The Immersion Poset and a Crystal Analysis of Claw-Free Graphs

By

DAVID KENEPP
DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Anne Schilling, Chair

Eugene Gorsky

Monica Vazirani

Committee in Charge

2025

To Chuck, David, Lila, Marion and Stanton, my grandparents who are no longer with us.

Contents

Abstract

This thesis consists of two independent parts. In the first part, we introduce the immersion poset $(\mathcal{P}(n), \leqslant_I)$ on partitions, defined by $\lambda \leqslant_I \mu$ if and only if $s_\mu(x_1, \ldots, x_N) - s_\lambda(x_1, \ldots, x_N)$ is monomial-positive. Relations in the immersion poset determine when irreducible polynomial representations of $GL_N(\mathbb{C})$ form an immersion pair, as defined by Prasad and Raghunathan [**PR22**]. We develop injections $\mathsf{SSYT}(\lambda, \nu) \hookrightarrow \mathsf{SSYT}(\mu, \nu)$ on semistandard Young tableaux given constraints on the shape of λ, and present results on immersion relations among hook and two column partitions. The standard immersion poset $(\mathcal{P}(n), \leqslant_{std})$ is a refinement of the immersion poset, defined by $\lambda \leqslant_{std} \mu$ if and only if $\lambda \leqslant_D \mu$ in dominance order and $f^\lambda \leqslant f^\mu$, where f^ν is the number of standard Young tableaux of shape ν. We classify maximal elements of certain shapes in the standard immersion poset using the hook length formula. Finally, we prove Schur-positivity of power sum symmetric functions p_{A_μ} on conjectured lower intervals in the immersion poset, addressing questions posed by Sundaram [**Sun19**].

In the second part, we use crystals to explore Schur positivity results of claw-free graphs. Crystals were introduced by Kashiwara [**Kas90**] and have often been used to prove Schur positivity. Using Kashiwara crystals, we give a type A crystal structure on the set of colorings of claw-free graphs. Previously, Ehrhard [**Ehr22**] had given a type A crystal structure on P-arrays, where P is a (3+1)-free poset, which is equivalent to having a crystal structure on the colorings of claw-free incomparability graphs. We show that our operators are isomorphic to Ehrhard's when confined to claw-free incomparability graphs. Stembridge showed that when a crystal satisfies certain local axioms, then the character of the crystal corresponds to the character of some representation [**Ste03**]. We show that the crystal structure satisfies these Stembridge axioms for the set of graphs which are unit interval graphs, but do not contain an induced sub graph isomorphic to the path graph of length 4. Finally, we end with a discussion of ways to prove Schur positivity on claw-free graphs which are not incomparability graphs using this crystal structure.

Acknowledgments

I'd like to begin by thanking my advisor, Anne Schilling for introducing me to crystals and for having the patience to guide me over these past few years. I'm sure I was difficult at times, but I'm glad I had someone who cared as much as you do about students.

I'd want to thank the math faculty at Penn State Altoona, in particular Pinaki Das, Juan Gil, Mike Weiner, Karl Lorenson and Mark Johnson, whose encouragement and guidance helped get me to where I am now. I'd also like to thank all the staff in the LRC for all the memorable years I had tutoring there. In addition I'd like to thank Gary Weisel and Joseph Petrulionis for the great conversations and encouragement to pursue grad school.

I'd like to thank all those I worked and collaborated with, Evuilynn Nguyen, Lisa Johnston, Mary Claire Simone, and Regina Zhou. In particular, I would like to thank Digjoy Paul for introducing us to the immersion poset and Joseph Pappe for introducing me to the crystal structure on unit interval graphs and providing the Sage code to get started on the problems tackled in Chapter 3.

I'd especially like to thank Erik Carlsson for serving on my qualifying committee, Monica Vazirani for serving on my dissertation committee, and Eugene Gorsky for serving on both committees, I truly appreciate your service.

To all my friends and colleagues in the math program who have helped me throughout the past 5 years: Acadia, Alex, Ander, Andrew, Angel, Brian, Connor, Darian, Elysee, Felix, Hannah, Jeshu, Jonathan, Kayleigh, Matt, Peyton, Sharon, Simon, Stephanie, Tait, and Vincent.

I would like to thank the National Science Foundation for partially supporting this dissertation with grant DMS–2053350. I would also like to thank the U.S. Department of Education for partially funding my dissertation as a GAANN fellow through award number P200A240025.

I would like to thank my wife Enisieh, my parents, my siblings and their families, and my best friend Justin and his family for all of the support they have given me over the years. We've certainly had our share of struggles over the years and I'm thankful for all the moments I've shared with all of you in those trying times.

CHAPTER 1

Introduction

1.1. Overview

Representation theory has been used as a powerful mathematical tool to study various applications, such as particle physics and quantum mechanics. Often, the study of certain aspects of representation theory can be rephrased in terms of symmetric functions. This is because the Frobenius characteristic map establishes an isometry between the ring of symmetric functions and the ring of characters of the symmetric group. Meaning we have a direct bridge between the orthonormal bases of each, which are the irreducible representations and the Schur functions; and both can be indexed by integer partitions. In the study of immersion pairs for finite-dimensional irreducible polynomial representations of the general linear group $GL_N(\mathbb{C})$, we can rephrase the language of representations into the language of symmetric functions and reduce this relationship to a combinatorial one. In the symmetric function perspective, the integer partitions now index Schur functions, and we say a partition λ is immersed in a partition μ if the difference in the Schur functions is monomial positive. This sets up a partial order on integer partitions which is called the immersion poset.

