On the Faces and Ehrhart Polynomials of Polytopes

By

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To my sister, mom, and dad

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Abstract

We investigate the face structure and Ehrhart polynomials of several families of polytopes that have been central to the author's research journey throughout graduate study, including generalized type A and type B permutohedra, hive polytopes associated with Littlewood–Richardson coefficients, and parking function polytopes. In Chapter 1, we present essential background on the foundational concepts that underpin the results of this dissertation. This includes an introduction to polytopes, their face structures and normal fans, as well as an overview of Ehrhart theory, which concerns the enumeration of lattice points in rational polytopes. Later in the chapter, we introduce type A generalized and type B permutohedra, and provide existing formulas for their Ehrhart polynomials. These core concepts and definitions provide the groundwork for the more specialized constructions and results developed in the chapters that follow.

In Chapter 2, we derive a formula for the Ehrhart polynomials of type B generalized permutohedra, providing a concise alternative to a recent formula obtained by Eur, Fink, Larson, and Spink from a study of delta-matroids. Our approach builds on the techniques and tools introduced by Postnikov around two decades ago in his work on type A generalized permutohedra—a rich family of polytopes deeply connected to many mathematical concepts such as matroids, graphs, and Weyl groups. Our approach suggests that many combinatorial models originally developed for type A generalized permutohedra may be used directly or suitably adapted to investigate the combinatoral properties of their type B analogues. We conclude the chapter by proposing several questions for future work on how the combinatorial properties of type B generalized permutohedra could be deduced from those of their type A counterparts.

In Chapter 3, we provide an alternative proof of the conjecture by King, Tollu, and Toumazet that stretched Littlewood-Richardson coefficient $c_{t\lambda,t\mu}^{t\nu}$ is a polynomial function in t. Note that the conjecture was shown to be true by Derksen and Weyman using semi-invariants of quivers. Later, Rassart used Steinberg's formula, the hive conditions, and the Kostant partition function to show a stronger result that $c_{\lambda,\mu}^{\nu}$ is indeed a polynomial in variables ν, λ, μ provided they lie in certain polyhedral cones. Motivated by Rassart's approach, we give a short alternative proof of the polynomiality of $c_{t\lambda,t\mu}^{t\nu}$ using Steinberg's formula and a simple argument about the chamber complex of the Kostant partition function. The main idea of our proof uses the hive conditions to

realize $c_{t\lambda,t\mu}^{t\nu}$ as the number of lattice points in the t-dilation of a rational polytope, which implies, by Ehrhart theory, that $c_{t\lambda,t\mu}^{t\nu}$ is a quasi-polynomial. We then employ Steinberg's formula and the chamber complex of the Kostant partition function to deduce that $c_{t\lambda,t\mu}^{t\nu}$ is a polynomial for sufficiently large t, and hence must be a polynomial for $t \geq 0$. Additionally, we discuss a connection to flow polytopes and outline potential research problems for future work regarding the coefficients of the polynomial $c_{t\lambda,t\mu}^{t\nu}$.

In Chapter 4, we extended the concept of parking function polytopes and investigated their normal fans, face posets, h-polynomials, and connections to other families of polytopes. This extension broadens the family to encompass several additional combinatorial types of polytopes. To describe their face structures and normal fans, we introduced generalized forms of ordered set partitions, which we refer to as binary partitions and skewed binary partitions. Using properties of preorder cones, we developed tools to characterize the family of skewed binary partitions that correspond bijectively to the normal fan–and thus the face poset–of a parking function polytope. This framework leads to several related findings, including the insight that the combinatorial type of a parking function polytope depends solely on its defining multiplicity vector, and a characterization of simple parking function polytopes.

Later in the chapter, we present a formula for the h-polynomials of simple parking function polytopes in terms of generalized Eulerian polynomials, and further refine it to express the h-polynomials as sums of products of classical Eulerian polynomials. At the end of the chapter, we discuss how parking function polytopes can be realized as polymatroids, type B generalized permutohedra, and type A generalized permutohedra. These connections allow us to derive formulas for the volume and Ehrhart polynomials of parking function polytopes.

Acknowledgments

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CHAPTER 1

Introduction

Polytopes play a central role in many branches of mathematics, serving as a unifying framework across various fields, due to their rich geometric, combinatorial, and algebraic structure. In geometry and topolygy, for example, their boundary complexes often appear as simplicial complexes, and their structure connects with important topological invariants [22, 50]. In algebra, many toric varieties, an important class of algebraic varieties, can be constructed from polytopes [12, 20]. In linear and integer programming, polytopes are fundamental to the study [33, 39]. The feasible region of a linear program is a polytope, and understanding its structure is a key to optimization algorithms.

In this dissertation, we focus on exploring the combinatorial aspects of polytopes by studying their face structures and the enumeration of their lattice points. The face structure of a polytope offers a rich source of combinatorial data. In many cases, trying to understand how to describe their faces naturally leads to connections with other combinatorial objects, such as partially order sets (posets), graphs, partitions, and matroids, allowing for a translation between geometric intuition and discrete structures.

On the other hand, counting the number of lattice points contained in rational polytopes gives rise to Ehrhart theory. The number of lattice points in every integral and some rational polytopes can be computed using a polynomial called Ehrhart polynomial. Beyond enumeration, Ehrhart polynomials is also a powerful invariants that encode both geometric and combinatorial information about the underlying polytope, including its volume, boundary structure, and symmetry. Thus, given a family of polytopes, it is common to seek a formula for the Ehrhart polynomials of the family. In some occasion, Ehrhart theory also provides a tool for proving that a combinatorial function is a polynomial: this can be done by interpreting the function as counting the number of lattice points in a rational polytope. Taken together, the study of face structures and Ehrhart polynomials offers deep insights into the interplay between discrete geometry and combinatorics.

Later in the dissertation, we will investigate the face structures and Ehrhart polynomials of several families of polytopes that have been central to the author's research journey throughout graduate study. These include type A and type B generalized permutohedra, hive polytopes associated with Littlewood–Richardson coefficients, and parking function polytopes. To lay the groundwork for these investigations, we begin this chapter by introducing the necessary notation and preliminary concepts.

1.1. Polyhedra

A *polyhedron* P in $\mathbb{R}^n$ is the solution to a finite set of linear inequalities, that is,

$$P = \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^n \;\middle|\; \mathbf{a}_i \cdot \mathbf{x} \leq b_i \text{ for } i \in I \right\} \tag{1.1.1}$$

for some $\mathbf{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, where the dot $\cdot$ is the usual dot product and I is a finite set of indices. The dimension of P, denoted by $\dim(P)$, is defined to be the dimension of $\operatorname{aff}(P)$ the affine span of P. We denote by P° the *relative interior* of the polyhedra P.

A *cone* σ is a polyhedron defined by a system of homogeneous linear inequalities, i.e. inequalities of the form $\mathbf{a} \cdot \mathbf{x} \leq 0$. If the only subspace of $\mathbb{R}^n$ contained in a cone σ is $\{\mathbf{0}\}$ the trivial subspace, then σ is said to be a *pointed* cone. A d-dimensional pointed cone is said to be *simplicial* if it can be spanned by d linearly independent rays, i.e., σ is a d-dimensional pointed cone of the form $\{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d \mid \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i \in [d] \text{ and } \mathbf{v}_1, \ldots, \mathbf{v}_d \text{ are linearly independent}\}$.

A *polytope* P is a bounded polyhedron. By the Minkowski-Weyl Theorem [31, 49], we can equivalently define a polytope in $\mathbb{R}^n$ as the convex hull of finitely many points in $\mathbb{R}^n$, i.e.,

$$P = \operatorname{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k) := \{\lambda_1 \mathbf{x}_1 \cdots + \lambda_k \mathbf{x}_k \mid \lambda_1 + \cdots + \lambda_k = 1, \lambda_i \geq 0 \text{ for all } i \in [k]\}.$$

A 2-dimensional polytope is also called a *polygon*. We denote by $\operatorname{Vol}(P)$ the volume of P with respect to the lattice $\mathbb{Z}^n \cap \operatorname{aff}(P)$ in the affine span of P. The *normalized volume* of a d-dimensional polytope is defined to be $\operatorname{NVol}(P) := d!\operatorname{Vol}(P)$.

For two nonempty polytopes P_1, P_2 in $\mathbb{R}^n$, the *Minkowski sum* of P_1 and P_2, denoted by $P_1 + P_2$, is the set $\{\mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^d \,|\, \mathbf{x}_1 \in P_1, \mathbf{x}_2 \in P_2\}$. If U and V are the set of vertices of P_1 and P_2, respectively, then $P_1 + P_2 = \operatorname{conv}(\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in U, \mathbf{v} \in V)$. This implies that a Minkowski sum of polytopes is a polytope. The *Minkowski difference* of P_2 in P_1, denoted by $P_1 - P_2$, is the set $\{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{x} + P_2 \subseteq P_1\}$. Since the vectors that translate P_2 to lie in P_1 form a polytope, it follows that a Minkowski difference of two polytopes is also a polytope. It is important to note that, in general, the Minkowski difference on polytopes is neither a commutative nor an associative operator. For example, while $(P_1 + P_2) - P_2 = P_1$ always holds, the expression $(P_1 - P_2) + P_2$ may not equal P_1, or may even be undefined if the difference $P_1 - P_2$ is empty. Thus, it is crucial to clearly specify the order of the sum (and difference). Given nonempty polytopes P_i in $\mathbb{R}^n$ and signs $\delta_i \in \{1, -1\}$ for $i \in [m]$, we define the Minkowski sum

$$\sum_{i=1}^{m} \delta_i P_i := Q_1 - Q_2 \text{ where } Q_1 := \sum_{\delta_i = 1} P_i \text{ and } Q_2 := \sum_{\delta_i = -1} P_i.$$

1.2. Faces and Normal Fans

A subset F of a polyhedron $P \subset \mathbb{R}^n$ is said to be a *face* of P if there exists a half-space H defined by $\mathbf{h} \cdot \mathbf{x} \leq a$ where $\mathbf{h} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, such that $P \subseteq H$ and $F = P \cap H$. One sees that this is equivalent to requiring $\mathbf{h}$ to satisfy

$$F = \{\mathbf{x} \in P \mid \mathbf{h} \cdot \mathbf{x} \geq \mathbf{h} \cdot \mathbf{y}, \text{ for all } \mathbf{y} \in P\}.$$

A face of dimension $\dim(P) - 1$ is called a *facet*, a face of dimension 1 is called an *edge*, and a face of dimension 0 is called a *vertex*. The empty set is also considered by convention as a face of P of dimension -1. We note that the (relative) interior of a face of P cannot intersect the (relative) interior of any other faces of P. In fact, two faces of a polytope can only intersect at their common face. The partially ordered set $\mathcal{F}(P)$ of all nonempty faces of P ordered by inclusion is called the *face poset* of P. When two polytopes have isomorphic face posets, we say that they are *combinatorially equivalent*.

A *facet-defining inequality* of a polyhedron P is an inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ such that the set $\{\mathbf{x} \in P \mid \mathbf{a} \cdot \mathbf{x} = b\}$ is a facet of P. A *minimal inequality description* of P is a system of minimal number of inequalities that defines P.

Let F be a nonempty face of a polytope $P \subset \mathbb{R}^n$. The *normal cone* of P at F is the set

$$\operatorname{ncone}(F, P) := \{\mathbf{c} \in \mathbb{R}^n \mid \mathbf{c} \cdot \mathbf{x} \geq \mathbf{c} \cdot \mathbf{y} \text{ for all } \mathbf{x} \in F \text{ and all } \mathbf{y} \in P\},$$

that is, $\operatorname{ncone}(F, P)$ is the set of all $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c} \cdot \mathbf{x}$ attains maximum value at F over all points in P.

The *normal fan* of P, denoted $\Sigma(P)$, is the set of normal cones of P at all of its nonempty faces. We say that a polytope Q is a *deformation* of another polytope P if $\Sigma(Q)$ is a coarsening of $\Sigma(P)$, i.e., every cone in $\Sigma(Q)$ is a union of cones in $\Sigma(P)$. A deformation Q of P can be obtained by parallel translations of the facets of P. We denote by $\mathcal{F}(\Sigma(P))$ the poset on $\Sigma(P)$ ordered by inclusion. The following well-known lemma gives a correspondence between the nonempty faces of polytope and the normal cones at the faces.

LEMMA 1.2.1. *Let P be a polytope. Then, the map $F \mapsto \operatorname{ncone}(F, P)$ for nonempty faces F is a poset isomorphism from the dual poset of $\mathcal{F}(P)$ to the poset $\mathcal{F}(\Sigma(P))$.*

The next result is a slight variation of [9, Lemma 2.4]. It provides a way to verify normal cones at vertices of polytope. We omit the proof as it is very similar to the proof of the original result.

LEMMA 1.2.2. *Suppose that $\mathcal{M} = \{\sigma_1, \dots, \sigma_k\}$ is a set of cones satisfying $\sigma_1 \cup \cdots \cup \sigma_k = \mathbb{R}^n$ and that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ is a set of points in which for every $i \in \{1, \dots, k\}$*

$$\mathbf{c} \cdot \mathbf{v}_i > \mathbf{c} \cdot \mathbf{v}_j \text{ for all } \mathbf{c} \in \sigma_i^\circ \text{ and all } j \neq i.$$

Let P be a polytope defined by $P := \operatorname{conv}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Then the set of vertices of P is $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. In addition, we have that $\sigma_i = \operatorname{ncone}(\mathbf{v}_i, P)$ for all $i \in \{1, \dots, k\}$. As a consequence, the set of cones in $\mathcal{M}$ and their faces form the normal fan $\Sigma(P)$ of P.

A d-dimensional polytope is a *simplex* if it is a convex hull of $d+1$ affinely independent points. When every facet of a polytope P is a simplex, we say that P is *simplicial.* A d-dimensional polytope is said to be *simple* if all of its vertices are incident to exactly d edges. If P is full-dimensional, that is, P is a d-dimensional polytope in $\mathbb{R}^d$, then one can show that being simple is equivalent to having the *normal cone* at every vertex being simplicial.

Given a d-dimensional polytope P, we let $f_i(P)$ be the number of its i-dimensional faces. The *f-vector* of P is defined to be the vector $(f_0(P), \dots, f_d(P))$, and the *f-polynomial* of P is given by $f_P(t) := f_0(P) + f_1(P)t + \cdots + f_d(P)t^d$. If a d-dimensional polytope P is also simple, we define its *h-polynomial* $h_P(t) := h_0(P) + h_1(P)t + \cdots + h_d(P)t^d$ and its *h-vector* $(h_0(P), \ldots, h_d(P))$ to satisfy the relation $f_P(t) = h_P(t+1)$. This is equivalent to having

$$f_j(P) = \sum_{i=j}^{d} \binom{i}{j} h_i(P) \quad \text{for all } j = 0, \ldots, d. \tag{1.2.1}$$

It is well-known that the h-polynomial $h_P(t)$ of a simple polytope P has nonnegative coefficients [50, Section 8.2] and is palindromic [14, 40] as it satisfies the Dehn-Sommerville symmetry. That is, $0 \leq h_i(P) = h_{d-i}(P)$ for all $i = 0, \ldots, d$. Thus, we only need to know half of the coefficients of $h_P(t)$ to recover the f-vector using equation (1.2.1).

1.3. Ehrhart Theory

A *lattice point* or an *integer point* in $\mathbb{R}^n$ is a point whose coordinates are integers. A polytope is said to be *rational* if all of its vertices have rational coordinates, and is said to be *integral* if all of its vertices have integral coordinates. For a polytope P in $\mathbb{R}^n$ and a non-negative integer t, the t^{th}-dilation tP is the set $\{tx \mid x \in P\}$. We define

$$i(P, t) := |\mathbb{Z}^n \cap tP|$$

to be the number of lattice points in the t^{th}-dilation tP.

Recall that a quasi-polynomial is a function $f : \mathbb{Z} \longrightarrow \mathbb{R}$ of the form $f(t) = a_d(t)t^d + \cdots a_1(t)t + a_0(t)$ where each of $a_d(t), \ldots, a_0(t)$ is a periodic function in $t \in \mathbb{Z}$. The *period* of $f(t)$ is the least common period of $a_d(t), \ldots, a_0(t)$. Clearly, a quasi-polynomial of period one is a polynomial.

We define the *denominator* of a rational polytope P to be the least positive integer m such that the m-dilation mP is an integral polytope. It is easy to see that the denominator of P equals the least common multiple of the denominators of the coordinates of its vertices (when the rational coordinates are written in the lowest terms). The behavior of the function $i(P, t)$ is described by the following theorem due to Ehrhart [17].

THEOREM 1.3.1 (Ehrhart Theory). *If P is a rational polytope of dimension d, then $i(P, t)$ is a quasi-polynomial in t of degree d with rational coefficients. Moreover, the period of $i(P, t)$ is a divisor of the denominator of P. In particular, if P is an integral polytope, then $i(P, t)$ is a polynomial in t.*

The polynomial (resp. quasi-polynomial) $i(P,t)$ is called the *Ehrhart polynomial* of P (resp. Ehrhart quasi-polynomial of P). When P is integral, the Ehrhart polynomial $i(P,t)$ of P encodes its geometric and combinatorial properties. For instance, the leading coefficient of $i(P,t)$ equals the volume of P, the second coefficient is a half of its normalized surface area, and the constant term is one. This means that these three coefficients of any Ehrhart polynomial are positive numbers. However, other coefficients can be negative and their general simple interpretations are less understood. When all of the coefficients are positive, we will say that the corresponding polytope P is *Ehrhart positive*.

Two integral polytopes P, Q such that $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are said to be *integrally equivalent* if there exists an invertible affine transformation from $\mathrm{aff}(P)$ to $\mathrm{aff}(Q)$ that preserves the lattice points in the two polytopes. When two integral polytopes are integrally equivalent, they have the same face poset, volume, and Ehrhart polynomials.

CHAPTER 2

Ehrhart Polynomials of Generalized Permutohehra From A to B

In this chapter, we derive a formula for the Ehrhart polynomials for type B generalized permutohedra, providing a concise alternative to the formula obtained recently by Eur, Fink, Larson, and Spink in [18, Theorem A] as a result from their study of delta-matroids. The approach presented here builds upon the existing notions and techniques introduced by Postnikov in his work on type A generalized permutohedra [35], a family of polytopes interconnected with many mathematical concepts such as matroids, graphs, and Weyl groups. Postnikov employed the Cayley trick to subdivide polytopes in such a way that each cell in the subdivision corresponds bijectively to the lattice points in the polytope. This method allowed Postnikov to enumerate the lattice points in type A generalized permutohedra in terms of what are called *G-draconian sequences*. By viewing a type B generalized permutohdron in each octant as a type A generalized permutohedron, we are able to apply Postnikov's approach to express its Ehrhart polynomial in terms of G-draconian sequences.

Chapter Organization. We begin by introducing type A and type B generalize permutohedra, along with a review of existing techniques for enumerating lattice points in these polytopes. We then develop a method for realizing part of type B generalized permutohedron as a type A generalize permutohedron. Using this realization, we show how the number of lattice points in type B generalized permutohedra can be computed in terms of G-draconian sequences. Lastly, we propose questions for future work on how the combinatorial properties of type B generalized permutohedra might be deduced from those of their type A counterparts.

2.1. Type A generalized Permutohedra

In this section, we introduce the family of polytopes known as *type A generalized permutohedra*, following the notation and framework established by Postnikov in [35] around two decades ago. For a more comprehensive treatment beyond what is presented here, we refer the reader to [35].

Let $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ satisfy $w_1 > \cdots > w_n \geq 0$. The *permutohedron* $\Pi(\mathbf{x})$ is the polytope defined as the convex hull of all permutations of the coordinates of $\mathbf{w}$. See Figure 2.1 for examples of permutohedra in $\mathbb{R}^2, \mathbb{R}^3$, and $\mathbb{R}^4$.

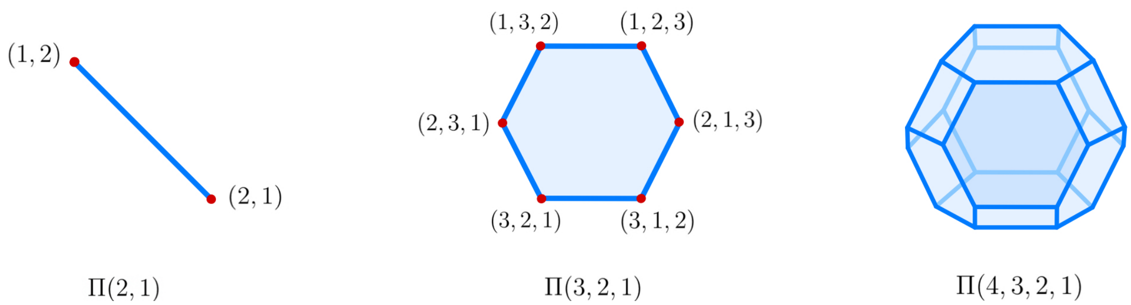


Figure 2.1. Examples of permutohedra in $\mathbb{R}^2, \mathbb{R}^3$, and $\mathbb{R}^4$, respectively

The normal fan of $\Pi(w_1, \ldots, w_n)$, known as the *braid fan* and denoted by $\Sigma_{A_{n-1}}$, is the fan whose maximum cones are the chambers in the arrangement of the hyperplanes

$$H_{i,j} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_i - c_j = 0\} \text{ for all } 1 \leq i < j \leq n,$$

which is known as the *braid arrangement*.

Definition 2.1.1. A *type A generalized permutohedron* P is a polytope that is a deformation of the permutohedron $\Pi(\mathbf{x})$.

Equivalently, one can defined a type A generalized permutohedron to be a polytope whose edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $1 \leq i < j \leq n$, where $\mathbf{e}_i$ is the standard basis vector in $\mathbb{R}^n$. Every type A generalized permutohedron $P \subset \mathbb{R}^n$ lies on a hyperplane $x_1 + \cdots x_n = a$ for some real number a. For instance, $\Pi(w_1, \ldots, w_n)$ lies on the hyperplane $x_1 + \cdots + x_n = w_1 + \cdots + w_n$. Thus, the dimension of P is at most $n - 1$.

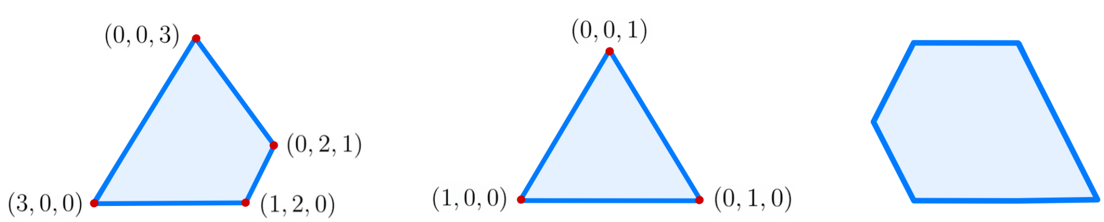


Figure 2.2. Examples of type A generalized permutohedra in $\mathbb{R}^3$

We note that these polytopes are also known simply as *generalized permutohedra*. We include "type A" in the name to emphasize the fact that the edge directions of these polytopes are parallel to some vectors in the type A positive root system $\{e_i - e_j \mid 1 \le i < j \le n\}$. Type A generalized permutohedra form a widely studied family of polytopes that interconnect with various combinatorial objects, such as matroids, graphs, and Weyl groups. Many well-known polytopes in the literature can be realized as type A generalized permutohedra, including zonotopes, associahedra, cyclohedra, and the Pitman-Stanley polytopes (see [35, Setion 8]).

For a nonempty subset $I \subseteq [n]$, we define Δ_I to be the simplex $\mathrm{conv}(e_i \mid i \in I)$. Postnikov showed in [35, Section 6] that every type A generalized permutohedron can be written as a Minkowski sum (and difference) of simplices. The following lemma states this more precisely.

Lemma 2.1.2. *Every type A generalized permutohedron has the form*

$$\sum_{I\subset[n], I\neq\emptyset} y_I \Delta_I \text{ for some } y_I \in \mathbb{R}. \tag{2.1.1}$$

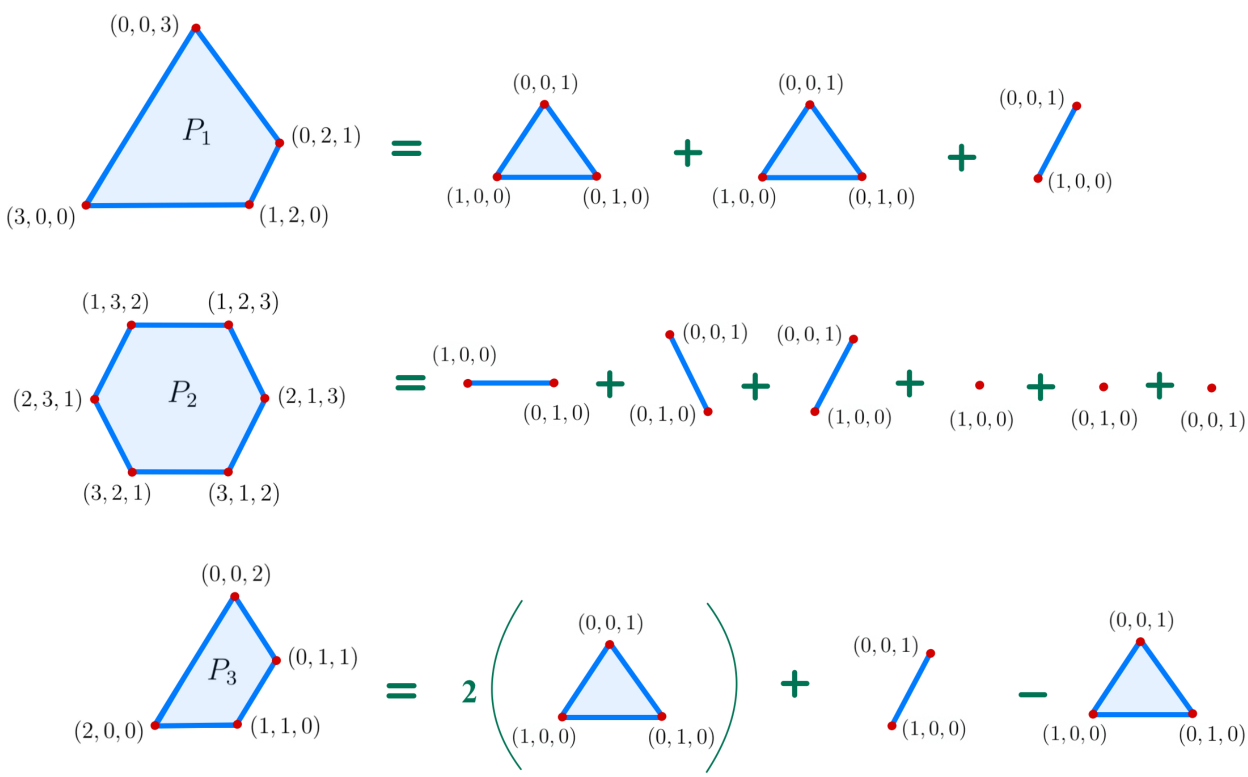


Figure 2.3. Type-A generalized permutohedra as Minkowski sums of simplices

Thus, we may write every permutohedron P as $P = P(\{y_I\})$. Note that if $P(\{y_I\})$ is integral (resp. rational), then y_I can be chosen to be integers (resp. rational numbers) for all nonempty subsets $I \subseteq [n]$.

EXAMPLE 2.1.3. In Firgure 2.3, Three examples of type-A generalized permutohedra in $\mathbb{R}^3$ are written as Minkowski sums of simplies. One sees that

$$P_1 = \Delta_{[3]} + \Delta_{[3]} + \Delta_{\{1,3\}}$$

$$P_2 = \Delta_{\{1,2\}} + \Delta_{\{2,3\}} + \Delta_{\{1,3\}} + \Delta_{\{1\}} + \Delta_{\{2\}} + \Delta_{\{3\}}$$

$$P_3 = 2\Delta_{[3]} + \Delta_{\{1,3\}} - \Delta_{[3]} = \Delta_{[3]} + \Delta_{\{1,3\}}.$$

LEMMA 2.1.4 (Proposition 6.3 of [35]). *Let $P(\{y_I\})$ be a type A generalized permutohedron where y_I are real numbers. Then, the point $(x_1, \dots, x_n)$ lies in $P(\{y_I\})$ if and only if*

$$x_1 + \cdots + x_n = \sum_{I \subseteq [n]} y_I \text{ and } \sum_{j \in J} x_j \geq \sum_{I \subseteq J} y_I \text{ for all nonempty subset } J \subseteq [n].$$

Given a bipartite graph G on m left vertices $\{\ell_1, \dots, \ell_m\}$ and n right vertices $\{r_1, \dots, r_n\}$, we denote by $N(v)$ the set of the vertices of G adjacent to the vertex v. We call $N(v)$ the set of neighbors of v. For $i \in [m]$, we let

$$I_i := \{j \in [n] \mid r_j \in N(\ell_i)\} \tag{2.1.2}$$

be the set of indices of the neighors of the left vertex ℓ_i. Then, one may define $P_G(y_1, \dots, y_m)$ to be the type A generalized permutohedron $y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m}$.

Conversely, given $P = y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m} \subseteq \mathbb{R}^n$, there is a bipartite graph G on m left and n right vertices such that $P_G(y_1, \dots, y_m) = P$. Thus, we may interchangeably write $P(\{y_I\})$ as $P_G(y_1, \dots, y_m)$ where G is a corresponding bipartite graph.

REMARK 2.1.5. If $P_G(y_1, \dots, y_m) \subseteq \mathbb{R}^n$ is $(n-1)$-dimensional, then G is a connected graph.

We now aim to describe how Postnikov derives a formula for the Ehrhart polynomials of type A generalized permutohedra. Thus, we only consider $P(\{y_I\})$ where y_I are integers throughout the rest of this section.

Definition 2.1.6. Given a bipartite graph G on m left vertices and n right vertices, we let I_i be defined as in equation (2.1.2). A sequence $(a_1, \ldots, a_m)$ of nonnegative integers is a *G-draconian* sequence if it satisfies the following inequalities: for every nonempty subset $J \subseteq [m]$

$$\sum_{i \in J} a_i \leq \Big| \bigcup_{i \in J} I_i \Big| - 1, \text{ and } a_1 + \cdots a_m = n - 1.$$

The set of all G-draconian sequences is denoted by $\mathrm{D}(G)$.

In [**35**, Theorem 11.3 and Remark 6.4], Postnikov gives the following formula for the number of lattice points in integral type A generalized permutohedra.

Lemma 2.1.7. *Let $P := P_G(y_1, \ldots, y_m) = y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m}$ where y_i are integers for all $i \in [m]$. Then, the number of lattice points in $P_G - \Delta_{[n]}$ is given by*

$$|(P_G - \Delta_{[n]}) \cap \mathbb{Z}^n| = \sum_{\mathbf{a} \in D(G)} \binom{y_1 + a_1 - 1}{a_1} \cdots \binom{y_m + a_m - 1}{a_m}. \tag{2.1.3}$$

Note that the Ehrhart polynomial of $P_G(y_1, \ldots, y_m)$ is given by replacing P_G in formula (2.1.3) by $P_G(y_1 t, \ldots, y_m t) + \Delta_{[n]} := y_1 t \Delta_{I_1} + \cdots + y_m t \Delta_{I_m} + \Delta_{[n]}$.

The rest of this section is devoted to outlining how Postnikov comes up with formula (2.1.3). Since every type A generalized permutohedron is a deformation of $\Pi(w_n, \ldots, w_1)$ for some $w_1 > \cdots > w_n$, it suffices by [**35**, Remark 6.4] to only show that the formula holds for $P_G(y_1, \ldots, y_m) = y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m}$ where y_i are positive integers for all $i \in [m]$. Because every simplex $y\Delta_I$ where y is a positive integer can be written as the Minkowski sum of y copies of the simplex Δ_I, Postnikov first shows that the formula holds for $P_G(1, \ldots, 1) = \Delta_{I_1} + \cdots + \Delta_{I_m}$.

Example 2.1.8. Let G_1, G_2, G_3, G_4 be bipartite graphs shown in Figure 2.4. Then, we can write the type-A generalized permutoheda P_1, P_2, and P_3 in Figure 2.3 as

$$P_1 = P_{G_1}(1,1,1)$$

$$P_2 = P_{G_2}(1,1,1,1,1,1)$$

$$P_3 = P_{G_3}(2,1,-1) = P_1 - \Delta_{[3]} = P_{G_4}(1,1,1,-1).$$

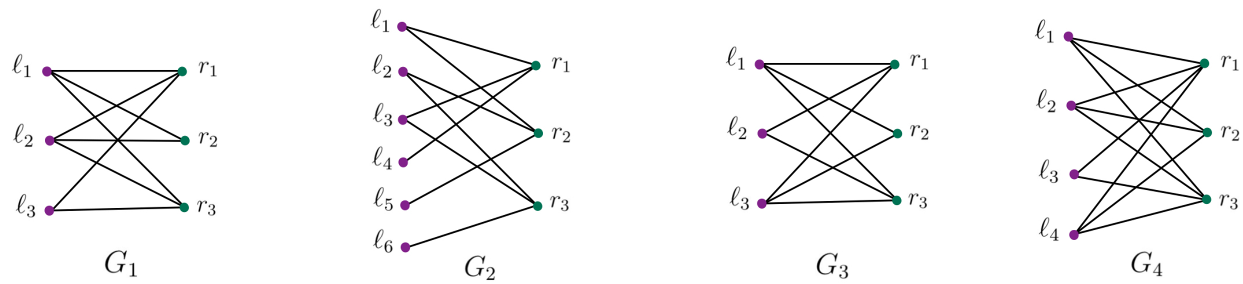


FIGURE 2.4. Bipartite graphs associated to polytopes in Figure 2.3

Postnikov uses the Cayley trick to obtain a *polyhedral subdivision* of $P_G(1, \ldots, 1)$ into *fine mixed cells* and use it as a key to counting lattice points in $P_G(1, \ldots, 1)$. We refer the reader to [**35**, Section] for more details regarding the fine mixed subdivision of $P_G(1, \ldots, 1)$.

Recall that a *polyhedral subdivision* of a polytope P is a subdivision of P into a union of cells of the same dimension as $\dim(P)$.

DEFINITION 2.1.9. Let $P_G(1, \ldots, 1) = \Delta_{I_1} + \cdots + \Delta_{I_m}$. Then, a *fine mix cell* of P_G is a polytope Π of the form $\Delta_{J_1} + \cdots + \Delta_{J_m}$ where $J_i \subseteq I_i$ for all $i \in [m]$ and satisfies $\dim(\Delta_{J_1}) + \cdots + \dim(\Delta_{J_m}) = \dim(P_G(1, \ldots, 1))$. A *fine mixed subdivision* of $P_G(1, \ldots, 1)$ is a polyhedral subdivision of $P_G(1, \ldots, 1)$ into fine mixed cells.

EXAMPLE 2.1.10. Let $P := P_G(1, 1, 1)$ be the polytope shown in Figure 2.5 (also shown as P_1 in Figure 2.3). A fine mixed subdivision of P is drawn inside of P. One sees that there are five fine mixed cells, labeled as $\Pi_1, \ldots, \Pi_5$ in the subdivision.

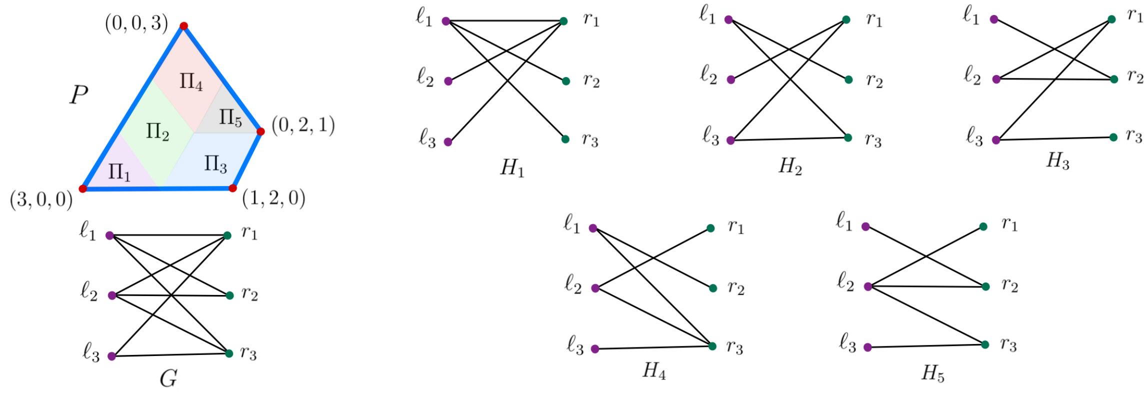


FIGURE 2.5. Fine mixed cells in a fine mixed subdivision of $P_G(1, 1, 1)$ and their corresponding spanning trees

A lattice point in a fine mixed cell of $P_G(1,\dots,1)$ has a simple form in the following sense.

Lemma 2.1.11. *Let H be a bipartite graph on m left vertices and n right vertices. If H is a forest, then every lattice point in the polytope $\Pi := P_H(1,\dots,1)$ is a vertex of Π. Moreover, every lattice point in Π has the form $\mathbf{e}_{j_1}+\cdots+\mathbf{e}_{j_m}$ for some j_i such that $r_{j_i}\in N(\ell_i)$ for all $i\in[m]$.*

Remark 2.1.12. In other words, Lemma 2.1.11 states that every lattice point in Π corresponds to a transversal $\{r_{j_1},\dots,r_{j_m}\}$ of the sequence $(N(\ell_1),\dots,N(\ell_m))$ the neighbors of the left vertices, i.e., $r_{j_i}\in N(\ell_i)$ for all $i\in[m]$.

