertex $\emptyset$, and the cluster variables in the initial seed are given by the ratios $\Delta_{I(a,i)}/\Delta_{n-k+1,\ldots,n}$. More precisely, we have the following:

PROPOSITION 4.2.2. *We have a quasi-equivalence of cluster varieties*

$$\widehat{\Pi}^\circ_{k,n} \simeq \mathbb{C}^* \times \Pi^\circ_{k,n} \tag{4.3}$$

and the corresponding quasi-equivalence of cluster algebras

$$\mathbb{C}[\widehat{\Pi}^\circ_{k,n}] \simeq \mathbb{C}[\Delta^{\pm}_{n-k+1,\ldots,n}] \otimes \mathbb{C}[\Pi^\circ_{k,n}].$$

PROOF. This result is well known, but we provide a proof for the sake of completeness.

Define the map $f : \widehat{\Pi}^\circ_{k,n} \to \mathbb{C}^* \times \Pi^\circ_{k,n}$ by sending each Plücker coordinate $\Delta_{I(a,i)}$ to $\widetilde{\Delta}_{I(a,i)} := \Delta_{I(a,i)}/\Delta_{n-k+1,\ldots,n}$. We note that this map is well defined since $\Delta_{n-k+1,\ldots,n} = \Delta_{I(n-k+1,k)}$ is nonzero by definition of $\widehat{\Pi}^\circ_{k,n}$. We show that this map defines a quasi-equivalence by verifying that it preserves exchange ratios in the cases illustrated in Figure 4.4.

(a) Left corner: In $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$, the mutable variable $\Delta_{I(n-k,1)}$ in the left corner has a total of two incoming arrows and two outgoing arrows. However, under the map f the cluster variable $\Delta_{I(n-k+1,k)}$ is mapped to 1, and therefore, the arrow from $\widetilde{\Delta}_{I(n-k+1,k)}$ to $\widetilde{\Delta}_{I(n-k,1)}$ vanishes. We now have that the mutable variable $\widetilde{\Delta}_{I(n-k,1)}$ in $\mathbb{C}[\Pi^\circ_{k,n}]$ has one incoming arrow and two outgoing arrows. However, we see that the exchange

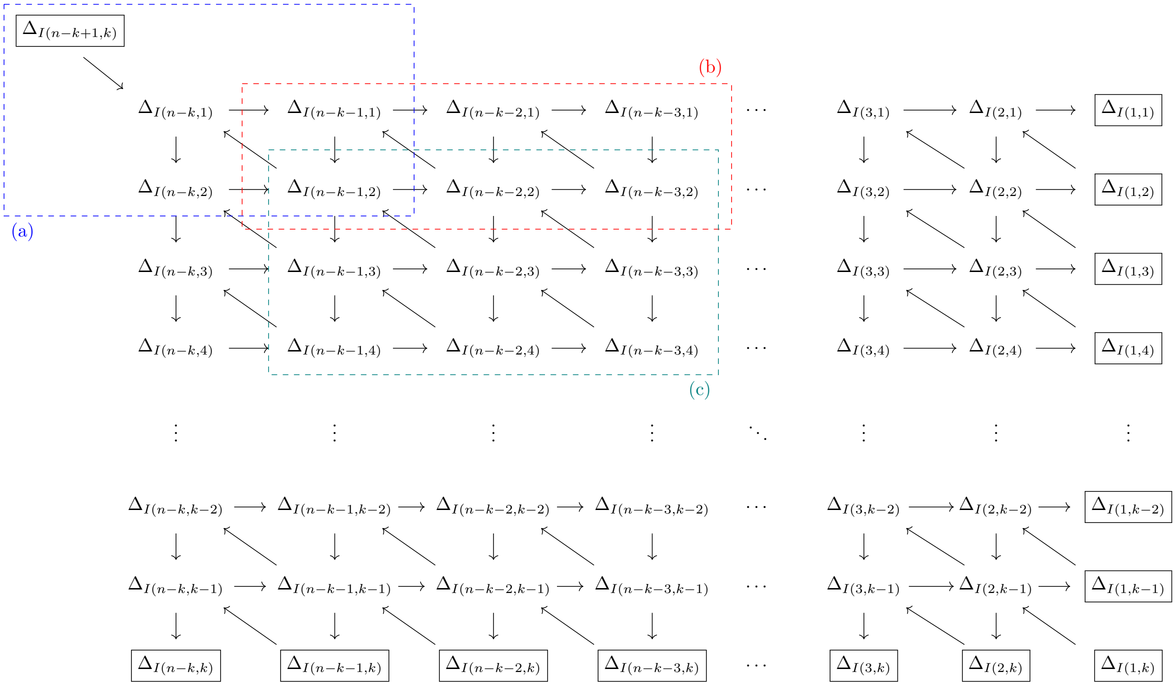


FIGURE 4.4. Cluster variables in $\widehat{\Pi}^\circ_{k,n}$ in the rectangle seed.

ratios under the map f are equivalent:

$$\widehat{y}_{\widetilde{\Delta}_{I(n-k,1)}} = \frac{\frac{\Delta_{I(n-k-1,2)}}{\Delta_{I(n-k+1,k)}}}{\frac{\Delta_{I(n-k-1,1)}}{\Delta_{I(n-k+1,k)}}\frac{\Delta_{I(n-k,2)}}{\Delta_{I(n-k+1,k)}}} = \frac{\Delta_{I(n-k-1,2)}\Delta_{I(n-k+1,k)}}{\Delta_{I(n-k-1,1)}\Delta_{I(n-k,2)}} = \widehat{y}_{\Delta_{I(n-k,1)}}.$$

(b) Boundary: Either $2 \le a \le n-k-1$ and $i = 1$, or $a = n-k$ and $2 \le i \le k-1$. In $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$, the mutable variable $\Delta_{I(a,i)}$ has two incoming arrows and two outgoing arrows. Under the map f, the mutable variable $\widetilde{\Delta}_{I(a,i)}$ in $\mathbb{C}[\Pi^\circ_{k,n}]$ still has two incoming and outgoing arrows where each of the corresponding variables have a factor of $(\Delta_{I(n-k+1,k)})^{-1}$ which cancels in the computation of the exchange ratio $y_{\widetilde{\Delta}_{I(a,i)}}$.

(c) Interior: Similarly to the boundary case, the mutable variable $\Delta_{I(a,i)}$ in $\mathbb{C}[\widehat{\Pi}^\circ_{k,n}]$ and $\widetilde{\Delta}_{I(a,i)}$ in $\mathbb{C}[\Pi^\circ_{k,n}]$ both have three incoming arrows and three outgoing arrows. Therefore, the factor $(\Delta_{I(n-k+1,k)})^{-1}$ cancels out in the computation of the exchange ratio $y_{\widetilde{\Delta}_{I(a,i)}}$.

□

Thanks to Proposition 4.2.2, we will freely translate various results and computations between the cluster structures on $\widehat{\Pi}^\circ_{k,n}$ and on $\Pi^\circ_{k,n}$. In particular, we will always compute the exchange ratios in $\widehat{\Pi}^\circ_{k,n}$, since they

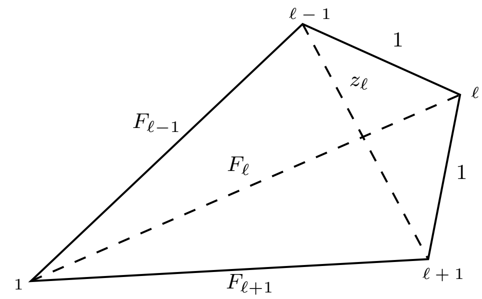


FIGURE 4.5. Section of the triangulation of $\mathrm{U}_{\mathrm{fan}}$, see Figure 4.6, between the vertices $1, \ell-1, \ell$ and $\ell+1$

coincide with the ones in $\Pi^\circ_{k,n}$. If we apply the additional condition that the Plücker coordinate associated to $\varnothing$ is equal to 1, we can neglect that particular cluster variable and obtain a cluster structure for $\mathbb{C}[\Pi^\circ_{k,n}]$. In [**21**] we generalize this cluster structure to skew shaped positroid varieties which are described by skew Young diagrams contained in the $k\times(n-k)$ rectangle R. Which is consistent with previous results. See [**16**] for a complete description of the construction of cluster structures for general positroid varieties using Postnikov diagrams.

THEOREM 4.2.3 (Scott[**29**], Galashin–Lam[**16**], Serhiyenko–Sherman-Bennett–Williams[**31**]). *Any open positroid variety has a cluster structure.*

4.3. Cluster structure for two-stranded braid varieties

For positroid varieties $\Pi^\circ_{2,\ell+1}$ we obtain a cluster variety of type $A_{\ell-2}$ with $\ell+1$ frozen variables. We assign the vectors v_i from Lemma 3.2.5 to the vertices of a regular polygon $\mathcal{P}$. The cluster charts in $\Pi^\circ_{2,\ell+1}$ are determined by triangulations of $\mathcal{P}$. Given a triangulation, the edges between the vertices i and j correspond to cluster variables determined by the Plücker coordinates $\Delta_{i,j}=\det(v_i,v_j)$.

LEMMA 4.3.1 (Hughes[**23**]). *In* $\Pi^{\circ,1}_{2,\ell+1}$ *for all* $i<j$ *we have*

$$\Delta_{i,j}=F_{j-i-1}(z_{i+1},\ldots,z_{j-1}).$$

In particular, $\Delta_{i,i+2}=z_{i+1}$.

Proof. Using the results from Lemma 3.2.3, we have the following relations

$$B(z_1)\dots B(z_i) = \begin{pmatrix} v_{i+1} & -v_i \end{pmatrix}$$

$$B(z_1)\dots B(z_j) = \begin{pmatrix} v_{j+1} & -v_j \end{pmatrix}$$

Given that $i < j$, we then rewrite

$$B(z_1)\dots B(z_i)B(z_{i+1})\dots B(z_j) = \begin{pmatrix} v_{j+1} & -v_j \end{pmatrix}$$

$$\begin{pmatrix} v_{i+1} & -v_i \end{pmatrix} B(z_{i+1})\dots B(z_j) = \begin{pmatrix} v_{j+1} & -v_j \end{pmatrix}$$

From Theorem 2.1.6, the product of the braid matrices from $i+1$ to j can be expressed as

$$B(z_{i+1})\dots B(z_j) = \begin{pmatrix} F_{j-i}(z_{i+1},\dots,z_j) & -F_{j-i-1}(z_{i+1},\dots,z_{j-1}) \\ F_{j-i-1}(z_{i+2},\dots,z_j) & -F_{j-i-2}(z_{i+2},\dots,z_{j-1}) \end{pmatrix}$$

Which allows us to rewrite the previous equation as

$$(v_{i+1} \quad -v_i) \begin{pmatrix} F_{j-i}(z_{i+1},\dots,z_j) & -F_{j-i-1}(z_{i+1},\dots,z_{j-1}) \\ F_{j-i-1}(z_{i+2},\dots,z_j) & -F_{j-i-2}(z_{i+2},\dots,z_{j-1}) \end{pmatrix} = (v_{j+1} \quad -v_j)$$

Here we obtain the equation

$$-v_j = -F_{j-i-1}(z_{i+1},\dots,z_{j-1})v_{i+1} + F_{j-i-2}(z_{i+2},\dots,z_{j-1})v_i$$

By finding an expression for v_j, we may now determine Δ_{ij}, since determinants are linear, we find that

$$\begin{aligned}
\Delta_{ij} &= \det \begin{pmatrix} v_i & v_j \end{pmatrix} \\
&= F_{j-i-1}(z_{i+1},\dots,z_{j-1}) \det \begin{pmatrix} v_i & v_{i+1} \end{pmatrix} - F_{j-i-2}(z_{i+2},\dots,z_{j-1}) \det \begin{pmatrix} v_i & v_i \end{pmatrix} \\
&= F_{j-i-1}(z_{i+1},\dots,z_{j-1})(1) - F_{j-i-2}(z_{i+2},\dots,z_{j-1})(0) = F_{j-i-1}(z_{i+1},\dots,z_{j-1})
\end{aligned}$$

To see that $\Delta_{i,i+2} = z_{i+1}$, we see that $\Delta_{ij} = F_{(i+2)-i-1}(z_{i+1}) = F_1(z_{i+1}) = z_{i+1}$ as desired. □

For $a < b < c < d$ we have the Plücker relation

$$\Delta_{ac}\Delta_{bd} = \Delta_{ab}\Delta_{cd} + \Delta_{ad}\Delta_{bc}. \tag{4.1}$$

A special case of (4.1) is

$$\Delta_{i,\ell}\Delta_{\ell-1,\ell+1} = \Delta_{i,\ell-1}\Delta_{\ell,\ell+1} + \Delta_{i,\ell+1}\Delta_{\ell-1,\ell}$$

which in $\Pi^{\circ,1}_{2,\ell+1}$ translates to

$$\Delta_{i,\ell}z_\ell = \Delta_{i,\ell-1} + \Delta_{i,\ell+1}$$

For $i = 1$ it is indeed equivalent to our recursion (2.1), see Figure 4.5

Outer edges of $\mathcal{P}$ correspond to frozen variables, while diagonals correspond to mutable variables. In particular, $\Pi^{\circ}_{2,\ell+1}$ has $\ell+1$ frozen variables, while in $\Pi^{\circ,1}_{2,\ell+1}$ we specialize ℓ frozen Plücker coordinates, $\Delta_{i,i+1} = 1$ for $1 \le i \le \ell$ and therefore can be neglected. Thus $\Pi^{\circ,1}_{2,\ell+1}$ has one frozen variable $\Delta_{1,\ell+1}$ which we denote by w. To generate the quiver, in each triangle of the triangulation we connect the cluster variables by arrows in clockwise order. Mutations correspond to flips of triangulations due to the Plucker relation.

Consider the special chart $\mathrm{U}_{\mathrm{fan}}$ in $\Pi^{\circ,1}_{2,\ell+1}$ corresponding to the "fan" triangulation where the $\ell-2$ diagonals are defined by $\Delta_{1,i}$ for $2 \le i \le \ell$, as seen in Figure 4.6. Equivalently, the chart $\mathrm{U}_{\mathrm{fan}}$ is given by inequalities

$$\mathrm{U}_{\mathrm{fan}} = \{F_{i-1}(z_2,\ldots,z_i) \neq 0, 1 \le i \le \ell\} \subset X(\sigma^\ell).$$

In this chart, the quiver is precisely $A_{\ell-2}$ with one frozen variable w. From Lemma **??** the mutable cluster variables are precisely $w_i = F_i(z_2,\ldots,z_{i+1})$ and the frozen variable is $w = w_{\ell-2} = F_\ell(z_2,\ldots,z_{\ell+1})$.

Remark 4.3.2. In order to obtain the cluster structure as in Section 4.2, we triangulate the $(\ell+1)$-gon with all $\ell-2$ diagonals are defined by $\Delta_{i,\ell+1}$ for $2 \le i \le \ell-1$.

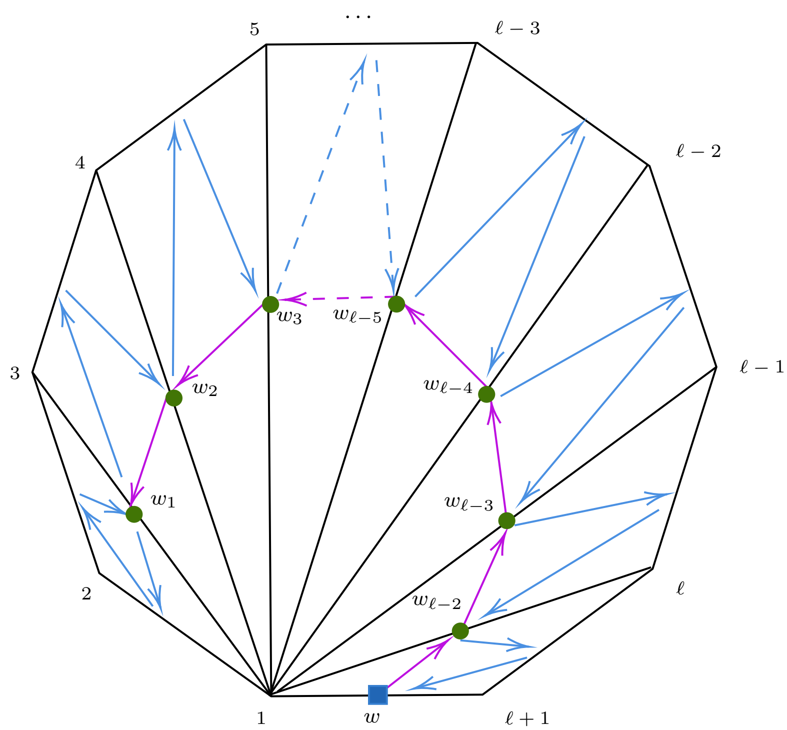


FIGURE 4.6. The special chart $\mathrm{U}_{\mathrm{fan}} \in \Pi^{\circ,1}_{2,\ell+1}$ where each of the $\ell-2$ diagonals are have fixed endpoint at v_1. The Plučker coordinates, or cluster variables, correspond to the weights of the edges given by either a blue square (frozen vertices) or a green circle (mutable vertices). The quiver of the cluster chart is generated by clockwise orientation of the colored arrows in each triangle of the triangulation. This procedure produces the quiver $A_{\ell-1}$, seen in purple, with w as the singular frozen variable. In the terminology of [**5**], this chart is given by the right inductive weave.

CHAPTER 5

Splicing map

In earlier chapters, we explored braid varieties from both algebraic and geometric perspectives, established their relationship with positroid varieties, and examined their cluster structures. These results help us have a detailed understanding of individual braid varieties, in particular two stranded braid varieties, and they also raise natural questions about how more complex braid varieties can be constructed from simpler components.

This chapter introduces a splicing map, a construction that formalizes how braid varieties can be "glued" together in a way that reflects the combinatorics of braid composition. The idea is motivated by the operation of multiplying braids, such as composing torus braids $T(k,s)\cdot T(k,t)\to T(k,s+t)$, which gives rise to larger braid diagrams from smaller ones. This multiplicative structure has a geometric counterpart in the world of braid and positroid varieties, and the splicing map aims to capture that structure explicitly.

From the perspective of positroid varieties, the splicing map provides a map between open positroid strata:

$$\Pi^{\circ}_{k,k+s}\times\Pi^{\circ}_{k,k+t}\to\Pi^{\circ}_{k,k+s+t},$$

and we show that this map preserves the expected dimension. The construction involves manipulating Plücker coordinates in a controlled way and reassembling flags or vector configurations across overlapping coordinate charts. To make this concrete, we define local charts $U_a\subset\Pi^{\circ}_{k,n}$ where the splicing map is well-behaved, and we carefully analyze how the map acts on vector columns in the associated matrices.

We also study the map from the point of view of the cluster structure. Although the splicing operation is not a cluster morphism in general, it interacts well with cluster coordinates and provides insight into how cluster variables behave under such gluings. In some cases, we can explicitly track how splicing affects frozen and mutable variables in cluster charts, especially those modeled on triangulations in the context of surfaces.

Throughout the chapter, we work through examples to illustrate the splicing map and the geometric intuition behind the map. They also serve to demonstrate the utility of the splicing map as a tool for constructing new varieties and for studying the recursive structure of braid varieties. This map will prove useful in Chapter 6 in studying the cohomological structure of braid varieties.

5.1. Splicing map for torus links

Let $\Pi^{\circ}_{k,n}$ be the open positroid variety in the Grassmannian $\mathrm{Gr}(k,n)$.