In Chapter 2, we analyze various properties of the immersion poset. We begin in Section 2.2 by defining the standard immersion poset. The relation $\lambda \leqslant_I \mu$ in the immersion poset for $\lambda, \mu \in \mathcal{P}(n)$ holds if the Kostka numbers $K_{\lambda,\alpha} \leqslant K_{\mu,\alpha}$ for all $\alpha \in \mathcal{P}(n)$. In the standard immersion poset, one only compares the number of standard tableaux of shape λ and μ (instead of semistandard tableaux of all content). Relations in the immersion poset imply relations for the standard immersion poset, but not vice versa. In Section 2.2, we study properties and maximal elements of the standard immersion poset. In particular, maximal elements in the standard immersion poset are also maximal elements in the immersion poset. In Section 2.3, we study properties of the immersion poset. In particular, in Section 2.3.2 we study relations and covers in the immersion poset using explicit injections between sets of semistandard tableaux. In Section 2.3.3, we analyze the immersion poset restricted to partitions of hook shape. In Section 2.3.4, we analyze the immersion relations on partitions with at most two columns. In Section 2.3.5 we conjecture the structure of certain lower intervals in the immersion poset and prove that $p_{[(1^n),(n-2,1,1)]}$ and $p_{[(1^n),(n-2,2)]}$ ($n \neq 7$) are Schur-positive. In Section 2.5, we prove the case when $k = 3$ of Conjecture 2.2.14, which guarantees the maximality of a partition in the immersion poset if certain inequalities are satisfied. We conclude in Section 2.4 with a discussion of open problems.

In Chapter 3, we introduce crystal operators on colorings of claw-free graphs. P-arrays, which were introduced by Gessel and Viennot in [**GV89**] and later used by Gasharov in [**Gas96**], are combinatorial objects which correspond to proper colorings of incomparability graphs. Crystal operators on P-arrays, where P is a finite $(3+1)$-free poset, were introduced by Ehrhard [**Ehr22**] and in Section 3.2.2, we show that our crystal operators are isomorphic to Ehrhard's crystal operators when restricted to claw-free incomparability graphs.

In [**Kas90**], Kashiwara introduced crystals, which have often been used as a means to achieve Schur positivity results that also have a strong connection to the representation theory of Lie groups. Later, in [**Ste03**], Stembridge showed that when a crystal satisfies certain local axioms, one can be sure that the character of the crystal corresponds to the character of a representation, thereby immediately proving Schur positivity. In Section 3.3, we show that in general, our operators do not satisfy the Stembridge axioms. However, in Section 3.4, we show that for unit interval graphs which do not contain an induced subgraph isomorphic to the path graph of length 4, these Stembridge axioms are satisfied. We end with Section 3.5, where we discuss our current research in how to prove Schur positivity of claw-free graphs which are not incomparability graphs.

1.2. Preliminaries

1.2.1. Schur functions. Let $x = (x_1, x_2, \dots)$ be a set of indeterminates. Suppose the formal power series $f(x)$ with coefficients in $\mathbb{C}$ satisfies the condition that for every permutation σ of the natural numbers $\mathbb{N}$ we have $f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$, then we say $f(x)$ is a *symmetric function*. Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a weak composition of $n \in \mathbb{N}$ and let $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. Consider the symmetric function

$$f(x) = \sum_\alpha c_\alpha x^\alpha,$$

where α varies over all weak compositions of n and $c_\alpha \in \mathbb{C}$. Here we call $f(x)$ a *homogeneous symmetric function of degree n*.

Suppose f and g are homogeneous symmetric functions of degree n. Then assuming we don't get 0, $f \pm g$ is also a homogeneous symmetric function of degree n. Similarly, any non-zero scalar multiple of a homogeneous symmetric function of degree n, remains so. Hence, we can form a vector space.

Definition 1.2.1. We let Λ^n be the vector space over $\mathbb{C}$ consisting of all homogeneous symmetric functions of degree n plus the zero symmetric function. Let $f \in \Lambda^n$ and $g \in \Lambda^m$. Then $fg \in \Lambda^{n+m}$. Hence we can form

$$\Lambda = \bigoplus_{n \in \mathbb{Z}_{\geq} 0} \Lambda^n.$$

We refer to Λ as the *algebra of symmetric functions*.

Definition 1.2.2. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition. Then the *monomial symmetric function that corresponds to λ* is

$$m_\lambda = \sum_{\alpha(\lambda)} x^\alpha$$

where the sum ranges over all distinct permutations α of the entries of λ.

Example 1.2.3.

$$m_{(1)} = \sum_i x_i$$

$$m_{(2)} = \sum_i x_i^2$$

$$m_{(2,1)} = \sum_{i,j} x_i x_j^2$$

Theorem 1.2.4. *Let n be a fixed non-negative integer. Let $Par(n) = \{\lambda | \lambda \vdash n\}$ and let $Par = \cup_{n \geq 0} Par(n)$. Then the set $\{m_\lambda | \lambda \in Par(n)\}$ is a basis for Λ^n, and the set $\{m_\lambda | \lambda \in Par\}$ is a basis for Λ.*