Definition 2.1.13. Given be a bipartite graph G on m left vertices $\{\ell_1,\dots,\ell_2\}$ and n right vertices $\{r_1,\dots,r_n\}$, we define the *left degree* $LD(G)$ and the *right degree* $RD(G)$ of G, respectively, as

$$LD(G) := (\deg(\ell_1)-1,\dots,\deg(\ell_m)-1) \text{ and } RD(G) := (\deg(r_1)-1,\dots,\deg(r_n)-1).$$

Fine mixed cells in a fine mixed subdivision $P_G(1,\dots,1)$ and G-draconian sequences are shown in [**35**, Lemma 12.6, Lemma 12.8, and Theorem 12.9] to correspond to spanning forests of G.

Lemma 2.1.14. *Suppose that $\mathcal{C}=\{\Pi_1,\dots,\Pi_p\}$ is the set of fine mixed cells in a subdivision of $P_G(1,\dots,1)$. Then, there exists a sequence of bipartite subgraphs $H_1,\dots,H_p$ of G such that H_i are spanning forests of G satisfying all of the following properties.*

(1) If $P_G(1,\dots,1)$ is $(n-1)-$dimensional, then $H_1,\dots,H_p$ are spanning trees of G.
(2) For all $i\in[p]$, we have $P_{H_i}(1,\dots,1)=\Pi_i$
(3) For $i,j\in[p]$ such that $i\neq j$, we have $LD(H_i)\neq LD(H_j)$ and $RD(H_i)\neq RD(H_j)$
(4) $(P_G(1,\dots,1)-\Delta_{[n]})\cap\mathbb{Z}^n=\{RD(H_1),\dots,RD(H_p)\}$ and $D(G)=\{LD(H_1),\dots,LD(H_p)\}$.

Let $P := P_G(1,1,1)$ be the polytope shown in Figure 2.5. The fine mixed cells in the given subdivision of P, labeled as $\Pi_1,\dots,\Pi_5$, and their corresponding spanning trees $H_1,\dots,H_5$ of G are also shown in the figure.

Example 2.1.15. Let $P := P_G(1,1,1)$ be the polytope in Figure 2.5. Since $P-\Delta_{[n]}=P_3$ where P_3 is the polytope shown in Figure 2.3, one has

$$(P_G(1,\dots,1)-\Delta_{[n]})\cap\mathbb{Z}^n=\{(2,0,0),(1,0,1),(1,1,0),(0,0,2),(0,1,1)\}.$$

One can also show that

$$D(G) = \{(2,0,0),(1,0,1),(0,1,1),(1,1,0),(0,2,0)\}.$$

Moreover, one sees that

$$(P_G(1,\dots,1) - \Delta_{[n]}) \cap \mathbb{Z}^n = \{RD(H_1), RD(H_2), RD(H_3), RD(H_4), RD(H_5)\}$$
$$D(G) = \{LD(H_1), LD(H_2), LD(H_3), LD(H_4), LD(H_5)\}.$$

We obtain the following result as an immediate consequence of Lemma 2.1.14.

COROLLARY 2.1.16. *Let $P_G := P_G(1,\dots,1)$. Suppose that $\mathcal{C}$ is the set of fine mixed cells in a mixed subdivision of P_G. Then,*

$$|(P_G - \Delta_{[n]}) \cap \mathbb{Z}^n| = |\mathcal{C}| = |D(G)|.$$

Corollary 2.1.16 implies that the number of lattice points in $P_G - \Delta_{[n]}$ is given by

$$|(P_G - \Delta_{[n]}) \cap \mathbb{Z}^n| = \sum_{\mathbf{a}\in D(G)} 1 = \sum_{\mathbf{a}\in D(G)} \binom{1+a_1-1}{a_1} \cdots \binom{1+a_m-1}{a_m} \tag{2.1.4}$$

Postnikov obtains formula (2.1.3) by expressing $P_G(y_1,\dots,y_m)$ as

$$P_{G'}(1,\dots,1) = \underbrace{\Delta^0_{I_1} + \cdots + \Delta^0_{I_1}}_{y_1 \text{ terms}} + \cdots + \underbrace{\Delta^0_{I_m} + \cdots + \Delta^0_{I_m}}_{y_m \text{ terms}}$$

for an appropriate G', and then applying a simple binomial identity to the right-hand side of equation (2.1.4).

2.2. Type B generalized Permutohedra

Recall that a type A generalized permutohedron can be defined as a polytope whose edges are parallel to some vectors in a type A positive root system. It is then natural to generalize this concept to root systems of other types.

DEFINITION 2.2.1. A *type B generalized permutohedron* in $\mathbb{R}^n$ is a polytope whose edges are parallel to $\mathbf{e}_i + \mathbf{e}_j, \mathbf{e}_i - \mathbf{e}_j$, or $\mathbf{e}_i$ for some $i, j \in [n]$ where $\mathbf{e}_i$ denotes the standard basis vector in $\mathbb{R}^n$.

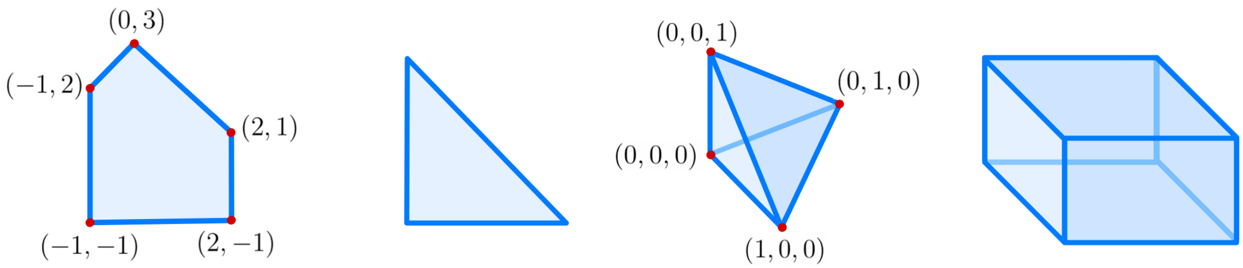


FIGURE 2.6. Examples of type B generalized permutohedra in $\mathbb{R}^2$ and $\mathbb{R}^3$

The name "type B" comes from the fact that the edge directions of these polytopes are parallel to some vectors in the type B positive root system $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i \leq j \leq n\}$. We note that these polytopes are also known as *generalized signed permtohedra* (see [18]). By definition, every type A generalized permutohedron is a type B generalized permutohedron.

One may equivalently define a type B generalized permutohedron in $\mathbb{R}^n$ as a polytope in which its normal fan coarsens the fan whose full dimensional cones are defined by the chambers of the arrangement of hyperplanes

$$\begin{cases} H_{i,j}^{+} & := \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_i + c_j = 0\} \text{ for all } i \neq j \in [n], \\ H_{i,j}^{-} & := \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_i - c_j = 0\} \text{ for all } i \neq j \in [n], \text{ and} \\ H_i & := \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_i = 0\} \text{ for all } i \in [n]. \end{cases} \tag{2.2.1}$$

The arrangement of hyperplanes in equation (2.2.1) is called the *type B Coxeter arrangement*. The fan whose full dimensional cones are defined by the chambers of the type B Coxeter arrangement is known as the B_n *permutohedral fan* and is denoted by Σ_{B_n}. Note that each chamber of the arrangement in (2.2.1) is a cone of the form

$$\{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid 0 \leq (-1)^{k_1} c_{i_1} \leq \cdots \leq (-1)^{k_n} c_{i_n}\} \tag{2.2.2}$$

where $\{i_1, \ldots, i_n\} = [n]$, and $k_j \in \{0, 1\}$ for all $j \in [n]$. Thus, there are $n!2^n$ of such chambers. We refer the reader to [3] and [18] for more details regarding type B generalized permutohedra beyond what are presented here.

Let $[\overline{n}] := \{\overline{1}, \ldots, \overline{n}\}$ and $[n, \overline{n}] := [n] \sqcup [\overline{n}] = \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$. For $i \in [n]$, we define $\mathbf{e}_{\overline{i}} := -\mathbf{e}_i$ where $\mathbf{e}_i$ is a standard basis vector of $\mathbb{R}^n$.

Definition 2.2.2. The set $\mathbf{AdS}$ of *admissible subsets* of $[n, \overline{n}]$ is defined to be

$$\mathbf{AdS} := \{S \subset [n, \overline{n}] \,|\, S \neq \emptyset \text{ and } \{i, \bar{i}\} \not\subset S \text{ for all } i\} \text{ and } \mathbf{AdS}_n := \{S \in \mathbf{AdS} \,|\, |S| = n\}.$$

Readers may view each $T \in \mathbf{AdS}_n$ as an octant of $\mathbb{R}^n$. For instance, $T = \{1, \ldots, n\}$ represents $\mathbb{R}^n_{\geq 0}$ the first octant, while $\{\overline{1}, \ldots, \overline{n}\}$ represents $\mathbb{R}^n_{\leq 0}$ the opposite octant. More precisely, we have the following definition.

Definition 2.2.3. Let $T \in \mathbf{AdS}_n$. We define $\mathbb{R}_T := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot e_i \geq 0 \text{ for all } i \in T\}$ to be the associated octant to T of $\mathbb{R}^n$.

Definition 2.2.4. Let $S \in \mathbf{AdS}$ and $T \in \mathbf{AdS}_n$. We denote by

(1) Δ^0_S the simplex $\operatorname{conv}(\mathbf{0}, \mathbf{e}_i \mid i \in S)$.
(2) ∇_S the simplex $\operatorname{conv}\left(\sum_{i \in S} \mathbf{e}_i, \sum_{i \in J} \mathbf{e}_i \mid J \subset S \text{ and } |J| = |S| - 1\right)$.
(3) $\square_T$ the unit cube $\sum_{i \in T} \Delta^0_{\{i\}}$ in the octant $\mathbb{R}_T$.

Bastidas showed from a study of Tits algebras in [3] that every type B generalized permutohedron can be written as Minkowski sum (and difference) of the simplices Δ^0_S. This result was also established later in [18] by Eur, Fink, Larson, and Spink from a study of delta-matroids.

Lemma 2.2.5. *Every type B generalized permutohedron in* $\mathbb{R}^n$ *can be expressed as*

$$\sum_{S \in \mathbf{AdS}} y_S \Delta^0_S \quad \textit{for some } y_S \in \mathbb{R}.$$

One sees that Lemma 2.2.5 generalizes to type B the type A result by Postnikov in Lemma 2.1.2. Thus, one may write every type B generalized permutohedron as $P(\{y_S\})$. If $P(\{y_S\})$ is integral (resp. rational), then y_S may be chosen to be integers (resp. rational numbers) for all $S \in \mathbf{AdS}$.

Example 2.2.6. In Figure 2.7, type-B generalized permutohedra P_1 and P_2 are written as

$$P_1 = \Delta^0_{[2]} + \Delta^0_{[2]} + \Delta^0_{\{\overline{1},2\}} + \Delta^0_{\{\overline{2}\}}$$
$$P_2 = P_1 - \square_{[2]} = 2\Delta^0_{[2]} + \Delta^0_{\{\overline{1},2\}} + \Delta^0_{\{\overline{2}\}} - \Delta^0_{\{1\}} - \Delta^0_{\{2\}}.$$

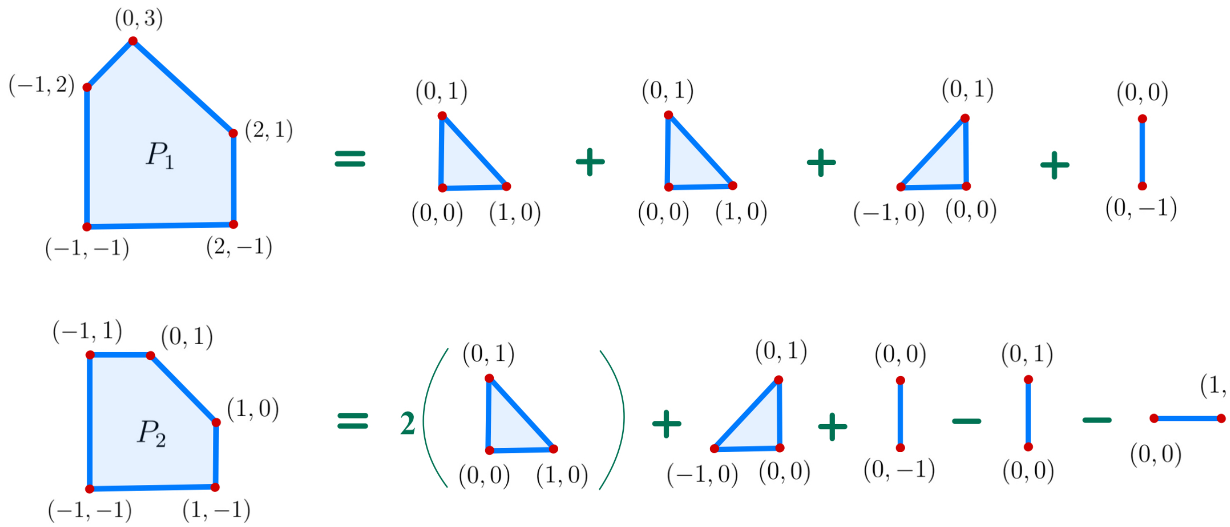


FIGURE 2.7. Type-B generalized permutohedra as Minkowski sums of simplices

DEFINITION 2.2.7. Let $S_1, \ldots, S_n$ be admissible subsets of $[n, \overline{n}]$. The *signed transversal* of $(S_1, \ldots, S_n)$ is an admissible subset $T \in \mathbf{AdS}_n$ such that there exists a bijection $g : [n] \longrightarrow T$ satisfying $g(i) \in S_i$ for all $i \in [n]$.

In [18], Eur, Fink, Larson, and Spink give formulas for the volume and the number of lattice points of type B generalized permutohedra in terms of sign transversals as follows.

LEMMA 2.2.8. *Suppose that* $P(\{y_S\}) = \sum_{S \in \mathbf{AdS}} y_S \Delta^0_S$ *where* y_S *are integers for all* $S \in \mathbf{AdS}$*.*

(1) The normalized volume of P *is given by*

$$\text{NVol}(P(\{y_S\})) = \sum_{(S_1,\ldots,S_n)} |\textit{sign transversals of } (S_1, \ldots, S_n)| \cdot y_{S_1} \cdots y_{S_n}. \tag{2.2.3}$$

(2) The number of lattice point in $P(\{y_S\}) - \square_{[n]}$ *is given by*

$$|(P(\{y_S\}) - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{(S_1,\ldots,S_n)} |\textit{sign transversals of } (S_1, \ldots, S_n)| \cdot \Psi\left(y_{S_1} \cdots y_{S_n}\right) \tag{2.2.4}$$

where Ψ *is the linear operator on the set of polynomials that maps each monomial* $x_1^{a_1} \cdots x_m^{a_m}$ *to* $\frac{a_1! \cdots a_m!}{(a_1 + \cdots + a_m)!}\binom{x_1}{a_1} \cdots \binom{x_m}{a_m}$.

Note that one can compute the Ehrhart polynomial of the integral polytope $P(\{y_S\})$ by plugging the polytope $P(\{ty_S\}) + \Delta^0_{\{1\}} + \cdots + \Delta^0_{\{n\}}$ into formula (2.2.4) in Lemma 2.2.8.

The rest of this section is devoted to establishing basic properties of type B generalized permutohedra necessarily for proving our results in the following two sections.

Definition 2.2.9. Let $P = y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m}$ where y_i are positive real numbers and $S_i \in \mathbf{AdS}$ for all $i \in [m]$. For $T \in \mathbf{AdS}_n$, we define

$$P_T := \sum_{i=1}^{m} y_i \Delta^0_{S_i \cap T}.$$

We highlight some basic properties of points in the simplex Δ^0_S in the next two remarks.

Remark 2.2.10. Since $V = \{\mathbf{e}_j \mid j \in S\} \cup \{\mathbf{0}\}$ is the set of all vertices of Δ^0_S, we can write every point in Δ^0_S as a convex combination of the points in V. Thus, every point $\mathbf{x}$ in Δ^0_S has the form

$$\mathbf{x} = \sum_{j\in S} \lambda_j \mathbf{e}_j \text{ where } \sum_{j \in S} \lambda_j \le 1 \text{ and } 0 \le \lambda_j \text{ for all } j \in S. \tag{2.2.5}$$

Remark 2.2.11. For $S, T \in \mathbf{AdS}$, let $\mathbf{x} \in \Delta^0_S$. It is easy to see from equation (2.2.5) that we can write $\mathbf{x} = \mathbf{a} + \mathbf{b}$ for some $\mathbf{a} \in \Delta^0_{S\cap T}$ and $\mathbf{b} \in \Delta^0_{S \setminus T}$. Thus, if $\mathbf{y} \in P = y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m}$, then we have $\mathbf{y} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in y_1\Delta^0_{S_1\cap T} + \cdots + y_m\Delta^0_{S_m \cap T}$ and $\mathbf{v} \in y_1\Delta^0_{S_1 \setminus T} + \cdots + y_m\Delta^0_{S_m\setminus T}$.

Lemma 2.2.12. *Let $P = y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m}$ where y_i are positive real numbers and $S_i \in \mathbf{AdS}$ for all $i \in [m]$. If the point $(x_1, \ldots, x_n)$ lies in P, then, for every $(r_1, \ldots, r_n) \in \mathbb{R}^n$ such that $0 \le r_k \le 1$ for all $k \in [n]$, the point $(r_1x_1, \ldots, r_nx_n)$ also lies in P.*

Proof. We first show that this property holds for the polytope $P_i = y_i\Delta^0_{S_i}$ for all $i \in [m]$. By Remark 2.2.10, every point $\mathbf{x} = (x_1, \ldots, x_n)$ in P_i has the form

$$\mathbf{x} = \sum_{j\in S_i} \lambda_j y_i \mathbf{e}_j \text{ where } \sum_{j \in S_i} \lambda_j \le 1 \text{ and } 0 \le \lambda_j \text{ for all } j \in S_i.$$

We note that for $k \in [n]$, the coordinate x_k of $\mathbf{x}$ is zero if neither k nor $\overline{k}$ lies in S_i. Let $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ satisfies $0 \le r_k \le 1$ for all $k \in [k]$ and let $\lambda'_j = r_k\lambda_j$ if $j \in \{k, \overline{k}\}$, for all $j \in S_i$. Then, $\sum_{j\in S_i} \lambda'_j \le 1$ and $0 \le \lambda'_j$ for all $j \in S_i$. Thus, the point

$$\sum_{j\in S_i} \lambda'_j y_j \mathbf{e}_j = (r_1x_1, \ldots, r_nx_n) = \mathbf{r}\cdot\mathbf{x} \text{ lies in } P_i.$$

Therefore, P_i has the desired property for all $i \in [m]$.

Now consider $\mathbf{x} = (x_1, \ldots, x_n) \in P$. Since $P = P_1 + \cdots + P_m$, we have $\mathbf{x} = \mathbf{z}_1 + \cdots + \mathbf{z}_m$ for some $\mathbf{z}_i \in P_i$, $i \in [m]$. Since, for every $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ such that $0 \leq r_k \leq 1$ for all $k \in [n]$ the points $\mathbf{r} \cdot \mathbf{z}_i$ lie in P_i for all $i \in [m]$, it follows that $(r_1 x_1, \ldots, r_n x_n) = \mathbf{r} \cdot \mathbf{x} = \mathbf{r} \cdot \mathbf{z}_1 + \cdots + \mathbf{r} \cdot \mathbf{z}_m$ also lies in P. □

LEMMA 2.2.13. *Let $P = y_1 \Delta^0_{S_1} + \cdots + y_m \Delta^0_{S_m}$ where y_i are positive real numbers and $S_i \in \mathbf{AdS}$ for all $i \in [m]$. Then, for $T \in \mathbf{AdS}_n$, we have $P_T = P \cap \mathbb{R}^n_T$. That is, P_T equals the polytope P intersecting with the octant associated to T.*

PROOF. Clearly, $P_T \subseteq P$. Since $y_i \Delta^0_{S_i \cap T} \subset \mathbb{R}^n_T$ for all i, it follows that $P_T \subseteq \mathbb{R}^n_T$. Thus, $P_T \subseteq P \cap \mathbb{R}^n_T$.

Next, we show that $P \cap \mathbb{R}^n_T \subseteq P_T$. Due to symmetry, it suffices to only show this for $T = \{1, \ldots, n\}$. That is, we only need to consider $P \cap \mathbb{R}^n_T = P \cap \mathbb{R}^n_{\geq 0}$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in P \cap \mathbb{R}^n_{\geq 0}$. Then, by Remark 2.2.11, $\mathbf{x} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} = (a_1, \ldots, a_n) \in P_T$ and $\mathbf{b} = (b_1, \ldots, b_n) \in y_1 \Delta^0_{S_1 \setminus T} + \cdots + y_m \Delta^0_{S_m \setminus T} = P_{T^c}$ where $T^c := [n, \overline{n}] \setminus T = \{\overline{1}, \ldots, \overline{n}\}$. This implies $a_i \geq 0$ and $b_i \leq 0$ for all $i \in [n]$. Since $\mathbf{x} = \mathbf{a} + \mathbf{b} \in \mathbb{R}^n_{\geq 0}$, it follows that $0 \leq x_i = a_i + b_i \leq a_i$ for all $i \in [n]$. Thus, $\mathbf{x} = (r_1 a_1, \ldots, r_n a_n)$ for some $(r_1, \ldots, r_n)$ with $0 \leq r_i \leq 1$ for all $i \in [n]$. Therefore, by Lemma 2.2.12, $\mathbf{x} \in P_T$. This shows $P \cap \mathbb{R}^n_T \subseteq P_T$ as desired. □

EXAMPLE 2.2.14. Let $P = \Delta^0_{[2]} + \Delta^0_{[2]} + \Delta^0_{\{\overline{1},2\}} + \Delta^0_{\{\overline{2}\}} \subset \mathbb{R}^2$ be the generalized permutohedron shown in Figure 2.8 (also shown as P_1 in Figure 2.7). Then, P_T for $T \in \mathbf{AdS}_2$ are also depicted in the same figure. One has that

$$P_{\{1,2\}} := \Delta^0_{[2]} + \Delta^0_{[2]} + \Delta^0_{\{2\}}$$

$$P_{\{1,\overline{2}\}} = \Delta^0_{\{1\}} + \Delta^0_{\{1\}} + \Delta^0_{\{\overline{2}\}}$$

$$P_{\{\overline{1},2\}} = \Delta^0_{\{2\}} + \Delta^0_{\{2\}} + \Delta^0_{\{\overline{1},2\}}$$

$$P_{\{\overline{1},\overline{2}\}} = \Delta^0_{\{\overline{1}\}} + \Delta^0_{\{\overline{2}\}}.$$

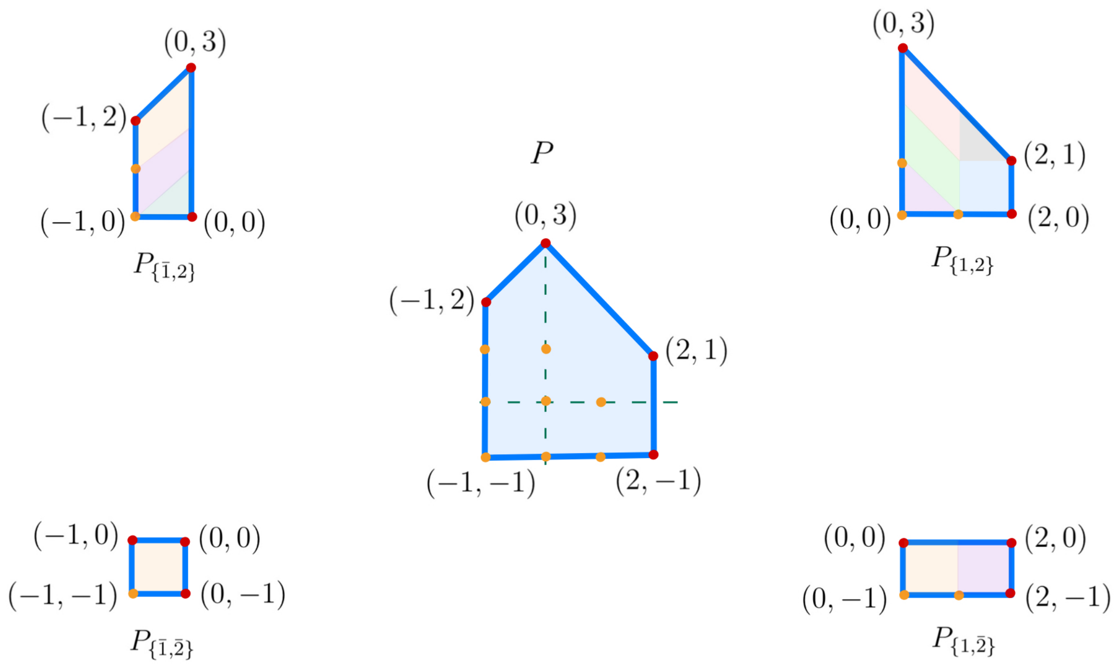


FIGURE 2.8. P and P_T, the lattice points in $P - \square_{[n]}$ and $P_T - \square_{[n]}$ (the orange points), and fine mixed subdivisions of P_T for $T \in \mathbf{AdS}_2$

2.3. Thinking of B from A

DEFINITION 2.3.1. For $i \in [n]$, we define $|\bar{i}| := |i| = i$. Additionally, for $S \in \mathbf{AdS}$, we define $||S|| := \{|i| \mid i \in S\}$.

For $T \in \mathbf{AdS}_n$, we let $\varphi_T : \mathbb{R}^{[0,n]} \longrightarrow \mathbb{R}^n$ be the projection from $\mathbb{R}^{[0,n]}$ onto $\mathbb{R}^n$ defined by

$$\varphi_T(x_0, x_1, \ldots, x_n) = \sum_{i \in T} x\mathbf{e}_i. \tag{2.3.1}$$

That is, φ_T is the projection that projects the first octant of $\mathbb{R}^{[0,n]}$ onto the octant $\mathbb{R}^T$ of $\mathbb{R}^n$. It is easy to see that, for every $S \in \mathbf{AdS}$, the projection φ_T is a bijection from the simplex $\Delta_{\{0\}\cup||S||} \subset \mathbb{R}^{[0,n]}$ to the simplex $\Delta^0_S \subset \mathbb{R}^n$.

LEMMA 2.3.2. *Let $T \in \mathbf{AdS}_n$ and $P = y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m}$ where y_i are positive real numbers and $S_i \in \mathbf{AdS}$ for all i. Suppose that Q_T is the type A generalized permutohedron in $\mathbb{R}^{[0,n]}$ defined by $Q_T = y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m}$ where $I_i := \{0\} \cup ||S_i \cap T||$. Then, the projection φ defined in (2.3.1) is a bijection from P_T to Q_T. Moreover, if P is integral, then P_T is integrally equivalent to Q_T.*

Proof. By Lemma 2.1.4, the type A generalized permutohedron Q_T can be equivalently expressed as the set of all $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{[0,n]}_{\geq 0}$ satisfying

$$x_0 + x_1 + \cdots + x_n = y_1 + \cdots + y_n \text{ and } \sum_{i \in I} x_i \leq \left(\sum_{j=1}^{n} y_j \right) - \left(\sum_{I_i \subseteq [0,n] \setminus I} y_i \right)$$

for all nonempty proper subset I of $[n]$

Let $\varphi : \mathbb{R}^{[0,n]} \longrightarrow \mathbb{R}^n$ be the projection from $\mathbb{R}^{[0,n]}$ onto $\mathbb{R}^n$ defined by $\varphi(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Then, the type B generalized permutohedron

$$R_T := y_1 \Delta^0_{||S_1 \cap T||} + \cdots + y_m \Delta^0_{||S_m \cap T||} \subseteq \mathbb{R}^n_{\geq 0}$$

is given by the set of all $(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ such that

$$x_1 + \cdots + x_n \leq n \text{ and } \sum_{i \in I} x_i \leq \left(\sum_{j=1}^{n} y_j \right) - \left(\sum_{I_i \subseteq [0,n] \setminus I} y_i \right)$$

for all nonempty proper subset I of $[n]$. Hence, φ is a bijection from Q_T onto R_T. Morevoer, we see that if P_T is integral ($y_1, \ldots, y_m$ are integers), then Q_T is integral and is integrally equivalent to R_T.

Let $\phi_T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the bijection on $\mathbb{R}^n$ defined by $\phi_T(x_1, \ldots, x_n) = \sum_{i \in T} x_i \mathbf{e}_i$. Then, ϕ_T is the map that rotates the first octant of $\mathbb{R}^n$ to the octant $\mathbb{R}_T$. Hence, ϕ_T is a bijection from R_T to P_T. Since $\varphi_T = \phi_T \circ \varphi$, it follows that φ_T is a bijection from Q_T onto P_T. Furthermore, when P_T is integral, then Q_T is integrally equivalent to P_T. □

Informally, Lemma 2.3.2 states that we can view the polytope obtained by intersecting P with the octant $\mathbb{R}_T$ as a type A generalized permutohedron. This is where we can apply some of the techniques and tools introduced by Postnikov in [35] to type B generalized permutohedra. This realization, in particular, allows us to associate a bipartite graph to P_T as follows.

Definition 2.3.3. With the same assumptions as given in Lemma 2.3.2, we define the corresponding bipartite graph $G(P_T)$ of P_T to be the bipartite graph $G(Q_T)$ of the type A permutohedron Q_T.

Example 2.3.4. Let $P = \Delta^0_{[2]} + \Delta^0_{[2]} + \Delta^0_{\{\bar{1},2\}} + \Delta^0_{\{\bar{2}\}} \subset \mathbb{R}^2$ be the type-B generalized permutohedron shown in Figure 2.8. By setting $T = \{1, 2\} \in \mathbf{AdS}_2$, one sees from Figure 2.9 that $P_{\{1,2\}}$ is integrally equivalent to the type-A generalized permutohedron $Q_{\{1,2\}} \subset \mathbb{R}^{[0,2]}$ (also shown as P in Figure 2.5) through the projection $\varphi_{\{1,2\}}(x_0, x_1, x_2) = (x_1, x_2)$. Moreover, the associated bipartite graph $G(P_{\{1,2\}})$ of $P_{\{1,2\}}$ (shown in Figure 2.9) is the graph $G(Q_{\{1,2\}}) = G$ where G is the bipartite graph shown in Figure 2.5, except that the right vertices r_1, r_2, r_3 are relabeled respectively as r_0, r_1, r_2. The fine mixed cells in $P_{\{1,2\}}$ are also obtained by projecting the fine mixed cells in $Q_{\{1,2\}}$.

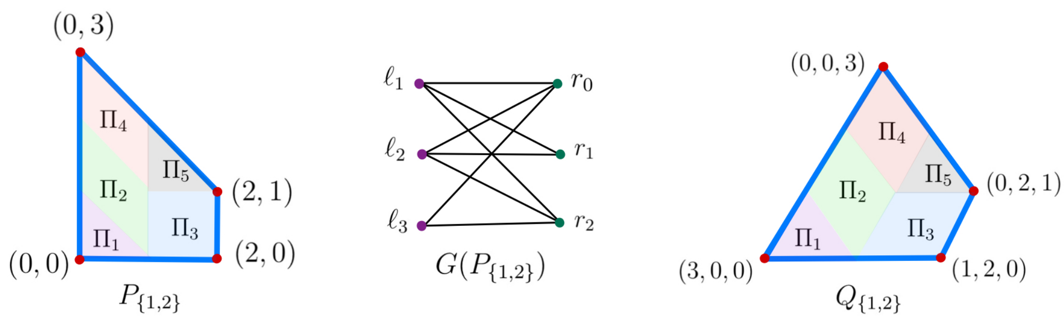


FIGURE 2.9. $P_{\{1,2\}} \subset \mathbb{R}^2$ and $Q_{\{1,2\}} \subset \mathbb{R}^{[0,2]}$ are integrally equivalent

It is important to note that when we view P_T as a type A generalized permutohedron, we always set $\mathbf{e}_0 = 0$.

2.4. Counting Lattice Points from A to B

In similar fashion to how Postnikov obtains the Ehrhart polynomial of integral generalized permutohedra, we only need to derive the Ehrhart polynomial of integral type B generalized permutohedra $P = \sum_{S \in \mathbf{AdS}} y_S \Delta^0_S$ that holds for nonnegative integers y_S, for all $S \in \mathbf{AdS}$. Once we get a formula that works for all nonnegative integers y_S, it will extend to hold for all integers y_S. Since every $y\Delta^0_S$ can be written as Minkowski sum of y copies of Δ^0_S, we deduce the problem to finding a formula for the number of lattice points in $P = \Delta^0_{S_1} + \cdots + \Delta^0_{S_m}$ where $S_i \in \mathbf{AdS}$ for all $i \in [m]$.

DEFINITION 2.4.1. Let $P = y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m}$ where y_i are nonzero integers and $S_i \in \mathbf{AdS}$ for all $i \in [m]$. For a given admissible set $T \in \mathbf{AdS}_n$, we define a *G-draconian* sequence of P_T to be a sequence $(a_1, \ldots, a_m)$ of nonnegative integers satisfying the following inequalities: for every nonempty subset $I \subseteq \{1, \ldots, m\}$

$$\sum_{i \in I} a_i \leq \Big|\bigcup_{i \in I} S_i \cap T\Big|, \text{ and } a_1 + \cdots a_m = n.$$

We denote by $\mathrm{D}(P_T)$ the set of all G-draconian sequences, and denote by $\mathrm{D}(P_T) \cap \square_{[m]}$ the set of G-draconian sequences in which $a_i \leq 1$ for all $i \in [m]$.

One observes that the definition of G-draconian sequence in Definition 2.4.1 is exactly the definition of $G(P_T)$-draconian sequence in Definiton 2.1.6. One also sees that that if P_T is not n-dimensional, $G(P_T)$ must have a right vertice of degree zero. Consequently, $a_1 + \cdots + a_m \leq \big|\bigcup_{i \in [m]} S_i \cap T\big| < n$. Thus, $\mathrm{D}(P_T) = \emptyset$ provided P_T is not n-dimensional.

Recall that Postnikov utilized the Cayley trick to subdivide polytopes into fine mixed cells to show that the lattice points in certain polytopes are in one-to-one correspondence with their associated G-draconian sequences, which consequently led to formula (2.1.3) in Lemma 2.1.7. Adapting Postnikov's approach, we found a similar correspondence which gives the next theorem as a result.

THEOREM 2.4.2 (Thawinrak). *Suppose that $P = \Delta^0_{S_1} + \cdots + \Delta^0_{S_m}$ where $S_i \in \mathbf{AdS}$ for all $i \in [m]$. Then, the number of lattice points in the polytope $P - \square_{[n]}$ equals*

$$|(P - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{T \in \mathbf{AdS}_n} |\mathrm{D}(P_T) \cap \square_{[m]}|. \tag{2.4.1}$$

The rest of this section is devoted to proving this theorem and its consequences.

DEFINITION 2.4.3. Let $T \in \mathbf{AdS}_n$, and $P = \Delta^0_{S_1} + \cdots + \Delta^0_{S_m}$ where $S_i \in \mathbf{AdS}$ for all i be a type B generalized permutohedron. Let $\Pi \in \mathcal{C}$ be a fine mixed cell of P_T, and H be the bipartite subgraph of $G(P_T)$ corresponding to Π, i.e., Π is integrally equivalent to the type A generalized permutohedron $P_H(1, \ldots, 1) \subset \mathbb{R}^{[0,n]}$. Suppose that $\Pi = \Delta_{I_1} + \cdots + \Delta_{I_m}$ where $I_i \subseteq \{0\} \cup S_i$ for all i. Let $K = \{I_i \mid i \in [m] \text{ and } |I_i| \geq 2\}$. Then, we define $\hat{\Pi}$ to be the polytope

$$\hat{\Pi} := \sum_{J \in K} \Delta_K$$

obtained by removing the translating factor from Π. We denote by $\hat{H}$ the induced bipartite subgraph of H corresponding to $\hat{\Pi}$.

We highlight some of the basic properties of $\hat{\Pi}$ in the following two Remarks.

REMARK 2.4.4. Let $K' = \{I_i \mid i \in [m] \text{ and } |I_i| = 1\}$. The fine mixed cell Π is the translation $\mathbf{x} + \hat{\Pi}$ of $\hat{\Pi}$ where $\mathbf{x}$ is the integral vector (translating factor)

$$\mathbf{x} = \sum_{J \in K'} \Delta_J \text{ lying in the octant } \mathbb{R}_T.$$

REMARK 2.4.5. If we assume further that P_T is n-dimensional ($G(P_T)$ is connected), then H is a spanning tree of $G(P_T)$ with m left and $n+1$ right vertices. Consequently, we have that $\hat{H}$ is a tree and a bipartite graph with $n+1$ right vertices in which every left vertex has degree at least two. Moreover, $\hat{H}$ has at most n left vertices. The tree $\hat{H}$ has exactly n left vertices if and only if every left vertex of $\hat{H}$ has degree two.

Unless stated otherwise, we assume throughout the rest of this section that $P = \Delta^0_{S_1} + \cdots + \Delta^0_{S_m}$ where $S_i \in \mathbf{AdS}$ for all i. In addition, for $T \in \mathbf{AdS}_n$, we denote by $G(P_T)$ the corresponding bipartite graph of P_T with m left vertices $\{\ell_1, \ldots, \ell_m\}$ and $n+1$ right vertices $\{r_0, r_1, \ldots, r_n\}$.