Definition 5.1.1. Given $2 \le a \le n-k$, we define an open subset $U_a \subset \Pi^{\circ}_{k,n}$ by the inequalities

$$U_a = \{V \in \Pi^{\circ}_{k,n} : \Delta_{I(a,i)}(V) \neq 0,\ 1 \le i \le k-1\}. \tag{5.1}$$

For $0 \le s \le i-1$ and $0 \le t \le k-i-1$ we define ordered subsets

$$I'(a,s,i) = \{a, \dots, a+s-1, a+i, a+s+1, \dots, a+i-1, n-k+i+1, \dots, n\},$$

$$I'_{\mathrm{sort}}(a,s,i) = \{a, \dots, a+s-1, a+s+1, \dots, a+i-1, a+i, n-k+i+1, \dots, n\},$$

and

$$I''(a,t,i) = \{a, \dots, a+i-1, n-k+i+1, \dots, n-t-1, a+i, n-t+1, \dots, n\},$$

$$I''_{\mathrm{sort}}(a,t,i) = \{a, \dots, a+i-1, a+i, n-k+i+1, \dots, n-t-1, n-t+1, \dots, n\},$$

Note that $I'(a,s,i)$ is obtained from $I(a,i)$ by replacing $a+s$ by $a+i$ (without changing the order), while $I''(a,t,i)$ is obtained from $I(a,i)$ by replacing $n-t$ by $a+i$. Also, $I'(a,s,i)$ and $I'_{\mathrm{sort}}(a,s,i)$ are related by an $(i-s)$-cycle while $I''(a,t,i)$ and $I''_{\mathrm{sort}}(a,t,i)$ are related by a $(k-i-1-t)$-cycle.

Lemma 5.1.2. *Given a matrix $V \in U_a$, for all $1 \le i \le k-2$ we have an identity*

$$\begin{aligned} v_{a+i} &= \sum_{s=0}^{i-1} \frac{\Delta_{I'(a,s,i)}}{\Delta_{I(a,i)}} v_{a+s} + \sum_{t=0}^{k-i-1} \frac{\Delta_{I''(a,t,i)}}{\Delta_{I(a,i)}} v_{n-t} \\ &= \sum_{s=0}^{i-1} (-1)^{i-s-1} \frac{\Delta_{I'_{\mathrm{sort}}(a,s,i)}}{\Delta_{I(a,i)}} v_{a+s} + \sum_{t=0}^{k-i-1} (-1)^{k-i-t} \frac{\Delta_{I''_{\mathrm{sort}}(a,t,i)}}{\Delta_{I(a,i)}} v_{n-t} \end{aligned}$$

Proof. Since $\Delta_{I(a,i)}(V) \neq 0$, the vectors $v_a, \dots, v_{a+i-1}, v_{n-k+i+1}, \dots, v_n$ span $\mathbb{C}^k$. We can uniquely write v_{a+i} as linear combination of these:

$$v_{a+i} = x_0 v_a + \dots + x_{i-1} v_{a+i-1} + y_{k-i-1} v_{n-k+i+1} + \dots + y_0 v_n.$$

Now the coefficients x_s, y_t are determined by Cramer's Rule:

$$x_s = \frac{\Delta_{I'(a,s,i)}}{\Delta_{I(a,i)}} = (-1)^{i-s-1}\frac{\Delta_{I'_{\text{sort}}(a,s,i)}}{\Delta_{I(a,i)}},\ y_t = \frac{\Delta_{I''(a,t,i)}}{\Delta_{I(a,i)}} = (-1)^{k-i-t}\frac{\Delta_{I''_{\text{sort}}(a,t,i)}}{\Delta_{I(a,i)}}.$$

□

EXAMPLE 5.1.3. For $a = 3$ the open subset $U_3 \subset \Pi^\circ_{3,8}$ is defined by $\Delta_{348}, \Delta_{378} \neq 0$, indicating that the vectors v_3, v_7, v_8 span $\mathbb{C}^3$ and the vectors v_3, v_4, v_8 span $\mathbb{C}^3$. Using Cramer's rule we may express the vector v_4 as

$$v_4 = \frac{\Delta_{478}}{\Delta_{378}}v_3 + \frac{\Delta_{348}}{\Delta_{378}}v_7 + \frac{\Delta_{374}}{\Delta_{378}}v_8 = \frac{\Delta_{478}}{\Delta_{378}}v_3 + \frac{\Delta_{348}}{\Delta_{378}}v_7 - \frac{\Delta_{347}}{\Delta_{378}}v_8$$

EXAMPLE 5.1.4. The open subset $U_3 \subset \Pi^\circ_{5,10}$ is defined by

$$U_3 = \{V \in \Pi^\circ_{5,10} : \Delta_{3,7,8,9,10}(V), \Delta_{3,4,8,9,10}(V), \Delta_{3,4,5,9,10}(V), \Delta_{3,4,5,6,10}(V) \neq 0\}.$$

From this collection of nonvanishing Plücker coordinates we have

$$\mathbb{C}^5 = \langle v_3, v_7, v_8, v_9, v_{10}\rangle = \langle v_3, v_4, v_8, v_9, v_{10}\rangle = \langle v_3, v_4, v_5, v_9, v_{10}\rangle.$$

By Cramer's Rule, we can expand v_4, v_5, v_6 in the respective bases:

$$v_4 = \frac{\Delta_{4,7,8,9,10}}{\Delta_{3,7,8,9,10}}v_3 + \frac{\Delta_{3,4,8,9,10}}{\Delta_{3,7,8,9,10}}v_7 - \frac{\Delta_{3,4,7,9,10}}{\Delta_{3,7,8,9,10}}v_8 + \frac{\Delta_{3,4,7,8,10}}{\Delta_{3,7,8,9,10}}v_9 - \frac{\Delta_{3,4,7,8,9}}{\Delta_{3,7,8,9,10}}v_{10}$$
$$v_5 = -\frac{\Delta_{4,5,8,9,10}}{\Delta_{3,4,8,9,10}}v_3 + \frac{\Delta_{3,5,8,9,10}}{\Delta_{3,4,8,9,10}}v_4 + \frac{\Delta_{3,4,5,9,10}}{\Delta_{3,4,8,9,10}}v_8 - \frac{\Delta_{3,4,5,8,10}}{\Delta_{3,4,8,9,10}}v_9 + \frac{\Delta_{3,4,5,8,9}}{\Delta_{3,4,8,9,10}}v_{10}$$
$$v_6 = \frac{\Delta_{4,5,6,9,10}}{\Delta_{3,4,5,9,10}}v_3 - \frac{\Delta_{3,5,6,9,10}}{\Delta_{3,4,5,9,10}}v_4 + \frac{\Delta_{3,4,6,9,10}}{\Delta_{3,4,5,9,10}}v_5 + \frac{\Delta_{3,4,5,6,10}}{\Delta_{3,4,5,9,10}}v_9 - \frac{\Delta_{3,4,5,6,9}}{\Delta_{3,4,5,9,10}}v_{10}$$

DEFINITION 5.1.5. Given a matrix $V = (v_1, \ldots, v_n)$ in U_a, we define two matrices V_1, V_2 as follows:

$$V_1 = (v_a, \ldots, v_n),\ V_2 = (v_1, \ldots, v_a, u_1 \ldots, u_{k-2}, v_n) \tag{5.2}$$

where

$$u_i = v_{a+i} - \sum_{s=0}^{i-1}\frac{\Delta_{I'(a,s,i)}}{\Delta_{I(a,i)}}v_{a+s} = \sum_{t=0}^{k-i-1}\frac{\Delta_{I''(a,t,i)}}{\Delta_{I(a,i)}}v_{n-t}. \tag{5.3}$$

The second equation in (5.3) follows from Lemma 5.1.2.

Example 5.1.6. Continuing Example 5.1.3, we decompose the matrix

$$V = (v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8)$$

into

$$V_1 = (v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8)\,, \quad V_2 = (v_1 \quad v_2 \quad v_3 \quad u \quad v_8)$$

where

$$u = v_4 - \frac{\Delta_{478}}{\Delta_{378}} v_3 = \frac{\Delta_{348}}{\Delta_{378}} v_7 - \frac{\Delta_{347}}{\Delta_{378}} v_8.$$

Lemma 5.1.7. *Assume that $V \in U_a$. Then for $1 \le i \le k-2$ the intersection of two subspaces*

$$\langle v_a, v_{a+1}, \dots, v_{a+i} \rangle \cap \langle v_{n-k+i+1}, v_{n-k+i+2}, \dots, v_n \rangle.$$

is one-dimensional and spanned by the vector u_i.

Proof. By (5.3) and Lemma 5.1.2 the vector u_i is indeed contained in this intersection. Since $\Delta_{I(a,i+1)} \neq 0$, the vectors $v_a, \dots, v_{a+i}$ are linearly independent and hence $u_i \neq 0$. Since $\Delta_{I(a,i)} \neq 0$, the vectors $v_{n-k+i+1}, \dots, v_n$ are linearly independent as well and altogether the two subspaces span $\mathbb{C}^k$. Now

$$\dim \langle v_a, v_{a+1}, \dots, v_{a+i} \rangle \cap \langle v_{n-k+i+1}, v_{n-k+i+2}, \dots, v_n \rangle = (i+1) + (k-i) - k = 1.$$

□

Lemma 5.1.8. *a) We have*

$$v_a \wedge u_1 \wedge \cdots \wedge u_i = v_a \wedge v_{a+1} \wedge \cdots \wedge v_{a+i}.$$

b) We have

$$u_i \wedge \cdots \wedge u_{k-2} \wedge v_n = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n.$$

Proof. a) By (5.3) we have

$$u_i \in v_{a+i} + \langle v_a, \dots, v_{a+i-1} \rangle,$$

so

$$v_a \wedge u_1 \wedge \cdots \wedge u_i = v_a \wedge (v_{a+1} + \dots) \wedge \cdots (v_{a+i} + \dots) = v_a \wedge v_{a+1} \wedge \cdots \wedge v_{a+i}.$$

b) Similarly, by the second equation in (5.3) we have

$$u_i \in \frac{\Delta_{I''(a,k-i-1,i)}}{\Delta_{I(a,i)}} v_{n-k+i+1} + \langle v_{n-k+i+2}, \ldots, v_n \rangle.$$

Note that $I''(a, k-i-1, i)$ is obtained from $I(a,i) = \{a, \ldots, a-i-1, n-k+i+1, \ldots, n\}$ by replacing $n-k+i+1$ with $a+i$, so in fact $I''(a, k-i-1, i) = I(a, i+1)$. Now

$$u_i \wedge \cdots \wedge u_{k-2} \wedge v_n = \left(\frac{\Delta_{I(a,i+1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} + \ldots\right) \wedge \cdots \wedge \left(\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,k-2)}} v_{n-1} + \ldots\right) \wedge v_n =$$

$$\frac{\Delta_{I(a,i+1)}}{\Delta_{I(a,i)}} \cdot \frac{\Delta_{I(a,i+2)}}{\Delta_{I(a,i+1)}} \cdots \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,k-2)}} v_{n-k+i+1} \wedge \cdots \wedge v_n.$$

The factors in the coefficient cancel pairwise except for $\Delta_{I(a,k-1)}/\Delta_{I(a,i)}$. □

Lemma 5.1.9. *If $V \in U_a$ then $V_1 \in \Pi^\circ_{k,n-a+1}$ and $V_2 \in \Pi^\circ_{k,a+k-1}$.*

Proof. The first statement is clear by the definition of U_a. To prove the second one, we need to compute the following minors:

1) $\Delta_{b,\ldots,b+k-1}(V_2)$, $b+k-1 \le a$. This minor does not change, so $\Delta_{b,\ldots,b+k-1}(V_2) = \Delta_{b,\ldots,b+k-1}(V) \neq 0$.

2) $\Delta_{b,\ldots,b+k-1}(V_2)$, $b < a < b+k-1$. Let $i = b+k-1-a$, then by Lemma 5.1.8(a) we get

$$\Delta_{b,\ldots,b+k-1}(V_2) = v_b \wedge \cdots \wedge v_a \wedge u_1 \wedge \cdots \wedge u_i =$$

$$v_b \wedge \cdots \wedge v_a \wedge v_{a+1} \wedge \cdots \wedge v_{a+i} = \Delta_{b,\ldots,b+k-1}(V) \neq 0.$$

3) $\Delta_{a,\ldots,a+k-1}(V_2) = v_a \wedge u_1 \wedge \cdots \wedge u_{k-2} \wedge v_n = v_a \wedge \cdots \wedge v_{a+k-2} \wedge v_n \neq 0$ by definition of U_a.

4) Finally, we need to consider the minor $u_i \wedge \cdots \wedge u_{k-2} \wedge v_n \wedge v_1 \cdots v_i$ which by Lemma 5.1.8(b) equals

$$\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} \wedge \cdots \wedge v_n \wedge v_1 \cdots v_i = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{n-k+i+1,\ldots,n,1,\ldots,i} \neq 0.$$

□

Theorem 5.1.10. *The map $\Phi_a : V \mapsto (V_1, V_2)$ defined by (5.2) is an isomorphism between $U_a \subset \Pi^\circ_{k,n}$ and the product $\Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$.*

Remark 5.1.11. We have $\dim U_a = \dim \Pi^\circ_{k,n} = k(n-k)$ while

$$\dim \Pi^\circ_{k,n-a+1} + \dim \Pi^\circ_{k,a+k-1} = k(n-a+1-k) + k(a-1) = k(n-k).$$

Proof. By Lemma 5.1.9 the map $\Phi_a : U_a \to \Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$ is well defined. We need to construct the inverse map, reconstructing V from V_1 and V_2. Since V_1 and V_2 are both defined up to row operations, we need to choose appropriate representatives in their equivalence classes and make sure that they glue correctly to V.

For V_1, choose a representative in the equivalence class arbitrarily and label the column vectors by $(v_a, \ldots, v_n)$. Since $V_1 \in \Pi^\circ_{k,n-a+1}$, we have $\Delta_{I(a,i)} \neq 0$. By Lemma 5.1.2, we can define the vectors $u_1, \ldots, u_{k-2}$ by (5.3). Applying row operations to V_1 is equivalent to the multiplication by an invertible $(k \times k)$ matrix A on the left. It transforms v_i to Av_i, multiplies all the minors of V_1 by $\det A$, and transforms u_i to

$$u_i \to Av_{a+i} - \sum_{j=0}^{i-1} \frac{\Delta_{I'(a,j,i)} \det(A)}{\Delta_{I(a,i)} \det(A)} (Av_{a+j}) = A\left[v_{a+i} - \sum_{j=0}^{i-1} \frac{\Delta_{I'(a,j,i)}}{\Delta_{I(a,i)}} v_{a+j}\right] = Au_i. \tag{5.4}$$

By Lemma 5.1.8(a) we get $v_a \wedge u_1 \cdots u_{k-2} \wedge v_n = v_a \wedge v_{a+1} \cdots v_{a+k-2} \wedge v_n$. This is nonzero since $V_1 \in \Pi^\circ_{k,n-a+1}$, so the vectors $v_a, u_1, \ldots, u_{k-2}, v_n$ form a basis of $\mathbb{C}^k$. Therefore we can uniquely find a representative for V_2 of the form $V_2 = (v_1, \ldots, v_{a-1}, v_a, u_1, \ldots, u_{k-2}, v_n)$. Indeed, if $V_2' = (v_1', \ldots, v_{a+k-1}')$ is some other representative then

$$V_2 = (v_a, u_1, \ldots, u_{k-2}, v_n)(v_a', \ldots, v_{a+k-1}')^{-1} V_2'.$$

By (5.4), row operations $V_1 \mapsto AV_1$ also change $V_2 \to AV_2$. Now we can define $V = (v_1, \ldots, v_{a-1}, v_a, \ldots, v_n)$ where the vectors $v_1, \ldots, v_{a-1}$ are the first $(a-1)$ columns of V_2 and $(v_a, \ldots, v_n) = V_1$. By the above, this is well defined up to row operations.

Similarly to the proof of Lemma 5.1.9, one can check that $V \in \Pi^\circ_{k,n}$, and $V_1 \in \Pi^\circ_{k,n-a+1}$ immediately implies that $V \in U_a$. This completes the proof. □

5.2. Cluster algebra interpretation

We would like to compare the quivers and cluster coordinates (4.2) for the matrices V, V_1 and V_2, which we denote by Q_V, Q_{V_1} and Q_{V_2}. By construction, the empty rectangle in both Q_V and Q_{V_1} corresponds to $\Delta_{n-k+1,\ldots,n}(V)$. On the other hand, by Lemma 5.1.8(a) the empty rectangle in Q_{V_2} corresponds to the minor

$$\Delta_{I(a,k-1)}(V_2) = v_a \wedge u_1 \wedge \cdots u_{k-2} \wedge v_n = v_a \wedge v_{a+1} \wedge \cdots v_{a+k-2} \wedge v_n = \Delta_{I(a,k-1)}(V) = \Delta_{I(1,k-1)}(V_1)$$

which is connected to $\Delta_{a-1,1}(V_2)$.

Clearly, the open subset $U_a \subset \Pi^\circ_{k,n}$ is defined by freezing the cluster variables $\Delta_{I(a,i)}(V)$ in Q_V, which are identified with $\Delta_{I(1,i)}(V_1)$. We need to analyze the behavior of all other minors in Q_V under Φ_a.

Lemma 5.2.1.

(a) If $b \geq a$, then $\Delta_{I(b,i)}(V) = \Delta_{I(b-a+1,i)}(V_1)$.

(b) If $b < a$, then $\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{I(b,i)}(V) = \Delta_{I(b,i)}(V_2)$.