Definition 1.2.5. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition. Then the *Young (or Ferrers) diagram of shape λ* is the finite collection of left justified boxes whose ith row contains λ_i boxes. T is a *semistandard Young tableau of shape λ* if T is a filling of the Young diagram of shape λ with entries from $\mathbb{N}$ where the entries are weakly increasing within a row and strictly increasing down a column. The *content μ of T* is the vector whose ith position contains the number of boxes in T containing the number i. We can define the weight of T, denoted x^T as follows

$$x^T = x^\mu$$

EXAMPLE 1.2.6. The following is a semistandard Young tableau of shape $\lambda = (5, 3, 1, 1)$, whose content is $\mu = (2, 4, 1, 1, 1, 1, 0, \dots)$

1	1	2	2	4
2	2	6		
3				
5				

DEFINITION 1.2.7. The *Schur function corresponding to λ* is

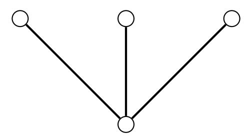
$$s_\lambda = \sum_T x^T$$

where the sum ranges over all possible semistandard Young tableaux of shape λ. The set of all Schur functions is also a basis for Λ.

For a more extensive treatment of Schur functions, or symmetric functions in general, see chapter 7 of [**Sta99**]. For those wishing to understand the intimate connection between Schur functions and irreducible representations of the symmetric group see [**Sag91**].

1.2.2. Graphs. Here we introduce the terminology used for graphs in Chapter 3.

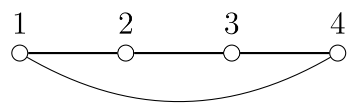
DEFINITION 1.2.8. A *simple graph* is an undirected graph containing no multiple edges or loops. All graphs in Chapter 3 will be simple graphs, so from now on, we will simply say graphs, though we really mean simple graphs. If G is a graph with edge set E and vertex set $V = \{1, 2, \dots, n\}$ and G has the property that for $i < j$, if $ij \in E$ (ij is the edge between vertices i and j), then for any k where $i < k < j$, we must have $ik \in E$ and $kj \in E$, then we say G is a *unit interval graph*. An *induced subgraph of G* is the graph obtained by deleting some subset of vertices of G and any edges shared by a deleted vertex. We say a graph is *claw-free* if it does not contained an induced subgraph isomorphic to the *claw graph*, shown below.



DEFINITION 1.2.9. Define P_n to be the *path graph of length n* with vertex set $V = \{1, 2, \dots, n\}$ and whose edge set is $E = \{12, 23, \dots, n-1n\}$. For example, P_4 is the following graph:

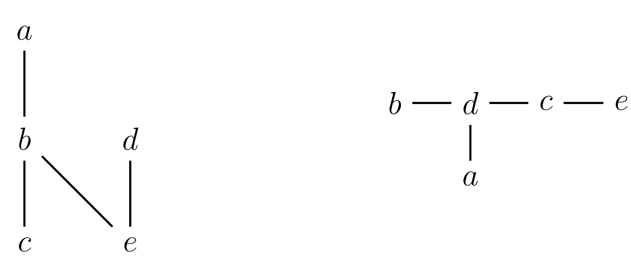
For much of Chapter 3 we will focus on unit interval graphs which do not contain an induced subgraph isomorphic to P_4, we refer to this set as $\mathcal{G}_4$.

Let $n \geq 3$. Define C_n to be the *cycle graph of length n* with vertex set $V = \{1, 2, \dots, n\}$ and whose edge set is $E = \{12, 23, \dots, n-1n, 1n\}$. For example, C_4 is the following graph:

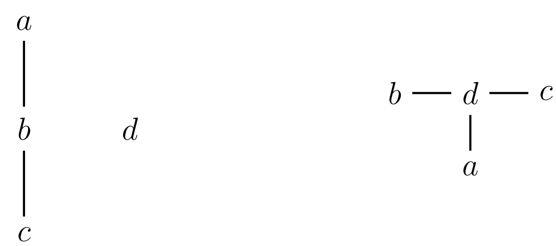


DEFINITION 1.2.10. Let $(P, \leq_P)$ be a finite poset. The *incomparability graph* of P, denoted *inc(P)* is the graph whose vertex set is P and vertices $a, b \in P$ share an edge when a and b are incomparable in P. We say a poset is *$(a+b)$-free* if it does not contain an induced subposet isomorphic to a disjoint union of chains with lengths a and b.

EXAMPLE 1.2.11. Consider the Hasse diagram of a poset P pictured on the left, with its corresponding incomparability graph on the right.



Notice that P is not $(3+1)$-free because if we delete the element e, what we obtain is a poset containing a chain of length 3 and a chain of length 1 that is disjoint. Notice that the corresponding incomparability graph of this induced poset is the claw graph, as shown below.



Example 1.2.11 illustrates the fact that saying G is the incomparability graph of a $(3+1)$-free poset is the same as saying G is a claw-free incomparability graph, so we may use these descriptors interchangeably, depending upon the context.

Let G be the incomparability graph of a $(3+1)$-free and $(2+2)$-free poset P, then G is a unit interval graph. The converse of this statement is also true. So once again, we can either characterize unit interval graphs from the incomparability graph perspective, or from the characterization given by Definition 1.2.8.

CHAPTER 2

The immersion poset on partitions

This chapter is based on work in collaboration with Lisa Johnston, Evuilynn Nguyen, Digjoy Paul, Anne Schilling, Mary Claire Simone, and Regina Zhou, published in [**JKN+25**]. Section 2.5 was based on work performed after the paper was published.