LEMMA 2.4.6. *Suppose that P_T is n-dimensional. Let $\mathcal{C}$ be the set of fine mixed cells in a fine mixed subdivision of P_T. If Π is a fine mixed cell in $\mathcal{C}$, then Π and $\mathbf{y} + \Pi$ have no common interior for all vectors $\mathbf{y} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$.*

PROOF. Suppose that $\Pi \in \mathcal{C}$ is a fine mixed cell of the form $\Pi = \Delta_{I_1} + \cdots + \Delta_{I_m}$ where $I_i \subseteq \{0\} \cup S_i$ for all i. Let $K = \{I_i \mid i \in [m] \text{ and } |I_i| \geq 2\}$. Then,

$$\hat{\Pi} := \sum_{J \in K} \Delta_J.$$

For each $J \in K$, let us write $J = \{j_1, \ldots, j_{|J|} \mid |j_i| < |j_{i+1}|, \text{ for all } 1 \leq i < |J|\}$, and define the corresponding set of intervals $\binom{J}{2}^* := \{[\mathbf{e}_{j_1}, \mathbf{e}_{j_2}], [\mathbf{0}, \mathbf{e}_{j_3} - \mathbf{e}_{j_1}], \ldots, [\mathbf{0}, \mathbf{e}_{j_{|J|}} - \mathbf{e}_{j_1}] \subset \mathbb{R}^n\}$ where we denote by $[\mathbf{x}, \mathbf{y}]$ the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ (the interval from $\mathbf{x}$ to $\mathbf{y}$), and set $\mathbf{e}_0 = 0 \in \mathbb{R}^n$. In addition, we define

$$B_K := \bigcup_{J \in K} \binom{J}{2}^*$$

to be the set of intervals in all $\binom{J}{2}^*$ for $J \in K$. It is easy to see that every vertex of the simplex Δ_J lies in the zonotope

$$\diamond_J := \sum_{\Delta \in \binom{J}{2}^*} \Delta$$

spanned by the intervals in $\binom{J}{2}^*$. This implies that $\Delta_J \subseteq \diamond_J$ for all $J \in K$. Thus, $\hat{\Pi}$ lies in the zonotope

$$\diamond_K := \sum_{\Delta \in B_K} \Delta = \sum_{J \in K} \diamond_J.$$

Since P_T is n-dimensional, by Remark 2.4.5, $\hat{H}$ is a tree and a bipartite graph with n left and $n+1$ right vertices. Because $|\binom{J}{2}^*| = |J| - 1$, it follows that

$$\sum_{J \in K} \left| \binom{J}{2}^* \right| = n.$$

Moreover, one sees that the set B contains translations of integral vectors that form an integral basis for $\mathbb{Z}^n$. Thus, the polytope $\diamond$ is the parallelepiped in $\mathbb{R}^n$ of volume one spanned by the intervals $\Delta \in B$. Hence, $\diamond$ and $\mathbf{y} + \diamond$ have no common interior for all nonzero integral vectors $\mathbf{y}$. Since $\hat{\Pi} \subset \diamond$ and $\mathbf{z} + \hat{\Pi} \subset \mathbf{z} + \diamond$ for all $\mathbf{z} \in \mathbb{R}^n$, it follows that Π has no common interior with $\mathbf{y} + \Pi$ for all vectors $\mathbf{y} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$. □

A *zonotope* in $\mathbb{R}^n$ is a polytope defined as a Minkowski sum of line segments (intervals) in $\mathbb{R}^n$. This means that a translation of a zonotope is also a zonotope. The next lemma characterize fine mixed cells that are zonotopes.

LEMMA 2.4.7. *Suppose that P_T is n-dimensional. Let $\mathcal{C}$ be the set of fine mixed cells in a fine mixed subdivision of P_T. Let Π be a fine mixed cell in $\mathcal{C}$ and $\mathbf{x} \in \mathbb{Z}^n$ be the integral vector satisfying $\mathbf{x} + \hat{\Pi} = \Pi$. Then, the following statements are equivalent.*

(1) The fine mixed cell Π satisfies $\nabla_T \subseteq \hat{\Pi}$.

(2) $\mathbf{x}$ is the unique integral vector in $(P_T - \square_T) \cap \mathbb{Z}^n$ such that $\mathbf{x} + \nabla_T \subseteq \Pi$.

(3) The fine mixed cell Π is a zonotope.

To prove this lemma, we need the following result regarding the existence of a perfect matching of a certain bipartite graph.

Lemma 2.4.8. *Let n be a positive integer. Suppose that B is a bipartite graph with n left vertices and n right vertices satisfying the following two conditions.*

(1) Every left and right vertex of B has degree at least one.

(2) Every left and right vertex of B is adjacent to at most one vertex of degree one.

Then, there is a perfect matching in B.

Proof. We will proceed by induction on $n \geq 1$. When $n = 1$, there is a perfect matching in B, since the left and right vertices are adjacent. This establishes the base case for induction. Now suppose that the statement is true for some $n \geq 1$. Consider a bipartite graph B with $n+1$ left vertices $\{\ell_1, \ldots, \ell_{n+1}\}$ and $n+1$ right vertices $\{r_1, \ldots, r_{n+1}\}$ satisfying the two conditions. Without loss of generality, we may assume that r_{n+1} is a right vertex of the least degree in B, and that the left vertex ℓ_{n+1} is adjacent to r_{n+1} and has the least degree among other left vertices adjacent to r_{n+1}.

Let $\hat{B}$ be the induced bipartite subgraph of B obtained by removing the vertices ℓ_{n+1} and r_{n+1} of B. Then, it is easy to see that $\hat{B}$ is a bipartite graph on n left and n right vertices satisfying the two conditions. Thus, by the induction hypothesis, there is a perfect matching in $\hat{B}$.

By matching ℓ_{n+1} with r_{n+1} and matching other vertices of B using a perfect matching in $\hat{B}$, we obtain a perfect matching in B. Therefore, by induction, the statement holds for all $n \geq 1$. $\square$

Proof of Lemma 2.4.7. Due to symmetry, we may assume for simplicity of notation without loss of generality that $T = [n]$. Let $\mathbf{x} \in \mathbb{Z}^n_{\geq 0}$ be the integral vector satisfying $\mathbf{x} + \hat{\Pi} = \Pi$.

Firstly, we show that (1) implies (2). Suppose that $\nabla_{[n]} \subseteq \hat{\Pi}$. Then, $\mathbf{x} + \nabla_{[n]} \subset \mathbf{x} + \hat{\Pi} = \Pi$. In particular, we have $\mathbf{x} + \mathbf{e}_1 + \cdots + \mathbf{e}_n \in \Pi \subseteq P_{[n]}$. By Lemma 2.2.12, we must have $\mathbf{x} + \square_{[n]} \subseteq P_{[n]}$. Thus, $\mathbf{x} \in (P_{[n]} - \square_{[n]}) \cap \mathbb{Z}^n$. Hence, $\mathbf{x} \in (P_{[n]} - \square_{[n]}) \cap \mathbb{Z}^n$ is an integral vector such that $\mathbf{x} + \nabla_{[n]} \subset \Pi$.

Suppose that $\mathbf{y} \in (P_{[n]} - \square_{[n]}) \cap \mathbb{Z}^n$ satisfies $\mathbf{y} + \nabla_{[n]} \subset \Pi$. Since $\Pi = \mathbf{x} + \hat{\Pi}$, we have $\nabla_{[n]} \subseteq \mathbf{x} - \mathbf{y} + \hat{\Pi}$. This implies that $\hat{\Pi}$ has common interior with $\mathbf{x} - \mathbf{y} + \hat{\Pi}$. By Lemma 2.4.6, we must have $\mathbf{x} = \mathbf{y}$. This shows the uniqueness of the integral vector $\mathbf{x} \in (P_{[n]} - \square_{[n]}) \cap \mathbb{Z}^n$ such that $\mathbf{x} + \nabla_{[n]} \subset \Pi$. Thus, (1) implies (2).

Next, we show that (2) implies (3). Suppose that $\mathbf{x} \in (P_{[n]} - \square_{[n]}) \cap \mathbb{Z}^n$ and $\mathbf{x} + \nabla_{[n]} \subseteq \Pi$. We claim that Π is a zonotope. Assume for the sake of contradiction that Π is not a zonotope.

Then, $\hat{\Pi}$ is also not a zonotope. This implies that $\hat{H}$ is a bipartite graph that is also a tree with at least one left vertex has degree at least three. Together with Remark 2.4.5, we conclude that $\hat{H}$ must have less than n left vertices. Thus, the sequence of right vertices $(r_1, \dots, r_n)$ cannot be a transversal of the neighbors of the left vertices of $\hat{H}$. Thus, By Lemma 2.1.11 and Remark 2.1.12, $\mathbf{e}_1 + \cdots + \mathbf{e}_n \notin \hat{\Pi}$. In particular, we must have $\nabla_{[n]} \not\subseteq \hat{\Pi}$. Therefore, $\mathbf{x} + \nabla_{[n]} \not\subset \mathbf{x} + \hat{\Pi} = \Pi$, a contradiction. This shows that (2) implies (3).

Lastly, we show that (3) implies (1). Suppose that Π is a zonotope. Then, $\hat{\Pi}$ is also a zonotope. Thus, $\hat{H}$ is a bipartite graph that is also a tree with $n+1$ right vertices in which every left vertex has degree two. Consequently, $\hat{H}$ must have exactly n left vertices. To see that $\nabla_{[n]} \subseteq \hat{\Pi}$, it suffices to show $\mathbf{e}_1 + \cdots + \mathbf{e}_n \in \hat{H}$ and $\mathbf{e}_1 + \cdots \mathbf{e}_{i-1} + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_n \in \hat{\Pi}$ for all $i \in [n]$. For every $i \in \{0, 1, \dots, , n\}$, let $\hat{H}_{(i)}$ be the induced bipartite subgraph of $\hat{H}$ obtained by removing the right vertex r_i of $\hat{H}$. Then, $\hat{H}_{(i)}$ is a bipartite graph on n left and n right vertices satisfying the two conditions in Lemma 2.4.8. Thus, there is a matching in $\hat{H}_{(i)}$ for all $i \in \{0, 1, \dots, , n\}$. By Remark 2.1.12, a matching of $\hat{H}_{(0)}$ implies that $\mathbf{e}_1 + \cdots + \mathbf{e}_n \in \hat{H}$ while a matching of $\hat{H}_{(i)}$ implies that $\mathbf{e}_1 + \cdots \mathbf{e}_{i-1} + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_n \in \hat{\Pi}$ for all $i \in [n]$. This completes the proof. □

The next lemma gives an analog result to Corollary 2.1.16.

LEMMA 2.4.9. *Suppose that P_T is n-dimensional. Let $\mathcal{C}^* = \{\Pi_1, \dots, \Pi_q\}$ be the subset of $\mathcal{C}$ consisting of fine mixed cells that are zonotopes. Then,*

$$|\mathrm{D}(P_T) \cap \square_{[m]}| = |\mathcal{C}^*| = |(P_T - \square_T) \cap \mathbb{Z}^n|.$$

PROOF. We first show that $|\mathrm{D}(P_T) \cap \square_{[m]}| = |\mathcal{C}^*|$. Since P_T is n-dimensional, its corresponding bipartite graph $G(P_T)$ is connected. Moreover, because all fine mixed cells in $\mathcal{C}^*$ are zonotopes, each cell $\Pi_i \in \mathcal{C}^*$ corresponds to a spanning tree of $G(P_T)$ whose left vertices have degree at most two. By [35, Theorem 12.2], $LD(\Pi_1), \dots, LD(\Pi_q)$ are distinct elements of $\mathrm{D}(P_T) \cap \square_{[m]}$. Thus, $|\mathrm{D}(P_T) \cap \square_{[m]} \geq |\mathcal{C}^*|$. Moreover, for every $\mathbf{a} \in \mathrm{D}(P_T) \cap \square_{[m]}$, there exists a fine mixed cell $\Pi \in \mathcal{C}$ such that $\mathbf{a} = LD(\Pi)$. This implies that the corresponding spanning tree of Π has left vertices of degree at most two. Thus, Π is a zonotope, i.e., $\Pi \in \mathcal{C}^*$. Therefore, $|\mathrm{D}(P_T) \cap \square_{[m]}| \leq |\mathcal{C}^*|$, which implies $|\mathrm{D}(P_T) \cap \square_{[m]}| = |\mathcal{C}^*|$.

Next, we show that $|\mathcal{C}^*| \leq |(P_T - \square_T) \cap \mathbb{Z}^n|$. By Lemma 2.4.7/(2), the map $\Pi_i \mapsto \mathbf{x}$ where $\mathbf{x} \in (P_T - \square_T) \cap \mathbb{Z}^n$ satisfying $\mathbf{x} + \nabla_T \subseteq \Pi_i$ is an injection from $\mathcal{C}^*$ to $(P_T - \square_T) \cap \mathbb{Z}^n$. Hence, $|\mathcal{C}^*| \leq |(P_T - \square_T) \cap \mathbb{Z}^n|$.

Lastly, we show that $|\mathcal{C}^*| \geq |(P_T - \square_T) \cap \mathbb{Z}^n|$. Let $\mathbf{x} \in (P_T - \square_T) \cap \mathbb{Z}^n$. Due to symmetry, we may assume for simplicity of notation without loss of generality that $T = [n]$. We claim that there exists a unique fine mixed cell $\Pi \in \mathcal{C}^*$ such that $\mathbf{x} + \nabla_{[n]} \subseteq \Pi$. Note that once the existence of such a fine mixed cell $\Pi \in \mathcal{C}^*$ is established, the uniqueness will automatically follows, since two distinct cells in $\mathcal{C}$ have no common interior. Thus, we only need to show the existence of $\Pi \in \mathcal{C}^*$ such that $\mathbf{x} + \nabla_{[n]} \subseteq \Pi$. To see this, let $\Pi \in \mathcal{C}$ be a fine mixed cell that has a common interior interior with $\mathbf{x} + \nabla_{[n]}$. Note that such a fine mixed cell Π exists because $\mathbf{x} + \nabla_{[n]} \subseteq \mathbf{x} + \square_{[n]} \subseteq P_T$. We now proceed to show that $\mathbf{x} + \nabla_{[n]} \subseteq \Pi$ by first establishing that Π is a zonotope.

Assume for the sake of contradiction that Π is not a zonotope. Then, $\hat{\Pi}$ is also not a zonotope. This implies that $\hat{H}$ is a bipartite graph that is also a tree with at least one left vertex has degree at least three. Together with Remark 2.4.5, we conclude that $\hat{H}$ must have less than n left vertices. Thus, the sequence of right vertices $(r_{i_1}, \dots, r_{i_n})$ cannot be a transversal of the neighbors of the left vertices of $\hat{H}$. By Lemma 2.1.11 and Remark 2.1.12, we deduce that $\hat{\Pi}$ lies in the half-space $x_1 + \cdots + x_n \leq n - 1$. Because $\hat{\Pi} \subseteq \mathbb{R}^n_{\geq 0}$, it follows that $\hat{\Pi}$ lies in the polytope Q given by

$$Q := \{\mathbf{x} \mid x_1 + \cdots + x_n \leq n - 1 \ \text{ and } 0 \leq x_i \text{ for all } i \in [n]\}.$$

Note that

$$\nabla_{[n]} := \{\mathbf{x} \mid x_1 + \cdots + x_n \geq n - 1 \ \text{ and } x_i \leq 1 \text{ for all } i \in [n]\}.$$

For $\mathbf{y} = (y_1, \dots, y_n), \mathbf{z} = (z_1, \dots, z_n)$ in $\mathbb{Z}^n$, the translations $\mathbf{y} + Q$ and $\mathbf{z} + \nabla_{[n]}$ are given by

$$\mathbf{y} + Q = \{\mathbf{x} \mid x_1 + \cdots + x_n \leq y_1 + \cdots + y_n + n - 1 \ \text{ and } y_i \leq x_i \text{ for all } i \in [n]\}$$

$$\mathbf{z} + \nabla_{[n]} = \{\mathbf{x} \mid x_1 + \cdots + x_n \geq z_1 + \cdots + z_n + n - 1 \ \text{ and } x_i \leq z_i + 1 \text{ for all } i \in [n]\}.$$

One sees that $\mathbf{y} + Q$ and $\mathbf{z} + \nabla_{[n]}$ have no common interior for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$. In fact, if $\mathbf{y} + Q$ and $\mathbf{z} + \nabla_{[n]}$ were to have a common interior, then we would have $z_1 + \cdots + z_n + n - 1 < y_1 + \cdots + y_n + n - 1$.

This would imply that there exists $j \in [n]$ such that $z_j + 1 \leq y_j$. However, this would mean that $\mathbf{y} + Q$ lies in the half-space $z_j + 1 \leq x_j$ while $\mathbf{z} + \nabla_{[n]}$ lies in the half-space $x_j \leq z_j + 1$. Hence, $\mathbf{y} + Q$ and $\mathbf{z} + \nabla_{[n]}$ would have no common interior, a contradiction. Thus, by setting $\mathbf{z} = \mathbf{x}$, we see that $\mathbf{y} + \hat{\Pi} = \Pi$ lies in $\mathbf{y} + Q$ and has no common interior with $\mathbf{x} + \nabla_{[n]}$, a contradiction. Therefore, Π is a zonotope.

Suppose that $\mathbf{y} \in \mathbb{Z}^n$ is the integral vector such that $\mathbf{y} + \hat{\Pi} = \Pi$. Since Π is a zonotope, by Lemma 2.4.7/(2), $\mathbf{y} + \nabla_{[n]} \subseteq \Pi$. Thus, $\mathbf{x} + \nabla_{[n]} \subseteq \mathbf{x} - \mathbf{y} + \Pi$. This means that $\mathbf{x} - \mathbf{y} + \Pi$ and Π have a common interior. By Lemma 2.4.6, we must have $\mathbf{x} = \mathbf{y}$. Therefore, $\mathbf{x} + \nabla_{[n]} \subseteq \Pi$ as claimed.

The map $\mathbf{x} \mapsto \Pi$ where Π is the fine mixed cell such that $\mathbf{x} + \nabla_{[n]} \subseteq \Pi$ is now seen to be an injection from $(P_T - \square_T) \cap \mathbb{Z}^n$ to $\mathcal{C}^*$. Therefore, $|\mathcal{C}^*| \geq |(P_T - \square_T) \cap \mathbb{Z}^n|$. ☐

We are now ready to give a proof of Theorem 2.4.2.

Proof of Theorem 2.4.2. We first observe that

$$(P - \square_{[n]}) \cap \mathbb{Z}^n = \bigsqcup_{T \in \mathbf{AdS}_n} (P_T - \square_{[n]}) \cap \mathbb{Z}^n.$$

Thus,

$$|(P - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{T \in \mathbf{AdS}_n} |(P_T - \square_{[n]}) \cap \mathbb{Z}^n|.$$

For $T \in \mathbf{AdS}_n$ such that P_T is not n-dimension, we have that $|(P_T - \square_T) \cap \mathbb{Z}^n| = 0 = |\mathrm{D}(P_T) \cap \square_{[m]}|$. Also, for $T \in \mathbf{AdS}_n$ such that P_T is n-dimensional, we have by Lemma 2.4.9 that $|(P_T - \square_T) \cap \mathbb{Z}^n| = |\mathrm{D}(P_T) \cap \square_{[m]}|$. This means $|(P_T - \square_T) \cap \mathbb{Z}^n| = |\mathrm{D}(P_T) \cap \square_{[m]}|$ for all $T \in \mathbf{AdS}_n$. Thus, to see that

$$|(P - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{T \in \mathbf{AdS}_n} |\mathrm{D}(P_T) \cap \square_{[m]}|, \tag{2.4.2}$$

it suffices to show that $|(P_T - \square_{[n]}) \cap \mathbb{Z}^n| = |(P_T - \square_T) \cap \mathbb{Z}^n|$ for all $T \in \mathbf{AdS}_n$.

For every $T \in \mathbf{AdS}_n$, we have $\square_{[n]} = \square_T - \sum_{i \in T \setminus [n]} \mathbf{e}_i$. Thus,

$$|(P_T - \square_{[n]}) \cap \mathbb{Z}^n| = \left| \left(\sum_{i \in T \setminus [n]} \mathbf{e}_i \right) + (P_T - \square_T) \cap \mathbb{Z}^n \right| = \left| (P_T - \square_T) \cap \mathbb{Z}^n \right|.$$

This gives (2.4.2) as desired. ☐

EXAMPLE 2.4.10. Figure 2.8 shows $P = \Delta^0_{[2]} + \Delta^0_{[2]} + \Delta^0_{\{\overline{1},2\}} + \Delta^0_{\{\overline{2}\}} \subset \mathbb{R}^2$ together with P_T for $T \in \mathbf{AdS}_2$, and the lattice points in $P - \square_{[2]}$ and $P_T - \square_{[2]}$ (the orange points). One sees that

$$(P - \square_{[2]}) \cap \mathbb{Z}^2 = \sum_{T \in \mathbf{AdS}_2} |(P_T - \square_{[2]}) \cap \mathbb{Z}^2|.$$

Moreover, one sees that, for every $T \in \mathbf{AdS}_2$, the number of lattice points in $P_T - \square_{[2]}$ equals the number of fine mixed cells in a subdivision of P_T that are zonotopes.

Theorem 2.4.2 implies that

$$|(P - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{T \in \mathbf{AdS}_n} \left(\sum_{\mathbf{a} \in \mathrm{D}(P_T)} \binom{1}{a_1} \cdots \binom{1}{a_m} \right), \tag{2.4.3}$$

since only the G-draconian sequences $\mathbf{a} = (a_1, \dots, a_m)$ with $a_i \leq 1$ make the summand in equation (2.4.1) nonzero, and equal to one. Together with a simple binomial identity, we derive the following key result as a consequence.

COROLLARY 2.4.11. *Suppose that $P = y_1 \Delta^0_{S_1} + \cdots + y_m \Delta^0_{S_m}$ where y_i are integers and $S_i \in \mathbf{AdS}$ for all i. Then,*

$$|(P - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{T \in \mathbf{AdS}_n} \left(\sum_{\mathbf{a} \in \mathrm{D}(P_T)} \binom{y_1}{a_1} \cdots \binom{y_m}{a_m} \right). \tag{2.4.4}$$

PROOF. As noted at the begining of the section, it suffices to show that the formula holds for positive integers $y_1, \dots, y_n$. Clearly, we can write any $y\Delta^0_S$ where $S \in \mathbf{AdS}$ and y is a positive integer as the Miknowski sum of y copies of Δ^0_S. By writing

$$P = \underbrace{\Delta^0_{S_1} + \cdots + \Delta^0_{S_1}}_{y_1 \text{ terms}} + \cdots + \underbrace{\Delta^0_{S_m} + \cdots + \Delta^0_{S_m}}_{y_m \text{ terms}}$$

and applying formula (2.4.3) to P, one can express the right-hand side of (2.4.3) as

$$\sum_{T \in \mathbf{AdS}_n} \left(\sum_{\mathbf{a} \in \mathrm{D}(P_T)} \binom{y_1}{a_1} \cdots \binom{y_m}{a_m} \right)$$

using the binomial identity

$$\binom{y}{a} = \sum_{\substack{b_1+\cdots+b_y=a\\ b_1,\ldots,\ b_y\ \in\mathbb{Z}^n_{\geq 0}}} \binom{1}{b_1}\cdots\binom{1}{b_y} \quad \text{for all } a, y \in \mathbb{Z}_{\geq 0}. \qquad \square$$

Note that the Ehrhart polynomial of $P = y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m}$ can be computed simply by replacing it in formula (2.4.4) with $tP + \square_{[n]} = ty_1\Delta^0_{S_1} + \cdots + ty_m\Delta^0_{S_m} + \Delta^0_{\{1\}} + \cdots + \Delta^0_{\{n\}}$.

Remark 2.4.12. Let

$$D(T) = \bigcup_{T\in\mathbf{AdS}_n} D(P_T).$$

Then, formula (2.4.4) can also be expressed as

$$|(P - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{\mathbf{a}\in D(P)} |\{T \in \mathbf{AdS}_n \mid \mathbf{a} \in D(P_T)\}| \binom{y_1}{a_1}\cdots\binom{y_m}{a_m}. \tag{2.4.5}$$

One observes that formula (2.4.5) bears a resemblance to formula (2.2.4) from Lemma 2.2.8 by Eur, Fink, Larson, and Spink. However, (2.4.5) is written in a more compact form: each summand $|\{T \in \mathbf{AdS}_n \mid \mathbf{a} \in D(P_T)\}|\binom{y_1}{a_1}\cdots\binom{y_m}{a_m}$ in (2.4.5) consolidates $\frac{(a_1+\cdots+a_m)!}{a_1!\cdots a_m!}$ individual terms that appear in (2.2.4).

Corollary 2.4.13. *Suppose that $P = y_1\Delta^0_{S_1}+\cdots+y_m\Delta^0_{S_m}$ where y_i are real numbers and $S_i \in \mathbf{AdS}$ for all i. Then, the volume of P is given by*

$$\mathrm{Vol}(P) = \sum_{T\in\mathbf{AdS}_n} \left(\sum_{\mathbf{a}\in \mathrm{D}(P_T)} \frac{y_1^{a_1}}{a_1!}\cdots\frac{y_m^{a_m}}{a_m!} \right). \tag{2.4.6}$$

Proof. To see that formula (2.4.6) gives the volume of P, it suffices to show that the formula holds for nonnegative integers $y_1, \ldots, y_m$. Let

$$Q^{(t)} := tP + \square_{[n]} = ty_1\Delta^0_{S_1} + \cdots + ty_m\Delta^0_{S_m} + \Delta^0_{\{1\}} + \cdots + \Delta^0_{\{n\}}.$$

We note that $\mathrm{D}(Q^{(t)}_T) \subseteq \mathbb{Z}^{m+n}_{\geq 0}$. Then, the Ehrhart polynomial of P is given by

$$i(t,P) = |(Q^{(t)} - \square_{[n]}) \cap \mathbb{Z}^n| = \sum_{T\in\mathbf{AdS}_n} \left(\sum_{\mathbf{a}\in\mathrm{D}(Q^{(t)}_T)} \binom{y_1t}{a_1}\cdots\binom{y_mt}{a_m}\binom{1}{a_{m+1}}\cdots\binom{1}{a_{n+m}} \right).$$

Since the volume of P equals the leading coefficient of $i(P,t)$, it follows that

$$\mathrm{Vol}(P) = \sum_{T \in \mathbf{AdS}_n} \left(\sum_{\substack{\mathbf{a} \in \mathrm{D}(Q_T^{(t)}) \\ a_i = 0,\ \forall i > m}} \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!} \right).$$

One observes that $\mathbf{a} \in \mathrm{D}(Q_T^{(t)})$ satisfies $a_i = 0$ for all $i > m$ if and only if $(a_1, \ldots, a_m) \in \mathrm{D}(P_T)$. Thus, we arrive at the desired formula

$$\mathrm{Vol}(P) = \sum_{T \in \mathbf{AdS}_n} \left(\sum_{\mathbf{a} \in \mathrm{D}(P_T)} \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!} \right) \qquad \square$$

2.5. More Problems from A to B

Our approach for computing the Ehrhart polynomials suggests that there seem to be many aspects of type B generalized permutohedra that can be explored using existing techniques and tools from the study of their type A counterparts. The following questions highlight some potential research directions.

PROBLEM 2.5.1. In [35, Section 7], Postnikov introduces building sets and nested complexes to describe the face posets of some type A generalized permutohedra. Can we give a combinatorial description of the faces of type B generalized permutohedra using similar combinatorial models as building sets and nested complexes?

PROBLEM 2.5.2. In [36], Postnikov, Reiner, and Williams compute the f and h-polynomials of a family of simple type A generalized permutohedra using building sets and the corresponding preposets. Can we employ a similar approach to compute the f and h-polynomials of type B generalized permutohedra?

PROBLEM 2.5.3. Bastidas shows in [3] that every type B generalized permutohedron can also be written as the Minkowski sum of the simplices Δ_S and Δ_S^0 where $S \in \mathbf{AdS}$ are admissible subsets such that $\min(|i| \mid i \in S) \in S$. That is, the family of admissible subsets Δ_S and Δ_S^0 where $\min(|i| \mid i \in S) \in S$ is a "basis" for the type B generalized permutohedra. Find a formula for the

Ehrhart polynomial of type B generalized permutohedra with respect to this basis (directly without transforming this basis to the basis used in this chapter).

PROBLEM 2.5.4. Postnikov's formula (2.1.3) implies that every type A generalized permutohedron of the form $y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m} \subset \mathbb{R}^n$ is Ehrhart positive provided y_i are positive integers for all $i \in [m]$. Our formula (2.4.4) in Corollary 2.4.11 does not make it immediately clear whether Ehrhart positivity holds for type B generalized permutohedra in a similar situation. This leads to the natural question: Is every type B generalized permutohedron of the form $y_1\Delta^0_{S_1} + \cdots + y_m\Delta^0_{S_m} \subset \mathbb{R}^n$, where each y_i is a positive integer for all $i \in [m]$, Ehrhart positive?

CHAPTER 3

Polynomiality of Stretched Littlewood-Richardson Coefficients

The Littlewood-Richardson coefficients appear in many areas of mathematics [**21**,**28**,**29**,**38**,**48**]. An example comes from the study of symmetric functions. The set of Schur functions s_λ, indexed by partitions λ, is a linear basis for the ring of symmetric functions. Thus, for any partitions λ and μ, the product of Schur functions s_λ and s_μ can be uniquely expressed as

$$s_\lambda \cdot s_\mu = \sum_{\nu : |\nu| = |\lambda| + |\mu|} c^\nu_{\lambda,\mu} s_\nu \tag{3.0.1}$$

for some real numbers $c^\nu_{\lambda,\mu}$, where $|\lambda|$ denotes the sum of the parts of λ. The coefficient $c^\nu_{\lambda,\mu}$ of s_ν in (3.0.1) is called the *Littlewood-Richardson coefficient*.

There are several ways to compute $c^\nu_{\lambda,\mu}$ such as the Littlewood-Richardson rule [**45**], the Littlewood-Richardson triangles [**34**], the Berenstein-Zelevinsky triangles [**6**], and the honeycombs [**26**]. In this chapter, we employ the hive model that was first introduced by Knutson and Tao [**26**]. The hive model imposes certain inequalities that allow us to compute $c^\nu_{\lambda,\mu}$ as the number of integer points in a rational polytope, which we call a hive polytope.

For fixed partitions λ, μ, ν such that $|\nu| = |\lambda| + |\mu|$, we define the *stretched* Littlewood-Richardson coefficients to be the function $c^{t\nu}_{t\lambda,t\mu}$ for non-negative integers t. The hive model implies that

$$c^{t\nu}_{t\lambda,t\mu} = \text{ the number of integer points in the } t^{\text{th}}\text{-dilation of the hive polytope.}$$

By Ehrhart theory (see Thoerem 1.3.1), $c^{t\nu}_{t\lambda,t\mu}$ is a quasi-polynolmial in $t \in \mathbb{Z}$, which means $c^{t\nu}_{t\lambda,t\mu}$ is a function of the form $a_d(t)t^d + \cdots + a_1(t)t + a_0(t)$ where each of $a_d(t), \ldots, a_0(t)$ is a periodic function in t with an integral period. The function $c^{t\nu}_{t\lambda,t\mu}$ was, however, observed and conjectured by King, Tollu, and Toumazet [**25**] to be a polynomial function in t (as opposed to a quasi-polynomial). The conjecture was then shown to be true by Derksen-Weyman [**15**], and Rassart [**37**]. More precisely, they proved the following theorem.

Theorem 3.0.1. *Let μ, λ, ν be partitions with at most k part such that $|\nu| = |\lambda| + |\mu|$. Then $c_{t\lambda,t\mu}^{t\nu}$ is a polynomial in t of degree at most $\binom{k-1}{2}$.*

The proof by Derksen and Weyman [15] makes use of semi-invariants of quivers. They proved a result on the structure of a ring of quivers and then derived the polynomiality of $c_{t\lambda,t\mu}^{t\nu}$ as a special case. Later, Rassart [37] proved a stronger result, which gives Theorem 3.0.1 as an easy consequence, by showing that $c_{\lambda,\mu}^{\nu}$ is a polynomial in variables λ, μ, ν provided that they lie in certain polyhedral cones of a chamber complex. The proof by Rassart employs Steinberg's formula, the hive conditions, and the Kostant partition function to give the chamber complex of cones in which $c_{\lambda,\mu}^{\nu}$ is a polynomial in variables λ, μ, ν. A considerably large portion of Rassart's paper was devoted to describing this chamber complex and showing its desired property, resulting in a fairly long justification. We note that although this chamber complex of cones was provided, it is in practice computationally hard to work out the cones.

Inspired by Rassart's approach, we ask if similar tools can be utilized to give a simple proof of Theorem 3.0.1 directly. We found that Steinberg's formula and a simple argument about the chamber complex of the Kostant partition function are indeed sufficient. The main objective of this chapter is to give a short alternative proof of Theorem 3.0.1 using this idea.

Chapter Organization. We begin by introducing necessary notations and describing the hive model for computing $c_{\lambda,\mu}^{\nu}$. The hive model will help us understand the behavior of stretched Littlewood-Richardson coefficients through properties of associated polytopes. We then introduce the Kostant partition functions and state Steinberg's formula and related results that will later be used for proving Theorem 3.0.1. Lastly, we describe a connection of stretched Littlewood-Richardson coefficients and flow polytopes, and outline potential research problems and directions for future work.

3.1. Littlewood-Richardson Coefficients

We say that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a *partition* of a non-negative integer m if $\lambda_1 \geq \cdots \geq \lambda_k$ are positive integers such that $\lambda_1 + \cdots + \lambda_k = m$. For convenience, we will abuse the notation by allowing λ_i to be zero. The positive numbers among $\lambda_1, \ldots, \lambda_k$ are called *parts* of λ. For example, $\lambda = (2, 2, 1, 0)$ is a partition of 5 with 3 parts. We write $|\lambda|$ to denote $\lambda_1 + \cdots + \lambda_k$.

A *hive* Δ_k of size k is an array of vertices h_{ij} arranged in a triangular grid consisting of k^2 small equilateral triangles as shown in Figure 3.1. Two adjacent equilateral triangles form a rhombus with two equal obtuse angles and two equal acute angles. There are three types of these rhombi: tilted to the right, left, and vertical as shown in Figure 3.1.

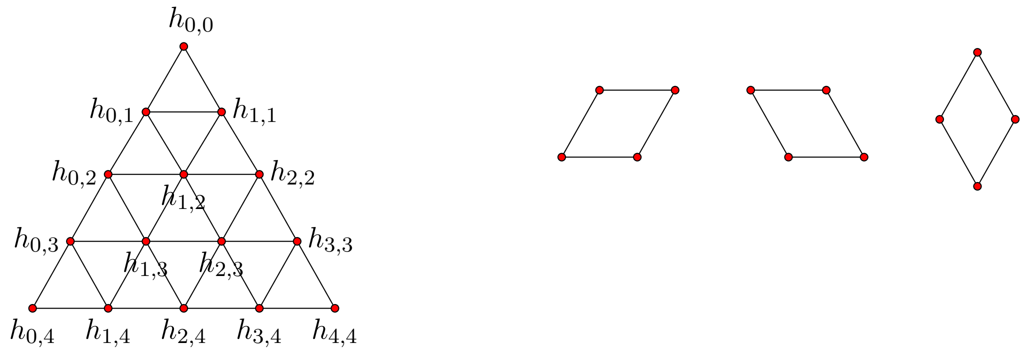


FIGURE 3.1. Hive of size 4 (left), and the three types of rhombi in a hive (right)

Let $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_k), \nu = (\nu_1, \ldots, \nu_k)$ be partitions with at most k parts such that $|\nu| = |\lambda| + |\mu|$. A *hive of type* (ν, λ, μ) is a labelling $(h_{i,j})$ of Δ_k that satisfies the following *hive conditions.*

(HC1) [Boundary condition] The labelings on the boundary are determined by λ, μ, ν in the following ways.

$$h_{0,0} = 0,\ h_{j,j} - h_{j-1,j-1} = \nu_j,\ h_{0,j} - h_{0,j-1} = \lambda_j, \qquad \text{for } 1 \leq j \leq k.$$

$$h_{i,k} - h_{i-1,k} = \mu_i, \qquad \text{for } 1 \leq i \leq k.$$

(HC2) [Rhombi condition] For every rhombus, the sum of the labels at obtuse vertices is greater than or equal to the sum of the labels at acute vertices. That is, for $1 \leq i < j \leq k$,

$$h_{i,j} - h_{i,j-1} \geq h_{i-1,j} - h_{i-1,j-1},$$

$$h_{i,j} - h_{i-1,j} \geq h_{i+1,j+1} - h_{i,j+1}, \text{ and}$$

$$h_{i-1,j} - h_{i-1,j-1} \geq h_{i,j+1} - h_{i,j}.$$

Let $H_k(\nu, \lambda, \mu)$ denote the set of all hive of type (ν, λ, μ). Then the hive conditions (HC1) and (HC2) imply that $H_k(\nu, \lambda, \mu)$ is a rational polytope in $\mathbb{R}^n$ where $n = \binom{k+2}{2}$. Hence, we will call

$H_k(\nu,\lambda,\mu)$ the *hive polytope of type* (ν,λ,μ). Knutson-Tao [26] and Buch [7] showed that

$$c^{\nu}_{\lambda,\mu} = \text{ the number of integer points in } H_k(\nu,\lambda,\mu).$$

EXAMPLE 3.1.1. Let $k=3$, $\nu=(4,3,1), \lambda=(2,1,0)$, and $\mu=(3,2,0)$. Then, in the hive Δ_3, we have by the boundary condition

$$h_{0,0}=0; h_{0,1}-h_{0,0}=2; h_{0,2}-h_{0,1}=1; h_{0,3}-h_{0,2}=0$$

$$h_{1,3}-h_{0,3}=3; h_{2,3}-h_{1,3}=2; h_{3,3}-h_{2,3}=0$$

$$h_{1,1}-h_{0,0}=3; h_{2,2}-h_{1,1}=3; h_{3,3}-h_{2,2}=1$$

Solving these equations, we obtain the boundaries of Δ_3 as shown in (one of) the hives in Figure 3.2. Thus, we only need to solve for $h_{1,2}$. Using the rhombi condition, one sees that the following two inequalities suffice for determining $h_{1,2}$:

$$h_{1,2} \geq h_{0,2}+h_{2,3}-h_{1,2}=5 \text{ and } h_{0,1}+h_{1,1}-h_{0,0}=6 \geq h_{1,2}.$$

Thus, the only integers $h_{1,2}$ satisfying the rhombi condition are 5 and 6. This implies that there are two integer points in $H_3(\nu,\lambda,\mu)$, each corresponds to an integer label of Δ_3 shown in Figure 3.2. Therefore, have that $c^{\nu}_{\lambda,\mu} = |H_3(\nu,\lambda,\mu)\cap\mathbb{R}^{10}| = 2$.