Proof. Part (a) is clear from (5.2). For part (b), we first assume $b + i - 1 \geq a$ and write

$$\Delta_{I(b,i)}(V_2) = v_b \wedge \cdots v_a \wedge (u_1 \wedge \cdots \wedge u_{i-(a-b+1)}) \wedge (u_i \wedge \cdots \wedge u_{k-2} \wedge v_n).$$

By Lemma 5.1.8 we get

$$v_a \wedge u_1 \wedge \cdots \wedge u_{i-(a-b+1)} = v_a \wedge v_{a+1} \wedge \cdots \wedge v_{b+i-1}$$

and

$$u_i \wedge \cdots \wedge u_{k-2} \wedge v_n = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n,$$

so

$$\Delta_{I(b,i)}(V_2) = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} (v_b \wedge \cdots v_{b+i-1}) \wedge (v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n) = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{I(b,i)}(V)$$

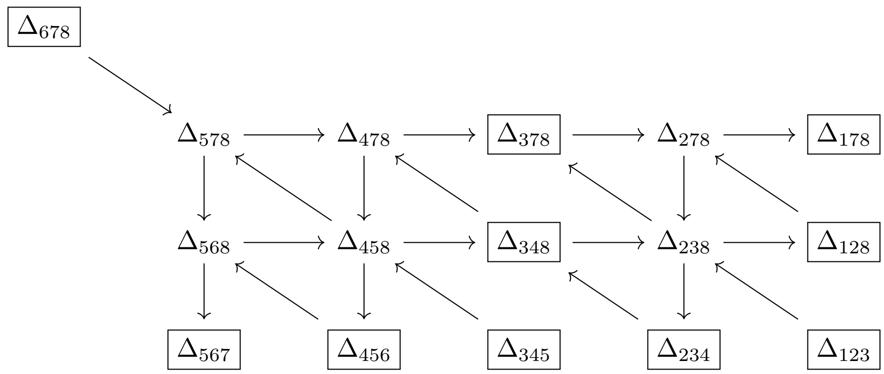
Similarly, if $b + i - 1 < a$ then

$$\Delta_{I(b,i)}(V_2) = v_b \wedge \cdots v_{b+i-1} \wedge (u_i \wedge \cdots \wedge u_{k-2} \wedge v_n) =$$

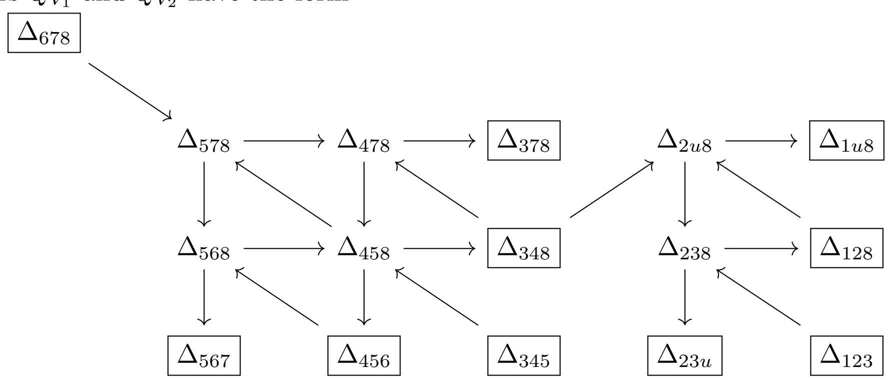
$$\frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} (v_b \wedge \cdots v_{b+i-1}) \wedge (v_{n-k+i+1} \wedge \cdots \wedge v_{n-1} \wedge v_n) = \frac{\Delta_{I(a,k-1)}}{\Delta_{I(a,i)}} \Delta_{I(b,i)}(V)$$

□

Example 5.2.2. For $a = 3,\ k = 3, n = 8$ we get $\Delta_{378}, \Delta_{348} \neq 0$, as in Example 5.1.6. We have $V_1 = (v_3, v_4, v_5, v_6, v_7, v_8)$ and $V_2 = (v_1, v_2, v_3, u, v_8)$. The quiver Q_V after freezing Δ_{378} and Δ_{348} has the form:



while the quivers Q_{V_1} and Q_{V_2} have the form



Note that we identified $\Delta_{3u8} = \Delta_{348}$. We claim that the two cluster structures are related by a quasi-equivalence. Indeed,

$$\Delta_{3u8} = \Delta_{348},\ \Delta_{23u} = \Delta_{234},\ \Delta_{2u8} = \alpha\Delta_{278},\ \Delta_{1u8} = \alpha\Delta_{178} \text{ where } \alpha = \frac{\Delta_{348}}{\Delta_{378}},$$

and all other cluster variables are unchanged. Therefore all cluster variables are the same up to monomials in frozen. We need to check the exchange ratios:

$$y_{2u8}(V_2) = \frac{\Delta_{348}\Delta_{128}}{\Delta_{1u8}\Delta_{238}} = \alpha^{-1}\frac{\Delta_{128}\Delta_{348}}{\Delta_{178}\Delta_{238}} = \frac{\Delta_{128}\Delta_{378}}{\Delta_{178}\Delta_{238}} = y_{278}(V).$$

while

$$y_{238}(V_2) = \frac{\Delta_{2u8}\Delta_{123}}{\Delta_{128}\Delta_{23u}} = \alpha\frac{\Delta_{278}\Delta_{123}}{\Delta_{128}\Delta_{234}} = \frac{\Delta_{278}\Delta_{123}\Delta_{348}}{\Delta_{128}\Delta_{234}\Delta_{378}} = y_{238}(V).$$

Since the exchange ratios agree, we indeed get a quasi-equivalence.

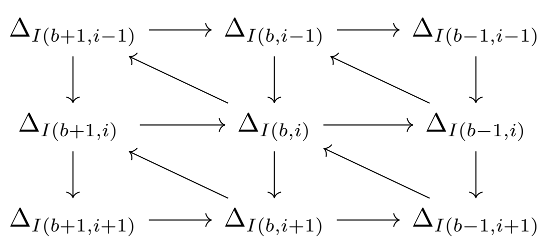
We are ready to state and prove our main result.

Theorem 5.2.3. *The map* $\Phi_a : V \mapsto (V_1, V_2)$ *defined by* (5.2) *is a cluster quasi-isomorphism between* $\widehat{U_a} \subset \widehat{\Pi}^\circ_{k,n}$ *and the product* $\widehat{\Pi}^\circ_{k,n-a+1} \times \widehat{\Pi}^\circ_{k,a+k-1}$ *with identified frozen variables* $\Delta_{I(a,k)}(V_2) = \Delta_{I(1,k-1)}(V_1)$*. As a consequence,* Φ_a *yields a cluster quasi-isomorphism between* U_a *and the product* $\Pi^\circ_{k,n-a+1} \times \Pi^\circ_{k,a+k-1}$*.*

Proof. The second statement follows from the first by Proposition 4.2.2, so we focus on $\widehat{U_a}$. By Lemma 5.2.1(a) Scott minors $\Delta_{I(b,i)}(V_1)$ are the same as the minors in the left half of Q_V.

We need to analyze the right half of Q_V. By Lemma 5.2.1(b) all minors in the right half are multiplied by some monomials in $\Delta_{I(a,i)}$ which are frozen on U_a. It remains to compute the exchange ratios. We have the following cases:

(a) Interior: $b < a$, $i > 1$. The piece of the quiver Q_V around $\Delta_{I(b,i)}$ has the form



The exchange ratios are equal to

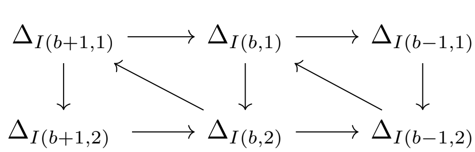
$$y_{I(b,i)} = \frac{\Delta_{I(b,i-1)}\Delta_{I(b+1,i)}\Delta_{I(b-1,i+1)}}{\Delta_{I(b-1,i)}\Delta_{I(b,i+1)}\Delta_{I(b+1,i-1)}}$$

so by Lemma 5.2.1 we get

$$\frac{y_{I(b,i)}(V)}{y_{I(b,i)}(V_2)} = \frac{\Delta_{I(a,i-1)}\Delta_{I(a,i)}\Delta_{I(a,i+1)}}{\Delta_{I(a,i)}\Delta_{I(a,i+1)}\Delta_{I(a,i-1)}} = 1.$$

and $y_{I(b,i)}(V) = y_{I(b,i)}(V_2)$. Note that $\Delta_{I(a,k-1)}$ cancels out.

(b) Top boundary $i = 1$:



The exchange ratios are equal to

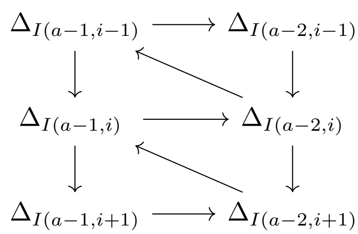
$$y_{I(b,1)} = \frac{\Delta_{I(b+1,1)}\Delta_{I(b-1,2)}}{\Delta_{I(b-1,1)}\Delta_{I(b,2)}}$$

so by Lemma 5.2.1 we get

$$\frac{y_{I(b,1)}(V)}{y_{I(b,1)}(V_2)} = \frac{\Delta_{I(a,1)}\Delta_{I(a,2)}}{\Delta_{I(a,1)}\Delta_{I(a,2)}} = 1.$$

and $y_{I(b,i)}(V) = y_{I(b,i)}(V_2)$. Note that $\Delta_{I(a,k-1)}$ cancels again.

(c) Left boundary, $b = a - 1$:



The exchange ratios are equal to

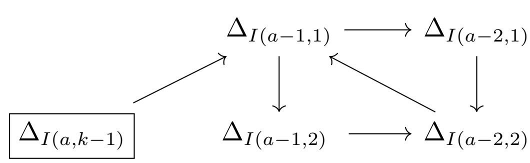
$$y_{I(a-1,i)}(V_2) = \frac{\Delta_{I(a-1,i-1)}(V_2)\Delta_{I(a-2,i+1)}(V_2)}{\Delta_{I(a-2,i)}(V_2)\Delta_{I(a-1,i+1)}(V_2)}$$

so by Lemma 5.2.1 we get

$$y_{I(a-1,i)}(V_2) = \left[\frac{\Delta_{I(a,i-1)}(V)\Delta_{I(a,i+1)}(V)}{\Delta_{I(a,i)}(V)\Delta_{I(a,i+1)}(V)}\right]^{-1} \cdot \frac{\Delta_{I(a-1,i-1)}(V)\Delta_{I(a-2,i+1)}(V)}{\Delta_{I(a-2,i)}(V)\Delta_{I(a-1,i+1)}(V)}$$

$$= \frac{\Delta_{I(a,i)}(V)\Delta_{I(a-1,i-1)}(V)\Delta_{I(a-2,i+1)}(V)}{\Delta_{I(a,i-1)}(V)\Delta_{I(a-2,i)}(V)\Delta_{I(a-1,i+1)}(V)} = y_{I(a-1,i)}(V).$$

(d) Corner, $b = a - 1, i = 1$:



Here we identify $\Delta_{I(a,k-1)}(V_2)$ with $\Delta_{I(a,k-1)}(V) = \Delta_{I(1,k-1)}(V_1)$ as above. The exchange ratio is equal to

$$y_{I(a-1,1)}(V_2) = \frac{\Delta_{I(a,k-1)}(V)\Delta_{I(a-2,2)}(V_2)}{\Delta_{I(a-2,1)}(V_2)\Delta_{I(a-1,2)}(V_2)}$$

so by Lemma 5.2.1 we get

$$\Delta_{I(a,k-1)}(V) \cdot \frac{\Delta_{I(a-2,2)}(V)\Delta_{I(a,k-1)}(V)}{\Delta_{I(a,2)}(V)} \cdot \frac{\Delta_{I(a,1)}(V)}{\Delta_{I(a-2,1)}(V)\Delta_{I(a,k-1)}(V)} \cdot \frac{\Delta_{I(a,2)}(V)}{\Delta_{I(a-1,2)}(V)\Delta_{I(a,k-1)}(V)}$$

$$= \frac{\Delta_{I(a-2,2)}(V)\Delta_{I(a,1)}(V)}{\Delta_{I(a-2,1)}(V)\Delta_{I(a-1,2)}(V)} = y_{I(a-1,1)}(V).$$

□

5.3. Relation to braid varieties

In this section, we describe the map Φ in terms of braid varieties. Recall from Section 3.2 that $\beta = (\sigma_1 \ldots \sigma_{k-1})^{n-k} w_0$ is the braid associated to $X(\beta) \simeq \Pi^{\circ,1}_{k,n}$. Then the process of freezing $\Delta_{I(a)}$ corresponds to severing the braid β at flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ where $A = (a-1)(k-1)+1$ and $N = (n-k)(k-1) + \binom{k}{2} + 1$. Upon severing the braid at the given flags, we disassemble the braid into two separate braids decorated by the flags

(5.1) $$\mathcal{F}^{(A)} \xrightarrow{s_1} \mathcal{F}^{(A+1)} \ldots \mathcal{F}^{(N-1)} \xrightarrow{s_1} \mathcal{F}^{(N)}$$

and

(5.2) $$\mathcal{F}^{(0)} \xrightarrow{s_1} \mathcal{F}^{(1)} \ldots \mathcal{F}^{(A-1)} \xrightarrow{s_{k-1}} \mathcal{F}^{(A)} \xrightarrow{s_1} \widetilde{\mathcal{F}}^{(A+1)} \xrightarrow{\cdots} \widetilde{\mathcal{F}}^{A+\binom{k}{2}-1} \xrightarrow{s_1} \mathcal{F}^{(N)}.$$

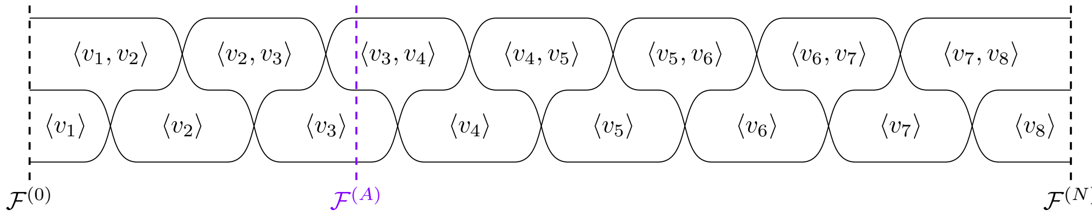
The first braid is decorated by the flags between $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ and is associated to $X(\beta_1)$ where $\beta_1 = (\sigma_1 \cdots \sigma_{k-1})^{n-k-a+1} w_0$. Note that the conditions defining the open subset U_a guarantee that the flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ are in position w_0, so as above there is a unique matrix M such that $M\mathcal{F}^{(A)}$ is the standard flag and $M\mathcal{F}^{(N)}$ is the antistandard flag.

For the second braid, we "splice" together the flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ by adding the sequences of flags $\widetilde{\mathcal{F}}$ associated with the half twist on k strands. See Figure **??** for an example of the decomposition of the braid β into its two separate components, and Figure 5.2 for a depiction of the local splicing effect on the flags. Stitching the flags $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$ together with the half twist fills the bottom row of the braid with $k-2$ vectors $u_1, \ldots, u_{k-2}$, and the intermediate flags $\widetilde{F}^{(A+j)}$ are uniquely determined by $\mathcal{F}^{(A)}$ and $\mathcal{F}^{(N)}$. Through this process the resulting braid is $\beta_2 = (\sigma_1 \cdots \sigma_{k-1})^{a-1} w_0$.

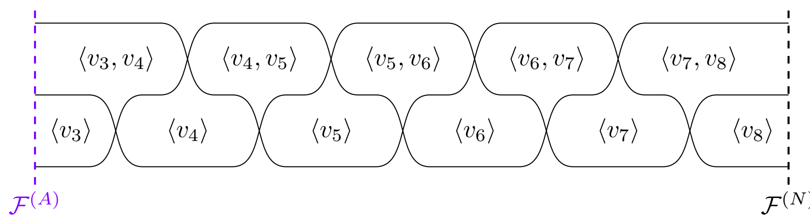
Finally, we can compare the cluster structures on braid varieties. The cluster structure on $\Pi^{\circ,1}_{k,n}$ is obtained from (4.2) by removing the frozen variables $\Delta_{I(b,k)}$ from Q_V.

THEOREM 5.3.1. *The map* $\overline{\Phi}_a : V \mapsto (V_1, \overline{V_2})$,

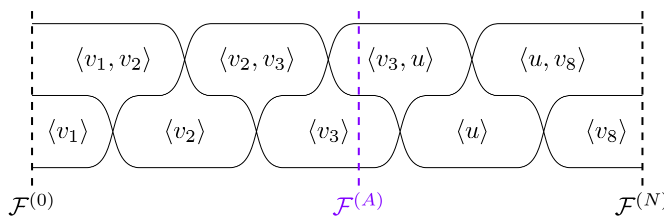
$$V_1 = (v_a, \ldots, v_n), \quad \overline{V_2} = \left(v_1, \ldots, v_a, u_1, \ldots, u_{k-2}, \frac{v_n}{\Delta_{I(a,k-1)}}\right)$$



(A) The braid associated to $\Pi^{\circ,1}_{3,8}$. We select the flag $\mathcal{F}^{(A)}$ given by $0 \subset \langle v_3\rangle \subset \langle v_3, v_4\rangle \subset \mathbb{C}^3$ to sever the braid for the splicing map since the cluster variables which intersect $\mathcal{F}^{(A)}$ are $\Delta_{348}, \Delta_{378}$.



(B) The braid corresponding to $V_1 = \begin{pmatrix} v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{pmatrix} \in \Pi^{\circ,1}_{3,6}$ under the splicing map $\overline{\Phi}_3$.



(C) The braid corresponding to $V_2 = \begin{pmatrix} v_1 & v_2 & v_3 & u & \frac{v_8}{\Delta_{378}(V)} \end{pmatrix} \in \Pi^{\circ,1}_{3,5}$ given $\langle u\rangle = \langle v_3, v_4\rangle \cap \langle v_7, v_8\rangle$ under the splicing map $\overline{\Phi}_3$.

Figure 5.1. Freezing $\Delta_{348}, \Delta_{378}$ in the braid associated to $X(\beta_{3,8}) \simeq \Pi^{\circ,1}_{3,8}$.

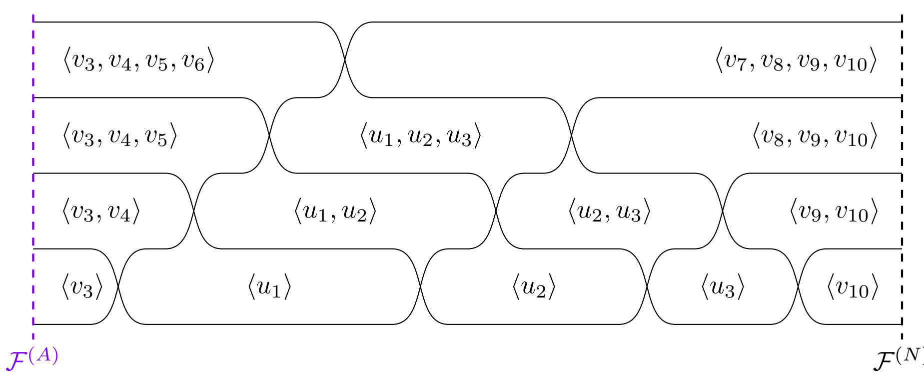


Figure 5.2. Braid diagram and flags for Example 5.1.4. Here $\langle u_1\rangle = \langle v_3, v_4\rangle \cap \langle v_7, v_8, v_9, v_{10}\rangle$, $\langle u_2\rangle = \langle v_3, v_4, v_5\rangle \cap \langle v_8, v_9, v_{10}\rangle$ and $\langle u_3\rangle = \langle v_3, v_4, v_5, v_6\rangle \cap \langle v_9, v_{10}\rangle$.

defines a quasi-cluster isomorphism between $U_a^1 = U_a \cap \Pi_{k,n}^{\circ,1}$ *and* $\Pi_{k,n-a+1}^{\circ,1} \times \Pi_{k,a+k-1}^{\circ,1}$.

Note that we do not need to change the matrix V_1 since all of its consecutive minors are still equal to 1. Also note that by (**??**) we get $\dim U_a^1 = \dim \Pi_{k,n}^{\circ,1} = (k-1)(n-k)$ while

$$\dim \Pi_{k,n-a+1}^{\circ,1} + \Pi_{k,a+k-1}^{\circ,1} = (k-1)(n-a+1-k) + (k-1)(a-1) = (k-1)(n-k).$$

5.4. Splicing two-stranded braid varieties

We now describe the splicing map for the case $k = 2$. Since $k - 2 = 0$, no u_i's are required in the splicing map. We refer to the two-stranded splicing map as a (diagonal) cut map.

Let $\mathcal{P}$ be the $(\ell+1)$-gon corresponding to the braid variety $X(\sigma^\ell)$. We can choose a diagonal D_{ij} which cuts the polygon $\mathcal{P}$ in two pieces, a $(j-i+1)$-gon $\mathcal{P}_1(i,j)$ and a $(\ell-j+i+2)$-gon $\mathcal{P}_2(i,j)$. These correspond to braid varieties $X(\sigma^{j-i})$ and $X(\sigma^{\ell-j+i+1})$, respectively. We will refer to this procedure as a diagonal cut. If we denote $a = j - i$ and $b = \ell - j + i + 1$ then $a + b = \ell + 1$. See Figure 5.3 for an example of the cut map on $\mathrm{U_{fan}}$.