2.1. Background and Definitions

2.1.1. Immersion of representations. Given two finite-dimensional representations $\pi_1 : G \to GL(W_1)$ and $\pi_2 : G \to GL(W_2)$ of a group G, we say that the representation π_1 is *immersed* in the representation π_2 if the eigenvalues of $\pi_1(g)$, counting multiplicities, are contained in the eigenvalues of $\pi_2(g)$ for all $g \in G$. In this case, we call (W_1, W_2) an *immersion pair* denoted by $W_1 \leqslant_I W_2$. Note that, if π_1 is a subrepresentation of π_2, then $W_1 \leqslant_I W_2$, but the converse is not true.

Question 2.1.1 (Prasad and Raghunathan [**PR22**]). Classify immersion of representations $W_1 \leqslant_I W_2$ for a given group.

Recently, some progress was made on the above problem for symmetric groups [**PPS24b**] and alternating groups [**PPS24a**]. In this thesis, we study immersion pairs for finite-dimensional irreducible polynomial representations of the general linear group $GL_N(\mathbb{C})$.

2.1.2. Polynomial representation theory of $GL_N(\mathbb{C})$ and symmetric polynomials. The polynomial representation theory of $GL_N(\mathbb{C})$ was developed by Schur [**Sch07**] and later popularized by Weyl [**Wey39**] in his expository book on the representation theory of the classical groups. Briefly, the homogeneous irreducible polynomial representations (of degree n) of $GL_N(\mathbb{C})$, also known as *Weyl modules* $W_\lambda(\mathbb{C}^N)$, are indexed by integer partitions λ (of size n) with at most N non-zero parts. The corresponding irreducible characters, known as *Schur polynomials* $s_\lambda(x_1, \dots, x_N)$, are homogeneous symmetric polynomials (of degree n) in N variables $x_1, \dots, x_N$.

2.1.3. Monomial positivity. Given a partition λ of n with at most N parts, the *monomial symmetric polynomials* are $m_\lambda(x_1,\ldots,x_N) := \sum_\alpha x_1^{\alpha_1}\cdots x_N^{\alpha_N}$, where the sum is over all distinct permutations α of the parts of λ. For example, $m_{(2,1)}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$.

The *Schur polynomials* $\{s_\lambda \mid \lambda \vdash n\}$ as well as the monomial symmetric polynomials $\{m_\lambda \mid \lambda \vdash n\}$ form a basis for the vector space of symmetric polynomials of degree n. A symmetric polynomial $f(x_1,\ldots,x_N)$ is called *monomial-positive* if

$$f(x_1,\ldots,x_N) = \sum_\lambda c_\lambda m_\lambda(x_1,\ldots,x_N),$$

where the coefficients c_λ are non-negative numbers.

2.1.4. Immersion of Weyl modules: the immersion poset. For a partition λ of n with length $\ell(\lambda) \leqslant N$, let

$$\rho_\lambda : GL_N(\mathbb{C}) \longrightarrow GL\big(W_\lambda(\mathbb{C}^N)\big)$$

be the irreducible polynomial representation of degree n of highest weight λ. It is a known fact that (for example, see [**Sta99**, Chapter 7]) if $g \in GL_N(\mathbb{C})$ has the eigenvalues $x_1,\ldots,x_N$, then the eigenvalues of $\rho_\lambda(g)$ are the monomials appearing in the Schur polynomial $s_\lambda(x_1,\ldots,x_N)$.

Thus, given two partitions λ,μ of n with $\ell(\lambda),\ell(\mu) \leqslant N$, the Weyl module $W_\lambda(\mathbb{C}^N)$ is immersed in $W_\mu(\mathbb{C}^N)$ if and only if $s_\mu(x_1,\ldots,x_N) - s_\lambda(x_1,\ldots,x_N)$ is monomial-positive. Hence studying the immersion of Weyl modules is equivalent to studying monomial positivity of the difference of Schur polynomials.

Let $\mathcal{P}(n)$ denote the set of integer partitions of n.

DEFINITION 2.1.2. We define a partial order on $\mathcal{P}(n)$ as follows. For $\lambda,\mu \in \mathcal{P}(n)$, we define $\lambda \leqslant_I \mu$ if $s_\mu(x_1,\ldots,x_N) - s_\lambda(x_1,\ldots,x_N)$ is monomial-positive. We call the poset $(\mathcal{P}(n),\leqslant_I)$ the *immersion poset*.

2.1.5. Representation theory of symmetric groups. The irreducible representations as well as the conjugacy classes of the symmetric group S_n are indexed by partitions of n. Let $\chi^\lambda(\mu)$ denote the character value of the irreducible character χ^λ evaluated at an element of cycle type μ. The character table of S_n is a square matrix encoding character values, whose rows are indexed by irreducible characters χ^λ and whose columns are indexed by conjugacy classes C_μ. The character values of S_n are all integers. Solomon [**Sol61**] proved that all row sums of the character table of S_n are non-negative integers. Finding a combinatorial interpretation of the row sums is still an open problem (see [**Sta99**, Exercise 7.71]).

2.1.6. Schur-positivity. A symmetric polynomial $f(x_1,\ldots,x_N)$ of degree n is called *Schur-positive* if

$$f(x_1,\ldots,x_N) = \sum_{\lambda\vdash n} c_\lambda s_\lambda(x_1,\ldots,x_N),$$

where the coefficients c_λ are non-negative numbers. Schur-positivity is intimately tied to representation theory. Namely, the symmetric function f is Schur-positive if it is the character of a representation W of $GL_N(\mathbb{C})$ which admits the decomposition into irreducibles $W \cong \bigoplus_{l(\lambda)\leqslant N} W_\lambda(\mathbb{C}^N)^{\oplus c_\lambda}$.