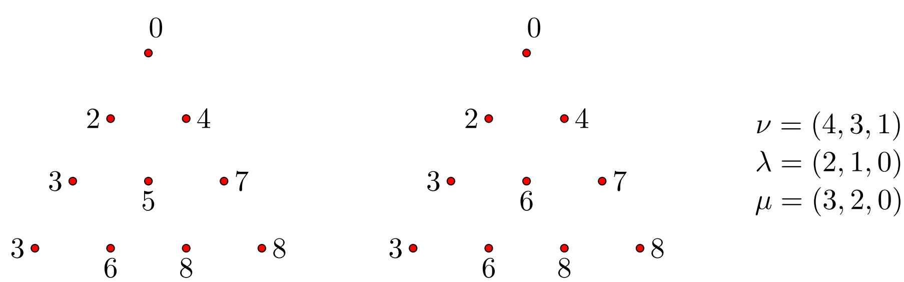


FIGURE 3.2. The only two integer points (integer labels) of $H_3(\nu,\lambda,\mu)$

For fixed partitions λ,μ,ν with at most k parts such that $|\nu|=|\lambda|+|\mu|$, we define the the *stretched* Littlewood-Richardson coefficient to be the function $c^{t\nu}_{t\lambda,t\mu}$ for non-negative integer t.

Because $H_k(t\nu, t\lambda, t\mu) = tH_k(\nu, \lambda, \mu)$, we have that

$$c^{t\nu}_{t\lambda,t\mu} = i(H_k(\nu, \lambda, \mu), t).$$

REMARK 3.1.2. Examples provided in [25] indicate that $H_k(\nu, \lambda, \mu)$ is in general not an integral polytope. Thus, by Ehrhart theory (Theorem 1.3.1), $c^{t\nu}_{t\lambda,t\mu}$ is a quasi-polynomial in t.

3.2. Kostant Partition Function and Steinberg's Formula

We will show the polynomiality of $c^{t\nu}_{t\lambda,t\mu}$ by using Steinberg's formula as derived in [37] by Rassart and the chamber complex of the Kostant partition function. To this end, we state the related notations and results for later reference.

Let $e_1, \dots, e_k$ be the standard basis vectors in $\mathbb{R}^k$, and let $\Delta_+ = \{e_i - e_j : 1 \le i < j \le k\}$ be the set of positive roots of the root system of type A_{k-1}. We define M to be the matrix whose columns consist of the elements of Δ_+. The *Kostant partition function* for the root system of type A_{k-1} is the function $K : \mathbb{Z}^k \longrightarrow \mathbb{Z}_{\ge 0}$ defined by

$$K(v) = \left| \left\{ b \in \mathbb{Z}_{\ge 0}^{\binom{k}{2}} \mid Mb = v \right\} \right|.$$

That is, $K(v)$ equals the number of ways to write v as nonnegative integer linear combinations of the positive roots in Δ_+.

An important property of the matrix M, when written in the basis of simple roots $\{e_i - e_{i+1} \mid i = 1, \dots, k-1\}$, is that it is totally unimodular, i.e., the determinant of every square submatrix equals $-1, 0$, or 1. Indeed, it is shown in [39] that a matrix A is totally unimodular if every column of A only consists of 0's and 1's in a way that the 1's come in a consecutive block. Let

$$\mathrm{cone}(\Delta_+) = \left\{ \sum \lambda_v v \mid v \in \Delta_+, \lambda_v \ge 0 \right\}$$

be the cone spanned by the vectors in Δ_+. The *chamber complex* is the polyhedral subdivision of $\mathrm{cone}(\Delta_+)$ that is obtained from the common refinement of cones $\mathrm{cone}(B)$ where B are the maximum linearly independent subsets of Δ_+. A maximum cell (a cone of maximum dimension) $\mathcal{C}$ in the chamber complex is called a *chamber*. Since M is totally unimodular, the behavior of $K(v)$ is given by the following lemma as a special case of [47, Theorem 1] due to Sturmfels.

LEMMA 3.2.1. *Let $\mathcal{C}$ be a chamber in the chamber complex of* $\mathrm{cone}(\Delta_+)$. *Then the Kostant partition function $K(v)$ is a polynomial in $v = (v_1, \ldots, v_k)$ on $\mathcal{C}$ of degree at most $\binom{k-1}{2}$.*

Steinberg's formula [46] expresses the tensor product of two irreducible representations of semisimple Lie algebras as the direct sum of other irreducible representations. When restricting the formula to $\mathrm{SL}_k\mathbb{C}$, we obtain the following version of Steinberg's formula for computing $c^{\nu}_{\lambda,\mu}$.

LEMMA 3.2.2 (Steinberg's Formula). *Let μ, λ, ν be partitions with at most k parts such that $|\nu| = |\lambda| + |\mu|$. Then*

$$c^{\nu}_{\lambda,\mu} = \sum_{\sigma,\tau\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma\tau)} K(\sigma(\lambda+\delta) + \tau(\mu+\delta) - (\nu + 2\delta))$$

where $\mathrm{inv}(\psi)$ *is the number of inversions of the permutation ψ and*

$$\delta = \frac{1}{2}\sum_{1\le i<j\le k}(e_i - e_j) = \frac{1}{2}(k-1, k-3, \ldots, -(k-3), -(k-1))$$

is the Weyl vector for type A_{k-1}.

Details of the derivation can be found in [37, section 1.1].

3.3. Proof of the Polynomiality

We are now ready to prove Theorem 3.0.1.

PROOF OF THEOREM 3.0.1. The hive conditions imply that $c^{t\nu}_{t\lambda,t\mu}$ is a quasi-polynomial in t. To see that $c^{t\nu}_{t\lambda,t\mu}$ is in fact a polynomial in t, it suffices to show that there exists an integer N such that $c^{t\nu}_{t\lambda,t\mu}$ is a polynomial in t for $t \ge N$.

For $\sigma, \tau \in \mathfrak{S}_k$, let

$$\begin{aligned} r^{\lambda,\mu,\nu}_{\sigma,\tau}(t) &:= \sigma(t\lambda+\delta) + \tau(t\mu+\delta) - (t\nu+2\delta) \\ &= t(\sigma(\lambda)+\tau(\mu)-\nu) + \sigma(\delta) + \tau(\delta) - 2\delta. \end{aligned}$$

Then $r^{\lambda,\mu,\nu}_{\sigma,\tau}(t)$ is a ray (when allowing t to be a non-negative real number) emanating from $\sigma(\delta) + \tau(\delta) - 2\delta$ in the direction of $\sigma(\lambda) + \tau(\mu) - \nu$.

By Steinberg's formula,

$$c_{t\lambda,t\mu}^{t\nu} = \sum_{\sigma,\tau\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma\tau)} K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t)).$$

Lemma 3.2.1 states that $K(v)$ is a polynomial in v when v stays in one particular cone (chamber) of the chamber complex of cone(Δ_+). Because there are only finitely many cones in the chamber complex, we have that for every pair $\sigma, \tau \in \mathfrak{S}_k$ there exists an integer $N_{\sigma,\tau}^{\lambda,\mu,\nu}$ such that exactly one of the following happens:

(1) The ray $r_{\sigma,\tau}^{\lambda,\mu,\nu}(t)$ lies in one particular cone of the chamber complex for all $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$

(2) The ray $r_{\sigma,\tau}^{\lambda,\mu,\nu}(t)$ lies outside cone(Δ_+) for all $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$.

If (1) is satisfied, then $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is a polynomial in t for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$. If (2) is satisfied, then $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is the zero polynomial for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$. In either case, $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is a polynomial in t for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$. Now let

$$N = \max_{\sigma,\tau\in\mathfrak{S}_k} \{N_{\sigma,\tau}^{\lambda,\mu,\nu}\}.$$

Then Steinberg's formula implies that $c_{t\lambda,t\mu}^{t\nu}$ is a polynomial in t for $t \geq N$. Therefore, $c_{t\lambda,t\mu}^{t\nu}$ is a polynomial in t.

By Lemma 3.2.1, each polynomial piece of $K(v)$ has degree at most $\binom{k-1}{2}$. Thus, for every σ, τ, we have that $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is a polynomial in t of degree at most $\binom{k-1}{2}$ for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$. Hence, $c_{t\lambda,t\mu}^{t\nu}$ is a polynomial in t of degree at most $\binom{k-1}{2}$. □

In the proof of Theorem 3.0.1, we showed that every $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ eventually becomes either the zero polynomial or a non-zero polynomial in t. A characterization of those $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ that eventually become non-zero polynomials will be given in Proposition 3.3.2. Its proof uses the following characterization of non-zero $K(v)$.

Lemma 3.3.1. *Let $v = (v_1, \ldots, v_k)$ be a vector in $\mathbb{Z}^k$ with $v_1 + \cdots + v_k = 0$. Then $K(v)$ is non-zero if and only if $v_1 + \cdots + v_i \geq 0$ for all $i = 1, \ldots, k$.*

Proof. Let M^* be the matrix M written using the simple roots $e_1 - e_2, \ldots, e_{k-1} - e_k$ as a basis. Then, the entries of M^* are only 0 and 1. Moreover, because the simple roots themselves are columns of M, we have that the identity matrix is a submatrix of M^*. Similarly, let v^* be the vector v written using the simple roots as a basis. Then, $v^* = (v_1, v_1 + v_2, \dots, v_1 + \cdots v_{k-1})$. The desired result is obtained by observing that

$$K(v) = \left| \left\{ b \in \mathbb{Z}_{\geq 0}^{\binom{k}{2}} \mid M^* b = v^* \right\} \right|.$$

□

PROPOSITION 3.3.2. *Let μ, λ, ν be partitions with at most k part such that $|\nu| = |\lambda| + |\mu|$. For $\sigma, \tau \in \mathfrak{S}_k$, let*

$$r_{\sigma,\tau}^{\lambda,\mu,\nu}(t) = t\beta + \gamma$$

where $\beta = \sigma(\lambda) + \tau(\mu) - \nu$ and $\gamma = \sigma(\delta) + \tau(\mu) - 2\delta$. Then there exists an integer $N_{\sigma,\tau}^{\lambda,\mu,\nu}$ such that $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is a non-zero polynomial in t for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$ if and only if for all $i = 1, \dots, k$ we have that

(1) $\beta_1 + \beta_2 + \cdots + \beta_i$ is positive, or

(2) $\beta_1 + \beta_2 + \cdots + \beta_i$ is zero and $\gamma_1 + \gamma_2 + \cdots + \gamma_i$ is non-negative.

PROOF. Let $r_{\sigma,\tau}^{\lambda,\mu,\nu}(t) = (r_1(t), \dots, r_k(t))$. Then $r_i(t) = t\beta_i + \gamma_i$. In the proof of Theorem 3.0.1, we showed that there exists a positive integer $N_{\sigma,\tau}^{\lambda,\mu,\nu}$ such that $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is a polynomial in t for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$. For every $i = 1, \dots, k$, the partial sum $r_1(t) + \cdots + r_i(t)$ is non-negative for all $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$ precisely when one of the two conditions meets for all $i = 1, \dots, k$. Thus, by Lemma 3.3.1, $K(r_{\sigma,\tau}^{\lambda,\mu,\nu}(t))$ is a non-zero polynomial for $t \geq N_{\sigma,\tau}^{\lambda,\mu,\nu}$. □

3.4. Connection to Flow Polytopes and Other Problems

After having gained a deeper understanding of the Ehrhart polynomial $c_{t\lambda,t\mu}^{t\nu}$, we now turn our attention to its coefficients. In [24], King, Tollu, and Taumazet proposed the following conjecture, which still remains unsolved.

CONJECTURE 3.4.1. *For $c_{\lambda,\mu}^{\nu} > 0$, every coefficient in the polynomial $c_{t\lambda,t\mu}^{t\nu}$ is positive, i.e., the hive polytope $H_k(\nu, \lambda, \mu)$ is Ehrhart positive.*

One difficulty in solving this statement is the lack of a general formula for $K(v)$ that allows us to see the positivity in Steinberg's formula. In fact, this is a common obstacle encountered when attempting to show Ehrhart positivity of any polytope.

The author's effort to develop mathematical tools for tackling this problem has revealed the connection of $c^{\nu}_{\lambda,\mu}$ to other families of polytopes. When plugging, for example, $\lambda = \mu = (k-1, k-2, \dots, 1)$ and $\nu = (2k-3, 2(k-2), 2(k-3), \dots, 2, 1)$ into Steinberg's formula, $c^{t\nu}_{t\lambda,t\mu}$ agrees with the Ehrhart polynomial of a *Chan-Robbins-Yuen (CRY) polytope*, which will be denoted by CRY_k. The polytope CRY_k is an example of a *flow polytope*, defined as the convex hull of the flows conserving a specific net flow at every vertex of the directed complete graph on k vertices (see [**30**]). It stands out as the simplest yet most significant example in the sense that other flow polytopes defined on complete graphs are Minkowski sums of CRY polytopes. Morales [**32**] conjectured that

CONJECTURE 3.4.2. *The polytope* CRY_k *is Ehrhart positive for all* k.

Note that there exist formulas of the Ehrhart polynomials for flow polytopes known as Baldoni–Vergne–Lidskii formulas [**2**]. However, they do not readily reveal the positivity, similar to the difficulty encountered with Steinberg's formula.

Beyond Ehrhart positivity, one can ultimately ask for combinatorial interpretations of Ehrhart polynomials' coefficients. It is known that the normalized volume of CRY_k, which equals a multiple of the leading coefficient of its Ehrhart polynomial, is a product of consecutive Catalan numbers [**11**]. It is then natural to ask if a similar phenomena occurs with other coefficients.

PROBLEM 3.4.3. Find combinatorial interpretations of the coefficients of CRY polytopes' Ehrhart polynomials.

Finally, we note that Steinberg's formula in [**46**] provides a method for computing the multiplicities $C^{\nu}_{\lambda,\mu}$, called the *Clebsch-Gordan coefficients*, of the tensor product of two irreducible representations of semisimple Lie algebras, and that the formula given in Lemma 3.2.2 is a special case where the formula in [**46**] is restricted to type A_{k-1} Lie algebras. When restricted to other classical Lie algebras (types B_k, C_k, and D_k Lie algebras), Berenstein and Zelevinsky showed in [**5**] that the stretched Clebsch-Gordan coefficient $C^{t\nu}_{t\lambda,t\mu}$ equals the Ehrhart quasi-polynomial of a rational polytope, referred to as *BZ-polytope*. Based on computational evidence, De Loera and McAllister later proposed a conjecture in [**13**] regarding the following property the quasi-polynomial $C^{t\nu}_{t\lambda,t\mu}$.

CONJECTURE 3.4.4. *Every coefficient of the stretched Clebsch-Gordan coefficient* $C^{t\nu}_{t\lambda,t\mu}$ *is nonnegative.*

CHAPTER 4

Parking Function Polytopes

Suppose that $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n_{\geq 0}$ is a vector satisfying $0 \leq u_1 \leq \cdots \leq u_n$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be the non-decreasing rearrangement of $a_1, \ldots, a_n$. We say that $\mathbf{a}$ is a $\mathbf{u}$-*parking function* if $b_i \leq u_i$ for all $i = 1, \ldots, n$. The *parking function polytope* associated to $\mathbf{u}$, denoted by $\mathrm{PF}(\mathbf{u})$, is defined to be the convex hull of all $\mathbf{u}$-parking functions. For a nonzero vector $\mathbf{u}$, the polytope $\mathrm{PF}(\mathbf{u})$ contains $n+1$ affinely independent points $\mathbf{0}$, $(u_n, 0, 0, \ldots, 0), (0, u_n, 0, \ldots, 0), \ldots,$ $(0, 0, \ldots, 0, u_n)$. This means that $\mathrm{PF}(\mathbf{u})$ is n-dimensional for all $\mathbf{u} \in \mathbb{R}^n_{\geq 0} \backslash \{\mathbf{0}\}$. Thus, for non-triviality, we will always assume that $\mathbf{u}$ is a nonzero vector.

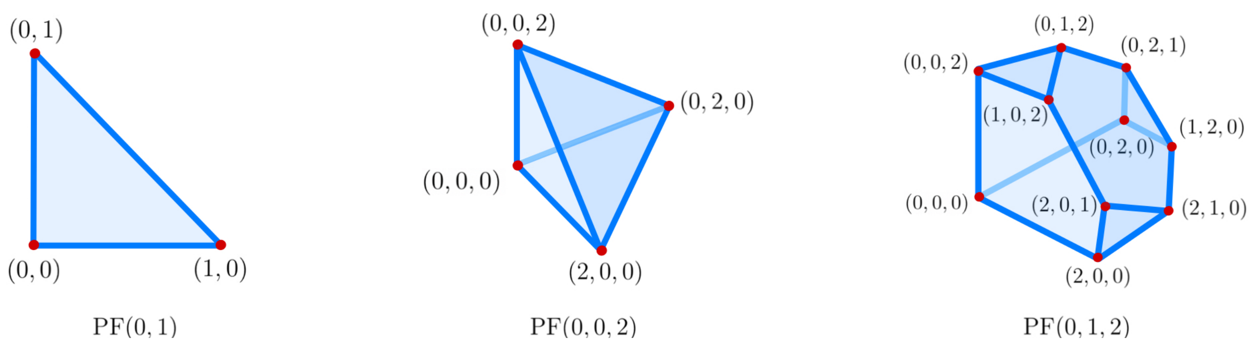


FIGURE 4.1. Three examples of parking function polytopes

We note that a parking function (of length n) was originally defined as a sequence of positive integers $(a_1, \ldots, a_n)$ such that its non-decreasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq i$ for all $i \in [n]$ where $[n] := \{1, 2, \ldots, n\}$. It is a fascinating combinatorial object closely connected to other combinatorial models such as labeled trees [10], hyperplane arrangement, and non-crossing partitions [41, 42]. The name "parking function" originates from Konheim and Weiss [27], who introduced it as a way to choose n spots for parking n cars. Stanley later defined parking function polytopes to be the convex hull of all such parking functions in [44, Problem 12191], which corresponds, in our notation, to $\mathrm{PF}(0, 1, \ldots, n-1)$. He also posed questions regarding their faces, volume, and number of lattice points, which were subsequently answered by Amanbayeva and Wang [1]. Recently, Hanada et al. [23], and Bayer et al. [4] examined a larger class of parking function polytopes $\mathrm{PF}(\mathbf{u})$ where $u_1, \dots, u_n$ are integers satisfying $0 \le u_1 < \cdots < u_n$. Their work focused on the combinatorial properties of these polytopes, providing formulas for volume and h-polynomials, and exploring connections to other polytopes. One sees that our definition of parking function polytopes further generalizes this notion by allowing $0 \le u_1 \le \cdots \le u_n$ to be any non-decreasing real numbers, rather than strictly increasing integers.

Chapter organization. In this chapter, we aim to describe the normal fans, face posets, h-polynomials, and Ehrhart polynomials of parking function polytopes, and present related findings. We begin with an overview of fundamental concepts related to preposets, and preorder cones, and then introduce binary partitions and skewed binary partitions as generalizations of ordered set partitions. Following this, we develop tools to characterize the family of skewed binary partitions that corresponds bijectively to the normal fan of a parking function polytope, and express the h-polynomials of simple parking function polytopes in terms of generalized Eulerian polynomials. In the last section, we describe connections between parking function polytopes and other families of polytopes, and deduce several results from these connections, including the formulas for volumes and Ehrhart polynomials.

4.1. Preposets and preorder cones

We introduce the notion of preposets which is, in a sense, a generalization of posets, and then introduce their associated preorder cones. Readers are expected to be familiar with basic notations regarding poset as appear, for example, in [43, Section 3.1].

A binary operator $\preceq$ on a finite set A is called a *preorder* if it is reflexive and transitive on A. A *preposet* is an ordered pair $(A, \preceq)$ of a finite set A and a preorder $\preceq$ on it. We write $i \equiv j$ if $i \preceq j$ and $j \preceq i$. The relation $\equiv$ is an equivalence relation on A and thus partitions A into equivalence classes. We denote by $A/_{\equiv}$ the set of equivalence classes of A and $\bar{i}$ the equivalence class of i. One sees we recover the definition of a poset if we require a preposet $(A, \preceq)$ satisfies that $i \equiv j$ if and only if $i = j$, i.e., the relation $\equiv$ is antisymmetric.

Note that the preorder $\preceq$ on A induces a partial order on $A/_\equiv$ by letting $\bar{i} \preceq \bar{j}$ if $i \preceq j$ in A, and thus defines a poset $(A/_\equiv, \preceq)$ which is closely related to the preposet $(A, \preceq)$. This allows us to define several concepts for the preposet $(A, \preceq)$ from the concepts for the poset $(A/_\equiv, \preceq)$. For instance, we say that j is a *cover* of i in the preposet $(A, \preceq)$, denoted $i \lessdot j$, if $\bar{i}$ is a cover of $\bar{j}$ in the poset $(A/_\equiv, \preceq)$. The *Hasse diagram* of a preposet $(A, \preceq)$ is the Hasse diagram of the poset $(A/_\equiv, \preceq)$ except that, when labeling each node by equivalence classes $\bar{i}$, we remove the parentheses about the set.

A preoder $\preceq_1$ on A is said to be a *contraction* of another preorder $\preceq_2$ on A if the Hasse diagram of $(A, \preceq_1)$ can be obtained by a sequence of edge contractions of the Hasse diagram and merges of the vertex labels of $(A, \preceq_2)$. If $(A, \preceq_1)$ and $(A, \preceq_2)$ are two distinct preposets on A, then $(A, \preceq_1)$ is a contraction of $(A, \preceq_2)$ if and only if $(A, \preceq_1)$ can be obtained by imposing additional relations $j \preceq_1 i$ on $(A, \preceq_1)$ for various $i \lessdot_1 j$.

Example 4.1.1. We draw in Figure 4.2 Hasse diagrams of three different preposets on $[0, 8]$, among which $([0, 8], \preceq_1)$ is a poset. The preorder $\preceq_2$ is a contraction of preorder $\preceq_1$ by contracting the edge $6 - 8$ and the edge $5 - 7$. The preorder $\preceq_3$ is a contraction of the preorder $\preceq_2$ by contracting the edge $3 - 0$. As a result, the preorder $\preceq_3$ is also a contraction of the preorder $\preceq_1$.

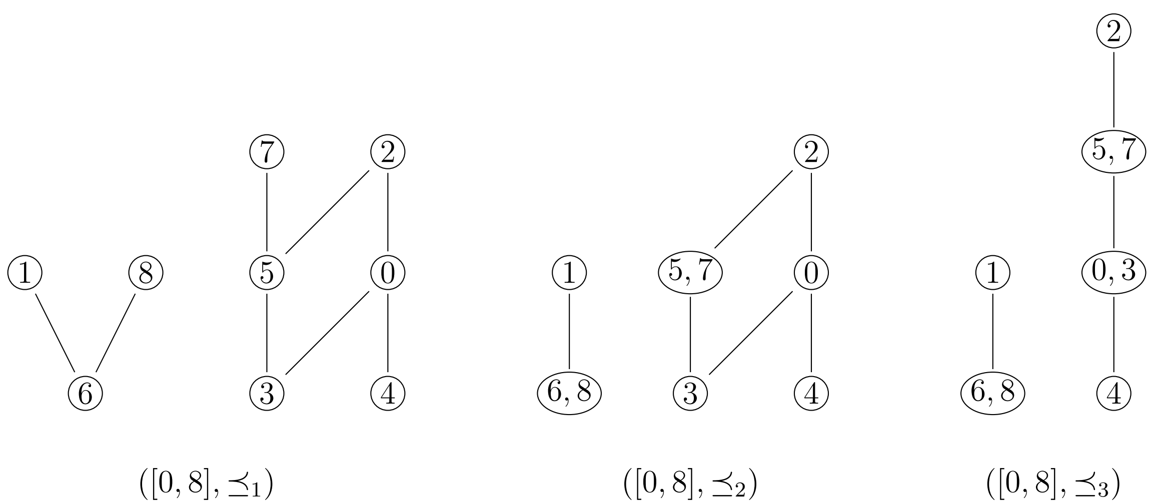


Figure 4.2. Both $\preceq_2$ and $\preceq_3$ are contractions of $\preceq_1$

A map f from a preposet $(A_1, \preceq_1)$ to another preposet $(A_2, \preceq_2)$ is *order-preserving* if for every $x, y \in A_1$ such that $x \preceq_1 y$ one has $f(x) \preceq_2 f(y)$. If an order-preserving map f is bijective and its inverse is also order-preserving, we say that f is an *isomorphism*.

The *dual* of a preposet $(A, \preceq)$ is the preposet $(A, \preceq^*)$ such that $i \preceq^* j$ if and only if $j \preceq i$. Clearly, the Hasse diagram of the dual poset $(A, \preceq^*)$ is obtained by turning the Hasse diagram of $(A, \preceq)$ upside down.

In [36, Section 3], Postnikov, Reiner, and Williams introduce a natural correspondence between cones in quotient space $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$ and preorders of the set $[n]$ in the study of faces of generalized permutahedra. Later, Castillo and Liu [9] call these cones *preorder cones*, indicating that they arise from some preposets. They also introduce variations of preorder cones, including ones that are defined in the first orththant of $\mathbb{R}^n$. In this chapter, we start with a preposet on $[0, n]$ and consider preorder cones without quotienting out $(1, \ldots, 1)\mathbb{R}$. More precisely, given a preposet $([0, n], \preceq)$, we define its associated *preorder cone* to be the cone

$$\sigma_{\preceq} := \{(c_0, c_1, \ldots, c_n) \in \mathbb{R}^{n+1} \mid c_i \leq c_j \text{ if } i \preceq j, i, j \in [0, n]\}. \tag{4.1.1}$$

We will utilize preorder cones to study the face structure of parking function polytopes. However, it turns out that the slice of $\sigma_{\preceq}$ at $c_0 = 0$ will mostly play an important role. This leads us to introduce the following definition.

Definition 4.1.2. Let $\preceq$ be a preorder on $[0, n]$. The *sliced preorder cone* $\tilde{\sigma}_{\preceq}$ associated to $\preceq$ is given by

$$\tilde{\sigma}_{\preceq} := \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_i \leq c_j \text{ if } i \preceq j, i, j \in [0, n]\}. \tag{4.1.2}$$

Example 4.1.3. Let $\preceq$ be the third preorder $\preceq_3$ on $[0, 8]$ shown in Figure 4.2. Then

$$\tilde{\sigma}_{\preceq} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_6 = c_8 \leq c_1 \text{ and } c_4 < 0 = c_3 < c_5 = c_7 < c_2\}.$$

A *linear extension* of the preposet $([0, n], \preceq)$ is a bijective order-preserving map from the preposet $([0, n], \preceq)$ to the poset $([0, n], \leq)$ where $\leq$ is the usual order of integers. We denote by $L(\preceq)$ the set of all linear extensions of the preposet $([0, n], \preceq)$.

The next lemma contains a variation of results from [36, Proposition 3.5] and [9] regarding sliced preorder cones.

Lemma 4.1.4. *Let $\preceq$ and $\preceq'$ be preorders on $[0,n]$. We have that*

(1) The sliced preorder cone $\tilde{\sigma}_{\preceq'}$ is a face of the sliced preorder cone $\tilde{\sigma}_{\preceq}$ if and only if $\preceq'$ is a contraction of $\preceq$.

(2) If a preposet $([0,n], \preceq)$ is a poset, then the associated sliced preorder cone has the following minimal inequality description:

$$\tilde{\sigma}_{\preceq} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_i \leq c_j \text{ if } i \lessdot j, i, j \in [0,n]\}$$

and, hence, the relative interior $\tilde{\sigma}^{\circ}_{\preceq}$ of $\tilde{\sigma}_{\preceq}$ is given by

$$\tilde{\sigma}^{\circ}_{\preceq} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_i < c_j \text{ if } i \lessdot j, i, j \in [0,n]\}.$$

(3) The dimension of the sliced preorder cone $\tilde{\sigma}_{\preceq}$ is the number of equivalence classes in $([0,n], \preceq)$ minus 1.

(4) The cone $\tilde{\sigma}_{\preceq}$ is pointed if and only if the Hasse diagram of $([0,n], \preceq)$ is a connected graph.

(5) The sliced preorder cone $\tilde{\sigma}_{\preceq}$ is an n-dimensional simplicial cone if and only if $([0,n], \preceq)$ is a poset and its Hasse diagram is a tree (a connected graph with no cycles).

(6) If a preposet $([0,n], \preceq)$ is a poset, then

$$\tilde{\sigma}_{\preceq} = \bigcup_{\pi \in L[\preceq]} \tilde{\sigma}(\pi)$$

where

$$\tilde{\sigma}(\pi) := \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_{\pi(0)} \leq c_{\pi(1)} \leq \cdots \leq c_{\pi(n)}\}. \tag{4.1.3}$$

The proofs of these results can be obtained by setting $c_0 = 0$ in the proofs of the original results from [36, Proposition 3.5], and [9]. We provide their proofs here for completeness.

Proof. (1) A face of $\tilde{\sigma}_{\preceq}$ is obtained by replacing some inequalities $c_i \leq c_j$, $i, j \in [0,n]$, defining $\tilde{\sigma}_{\preceq}$ with equalities $c_i = c_j$, or equivalently, by adding the opposite inequalities $c_i \geq c_j$. Suppose that $i = i_0 \lessdot i_1 \lessdot \cdots \lessdot i_k = j$ be a maximal chain from i to j, i.e., i_t is a cover of i_{t-1} for all $t \in [k]$.

Then, by adding the inequality $c_i \geq c_j$ to $\tilde{\sigma}_{\preceq}$, the resulting sliced preorder cone corresponds to the preorder on $[0, n]$ that is obtained by contracting the edges of connecting i_{t-1} and i_t for all $i \in [k]$ in the Hasse diagram of $([0, n], \preceq)$. This implies that $\tilde{\sigma}_{\preceq'}$ is a face of $\tilde{\sigma}_{\preceq}$.

Conversely, suppose that $\tilde{\sigma}_{\preceq'}$ is a contraction of $\tilde{\sigma}_{\preceq}$. Then, $([0, n], \preceq')$ can be obtained by imposing additional relations $j \preceq_1 i$ on $([0, n], \preceq)$ for various $i \lessdot j$. Thus, $\tilde{\sigma}_{\preceq'}$ is obtained by adding the inequalities $c_i \geq c_j$ to $\tilde{\sigma}_{\preceq}$ for all such $i \preceq j$. Thus, $\tilde{\sigma}_{\preceq'}$ is a face of $\tilde{\sigma}_{\preceq}$.

(2) Suppose that $([0, n], \preceq)$ is a poset. That is, every equivalence class of $(A/_{\equiv}, \preceq)$ is a singleton. We note that a set of covering relations uniquely defines a poset. This implies that

$$\tilde{\sigma}_{\preceq} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_i \leq c_j \text{ if } i \lessdot j, i, j \in [0, n]\}$$

and that this is the least number of inequalities to define $\tilde{\sigma}_{\preceq}$. Thus, for every i, j such that $i \lessdot j$, the inequality $c_i \leq c_j$ is facet-defining. Hence, the relative interior of $\tilde{\sigma}_{\preceq}$ is given by

$$\tilde{\sigma}^{\circ}_{\preceq} = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_i < c_j \text{ if } i \lessdot j, i, j \in [0, n]\}.$$

(3) We first show that if $([0, n], \preceq)$ is a poset, then $\tilde{\sigma}_{\preceq}$ is (full) n-dimensional. For $\tilde{\sigma}_{\preceq}$ to be full dimensional, its defining relations must not include any $c_i = c_j$ for $i \neq j$. This is equivalent to requiring that every equivalence class of $(A/_{\equiv}, \preceq)$ is a singleton, i.e., $([0, n], \preceq)$ is a poset. Thus, if $([0, n], \preceq)$ is a poset, then the dimension of $\tilde{\sigma}_{\preceq}$ is the number of equivalence classes in $([0, n], \preceq)$ minus 1.

Now suppose that $([0, n], \preceq)$ is not a poset. Let $S \subseteq [0, n]$ be a set of representative of the equivalence classes of $(A/_{\equiv}, \preceq)$ such that $0 \in S$. We define the $(S, \preceq_S)$ to be the induced preposet on S, i.e., $i \preceq_S j$ if and only if $i \preceq j$. The sliced preorder cones $\tilde{\sigma}_{\preceq}$ and $\tilde{\sigma}_{\preceq_S} := \{(c_i)_{i \in S} \in \mathbb{R}^S \mid c_0 = 0 \text{ and } c_i \leq c_j \text{ if } i \preceq_S j, i, j \in S\} \subset \mathbb{R}^S$ have the same dimension, since the map $\varphi : R^n \longrightarrow \mathbb{R}^S$ be given by $\varphi(x_1, \ldots, x_n) = (x_i)_{i \in S} \in R^S$ is a linear bijection between $\tilde{\sigma}_{\sigma}$ and $\tilde{\sigma}_{\sigma_S}$. By the construction, $(S, \preceq_S)$ is a poset and, hence, we must have

$$\dim(\sigma_{\preceq}) = \dim(\sigma_{\preceq_S}) = |S| - 1 = \#(\text{ equivalence classes in } ([0, n], \preceq)) - 1.$$

(4) The maximal subspace in the half-space $H := \{(c_1, \ldots, c_n) \mid c_i \leq c_j\}$ is given by $c_i = c_j$. Thus, the maximal subspace contained in the cone $\tilde{\sigma}_{\preceq}$ is the intersection of all subspaces defined

by $c_i = c_j$ for $i \preceq j$. Suppose that the Hasse diagram of $([0,n], \preceq)$ is a connected graph. Then, the maximal subspace in $\tilde{\sigma}_{\preceq}$ is given by $0 = c_0 = c_1 = \cdots = c_n$. That is, the only subspace of $\mathbb{R}^n$ contained in $\tilde{\sigma}_{\preceq}$ is the trivial subspace. Hence, $\tilde{\sigma}_{\preceq}$ is pointed.

Conversely, suppose that Hasse diagram of $([0,n], \preceq)$ is not a connected graph. Let $S_1, \ldots, S_k$, where $k \geq 2$, be its connected components, i.e., $S_1, \ldots, S_k$ are disjoint nonempty subset of $[0,n]$ such that $S_1 \cup \cdots \cup S_k = [0,n]$. Then, the maximal subspace U contained in $\tilde{\sigma}_{\preceq}$ is the intersection of the subspaces $U_1, \ldots, U_k$ where $U_t := \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_0 = 0 \text{ and } c_i = c_j \text{ for } i,j \in S_t\}$. Note that, for $t \in [k]$, the dimension of U_t equals $n - |S_t| + 1$ for all $t \in [k]$. Thus, the dimension of U equals $n - (|S_1| + \cdots |S_k| - k) = k - 1 \geq 1$. Hence, $\tilde{\sigma}$ contains the subspace U of dimension at least 1, implying that $\tilde{\sigma}$ is not pointed.

(5) Suppose that $\tilde{\sigma}_{\preceq}$ is an n-dimensional simplicial cone. Then, $\tilde{\sigma}_{\preceq}$ is pointed. Thus, by (4), the Hasse diagram of $([0,n], \preceq)$ is a connected graph. Since the dimension of $\tilde{\sigma}_{\preceq}$ is n, it follows from (3) that there are exactly $n+1$ equivalence classes in $([0,n], \preceq)$. This implies that $([0,n], \preceq)$ is a poset. We note that the simplicity of $\tilde{\sigma}_{\preceq}$ implies that $\tilde{\sigma}_{\preceq}$ can be described by exactly n inequalities. Using (2), we see that the poset $([0,n], \preceq)$ must have exactly n distinct covering relations. Hence, the Hasse diagram of $([0,n], \preceq)$ must be a tree. The converse of the statement also follows from a similar argument.

(6) This follows from (2) and the definition of $\sigma_{\preceq}$.