THEOREM 5.4.1. *Performing one diagonal cut on* P *along* D_{ij} *defines an injective map*

$$\Phi_{ij}^{-1} : X(\sigma^a) \times X(\sigma^b) \longrightarrow X(\sigma^{a+b-1})$$

and its image is the open subset $\{\Delta_{ij} \neq 0\}$ *in* $X(\sigma^{a+b-1})$.

PROOF. We use the isomorphism $\Pi_{2,\ell+1}^{\circ,1} \simeq X(\sigma^\ell)$ from Theorem 3.2.5. We first describe the inverse map

$$\Phi_{ij} : \{\Delta_{ij} \neq 0\} \to X(\sigma^a) \times X(\sigma^b).$$

Let $V \in \Pi_{2,\ell+1}^{\circ,1}$ be a $2\times(\ell+1)$ matrix, choose some i, j such that $1 \leq i < j \leq \ell+1$ where $(i,j) \neq (1, \ell+1)$, to perform the diagonal cut of the $(\ell+1)$-gon resulting in two polygons $\mathcal{P}_a$ and $\mathcal{P}_b$ where $\mathcal{P}_a$ is a $(j-i+1)$-gon and $\mathcal{P}_b$ is a $(\ell-j+i+2)$-gon. Assume that $\Delta_{ij}(V) \neq 0$. Then we can decompose the matrix V into two matrices:

$$V_1 = \begin{pmatrix} v_i & \ldots & v_j \end{pmatrix} \in \mathrm{Mat}(2, a+1)$$

$$V_2 = \begin{pmatrix} v_1 & \ldots & v_i & v_j & \ldots & v_{\ell+1} \end{pmatrix} \in \mathrm{Mat}(2, b+1)$$

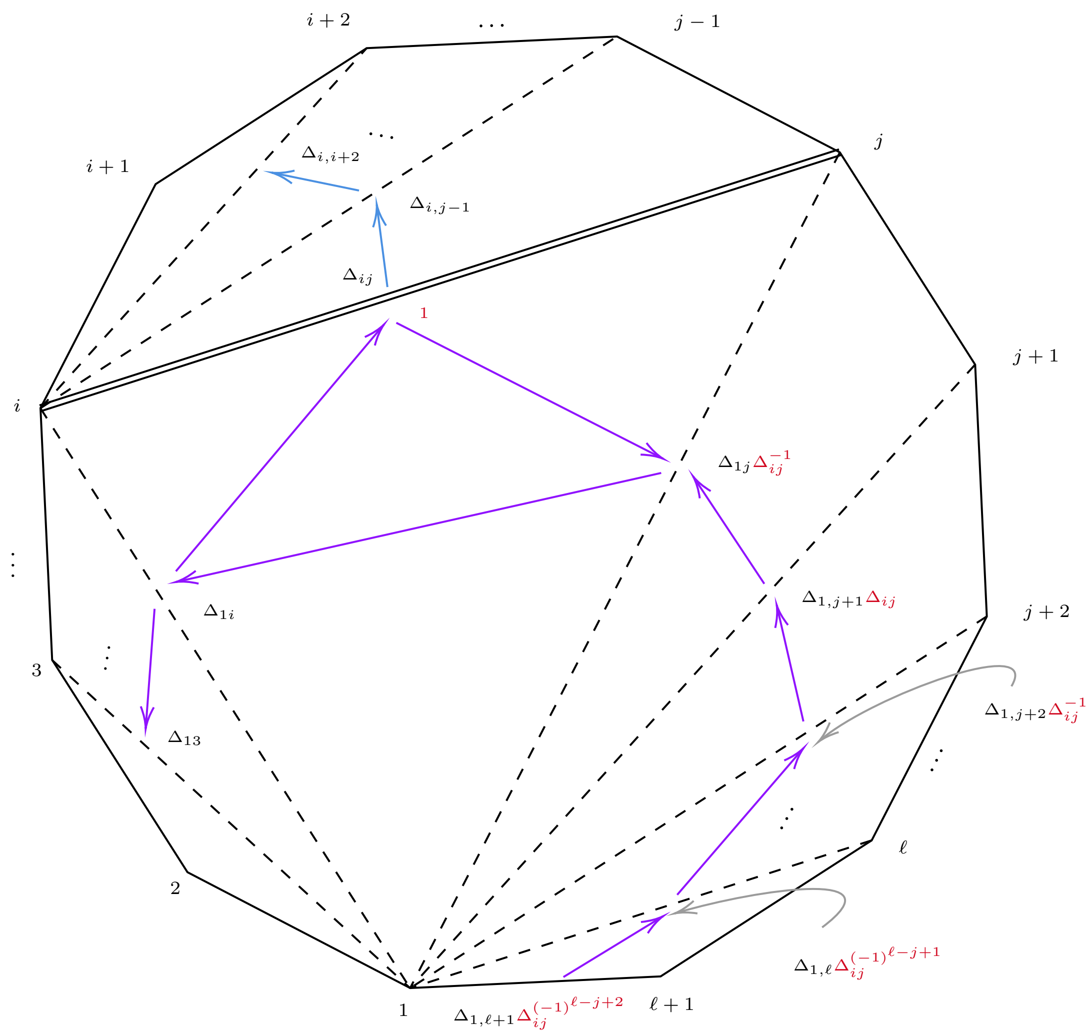


FIGURE 5.3. Triangulation of $(\ell+1)$-gon corresponding to the braid variety $X(\sigma^\ell)$ with its associated quiver. A cut Δ_{ij} is depicted between vertices i and j. The cluster variables from the particular triangulation are the written in black and the rescaling factor of the cluster variables from the cut Δ_{ij} are written in red.

Let us prove that $V_1 \in \Pi^{\circ,1}_{2,a+1}$. As it happens $\Delta_{m,m+1}(V_1) = \Delta_{m+i-1,m+i}(V) = 1$ for $1 \leq m \leq a$, and $\Delta_{1,a+1}(V_1) = \Delta_{ij}(V) \neq 0$. We use the isomorphism $\Pi^{\circ,1}_{2,a+1} \simeq X(\sigma^a)$ from Theorem 3.2.5 to obtain a point in $X(\sigma^a)$ from V_1.

Next, we study the matrix V_2. We have

$$\Delta_{m,m+1}(V_2) = \begin{cases} \Delta_{m,m+1}(V) = 1 & \text{if } m < i \\ \Delta_{ij}(V) & \text{if } m = i \\ \Delta_{m+j-i-1,m+j-i}(V) = 1 & \text{if } i < m \leq \ell - j + i + 1. \end{cases}$$

Furthermore, $\Delta_{1,b+1}(V_2) = \Delta_{1,\ell+1}(V) \neq 0$, so $V_2 \in \Pi^{\circ}_{2,b+1}$. We would like to use Lemmas 3.2.7 and 3.2.8 to change V_2 to a different matrix $V_2' \in \Pi^{\circ,1}_{2,b+1}$. We have two cases:

Case 1: If $i = 1$, then we first apply Lemma 3.2.7. Since $S = (v_1 \ v_{\ell+1})^{-1}$ is diagonal, we simply rescale the second row of V_2 by Δ_{1j}^{-1} to get V_2 to the form (3.4). Next, we apply Lemma 3.2.8 to rescale the vectors, and get $V_2' = (v_1, v_2', \ldots, v_{b+1}')$ where

$$v_m' = \begin{cases} (v^1_{m+j-2}, v^2_{m+j-2}\Delta_{1j}^{-1}) & \text{if } m \text{ is even} \\ (v^1_{m+j-2}\Delta_{1j}, v^2_{m+j-2}) & \text{if } m \text{ is odd.} \end{cases}$$

Case 2: If $i \geq 2$, then we do not need to apply Lemma 3.2.7, we rescale the vectors v_m for $m \geq j$. As a result, we get a matrix $V_2' = (v_1, \ldots, v_i, v_j', v_{j+1}', \ldots, v_{\ell+1}')$ where

$$v_m' = \Delta_{ij}^{(-1)^{m-j+1}} v_m.$$

Now we can describe the desired map $\Phi_{ij}^{-1} : X(\sigma^a) \times X(\sigma^b) \to \{\Delta_{ij} \neq 0\}$ as follows. Given two matrices $V_1 \in \Pi^{\circ,1}_{2,a+1}, V_2' \in \Pi^{\circ,1}_{2,b+1}$, we can read off $\Delta_{ij}(V) = \Delta_{1,a+1}(V_1)$ which is nonzero by assumption. The matrix V_2' was obtained from V_2 above using multiplication by $\Delta_{ij}^{\pm 1}$, and hence is invertible, so given V_2' and Δ_{ij} we can reconstruct V_2.

Now we can reconstruct V by simply inserting V_1 into V_2. Note that if the vectors v_i and v_j from V_1 do not agree with the ones from V_2, we can always use row operations to make them agree since $\det(v_i \quad v_j) = \Delta_{ij} \neq 0$.

□

THEOREM 5.4.2. *The map Φ_{ij} defines a quasi-equivalence of cluster varietes $\{\Delta_{ij} \neq 0\} \subset X(\sigma^{a+b-1})$ and $X(\sigma^a) \times X(\sigma^b)$. The latter has a cluster structure obtained by freezing Δ_{ij} in the cluster structure from $X(\sigma^{a+b-1})$.*

PROOF. We use the clusters in $X(\sigma^a), X(\sigma^b)$ and $X(\sigma^{a+b-1})$ defined by the triangulation in Figure 5.3. In particular, we get fan triangulations for $X(\sigma^a), X(\sigma^b)$.

By construction, all cluster variables corresponding to diagonals are multiplied by monomials in Δ_{ij}, but we still need to check that the exchange ratios (as in Equation 4.1) are preserved. All diagonals above Δ_{ij} are unchanged, so we need to verify that the exchange ratios do not change for diagonals $\Delta_{1,m}$. For $m < i$, this is clear. For $m = i$, the exchange ratio is

$$\frac{\Delta_{1j}}{\Delta_{ij}\Delta_{1,j-1}} = \frac{\Delta_{1j}\Delta_{ij}^{-1}}{1 \cdot \Delta_{1,j-1}}.$$

For $m = j$, the exchange ratio is

$$\frac{\Delta_{ij}\Delta_{1,j+1}}{\Delta_{1i}} = \frac{1 \cdot (\Delta_{1,j+1}\Delta_{ij})}{\Delta_{1i}}.$$

Finally, for $m > j$ we get

$$\frac{\Delta_{1,m+1}}{\Delta_{1,m-1}} = \frac{\Delta_{1,m+1}\Delta_{ij}^{(-1)^{m+1-j+1}}}{\Delta_{1,m-1}\Delta_{ij}^{(-1)^{m-1-j+1}}}$$

since $m + 1 - j$ and $m - 1 - j$ have the same parity. □

5.4.1. Quasi-associativity of splicing two-stranded braid varieties. Suppose $a + b + c - 2 = \ell$. We will study the associativity properties of our cuts along two non-intersecting diagonals D_{ij} and $D_{i'j'}$, see Figure 5.4. There are two general cases to consider when performing two cuts which we label as Type A or Type B. The two cuts occur at D_{ij} and $D_{i'j'}$ and will be denoted Φ_{ij} and $\Phi_{i'j'}$, respectively. Type A cuts are diagonal cuts of the form $1 \leq i' \leq i < j \leq j' \leq \ell + 1$ given that the cuts do not degenerate to the one cut case, whereas, Type B cuts are diagonal cuts of the form $1 \leq i < j \leq i' < j' \leq \ell + 1$, see Figure 5.4.

THEOREM 5.4.3. *For Type A cuts we have a commutative diagram*

$$\begin{array}{ccc} X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) & \xrightarrow{\Phi_{ij}^{-1} \times \mathrm{Id}} & X(\sigma^{a+b-1}) \times X(\sigma^c) \\ \Big\downarrow {\scriptstyle \mathrm{Id} \times \Phi_{i'j'}^{-1}} & & \Big\downarrow {\scriptstyle \Phi_{i'j'}^{-1}} \\ X(\sigma^a) \times X(\sigma^{b+c-1}) & \xrightarrow[\Phi_{ij}^{-1}]{} & X(\sigma^{a+b+c-2}) \end{array}$$

PROOF. Let $V \in \Pi_{2,\ell+1}^{\circ,1}$ by Theorem 3.2.5 V corresponds to a point in $X(\sigma^\ell)$.

For Type A cuts, choose some i, j, i', j' such that $1 \leq i' \leq i < j < j' \leq \ell$. Similar to Theorem 5.4.1 involving a single diagonal cut, we describe the inverse maps then produce the desired map. Here $a = j - i, b =$

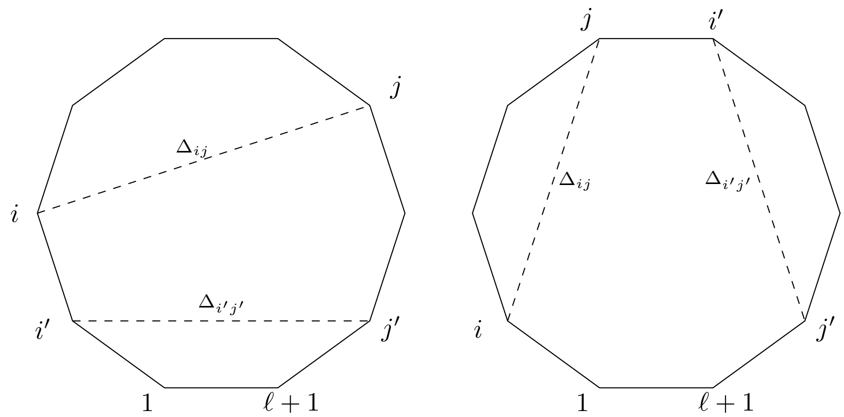


FIGURE 5.4. The possible cuts when performing two diagonal cuts, the dashed lines indicate these potential cuts. The polygon on the left depicts cuts of Type A and the polygon on the right depicts cuts of Type B.

$j' - j + i - i' + 1$ and $c = \ell - j' + i' + 1$. Define the matrix $V \in \Pi^{\circ,1}_{2,\ell+1}$ associated to $X(\sigma^\ell)$ as

$$V = (v_1 \quad \dots \quad v_{i'} \quad \dots \quad v_i \quad \dots \quad v_j \quad \dots \quad v_{j'} \quad \dots \quad v_{\ell+1})$$

We will be dealing with minors in several different matrices, as such we will include the matrices in the notations.

(i) First, we consider the case where we cut at along $\Delta_{ij}(V)$ then $\Delta_{i'j'}(V)$ which is described in Figure 5.5a by

$$X(\sigma^{a+b+c-2}) \xrightarrow{\Phi_{ij}} X(\sigma^a) \times X(\sigma^{b+c-1}) \xrightarrow{\mathrm{Id}\times\Phi_{i'j'}} X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c)$$

By performing the initial cut $\Delta_{ij}(V)$, given by $\Phi_{ij} : X(\sigma^{a+b+c-2}) \to X(\sigma^a) \times X(\sigma^{b+c-1})$, we decompose the matrix V into the two following matrices

$$V_1 = (v_i \quad \dots \quad v_j) \in \mathrm{Mat}(2, a+1)$$

$$V_2 = (v_1 \quad \dots \quad v_{i'} \quad \dots \quad v_i \quad v_j \quad \dots \quad v_{j'} \quad \dots \quad v_{\ell+1}) \in \mathrm{Mat}(2, b+c)$$

Similar to the argument in Theorem 5.4.1, $\Delta_{ij}(V) \neq 0$ and we find that $V_1 \in \Pi^{\circ,1}_{2,a+1} \simeq X(\sigma^a)$. Here, the rescaling of vectors v_m for $m \geq j$ in

$$V_3 = \left(v_1 \quad \dots \quad v_{i'} \quad \dots \quad v_i \quad v'_j \quad \dots \quad v'_{j'} \quad \dots \quad v'_{\ell+1}\right)$$

is given by

$$v'_m = v_m \Delta_{ij}(V)^{(-1)^{m-j+1}} \tag{5.1}$$

Therefore, $V_3 \in \Pi^{\circ,1}_{2,b+c} \simeq X(\sigma^{b+c-1})$ and Φ_{ij} is well-defined. Note that during the rescaling of the matrix V_2 into V_3 the minors of V_3 also experience rescaling by a factor of $\Delta_{ij}(V)$, hence given that $v'_{j'} = v_{j'}\Delta_{ij}(V)^{(-1)^{j'-j+1}}$

$$\Delta_{i'j'}(V_3) = \Delta_{i'j'}(V)\Delta_{ij}(V)^{(-1)^{j'-j+1}} \tag{5.2}$$

Now we perform the second cut $\Delta_{i'j'}(V)$ given by the map $X(\sigma^a) \times X(\sigma^{b+c-1}) \xrightarrow{\mathrm{Id}\times\Phi^{-1}_{i'j'}} X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c)$. The matrix V_1 remains unchanged whereas V_3 decomposes into

$$V_4 = \begin{pmatrix} v_{i'} & \dots & v_i & v'_j & \dots & v'_{j'} \end{pmatrix} \in \mathrm{Mat}(2, b+1)$$

$$V_5 = \begin{pmatrix} v_1 & \dots & v_{i'} & v'_{j'} & \dots & v'_{\ell+1} \end{pmatrix} \in \mathrm{Mat}(2, c+1)$$

By the rescaling of matrix V_3 in the previous cutting and $\Delta_{i'j'}(V_3) \neq 0$, then $V_4 \in \Pi^{\circ,1}_{2,b+1} \simeq X(\sigma^b)$. After performing the second cut there is again a rescaling, this time of the matrix V_5 which is given by the new matrix

$$V_6 = \begin{pmatrix} v_1 & \dots & v_{i'} & v''_{j'} & \dots & v''_{\ell+1} \end{pmatrix}$$

where for $m \geq j'$ the vectors are

$$v''_m = v'_m \Delta_{i'j'}(V_3)^{(-1)^{m-j'+1}} = v_m \Delta_{ij}(V)^{(-1)^{m-j+1}} \Delta_{i'j'}(V_3)^{(-1)^{m-j'+1}}.$$

Given that

$$\begin{aligned} \Delta_{i'j'}(V_3)^{(-1)^{m-j'+1}} &= \Delta_{i'j'}(V)^{(-1)^{m-j'+1}} \Delta_{ij}(V)^{(-1)^{j'-j+1}(-1)^{m-j'+1}} \\ &= \Delta_{i'j'}(V)^{(-1)^{m-j'+1}} \Delta_{ij}(V)^{(-1)^{m-j}} \end{aligned}$$

and $(-1)^{m-j+1} + (-1)^{m-j} = 0$ we conclude that

$$v''_m = v_m \Delta_{ij}(V)^{(-1)^{m-j+1}} \Delta_{i'j'}(V)^{(-1)^{m-j'+1}} \Delta_{ij}(V)^{(-1)^{m-j}} = v_m \Delta_{i'j'}(V)^{(-1)^{m-j'+1}}. \tag{5.3}$$

As such $V_6 \in \Pi^{\circ,1}_{2,c+1} \simeq X(\sigma^c)$. This concludes the construction of the inverse map.