The *Frobenius characteristic map* is a bridge between characters of the symmetric group and symmetric polynomials. The irreducible character χ^λ maps to s_λ under the Frobenius characteristic map. Via the Frobenius map, Schur-positivity of f implies that there exists a representation V of S_n such that $V \cong \bigoplus_{\lambda\vdash n} V_\lambda^{\oplus c_\lambda}$, where V_λ is the irreducible representation of S_n indexed by λ.

2.1.7. Power sum symmetric polynomials and restricted row sums of character table. Define the r-th power sum symmetric polynomial as

$$p_r(x_1,\ldots,x_N) := \sum_{i=1}^{N} x_i^r.$$

For a partition $\mu = (\mu_1,\mu_2,\ldots)\vdash n$, define the *power sum symmetric polynomial* as $p_\mu := p_{\mu_1}p_{\mu_2}\cdots$.

Given a subset A_n of partitions of n, consider the sum of power sum symmetric polynomials

$$p_{A_n} := \sum_{\mu\in A_n} p_\mu. \tag{2.1.1}$$

By the Murnaghan–Nakayama rule [**Sta99**, Corollary 7.17.4], p_μ can be expressed in the basis of Schur polynomials as

$$p_\mu = \sum_{\lambda\vdash n} \chi^\lambda(\mu) s_\lambda.$$

Observe that the coefficient of s_λ in the expansion of p_{A_n} is $\sum_{\mu\in A_n}\chi^\lambda(\mu)$. This is precisely the restricted row sum (ignoring the columns not in A_n) of the character table of S_n. These values need not always be non-negative integers, that is, p_{A_n} need not be Schur-positive. For example, if $A_4 = \{(1^4), (2,1,1), (4)\}$, then one can deduce from the character table of S_4 that $p_{A_4} = 3s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} - s_{(1^4)}$ is not Schur-positive.

Question 2.1.3 (Sundaram [**Sun18**]). For which choices of A_n is the symmetric polynomial p_{A_n} Schur-positive? In other words, which subsets A_n of columns in the character table of S_n result in non-negative row sums?

In pursuit of Sundaram's question, we explore the immersion poset in detail, hence understanding the immersion of polynomial representations for $GL_N(\mathbb{C})$. Given a partition μ of n, consider the interval in the immersion poset $[(1^n), \mu] \coloneqq \{\lambda \mid (1^n) \leqslant_I \lambda \leqslant_I \mu\}$. One may ask for what choices of μ, the symmetric polynomial $p_{[(1^n),\mu]}$ defined in Equation (2.1.1) is Schur-positive. Assuming Conjectures 2.3.40 and 2.3.43, we prove that:

(1) $p_{[(1^n),(n-2,1,1)]}$ is Schur-positive;
(2) $p_{[(1^n),(n-2,2)]}$ is Schur-positive for $n \neq 7$.

One natural question which arises from the Schur-positivity of the above symmetric functions is to explore the representation theory behind it. It would be interesting to construct a natural representation V of the symmetric group such that its character maps to the symmetric polynomial $p_{[(1^n),\mu]}$ under the Frobenius map, when $\mu = (n-2,1,1)$ or $(n-2,2)$.

2.2. Standard immersion poset

In this section, we introduce the standard immersion poset, which is a refinement of the immersion poset. The definition is given in Section 2.2.1. Basic properties of the standard immersion poset are proved in Section 2.2.2. In Section 2.2.3, the maximal elements of the standard immersion poset are studied. We follow the notational conventions in [**Sta99**, Chapter 6,7].

2.2.1. Definition of the standard immersion poset.

The *Schur polynomial* s_λ for $\lambda \vdash n$ is defined as

$$s_\lambda(x_1,\dots,x_N) = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu(x_1,\dots,x_N), \tag{2.2.1}$$

where $K_{\lambda,\mu}$ are the *Kostka numbers* which count the number of semistandard Young tableaux of shape λ and content μ. Note that with this definition the Schur polynomials are zero unless

$N \geqslant \ell(\lambda)$, that is, the number of variables needs to be at least as large as the number of parts in λ.

LEMMA 2.2.1. *For $\lambda, \mu \in \mathcal{P}(n)$, $\lambda \leqslant_I \mu$ if $K_{\lambda,\alpha} \leqslant K_{\mu,\alpha}$ for all $\alpha \in \mathcal{P}(n)$.*

PROOF. By Definition 2.1.2, two partitions $\lambda, \mu \in \mathcal{P}(n)$ are comparable in the immersion poset $\lambda \leqslant_I \mu$ if

$$s_\mu(x_1, \ldots, x_N) - s_\lambda(x_1, \ldots, x_N)$$

is monomial-positive. Using (2.2.1), this can be restated as saying $\lambda \leqslant_I \mu$ if $K_{\lambda,\alpha} \leqslant K_{\mu,\alpha}$ for all $\alpha \in \mathcal{P}(n)$. □

In particular, Lemma 2.2.1 implies that a necessary condition for $\lambda \leqslant_I \mu$ is that $K_{\lambda,(1^n)} \leqslant K_{\mu,(1^n)}$, which count the standard Young tableaux of shape λ and μ, respectively. Note that $f^\lambda := K_{\lambda,(1^n)}$ is also the dimension of the Specht module V_λ (the irreducible representation of S_n) indexed by λ.