□

4.2. Binary partition and contraction

In this section, we consider a special family of preorders on $[0,n]$ that can be represented by what we call *binary partitions* of $[0,n]$. We will then characterize the contractions of these preorders in terms of binary partitions. In the next section, we will consider special cases of these partitions that will be useful for describing the normal cones of parking function polytopes.

Recall that an ordered partition of a nonempty set S is a tuple $\mathcal{B} = (B_1, \ldots, B_k)$ of nonempty disjoint subsets of S such that $B_1 \sqcup \cdots \sqcup B_k = S$. Each subset B_i is called a *block*. To represent a special family of preorders on $[0,n]$, we introduce an analogue of ordered partition called "binary partition" of the set $S = [0, n]$ by separating blocks into two different types: *homogeneous*, and *non-homogeneous*. These block types will be useful for expressing the inequality description of preorder cones. A homogeneous block is marked with a superscript $\star$ for differentiation; for example, $\{1, 3\}^\star$ is a homogeneous block. We are allowed to apply usual set operations such as union and intersection to homogeneous and non-homogeneous blocks as we normally do to regular sets.

DEFINITION 4.2.1. Let $k \in \mathbb{P}$. A *binary partition* of $[0, n]$ into k blocks is an ordered tuple $(B_1, \ldots, B_k)$ of nonempty disjoint subsets of $[0, n]$ such that $B_1 \sqcup \cdots \sqcup B_k = [0, n]$ and satisfies the following additional properties.

(1) Every block is either homogeneous or non-homogeneous.
(2) Every singleton block is non-homogeneous.

REMARK 4.2.2. For a singleton block, it has the property of both homogenous and non-homogeneous blocks. However, because we do not want to allow both kinds, we make a choice to make it always non-homogenous. Hence, we have Condition (2) in the above definition.

DEFINITION 4.2.3. For each binary partition $\mathcal{B} = (B_1, \ldots, B_k)$ of $[0, n]$, we associate the preorder $\preceq_\mathcal{B}$ on the set $[0, n]$ by letting

$$\begin{aligned} p &\preceq_\mathcal{B} q && \text{if } p \in B_i \text{ and } q \in B_j \text{ and } i < j \\ p &\equiv_\mathcal{B} q && \text{if } p, q \in B_i \text{ for some homogeneous block } B_i. \end{aligned}$$

If a preposet $([0, n], \preceq)$ satisfies $([0, n], \preceq) = ([0, n], \preceq_\mathcal{B})$ for some binary partition $\mathcal{B}$, then we say that the preorder $\preceq$ is *representable*.

A binary partition $\mathcal{C}$ is a *contraction* of another binary partition $\mathcal{B}$, denoted by $\mathcal{C} \leq \mathcal{B}$, if $\preceq_\mathcal{C}$ is a contraction of $\preceq_\mathcal{B}$.

EXAMPLE 4.2.4. Figure 4.3 shows preorders $\preceq_\mathcal{B}, \preceq_\mathcal{C}$ and $\preceq_\mathcal{D}$ associated to the binary paritions

$$\begin{aligned} \mathcal{B} &= (\{0, 2, 3\}, \{1, 6, 7\}, \{8\}, \{4, 5\}), \\ \mathcal{C} &= (\{1, 2, 5\}, \{3, 6\}^\star, \{7\}, \{0, 4\}^\star, \{8\}), \text{ and} \\ \mathcal{D} &= (\{2, 3\}, \{0, 7\}^\star, \{6\}, \{1, 8\}^\star, \{4, 5\}), \text{ respectively.} \end{aligned}$$

It is not difficult to see that $\preceq_{\mathcal{D}}$ is a contraction of $\preceq_{\mathcal{B}}$ and so $\mathcal{D} \leq \mathcal{B}$.

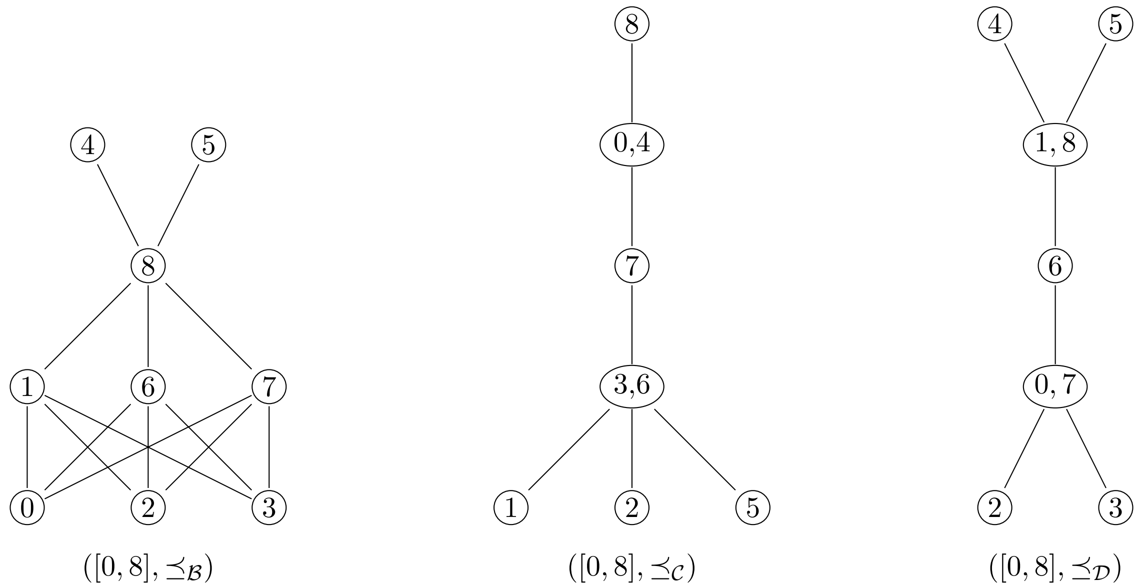


FIGURE 4.3. The preorders associated to $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ in Example 4.2.4

One sees that labeling a block as being homogeneous is simply a way to represent an equivalent class of a preposet. Not every preorder on $[0, n]$ is representable by a binary partition. In fact, a preorder on $[0, n]$ is representable by a binary partition if and only if it induces a graded poset with the properties $\bar{i} \preceq \bar{j}$ if the rank of $\bar{j}$ is higher than the rank of $\bar{i}$, and every equivalent class of size at least two is comparable with all other equivalent classes.

LEMMA 4.2.5. *Every contraction of a representable preorder on* $[0, n]$ *is representable.*

PROOF. Suppose that a preorder is representable by $\mathcal{B} = (B_1, \ldots, B_k)$. To prove the statement, it suffices to show that contracting one edge of the Hasse diagram of $([0, n], \preceq_{\mathcal{B}})$ gives a preorder $([0, n], \preceq_C)$ for some binary partition $\mathcal{C}$.

Recall that the nodes of the Hasse diagram of a preorder are equivalent classes. Consider the preorder $([0, n], \preceq)$ obtained by contracting an edge $\overline{g} - \overline{h}$ of the Hasse diagram of $(\preceq_{\mathcal{B}}, [0, n])$ where $\overline{g}$ and $\overline{h}$ are two equivalent classes of $[0, n]/_{\equiv_{\mathcal{B}}}$. As an edge is contracted, we see that $\overline{g}$ and $\overline{h}$ come from two consecutive blocks of $\mathcal{B}$, that is, there is a positive integer i such that $\overline{g} \subseteq B_i$ and $\overline{h} \subseteq B_{i+1}$. We note that $B_i \backslash \overline{g} = \emptyset$ (resp. $B_{i+1} \backslash \overline{h} = \emptyset$) if and only if B_i is a homogeneous or singleton block.

If both B_i and B_{i+1} are neither homogeneous nor singleton blocks, we let $X = B_i \backslash \overline{g}$ be a block of the same type as B_i, and $Y = B_{i+1} \backslash \overline{h}$ be a block of the same type as B_{i+1}. We define a binary partition $\mathcal{C}$ having $(\overline{g} \cup \overline{h})^\star$ as its homogeneous block as follows.

$$\mathcal{C} = \begin{cases} (B_1, \ldots, B_{i-1}, (\overline{g} \cup \overline{h})^\star, B_{i+2}, \ldots, B_p) & \text{if } B_i \backslash \overline{g} = B_{i+1} \backslash \overline{h} = \emptyset \\ (B_1, \ldots, B_{i-1}, X, (\overline{g} \cup \overline{h})^\star, B_{i+2}, \ldots, B_p) & \text{if } B_i \backslash \overline{g} \neq \emptyset \text{ and } B_{i+1} \backslash \overline{h} = \emptyset \\ (B_1, \ldots, B_{i-1}, (\overline{g} \cup \overline{h})^\star, Y, B_{i+2}, \ldots, B_p) & \text{if } B_i \backslash \overline{g} = \emptyset \text{ and } B_{i+1} \backslash \overline{h} \neq \emptyset \\ (B_1, \ldots, B_{i-1}, X, (\overline{g} \cup \overline{h})^\star, Y, B_{i+2}, \ldots, B_p) & \text{if } B_i \backslash \overline{g} \neq \emptyset \text{ and } B_{i+1} \backslash \overline{h} \neq \emptyset \end{cases} \tag{4.2.1}$$

It's then easy to check that $([0, n], \preceq) = ([0, n], \preceq_{\mathcal{C}})$. □

The set of all binary partitions of $[0, n]$ becomes a poset when partially ordered by contraction. We now aim to characterize contraction in terms of graphs defined by binary partitions.

Given two binary partitions $\mathcal{B} = (B_1, \ldots, B_p)$ and $\mathcal{C} = (C_1, \ldots, C_q)$ of $[0, n]$, we associate the bipartite graph $G(\mathcal{B}, \mathcal{C})$ whose two disjoint sets of vertices are $V_1 = \{B_1, \ldots B_p\}$ and $V_2 = \{C_1, \ldots, C_q\}$ (written in this order), and a vertex $B_i \in V_1$ is adjacent to a vertex $C_j \in V_2$ if $B_i \cap C_j \neq \emptyset$. The vertices in V_1 will be called *left vertices* and the vertices in V_2 will be called *right vertices*. A vertex of $G(\mathcal{B}, \mathcal{C})$ is said to be *non-homogeneous* (resp. *homogeneous*) if it corresponds to a non-homogeneous (resp. *homogeneous*) block of either $\mathcal{B}$ or $\mathcal{C}$. When the edges of $G(\mathcal{B}, \mathcal{C})$ are not crossing, we say that $G(\mathcal{B}, \mathcal{C})$ is *non-crossing*. See Figure 4.4 for examples of crossing and non-crossing bipartite graphs.

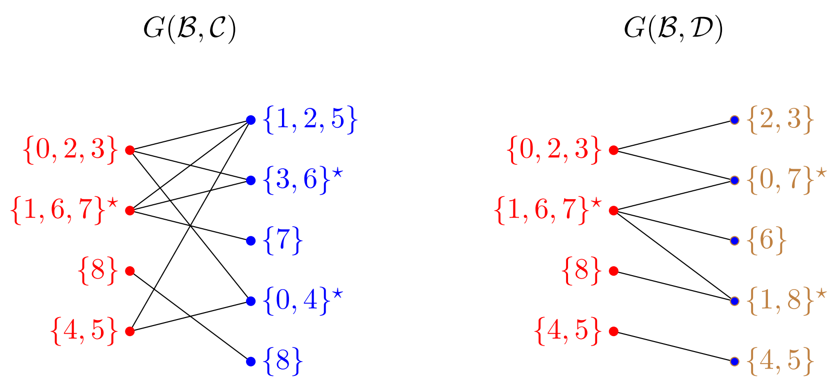


FIGURE 4.4. $G(\mathcal{B}, \mathcal{C})$ is crossing but $G(\mathcal{B}, \mathcal{D})$ is non-crossing

When $G(\mathcal{B},\mathcal{C})$ is non-crossing, it is not difficult to verify the following result regarding the intersections of the blocks of $\mathcal{B}$ and $\mathcal{C}$.

LEMMA 4.2.6. *Let $\mathcal{B} = (B_1, \ldots, B_p)$ and $\mathcal{C} = (C_1, \ldots, C_q)$ be two binary partitions of $[0, n]$. Suppose that $G(\mathcal{B},\mathcal{C})$ is non-crossing. Then, for $i \in [p]$ and $j \in [q]$, we have*

(1) If $|C_1| + \cdots |C_j| < |B_1| + \cdots + |B_i|$, then there exist positive integers s and t such that $s \leq i$, $t > j$, and both $C_j \cap B_s$ and $C_t \cap B_i$ are non-empty.

(2) $C_i \cap B_j$ is nonempty if and only if

$$|C_1| + |C_2| + \cdots + |C_{j-1}| < |B_1| + |B_2| + \cdots + |B_i|, \text{ and}$$

$$|B_1| + |B_2| + \cdots + |B_{i-1}| < |C_1| + |C_2| + \cdots + |C_j|.$$

(3) $B_i \subseteq C_j \neq \emptyset$ if and only if

$$|C_1| + |C_2| + \cdots + |C_{i-1}| \leq |B_1| + |B_2| + \cdots + |B_{j-1}|, \text{ and}$$

$$|B_1| + |B_2| + \cdots + |B_i| \leq |C_1| + |C_2| + \cdots + |C_j|.$$

For a vertext v of $G(\mathcal{B},\mathcal{C})$, we define

$$\deg^\star(v) := \#(\text{homogeneous vertices adjacent to } v),$$

$$\deg^\vee(v) := \#(\text{non-homogeneous vertices adjacent to } v).$$

Clearly, $\deg(v) = \deg^\star(v) + \deg^\vee(v)$.

The next theorem is the main result of this section. It provides a characterization of binary partition contractions in terms of bipartite graphs and their vertex degrees. We will devote the rest of this section to proving it.

THEOREM 4.2.7. *Let $\mathcal{B}$ and $\mathcal{C}$ be binary partitions of $[0, n]$. We have that $\mathcal{C} \leq \mathcal{B}$ if and only if $G(\mathcal{B},\mathcal{C})$ satisfies the following conditions.*

(1) $G(\mathcal{B},\mathcal{C})$ is non-crossing.

(2) Every left non-homogeneous vertex v satisfies $\deg^\vee(v) \leq 1$.

(3) Every left homogeneous vertex v satisfies $\deg^\star(v) = 1$ *and* $\deg^\vee(v) = 0$

(4) *Every right non-homogeneous vertex v satisfies* $\deg^{\star}(v) = 0$ *and* $\deg^{\vee}(v) = 1$.

(5) *If a right homogeneous vertex v satisfies* $\deg(v) = 1$, *then* $\deg^{*}(v) = 1$.

To prove Theorem 4.2.7, we first need to develop several observations and lemmas. Let us start by describing the covering relations $\mathcal{C} \lessdot \mathcal{B}$. Recall that the nodes of the Hasse diagram of a preorder are equivalent classes. Lemma 4.2.5 implies that $([0,n], \preceq_{\mathcal{C}})$ is obtained by contracting an edge $\overline{g} - \overline{h}$ of the Hasse diagram of $([0,n], \preceq_{\mathcal{B}})$ where $\overline{g}$ and $\overline{h}$ are two equivalent classes of $[0,n]/_{\equiv_{\mathcal{B}}}$. As in the proof of Lemma 4.2.5, $\mathcal{C}$ can be written as in equation (4.2.1). We can describe this in terms of $G(\mathcal{B}, \mathcal{C})$ as follows.

Lemma 4.2.8. *We have that $\mathcal{B} = (B_1, \ldots, B_p)$ is a cover of $\mathcal{C} = (C_1, \ldots, C_q)$ if and only if $G(\mathcal{B}, \mathcal{C})$ is a non-crossing bipartite graph with a unique right vertex C_j of degree two satisfying the following properties.*

(1) *The vertex C_j is homogeneous.*

(2) *Every right vertex that is not C_j has degree one and is adjacent to a vertex of the same type*

(3) *Every left non-homogeneous vertex B_i that is adjacent to C_j has degree at most two and satisfies $|B_i \cap C_j| = 1$.*

(4) *Every left vertex that is not a non-homogeneous vertex adjacent to C_j has degree one and is adjacent to a vertex of the same type.*

Proof. Suppose that $\mathcal{C} \lessdot \mathcal{B}$. Then, there are four possible cases of $\mathcal{C}$ to check as shown in equation (4.2.1). We note that the unique right homogeneous vertex C_j of degree two in $G(\mathcal{B}, \mathcal{C})$ corresponds to the block $(\overline{g} \cup \overline{h})^{\star}$ of $\mathcal{C}$. It is easy to verify using these four cases that $G(\mathcal{B}, \mathcal{C})$ satisfies the non-crossing property and the two conditions. Conversely, suppose that $G(\mathcal{B}, \mathcal{C})$ is non-crossing and satisfies the four conditions. Then, it is also not difficult to check that $\mathcal{C}$ can only have the form shown in equation (4.2.1). Thus, $\mathcal{C} \lessdot \mathcal{B}$. □

Lemma 4.2.9. *If $\mathcal{C} \leq \mathcal{B}$, then $G(\mathcal{B}, \mathcal{C})$ satisfies conditions (2)-(5) in Theorem 4.2.7*

Proof. To prove this statement, we first establish the following two steps. The first step is to show that the statement holds for every $G(\mathcal{B}, \mathcal{C})$ such that $\mathcal{B}$ is a cover of $\mathcal{C}$. This is a straightforward application of Lemma 4.2.8 and is left to the reader to verify. The second step is to show that for all $\mathcal{B}, \mathcal{B}', \mathcal{C}$ such that $G(\mathcal{B}, \mathcal{B}')$ and $G(\mathcal{B}', \mathcal{C})$ satisfy conditions (2)-(5), we must have that $G(\mathcal{B}, \mathcal{C})$ also satisfy these conditions. To see this, suppose that $\mathcal{B}, \mathcal{B}', \mathcal{C}$ are binary partitions in which $G(\mathcal{B}, \mathcal{B}')$ and $G(\mathcal{B}', \mathcal{C})$ satisfy conditions (2)-(5). Let B_i be a left homogeneous vertex of $G(\mathcal{B}, \mathcal{B}')$. By condition (3), B_i is adjacent to exactly one homogeneous vertex B'_j. Thus, $B_i \subseteq B'_j$. Similarly for $G(\mathcal{B}', \mathcal{C})$, we have $B'_j \subseteq C_k$ for some right homogeneous vertex C_k. Hence, $B_i \subseteq C_k$. Therefore, in $G(\mathcal{B}, \mathcal{C})$, the left homogeneous vertex B_i has degree one and is adjacent to the right homogeneous vertex C_k. This shows that condition (3) holds for $G(\mathcal{B}, \mathcal{C})$.

Now let C_k be a right non-homogeneous vertex of $G(\mathcal{B}', \mathcal{C})$. Since $G(\mathcal{B}, \mathcal{B}')$ and $G(\mathcal{B}', \mathcal{C})$ satisfy condition (4), one can use a similar argument to show that, in $G(\mathcal{B}, \mathcal{C})$, the left non-homogeneous vertex C_k has degree one and is adjacent to a non-homogeneous vertex. Thus, condition (4) holds for $G(\mathcal{B}, \mathcal{C})$.

Assume for the sake of contradiction that $G(\mathcal{B}, \mathcal{C})$ doesn't satisfy condition (2). Then, there are a left non-homogeneous vertex B_i and two right non-homogeneous vertices C_{k_1} and C_{k_2} adjacent to B_i in $G(\mathcal{B}, \mathcal{C})$. We deduce from conditions (4) and (2) for $G(\mathcal{B}', \mathcal{C})$ that $C_{k_1} \subseteq B'_{j_1}$ and $C_{k_2} \subseteq B'_{j_2}$ for some distinct left non-homogeneous vertices B'_{j_1} and B'_{j_2} of $G(\mathcal{B}', \mathcal{C})$. Since $B_i \cap C_{k_1} \neq \emptyset$ and $B_i \cap C_{k_2} \neq \emptyset$, it follows that $B_i \cap B'_{j_1} \neq \emptyset$ and $B_i \cap B'_{j_2} \neq \emptyset$. Hence, in $G(\mathcal{B}, \mathcal{B}')$, the left non-homogeneous B_i is adjacent to the two non-homogeneous vertices B'_{j_1} and B'_{j_2}, a contradiction to condition (2) for $G(\mathcal{B}, \mathcal{B}')$.

Now assume for the sake of contradiction that $G(\mathcal{B}, \mathcal{C})$ doesn't satisfy condition (5). Then, there is a right homogeneous vertex C_k of degree one and a left non-homogeneous vertex B_i adjacent to C_k. Thus, $C_k \subseteq B_i$. Consider the following two cases. Cases I: C_k is adjacent to a left homogeneous vertex B'_j in $G(\mathcal{B}', \mathcal{C})$. Applying condition (3) to $G(\mathcal{B}', \mathcal{C})$, we have $B'_j \subseteq C_k$ and hence $B'_j \subseteq B_i$. Thus, in $G(\mathcal{B}, \mathcal{B}')$, the the right non-homogeneous vertex B'_j has degree one and is adjacent to the non-homogeneous vertex B_i, a contradiction to condition (5) for $G(\mathcal{B}, \mathcal{B}')$. Case II: C_k is not adjacent to any left homogeneous vertex in $G(\mathcal{B}', \mathcal{C})$. By condition (5), we deduce that C_k must be adjacent to two non-homogeneous B'_{j_1} and B'_{j_2} of $G(\mathcal{B}', \mathcal{C})$. Consequently, by condition (4), both B'_{j_1} and B'_{j_2} are vertices of $G(\mathcal{B}, \mathcal{B}')$ of degree one and are adjacent to B_i. Hence, $B'_{j_1} \subseteq B_i$ and $B'_{j_2} \subseteq B_i$. In particular, this implies that, in $G(\mathcal{B}, \mathcal{B}')$, the left non-homogeneous vertex B_i is adjacent to at least two non-homogeneous vertices, a contradiction to (2) for $G(\mathcal{B}, \mathcal{B}')$. Since both cases lead to contradictions, we must have that $G(\mathcal{B}, \mathcal{C})$ satisfies condition (5).

Suppose that $\mathcal{C} \leq \mathcal{B}$. Then, $\mathcal{C} = \mathcal{B}^t \lessdot \cdots \lessdot \mathcal{B}^1 \lessdot \mathcal{B}$ for some $\mathcal{B}^1, \ldots, \mathcal{B}^t$. The two steps we established above implies that $G(\mathcal{B}, \mathcal{C})$ satisfies conditions (2)-(5) in Theorem 4.2.7. ☐

The proof of Lemma 4.2.9 shows that conditions (2)-(5) in Theorem 4.2.7 define a transitive relation on the set of all binary partitions on $[0, n]$. That is, we can define a transitive relation $\rightsquigarrow$ on the set of all binary partitions on $[0, n]$ by letting $\mathcal{B} \rightsquigarrow \mathcal{C}$ if $G(\mathcal{B}, \mathcal{C})$ satisfies conditions (2)-(5) in Theorem 4.2.7.

Lemma 4.2.10. *Suppose that $\mathcal{B}$, $\mathcal{C} = (C_1, \ldots, C_q)$, and $\mathcal{D}$ be binary partitions of $[0, n]$ in which both $G(\mathcal{B}, \mathcal{C})$ and $G(\mathcal{C}, \mathcal{D})$ are non-crossing. If $G(\mathcal{B}, \mathcal{D})$ is crossing, then there exists an integer j such that C_j is a vertex of degree at least two in $G(\mathcal{B}, \mathcal{C})$ and a vertex of degree at least two in $G(\mathcal{C}, \mathcal{D})$.*

Proof. Suppose that $G(\mathcal{B}, \mathcal{D})$ is crossing. Then there exists an i such that B_i is adjacent to D_{k_1} and B_{i+1} is adjacent to D_{k_2} for some $k_1 > k_2$. Moreover, because $G(\mathcal{B}, \mathcal{C})$ is non-crossing, there must exists $j_1 \leq j_2$ such that such that B_i is adjacent to C_{j_1} in $G(\mathcal{B}, \mathcal{C})$ and C_{j_1} is adjacent to D_{k_1} in $G(\mathcal{C}, \mathcal{D})$, and B_{i+1} is adjacent to C_{j_2} in $G(\mathcal{B}, \mathcal{C})$ and C_{j_2} is adjacent to D_{k_2} in $G(\mathcal{C}, \mathcal{D})$. Since $G(\mathcal{C}, \mathcal{D})$ is non-crossing, we deduce that $j_1 \geq j_2$. Thus, $j_1 = j_2$. Let $j = j_1 = j_2$. Then C_j is adjacent to both B_i and B_{i+1} in $G(\mathcal{B}, \mathcal{C})$ and is adjacent to both D_{k_1} and D_{k_2} in $G(\mathcal{C}, \mathcal{D})$. Hence, we found an integer j with the desired property. ☐

We can now give a proof of the characterization in Theorem 4.2.7.

Proof of Theorem 4.2.7. Suppose that $\mathcal{C} \leq \mathcal{B}$. Then by Lemma 4.2.9, $G(\mathcal{B}, \mathcal{C})$ must satisfy conditions (2)-(5). Thus, it only remains to be shown that $G(\mathcal{B}, \mathcal{C})$ is non-crossing. Clearly, the non-crossing condition is satisfied when $\mathcal{C} = \mathcal{B}$. Thus, we may assume that $\mathcal{B} \neq \mathcal{C}$. Let $\mathcal{C} = \mathcal{B}^t \lessdot \mathcal{B}^{t-1} \lessdot \cdots \lessdot \mathcal{B}^0 = \mathcal{B}$ be a maximal chain of strictly decreasing binary partitions from $\mathcal{B}$ to $\mathcal{C}$, i.e., $\mathcal{B}^{i-1}$ is a cover of $\mathcal{B}^i$ for all $i \in [t]$. To see that $G(\mathcal{B}, \mathcal{C})$ is non-crossing, we proceed by induction on t. When $t = 1$, $\mathcal{B}^0$ is a cover of $\mathcal{B}^1$. Thus, by Lemma 4.2.8, $G(\mathcal{B}^0, \mathcal{B}^1)$ is non-crossing. This establishes the base case. Now suppose that $G(\mathcal{B}^0, \mathcal{B}^t)$ is non-crossing for a positive integer t. Assume for the sake of contradiction $G(\mathcal{B}^0, \mathcal{B}^{t+1})$ is crossing. Then by Lemma 4.2.10, there exists an integer j such that B_j^1 is a right vertex of degree at least two of $G(\mathcal{B}^0, \mathcal{B}^1)$ and a left vertex of degree at least two of $G(\mathcal{B}^1, \mathcal{B}^{t+1})$. Because $\mathcal{B}^1$ is a cover of $\mathcal{B}^0$, it follows from Lemma 4.2.8 that B_j^1 is a right homogeneous vertex of $G(\mathcal{B}^0, \mathcal{B}^1)$. Condition 4.2.9/(3) then implies that the vertex B_j^t in $G(\mathcal{B}^t, \mathcal{B}^{t+1})$ has degree one, a contradiction. Thus, $G(\mathcal{B}^0, \mathcal{B}^{t+1})$ must be non-crossing. By induction, $G(\mathcal{B}^0, \mathcal{B}^t) = G(\mathcal{B}, \mathcal{C})$ is non-crossing.

Conversely, suppose that $G(\mathcal{B}, \mathcal{C})$ meets these conditions. Let us first consider $G(\mathcal{B}, \mathcal{C})$ without any right vertex of $G(\mathcal{B}, \mathcal{C})$ of degree greater than one. We claim that in this case $\mathcal{C} = \mathcal{B}$, which will automatically gives $\mathcal{C} \leq \mathcal{B}$ as desired. To see this, we show that every left vertex of $G(\mathcal{B}, \mathcal{C})$ has degree one and is adjacent to a vertex of the same type. Assume that there is a left vertex, say B_i, of degree at least two. Then by condition (3), the left vertex B_i must be non-homogeneous. By condition (2), one of the right vertex adjacent to B_i, say C_j, has to be homogeneous. However, by condition (5), C_j must have degree at least two, a contradiction to the assumption $\deg(C_j) = 1$. Hence, every left vertex of $G(\mathcal{B}, \mathcal{C})$ has degree one. From here, one can easily verify using conditions (3) and (4) that every edge of $G(\mathcal{B}, \mathcal{C})$ connects two vertices of the same types. This implies $\mathcal{C} = \mathcal{B}$ as claimed.

Now we consider $G(\mathcal{B}, \mathcal{C})$ with at least one right vertex of degree at least two. To see that $\mathcal{C} \leq \mathcal{B}$, we will construct a maximal chain of strictly increasing binary partitions $\mathcal{C} = \mathcal{B}^t \lessdot \mathcal{B}^{t-1} \lessdot \cdots \lessdot \mathcal{B}^0 = \mathcal{B}$.

Let j be an integer such that C_j is a right vertex of degree at least two. We note that condition (4) implies that C_j is a homogeneous vertex. Because $G(\mathcal{B}, \mathcal{C})$ is a non-crossing bipartite graph, C_j is adjacent to two consecutive blocks, say B_i and B_{i+1}. This implies that there exist an equivalent class $\overline{g} \subseteq B_i$ such that $\overline{g} \subseteq B_i \cap C_j$ and an equivalent class $\overline{h} \subseteq B_{i+1}$ such that $\overline{h} \subseteq B_{i+1} \cap C_j$. Let $\mathcal{B}^1$ be the binary partition corresponding to contracting the edge $\overline{g} - \overline{h}$ in the Hasse diagram of the preposet $([0, n], \preceq_{\mathcal{B}})$. Hence, by construction, $\mathcal{B}^1 \lessdot \mathcal{B}$. One sees that $\overline{g} \cup \overline{h}$ is a homogeneous block of $\mathcal{B}^1$ and that $G(\mathcal{B}^1, \mathcal{C})$ has $\overline{g} \cup \overline{h}$ as a left vertex of degree one and adjacent C_j. It is then easy to see by considering the four cases described in equation (4.2.1) that $G(\mathcal{B}^1, \mathcal{C})$ meets all of the five conditions. Now repeat the same construction with $G(\mathcal{B}^1, \mathcal{C})$ to produce $\mathcal{B}^2$ such that $\mathcal{B}^2 \lessdot \mathcal{B}^1$. By repeatedly applying this procedure, we can eventually produce $\mathcal{B}^t \lessdot \mathcal{B}^{t-1} \lessdot \cdots \lessdot \mathcal{B}^0 = \mathcal{B}$ such that the every right vertex of $G(\mathcal{B}^t, \mathcal{C})$ has degree one. This implies $\mathcal{C} = \mathcal{B}^t \leq \mathcal{B}$ as desired. $\square$

4.3. Skewed binary composition and skewed binary partition

We now introduce "skewed binary partition" which is a special case of binary partition, and "skewed binary composition". These two combinatorial objects will provide us with sufficient information to describe the normal fans of parking function polytopes. Similarly to how a composition records the sizes of blocks in an ordered partition, a skewed binary composition will be used for storing information of the blocks of a skewed binary partition. We begin by describing the skewed binary composition notation. First, the entries of our composition are nonzero integers (as opposed to being positive integers). Additionally, we allow two different variations of the entries: i° and $i^\star$. We consider these two variations to have the same numerical values as i, and use the absolute value sign to take their numerical values. Hence, $|i^\circ| = |i^\star| = i = |i|$.

For convenience, we let $\mathbb{N}^\circ := \{i^\circ \mid i \in \mathbb{N}\}$, $\mathbb{P} := \mathbb{N}_{>0}$ and $\mathbb{P}^\star_{\geq 2} := \{i^\star \mid i \in \mathbb{P}, i \geq 2\}$.

Definition 4.3.1. Let $n \in \mathbb{P}$ and $k \in \mathbb{N}$. A *skewed binary composition* of n into $k+2$ parts is an ordered tuple $\mathbf{b} = (b_{-1}, b_0, b_1, \ldots, b_k)$ such that $\sum_{i=-1}^{k} |b_i| = n$ and the entries of $\mathbf{b}$ satisfy

$$(b_{-1}, b_0) \in (\mathbb{N} \times \mathbb{N}^\circ) \cup (\mathbb{P} \times \{0\}) \text{ and } b_i \in \mathbb{P} \cup \mathbb{P}^\star_{\geq 2} \text{ for all } 1 \leq i \leq k.$$

Example 4.3.2. The following are all possible skewed binary compositions of $n = 3$.

$$(0, 0^\circ, 1, 1, 1), (0, 0^\circ, 1, 2), (0, 0^\circ, 1, 2^\star), (0, 0^\circ, 2, 1), (0, 0^\circ, 2^\star, 1), (0, 0^\circ, 3), (0, 0^\circ, 3^\star),$$

$$(0, 1^\circ, 1, 1), (0, 1^\circ, 2), (0, 1^\circ, 2^\star),$$

$$(1, 0^\circ, 1, 1), (1, 0^\circ, 2), (1, 0^\circ, 2^\star),$$

$$(1, 0, 1, 1), (1, 0, 2), (1, 0, 2^\star),$$

$$(0, 2^\circ, 1), (1, 1^\circ, 1), (2, 0^\circ, 1), (2, 0, 1)$$

$$(0, 3^\circ), (1, 2^\circ), (2, 1^\circ), (3, 0^\circ), (3, 0).$$

Next, we introduce a similar notion to binary partition called *ordered skewed binary partition* of the set $S = [0, n]$ by allowing empty blocks together with additional restrictions.

ordered bi-weekly partition $\mathcal{B}$	type($\mathcal{B}$)
$(\{0,2,3\},\emptyset,\{1,6,7\},\{8\},\{4,5\})$	$(2,0,3,1,2)$
$(\{2,3\},\{0,7\}^\star,\{6\},\{1,8\}^\star,\{4,5\})$	$(2,1^\circ,1,2^\star,2)$
$(\{1,3,4,5,8\},\{0\},\{2\},\{6,7\})$	$(5,0^\circ,1,2)$
$(\emptyset,\{0\},\{2,3,8\},\{1,6,7\}^\star,\{4,5\})$	$(0,0^\circ,3,3^\star,2)$
$(\{5,7\},\{0,1,3\}^\star,\{2,4\}^\star,\{6,8\}^\star)$	$(2,2^\circ,2^\star,2^\star)$
$(\emptyset,\{0,1,2,3,4,5,6,7,8\}^\star)$	$(0,8^\circ)$

Table 4.1. Examples of skewed binary partitions and their types

Definition 4.3.3. Let $n \in \mathbb{P}$ and $k \in \mathbb{N}$. An *(ordered) skewed binary partition* of $[0,n]$ into $k+2$ blocks is an ordered tuple $(B_{-1}, B_0, \ldots, B_k)$ of disjoint subsets of $[0,n]$ such that $B_{-1} \sqcup B_0 \sqcup B_1 \sqcup \cdots \sqcup B_k = [0,n]$ satisfying the following conditions:

(1) B_0 is homogeneous, provided $|B_0| \geq 2$, and B_{-1} is non-homogeneous.

(2) $0 \in B_{-1}$ or $0 \in B_0$. If $0 \in B_{-1}$, then B_{-1} contains at least another element and $B_0 = \emptyset$. Hence, if $0 \in B_{-1}$, then $|B_{-1}| \geq 2$ and $|B_0| = 0$.

(3) For each $0 \leq i \leq k$, if B_i is a singleton, then it is non-homogeneous.

(4) $B_i \neq \emptyset$ for all $1 \leq i \leq k$.

See the first column of Table 4.1 for examples of skewed binary partitions of $[0,8]$. Comparing Definition 4.3.3 to Definition 4.2.1, one sees that a skewed binary partition is simply a binary partition with extra requirements (conditions (1) and (2)). In fact, removing empty blocks from a skewed binary partition yields a binary partition. For instance, removing the empty block from the skewed binary partition shown at the top of Table 4.1 gives a binary partition in Example 4.2.4. Thus, properties of binary partitions extend naturally to skewed binary partitions when regarded in this way.

Definition 4.3.4. For a skewed binary partition $\mathcal{B}$ of $[0,n]$, let $\hat{\mathcal{B}}$ be the binary partition obtained by removing the empty blocks from $\mathcal{B}$. We define the associate preorder $\preceq_{\mathcal{B}}$ on the set $[0,n]$ to be the preorder $\preceq_{\hat{\mathcal{B}}}$. We also say that a skewed binary partition $\mathcal{C}$ is a contraction of another skewed binary partition $\mathcal{B}$ if $\preceq_{\mathcal{C}}$ is a contraction of $\preceq_{\mathcal{B}}$.

One notices that we also include a column of "type($\mathcal{B}$)" on the right of Table 4.1. We introduce this concept in the definition below.

Definition 4.3.5. Let $\mathcal{B} = (B_{-1}, B_0, B_1, \ldots, B_k)$ be a skewed binary partition of $[0, n]$. We associate a skewed binary composition $\mathbf{b} = (b_{-1}, b_0, \ldots, b_k)$ of n to it in the following way:

(1) For $1 \leq i \leq k$, let $b_i = |B_i|$ if B_i is non-homogeneous, and $b_i = |B_i|^\star$ if B_i is homogenous.

(2) If $0 \in B_0$, then let $b_0 = h^\circ$, where $h = |B_0| - 1$, and let $b_{-1} = |B_{-1}|$.

(3) If $0 \in B_{-1}$, then let $b_0 = |B_0| = 0$ and let $b_{-1} = |B_{-1}| - 1$.

We say this vector $\mathbf{b}$ is the *type* of $\mathcal{B}$ and denote it by type($\mathcal{B}$).

It is easy to see that two skewed binary partitions are of the same type if and only if they differ from one another by a permutation of nonzero numbers between blocks.