To construct the desired map

$$\Phi_{ij}^{-1} \circ (\mathrm{Id} \times \Phi_{i'j'}^{-1}) : X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \to X(\sigma^{a+b+c-2})$$

We reconstruct V by taking $V_1 \in \Pi^{\circ,1}_{2,a+1}$, $V_4 \in \Pi^{\circ,1}_{2,b+1}$, $V_6 \in \Pi^{\circ,1}_{2,c+1}$. We can read off $\Delta_{i'j'}(V) = \Delta_{1,b+1}(V_4)$ which is nonzero by assumption. The matrix V_5 is obtained from V_6 by multiplication of $\Delta_{i'j'}(V)^{\pm 1}$ to the vectors v_l for $l \geq i'+1$, which is well-defined since $\Delta_{i'j'}(V)$ is invertible. We reconstruct the matrix $V_3 \in \mathrm{Mat}(2, b+c)$ by inserting the matrix V_4 into V_5 in the appropriate location. Furthermore, $V_3 \in \Pi^{\circ,1}_{2,b+c} \simeq X(\sigma^{b+c-1})$ by construction. This concludes the construction of the map

$$X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \xrightarrow{\mathrm{Id}\times\Phi_{i'j'}^{-1}} X(\sigma^a) \times X(\sigma^{b+c-1})$$

Continuing the construction of the desired map, we read off $\Delta_{ij}(V) = \Delta_{1,a+1}(V_1)$ which is again nonzero by assumption. The matrix V_2 is obtained from V_3 by multiplication of $\Delta_{ij}(V)^{\pm 1}$ to the vectors v_l for $l \geq i+1$. We reconstruct V by inserting V_1 into V_2 at the appropriate location, completing the construction of the map

$$X(\sigma^a) \times X(\sigma^{b+c-1}) \xrightarrow{\Phi_{ij}^{-1}} X(\sigma^{a+b+c-2})$$

and producing the desired map.

(ii) Now, for the case where we cut along $\Delta_{i'j'}(V)$ then $\Delta_{ij}(V)$, described in Figure 5.5b by

$$X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \xrightarrow{\Phi_{i'j'}^{-1}} X(\sigma^{a+b-1}) \times X(\sigma^c) \xrightarrow{\Phi_{ij}^{-1}\times\mathrm{Id}} X(\sigma^{a+b+c-2}).$$

Perform the initial cut $\Delta_{i'j'}(V)$, to decompose V into the matrices

$$W_1 = (v_1 \quad \dots \quad v_{i'} \quad v_{j'} \quad \dots \quad v_{\ell+1}) \in \mathrm{Mat}(2, c+1)$$

$$W_2 = (v_{i'} \quad \dots \quad v_i \quad \dots \quad v_j \quad \dots \quad v_{j'}) \in \mathrm{Mat}(2, a+b)$$

By the same argument as in Theorem 5.4.1 $\Delta_{i'j'} \neq 0$ and $W_2 \in \Pi^{\circ,1}_{2,a+b} \simeq X(\sigma^{a+b-1})$. Now, the matrix W_1 requires rescaling of the vectors v_m for $m \geq j'$, producing the matrix

$$W_3 = \begin{pmatrix} v_1 & \dots & v_{i'} & v'_{j'} & \dots & v'_{\ell+1} \end{pmatrix}$$

here

$$v'_m = v_m \Delta_{i'j'}(V)^{(-1)^{m-j'+1}}$$

which is in agreement with (5.3). Hence, $W_3 \in \Pi^{\circ,1}_{2,c+1} \simeq X(\sigma^c)$.

We perform the second cut $\Delta_{ij}(V)$, which separates W_2 into

$$W_4 = (v_{i'} \quad \dots \quad v_i \quad v_j \quad \dots \quad v_{j'}) \in \mathrm{Mat}(2, b+1)$$

$$W_5 = (v_i \quad \dots \quad v_j) \in \mathrm{Mat}(2, a+1)$$

In this case, $W_5 \in \Pi^{\circ,1}_{2,a+1} \simeq X(\sigma^a)$, whereas the matrix $W_4 \in \Pi^{\circ}_{2,b+1}$ requires a rescaling for the vectors v_m for $j \leq m \leq j'$. Let

$$W_6 = \begin{pmatrix} v_{i'} & \dots & v_i & v''_j & \dots & v''_{j'} \end{pmatrix}$$

with the vectors

$$v''_m = v_m \Delta_{ij}(V)^{m-j+1}$$

which agrees with (5.1). Therefore, $W_6 \in \Pi^{\circ,1}_{2,b+1} \simeq X(\sigma^b)$, completing the construction of the inverse maps.

Finally, we construct the desired map

$$\Phi^{-1}_{i'j'} \circ (\Phi^{-1}_{ij} \times \mathrm{Id}) : X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \to X(\sigma^{a+b+c-2})$$

We reconstruct V by taking $W_5 \in \Pi^{\circ,1}_{2,a+1}$, $W_6 \in \Pi^{\circ,1}_{2,b+1}$, $W_3 \in \Pi^{\circ,1}_{2,c+1}$. We read off $\Delta_{ij}(V) = \Delta_{1,a+1}(W_5)$ which is nonzero by assumption. The matrix W_4 is recovered from W_6 by multiplication of $\Delta^{\pm 1}_{ij}$ to the vectors v_l for $l \geq i - i' + 1$, which is well-defined since Δ_{ij} is invertible. We reconstruct $W_2 \in \Pi^{\circ,1}_{2,a+b} \simeq X(\sigma^{a+b-1})$ by inserting the matrix W_5 into W_4 in the appropriate position. Concluding the construction of the map

$$X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \xrightarrow{\Phi^{-1}_{ij} \times \mathrm{Id}} X(\sigma^{a+b-1}) \times X(\sigma^c)$$

To complete the construction, we read off $\Delta_{i'j'}(v) = \Delta_{1,c+1}(W_3)$ which is also nonzero by construction. The matrix $W_1 \in \Pi^{\circ,1}_{1,c+1} \simeq X(\sigma^c)$ is recovered from the matrix W_3 by multiplication of $\Delta_{i'j'}(V)^{\pm 1}$ to the vectors v_l for $l \geq i' + 1$. We reconstruct V by inserting W_1 into W_2 at the appropriate location, concluding the construction of the map

$$X(\sigma^{a+b-1}) \times X(\sigma^c) \xrightarrow{\Phi^{-1}_{i'j'}} X(\sigma^{a+b+c-2})$$

which produces the desired map showing associativity of Type A cuts.

$\square$

Lemma 5.4.4. *For $A, A' \in \Pi^{\circ}_{2,n}$ we define the map T_λ for some fixed j and $\lambda \neq 0$ as*

$$T_\lambda : A \to A', \quad a_m \mapsto a_m \lambda^{(-1)^{m-j}}.$$

Then T_λ preserves $\Pi^{\circ,1}_{2,n}$ and defines $\mathbb{C}^*$ actions on $\Pi^{\circ}_{2,n}$ and $\Pi^{\circ,1}_{2,n}$.

Proof. Let a_m, a'_m be column vectors of $A, A' \in \Pi^{\circ}_{2,n}$, respectively. Since $A \in \Pi^{\circ}_{2,n}$, we have $\det\begin{pmatrix} a_m & a_{m+1}\end{pmatrix} \neq 0$; therefore, under the map T_λ, we have

$$\begin{aligned}\det\begin{pmatrix} a'_m & a'_{m+1}\end{pmatrix} &= \det\begin{pmatrix} a_m\lambda^{(-1)^{m-j}} & a_{m+1}\lambda^{(-1)^{m-j+1}}\end{pmatrix} = \lambda^{(-1)^{m-j}}\lambda^{(-1)^{m-j+1}}\det\begin{pmatrix} a_m & a_{m+1}\end{pmatrix}\\ &= \det\begin{pmatrix} a_m & a_{m+1}\end{pmatrix}.\end{aligned}$$

In addition, if $A, A' \in \Pi^{\circ,1}_{2,n}$, then

$$\det\begin{pmatrix} a'_m & a'_{m+1}\end{pmatrix} = \det\begin{pmatrix} a_m\lambda^{(-1)^{m-j}} & a_{m+1}\lambda^{(-1)^{m-j+1}}\end{pmatrix} = \lambda^{(-1)^{m-j}}\lambda^{(-1)^{m-j+1}}\det\begin{pmatrix} a_m & a_{m+1}\end{pmatrix} = 1,$$

therefore, preserving $\Pi^{\circ,1}_{2,n}$. Moreover, the maps T_λ define a $\mathbb{C}^*$ action since $T_{\lambda_1} \circ T_{\lambda_2} = T_{\lambda_1\lambda_2}$ and $T_1 = \mathrm{Id}$. $\square$

Theorem 5.4.5. *For Type B cuts we have a commutative diagram*

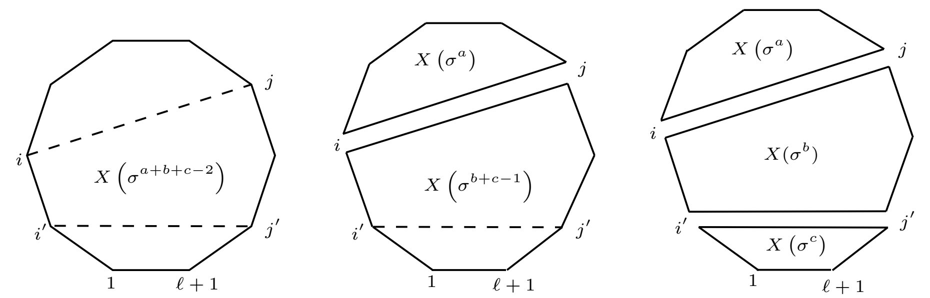
$$\begin{array}{ccccc} X(\sigma^a)\times X(\sigma^b)\times X(\sigma^c) & \xrightarrow{\mathrm{Id}\times\Phi^{-1}_{i'j'}} & X(\sigma^a)\times X(\sigma^{b+c-1}) & \xrightarrow{\Phi^{-1}_{ij}} & X(\sigma^{a+b+c-2}) \\ \downarrow{\scriptstyle \mathrm{Id}\times\mathrm{Id}\times T_{\Delta_{ij}}} & & & & \uparrow{\scriptstyle \Phi^{-1}_{i'j'}} \\ X(\sigma^a)\times X(\sigma^b)\times X(\sigma^c) & & \xrightarrow{\Phi^{-1}_{ij}\times\mathrm{Id}} & & X(\sigma^{a+b-1})\times X(\sigma^c) \end{array}$$

Here $T_{\Delta_{ij}}$ is defined as in Lemma 5.4.4 with $\lambda = \Delta_{ij}$. Informally, we can say that the gluing P from smaller polygons is associative only up to the additional transformation $T_{\Delta_{ij}}$.

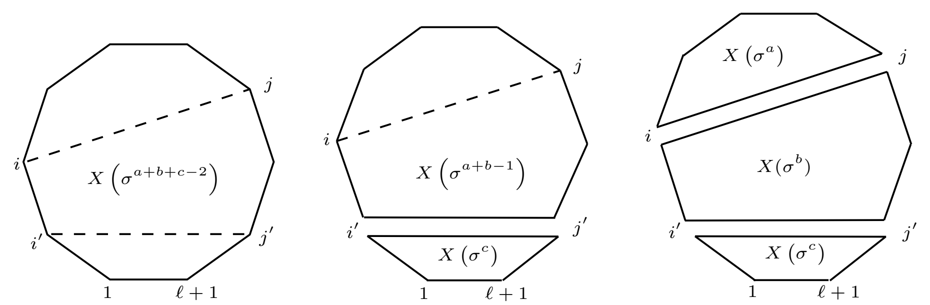
Proof. Let $V \in \Pi^{\circ,1}_{2,\ell+1}$ by Theorem 3.2.5.

For Type B cuts, choose some i, j, i', j' such that $1 \leq i < j \leq i' < j' \leq \ell+1$. Similar to Theorem 5.4.3, we describe the inverse maps then produce the desired map. Here $a = j-i$, $b = \ell - j' + i' - j + i + 2$, $c = j' - i'$. Define the matrix $V \in \Pi^{\circ,1}_{2,\ell+1}$ associated to $X(\sigma^\ell)$ as

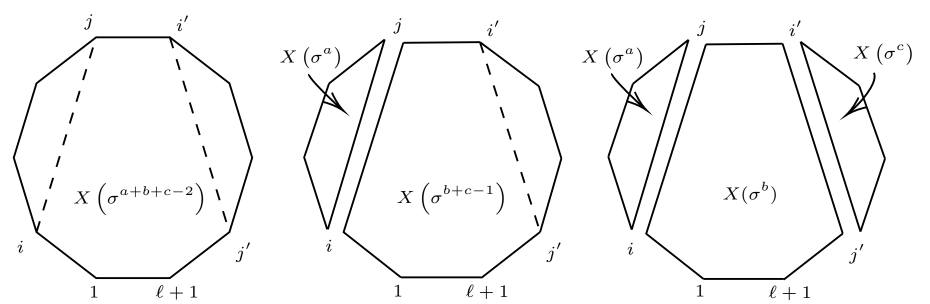
$$V = \begin{pmatrix} v_1 & \dots & v_i & \dots & v_j & \dots & v_{i'} & \dots & v_{j'} & \dots & v_{\ell+1}\end{pmatrix}$$



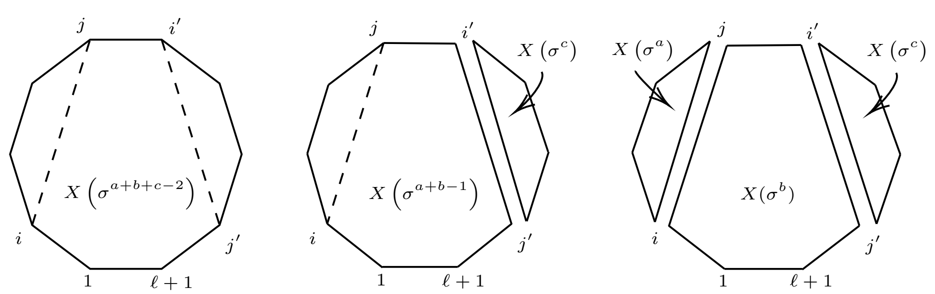
(A) Type A: Initial cut at Δ_{ij} followed by $\Delta_{i'j'}$.



(B) Type A: Initial cut at $\Delta_{i'j'}$ followed by Δ_{ij}.



(C) Type B: Initial cut at Δ_{ij} followed by $\Delta_{i'j'}$.



(D) Type B: Initial cut at $\Delta_{i'j'}$ followed by Δ_{ij}.

FIGURE 5.5. All possible variations of Type A and B cuts.

Similar to Theorem 5.4.3 we will be dealing with minors in several different matrices and will include the matrices in the notations.

(i) We first consider the case where we cut along $\Delta_{ij}(V)$ then $\Delta_{i'j'}(V)$, see Figure 5.5c, given by the map

$$X(\sigma^{a+b+c-2}) \xrightarrow{\Phi_{ij}} X(\sigma^a) \times X(\sigma^{b+c-1}) \xrightarrow{\mathrm{Id}\times\Phi_{i'j'}} X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c)$$

Performing the initial cut $\Delta_{ij}(V)$, given by $\Phi_{ij} : X(\sigma^{a+b+c-2}) \to X(\sigma^a) \times X(\sigma^{b+c-1})$, decomposes V into the two matrices

$$V_1 = (v_i \quad \dots \quad v_j) \in \mathrm{Mat}(2, a+1)$$

$$V_2 = (v_1 \quad \dots \quad v_i \quad v_j \quad \dots \quad v_{i'} \quad \dots \quad v_{j'} \quad \dots \quad v_{\ell+1}) \in \mathrm{Mat}(2, b+c)$$

By the same argument as in Theorem 5.4.1, $V_1 \in \Pi^{\circ,1}_{2,a+1} \simeq X(\sigma^a)$ whereas $V_2 \in \Pi^{\circ}_{2,b+c}$ and requires rescaling by $\Delta_{ij}(V)$ for the vectors v_m for $m \geq j$, resulting in the matrix

$$V_3 = \begin{pmatrix} v_1 & \dots & v_i & v'_j & \dots & v'_{i'} & \dots & v'_{j'} & \dots & v'_{\ell+1} \end{pmatrix}$$

where

$$v'_m = v_m \Delta_{ij}(V)^{(-1)^{m-j+1}} \tag{5.4}$$

Note that $\Delta_{i'j'}(V_3)$ experiences a rescaling by factor of $\Delta_{ij}(V)$, given that $v'_{i'} = V_{i'}\Delta_{ij}^{(-1)^{i'-j+1}}$ and $v'_{j'} = v_{j'}\Delta_{ij}^{(-1)^{j'-j+1}}$, the rescaled determinant is given by

$$\begin{aligned} \Delta_{i'j'}(V_3) &= \Delta_{i'j'}(V)\Delta_{ij}(V)^{(-1)^{i'-j+1}}\Delta_{ij}(V)^{(-1)^{j'-j+1}} \\ &= \Delta_{i'j'}(V)\Delta_{ij}(v)^{(-1)^{i'-j+1}+(-1)^{j'-j+1}} \end{aligned} \tag{5.5}$$

This completes the construction of the map $\Phi_{ij} : X(\sigma_{a+b+c-2}) \to X(\sigma^a) \times X(\sigma^{b+c-1})$. Applying the second cut $\Delta_{i'j'}(V)$ to the matrix V_3 produces the two matrices

$$V_4 = \begin{pmatrix} v'_{i'} & \dots & v'_{j'} \end{pmatrix} \in \mathrm{Mat}(2, c+1)$$

$$V_5 = \begin{pmatrix} v_1 & \dots & v_i & v'_j & \dots & v'_{i'} & v'_{j'} & \dots & v'_{\ell+1} \end{pmatrix} \in \mathrm{Mat}(2, b+1)$$

Here, $V_4 \in \Pi^{\circ,1}_{2,c+1} \simeq X(\sigma^c)$. Since $V_5 \in \Pi^{\circ}_{2,b+1}$ we applying a rescaling of the vectors v'_m for $m \geq j'$ into the matrix

$$V_6 = \begin{pmatrix} v_1 & \dots & v_i & v'_j & \dots & v'_{i'} & v''_{j'} & \dots & v''_{\ell+1} \end{pmatrix}$$

where

$$v''_m = v'_m \Delta_{i'j'}(V_3)^{(-1)^{m-j'+1}} \tag{5.6}$$

Using (5.5) we find that

$$\begin{aligned} \Delta_{i'j'}(V_3)^{(-1)^{m-j'+1}} &= (\Delta_{i'j'}(V)\Delta_{ij}(V)^{(-1)^{i'-j+1}+(-1)^{j'-j+1}})^{(-1)^{m-j'+1}} \\ &= \Delta_{i'j'}(V)^{(-1)^{m-j'+1}}\Delta_{ij}(V)^{(-1)^{i'-j+1}(-1)^{m-j'+1}+(-1)^{j'-j+1}(-1)^{m-j'+1}} \end{aligned}$$

and $(-1)^{i'-j+1}(-1)^{m-j'+1} + (-1)^{j'-j+1}(-1)^{m-j'+1} = (-1)^{m-j'+i'-j} + (-1)^{m-j}$. Therefore

$$\begin{aligned} v''_m &= v_m \Delta_{ij}(V)^{(-1)^{m-j+1}}\Delta_{i'j'}(V)^{(-1)^{m-j'+1}}\Delta_{ij}(V)^{(-1)^{m-j'+i'+j}}\Delta_{ij}^{(-1)^{m-j}} \\ &= v_m \Delta_{i'j'}(V)^{(-1)^{m-j'+1}}\Delta_{ij}(V)^{(-1)^{m-j'+i'-j}} \end{aligned} \tag{5.7}$$

Now, $V_6 \in \Pi^{\circ,1}_{2,b+1} \simeq X(\sigma^b)$. This concludes the construction of the inverse map, now we proceed to the construction of the desired map

$$X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \xrightarrow{\mathrm{Id}\times\Phi^{-1}_{i'j'}} X(\sigma^a) \times X(\sigma^{b+c-1}) \xrightarrow{\Phi^{-1}_{ij}} X(\sigma^{a+b+c-2})$$

Given $V_1 \in \Pi^{\circ,1}_{2,a+1}$, $V_6 \in \Pi^{\circ,1}_{2,b+1}$, $V_4 \in \Pi^{\circ,1}_{2,c+1}$ we reconstruct the matrix V. First, we determine that $\Delta_{i'j'}(V) = \Delta_{1,c+1}(V_4) \neq 0$.The matrix V_5 is found by multiplication of $\Delta_{i'j'}(V)^{\pm 1}$ to the vectors v_l for $l \geq i+i'-j+2$ in matrix V_6. We then reconstruct $V_3 \in \Pi^{\circ,1}_{2,b+c} \simeq X(\sigma^{b+c-1})$ by inserting the matrix V_4 into the appropriate position in the matrix V_5. This completes the map $\mathrm{Id} \times \Phi^{-1}_{i'j'} : X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \to X(\sigma^a) \times X(\sigma^{b+c-1})$. Now we continue our construction of the matrix V by reading off $\Delta_{ij}(V) = \Delta_{1,a+1}(V_1)$ which is nonzero by assumption. We rescale the vectors v_l for $l \geq i+1$ in the matrix V_3 by multiplication of $\Delta_{ij}(V)^{\pm 1}$ which is invertible, to obtain the matrix V_2. Finally, we insert the matrix V_1 into V_2 to obtain V. Therefore, giving us the desired map above.