Let $\lambda, \mu \in \mathcal{P}(n)$. Define $\lambda \leqslant_D \mu$ in *dominance order* on partitions by requiring that

$$\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i \qquad \text{for all } k \geqslant 1.$$

The Kostka matrix $(K_{\lambda,\alpha})_{\lambda,\alpha \in \mathcal{P}(n)}$ is unit upper-triangular with respect to dominance order, that is, $K_{\lambda,\lambda} = 1$ and $K_{\lambda,\alpha} = 0$ unless $\alpha \leqslant_D \lambda$. This implies another necessary condition for $\lambda \leqslant_I \mu$, namely $\lambda \leqslant_D \mu$. This motivates the definition of the standard immersion poset.

DEFINITION 2.2.2. On $\mathcal{P}(n)$, define $\lambda \leqslant_{std} \mu$ if $\lambda \leqslant_D \mu$ in dominance order and $f^\lambda \leqslant f^\mu$. We call this poset the *standard immersion poset*.

As argued above, the standard immersion poset is a refinement of the immersion poset, that is, $\lambda \leqslant_I \mu$ implies that $\lambda \leqslant_{std} \mu$. The converse is not always true. For $n \geqslant 12$, there are examples of $\lambda \leqslant_{std} \mu$, which do not satisfy $\lambda \leqslant_I \mu$. For example $(5,3,1,1,1,1)$ covers $(4,2,2,2,1,1)$ in the standard immersion poset for $n = 12$, but not in the immersion poset.

EXAMPLE 2.2.3. The immersion poset for $n = 8$ is given in Figure 2.1. It is equal to the standard immersion poset.

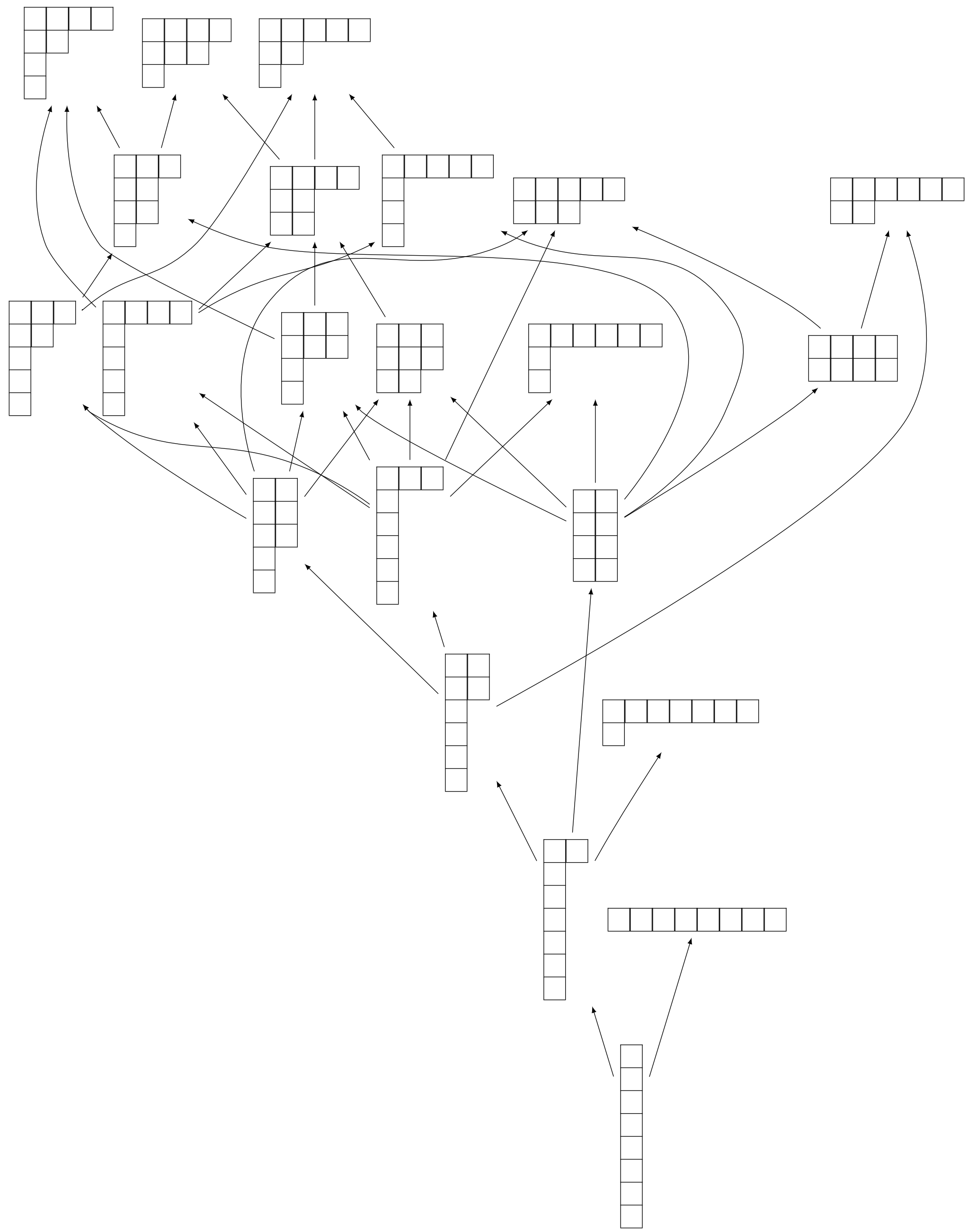


FIGURE 2.1. The (standard) immersion poset for $n = 8$.