Remark 4.3.6. Suppose $B = (B_{-1}, B_0, B_1, \ldots, B_k)$ has type $\mathbf{b} = (b_{-1}, b_0, \ldots, b_k)$. It is easy to see that for $1 \leq i \leq k$, the number b_i tells us the cardinality of B_i and whether B_i is homogeneous or not. In particular, Condition (3) of Definition 4.3.3 implies that $b_i \neq 1^\star$, and thus $b_i \in \mathbb{P} \cup \mathbb{P}^\star_{\geq 2}$.

For $i = -1$ or 0, the number $|b_i|$ is the cardinality of $B_i \setminus \{0\}$. Furthermore, one checks that

$$0 \in B_0 \quad \text{if and only if} \quad (b_{-1}, b_0) \in \mathbb{N} \times \mathbb{N}^\circ, \quad \text{and} \tag{4.3.1}$$

$$0 \in B_{-1} \quad \text{if and only if} \quad (b_{-1}, b_0) \in \mathbb{P} \times \{0\}. \tag{4.3.2}$$

Hence, the type of each skewed binary partition of $[0, n]$ is a skewed binary composition of n.

By (4.3.1), we have that $0 \in B_0$ if and only if $b_0 \in \mathbb{N}^\circ$. Moreover, when $b_0 = h^\circ$, we know that B_0 consists of h positive integers and 0. This is the reason we use the notation h° in which the superscript $\circ$ indicates that 0 needs to be included.

Example 4.3.7. See Figure 4.6 for examples of three skewed binary partitions $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ of $[0, 8]$ together with their respective types and associated preorder cones. (Note that $\mathcal{B}$ and $\mathcal{D}$ are the first two skewed binary partitions given in Table 4.1.)

Applying Lemma 4.1.4/(3), we can compute the dimension of the sliced preorder cone $\tilde{\sigma}_{\mathcal{B}}$ using its type vector type($\mathcal{B}$) as stated in the next proposition.

Proposition 4.3.8. *Suppose that* $\mathcal{B} = (B_{-1}, B_0, B_1, \ldots, B_k)$ *is a skewed binary partitions of* $[0, n]$ *with* $\text{type}(\mathcal{B}) = (b_{-1}, b_0, \ldots, b_k)$. *Then the dimension of the slice preorder cone* $\tilde{\sigma}_{\mathcal{B}}$ *is*

$$\dim(\tilde{\sigma}_{\mathcal{B}}) = b_{-1} + \left(\sum_{b_i \in \mathbb{P}} b_i\right) + \#(b_i \in \mathbb{P}^{\star}_{\geq 2}) \tag{4.3.3}$$

and the co-dimension of $\tilde{\sigma}_{\mathcal{B}}$ *(with respect to the space* $\mathbb{R}^n$*) is*

$$|b_0| + \sum_{b_i \in \mathbb{P}^{\star}_{\geq 2}} (|b_i| - 1).$$

Proof. By Lemma 4.1.4/(3), the dimension of $\tilde{\sigma}_{\mathcal{B}}$ equals the number of equivalence classes of the preposet $([0, n], \preceq_{\mathcal{B}})$ minus one. One observes that for each nonempty homogeneous block B_i, it gives arise one equivalence class of the preposet $([0, n], \preceq_{\mathcal{B}})$, and for each non-homogeneous block B_i, each singleton subset of B_i is an equivalence class and hence it gives arise $|B_i|$ equivalence classes. Thus, the total number of equivalences classes of $([0, n], \preceq_{\mathcal{B}})$ arising from $B_1, \ldots, B_k$ is

$$\left(\sum_{b_i \in \mathbb{P}} b_i\right) + \#(b_i \in \mathbb{P}^{\star}_{\geq 2})$$

and the total number equivalence classes arising from B_{-1} and B_0 is given by $|B_{-1}| + \chi_{B_0 \neq \emptyset}$, where $\chi_{B_0 \neq \emptyset}$ is 1 if B_0 is not the empty block $\emptyset$, and is 0 otherwise. However, by Condition (2) of Definition 4.3.3, we have that $B_0 = \emptyset$ if and only if $0 \in B_{-1}$. Then it follows from Definition 4.3.5 that $b_{-1} = |B_{-1}| - (1 - \chi_{B_0 \neq \emptyset})$. One sees that (4.3.3) follows from all the discussion above.

Finally, the co-dimension formula for $\tilde{\sigma}_{\mathcal{B}}$ follows from (4.3.3) and the fact that $n = \sum_{i=-1}^{k} |b_i|$. □

The following result is an immediate consequence of Lemma 4.1.4/(6).

Corollary 4.3.9. *Let* $\mathcal{B}$ *be a skewed binary partition of* $[0, n]$ *and* $\tilde{\sigma}_B$ *be the sliced preorder cone associated to* $\mathcal{B}$. *If the preorder* $\preceq_B$ *defines a poset on* $[0, n]$, *then*

$$\tilde{\sigma}_{\mathcal{B}} = \bigcup_{\pi \in L[\preceq_{\mathcal{B}}]} \tilde{\sigma}(\pi),$$

where $L[\preceq_{\mathcal{B}}]$ *is the set of linear extensions of the poset* $([0, n], \preceq_{\mathcal{B}})$ *and* $\tilde{\sigma}(\pi)$ *is defined as in* (4.1.3)

4.4. Face Structure

Recall that we define the parking function polytope $\mathrm{PF}(\mathbf{u})$ as the convex hull of all $\mathbf{u}$-parking functions. Although named a polytope, it is not immediately evident that $\mathrm{PF}(\mathbf{u})$ qualifies as one, since it is defined as the convex hull of infinitely many points. The following proposition justifies its name.

PROPOSITION 4.4.1. *The parking function polytope* $\mathrm{PF}(\mathbf{u})$ *is indeed a polytope, that is, it is a convex hull of finitely many points.*

DEFINITION 4.4.2. A point $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is $\mathbf{u}$-*extreme* if it is a permutation of a point of the form

$$(\underbrace{0, \ldots, 0}_{k}, u_{k+1}, \ldots, u_n) \tag{4.4.1}$$

for some $0 \leq k \leq n$. We denote by $\mathcal{X}(\mathbf{u})$ the set of all $\mathbf{u}$-extreme points.

EXAMPLE 4.4.3. Let $\mathbf{u} = (0, 0, 4, 4, 4, 6, 8, 8)$. Then $\mathcal{X}(\mathbf{u})$ is the set of all permutations of the 7 points:

$$(0,0,4,4,4,6,8,8),\ (0,0,0,4,4,6,8,8),\ (0,0,0,0,4,6,8,8),\ (0,0,0,0,0,6,8,8),$$
$$(0,0,0,0,0,0,8,8),\ (0,0,0,0,0,0,0,8), \text{ and } (0,0,0,0,0,0,0,0).$$

PROOF OF PROPOSITION 4.4.1. We claim that $\mathrm{PF}(\mathbf{u}) = \mathrm{conv}(\mathcal{X}(\mathbf{u}))$ the convex hull of $\mathbf{u}$-extreme points. Since there are only finitely many $\mathbf{u}$-*extreme* points, this will show that $\mathrm{PF}(\mathbf{u})$ is indeed a polytope.

Since every $\mathbf{u}$-extreme point is a $\mathbf{u}$-parking function, we have that $\mathrm{conv}(\mathcal{X}(\mathbf{u})) \subseteq \mathrm{PF}(\mathbf{u})$. It remains to be shown that $\mathrm{PF}(\mathbf{u}) \subseteq \mathrm{conv}(\mathcal{X}(\mathbf{u}))$. To see this, it suffices to show that every $\mathbf{u}$-parking function is a convex combination of $\mathbf{u}$-extreme points.

Suppose $\mathbf{a} = (a_1, \ldots, a_n)$ is a $\mathbf{u}$-parking function and $b_1 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$. Let $I_{\mathbf{a}}$ be the set of indices i such that $0 < b_i < u_i$. If $I_{\mathbf{a}}$ is empty, we define the *inner width* of $\mathbf{a}$ to be 0; otherwise, we let $t = \max(I_{\mathbf{a}})$ and s be the least integer such that $0 < u_s$, and define the inner width of $\mathbf{a}$ to be $t - s + 1$. One sees that the inner width of $\mathbf{a}$ is always nonnegative, and is 0 if and only if $\mathbf{a}$ is $\mathbf{u}$-extreme. We will prove that $\mathbf{a}$ is a convex combination of $\mathbf{u}$-extreme points by induction on the inner width of $\mathbf{a}$.

The base case when the inner width of $\mathbf{a}$ is 0 is true, since $\mathbf{a}$ itself is a $\mathbf{u}$-extreme. Now suppose that $\mathbf{a}$ has inner width $t - s + 1 \geq 1$, and that every $\mathbf{u}$-parking function of inner width less than $t + s - 1$ is a convex combination of $\mathbf{u}$-extreme points. Let $\tau \in \mathfrak{S}_n$ be a permutation such that $(a_{\tau(1)}, \ldots, a_{\tau} n) = (b_1, \ldots, b_n)$ and let $\tau(k) = t$. Then

$$(a_1, \ldots, a_n) = \frac{u_t - a_k}{u_t}(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n) + \frac{a_k}{u_t}(a_1, \ldots, a_{k-1}, u_t, a_{k+1}, \ldots, a_n)$$

is a convex combination of $\mathbf{u}$-parking functions $\mathbf{a}^* := (a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)$ and $\mathbf{a}' := (a_1, \ldots, a_{k-1}, u_t, a_{k+1}, \ldots, a_n)$. Notice that the inner width of both $\mathbf{a}^*$ and $\mathbf{a}'$ decrease from the inner width of $\mathbf{a}$ by at least one. Thus, by our induction hypothesis, we have that both $\mathbf{a}^*$ and $\mathbf{a}'$ are convex combinations of $\mathbf{u}$-extreme points. Hence, $\mathbf{a}$ is also a convex combination of $\mathbf{u}$-extreme points, completing the proof. □

It turns out that $\mathcal{X}(\mathbf{u})$ is also the set of all vertices of $\mathrm{PF}(\mathbf{u})$. Although one can apply Lemma 1.2.2 to compute the normal cones of $\mathrm{PF}(\mathbf{u})$ and get this as a consequence later in Theorem 4.4.14, we give a direct proof of it here.

Proposition 4.4.4. *The point $\mathbf{v} \in \mathrm{PF}(\mathbf{u})$ is a vertex of $\mathrm{PF}(\mathbf{u})$ if and only if $\mathbf{v} \in \mathcal{X}(\mathbf{u})$.*

Proof. Since $\mathrm{PF}(\mathbf{u}) = \mathrm{conv}(\mathcal{X}(u))$, every vertex of $\mathrm{PF}(\mathbf{u})$ must be a $\mathbf{u}$-extreme point. Conversely, suppose that $\mathbf{v} = (v_1, \ldots, v_n)$ is a $\mathbf{u}$-extreme point. For simplicity of notations, we may assume without loss of generality due to symmetry that $v_1 \leq \cdots \leq v_n$. Thus, $\mathbf{v}$ is in the form of (4.4.1) for some $0 \leq k \leq n$. Suppose

$$\mathbf{v} = (0, \ldots, 0, u_{k+1}, \ldots, u_n) = \sum_{i=1}^{r} \lambda_i \mathbf{a}_i$$

for some $\mathbf{u}$-extreme points $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,n})$, $i \in [r]$, and some nonnegative real numbers λ_i such that $\lambda_1 + \cdots + \lambda_r = 1$. To see that $\mathbf{v}$ is a vertex of $\mathrm{PF}(\mathbf{u})$, it suffices to show that $\mathbf{v} = \mathbf{a}_1 = \cdots = \mathbf{a}_r$.

Since 0 is the least value that the coordinates of a $\mathbf{u}$-extreme point can be, it follows that the first k coordinates of $\mathbf{a}_i$ are zero for all $i \in [r]$, i.e., $a_{i,j} = 0$ for $i \in [r]$ and $j \in [k]$. Thus, it is left to show that the last $n - k$ coordinates of $\mathbf{a}_i$ agree with the last $n - k$ coordinates of $\mathbf{v}$. However, because u_n is the greatest value of a **u**-extreme point's coordinates and $\sum_{i=1} \lambda_i a_{i,n} = v_n = u_n$, we must have that $a_{i,n} = v_n$ for all $i \in [r]$. Similarly, because $v_{n-1} = u_{n-1}$ is the greatest value after u_n that the coordinates of a **u**-extreme point can be, we have $a_{i,n-1} = v_{n-1}$ for all $i \in [r]$. We continue this arguments and can show that $a_{i,j} = v_j$ for all $j \in \{k+1, \ldots, n-2\}$ and $i \in [r]$. This completes the proof. □

Hence, the parking function polytope is integral (resp. rational) if and only if **u** is an integral (resp. rational) vector in $\mathbb{R}^n$.

Since permuting the coordinates of a **u**-parking function $(a_1, \ldots, a_n)$ still gives a **u**-parking function, the polytope $\mathrm{PF}(\mathbf{u})$ itself also inherits this symmetric property. The next proposition restates this observation more precisely.

PROPOSITION 4.4.5. *For every parking function polytope* $\mathrm{PF}(\mathbf{u})$, *we have that*

(1) If $(a_1, \ldots, a_n)$ *lies in* $\mathrm{PF}(\mathbf{u})$*, then so does every permutation of* $(a_1, \ldots, a_n)$*.*

(2) If F *is a face of* $\mathrm{PF}(\mathbf{u})$, *then for every permutation* $\tau \in \mathfrak{S}_n$ *the set*

$$F_\tau := \{(a_{\tau(1)}, \ldots, a_{\tau(n)}) \mid (a_1, \ldots, a_n) \in F\}$$

is also a face of $\mathrm{PF}(\mathbf{u})$*.*

(3) If σ *is a normal cone of* $\mathrm{PF}(\mathbf{u})$ *at a face* F, *then for every permutation* $\tau \in \mathfrak{S}_n$ *the set* $\sigma_\tau := \{(c_{\tau(1)}, \ldots, c_{\tau(n)}) \mid (c_1, \ldots, c_n) \in \sigma\}$ *is the normal corne of* $\mathrm{PF}(\mathbf{u})$ *at the face* F_τ*.*

4.4.1. Face Poset and Normal Fan. In this part, we study the face poset and the normal fan of the parking polytope $\mathrm{PF}(\mathbf{u})$. By Lemma 1.2.1, for every polytope P, the dual poset of $\mathcal{F}(P)$ is isomorphic to the poset $\mathcal{F}(\Sigma(P))$. Therefore, rather than describing the face poset of parking function polytopes, we can alternatively describe their normal fans. It turns out that these fans only depend on the *multiplicity vector* of **u**.

DEFINITION 4.4.6. Assume that there are ℓ positive integers appearing in **u**: $d_1 < d_2 < \cdots < d_\ell$. We define $m_0(\mathbf{u})$ to be the number of 0's in **u**, and $m_i(\mathbf{u})$ be the number of d_i's in **u** for each $1 \leq i \leq \ell$. We then define the *multiplicity vector* of **u** to be $\mathbf{m}(\mathbf{u}) = (m_0(\mathbf{u}), m_1(\mathbf{u}), \ldots, m_\ell(\mathbf{u}))$ and the *data vector* of **u** to be $\mathbf{d}(\mathbf{u}) = (d_1, d_2, \ldots, d_\ell)$. We call $(\mathbf{m}, \mathbf{d})$ the *MD pair* of **u**.

EXAMPLE 4.4.7. If $\mathbf{u} = (0,0,4,4,4,6,8,8)$, then $\mathbf{m}(\mathbf{u}) = (2,3,1,2)$ and $\mathbf{d}(\mathbf{u}) = (4,6,8)$.

We say that $\mathbf{d} = (d_1,\dots,d_\ell)$ is a *data vector* if it is a data vector of some $\mathbf{u}$, and $\mathbf{m} = (m_0,m_1,\dots,m_\ell)$ is a *multiplicity vector* of *magnitude* $n = m_0 + m_1 + \cdots m_\ell$ if it is a multiplicity vector of some $\mathbf{u}$. (Note that the magnitude is the length of $\mathbf{u}$.) Clearly, $\mathbf{m} = (m_0,m_1,\dots,m_\ell)$ is a multiplicity vector if and only if $m_0,m_1,\dots,m_\ell$ are integers such that $m_0 \geq 0$ and $m_i \geq 1$ for $i = 1,\dots,\ell$. Finally, we say $(\mathbf{m},\mathbf{d})$ is an *MD pair* if $(\mathbf{m},\mathbf{d})$ is the MD pair of some $\mathbf{u}$.

It is clear that starting with an MD pair $(\mathbf{m},\mathbf{d})$, there exists a unique $\mathbf{u}$ such that $(\mathbf{m},\mathbf{d})$ is its MD pair. Hence, we may interchangeably use $(\mathbf{m},\mathbf{d})$ for $\mathbf{u}$ and write $\mathrm{PF}(\mathbf{m},\mathbf{d})$ as $\mathrm{PF}(\mathbf{u})$. As we mentioned above, the face poset and the normal fan of the parking polytope $\mathrm{PF}(\mathbf{m},\mathbf{d})$ only depend on the multiplicity vector $\mathbf{m}$. Therefore, we will mostly use the notation $\mathrm{PF}(\mathbf{m},\mathbf{d})$ in this section.

DEFINITION 4.4.8. Suppose $\mathbf{m} = (m_0,m_1,\dots,m_\ell)$ is a multiplicity vector of magnitude n. Let $r = n - m_0 = m_1 + \cdots m_\ell$. We let $\mathbf{b}_0,\dots,\mathbf{b}_r$ be the following $r+1$ skewed binary compositions of n:

(1) We let

$$\mathbf{b}_0 := \begin{cases} (m_0,0,m_1,\dots,m_\ell) & \text{if } m_0 > 0 \\ (0,0^\circ,m_1,\dots,m_\ell) & \text{if } m_0 = 0 \end{cases}.$$

(2) Suppose $1 \leq k \leq r$. Let j be the unique integer in which $m_1 + \cdots + m_{j-1} < k \leq m_1 + \cdots + m_j$. We define $\mathbf{b}_k := (m_0 + k, 0^\circ, m_1 + \cdots + m_j - k, m_{j+1},\dots,m_\ell)$.

We denote by $\Omega_{\mathbf{m}}$ the set of these $r+1$ skewed binary compositions of n.

EXAMPLE 4.4.9. Let $\mathbf{m} = (2,3,1,2)$ and $\mathbf{m}' = (0,3,5)$ be two multiplicity vectors of magnitude 8. Then

$$\begin{aligned}\Omega_{\mathbf{m}} &= \{(2,0,3,1,2),(3,0^\circ,2,1,2),(4,0^\circ,1,1,2),(5,0^\circ,1,2),(6,0^\circ,2),(7,0^\circ,1),(8,0^\circ)\}\\ \Omega_{\mathbf{m}'} &= \{(0,0^\circ,3,5),(1,0^\circ,2,5),(2,0^\circ,1,5),(3,0^\circ,5),(4,0^\circ,4),(5,0^\circ,3),(6,0^\circ,2),\\ &\qquad (7,0^\circ,1),(8,0^\circ)\}.\end{aligned}$$

Remark 4.4.10. It is easy to see that every preposet $([0,n],\preceq_{\mathcal{B}})$ where $\mathrm{type}(\mathcal{B})\in\Omega_{\mathbf{m}}$ is a poset (every equivalent class in $([0,n],\preceq_{\mathcal{B}})$ is a singleton) whose Hasse diagram is a connected graph. Hence, by Lemma 4.1.4/(3)–(4) that every sliced preorder cone $\tilde{\sigma}_{\mathcal{B}}$ in which $\mathrm{type}(\mathcal{B})\in\Omega_{\mathbf{m}}$ is an n-dimensional pointed cone.

Proposition 4.4.11. *Suppose that $\mathbf{m}=(m_0,m_1,\dots,m_\ell)$ is a multiplicity vector of magnitude n. Let*

$$\mathcal{M}:=\{\tilde{\sigma}_{\mathcal{B}}\subseteq\mathbb{R}^n \;\Big|\; \mathrm{type}(\mathcal{B})\in\Omega_{\mathbf{m}}\}$$

be the collection of sliced preorder cones corresponding to skewed binary partitions whose types are in $\Omega_{\mathbf{m}}$. Then we have that

$$\bigcup_{\tilde{\sigma}\in\mathcal{M}}\tilde{\sigma}=\mathbb{R}^n.$$

Proof. Let $\mathbf{w}=(w_1,\dots,w_n)\in\mathbb{R}^n$. We need to show that there exists $\tilde{\sigma}_{\mathcal{B}}\in\mathcal{M}$ such that $\mathbf{w}\in\tilde{\sigma}_{\mathcal{B}}$. To see this, it suffices, due to symmetry, to only consider $w_1\le\cdots\le w_n$.

If $w_n\le 0$, then $\mathbf{w}$ lies in the cone

$$\{\mathbf{c}\in\mathbb{R}^n \mid c_1,\dots,c_n\le 0\}=\tilde{\sigma}_{\mathcal{B}}$$

where $\mathcal{B}=(\{1,\dots,n\},\{0\})$. Since $\mathrm{type}(\mathcal{B})=(n,0^\circ)\in\Omega_{\mathbf{m}}$, it follows that $\tilde{\sigma}_{\mathcal{B}}\in\mathcal{M}$.

If $w_n>0$, we let $i\in[n]$ be the least positive integer such that $w_i>0$. For $s\in[0,\ell]$, let $t_s:=m_0+m_1+\cdots+m_s$. If $m_0<i$, by letting j be the least integer such that $i\le t_j$, we see that $j\ge 1$ and that $\mathbf{w}$ lies in the cone

$$\{\mathbf{c}\in\mathbb{R}^n \mid c_1,\dots,c_{i-1}\le 0\le c_i,c_{i+1},\dots,c_{t_j}\le c_{t_j+1},\dots,c_{t_{j+1}}\le\cdots\le c_{t_{\ell-1}+1},\dots,c_{t_\ell}\}=\tilde{\sigma}_{\mathcal{B}}$$

where $\mathcal{B}=(\{1,\dots,i-1\},\{0\},\{i,i+1,\dots,t_j\},\{t_j+1,\dots,t_{j+1}\},\dots,\{t_{\ell-1}+1,\dots,t_\ell\})$. Because $\mathrm{type}(\mathcal{B})=(i-1,0^\circ,m_1+\cdots+m_j-i,m_{j+1},\dots,m_\ell)\in\Omega_{\mathbf{m}}$, we see that $\tilde{\sigma}_{\mathcal{B}}\in\mathcal{M}$. If $i\le m_0$, then $0<m_0=t_0$ and $\mathbf{w}$ lies in the cone

$$\{\mathbf{c}\in\mathbb{R}^n \mid 0,c_1,\dots,c_{t_0}\le c_{t_0+1},\dots,c_{t_1}\le\cdots\le c_{t_{\ell-1}+1},\dots,c_{t_\ell}\}=\tilde{\sigma}_{\mathcal{B}}$$

where $\mathcal{B}=(\{0,1,\dots,t_0\},\emptyset,\{t_0+1,\dots,t_1\},\dots,\{t_{\ell-1}+1,\dots,t_\ell\})$. As $\mathrm{type}(\mathcal{B})=(m_0,0,m_1,\dots,m_\ell)\in\Omega_{\mathbf{m}}$, we have $\tilde{\sigma}_{\mathcal{B}}\in\mathcal{M}$. This shows that $\mathbf{w}$ lies in some $\tilde{\sigma}\in\mathcal{M}$. Hence, $\bigcup_{\tilde{\sigma}\in\mathcal{M}}\tilde{\sigma}=\mathbb{R}^n$. ∎

DEFINITION 4.4.12. Let $(\mathbf{m}, \mathbf{d})$ be an MD pair such that $\mathbf{d} = (d_1, \ldots, d_\ell)$ and $\mathbf{m} = (m_0, m_1, \ldots, m_\ell)$ is a multiplicity vector of magnitude n. Suppose that $\mathcal{B} = (B_{-1}, B_0, B_1, \ldots, B_k)$ is a skewed binary partition such that $\text{type}(\mathcal{B}) = (b_{-1}, b_0, b_1, \ldots, b_k) \in \Omega_{\mathbf{m}}$. We define $\mathbf{v}_{\mathcal{B}} = (v_1, \ldots, v_n)$ to be the point in $\mathbb{R}^n$ given by

$$v_i = \begin{cases} 0 & \text{if } i \in B_{-1} \\ d_{\ell-k+j} & \text{if } i \in B_j \text{ for some } j \in [\ell] \end{cases}.$$

It is easy to see that the set $\{\mathbf{v}_{\mathcal{B}} \mid \text{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$ is exactly the set of $(\mathbf{m}, \mathbf{d})$-*extreme points* in $\text{PF}(\mathbf{m}, \mathbf{d})$ (see Definition 4.4.2 where we replace $\mathbf{u}$ with $(\mathbf{m}, \mathbf{d})$). Suppose that $(\mathbf{m}, \mathbf{d})$ is the MD pair of $\mathbf{u} = (u_1, \ldots, u_n)$ and $\text{type}(\mathcal{B}) = (b_{-1}, b_0, b_1, \ldots, b_k)$. Then, we can equivalently define $\mathbf{v}_{\mathcal{B}} = (v_1, \ldots, v_n)$ as the point that is the permutation of the point $(w_1, \ldots, w_n)$ $= (0, \ldots, 0, u_{b_{-1}+1}, \ldots, u_n)$ satisfying $v_i = w_t$ if $i \in B_j$, where $t = |b_{-1}| + \cdots + |b_j|$.

EXAMPLE 4.4.13. Consider the MD pair $(\mathbf{m}, \mathbf{d})$ where $\mathbf{m} = (2, 3, 1, 2)$ and $\mathbf{d} = (4, 6, 8)$. Let $\mathcal{B}, \mathcal{C}$ be the two skewed binary partitions given by

$$\mathcal{B} = (\{0, 2, 3\}, \emptyset, \{1, 6, 7\}, \{8\}, \{4, 5\}) \text{ and } \mathcal{C} = (\{1, 3, 4, 5, 8\}, \{0\}, \{2\}, \{6, 7\}).$$

By Example 4.4.9, one has that $\text{type}(\mathcal{B}), \text{type}(\mathcal{C}) \in \Omega_{\mathbf{m}}$ and that

$$\mathbf{v}_{\mathcal{B}} = (4, 0, 0, 8, 8, 4, 4, 6) \text{ and } \mathbf{v}_{\mathcal{C}} = (0, 6, 0, 0, 0, 8, 8, 0)$$

are two $(\mathbf{m}, \mathbf{d})$-extreme points in $\text{PF}(\mathbf{m}, \mathbf{d})$. In fact, one sees the set of $(\mathbf{m}, \mathbf{d})$-extreme points given in Example 4.4.3 is exactly the set $\{\mathbf{v}_{\mathcal{B}} \mid \text{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$.

The next theorem shows that $\{\mathbf{v}_{\mathcal{B}} \mid \text{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$ is precisely the set of vertices of $\text{PF}(\mathbf{m}, \mathbf{d})$ and describes the normal cone at each vertex.

THEOREM 4.4.14. *Let $(\mathbf{m}, \mathbf{d})$ be an MD pair. Then, the map $\mathcal{B} \mapsto \mathbf{v}_{\mathcal{B}}$ defines a bijection between the set $\{\mathcal{B} \mid \text{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$ and the set of the vertices of $\text{PF}(\mathbf{m}, \mathbf{d})$. Moreover, we have that $\text{ncone}(\mathbf{v}_{\mathcal{B}}, \text{PF}(\mathbf{m}, \mathbf{d})) = \tilde{\sigma}_{\mathcal{B}}$.*

Proof. Suppose that $\mathcal{B} = (B_{-1}, B_0, B_1, \dots, B_k)$ satisfies $\text{type}(\mathcal{B}) = (b_{-1}, b_0, \dots, b_k) \in \Omega_{\mathbf{m}}$. Since $\{\mathbf{v}_{\mathcal{B}} \mid \text{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$ equals the set of $(\mathbf{m}, \mathbf{d})$-extreme points, it follows from Proposition 4.4.1 that $\text{PF}(\mathbf{m}, \mathbf{d}) = \text{conv}(\mathbf{v}_{\mathcal{B}} \mid \text{type}(\mathcal{B}) \in \Omega_{\mathbf{m}})$. To prove the statement, we first show that $\mathbf{v}_{\mathcal{B}}$ is a vertex of $\text{PF}(\mathbf{m}, \mathbf{d})$ and $\text{ncone}(\mathbf{v}_{\mathcal{B}}, \text{PF}(\mathbf{m}, \mathbf{d})) = \tilde{\sigma}_{\mathcal{B}}$. By Theorem 4.4.11 and Lemma 1.2.2, it suffices to show that, for every $\mathcal{B}'$ such that $\text{type}(\mathcal{B}') \in \Omega_{\mathbf{m}}$ and $\mathcal{B}' \neq \mathcal{B}$, one has

$$\mathbf{c} \cdot \mathbf{v}_{\mathcal{B}} > \mathbf{c} \cdot \mathbf{v}_{\mathcal{B}'} \text{ for all } \mathbf{c} \in \tilde{\sigma}^{\circ}_{\mathcal{B}}.$$

Let $\mathbf{c} = (c_1, \dots, c_n) \in \tilde{\sigma}^{\circ}_{\mathcal{B}}$. Then, by Theorem 4.1.4/(2), the vector $\mathbf{c}$ satisfies

$$c_p < c_q \text{ if } p \in B_i \text{ and } q \in B_j \text{ and } i < j$$

where we set $c_0 = 0$. Similarly, by definition, the point $\mathbf{v}_{\mathcal{B}} = (v_1, \dots, v_n)$ satisfies

$$v_p = 0 \text{ if } p \in B_{-1}, \text{ and } 0 < v_p < v_q \text{ if } p \in B_i \text{ and } q \in B_j \text{ and } 1 \leq i < j.$$

One sees that $c_p < c_q$ provided $v_p < v_q$, and that $0 < c_p$ provided $0 < v_p$. Moreover, the non-decreasing rearrangement of $v_1, \dots, v_n$ is given by the sequence

$$\underbrace{0, \dots, 0}_{|b_{-1}| \text{ terms}}, \underbrace{d_{\ell-k+1}, \dots, d_{\ell-k+1}}_{|b_1| \text{ terms}}, \dots, \underbrace{d_{\ell-k+k}, \dots, d_{\ell-k+k}}_{|b_k| \text{ terms}}.$$

Let $\mathcal{B}' = (B'_{-1}, B'_0, B'_1, \dots, B'_t)$ be a skewed binary partition satisfying $\text{type}(\mathcal{B}') = (b'_{-1}, b'_0, \dots, b'_t) \in \Omega_{\mathbf{m}}$ and $\mathcal{B}' \neq \mathcal{B}$. Then, $\mathbf{v}_{\mathcal{B}'} = (v'_1, \dots, v'_n)$ is a point such that $\mathbf{v}_{\mathcal{B}} \neq \mathbf{v}_{\mathcal{B}'}$ and the non-decreasing rearrangement of $v'_1, \dots, v'_n$ is given by

$$\underbrace{0, \dots, 0}_{|b'_{-1}| \text{ terms}}, \underbrace{d_{\ell-t+1}, \dots, d_{\ell-t+1}}_{|b'_{-1}| \text{ terms}}, \dots, \underbrace{d_{\ell-t+t}, \dots, d_{\ell-t+t}}_{|b'_t| \text{ terms}}.$$

If $|b_{-1}| \leq |b'_{-1}|$, then, by the definition of $\Omega_{\mathbf{m}}$, the sequence

$$\underbrace{d_{\ell-k+1}, \dots, d_{\ell-k+1}}_{|b_1| \text{ terms}}, \dots, \underbrace{d_{\ell-k+k}, \dots, d_{\ell-k+k}}_{|b_k| \text{ terms}} \tag{4.4.2}$$

can be obtained by removing the first $(|b'_{-1}| - |b_{-1}|)$ terms of the sequence

$$\underbrace{d_{\ell-t+1}, \dots, d_{\ell-t+1}}_{|b'_1| \text{ terms}}, \dots, \underbrace{d_{\ell-t+t}, \dots, d_{\ell-t+t}}_{|b'_t| \text{ terms}}. \tag{4.4.3}$$

Thus, by applying the rearrangement inequality, one sees that $\mathbf{c} \cdot \mathbf{v}_{\mathcal{B}} > \mathbf{c} \cdot \mathbf{v}_{\mathcal{B}'}$.

Similarly, if $|b_{-1}| > |b'_{-1}|$, then the sequence in (4.4.3) can be obtained by removing the first $(|b'_{-1}| - |b_{-1}|)$ terms of the sequence in (4.4.2). In this case, we must have that $|b_{-1}| > m_0$ and that

$$c_p < 0 \text{ if } p \in B_{-1} \backslash \{0\}, \text{ and } 0 < c_p \text{ if } p \in B_j \text{ for some } j \geq 1.$$

Thus, the rearrangement inequality implies that $\mathbf{c} \cdot \mathbf{v}_{\mathcal{B}} > \mathbf{c} \cdot \mathbf{v}_{\mathcal{B}'}$.

Therefore, by Theorem 4.4.11 and Lemma 1.2.2, the point $\mathbf{v}_{\mathcal{B}}$ is a vertex of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ and $\mathrm{ncone}(\mathbf{v}_{\mathcal{B}}, \mathrm{PF}(\mathbf{m}, \mathbf{d})) = \tilde{\sigma}_{\mathcal{B}}$. Consequently, the map $\mathcal{B} \mapsto \mathbf{v}_{\mathcal{B}}$ is a bijection between the set $\{\mathcal{B} \mid \mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$ and the set of the vertices of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$. □

It is apparent that every polytope of dimension at most 2 is simple. When a polytope has dimension greater than 2, it becomes nontrivial to verify its simplicity. Theorem 4.4.14 allows us to determine exactly when $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is a simple polytope.

Corollary 4.4.15. *Let $(\mathbf{m}, \mathbf{d})$ be an MD pair where $\mathbf{m} = (m_0, m_1, \dots, m_\ell)$. Then $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is simple if and only if either $\mathbf{m} = (0, n)$ or $(n-1, 1)$ or $m_1 = \cdots = m_{\ell-1} = 1$ for some $\ell \geq 2$.*

Proof. Recall that an n-dimensional polytope in $\mathbb{R}^n$ is simple if and only if, for every vertex of the polytope, the normal cone at the vertex is simplicial. Since $\{\mathbf{v}_{\mathcal{B}} \mid \mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}\}$ is the set of vertices of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ and $\mathrm{ncone}(\mathbf{v}_{\mathcal{B}}, \mathrm{PF}(\mathbf{m}, \mathbf{d})) = \tilde{\sigma}_{\mathcal{B}}$, it suffices by Lemma 4.1.4/(5) to show that the preorder $\preceq_{\mathcal{B}}$ defines a poset whose Hasse diagram is a tree for all $\mathcal{B}$ satisfying $\mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}$ if and only if either $\mathbf{m} = (0, n)$ or $(n-1, 1)$ or $m_1 = \cdots = m_{\ell-1} = 1$ for some $\ell \geq 2$.

If $\ell = 1$, then it's easy to check that $([0, n], \preceq_{\mathcal{B}})$ is a poset whose Hasse diagram is a tree for all $\mathcal{B}$ with $\mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}$ if and only if $\mathbf{m} = (0, n)$ or $(n-1, 1)$. Similarly, if $\ell \geq 2$, then $([0, n], \preceq_{\mathcal{B}})$ is a poset whose Hasse diagram is a tree for all $\mathcal{B}$ satisfying $\mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}$ if and only if $m_1 = \cdots = m_{\ell-1} = 1$. □

Theorem 4.4.14 describes the full dimensional cones in $\Sigma(\mathrm{PF}(\mathbf{u}))$, the normal fan of $\mathrm{PF}(\mathbf{u})$. Since the lower dimensional cones are faces of the full dimensional cones in $\Sigma(\mathrm{PF}(\mathbf{u}))$, it follows from Lemma 4.1.4/(1) that they correspond to the contractions of the preorders $\preceq_{\mathcal{B}}$ on $[0, n]$ where $\mathcal{B}$ is a skewed binary partition such that $\mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}$.

Definition 4.4.16. Let $\mathbf{m}$ be a multiplicity vector. We define $\mathcal{BWP}(\mathbf{m})$ to be the poset of all skewed binary partitions $\mathcal{B}$ such that $\mathrm{type}(\mathcal{B}) \in \Omega_{\mathbf{m}}$ and their contractions, ordered by contraction, i.e., $\mathcal{C}, \mathcal{B} \in \mathcal{BWP}(\mathbf{m})$ satisfty $\mathcal{C} \leq \mathcal{B}$ if $\mathcal{C}$ is a contraction of $\mathcal{B}$.

The following result is then an immediate consequence of Theorem 4.4.14 and Lemma 4.1.4/(1).