(ii) Now, we consider the case were we first cut along $\Delta_{i'j'}(V)$ followed by the cut $\Delta_{ij}(V)$ and subsequently, a rescaling of $X(\sigma^c)$ by the torus action $T_{\Delta_{ij}}$, illustrated in Figure 5.5d, given by

$$X(\sigma^{a+b+c-2}) \xrightarrow{\Phi_{i'j'}} X(\sigma^{a+b-1}) \times X(\sigma^c) \xrightarrow{\Phi_{ij}\times \mathrm{Id}} X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c) \xrightarrow{\mathrm{Id}\times\mathrm{Id}\times T_{\Delta_{ij}}} X(\sigma^a) \times X(\sigma^b) \times X(\sigma^c)$$

We perform the initial cut $\Delta_{i'j'}(V)$ to V resulting in the matrices

$$W_1 = (v_{i'} \quad \dots \quad v_{j'}) \in \mathrm{Mat}(2, c+1)$$

$$W_2 = (v_1 \quad \dots \quad v_i \quad \dots \quad v_j \quad \dots \quad v_{i'} \quad v_{j'} \quad \dots \quad v_{\ell+1}) \in \mathrm{Mat}(2, a+b)$$

By the same argument in Theorem 5.4.1, $V_1 \in \Pi_{2,c+1} \simeq X(\sigma^c)$, whereas the matrix $V_2 \in \Pi^{\circ}_{2,a+b}$ requires as rescaling of the vectors v_m for $m \geq j'$ to obtain the matrix

$$W_3 = (v_1 \quad \dots \quad v_i \quad \dots \quad v_j \quad \dots \quad v_{i'} \quad \widetilde{v}_{j'} \quad \dots \quad \widetilde{v}_{\ell+1})$$

given by

$$\widetilde{v}_m = v_m \Delta_{i'j'}^{(-1)^{m-j'+1}} \tag{5.8}$$

Now, $W_3 \in \Pi^{\circ,1}_{2,a+b} \simeq X(\sigma^{a+b-1})$, completing the construction the first map.

We now perform the second cut $\Delta_{ij}(V) = \Delta_{ij}(W_2)$ by decomposing the matrix W_3 into the matrices

$$W_4 = (v_i \quad \dots \quad v_j) \in \mathrm{Mat}(2, a+1)$$

$$W_5 = (v_1 \quad \dots \quad v_i \quad v_j \quad \dots \quad v_{i'} \quad \widetilde{v}_{j'} \quad \dots \quad \widetilde{v}_{\ell+1}) \in \mathrm{Mat}(2, b+1)$$

Given that $W_4 \in \Pi^{\circ,1}_{2,a+1} \simeq X(\sigma^a)$ and $W_5 \in \Pi^{\circ}_{2,b+1}$, the matrix vectors v_m for $m \geq j$ in W_5 are rescaled into the matrix

$$W_6 = \big(v_1 \quad \dots \quad v_i \quad v'_j \quad \dots \quad v'_{i'} \quad \widetilde{v}'_{j'} \quad \dots \quad \widetilde{v}'_{\ell+1}\big)$$

where for $j \leq m \leq i'$

$$v'_m = v_m \Delta_{ij}^{(-1)^{m-j+1}} \tag{5.9}$$

and for $m \geq j'$

$$\widetilde{v}'_m = \widetilde{v}_m \Delta_{ij}(V)^{(-1)^{m-j'+i'-j}}$$

$$= v_m \Delta_{i'j'}(V)^{(-1)^{m-j'+1}} \Delta_{ij}(V)^{(-1)^{m-j'+i'-j}} \tag{5.10}$$

Now, $W_6 \in \Pi^{\circ,1}_{2,b+1} \simeq X(\sigma^b)$. Note that for the vectors v'_m for $j \le m \le i'$ (5.4) agrees with (5.9), for $j' \le m$ (5.7) agrees with (5.10). However, the vectors v_m found in W_1 for $i' < m < j'$ do not agree with (5.4) and differ by a factor of $\Delta_{ij}(V)^{(-1)^{m-j+1}}$. Since $\Delta_{ij} \neq 0$, we can then apply a torus action to the matrix $W_1 \in \Pi^{\circ,1}_{2,c+1} \simeq X(\sigma^c)$ using Lemma 3.2.8. Let $W_1, W_7 \in \Pi^{\circ,1}_{2,c+1}$, define the torus action by the map

$$\begin{aligned} T^{-1}_{\Delta_{ij}} : W_1 &\longrightarrow W_7 \\ v_m &\longmapsto v_m \Delta_{ij}^{(-1)^{m-j-1}} \end{aligned}$$

Thus concluding the construction of the inverse maps.

Now, we construct the suitable map to establish associativity up to an additional transformation $T_{\Delta_{ij}}$, given by

$$X(\sigma^a)\times X(\sigma^b)\times X(\sigma^c) \xrightarrow{\mathrm{Id}\times\mathrm{Id}\times T_{\Delta_{ij}}} X(\sigma^a)\times X(\sigma^b)\times X(\sigma^c) \xrightarrow{\Phi^{-1}_{ij}\times\mathrm{Id}} X(\sigma^{a+b-1})\times X(\sigma^c) \xrightarrow{\Phi^{-1}_{i'j'}} X(\sigma^{a+b+c-2})$$

We reconstruct the matrix V using $W_4 \in \Pi^{\circ,1}_{2,a+1}$, $W_6 \in \Pi^{\circ,1}_{2,b+1}$, $W_7 \in \Pi^{\circ,1}_{2,c+1}$. First, we read off $\Delta_{ij}(V) = \Delta_{1,a+1}(W_4) \neq 0$ by assumption. We apply the toric action $T_{\Delta_{ij}}(W_7) = W_1 \in \Pi^{\circ,1}_{2,c+1}$. Now, we rescale the matrix W_6 by multiplication of $\Delta_{ij}^{\pm 1}$ to the vectors v_l for $l \geq j$, producing the matrix W_5. We then reinsert the matrix W_4 into W_5 at the appropriate location, arriving at the matrix $W_3 \in \Pi^{\circ,1}_{2,a+b} \simeq X(\sigma^{a+b-1})$. We then read of $\Delta_{i'j'}(V) = \Delta_{1,c+1}(W_1) \neq 0$ and multiply W_3 by a factor of $\Delta_{i'j'}(V)^{\pm 1}$ for vectors v_l for $l \geq j'$ to produce W_2. We then reinsert the matrix W_1 into W_2 arriving at the desired matrix V. Thereby, completing the construction of the desired map. □

Remark 5.4.6. We can also study the composition of splicing maps on higher-strand braid varieties, and we expect a similar quasi-associative behavior, where the resulting subvarieties will differ by explicit monomials in frozen cluster variables.

CHAPTER 6

Cohomology

In this final chapter, we turn to the topological aspects of braid varieties and focus our attention on two-stranded braid varieties, more specifically, on the computation of their singular cohomology and give an explicit presentation of its cohomology ring in terms of generators and relations.

One of the main motivations for studying the homologies of braid varieties is their relation to the Khovanov-Rozansky homology of the corresponding link. The Khovanov-Rozansky homology, denoted HHH, is a triply graded link homology that generalizes the HOMFLY-PT polynomial which is relatively difficult to compute. Refer to [**22**] for additional details. The relation between braid varieties and HHH was established by Trinh.

THEOREM 6.0.1 (Trinh[**34**]). *For all r-strand braids $\beta \in Br_W^+$ we have*

$$\mathrm{HHH}^{r,r+j,k}(\beta w_0)^\vee \simeq \mathrm{gr}^w_{j+2(r-N)} \mathrm{H}^{!,G}_{-(j+k+2(r-N))}(X(\beta)).$$

Equivalently, by Gorsky-Hogancamp-Mellit-Nakagane [**11**]*, $\mathrm{H}^*(X(\beta)) \simeq \mathrm{HHH}^{0,*,*}(\beta w_o^{-1})^\vee$ where w_0 is the half-twist (aka longest word). Here gr^w denotes the associated graded with respect to the weight filtration in cohomology.*

In particular, the work of Galashin-Lam [**17**] related the equivariant cohomology of the open positroid variety $\Pi_{k,n}$ to the Khovanov-Rozansky homology of the torus link $T(k, n-k)$. On two strands this equivalence simplifies to

$$H^*(X(\sigma^\ell)) \simeq \mathrm{HHH}^{0,*,*}(\sigma^{\ell-1})$$

where the braid $\sigma^{\ell-1}$ closes up to the torus link $T(2, \ell-1)$.

Our approach combines techniques from algebraic topology, algebraic geometry, and cluster algebras. We first utilize the recursive structure of the braid variety defined in Corollary 2.1.8. We then apply Alexander duality to relate the cohomology of a braid variety to the homology of its complement. Finally, we apply

Poincaré duality to relate the cohomology of the compactification to the homology of the original noncompact space. For two-stranded braid varieties, this yields a concrete description of their cohomology groups in terms of complements of hypersurfaces in affine space. Our computation agrees with that of Lam-Speyer in [**26**]. We then use algebraic de Rham theory to compute the ring structure, expressing cohomology classes as differential forms and leveraging the defining equations of the varieties to identify relations among them.

We also make use of the splicing construction to understand how the cohomology of more complicated braid varieties relates to the cohomology of simpler ones. This viewpoint reveals how topological invariants behave under gluing operations and provides insight into the recursive structure observed in earlier chapters. We expect that the splicing map and techniques from previous chapters may lead to a deeper understanding of higher strand braid varieties.

Altogether, this chapter serves as a topological complement to the algebraic and geometric structures explored earlier in the thesis. The cohomology ring of a braid variety encodes rich information about its global structure, and these computations serve as a foundation for future work connecting braid varieties to representation theory, link homology, and mirror symmetry.

6.1. Cohomology using Alexander and Poincaré duality

Given the inductive definition of the two strand braid variety $X(\beta)$ we may determine the homology in terms of the vector space with Alexander and Poincaré duality. Our varieties are non-compact, so we have to be careful and sometimes use cohomology with compact support, for further information see [**28**, Section 3.3].

Theorem 6.1.1. *(Alexander Duality) If K is a locally contractible, nonempty, proper subspace of $\mathbb{R}^n$, then $\widetilde{H}_i(\mathbb{R}^n - K; \mathbb{C}) \simeq \widetilde{H}_c^{n-i-1}(K; \mathbb{C})$ for all i.*

Theorem 6.1.2. *(Poincaré Duality) If M is an orientable n-manifold then we have an isomorphism $\widetilde{H}_c^k(M; \mathbb{C}) \simeq \widetilde{H}_{n-k}(M; \mathbb{C})$ for all k.*

The cohomology of two-strand braid varieties was computed in [**26**, Section 6.2, Proposition 9.13] using cluster algebra methods (compare with Theorem 6.2.5 below). Here we give a simpler inductive proof using Poincaré and Alexander dualities.

THEOREM 6.1.3. *Let $\beta = \sigma^n$, then the homology of the two-strand braid variety is given by:*

$$H^i(X(\beta)) = \begin{cases} \mathbb{C} & \text{for } 0 \le i \le n-1 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We proceed by induction on n. Given Corollary 2.1.8, then

$$H^i(X(\sigma^2)) = H^i(\{z_1 z_2 - 1 = 0\}) = H^i(\{z_1 \neq 0\}) = H^i(\mathbb{C}^*)$$

Since $H^i(\mathbb{C}^*) = \mathbb{C}$ for $i = 0, 1$, then the theorem is true for $n = 2$. Supposing the statement holds for $n = \ell$ we determine that

$$\begin{aligned} \widetilde{H}_i(X(\sigma^{\ell+1})) &= \widetilde{H}_i(\{F_{\ell+1} = 0\}) = \widetilde{H}_i(\{F_\ell \neq 0\}) \text{ (by Corollary 2.1.8)} \\ &= \widetilde{H}_c^{2\ell-1-i}(\{F_\ell = 0\}) \quad \text{(by Theorem 6.1.1)} = \widetilde{H}_c^{2\ell-1-i}(X(\sigma^\ell)) \\ &= \widetilde{H}_{2\ell-2-(2\ell-1-i)}(X(\sigma^\ell)) \text{ (by Theorem 6.1.2)} = \widetilde{H}_{i-1}(X(\sigma^\ell)). \end{aligned}$$

Since $\widetilde{H}_i(X(\sigma^{\ell+1})) = \begin{cases} \mathbb{C} & 1 \le i \le \ell+1 \\ 0 & \text{otherwise} \end{cases}$, we obtain $H_i(X(\sigma^{\ell+1})) = \begin{cases} \mathbb{C} & 0 \le i \le \ell+1 \\ 0 & \text{otherwise.} \end{cases}$ □

6.2. Ring structure on cohomology using (algebraic) deRham cohomology

6.2.1. Constructing the forms. Define the one-form $\alpha = \frac{dw}{w}$ where $w = \Delta_{1,\ell+1}$ is the frozen cluster variable. Since $w \neq 0$ everywhere, α is regular everywhere.

Define the two-form as

$$\omega = \sum \widetilde{B}_{ij} \frac{dw_i}{w_i} \wedge \frac{dw_j}{w_j} \tag{6.1}$$

on some cluster chart with extended exchange matrix $\widetilde{B}$. By [**19**, Section 2.3] (see also [**26**]) the form ω is well-defined in any other cluster chart and is given by a similar equation (6.1) for the mutated quiver. The cluster charts cover $X(\sigma^\ell)$ up to codimension 2 and $X(\sigma^\ell)$ is smooth, so ω extends to a regular form on $X(\sigma^\ell)$.

For the special chart $\mathrm{U}_{\mathrm{fan}}$ we get

$$\omega = \frac{dw}{w} \wedge \frac{dw_{\ell-2}}{w_{\ell-2}} + \sum_{i=1}^{\ell-3} \frac{dw_{i+1}}{w_{i+1}} \wedge \frac{dw_i}{w_i} \tag{6.2}$$

where $w_i = \Delta_{1,i+2}$.

We can also write the forms α and ω explicitly in the coordinates z_i. Thus far, we have expressed $X(\sigma^\ell)$ as an open subset in the affine space with coordinates $z_1, \ldots, z_{\ell-1}$ with z_ℓ expressed as some function of these. Similarly, we may also have expressed $X(\sigma^\ell)$ is an open subset in the affine space with coordinates $z_2, \ldots, z_\ell$ with z_1 expressed as some function of these, i.e., $F_\ell(z_1, \ldots, z_\ell) = z_1 F_{\ell-1}(z_2, \ldots, z_\ell) - F_{\ell-2}(z_3, \ldots, z_\ell)$ where $F_{-1} \equiv 0$, $F_0 \equiv 1$, and $F_1(z_2) = z_2$. We will use $z_2, \ldots, z_k$ as a coordinate system on $X(\sigma^\ell)$ below.

Lemma 6.2.1. *For all $2 \le i \le \ell$ and $2 \le n \le \ell + 1$ we have*

$$\frac{\partial \Delta_{1n}}{\partial z_i} = \Delta_{1i}\Delta_{in}.$$

Proof. We have the matrix identity

$$\begin{pmatrix} F_n(z_1, \ldots, z_n) & -F_{n-1}(z_1, \ldots, z_{n-1}) \\ F_{n-1}(z_2, \ldots, z_n) & -F_{n-2}(z_2, \ldots, z_{n-1}) \end{pmatrix} = C \begin{pmatrix} z_i & -1 \\ 1 & 0 \end{pmatrix} \widetilde{C}$$

where

$$C = \begin{pmatrix} F_{i-1}(z_1, \ldots, z_{i-1}) & -F_{i-2}(z_1, \ldots, z_{i-2}) \\ F_{i-2}(z_2, \ldots, z_{i-1}) & -F_{i-3}(z_2, \ldots, z_{i-2}) \end{pmatrix}, \widetilde{C} = \begin{pmatrix} F_{n-i}(z_{i+1}, \ldots, z_n) & -F_{n-i-1}(z_{i+2}, \ldots, z_{n-1}) \\ F_{n-i-1}(z_{i+2}, \ldots, z_n) & -F_{n-i-2}(z_{i+2}, \ldots, z_{n-1}) \end{pmatrix},$$

which implies

$$\begin{aligned} F_{n-2}(z_2, \ldots, z_{n-1}) =& (F_{i-2}(z_2, \ldots, z_{i-1}) z_i - F_{i-3}(z_2, \ldots, z_{i-2})) F_{n-i-1}(z_{i+2}, \ldots, z_{n-1}) \\ & - F_{i-2}(z_2, \ldots, z_{i-1}) F_{n-i-2}(z_{i+2}, \ldots, z_{n-1}) \end{aligned}$$

and

$$\frac{\partial F_{n-2}(z_2, \ldots, z_{n-1})}{\partial z_i} = F_{i-2}(z_2, \ldots, z_{i-1}) F_{n-i-1}(z_{i+2}, \ldots, z_{n-1}).$$

Now by Lemma 4.3.1 we have $\Delta_{1,n} = F_{n-2}(z_2, \ldots, z_{n-1})$ and

$$\frac{\partial \Delta_{1,n}}{\partial z_i} = F_{i-2}(z_2, \ldots, z_{i-1}) F_{n-i-1}(z_{i+2}, \ldots, z_{n-1}) = \Delta_{1,i}\Delta_{i,n}.$$

□

Corollary 6.2.2. *We have*

$$\alpha = \frac{d\Delta_{1,\ell+1}}{\Delta_{1,\ell+1}} = \frac{1}{\Delta_{1,\ell+1}} \sum_{i=2}^{\ell} \Delta_{1,i}\Delta_{i,\ell+1} dz_i.$$

Lemma 6.2.3. *For $i \le \ell$ we have*

$$\frac{1}{\Delta_{1,i}\Delta_{1,i+1}} + \ldots + \frac{1}{\Delta_{1,\ell}\Delta_{1,\ell+1}} = \frac{\Delta_{i,\ell+1}}{\Delta_{1,i}\Delta_{1,\ell+1}}.$$