2.2.2. Properties of the standard immersion poset. We now state and prove properties of the standard immersion poset. Our main tool is the *hook length formula* for $\lambda \in \mathcal{P}(n)$

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}, \tag{2.2.2}$$

where $h(u)$ is the hook length of the cell u in λ which counts the cells weakly to the right of u and strictly below u (in English notation for partitions).

We write $\lambda \lessdot_{std} \mu$ if μ covers λ in the standard immersion poset. More precisely, $\lambda \lessdot_{std} \mu$ if $\lambda <_{std} \mu$ and there does not exist any ν such that $\lambda <_{std} \nu <_{std} \mu$.

Lemma 2.2.4. *The partition* (1^n) *is the unique minimal element in the standard immersion poset.*

Proof. The partition (1^n) is the unique minimal element in dominance order. Furthermore, $f^{(1^n)} = 1 \leqslant f^\lambda$ for all $\lambda \in \mathcal{P}(n)$. This proves the claim. ☐

Lemma 2.2.5. *We have*

(1) $(1^n) \lessdot_{std} (n)$ *for all* n *and*

(2) $(2, 1^{n-2}) \lessdot_{std} (n-1, 1)$ *for all* $n \geqslant 3$.

Proof. We have $(1^n) <_D (n)$ and $f^{(1^n)} = f^{(n)} = 1$. There is no other partition λ with $f^\lambda = 1$. This implies $(1^n) \lessdot_{std} (n)$. Similarly, $(2, 1^{n-2}) <_D (n-1, 1)$ and $f^{(2,1^{n-2})} = f^{(n-1,1)} = n-1$. There is no other partition λ with $f^\lambda = n-1$. This implies $(2, 1^{n-2}) \lessdot_{std} (n-1, 1)$. ☐

Remark 2.2.6.

(1) Let $\lambda <_{std} \mu$. If μ covers λ in dominance order, then μ covers λ with respect to $<_{std}$. The converse is not true. Take $\lambda = (1^n)$ and $\mu = (n)$.

(2) For a given partition λ with transpose λ^t, if $\lambda <_D \lambda^t$, then $\lambda <_{std} \lambda^t$ as both representations have the same dimension, that is, $f^\lambda = f^{\lambda^t}$. In general, λ^t does not cover λ.

Given a partition λ such that $\lambda <_D \lambda^t$, it would be interesting to find all partitions $\lambda <_D \nu <_D \lambda^t$ satisfying $f^\lambda = f^\nu$. This would help to understand when the transpose of λ covers λ in the immersion poset.

Lemma 2.2.7. *Let* $\lambda = (2^a, 1^b)$ *and* $\mu = (2^{a+1}, 1^{b-2})$. *Then* $\lambda \lessdot_{std} \mu$ *if and only if* $\frac{b(b-1)}{2} > a$.

Proof. We have $\lambda \lessdot_D \mu$. Hence by Remark 2.2.6(1), it suffices to show that $\lambda <_{std} \mu$. By the hook length formula, this is true if $\frac{f^\lambda}{f^\mu} = \frac{(b+1)(a+1)}{(b-1)(a+b+1)} \leqslant 1$, which is equivalent to the condition $\frac{b(b-1)}{2} > a$. ☐

2.2.3. Classifying maximal elements. In this section, we study the maximal elements of the standard immersion poset. Recall that the standard immersion poset is a refinement of the immersion poset. This implies that if a partition is maximal in the standard immersion poset, then it is also maximal in the immersion poset.

PROPOSITION 2.2.8. *The partition $(a+b,a)$ is a maximal element in the standard immersion poset if and only if $\frac{b(b+3)}{2} \geqslant a$.*

PROOF. Let $\lambda = (a+b,a)$. Any partition ν which dominates λ, that is, $\nu >_D \lambda$, must have the form $\nu = \nu^{(i)} = (a+b+i, a-i)$ for some $i \geqslant 1$. Note that $\frac{f^{\nu^{(1)}}}{f^{\lambda}} = \frac{a(b+3)}{(b+1)(a+b+2)}$. Hence $f^{\nu^{(1)}} < f^{\lambda}$ if and only if $\frac{b(b+3)}{a-1} > 2$ (which is equivalent to $\frac{b(b+3)}{2} \geqslant a$). Thus, the condition is necessary.

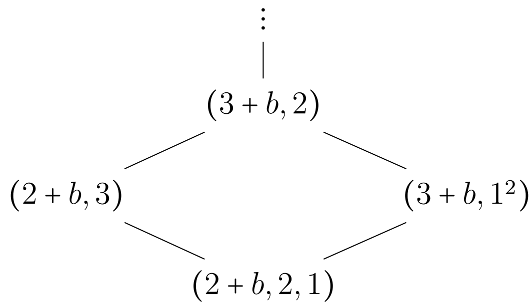
To prove that the condition $\frac{b(b+3)}{a-1} > 2$ is sufficient, note that

$$\frac{f^{\nu^{(i+1)}}}{f^{\nu^{(i)}}} = \frac{(a-i)(b+3+2i)}{(b+1+2i)(a+b+i+2)} < \frac{a(b+3)}{(b+1)(a+b+2)} = \frac{f^{\nu^{(1)}}}{f^{\lambda}}.$$