Corollary 4.4.17. *Let $(\mathbf{m}, \mathbf{d})$ be an MD pair. Then the posets $\mathcal{BWP}(\mathbf{m})$ and $\mathcal{F}(\Sigma(\mathrm{PF}(\mathbf{m}, \mathbf{d})))$ are isomorphic. Moreover, if F is the face of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ in which $\mathrm{ncone}(F, \mathrm{PF}(\mathbf{m}, \mathbf{d}))$ corresponds to the skewed binary partition $\mathcal{B}$, then $\mathrm{ncone}(F, \mathrm{PF}(\mathbf{m}, \mathbf{d})) = \tilde{\sigma}_{\mathcal{B}}$.*

Thus, the combinatorial types of parking function polytopes only depend on the multiplicity vector, i.e., two parking functions polytopes $\mathrm{PF}(\mathbf{u}_1)$ and $\mathrm{PF}(\mathbf{u}_2)$ have isomorphic face posets if $\mathbf{m}(\mathbf{u}_1) = \mathbf{m}(\mathbf{u}_2)$. It is not difficult to see that there are $2^n - 1$ distinct multiplicity vectors $\mathbf{m}$ of magnitude n such that $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is n-dimensional. Figure 4.5 shows that there are exactly three different combinatorial types of 2-dimensional parking function polytopes $\mathrm{PF}(\mathbf{m}, \mathbf{d})$, and two distinct multiplicity vectors correspond to different types.

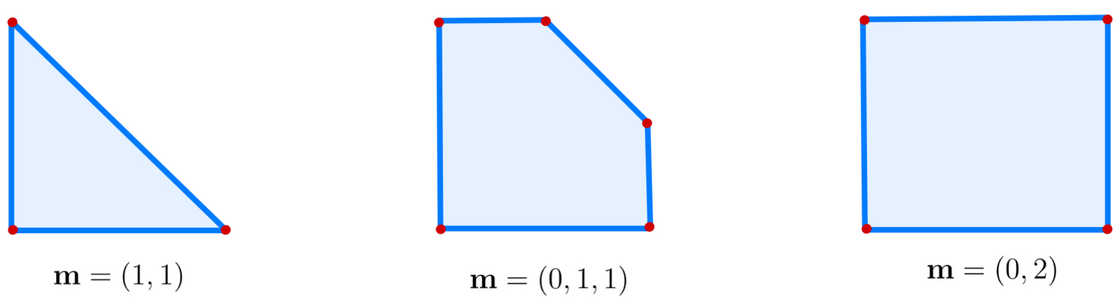


Figure 4.5. Three different combinatorial types of 2-dimensional $\mathrm{PF}(\mathbf{m}, \mathbf{d})$

Remark 4.4.18. Corollary 4.4.17 implies that the normal fan of every parking function polytope $\mathrm{PF}(\mathbf{u})$ is a coarsening of the normal fan of $\mathrm{PF}(1, 2, \ldots, n)$, meaning $\mathrm{PF}(\mathbf{u})$ can be viewed as a deformation of $\mathrm{PF}(1, 2, \ldots, n)$.

It is then natural to ask how we can describe skewed binary partitions in the correspondence. Due to the symmetry of parking function polytope, we have that if a skewed binary partition $\mathcal{B}$ corresponds to a normal cone in $\Sigma(\mathrm{PF}(\mathbf{m},\mathbf{d}))$, then every skewed binary partition of the same type $\mathrm{type}(\mathcal{B})$ also corresponds to a normal cone in $\Sigma(\mathrm{PF}(\mathbf{m},\mathbf{d}))$. Thus, we can describe these skewed binary partitions by their types. To have a better idea of how we can describe them, we present an example first.

Example 4.4.19. Consider the MD pair $(\mathbf{m},\mathbf{d})$ where $\mathbf{m} = (2,3,1,2)$ and $\mathbf{d} = (4,6,8)$. The skewed binary partition $\mathcal{B} = (\{0,2,3\},\emptyset,\{1,6,7\},\{8\},\{4,5\})$ is from $\mathcal{BWP}(\mathbf{m})$. We saw in Example 4.3.7 and Example 4.4.13 that $\mathcal{B}$ corresponds to the sliced preorder cone

$$\tilde{\sigma}_{\mathcal{B}} = \{(c_1,\ldots,c_8) \in \mathbb{R}^8 \mid 0, c_2, c_3 \le c_1, c_6, c_7 \le c_8 \le c_4, c_5\},$$

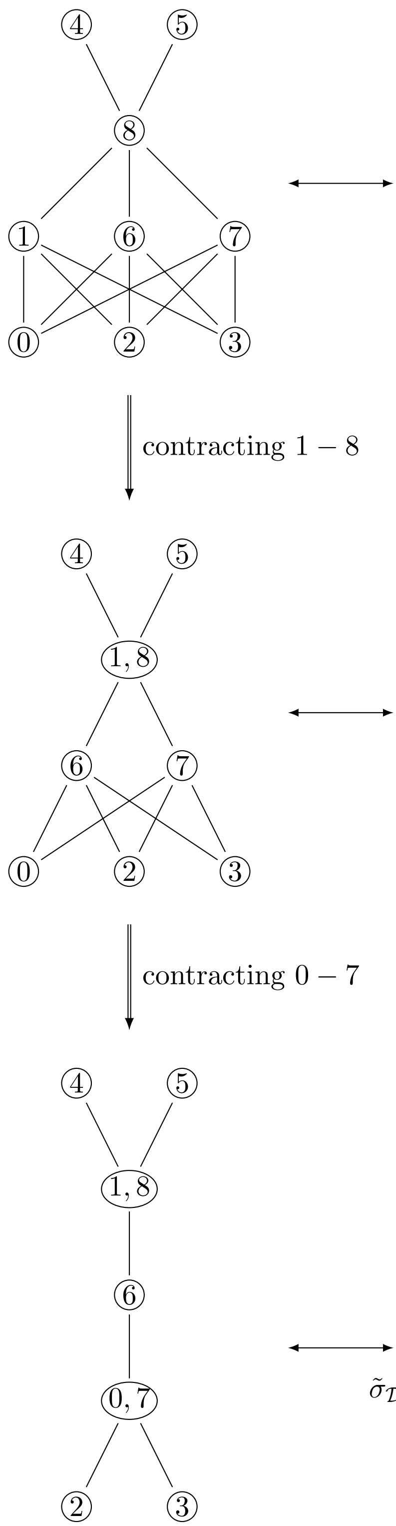
which is also the normal cone at the vertex $\mathbf{v}_{\mathcal{B}} = (4,0,0,8,8,4,4,6)$. The table below shows examples of skewed binary partitions in a maximal chain of $\mathcal{BWP}(\mathbf{m})$, each of which corresponds to contractions of $\preceq_{\mathcal{B}}$. They also correspond to the normal cones in $\Sigma(\mathrm{PF}(\mathbf{m},\mathbf{d}))$ of dimensions 8 to 0 (in decreasing order of dimensions). The first three skewed binary partitions are also displayed in Figure 4.6.

skewed binary partition	type
$(\{0,2,3\},\emptyset,\{1,6,7\},\{8\},\{4,5\})$	$(2,0,3,1,2)$
$(\{0,2,3\},\emptyset,\{6,7\},\{1,8\}^\star,\{4,5\})$	$(2,0,2,2^\star,2)$
$(\{2,3\},\{0,7\}^\star,\{6\},\{1,8\}^\star,\{4,5\})$	$(2,1^\circ,1,2^\star,2)$
$(\{3\},\{0,2,7\}^\star,\{6\},\{1,8\}^\star,\{4,5\})$	$(1,2^\circ,1,2^\star,2)$
$(\{3\},\{0,2,7\}^\star,\{1,6,8\}^\star,\{4,5\})$	$(1,2^\circ,3^\star,2)$
$(\{3\},\{0,2,7\}^\star,\{1,5,6,8\}^\star,\{4\})$	$(1,2^\circ,4^\star,1)$
$(\{3\},\{0,1,2,5,6,7,8\}^\star,\{4\})$	$(1,6^\circ,1)$
$(\emptyset,\{0,1,2,3,5,6,7,8\}^\star,\{4\})$	$(0,7^\circ,1)$
$(\emptyset,\{0,1,2,3,4,5,6,7,8\}^\star)$	$(0,8^\circ)$

We characterize the types of the ordered skewed binary partitions corresponding to the normal cones in $\Sigma(\mathrm{PF}(\mathbf{m},\mathbf{d}))$ in the following proposition.

$([0,8], \preceq_{\mathcal{B}})$

$$\mathcal{B} = (\{0,2,3\}, \emptyset, \{1,6,7\}, \{8\}, \{4,5\})$$
$$\text{type}(\mathcal{B}) = (2,0,3,1,2)$$
$$\tilde{\sigma}_{\mathcal{B}} = \{0, c_2, c_3 \leq c_1, c_6, c_7 \leq c_8 \leq c_4, c_5\}$$



$([0,8], \preceq_{\mathcal{C}})$

$$\mathcal{C} = (\{0,2,3\}, \emptyset, \{6,7\}, \{1,8\}^{\star}, \{4,5\})$$
$$\text{type}(\mathcal{C}) = (2,0,2,2^{\star},2)$$
$$\tilde{\sigma}_{\mathcal{C}} = \{0, c_2, c_3 \leq c_6, c_7 \leq c_1 = c_8 \leq c_4, c_5\}$$

$([0,8], \preceq_{\mathcal{D}})$

$$\mathcal{D} = (\{2,3\}, \{0,7\}^{\star}, \{6\}, \{1,8\}^{\star}, \{4,5\})$$
$$\text{type}(\mathcal{D}) = (2,1^{\circ},1,2^{\star},2)$$
$$\tilde{\sigma}_{\mathcal{D}} = \{c_2, c_3 \leq 0 = c_7 \leq c_6 \leq c_1 = c_8 \leq c_4, c_5\}$$

FIGURE 4.6. Both $\preceq_{\mathcal{C}}$ and $\preceq_{\mathcal{D}}$ are contractions $\preceq_{\mathcal{B}}$

Proposition 4.4.20. *Suppose that* $\mathbf{m} = (m_0, \ldots, m_\ell)$ *is a multiplicity vector of magnitude* n*, and* $\mathcal{B}$ *is a skewed binary partition of* $[0,n]$*. Then* $\mathcal{B}$ *is in* $\mathcal{BWP}(\mathbf{m})$ *if and only if* $\text{type}(\mathcal{B}) = (b_{-1}, b_0, \ldots, b_p)$ *is a skewed binary composition satisfying the following conditions.*

(1) $0 < |b_{-1}| + |b_0| \leq m_0$ *if and only if* $b_{-1} \neq 0$ *and* $b_0 = 0$*.*

(2) $m_0 < |b_{-1}| + |b_0| + |b_1|$ *and for every positive integer* $i \leq \ell$*, there exists at most one positive integer* j *such that*

$$m_0 + \cdots + m_{i-1} \leq |b_{-1}| + \cdots + |b_{j-1}| < |b_{-1}| + \cdots + |b_j| \leq m_0 + \cdots + m_i. \tag{4.4.4}$$

(3) If j *is a positive integer such that there exists a positive integer* i *satisfying* (4.4.4)*, then* $b_j \in \mathbb{P}$*. Otherwise,* $b_j \in \mathbb{P}^\star$ *for* $1 \leq j \leq p$*.*

Before proving Proposition 4.4.20, we first establish a few auxiliary results.

Lemma 4.4.21. *Let* $\mathbf{m} = (m_0, \ldots, m_\ell)$ *be a multiplicity vector, and* $\mathcal{A} = (A_{-1}, A_0, \ldots, A_q)$ *be a skew binary partition such that* $\text{type}(\mathcal{A}) = (a_{-1}, a_0, \ldots, a_q) \in \Omega_{\mathbf{m}}$*. If* $\mathcal{B} = (B_{-1}, B_0, \ldots, B_p)$ *is a contraction of* $\mathcal{A}$*, then* $\text{type}(\mathcal{B}) = (b_{-1}, b_0, \ldots, b_p)$ *satisfies* $m_0 < |b_{-1}| + |b_0| + |b_1|$*.*

Proof. Let us first consider the case where $\mathcal{B} = (B_{-1}, B_0)$. Then, $B_{-1} \cup B_0 = [0, n]$ and so $n = |b_{-1}| + |b_0|$. Since $\text{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$, we have $m_0 \leq |a_{-1}|$. If $m_0 = |a_{-1}|$, then $0 < m_0$ and $\text{type}(\mathcal{A}) = (m_0, 0, m_1, \ldots, m_\ell)$ for some $\ell \geq 1$. Thus, we obtain $m_0 = |a_{-1}| < n = |b_{-1}| + |b_0|$ as desired. If $m_0 < |a_{-1}|$, then also have $m_0 < |a_{-1}| \leq n = |b_{-1}| + |b_0|$.

Now consider $\mathcal{B} = (B_{-1}, B_0, \ldots, B_p)$ for some $p \geq 1$. Note that since one of B_{-1} and B_0 can be empty, one of them may not be a vertex of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$. We claim that

$$A_{-1} \text{ is a proper subset of } (B_{-1} \cup B_0 \cup B_1) \backslash \{0\}, \tag{4.4.5}$$

which then gives $m_0 \leq |a_{-1}| < |b_{-1}| + |b_0| + |b_1|$ as desired. To see this, it suffices by the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ to show that the vertex B_1 is adjacent to A_s for some positive integer $s \in [p]$. Since $0 \notin B_1$ and A_0 is either $\{0\}$ or $\emptyset$, the vertex B_1 of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is not adjacent to A_0. Thus, if $A_{-1} = \emptyset$, then clearly the vertex B_1 must be adjacent to A_s for some positive integer $s \in [p]$. Hence, we may suppose that A_{-1} is nonempty. Assume for the sake of contradiction that the vertex B_1 of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is only adjacent to the non-homogeneous vertex A_{-1}. Then, the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ implies that B_{-1} and B_0 are only adjacent to A_{-1} as well. Thus, by Theorem 4.2.7/(5), the vertices B_0 (if nonempty) and B_1 are non-homogeneous. However, this means that the left non-homogeneous vertex A_{-1} are adjacent to at least two non-homogeneous vertices in $\{B_{-1}, B_0, B_1\}$, a contradiction to Theorem 4.2.7/(2). Therefore, the claim must hold. $\square$

LEMMA 4.4.22. *Let* $\mathbf{m} = (m_0, \ldots, m_\ell)$ *be a multiplicity vector, and* $\mathcal{A}$ *be a skew binary partition such that* $\operatorname{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$. *Suppose that* p *is a positive integer, and* $\mathcal{B} = (B_{-1}, B_0, \ldots, B_p)$ *is a skew binary partition that is a contraction of* $\mathcal{A}$. *Let* $j \in [p]$ *be a positive integer. Then, the right vertex* B_j *of* $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ *has degree at least two if and only if there exists a nonnegative integer* $k \le \ell$ *such that* $\operatorname{type}(\mathcal{B}) = (b_{-1}, \ldots, b_p)$ *satisfies*

$$|b_{-1}| + |b_0| + \cdots + |b_{j-1}| < m_0 + \cdots m_k < |b_{-1}| + |b_0| + \cdots + |b_j|.$$

PROOF. Let $j \in [p]$ be a positive integer. Let us write $\mathcal{A} = (A_{-1}, A_0, \ldots, A_q)$ and $\operatorname{type}(\mathcal{A}) = (a_{-1}, a_0, \ldots, a_q)$. Note that since $\operatorname{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$, one has $|a_{-1}| \ge m_0$, and A_0 is either $\{0\}$ or $\emptyset$.

($\Longrightarrow$) Suppose that the right vertex B_j of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ has degree at least two. Then, by (4.4.5) in the proof of Lemma 4.4.21, one has that A_{-1} is a proper subset of $(B_{-1} \cup B_0 \cup B_1)\backslash\{0\}$, and

$$|a_{-1}| = |a_{-1}| + |a_0| < |b_{-1}| + |b_0| + |b_1| \le |b_{-1}| + |b_0| + \cdots + |b_{j-1}|.$$

By the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the right vertex B_j must be adjacent to a vertex A_s for some positive integer $s \in [q]$. Since B_j has degree at least two, we may let A_t where $t \ne s$ be another left vertex of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ adjacent to B_j. We first consider $t \ge 1$. Then, we may assume without loss of generality that $1 \le s < t$. Since $\operatorname{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$, we may write

$$(a_{-1}, a_0, \ldots, a_q) = (a_{-1}, a_0, m_0 + \cdots + m_g - a_{-1}, m_{g+1}, \ldots, m_\ell), \tag{4.4.6}$$

where $a_0 \in \{0, 0^\circ\}$ and $g < \ell$ is a positive integer such that $m_0 + \cdots + m_{g-1} \le a_{-1} < m_0 + \cdots + m_g$. Note that, for $f \in [0, q]$ and $h \in [0, p]$, one has

$$\sum_{r=0}^{f} |A_r| = 1 + \sum_{r=0}^{f} |a_r| \quad \text{and} \quad \sum_{r=-1}^{h} |B_r| = 1 + \sum_{r=-1}^{h} |b_r|. \tag{4.4.7}$$

Since both $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ and $G(\hat{\mathcal{B}}, \hat{\mathcal{A}})$ are non-crossing, and the intersections $B_j \cap A_s$ and $A_t \cap B_j$ are nonempty, we deduce from (4.4.7) and Lemma 4.2.6/(2) that

$$\sum_{r=-1}^{j-1} |b_r| < \sum_{r=-1}^{s} |a_r| \leq \sum_{r=-1}^{t-1} |a_r| < \sum_{r=-1}^{j} |b_r|.$$

Because $g + s - 1 \geq 1$ and $\sum_{r=-1}^{s} |a_r| = \sum_{r=0}^{g+s-1} m_r$, we obtain as desired

$$|b_{-1}| + |b_0| + \cdots + |b_{j-1}| < m_0 + \cdots m_{g+s-1} < |b_{-1}| + |b_0| + \cdots + |b_j|.$$

Now consider $t \leq 0$. Since $0 \notin B_j$ and A_0 is either $\{0\}$ or $\emptyset$, it follows that $t = -1$. By the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the vertices B_{-1}, $B_0, \ldots, B_{j-1}$ have degree one and are adjacent to A_{-1}, and the block $A_0 = \emptyset$. Thus, $B_{-1} \cup B_0 \cup \cdots \cup B_{j-1}$ is a proper subset of A_{-1}. As $\operatorname{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$, we must have $m_0 > 0$ and $\operatorname{type}(\mathcal{A}) = (m_0, 0, m_1, \ldots, m_\ell)$. Moreover, by Theorem 4.2.7/(5), the right vertices B_{-1}, $B_0, \ldots, B_{j-1}$ are non-homogeneous and are adjacent to the left non-homogeneous vertex A_{-1}. By Theorem 4.2.7/(2), we must have $j = 1$. Hence,

$$|b_{-1}| + |b_0| + \cdots + |b_{j-1}| = |b_{-1}| + |b_0| < |a_{-1}| = m_0 < |b_{-1}| + |b_0| + \cdots + |b_j|.$$

($\Longleftarrow$) Conversely, suppose that there exists a nonnegative integer $k \leq \ell$ such that $\operatorname{type}(\mathcal{B}) = (b_{-1}, \ldots, b_p)$ satisfies

$$|b_{-1}| + |b_0| + \cdots + |b_{j-1}| < m_0 + \cdots m_k < |b_{-1}| + |b_0| + \cdots + |b_j|.$$

If $g \leq k$, then one sees from (4.4.7) together with the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ and Lemma 4.2.6/(1) that B_j is adjacent to A_{k-g+1} and A_i for some $i > k - g + 1$. Thus, in this case, the vertex B_j has degree at least two. On the other hand, if $k < g$, then the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ implies that B_j is adjacent to A_{-1}. Moreover, by (4.4.5) in the proof of Lemma 4.4.21 and the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, we must have that B_j is adjacent to a vertex A_s for some positive integer $s \in [q]$. Hence, the vertex B_j also has degree at least two in this case. This completes the proof. □

Definition 4.4.23. A skewed binary partition $\mathcal{B} = (B_{-1}, B_0, B_1, \ldots, B_p)$ is *standard* if for every nonnegative integer $i \leq p$ every positive integer in B_i is greater than every positive integer in B_{i-1}.

We make the following remarks of useful properties of standard skewed binary partition for later reference.

Remark 4.4.24. If $\mathcal{B} = (B_{-1}, B_0, B_1, \ldots, B_p)$ is standard with type$(\mathcal{B}) = (b_{-1}, \cdots, b_p)$, then

$$|b_{-1}| + \cdots + |b_i| = \text{ the maximum integer in } B_i \backslash \{0\}$$

for all i such that $B_i \backslash \{0\} \neq \emptyset$.

Proof of Proposition 4.4.20. Let $\mathcal{B}$ be a skewed binary partition of $[0, n]$.

($\Longrightarrow$) Suppose that $\mathcal{B} \in \mathcal{BWP}(\mathbf{m})$ and type$(\mathcal{B}) = (b_{-1}, b_0, \ldots, b_p)$ is a skew binary composition. Thus, $\mathcal{B}$ is a contraction of $\mathcal{A} = (A_{-1}, \ldots, A_q)$ for some $\mathcal{A}$ such that type$(\mathcal{A}) = (a_{-1}, a_0, \ldots, a_q) \in \Omega_{\mathbf{m}}$. Note that $|a_{-1}| \geq m_0$, and A_0 is either $\{0\}$ or $\emptyset$.

We first show that $(b_{-1}, \ldots, b_p)$ satisfies condition (1). Suppose that $0 < |b_{-1}| + |b_0| \leq m_0$. Since $|a_{-1}| \geq m_0 > 0$, the block A_{-1} is nonempty. Assume for the sake of contradiction that $b_0 \neq 0$. Then $0 \in B_0$. Since $0 < |b_{-1}| + |b_0| = |(B_{-1} \cup B_0) \backslash \{0\}|$, there exists a positive integer in B_{-1} or B_0. If a positive integer is in B_0, then B_0 is homogeneous. By Theorem 4.2.7/(5), the right homogeneous vertex B_0 of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ must be adjacent to a vertex other than A_{-1}. The non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ then implies that A_{-1} can only be adjacent to B_{-1} or B_0 (or both). Thus, A_{-1} is a proper subset of $B_{-1} \cup B_0$. However, this is not possible, since it would give $|b_{-1}| + |b_0| = |b_0| > |a_{-1}| \geq m_0$. Hence, we must have $B_0 = \{0\}$ and that B_{-1} contains a positive integer. In particular, B_{-1} is a nonempty block and is a right non-homogeneous vertex of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ adjacent to A_{-1}. Since, by 4.2.7/(2), the left non-homogeneous vertex A_{-1} of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ cannot be adjacent to both non-homogeneous vertices B_{-1} and B_0, it follows that $0 \in A_0$. Because type$(\mathcal{A}) \in \Omega_{\mathbf{m}}$, it follows that $|a_{-1}| > m_0$. Moreover, in $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the vertices A_0 and B_0 are adjacent. By the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the vertex A_{-1} can only be adjacent to B_{-1} or B_0 (or both). Thus, $A_{-1} \subseteq B_{-1} \cup B_0$. This means $|b_{-1}| + |b_0| \geq |a_{-1}| > m_0$, a contradiction. Therefore, we must have $b_0 = 0$ as desired. It then follows from the definition of skew binary partition that $B_0 = \emptyset$, and B_{-1} contains 0 and another positive integer. Hence, we have $b_{-1} \neq 0$.

Conversely, suppose that $b_{-1} \neq 0$ and $b_0 = 0$. Then, $B_0 = \emptyset$, and B_{-1} contains 0 and another positive integer. By Theorem 4.2.7/(4), the left non-homogeneous B_{-1} of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ has degree one.

This implies that B_{-1} is a subset of a block A_j for some $j \in [-1, q]$. In particular, we must have that $A_0 \neq \{0\}$, since $A_0 = \{0\}$ would intersect but would not contain B_{-1}. Because type$(\mathcal{A}) \in \Omega_{\mathbf{m}}$, it follows that $m_0 > 0$, $A_0 = \emptyset$, and $|a_{-1}| = m_0$. By the non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ and Theorem 4.2.7/(4), the vertex B_{-1} is only adjacent to A_{-1}. Hence, we have $B_{-1} \subseteq A_{-1}$, which then implies $0 < |b_{-1}| = |b_{-1}| + |b_0| \leq |a_{-1}| = m_0$ as desired.

Next, we show that type$(\mathcal{B})$ satisfies condition (2). By Lemma 4.4.21, we have $m_0 < |b_{-1}| + |b_0| + |b_1|$. Assume by way of contradiction that there exists a positive integer $i \leq \ell$ together with two consecutive positive integers $j, j+1$ satisfying inequality (4.4.4). Let $k_t = |b_{-1}| + \cdots + |b_{j+t-1}|$ for $t = 0, 1, 2$. The assumption implies that

$$m_0 + \cdots + m_{i-1} \leq k_0 < k_1 < k_2 \leq m_0 + \cdots + m_i. \tag{4.4.8}$$

Then, by Lemma 4.4.22, both vertices B_j and B_{j+1} of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ must have degree one. Because every left vertex of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is non-homogeneous, it follows from Theorem 4.2.7/(5) that both B_j and B_{j+1} are non-homogeneous. As shown in (4.4.5) in the proof of Lemma 4.4.21, one has that A_{-1} is a proper subset of $(B_{-1} \cup B_0 \cup B_1) \backslash \{0\}$, and

$$|a_{-1}| = |a_{-1}| + |a_0| < |b_{-1}| + |b_0| + |b_1| \leq |b_{-1}| + |b_0| + \cdots + |b_{j-1}|.$$

The non-crossing property of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ then implies that the vertex B_j must be adjacent to A_s for some positive integer $s \in [q]$, and the vertex B_{j+1} must be adjacent to A_t for some positive integer $t \in [q]$ such that $s \leq t$. Thus, $B_j \subseteq A_s$ and $B_{j+1} \subseteq A_t$. Moreover, because both B_j and B_{j+1} are non-homogeneous, Theorem 4.2.7/(3) implies that $A_s \neq A_t$, i.e., $1 \leq s \neq t$. Then, we have by Lemma 4.2.6/(3)

$$k_1 = \sum_{r=-1}^{j} |b_r| < \sum_{r=-1}^{s} |a_r| = \sum_{r=0}^{g+s-1} m_r \leq \sum_{r=-1}^{t-1} |a_r| < \sum_{r=-1}^{j+1} |b_r| = k_2,$$

a contradiction to the assumption (4.4.8). This shows that type$(\mathcal{B})$ satisfies condition (2).

We now show that type$(\mathcal{B})$ satisfies condition (3). Suppose that j is a positive integer such that there exists a positive integer i satisfying inequality (4.4.4). Let us define k_0, k_1 as before and write $(a_{-1}, a_0, \ldots, a_q)$ as in equation (4.4.6). Then,

$$m_0 + \cdots + m_{i-1} \le k_0 < k_1 \le m_0 \cdots + m_i.$$

By Lemma 4.4.22, the vertex B_j of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ has degree one. Since every left vertex of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is non-homogeneous, we see from Theorem 4.2.7/(5) that B_j is non-homogeneous. Therefore, $b_j \in \mathbb{P}$.

Now suppose that j is a positive integer such that there is no positive integer i satisfying inequality (4.4.4). As shown in (4.4.5) in the proof of Lemma 4.4.21, we have $m_0 \le |a_{-1}| + |a_0| < |b_{-1}| + |b_0| + |b_1| \le k_1$, and the vertex B_j of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is adjacent to A_s for some positive integer $s \in [q]$. This implies that there exists a nonnegative integer $i \le \ell$ such that

$$k_0 < m_0 + \cdots + m_i < k_1.$$

Thus, by Lemma 4.4.22, the vertex B_j of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ also has at least degree two. Thus, by Theorem 4.2.7/(4), the block B_j is homogeneous. Hence, $b_j \in B^\star$.

($\Longleftarrow$) Conversely, suppose that $\mathcal{B}$ is a skew binary partition such that $\mathrm{type}(\mathcal{B}) = (b_{-1}, \ldots, b_p)$ satisfies conditions (1) - (3). By symmetry, we may assume without loss of generality that $\mathcal{B}$ is standard. To see that $\mathcal{B} \in \mathcal{BWP}(\mathbf{m})$, we will construct $\mathcal{A}$ such that $\mathcal{B} \le \mathcal{A}$ and $\mathrm{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$.

Case 1: $|b_{-1}| = n$. Then, $|b_{-1}| + |b_0| > m_0$. Thus, by condition (1), we have $\mathrm{type}(\mathcal{B}) = (n, 0^\circ)$, that is, $\mathcal{B} = ([n], \{0\})$. Let $\mathcal{A} = \mathcal{B}$. Since $\mathrm{type}(\mathcal{A}) = \mathrm{type}(\mathcal{B}) = (n, 0^\circ) \in \Omega_{\mathbf{m}}$, it follows that $\mathcal{B} \in \mathcal{BWP}(\mathbf{m})$.

Case 2: $m_0 < |b_{-1}| < n$. Let j be the unique positive integer in which $m_0 + m_1 + \cdots + m_{j-1} < |b_{-1}| \le m_1 + \cdots + m_j$ and define $\mathcal{A}$ to be the standard skewed binary partition such that

$$\mathrm{type}(\mathcal{A}) := (b_{-1}, 0^\circ, m_0 + m_1 + \cdots + m_j - |b_{-1}|, m_{j+1}, \ldots, m_\ell).$$

Case 3: $|b_{-1}| \le m_0$. Then we defined $\mathcal{A}$ to be the standard skewed binary partition with

$$\mathrm{type}(\mathcal{A}) = \begin{cases} (m_0, 0, m_1, \ldots, m_\ell) & \text{if } m_0 \neq 0 \\ (m_0, 0^\circ, m_1, \ldots, m_\ell) & \text{if } m_0 = 0 \end{cases}.$$

Clearly, in both Case 2 and Case 3, $\text{type}(\mathcal{A}) \in \Omega_{\mathbf{m}}$. To see that $\mathcal{B} \leq \mathcal{A}$, we will show that $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ satisfies all of the conditions in Theorem 4.2.7. As both cases can be shown using a similar argument, we will only discuss the proof for case 2.

Suppose that $m_0 < |b_{-1}| < n$. Then, by condition (1), $b_0 \neq 0$, i.e., $0 \in B_0$. Note that $\hat{\mathcal{A}} = \mathcal{A}$ and $\hat{\mathcal{B}} = \mathcal{B}$, since there is no empty block. Because every block of $\mathcal{A}$ is non-homogeneous, $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ automatically satisfies condition 4.2.7/(3). By construction, $A_{-1} = B_{-1}$ and $A_0 = \{0\} \subseteq B_0$. Because $\mathcal{A}$ and $\mathcal{B}$ are standard, it is then easy to see that $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is non-crossing. Thus, condition 4.2.7/(1) is satisfied.

Let j be a positive integer. Suppose that B_j is a non-homogeneous block. Then B_j are not adjacent to A_{-1} nor A_0, since $A_{-1} = B_{-1}$ and $A_0 = \{0\}$ are disjoint with B_j. Let A_k be the block of $\mathcal{A}$ that contains $m_0 + \cdots + m_i$. Together with conditions (2) and (3), we deduce that B_j is a subset of A_k. This means, in $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the right non-homogeneous vertex B_j has degree one and is adjacent to the non-homogeneous block A_k. If B_0 is non-homogeneous, i.e., $B_0 = \{0\}$, then we also see that, in $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the vertex B_0 has degree one and is adjacent to the non-homogeneous vertex A_0. Thus, $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ satisfies 4.2.7/(4). Moreover, condition (2) implies that for every positive integer k the only possible non-homogeneous vertex that A_k can be adjacent to (if there is any) is B_j. Hence, $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ also satisfies 4.2.7/(2).

It only remains to be shown that $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ satisfies 4.2.7/(5). To see this, let us suppose that j is a positive integer in which B_j is a homogeneous block. Then conditions (2) and (3) imply that there exists a nonnegative integer i such that

$$|b_{-1}| + \cdots + |b_{j-1}| < m_0 + \cdots m_i < |b_{-1}| + \cdots + |b_j|.$$

This means, in $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, the right homogeneous vertex B_j is adjacent to two left non-homogeneous vertices whose corresponding blocks contain $m_0 + \cdots m_i$ and $m_0 + \cdots m_i + 1$, respectively. If B_0 is homogeneous, then B_0 must contain 0 and another positive integer. Thus, the right homogeneous vertex B_0 of $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is adjacent to $A_0 = \{0\}$ and A_1. This shows that $G(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ satisfies 4.2.7/(5). Therefore, $\mathcal{B} \leq \mathcal{A}$. □

By Proposition 4.3.8, we have that the one dimensional cones in $\Sigma(\text{PF}(\mathbf{m}, \mathbf{d}))$ (and hence the facets of $\text{PF}(\mathbf{m}, \mathbf{d})$) correspond to the skew binary partitions $\mathcal{B} \in \mathcal{BWP}(\mathbf{m})$ such that $\text{type}(\mathcal{B}) =$

$(b_{-1}, b_0, \ldots, b_k)$ satisfies

$$1 = b_{-1} + \left(\sum_{b_i \in \mathbb{P}} b_i\right) + \#(b_i \in \mathbb{P}^\star_{\geq 2}).$$

This implies that

$$\text{(4.4.9)} \quad \operatorname{type}(\mathcal{B}) \in \{(1, (n-1)^\circ), (0, 0^\circ, n^\star), (0, 1^\circ, (n-1)^\star), \ldots, (0, (n-2)^\circ, 2^\star), (0, (n-1)^\circ, 1)\}.$$

Together with Proposition 4.4.20, we obtain the following characterization of the one-dimensional cones in $\Sigma(\mathrm{PF}(\mathbf{m}, \mathbf{d}))$.

Corollary 4.4.25. *Let $n \geq 2$ be a positive integer. Suppose that $(\mathbf{m}, \mathbf{d})$ is an MD pair where $\mathbf{m} = (m_0, \ldots, m_\ell)$ is a multiplicity vector of magnitude n. Then, the cone $\sigma \in \Sigma(\mathrm{PF}(\mathbf{m}, \mathbf{d}))$ has dimension one if and only if $\sigma = \tilde{\sigma}_{\mathcal{B}}$ for some skewed binary partition $\mathcal{B}$ of $[0, n]$ such that* $\operatorname{type}(\mathcal{B}) =$ *$(b_{-1}, b_0, \ldots, b_k)$ satisfies the following conditions.*

1. *If $\ell = 1$ and $m_1 = n$, then* $\operatorname{type}(\mathcal{B}) \in \{(1, (n-1)^\circ), (0, (n-1)^\circ, 1)\}$.
2. *If $\ell = 1$ and $m_1 = 1$ then* $\operatorname{type}(\mathcal{B}) \in \{(1, (n-1)^\circ), (0, 0^\circ, n^\star)\}$.
3. *If $(\ell = 1$ and $2 \leq m_1 \leq n-1)$ or $(\ell = 2$ and $m_1 = 1)$, then*

$$\operatorname{type}(\mathcal{B}) \in \{(1, (n-1)^\circ), (0, (n-1)^\circ, 1), (0, 0^\circ, n^\star)\}.$$

4. *If $\ell \geq 2$ and $(\ell, m_1) \neq (2, 1)$, then* $\operatorname{type}(\mathcal{B}) \in X \cup Y$ *where*

$$X := \{(1, (n-1)^\circ), (0, (n-1)^\circ, 1), (0, 0^\circ, n^\star)\}$$

$$Y := \{(0, (n - m_\ell - 1)^\circ, (m_\ell + 1)^\star), \ldots, (0, (m_0 + 1)^\circ, (n - m_0 - 1)^\star)\}.$$

Utilizing (4.4.9) and Corollary 4.4.25 to describe the one dimensional cones in $\Sigma(\mathrm{PF}(\mathbf{u}))$, we deduce the following inequality description for $\mathrm{PF}(\mathbf{u})$.

Corollary 4.4.26. *A point $\mathbf{x} = (x_1, \ldots, x_n)$ lies in the $\mathbf{u}$-parking function polytope* $\mathrm{PF}(\mathbf{u})$ *if and only if $x_i \geq 0$ for all $i \in [n]$ and for every nonempty subset $I \subseteq [n]$*

$$\sum_{i \in I} x_i \leq \sum_{i=0}^{|I|-1} u_{n-i}.$$

We will see later in Section 4.6 that the inequality description in Corollary 4.4.26 can also be deduced by viewing $\mathrm{PF}(\mathbf{u})$ as a *polymatroid*.

REMARK 4.4.27. By Corollary 4.4.25, for every multiplicity vector $\mathbf{m} \neq (0, n)$ of magnitude n, the skewed binary partition $\mathcal{B} = (\emptyset, \{0\}, \{1, \ldots, n\}^\star)$ is in $\mathcal{BWP}(\mathbf{m})$, since $\mathrm{type}(\mathcal{B}) = (0, 0^\circ, n^*)$.

Note that every polytope of dimension at most two is simplicial. The next corollary provides a characterization of the simplicial polytopes of dimension greater than two.

COROLLARY 4.4.28. *Let $n \geq 3$ be an integer. Suppose that $\mathbf{m}$ is a multiplicity vector of magnitude n and $(\mathbf{m}, \mathbf{d})$ is an MD pair. Then, $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is an n-dimensional simplicial polytope if and only if $\mathbf{m} = (n-1, 1)$, i.e., $\mathrm{PF}(\mathbf{m})$ is a simplex.*

PROOF. Let $(\mathbf{m}, \mathbf{d})$ be the MD pair of the vector $\mathbf{u}$. Suppose that $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is simplicial. When $\mathbf{m} = (0, n)$, the parking function polytope $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is an n-dimensional cube and is not simplicial for $n \geq 3$. Thus, $\mathbf{m} \neq (0, n)$. For $\mathbf{m} \neq (0, n)$, we have from Remark 4.4.27 that the skew binary partition $\mathcal{B} = (\emptyset, \{0\}, \{1, \ldots, n\}^\star)$ is in $\mathcal{BWP}(\mathbf{m})$. Moreover, $\mathcal{B} = (\emptyset, \{0\}, \{1, \ldots, n\}^\star)$ corresponds to the face $F_{\mathcal{B}} = \mathfrak{S}_n(\mathbf{u}) := \mathrm{conv}(\tau(\mathbf{u}) \mid \tau \in \mathfrak{S}_n)$ of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$. Thus, $F_{\mathcal{B}}$ is a permutohedron, and is a simplex if and only if $\mathbf{m} = (n-1, 1)$. Conversely, if $\mathbf{m} = (n-1, 1)$, then $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is an n-dimensional simplex and is simplicial. $\square$

4.5. h-vectors

Given a poset $(Q, \leq_Q)$ where $Q \subset \mathbb{N}$, we say that the ordered pair (i, j) is a *descent* of $(Q, \leq_Q)$ if $i \lessdot_Q j$ and $j < i$, and say that (i, j) is an *ascent* if $i \lessdot_{\mathcal{B}} j$ and $j > i$.