Proof. We prove it by induction in ℓ, for $\ell = i$ the statement is clear since $\Delta_{i,i+1} = 1$. For the step of induction, suppose that it is true for $\ell - 1$, then

$$\begin{aligned}\frac{1}{\Delta_{1,i}\Delta_{1,i+1}} + \ldots + \frac{1}{\Delta_{1,\ell-1}\Delta_{1,\ell}} + \frac{1}{\Delta_{1,\ell}\Delta_{1,\ell+1}} &= \frac{\Delta_{i,\ell}}{\Delta_{1,i}\Delta_{1,\ell}} + \frac{1}{\Delta_{1,\ell}\Delta_{1,\ell+1}} \\ &= \frac{\Delta_{i,\ell}\Delta_{1,\ell+1} + \Delta_{1,i}\Delta_{\ell,\ell+1}}{\Delta_{1,i}\Delta_{1,\ell}\Delta_{1,\ell+1}},\end{aligned}$$

which by Plücker relation simplifies to

$$\frac{\Delta_{1,\ell}\Delta_{i,\ell+1}}{\Delta_{1,i}\Delta_{1,\ell}\Delta_{1,\ell+1}} = \frac{\Delta_{i,\ell+1}}{\Delta_{1,i}\Delta_{1,\ell+1}}.$$

□

Lemma 6.2.4. *We have*

$$\omega = \frac{1}{\Delta_{1,\ell+1}} \sum_{2 \le i < j \le \ell} \Delta_{1,i}\Delta_{i,j}\Delta_{j,\ell+1} dz_i \wedge dz_j.$$

Proof. By Lemma 6.2.1 we can write

$$d\Delta_{1,s} \wedge d\Delta_{1,s+1} = \sum_{i<j\le s} (\Delta_{1,i}\Delta_{i,s}\Delta_{1,j}\Delta_{j,s+1} - \Delta_{1,i}\Delta_{i,s+1}\Delta_{1,j}\Delta_{j,s}) dz_i \wedge dz_j =$$

$$\sum_{i<j\le s} \Delta_{1,i}\Delta_{1,j}(\Delta_{i,s}\Delta_{j,s+1} - \Delta_{i,s+1}\Delta_{j,s}) dz_i \wedge dz_j.$$

By Plücker relation we have

$$\Delta_{i,s}\Delta_{j,s+1} - \Delta_{i,s+1}\Delta_{j,s} = \Delta_{ij},$$

hence

$$d\Delta_{1,s} \wedge d\Delta_{1,s+1} = \sum_{i<j\le s} \Delta_{1,i}\Delta_{1,j}\Delta_{i,j} dz_i \wedge dz_j.$$

The coefficient at $dz_i \wedge dz_j$ does not depend on ℓ, so we get

$$\omega = \sum_{s=1}^{\ell} \frac{d\Delta_{1,s} \wedge d\Delta_{1,s+1}}{\Delta_{1,s}\Delta_{1,s+1}} = \sum_{i<j} \Delta_{1,i}\Delta_{1,j}\Delta_{i,j} dz_i \wedge dz_j \left(\frac{1}{\Delta_{1,j}\Delta_{1,j+1}} + \ldots + \frac{1}{\Delta_{1,k}\Delta_{1,\ell+1}} \right).$$

By Lemma 6.2.3 this simplifies to

$$\sum_{i<j} \frac{\Delta_{1,i}\Delta_{1,j}\Delta_{i,j}\Delta_{j,\ell+1} dz_i \wedge dz_j}{\Delta_{1,j}\Delta_{1,\ell+1}} = \sum_{i<j} \frac{\Delta_{1,i}\Delta_{i,j}\Delta_{j,\ell+1} dz_i \wedge dz_j}{\Delta_{1,\ell+1}}.$$

□

In particular, Lemma 6.2.4 gives a direct proof that ω is regular everywhere on $X(\sigma^\ell)$. See Section 6.2.3 for explicit examples and computations.

6.2.2. de Rham cohomology. By construction, $d\alpha = d\omega = 0$, so they represent some de Rham cohomology classes. The following theorem shows that these are in fact nonzero in cohomology and generate $H^*(X(\sigma^\ell))$ as an algebra.

THEOREM 6.2.5. *The forms α and ω generate $H^*(X(\sigma^\ell))$ as an algebra, modulo the following relations:*

1) If ℓ is even, the only relation is $\omega^{\frac{\ell}{2}} = 0$. The basis in cohomology is given by:

$$1, \alpha, \omega, \alpha\omega, \ldots, \omega^{\frac{\ell}{2}-1}, \alpha\omega^{\frac{\ell}{2}-1}. \tag{6.3}$$

2) If ℓ is odd, the relations are $\alpha\omega^{\frac{\ell-1}{2}} = \omega^{\frac{\ell+1}{2}} = 0$. The basis in cohomology is given by:

$$1, \alpha, \omega, \alpha\omega, \ldots, \alpha\omega^{\frac{\ell-3}{2}}, \omega^{\frac{\ell-1}{2}}. \tag{6.4}$$

PROOF. We work in the chart $\mathrm{U}_{\mathrm{fan}}$, there is a natural inclusion map $i : \mathrm{U}_{\mathrm{fan}} \to X(\sigma^\ell)$ and the corresponding restriction map in cohomology: $i^* : H^*(X(\sigma^\ell)) \to H^*(\mathrm{U}_{\mathrm{fan}})$.

We want to first prove that the restrictions of all the forms (6.3) and (6.4) to $H^*(\mathrm{U}_{\mathrm{fan}})$ do not vanish, this would imply that these forms do not vanish in $H^*(X(\sigma^\ell))$. Recall that $\mathrm{U}_{\mathrm{fan}} \simeq (\mathbb{C}^*)^{\ell-1}$ with coordinates $w_1, \ldots, w_{\ell-2}, w = w_{\ell-1}$, so $H^*(\mathrm{U}_{\mathrm{fan}})$ is isomorphic to an exterior algebra in $\frac{dw_i}{w_i}$.

Suppose ℓ is odd then

$$\omega^{\frac{\ell-1}{2}} = \left(\frac{dw_2}{w_2} \wedge \frac{dw_1}{w_1} + \ldots + \frac{dw}{w} \wedge \frac{dw_{\ell-2}}{w_{\ell-2}} \right)^{(\ell-1)/2}$$

$$= (\ell-1)/2)!\frac{dw_1}{w_1} \wedge \dots \wedge \frac{dw_{\ell-2}}{w_{\ell-2}} \wedge \frac{dw}{w}$$

and

$$\begin{aligned}
\alpha\omega^{\frac{\ell-3}{2}} &= \frac{dw}{w} \wedge \left(\frac{dw_2}{w_2} \wedge \frac{dw_1}{w_1} + \ldots + \frac{dw}{w} \wedge \frac{dw_{\ell-2}}{w_{\ell-2}}\right)^{(\ell-3)/2} \\
&= \frac{dw}{w} \wedge \left(\frac{dw_2}{w_2} \wedge \frac{dw_1}{w_1} + \ldots + \frac{dw_{\ell-2}}{w_{\ell-2}} \wedge \frac{dw_{\ell-3}}{w_{\ell-3}}\right)^{(\ell-3)/2} \\
&= \frac{dw}{w} \wedge ((\ell-3)/2)! \sum_{j=0}^{(\ell-5)/2} \frac{dw_1}{w_1} \wedge \ldots \widehat{\frac{dw_{2j+1}}{w_{2j+1}}} \cdots \wedge \frac{dw_{\ell-3}}{w_{\ell-3}}
\end{aligned}$$

In particular, these are nonzero. Suppose ℓ is even, then similarly

$$\omega^{\frac{\ell}{2}-1} = \sum_{j=0}^{\ell/2-2} \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \cdots \wedge \widehat{\frac{dw_{2j+1}}{w_{2j+1}}} \wedge \cdots \wedge \frac{dw_{\ell-2}}{w_{\ell-2}} \wedge \frac{dw}{w}.$$

and

$$\alpha\omega^{\frac{\ell-3}{2}} = ((\ell-1)/2)!\frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \cdots \wedge \frac{dw_{\ell-3}}{w_{\ell-3}} \wedge \frac{dw_{\ell-2}}{w_{\ell-2}} \wedge \frac{dw}{w}$$

This implies that all the forms in (6.3) and (6.4) are nonzero in $H^*(\mathrm{U}_{\mathrm{fan}})$ and hence nonzero in $H^*(X(\sigma^\ell))$. On the other hand, by Theorem 6.1.3 the corresponding cohomology groups of $X(\sigma^\ell)$ are one-dimensional in each degree; therefore, we obtain a basis. □

6.2.3. Examples.

Example 6.2.6. Braid variety associated to $\beta = \sigma^3$

$$\begin{aligned}
X(\sigma^3) &= \{z_1z_2z_3 - z_3 - z_1 = 0\} \\
&= \{z_1z_2 - 1 \neq 0\}
\end{aligned}$$

Using row operations and scaling the columns, we can transform any matrix in $\Pi^\circ_{2,4}$ to the form

$$V = \begin{pmatrix} 1 & z_1 & z_1z_2 - 1 & z_1z_2z_3 - z_1 - z_3 \\ 0 & 1 & z_2 & z_2z_3 - 1. \end{pmatrix} \in \Pi^{\circ,1}_{2,4}$$

Using the correspondence of cluster algebras and Grassmannians, we obtain two cluster charts, as seen in Figure 6.1:

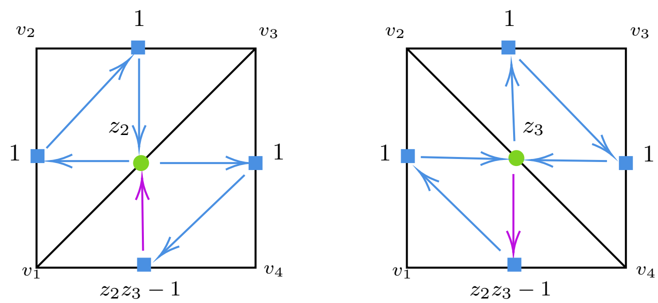


FIGURE 6.1. The two cluster charts for the braid variety $X(\sigma^3)$. On the left is chart U_1 where the vectors $v_i \in \Pi^{\circ,1}_{2,4}$ for $1 \le i \le 4$ correspond to the vertices of the polygon. The purple arrow depicts the Dynkin diagram A_1 with a frozen. On the right is chart U_2 which corresponds to the mutation of chart U_1.

$$U_1 = \{z_2 \neq 0\} \text{ with coordinates } (w_1 = z_2,\ w = z_2z_3 - 1)$$

$$U_2 = \{z_3 \neq 0\} \text{ with coordinates } (w_1' = z_3,\ w = z_2z_3 - 1)$$

We compute the cohomology of $X(\beta)$ using the (algebraic) de Rham cohomology on chart 1. Let $U_1 = \{w_1 = z_2 \neq 0,\ w = z_2z_3 - 1 \neq 0\}$. Then all possible forms

$$H^*(U_1) = H^*\big((\mathbb{C}^*)^2\big) = \left\langle 1, \frac{dw_1}{w_1}, \frac{dw}{w}, \frac{dw}{w} \wedge \frac{dw_1}{w_1} \right\rangle$$

To determine the cohomology, it suffices to determine which of the above forms extend to $X(\sigma^3)$. The forms which extend are

- 1
- $\dfrac{dw}{w} = \dfrac{z_2dz_3 + z_3dz_2}{z_2z_3 - 1}$
- $\dfrac{dw}{w} \wedge \dfrac{dw_1}{w_1} = \dfrac{dz_3 \wedge dz_2}{z_2z_3 - 1}$

The 2-form can be deduced from the quiver shown in Figure 6.1 which agrees with [**26**]. Therefore, $H^0(X(\sigma^3)) = H^1(X(\sigma^3)) = H^2(X(\sigma^3)) = \mathbb{C}$, which agrees with Theorem 6.1.3.

In addition, on the chart $U_2 = \{w_1' = z_3 \neq 0,\ w = z_2z_3 - 1 \neq 0\}$, with possible forms

$$H^*(U_2) = H^*\big((\mathbb{C}^*)^2\big) = \left\langle 1, \frac{dw_1'}{w_1'}, \frac{dw}{w}, \frac{dw}{w} \wedge \frac{dw_1'}{w_1'} \right\rangle$$

the forms which extend are

- 1
- $\dfrac{dw}{w} = \dfrac{z_2dz_3 + z_3dz_2}{z_2z_3 - 1}$
- $\dfrac{dw}{w} \wedge \dfrac{dw_1'}{w_1'} = \dfrac{dz_2 \wedge dz_3}{z_2z_3 - 1}$

Indeed, the cohomology of $X(\sigma^3)$ is independent from the choice of a chart.

EXAMPLE 6.2.7. The braid variety associated to $\beta = \sigma^4$

$$\begin{aligned} X(\sigma^4) &= \{z_1z_2z_3z_4 - z_1z_2 - z_1z_4 - z_3z_4 + 1 = 0\} \\ &= \{z_1z_2z_3 - z_3 - z_1 \neq 0\} \end{aligned}$$

with open positroid variety of the form

$$V = \begin{pmatrix} 1 & z_1 & z_1z_2 - 1 & z_1z_2z_3 - z_1 - z_3 & z_1z_2z_3z_4 - z_1z_2 - z_1z_4 - z_3z_4 + 1 \\ 0 & 1 & z_2 & z_2z_3 - 1 & z_2z_3z_4 - z_2 - z_4 \end{pmatrix} \in \Pi_{2,5}^{\circ,1}$$

Using the correspondence of cluster algebras and Grassmannians, we obtain one of five cluster charts, see Figure 6.2: Here

$$U = \{w_1 := \Delta_{13} = z_2 \neq 0, w_2 := \Delta_{14} = z_2z_3 - 1 \neq 0, w := \Delta_{15} = z_2z_3z_4 - z_4 - z_2 \neq 0\}$$

Using the de Rham cohomology

$$\begin{aligned} H^*(U) &= H^*((\mathbb{C}^*)^3) \\ &= \left\langle 1, \frac{dw_1}{w_1}, \frac{dw_2}{w_2}, \frac{dw}{w}, \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2}, \frac{dw_1}{w_1} \wedge \frac{dw}{w}, \frac{dw_2}{w_2} \wedge \frac{dw}{w}, \frac{dw}{w} \wedge \frac{dw_2}{w_2} \wedge \frac{dw_1}{w_1} \right\rangle \end{aligned}$$

The forms which extend to $X(\sigma^4)$ are:

- 1
- $\dfrac{dw}{w} = \dfrac{(z_3z_4 - 1)dz_2 + z_2z_4dz_3 + (z_2z_3 - 1)dz_4}{z_2z_3z_4 - z_4 - z_2}$

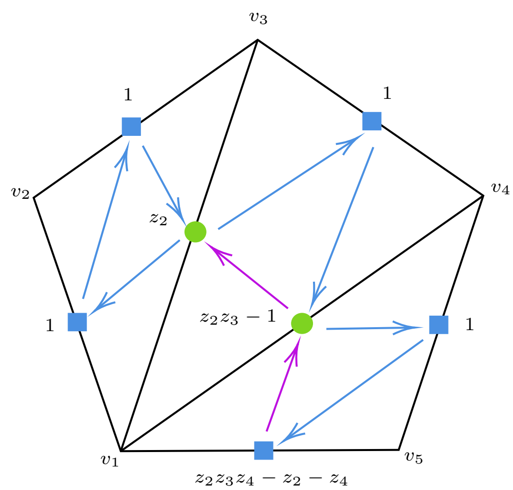


FIGURE 6.2. The cluster chart U_1 of $X(\sigma^4)$. One of the five possible charts given by the triangulation of the pentagon.

- $\frac{dw}{w}\wedge\frac{dw_2}{w_2}+\frac{dw_2}{w_2}\wedge\frac{dw_1}{w_1}=\frac{z_4dz_3\wedge dz_2+z_3dz_4\wedge dz_2+z_2dz_4\wedge dz_3}{z_2z_3z_4-z_4-z_2}$
- $\frac{dw}{w}\wedge\frac{dw_2}{w_2}\wedge\frac{dw_1}{w_1}=\frac{dz_4\wedge dz_3\wedge dz_2}{z_2z_3z_4-z_4-z_2}$

Therefore, $H^0(X(\sigma^4))=H^1(X(\sigma^4))=H^2(X(\sigma^4))=H^3(X(\sigma^4))=\mathbb{C}$ which agrees with Theorem 6.1.3.

6.2.4. Cuts, forms and cohomology. Now we can study the effect of the cuts on the forms α and ω. More precisely, we use the map $\Phi_{ij}^{-1}: X(\sigma^{j-i})\times X(\sigma^{\ell-j+i+1})\longrightarrow X(\sigma^\ell)$ to compute the pullbacks $\left(\Phi_{ij}^{-1}\right)^*\alpha$ and $\Phi_{ij}^*\omega$. The forms α and ω are equivalent under cluster mutation by [**26**]; hence, we choose an arbitrary cluster chart, see Figure 5.3, and determine the how the forms interact with cuts.

We will denote the forms from $X(\sigma^{j-i})$ by α_1 and ω_1, and the forms from $X(\sigma^{\ell-j+i+1})$ by α_2 and ω_2. As an abuse of notation we use the labeling from the larger positroid $\Pi^{o,1}_{2,a+b-1}$ identified with $X(\sigma^{a+b-1})$. Technically, under the isomorphism $\Delta_{1,j-i+1}=\left(\Phi_{ij}^{-1}\right)^*(\Delta_{ij})$, therefore, $\alpha_1=\left(\Phi_{ij}^{-1}\right)^*(d\log\Delta_{ij})$, similarly, $\alpha_2=\left(\Phi_{ij}^{-1}\right)^*(d\log\Delta_{ij}^{(-1)^{\ell-j+1}}\Delta_{1,\ell+1})$ with similar considerations made to ω_1 and ω_2.

LEMMA 6.2.8. *We have*

$$\left(\Phi_{ij}^{-1}\right)^*\alpha=\alpha_2+(-1)^{\ell-j}\alpha_1.$$

PROOF. Recall that $\alpha=d\log(\Delta_{1,\ell+1})$. By [**26**] let $\alpha_1=d\log(\Delta_{ij})$ be the 1-form associated to $X(\sigma^{j-i})$ and $\alpha_2=d\log(\Delta_{ij}^{(-1)^{\ell-j+1}}w)=d\log(\Delta_{ij}^{(-1)^{\ell-j+1}}\Delta_{1,\ell+1})$ be the 1-form associated to $X(\sigma^{\ell-j+i+1})$. Given these conditions we find that

$$\begin{aligned}\alpha_2 + (-1)^{\ell-j}\alpha_1 &= d\log(\Delta_{ij}^{(-1)^{\ell-j+1}}\Delta_{1,\ell+1}) + (-1)^{\ell-j} d\log(\Delta_{ij})\\ &= d\log(\Delta_{ij}^{(-1)^{\ell-j+1}}) + d\log(\Delta_{1,\ell+1}) + (-1)^{\ell-j} d\log(\Delta_{ij})\\ &= (-1)^{\ell-j+1} d\log(\Delta_{ij}) + d\log(\Delta_{1,\ell+1}) + (-1)^{\ell-j} d\log(\Delta_{ij})\\ &= d\log(\Delta_{1,\ell+1}) = \alpha\end{aligned}$$

□

LEMMA 6.2.9. *We have*

$$\left(\Phi_{ij}^{-1}\right)^{*}\omega = \omega_1 + \omega_2 + (-1)^{\ell-j}\alpha_1 \wedge \alpha_2.$$

PROOF. Consider the quiver associated to the triangulation of $X(\sigma^{\ell})$ in Figure 5.3 prior to the rescaling given by the cut Δ_{ij}, by (6.1), the two-form ω is described as

$$\begin{aligned}\omega = {} & d\log\Delta_{1,\ell+1} \wedge d\log\Delta_{1,\ell} + d\log\Delta_{1,\ell} \wedge d\log\Delta_{1,\ell-1}\\ & + \cdots + d\log\Delta_{1,j+1} \wedge d\log\Delta_{1,j} + d\log\Delta_{1,j} \wedge d\log\Delta_{1,i}\\ & + d\log\Delta_{1,i} \wedge d\log\Delta_{1,i-1} + d\log\Delta_{1,i-1} \wedge d\log\Delta_{1,i-2}\\ & + \cdots + d\log\Delta_{14} \wedge d\log\Delta_{13} + d\log\Delta_{1,i} \wedge d\log\Delta_{ij}\\ & + d\log\Delta_{ij} \wedge d\log\Delta_{1,j} + d\log\Delta_{i,j-1} \wedge d\log\Delta_{i,j-2}\\ & + d\log\Delta_{i,j-2} \wedge d\log\Delta_{i,j-3} + \cdots + d\log\Delta_{i,i+3} \wedge d\log\Delta_{i,i+2}\end{aligned}$$

Let α_1, α_2 be the 1-form and ω_1, ω_2 be the 2-form associated to $X(\sigma^{j-i})$ and $X(\sigma^{\ell-j+i+1})$, respectively. By Figure 5.3, we define the forms associated to $X(\sigma^{j-i})$ and $X(\sigma^{\ell-j+i+1})$ directly from quivers as follows:

$$\alpha_1 = d\log\Delta_{ij} \tag{6.5}$$

$$\alpha_2 = d\log(\Delta_{1,\ell+1}\Delta_{ij}^{(-1)^{\ell-j+2}}) = d\log\Delta_{1,\ell+1} + (-1)^{\ell-j+2} d\log\Delta_{ij} \tag{6.6}$$

$$\begin{aligned}\omega_1 = {} & d\log\Delta_{i,j-1} \wedge d\log\Delta_{i,j-2} + d\log\Delta_{i,j-2} \wedge d\log\Delta_{i,j-3}\\ & + \cdots + d\log\Delta_{i,i+3} \wedge d\log\Delta_{i,i+2}\end{aligned}$$

While α_1, α_2, ω_1 can be easily read from the cluster chart seen in Figure 5.3, the 2-form ω_2 requires a bit more finesse. We notice that there is a triangle formed between the vertices $1, i, j$, to simplify the computation of ω_2, which agrees with (6.1), we decompose the form into parts and call them pre-triangle $\omega_{2,pre}$ for vertices between 1 and i, triangle $\omega_{2,tri}$ for the special vertices $1, i, j$ and post-triangle $\omega_{2,post}$ for vertices between j and $\ell+1$. By Theorem 5.4.1 in the rescaled braid variety $X(\sigma^{\ell-j+i+1})$ the Plučker coordinate $\Delta'_{ij} = \Delta_{ij}\Delta_{ij}^{-1} = 1$ resulting in $d\log\Delta'_{ij} = d\log 1 = 0$, whereas Δ_{ij} shall remain the nonzero polynomial w describing $X(\sigma^{j-i})$. Using this decomposition, $\omega_2 = \omega_{2,pre} + \omega_{2,tri} + \omega_{2,post}$ is defined by

$$\omega_{2,pre} = d\log\Delta_{1i} \wedge d\log\Delta_{1,i-1} + d\log\Delta_{1,i-1} \wedge d\log\Delta_{1,i-2} + \cdots + d\log\Delta_{14} \wedge d\log\Delta_{13}$$

$$\begin{aligned}\omega_{2,tri} &= d\log\left(\Delta_{1j}\Delta_{ij}^{-1}\right) \wedge d\log\Delta_{1i} + d\log\Delta_{1i} \wedge d\log\Delta'_{ij} + d\log\Delta'_{ij} \wedge d\log\left(\Delta_{1j}\Delta_{ij}^{-1}\right) \\ &= (d\log\Delta_{1j} - d\log\Delta_{ij}) \wedge d\log\Delta_{1i} \\ &= d\log\Delta_{1j} \wedge d\log\Delta_{1i} - d\log\Delta_{ij} \wedge d\log\Delta_{1i}\end{aligned}$$

$$\begin{aligned}\omega_{2,post} &= d\log\Delta_{1,\ell+1}\Delta_{ij}^{(-1)^{\ell-j+2}} \wedge d\log\Delta_{1,\ell}\Delta_{ij}^{(-1)^{\ell-j+1}} \\ &\quad + d\log\Delta_{1,\ell}\Delta_{ij}^{(-1)^{\ell-j+1}} \wedge d\log\Delta_{1,\ell-1}\Delta_{ij}^{(-1)^{\ell-j}} \\ &\quad + \cdots + d\log\Delta_{1,j+2}\Delta_{ij}^{-1} \wedge d\log\Delta_{1,j+1}\Delta_{ij} + d\log\Delta_{1,j+1}\Delta_{ij} \wedge d\log\Delta_{1,j}\Delta_{ij}^{-1}\end{aligned}$$

$$\begin{aligned}&= (d\log\Delta_{1,\ell+1} + (-1)^{\ell-j+2} d\log\Delta_{ij}) \wedge (d\log\Delta_{1,\ell} + (-1)^{\ell-j+1} d\log\Delta_{ij}) \\ &\quad + (d\log\Delta_{1,\ell} + (-1)^{\ell-j+1} d\log\Delta_{ij}) \wedge (d\log\Delta_{1,\ell-1} + (-1)^{\ell-j} d\log\Delta_{ij}) \\ &\quad + \cdots + (d\log\Delta_{1,j+2} - d\log\Delta_{ij}) \wedge (d\log\Delta_{1,j+1} + d\log\Delta_{ij}) \\ &\quad + (d\log\Delta_{1,j+1} + d\log\Delta_{ij}) \wedge (d\log\Delta_{1,j} - d\log\Delta_{ij})\end{aligned}$$

$$\begin{aligned}&= d\log\Delta_{1,\ell+1} \wedge d\log\Delta_{1,\ell} + (-1)^{\ell-j+1} d\log\Delta_{1,\ell+1} \wedge d\log\Delta_{ij} \\ &\quad + (-1)^{\ell-j+2} d\log\Delta_{ij} \wedge d\log\Delta_{1,\ell} + d\log\Delta_{1,\ell} \wedge d\log\Delta_{1,\ell-1} \\ &\quad + (-1)^{\ell-j} d\log\Delta_{1,\ell} \wedge d\log\Delta_{ij} + (-1)^{\ell-j+1} d\log\Delta_{ij} \wedge d\log\Delta_{1,\ell-1} \\ &\quad + \cdots + d\log\Delta_{1,j+2} \wedge d\log\Delta_{1,j+1} + d\log\Delta_{1,j+2} \wedge d\log\Delta_{ij}\end{aligned}$$

$$- d\log\Delta_{i,j}\wedge d\log\Delta_{1,j+1} + d\log\Delta_{1,j+1}\wedge d\log\Delta_{1,j}$$

$$- d\log\Delta_{1,j+1}\wedge d\log\Delta_{ij} + d\log\Delta_{ij}\wedge d\log\Delta_{1,j}$$

$$= d\log\Delta_{1,\ell+1}\wedge d\log\Delta_{1,\ell} + d\log\Delta_{1,\ell}\wedge d\log\Delta_{1\ell-1}$$

$$+\cdots + d\log\Delta_{1,j+2}\wedge d\log\Delta_{1,j+1} + d\log\Delta_{1,j+1}\wedge d\log\Delta_{1,j}$$

$$+ (-1)^{\ell-j+1} d\log\Delta_{1,\ell+1}\wedge d\log\Delta_{ij}$$

Note that from (6.5) and (6.6), $\alpha_1\wedge\alpha_2 = d\log\Delta_{ij}\wedge d\log\Delta_{1,\ell+1}$. Therefore, the additional term $(-1)^{\ell-j+1} d\log\Delta_{1,\ell+1}\wedge d\log\Delta_{ij}$ from $\omega_{2,post}$ may be negated by $(-1)^{\ell-j}\alpha_1\wedge\alpha_2$, providing the necessary adjustment to acquire $\left(\Phi_{ij}^{-1}\right)^*\omega$ as stated. □

THEOREM 6.2.10. *The pullback map*

$$\left(\Phi_{ij}^{-1}\right)^* : H^*(X(\sigma^\ell)) \to H^*(X(\sigma^{j-i}))\otimes H^*(X(\sigma^{\ell-j+i+1}))$$

is injective and can be described by Lemmas 6.2.8 and 6.2.9

PROOF. Similar to Theorem 6.2.5, we want to prove that the restrictions of all forms in (6.3) and (6.4) do not vanish in $H^*(X(\sigma^{j-i}))\otimes H^*(X(\sigma^{\ell-j+i+1}))$, here we use the formulas from Lemmas 6.2.8 and 6.2.9. Suppose ℓ is odd, then we want to show that $\left(\Phi_{ij}^{-1}\right)^*\left[\alpha\omega^{\frac{\ell-3}{2}}\right]$ and $\left(\Phi_{ij}^{-1}\right)^*\left[\omega^{\frac{\ell-1}{2}}\right]$ are both nonzero. Since $\ell = a+b-1$ is odd, then either a, b are both even or both odd.

(i) Suppose a and b are both even. Given that $\omega_1^{\frac{a}{2}-1}$, $\alpha_1\omega_1^{\frac{a}{2}-1}$, $\omega_2^{\frac{b}{2}-1}$, $\alpha_2\omega_2^{\frac{b}{2}-1}$ are nonzero by definition, then

$$\begin{aligned}
\left(\Phi_{ij}^{-1}\right)^*\left[\alpha\omega^{\frac{\ell-3}{2}}\right] &= (\alpha_2 + (-1)^{\ell-j}\alpha_1)(\omega_1+\omega_2+(-1)^{\ell-j}\alpha_1\wedge\alpha_2)^{\frac{\ell-3}{2}}\\
&= (\alpha_2 + (-1)^{\ell-j}\alpha_1)(\omega_1+\omega_2+(-1)^{\ell-j}\alpha_1\wedge\alpha_2)^{\frac{a+b-4}{2}}\\
&= (\alpha_2 + (-1)^{\ell-j}\alpha_1)\sum_{l_1+l_2+l_3=\frac{a+b-4}{2}}\binom{\frac{a+b-4}{2}}{l_1,l_2,l_3}\omega_1^{l_1}\omega_2^{l_2}\left((-1)^{\ell-j}\alpha_1\wedge\alpha_2\right)^{l_3}\\
&= \binom{\frac{a+b-4}{2}}{\frac{a}{2}-1,\frac{b}{2}-1,0}\alpha_2\omega_1^{\frac{a}{2}-1}\omega_2^{\frac{b}{2}-1}+\dots
\end{aligned}$$

with $\alpha_2\omega_2^{\frac{b}{2}-1}$, $\omega_1^{\frac{a}{2}-1}\neq 0$, then $\left(\Phi_{ij}^{-1}\right)^*\left[\alpha\omega^{\frac{\ell-3}{2}}\right]$ is nonvanishing. Furthermore,

$$\left(\Phi_{ij}^{-1}\right)^*\left[\omega^{\frac{\ell-1}{2}}\right] = (\omega_1+\omega_2+(-1)^{\ell-j}\alpha_1\wedge\alpha_2)^{\frac{\ell-1}{2}}$$

$$= (\omega_1 + \omega_2 + (-1)^{\ell-j}\alpha_1 \wedge \alpha_2)^{\frac{a+b-2}{2}}$$

$$= \sum_{l_1+l_2+l_3=\frac{a+b-2}{2}} \binom{\frac{a+b-2}{2}}{l_1, l_2, l_3} \omega_1^{l_1}\omega_2^{l_2} \left((-1)^{\ell-j}\alpha_1 \wedge \alpha_2\right)^{l_3}$$

$$= \binom{\frac{a+b-2}{2}}{\frac{a}{2}-1, \frac{b}{2}-1, 1} \left((-1)^{\ell-j}\alpha_1 \wedge \alpha_2\right) \omega_1^{\frac{a}{2}-1}\omega_2^{\frac{b}{2}-1} + \dots$$

where $\alpha_1\omega_1^{\frac{a}{2}-1}$, $\alpha_2\omega_2^{\frac{b}{2}-1} \neq 0$. Then $\left(\Phi_{ij}^{-1}\right)^* \left[\omega^{\frac{\ell-1}{2}}\right]$ is nonvanishing.

(ii) Suppose a and b are both odd. Given that $\alpha_1\omega_1^{\frac{a-3}{2}}$, $\omega_1^{\frac{a-1}{2}}$, $\alpha_2\omega_2^{\frac{b-3}{2}}$, $\omega_2^{\frac{b-1}{2}}$ are nonzero, then

$$\left(\Phi_{ij}^{-1}\right)^* \left[\alpha\omega^{\frac{\ell-3}{2}}\right] = (\alpha_2 + (-1)^{\ell-j}\alpha_1) \sum_{l_1+l_2+l_3=\frac{a+b-4}{2}} \binom{\frac{a+b-4}{2}}{l_1, l_2, l_3} \omega_1^{l_1}\omega_2^{l_2} \left((-1)^{\ell-j}\alpha_1 \wedge \alpha_2\right)^{l_3}$$

$$= (\alpha_2 + (-1)^{\ell-j}\alpha_1) \binom{\frac{a+b-4}{2}}{\frac{a-3}{2}, \frac{b-}{2}, 0} \omega_1^{\frac{a-3}{2}}\omega_2^{\frac{b-1}{2}} + \dots$$

$$= (-1)^{\ell-j} \binom{\frac{a+b-4}{2}}{\frac{a-3}{2}, \frac{b-}{2}, 0} \alpha_1\omega_1^{\frac{a-3}{2}}\omega_2^{\frac{b-1}{2}} + \dots$$

Given $\alpha_1\omega_1^{\frac{a-3}{2}}$, $\omega_2^{\frac{b-1}{2}} \neq 0$, then $\left(\Phi_{ij}^{-1}\right)^* \left[\alpha\omega^{\frac{\ell-3}{2}}\right]$ is nonvanishing. Furthermore,

$$\left(\Phi_{ij}^{-1}\right)^* \left[\omega^{\frac{\ell-1}{2}}\right] = \sum_{l_1+l_2+l_3=\frac{a+b-2}{2}} \binom{\frac{a+b-2}{2}}{l_1, l_2, l_3} \omega_1^{l_1}\omega_2^{l_2} \left((-1)^{\ell-j}\alpha_1 \wedge \alpha_2\right)^{l_3}$$

$$= \binom{\frac{a+b-2}{2}}{\frac{a-1}{2}, \frac{b-1}{2}, 0} \omega_1^{\frac{a-1}{2}}\omega_2^{\frac{b-1}{2}} + \dots$$

Since $\omega_1^{\frac{a-1}{2}}$, $\omega_2^{\frac{b-1}{2}} \neq 0$, then $\left(\Phi_{ij}^{-1}\right)^* \left[\omega^{\frac{\ell-1}{2}}\right]$ is nonvanishing.

Now, suppose ℓ is even, then we want to show that $\left(\Phi_{ij}^{-1}\right)^* \left[\omega^{\frac{\ell}{2}-1}\right]$ and $\left(\Phi_{ij}^{-1}\right)^* \left[\alpha\omega^{\frac{\ell}{2}-1}\right]$ are both nonzero. Since $\ell = a + b - 1$ is even, without loss of generality a is even and b is odd. Since a is even and b is odd, then $\omega_1^{\frac{a}{2}-1}$, $\alpha_1\omega_1^{\frac{a}{2}-1}$, $\alpha_2\omega_2^{\frac{b-3}{2}}$, $\omega_2^{\frac{b-1}{2}}$ are nonzero, then

$$\left(\Phi_{ij}^{-1}\right)^* \left[\omega^{\frac{\ell}{2}-1}\right] = (\omega_1 + \omega_2 + (-1)^{\ell-j}\alpha_1 \wedge \alpha_2)^{\frac{\ell}{2}-1}$$

$$= (\omega_1 + \omega_2 + (-1)^{\ell-j}\alpha_1 \wedge \alpha_2)^{\frac{a+b-3}{2}}$$

$$= \sum_{l_1+l_2+l_3=\frac{a+b-3}{2}} \binom{\frac{a+b-3}{2}}{l_1, l_2, l_3} \omega_1^{l_1}\omega_2^{l_2} \left((-1)^{\ell-j}\alpha_1 \wedge \alpha_2\right)^{l_3}$$

$$= \binom{\frac{a+b-3}{2}}{\frac{a}{2}-1, \frac{b-1}{2}, 0} \omega_1^{\frac{a}{2}-1}\omega_2^{\frac{b-1}{2}} + \dots$$

Since $\omega_1^{\frac{a}{2}-1}$, $\omega_2^{\frac{b-1}{2}} \neq 0$, then $\left(\Phi_{ij}^{-1}\right)^* \left[\omega^{\frac{\ell}{2}-1}\right]$ is nonvanishing. Next,

$$\begin{aligned}\left(\Phi_{ij}^{-1}\right)^* \left[\alpha\omega^{\frac{\ell}{2}-1}\right] &= (\alpha_2 + (-1)^{\ell-j}\alpha_1)(\omega_1 + \omega_2 + (-1)^{\ell-j}\alpha_1 \wedge \alpha_2)^{\frac{\ell}{2}-1} \\ &= (\alpha_2 + (-1)^{\ell-j}\alpha_1)(\omega_1 + \omega_2 + (-1)^{\ell-j}\alpha_1 \wedge \alpha_2)^{\frac{a+b-3}{2}} \\ &= (\alpha_2 + (-1)^{\ell-j}\alpha_1) \sum_{l_1+l_2+l_3=\frac{a+b-3}{2}} \binom{\frac{a+b-3}{2}}{l_1, l_2, l_3} \omega_1^{l_1}\omega_2^{l_2} \left((-1)^{\ell-j}\alpha_1 \wedge \alpha_2\right)^{l_3} \\ &= (\alpha_2 + (-1)^{\ell-j}\alpha_1) \binom{\frac{a+b-3}{2}}{\frac{a}{2}-1, \frac{b-1}{2}, 0} \omega_1^{\frac{a}{2}-1}\omega_2^{\frac{b-1}{2}} + \dots \\ &= \binom{\frac{a+b-3}{2}}{\frac{a}{2}-1, \frac{b-1}{2}, 0} \alpha_2\omega_1^{\frac{a}{2}-1}\omega_2^{\frac{b-1}{2}} + \dots\end{aligned}$$

Since $\omega_1^{\frac{a}{2}-1}$, $\alpha_2\omega_2^{\frac{b-1}{2}} \neq 0$, then $\left(\Phi_{ij}^{-1}\right)^* \left[\alpha\omega^{\frac{\ell}{2}-1}\right]$ is nonvanishing.

This implies that all the forms in (6.3) and (6.4) are nonzero in $H^*(X(\sigma^{j-i})) \otimes H^*(X(\sigma^{\ell-j+i+1}))$ and hence nonzero in $H^*(X(\sigma^{\ell}))$. $\square$

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