As noted in Corollary 4.4.15, $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is simple if and only if either $\mathbf{m} = (0, n)$ or $(n-1, 1)$ or $m_1 = \cdots = m_{\ell-1} = 1$ for some $\ell \geq 2$. This implies that for every $\mathcal{B} \in \Omega_{\mathbf{m}}$, the preorder $\leq_{\mathcal{B}}$ is a poset and its Hasse diagram is a tree. We will denote the number of descents and ascents of the poset $([0, n], \leq_{\mathcal{B}})$ by $\mathrm{des}(\mathcal{B})$ and $\mathrm{asc}(\mathcal{B})$, respectively. The following lemma, which is a slight variation of [36, Theorem 4.2], expresses the h-polynomials of simple parking function polytopes in terms of descents and ascents.

LEMMA 4.5.1. *If* $\mathrm{PF}(\mathbf{m},\mathbf{d})$ *is an* n*-dimensional simple polytope, then its* $\mathbf{h}$*-polynomial equals*

$$h(t) = \sum_{\mathrm{type}(\mathcal{B})\in\Omega_{\mathbf{m}}} t^{\mathrm{des}(\mathcal{B})} = \sum_{\mathrm{type}(\mathcal{B})\in\Omega_{\mathbf{m}}} t^{\mathrm{asc}(\mathcal{B})}. \tag{4.5.1}$$

COROLLARY 4.5.2. *Let* $\mathbf{m} = (m_0, m_1, \ldots, m_\ell)$ *and* $r = m_1 + \cdots + m_\ell$. *If* $\mathrm{PF}(\mathbf{m},\mathbf{d})$ *is an* n*-dimensional simple polytope, then its* $\mathbf{h}$*-polynomial equals*

$$h(t) = \sum_{i=0}^{r}\left(\sum_{\mathrm{type}(\mathcal{B})=\mathbf{b}_i} t^{\mathrm{des}(\mathcal{B})}\right) = \sum_{i=0}^{r}\left(\sum_{\mathrm{type}(\mathcal{B})=\mathbf{b}_i} t^{\mathrm{asc}(\mathcal{B})}\right) \tag{4.5.2}$$

where $\mathbf{b}_i$ *is given as in Definition 4.4.8.*

Let Q be a poset on $[n]$. We define $\mathfrak{S}_n(Q) := \{\sigma(Q) \mid \sigma \in \mathfrak{S}_n\}$ to be the set of all posets on $[n]$ having the same Hasse diagram as Q.

DEFINITION 4.5.3. For $p, q \in \mathbb{N}$, let $T(p,q)$ be the poset on $[p+q]$ defined by the covering relations $j \lessdot j+1$ for all $j \in [p-1]$ and $p \lessdot k$ for all $k \in [p+1, q]$.

DEFINITION 4.5.4. Let $(Q, \leq_T)$ be a poset on $[n]$ whose Hasse diagram is a tree (a graph with no cycle). We define the *generalized Eulerian polynomial* on Q to be

$$A(Q,t) := \sum_{T\in\mathfrak{S}_n(Q)} t^{\mathrm{asc}(T)}.$$

Let $\tau = n\cdots 21 \in \mathfrak{S}_n$. Then $\mathrm{asc}(\tau(T)) = \mathrm{des}(T)$ for all $T \in \mathfrak{S}_n(Q)$. Thus,

$$A(Q,t) := \sum_{T\in\mathfrak{S}_n(Q)} t^{\mathrm{asc}(T)} = \sum_{T\in\mathfrak{S}_n(Q)} t^{\mathrm{asc}(\tau(T))} = \sum_{T\in\mathfrak{S}_n(Q)} t^{\mathrm{des}(T)}.$$

The generalized Eulerian polynomial $A(Q,t)$ has degree $n-1$ and is palindromic. We also have that $A(T(p,1),t)$ is the (usual) Eulerian polynomial $A_{p+1}(t)$ of degree p.

REMARK 4.5.5. $\mathrm{PF}(\mathbf{m},\mathbf{d})$ is an n-cube if $\mathbf{m} = (0,n)$, and is an n-simplex if $\mathbf{m} = (n-1,1)$. In these cases, their h-polynomials are known to be $(1+t)^n$ and $1 + t + \cdots t^n$, respectively.

We now describe how to obtain a more explicitly formula for the h-polynomials of all other simple parking function polytopes, i.e., $\mathrm{PF}(\mathbf{m},\mathbf{d})$ with $m_1 = \cdots = m_{\ell-1} = 1$ for some $\ell \geq 2$. Note that $n = m_0 + \ell - 1 + m_\ell$ and $r = m_1 + \cdots + m_\ell = \ell - 1 + m_\ell$. Thus, by Corollary 4.5.2, we may write

$$h(t) = \sum_{i=0}^{\ell-1+m_\ell} g_i(t) \text{ where } g_i(t) := \sum_{\text{type}(\mathcal{B})=\mathbf{b}_i} t^{\text{asc}(\mathcal{B})}.$$

For $\ell - 1 \le i \le \ell - 1 + m_\ell$, the poset $([0,n], \preceq_{\mathcal{B}})$ with $\text{type}(\mathcal{B}) = \mathbf{b}_i$ is given on the right of Figure 4.7. Thus,

$$g_i(t) = \binom{n}{m_0 + i} t^{\ell-1+m_\ell-i} \text{ for } \ell - 1 \le i \le \ell - 1 + m_\ell. \tag{4.5.3}$$

For $1 \le i \le \ell - 2$, the poset $([0,n], \preceq_{\mathcal{B}})$ with $\text{type}(\mathcal{B}) = \mathbf{b}_i$ is given on the left of Figure 4.7. Thus,

$$g_i(t) = \binom{n}{m_0 + i} tA(T(\ell - i - 1, m_\ell), t) \text{ for } 1 \le i \le \ell - 2. \tag{4.5.4}$$

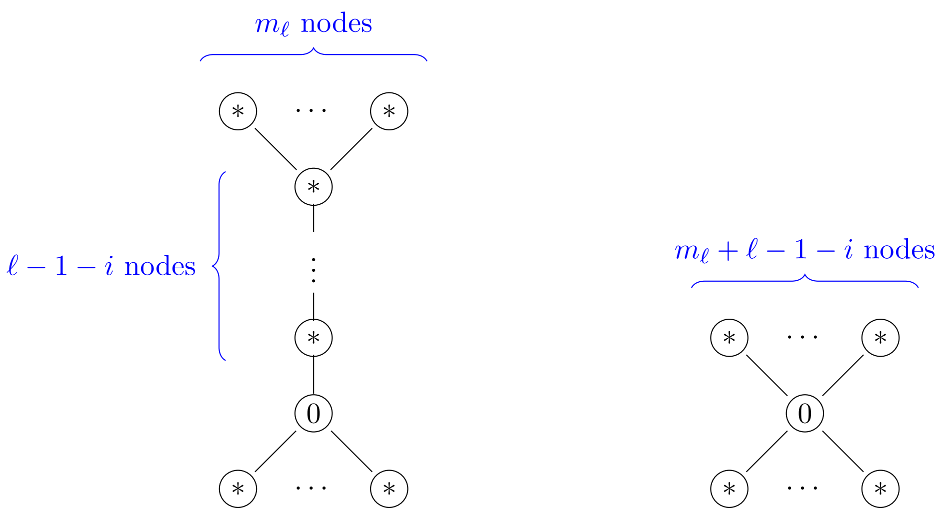


FIGURE 4.7. $([0,n], \preceq_{\mathcal{B}})$ with $\text{type}(\mathcal{B}) = \mathbf{b}_i \in \Omega_{\mathbf{m}}$ satisfying $1 \le i \le \ell - 2$ (left) and $\ell - 1 \le i \le m_\ell$ (right), where $*$ denotes an integer in $[n]$

The polynomial $g_0(t) = tA(Q, t)$ where Q is a poset on $[n]$ depending on whether $\mathbf{m}$ satisfies $m_0 = 0$ or not. If $m_0 = 0$, then the poset $([0,n], \preceq_{\mathcal{B}})$ with $\text{type}(\mathcal{B}) = \mathbf{b}_0$ is given on the left of Figure 4.8. Thus,

$$g_0(t) = tA(T(\ell - 1, m_\ell), t) \text{ if } m_0 = 0. \tag{4.5.5}$$

If $m_0 \neq 0$, then the poset $([0,n], \preceq_{\mathcal{B}})$ with $\text{type}(\mathcal{B}) = \mathbf{b}_0$ is given on the right of Figure 4.8. Thus,

$$g_0(t) = tA(Q,t) \text{ if } m_0 \neq 0 \tag{4.5.6}$$

where Q is the induced poset $[n]$ of the poset on the right of Figure 4.8.

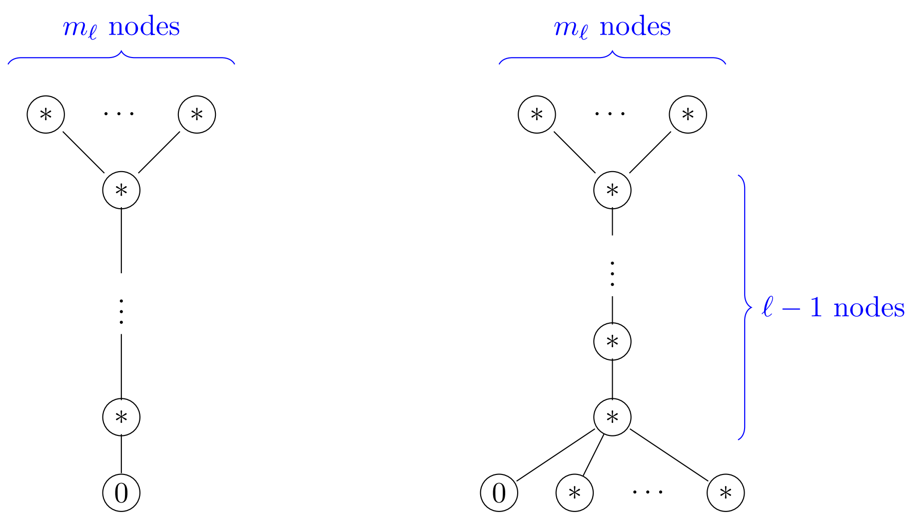


FIGURE 4.8. $([0,n], \preceq_{\mathcal{B}})$ with $\text{type}(\mathcal{B}) = \mathbf{b}_0 \in \Omega_{\mathbf{m}}$ satisfying $m_0 = 0$ (left) and $m_0 \neq 0$ (right), where $*$ denotes an integer in $[n]$

Consequently, by equations (4.5.4)–(4.5.6),

$$\begin{aligned}
h(t) &= g_0(t) + \left(\sum_{i=1}^{\ell-2} g_i(t)\right) + \sum_{i=\ell-1}^{\ell-1+m_\ell} g_i(t) \\
&= tA(Q,t) + \left(\sum_{i=1}^{\ell-2} \binom{n}{m_0+i} tA(T(\ell-i-1, m_\ell), t)\right) + \sum_{i=\ell-1}^{\ell-1+m_\ell} \binom{n}{m_0+i} t^{\ell-1+m_\ell-i} \\
&= tA(Q,t) + \left(\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} tA(T(i, m_\ell), t)\right) + \sum_{i=0}^{m_\ell} \binom{n}{i} t^i. \qquad (4.5.7)
\end{aligned}$$

Hence, if $m_0 = 0$, then

$$\begin{aligned}
h(t) &= tA(Q,t) + \left(\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} tA(T(i,m_\ell),t)\right) + \sum_{i=0}^{m_\ell} \binom{n}{i} t^i \\
&= tA(T(\ell-1,m_\ell),t) + \left(\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} tA(T(i,m_\ell),t)\right) + \sum_{i=0}^{m_\ell} \binom{n}{i} t^i \\
&= \left(\sum_{i=0}^{m_\ell} \binom{n}{i} t^i\right) + \sum_{i=1}^{\ell-1} \binom{n}{i+m_\ell} tA(T(i,m_\ell),t). \qquad \text{(M0)}
\end{aligned}$$

Since h-polynomials and generalized Eulerian polynomials are palindromic, we have

$$h(t) = t^n h(t^{-1}) \tag{4.5.8}$$

$$A(Q,t) = t^{n-1} A(Q,t^{-1}) \tag{4.5.9}$$

$$A(T(i,m_\ell),t) = t^{i+m_\ell-1} A(T(i,m_\ell),t^{-1}). \tag{4.5.10}$$

Thus, by equations (4.5.7) and (4.5.8)–(4.5.10),

$$\begin{aligned}
h(t) &= t^n h(t^{-1}) \\
&= t^n \left[t^{-1}A(Q,t^{-1}) + \left(\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} t^{-1}A(T(i,m_\ell),t^{-1})\right) + \sum_{i=0}^{m_\ell} \binom{n}{i} t^{-i} \right] \\
&= t^{n-1}A(Q,t^{-1}) + \left(\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} t^{n-1}A(T(i,m_\ell),t^{-1})\right) + \sum_{i=0}^{m_\ell} \binom{n}{i} t^{n-i} \\
&= A(Q,t) + \left(\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} t^{m_0+\ell-i-1}A(T(i,m_\ell),t)\right) + \sum_{i=0}^{m_\ell} \binom{n}{i} t^{n-i} \qquad (4.5.11)
\end{aligned}$$

Equations (4.5.7) and (4.5.11) allows us to express $A(Q,t)$ as

$$A(Q,t) = \frac{1}{t-1}\left[\left[\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} (t^{m_0+\ell-i-1} - t)A(T(i,m_\ell),t)\right] + \sum_{i=0}^{m_\ell} \binom{n}{i} (t^{n-i} - t^i)\right].$$

Hence, if $m_0 \neq 0$, then

$$h(t) = g(t) + \left(\sum_{j=0}^{m_\ell} \binom{n}{j} t^j\right) + t\sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} A(T(i,m_\ell),t), \tag{M1}$$

where

$$g(t) = g_0(t) = tA(Q,t)$$

$$\text{(G0)} \qquad = \left[\sum_{i=0}^{z}\binom{n}{i}\left(\sum_{j=i+1}^{n-i} t^j\right)\right] + \sum_{i=1}^{\ell-2}\binom{n}{i+m_\ell}\left(\sum_{j=2}^{n-i-m_\ell} t^j\right) A(T(i,m_\ell),t)$$

and $z = \min(m_\ell, n-m_\ell-1)$.

Theorem 4.5.6. *Let* $(\mathbf{m},\mathbf{d})$ *be an MD pair where* $\mathbf{m} = (m_0, m_1, \ldots, m_\ell)$ *for some* $\ell \geq 2$. *Suppose that* $\mathrm{PF}(\mathbf{m},\mathbf{d})$ *is* n*-dimensional and simple. Then its* h*-polynomial is given by*

$$\text{(H)} \qquad h(t) = \begin{cases} \left(\sum_{j=0}^{m_\ell}\binom{n}{j}t^j\right) + t\sum_{i=1}^{\ell-1}\binom{n}{i+m_\ell}A(T(i,m_\ell),t) & \text{if } m_0 = 0 \\ g(t) + \left(\sum_{j=0}^{m_\ell}\binom{n}{j}t^j\right) + t\sum_{i=1}^{\ell-2}\binom{n}{i+m_\ell}A(T(i,m_\ell),t) & \text{otherwise} \end{cases}$$

where

$$g(t) = \left[\sum_{i=0}^{z}\binom{n}{i}\left(\sum_{j=i+1}^{n-i} t^j\right)\right] + \sum_{i=1}^{\ell-2}\binom{n}{i+m_\ell}\left(\sum_{j=2}^{n-i-m_\ell} t^j\right) A(T(i,m_\ell),t)$$

and $z = \min(m_\ell, n-m_\ell-1)$.

We now aim to express $A(T(p,q),t)$ in terms of Eulerian polynomials. This will allow us to express equation (H) in Theorem 4.5.6 in terms of Eulerian polynomials. To do this, we first observe that, by symmetry, $A(Q,t) = A(Q^*,t)$ for all poset Q on $[n]$ whose Hasse diagram is a tree, where Q^* denotes the dual poset of Q.

Let $p,q,n \in \mathbb{P}$ satisfy $p+q=n$. If $p=1$, one can easily compute by a direct counting argument that $A(T(1,q),t) = 1+t+\cdots t^{n-1}$. If $p \geq 2$, then $T(p,q)^*$ is the induced poset on $[n]$ of a poset $(\leq_{\mathcal{B}}, [0,n])$ satisfying $\mathrm{type}(\mathcal{B}) = \mathbf{b}_0 \in \Omega_{\mathbf{m}}$ where $\mathbf{m} = (q,1,\ldots,1)$. Thus, we can deduce

$A(T(p,q),t)$ by applying equation (G0) to $\mathbf{m} = (q, 1, \ldots, 1)$ to get

$$\begin{aligned} A(T(p,q),t) &= A(T(p,q)^*,t) \\ &= \frac{g(t)}{t} \\ &= \left[\sum_{i=0}^{1}\binom{n}{i}\left(\sum_{j=i}^{n-i-1} t^j\right)\right] + \sum_{i=1}^{p-2}\binom{n}{i+1}\left(\sum_{j=1}^{n-i-2} t^j\right) A(T(i,1),t), \text{ if } p \geq 2. \end{aligned}$$

Therefore, we have the following result.

Lemma 4.5.7. *Let $p, q, n \in \mathbb{P}$ satisfy $p + q = n$. Then*

$$A(T(p,q),t) = \left[\sum_{i=0}^{y}\binom{n}{i}\left(\sum_{j=i}^{n-i-1} t^j\right)\right] + \sum_{i=1}^{p-2}\binom{n}{i+1}\left(\sum_{j=1}^{n-i-2} t^j\right) A_{i+1}(t) \tag{4.5.12}$$

where $y = \min(1, p-1)$ and $A_k(t)$ denotes the Eulerian polynomial of degree $k-1$.

Using equation (4.5.12) to express $A(T(i, m_\ell), t)$ in equation (H) of Theorem 4.5.6, we can write the h-polynomial of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ in terms of Eulerian polynomials. Together with Remark 4.5.5, we consequently have the following corollary.

Corollary 4.5.8. *Suppose that $\mathrm{PF}(\mathbf{u})$ is n-dimensional and simple. Then its h-polynomial has the form $h(t) = r_0(t) + \sum_{i=1}^{n} r_i(t) A_i(t)$ where $A_k(t)$ is the Eulerian polynomial of degree $k-1$ and $r_k(t)$ is a polynomial with nonnegative integral coefficients of degree $\leq n$.*

For instance, the h-polynomials of $\mathrm{PF}(1, \ldots, n)$ and $\mathrm{PF}(0, \ldots, n-1)$ equal

$$1 + \sum_{k=1}^{n}\binom{n}{k} tA_k(t) \text{ and } 1 + tA_n(t) + \sum_{k=1}^{n-2}\binom{n}{k} tA_k(t), \text{ respectively.}$$

4.6. Connection to other polytopes

Let $\mathfrak{S}_n(\mathbf{u}) := \mathrm{conv}(\tau(\mathbf{u}) \mid \tau \text{ is a permutation in } \mathfrak{S}_n)$ be the $\mathfrak{S}_n$-permutohedron generated by $\mathbf{u}$, where $\tau(\mathbf{u}) := (u_{\tau(1)}, \ldots, u_{\tau(n)})$. It is not difficult to show that the parking function polytope

PF($\mathbf{u}$) can be equivalently defined as

$$\mathrm{PF}(\mathbf{u}) = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \mid \exists\, \mathbf{w} \in \mathfrak{S}_n(\mathbf{u}) \text{ such that } \mathbf{w} - \mathbf{x} \in \mathbb{R}^n_{\geq 0}\} \tag{4.6.1}$$

$$= (\mathbb{R}^n_{\leq 0} + \mathfrak{S}_n(\mathbf{u})) \cap \mathbb{R}^n_{\geq 0}. \tag{4.6.2}$$

Viewed as polymatroid. Equations (4.6.1)–(4.6.2) allow us to see that every PF($\mathbf{u}$) is a polymatroid introduced by Edmonds in [16]. We note that a polymatroid in $\mathbb{R}^n$ is also commonly defined as the set of all $(x_1, \dots, x_n)$ such that $0 \leq \sum_{i \in I} x_i \leq w(I)$ for all nonempty subset I of $[n]$ where $w : 2^{[n]} \longrightarrow \mathbb{R}$ is a function defined on the power set of $[n]$ satisfying the following conditions:

(1) (Nonnegative) $0 \leq w(I)$ for all $I \subseteq [n]$.
(2) (Non-decreasing) $w(I_1) \leq w(I_2)$ for all $I_1 \subseteq I_2 \subseteq [n]$
(3) (Submodular) $w(A) + w(B) \geq w(A \cup B) + w(A \cap B)$ for all $A, B \subseteq [n]$.

The name "polymatroid" comes from its direct relation to matroids, the study concerning the abstraction of independent sets. This means that each polymatroid encodes some information about its corresponding matroid.

For a given $\mathbf{u}$, let $w_{\mathbf{u}} : 2^{[n]} \to \mathbb{R}$ be the function defined by

$$w_{\mathbf{u}}(\emptyset) = 0 \text{ and } w_{\mathbf{u}}(I) = \sum_{i=0}^{|I|} u_{n-i} \text{ for all nonempty } I \subseteq [n].$$

It is easy to verify that $w_{\mathbf{u}}$ satisfies conditions (1)–(3). We can then define PF($\mathbf{u}$) to be the polymatroid consisting of all $(x_1, \dots, x_n)$ satisfying

$$0 \leq \sum_{i \in I} x_i \leq w_{\mathbf{u}}(I) \text{ for all nonempty } I \subseteq [n].$$

This inequality description of PF($\mathbf{u}$) is equivalent to what is given in Corollary 4.4.26.

When $\mathbf{u} = (1, 2, \dots, n)$ or any strictly increasing sequence of positive numbers, the polytope PF($\mathbf{u}$) becomes a stellahedron, which is the graph associahedron of a star graph originally introduced by Carr and Devadoss [8]. It follows from Remark 4.4.18 that the normal fan of every parking function polytope is a coarsening of the normal fan of a stellahedron. Recent work by Eur, Huh, and Larson [19] leverages the geometry of the stellahedral toric variety to study matroids and explore the connections between deformations of PF$(1, 2, \dots, n)$ and polymatroids.

Viewed as type B generalized permutohedra. It is not difficult to see that every sliced preorder cone defined by (skewed) binary partition that is full-dimensional is a union of cones the B_n permutohedral fan Σ_{B_n} in (2.2.2). Hence, the normal fan of $\mathrm{PF}(\mathbf{u})$ coarsens the fan Σ_{B_n}. This means that every parking function polytope is a type B generalized permutohedron. In particular, we have that the edges of $\mathrm{PF}(\mathbf{u})$ are parallel to $\mathbf{e}_i + \mathbf{e}_j, \mathbf{e}_i - \mathbf{e}_j$, or $\mathbf{e}_i$ for some $i, j \in [n]$. Using the inequality description in Corollary 4.4.17, one sees further that every $\mathrm{PF}(\mathbf{u})$ has no edge parallel to $\mathbf{e}_i + \mathbf{e}_j$ for some $i < j$. Hence, we get the following proposition.

PROPOSITION 4.6.1. *Let* $\mathrm{PF}(\mathbf{u})$ *be a parking function polytope in* $\mathbb{R}^n$*. Then, every edge of* $\mathrm{PF}(\mathbf{u})$ *is either parallel to* $\mathbf{e}_i$ *or* $\mathbf{e}_i - \mathbf{e}_j$ *for some* $i, j \in [n]$ *such that* $i < j$*.*

PROOF. Since every parking function polytope is a type B generalized permutohedron, it's edges are either parallel to $\mathbf{e}_i + \mathbf{e}_j, \mathbf{e}_i - \mathbf{e}_j$, or $\mathbf{e}_i$ for some $i, j \in [n]$. Thus, it suffices to show that $\mathrm{PF}(\mathbf{u})$ has no edge parallel to $\mathbf{e}_i + \mathbf{e}_j$ for some $i < j$. Assume for the sake of contradiction that $\mathrm{PF}(\mathbf{u})$ has an edge (a line segment) parallel to $\mathbf{e}_i + \mathbf{e}_j$ for some $i < j$, connecting two vertices. Let $\mathbf{x}, \mathbf{y}$ be the two vertices of $\mathrm{PF}(\mathbf{u})$. Then, we may write $\mathbf{y} = \mathbf{x} + r(\mathbf{e}_i + \mathbf{e}_j)$ for some $r > 0$. The inequality description of $\mathrm{PF}(\mathbf{u})$ given in Corollary 4.4.17 implies that the points $\mathbf{x} + r\mathbf{e}_i$ and $\mathbf{x} + r\mathbf{e}_j$ also lie in $\mathrm{PF}(\mathbf{u})$. One then sees that the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ lies in the (relative) interior of the convex hull $\mathrm{conv}(\mathbf{x}, \mathbf{y}, \mathbf{x} + r\mathbf{e}_i, \mathbf{x} + r\mathbf{e}_j)$. Since $\mathrm{conv}(\mathbf{x}, \mathbf{y}, \mathbf{x} + r\mathbf{e}_i, \mathbf{x} + r\mathbf{e}_j)$ is a subset of $\mathrm{PF}(\mathbf{u})$ and a polytope of dimension two, the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ cannot be an edge of $\mathrm{PF}(\mathbf{u})$, a contradiction. □

Viewed as type A generalized permutohedra. Every parking function polytope can be realized as a projection of a type A generalized permutohedron. Moreover, every integral parking function polytope is integrally equivalent to an integral type A generalized permutohedron. This realization was also pointed out in [4] for $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ where $\mathbf{m} = (0, 1, \ldots, 1)$ and $(1, \ldots, 1)$. Thus, the properties of generalized permutohedra apply to parking function polytopes as well. In particular, one can compute the Ehrhart polynomials and the volume of parking function polytopes using existing formulas for type A generalized permutohedra.

Recall from Lemma 2.1.2 that every type A generalized permutohedron has the form

$$\sum_{I\subset[n],I\neq\emptyset} y_I\Delta_I \text{ for some } y_I\in\mathbb{R}, \tag{4.6.3}$$

where $\Delta_I := \mathrm{conv}(e_i \mid i\in I)$. Moreover, in Sections 9 and 11 of [35], Postnikov gave formulas for the volume and the Ehrhart polynomial of type A generalized permutohedron in (4.6.3). We will apply Postnikov's formulas to parking function polytopes to obtain their volume and Ehrhart polynomials. To do this, we first write $\mathrm{PF}(\mathbf{u})$ as in (4.6.3).

Proposition 4.6.2. *The parking polytope*

$$\mathrm{PF}(\mathbf{u}) = \sum_{I\in 2^{[n]}\setminus\{\emptyset\}} y_I\Delta_I^0$$

where

$$y_I = \sum_{j=0}^{|I|-1}\binom{|I|-1}{j}(-1)^j u_{|I|-j}. \tag{4.6.4}$$

Equation (4.6.4) implies that for $I_1, I_2 \in 2^{[n]}\setminus\{\emptyset\}$ such that $|I_1| = |I_2|$, one has $y_{I_1} = y_{I_2}$. Moreover, if $\mathrm{PF}(\mathbf{u})$ is integral, then y_I is an integer for all $I\in 2^{[n]}\setminus\{\emptyset\}$.

Example 4.6.3. Let $n\geq 2$ be an integer. Suppose that the entries of $\mathbf{u}$ form an arithmetic sequence of nonnegative real numbers. That is, $\mathbf{u} = (p, p+q, p+2q, \dots, p+(n-1)p)$ for some nonnegative real numbers p, q. Then,

$$y_I = \begin{cases} p & \text{if } |I| = 1 \\ q & \text{if } |I| = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Thus,

$$\mathrm{PF}(\mathbf{u}) = \sum_{i=1}^{n} p\Delta_{\{i\}}^0 + \sum_{1\leq i<j\leq n} q\Delta_{\{i,j\}}^0.$$

Proof of Proposition 4.6.2. Consider the type A generalized permutohedron $P(\{y_I\})$ in $\mathbb{R}^{[0,n]}$ defined by

$$P(\{y_I\}) = \sum_{I \in 2^{[0,n]}} y_I \Delta_I \tag{4.6.5}$$

where

$$y_I = \begin{cases} \sum_{j=0}^{|I|-1} \binom{|I|-1}{j} (-1)^j u_{|I|-j} & \text{if } 0 \in I \\ 0 & \text{otherwise} \end{cases}.$$

Then, it's not difficult to check that for every proper subset I of $[n]$

$$\sum_{J \subseteq [0,n] \setminus I} y_I = \sum_{j=1}^{n-|I|-1} u_j.$$

Then, by Section 6 of [35], $P(\{y_I\})$ can be equivalently expressed as the set of all $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{[0,n]}$ satisfying

$$x_0 + \cdots + x_n = \sum_{I \subseteq [0,n]} y_I = u_1 + \cdots + u_n, \text{ and}$$

$$\sum_{i \in I} x_i \leq \left(\sum_{J \subseteq [0,n]} y_J \right) - \left(\sum_{J \subseteq [0,n] \setminus I} y_J \right) = \sum_{j=0}^{|I|-1} u_{n-j}$$

for all nonempty proper subset I of $[n]$. Let $\pi : \mathbb{R}^{[0,n]} \longrightarrow \mathbb{R}^n$ be the linear projection of $\mathbb{R}^{[0,n]}$ onto $\mathbb{R}^n$ defined by $\pi(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. It is then easy to see that $\pi(P(\{y_I\}))$ is a polytope in $\mathbb{R}^n$ defined by

$$0 \leq x_i \text{ for all } i \in [n] \text{ and } \sum_{i \in I} x_i \leq \sum_{j=0}^{|I|-1} u_{n-j} \text{ for all nonempty } I \subseteq [n].$$

This is the inequality description for $\mathrm{PF}(\mathbf{u})$ given in Corollary 4.4.26. Thus, $\pi(P(\{y_I\})) = \mathrm{PF}(\mathbf{u})$ and so we have

$$\mathrm{PF}(\mathbf{u}) = \sum_{I \in 2^{[0,n]}} \pi(y_I \Delta_I) = \sum_{I \in 2^{[n]} \setminus \{\emptyset\}} y_I \Delta_I^0$$

as desired. □

The projection π defined in the proof of Proposition 4.6.2 is a linear bijection from $P(\{y_I\})$ to $\mathrm{PF}(\mathbf{u})$. This means that every integral parking function polytope $\mathrm{PF}(\mathbf{u})$ is integrally equivalent the type A generalized permutohedral $\mathbb{P}(\{y_I\})$ defined in equation (4.6.5). In particular, their t-dilations have the same number of lattice points for all nonnegative integers t. Hence, they have the same Ehrhart polynomial. This allows us to compute the Ehrhart polynomials of parking function polytopes using Postnikov's formula in Theorem 11.3 of [35].

Corollary 4.6.4. *Suppose that* $\mathrm{PF}(\mathbf{u}) = y_1\Delta^0_{I_1} + \cdots y_m\Delta^0_{I_m}$ *where* $I_1, \ldots, I_m$ *are distinct nonempty subsets of* $[n]$ *such that* $y_1, \ldots, y_m$ *given in equation* (4.6.4) *are all nonzero integers. Then the Ehrhart polynomial of* $\mathrm{PF}(\mathbf{u})$ *is given by*

$$i(\mathrm{PF}(\mathbf{u}), t) = \sum_{\mathbf{a}\in D(\mathrm{PF}(\mathbf{u}))} \binom{ty_1 + a_1 - 1}{a_1} \cdots \binom{ty_m + a_m - 1}{a_m} \tag{4.6.6}$$

where $D(\mathrm{PF}(\mathbf{u}))$ *is the set of all* $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0}$ *satisfying* $\sum_{j\in J} a_j \leq |\bigcup_{j\in J} I_j|$ *for all nonempty subsets* $J \subseteq [m]$.

Proof. The parking function polytope $\mathrm{PF}(\mathbf{u}) \subset \mathbb{R}^n$ and the type A generalized permutohedron $P(\{y_I\}) \subset \mathbb{R}^{[0,n]}$ defined by

$$P(\{y_I\}) = y_1\Delta_{\{0\}\cup I_1} + \cdots y_m\Delta_{\{0\}\cup I_m}$$

have the same Ehrhart polynomials. Let $y_0 = 1$ and $I_0 = [n]$, and define

$$P^+_t(\{y_I\}) = y_0\Delta_{\{0\}\cup I_0} + ty_1\Delta_{\{0\}\cup I_1} + \cdots ty_m\Delta_{\{0\}\cup I_m}.$$

Applying Theorem 11.3 of [35] to $\mathbb{P}^+_t(\{y_I\})$, we deduce that

$$\begin{aligned} i(\mathrm{PF}(\mathbf{u}), t) &= i(P(\{y_I\}), t) \\ &= \sum_{\mathbf{a}\in \overline{D}(P^+_t(\{y_I\}))} \binom{1 + a_0 - 1}{a_0}\binom{ty_1 + a_1 - 1}{a_1} \cdots \binom{ty_m + a_m - 1}{a_m} \end{aligned} \tag{4.6.7}$$

where $\overline{D}(P^+_t(\{y_I\}))$ is the set of "G-draconian sequences of $P^+_t(\{y_I\})$" $\mathbf{a} = (a_0, a_1, \ldots, a_m) \in \mathbb{Z}^n_{\geq 0}$ satisfying $a_0 + a_1 + \cdots + a_m = n$ and $\sum_{j\in J} a_j \leq |\bigcup_{j\in J} I_j|$ for all nonempty subsets $J \subseteq [m]$. Since

$\binom{1+a_0-1}{a_0} = 1$ for all nonnegative integer a_0, we can rewrite equation (4.6.7) as

$$i(\mathrm{PF}(\mathbf{u}), t) = \sum_{\mathbf{a} \in D(P(\{y_I\}))} \binom{ty_1 + a_1 - 1}{a_1} \cdots \binom{ty_m + a_m - 1}{a_m}$$

where $D(\mathrm{PF}(\mathbf{u}))$ is the set of all $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ satisfying $\sum_{j \in J} a_j \leq |\bigcup_{j \in J} I_j|$ for all nonempty subsets $J \subseteq [m]$. This gives the formula in equation (4.6.6) as desired. □

We note that one can use Proposition 4.6.2 to derive another formula for the Ehrhart polynomials of parking function polytopes by applying Theorem $a/(c)$ in [18] given by Eur et al. for type B generalized permutohedra. However, we find that Postnikov's formula for type A generalized permutohedra gives a more desirable expression in the sense that when all y_I given in equation (4.6.4) are nonnegative, one easily sees from formula (4.6.6) that all the coefficients of $i(\mathrm{PF}(\mathbf{u}), t)$ are positive. This positivity of coefficients is, however, not easily seen when expressing $i(\mathrm{PF}(\mathbf{u}), t)$ using the formula provided by Eur et al.

Next, we apply the volume formula provided by Postnikov in Theorem 9.3 of [35] to compute the volume of parking function polytopes.

Corollary 4.6.5. *Suppose that* $\mathrm{PF}(\mathbf{u})$ *is* n*-dimensional and that* $\mathrm{PF}(\mathbf{u}) = y_1\Delta^0_{I_1} + \cdots y_m \Delta^0_{I_m}$ *where* $I_1, \ldots, I_m$ *are distinct nonempty subsets of* $[n]$ *such that* $y_{I_1}, \ldots, y_{I_m}$ *given in equation* (4.6.4) *are all nonzero. Then, the volume of* $\mathrm{PF}(\mathbf{u})$ *is given by*

$$\sum_{\substack{\mathbf{a} \in D(\mathrm{PF}(\mathbf{u})) \\ a_1 + \cdots + a_m = n}} \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!}. \tag{4.6.8}$$

Proof. Suppose that $\mathrm{PF}(\mathbf{u})$ is n-dimensional. We apply Theorem 9.3 of [35] to $P(\{y_I\}) = y_1 \Delta_{\{0\} \cup I_1} + \cdots y_m \Delta_{\{0\} \cup I_m}$ to deduce the volume of $\mathrm{PF}(\mathbf{u})$ as

$$\mathrm{Vol}(\mathrm{PF}(\mathbf{u})) = \mathrm{Vol}(P(\{y_I\})) = \sum_{\mathbf{a} \in \overline{D}(P(\{y_I\}))} \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!}$$

where $\overline{D}(P(\{y_I\}))$ is the set of G-draconian sequences of $P(\{y_I\})$, i.e., $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0}$ satisfying $a_1 + \cdots + a_m = n$ and $\sum_{j \in J} a_j \leq |\bigcup_{j \in J} I_j|$ for all nonempty subsets $J \subseteq [m]$. This is equivalent to what is given in (4.6.8). □

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