Since $f^{\nu^{(1)}} < f^{\lambda}$ when $\frac{b(b+3)}{a-1} > 2$, we must have $\frac{f^{\nu^{(i+1)}}}{f^{\nu^{(i)}}} < 1$. This is true for each i. Hence λ is a maximal element. □

PROPOSITION 2.2.9. *Let $\lambda = (a+b,a,1)$ where $a \geqslant 2$. Then λ is maximal in the standard immersion poset if and only if $a \leqslant \frac{(b+1)(b+2)}{2}$.*

PROOF. We first prove the reverse direction by inducting on a. For our base case, let $a = 2$ and $2 \leqslant \frac{(b+1)(b+2)}{2}$. To prove that $\lambda = (2+b,2,1)$ is maximal, we show that there exists no partition ν such that $\lambda <_D \nu$ and $f^{\lambda} < f^{\nu}$. We start by classifying all partitions ν such that $\lambda <_D \nu$. It is known that $\lambda \lessdot_D \nu$ if and only if the Young diagram of ν can be obtained from the Young diagram of λ by moving a single box in row k to row $k-1$ or by moving a single box in column k to column $k+1$. This means that the partition $(2+b,2,1)$ has exactly two covers: $(2+b,3)$ and $(3+b,1^2)$. The former is obtained by moving the box in row 3 to row 2, and the latter is obtained by moving the box at the end of row 2 to row 1. Furthermore, $(2+b,3)$ and $(3+b,1^2)$ are only covered by $(3+b,2)$. Below is the Hasse diagram in dominance order summarizing the specific covering relations:



Let ν be any partition such that $\lambda <_D \nu$. By our covering relations, we have that either $\nu = (2+b,3)$ or ν is contained in some chain $\lambda <_D (3+b,1^2) <_D \cdots <_D \nu$.

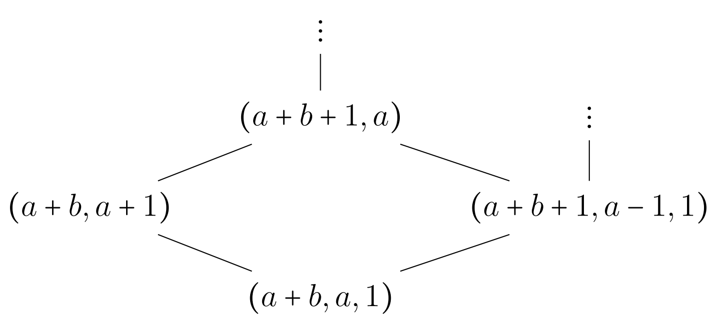
Now, we will show that for $\lambda = (2+b,2,1)$, $f^\lambda > f^\nu$ for all ν such that $\lambda <_D \nu$.

By Proposition 2.2.12, we know that $(3+b,1^2)$ is maximal in the standard immersion poset. That is, if $(3+b,1^2) <_D \nu$ then $f^{(3+b,1^2)} > f^\nu$. Note that our assumption that $2 \leqslant \frac{(b+1)(b+2)}{2}$ implies $b \geqslant 1$. By this fact and the hook length formula,

$$\frac{f^{(2+b,3)}}{f^\lambda} = \frac{b(b+4)}{2(b+1)(b+3)} < 1 \qquad \text{and} \qquad \frac{f^{(3+b,1^2)}}{f^\lambda} = \frac{3(b+4)}{2(b+1)(b+5)} < 1.$$

Since $f^\lambda > f^{(2+b,3)}$ and $f^\lambda > f^{(3+b,1^2)} > f^\nu$, we have shown that $f^\lambda > f^\nu$ for all ν such that $\lambda <_D \nu$.

Now for the remainder of the proof, let $\lambda = (a+b,a,1)$ where $a \leqslant \frac{(b+1)(b+2)}{2}$ and suppose that for some $a \geqslant 2$, the partition $(c+b,c,1)$ is maximal when $c < a \leqslant \frac{(b+1)(b+2)}{2}$. We follow a similar argument as the base case and show that $f^\lambda > f^\nu$ for $\lambda <_D \nu$. Observe that the Hasse diagram in dominance order around λ looks as follows:



We first consider the partition $(a+b,a+1)$. Then by the hook length formula,

$$\frac{f^{(a+b,a+1)}}{f^\lambda} = \frac{(a+b+2)(b)}{(a+b+1)(a)(b+1)} < \frac{(a+b+2)}{(a+b+1)(a)} < 1$$

where the last inequality follows since $a \geqslant 2$.

Next, consider the partition $(a+b+1, a-1, 1)$. By our inductive hypothesis, $(a+b+1, a-1, 1) = ((a-1)+(b+2), a-1, 1)$ is maximal since $a-1 < a \leqslant \frac{(b+1)(b+2)}{2} < \frac{(b+3)(b+4)}{2}$. Suppose that a is the upper bound of our inequality $a \leqslant \frac{(b+1)(b+2)}{2}$, that is, $a = \frac{(b+1)(b+2)}{2}$. Then by the hook length formula,

$$\begin{aligned}
\frac{f^{(a+b+1,a-1,1)}}{f^{\lambda}} &= \frac{(a+b+2)(a+1)(a-1)(b+3)}{(a+b+3)(a+b+1)(a)(b+1)} \\
&= \frac{((b+1)(b+2)+2b+4)\,((b+1)(b+2)+2)\,((b+1)(b+1-2))\,(2b+6)}{((b+1)(b+2)+2b+6)\,((b+1)(b+2)+2b+2)\,((b+1)(b+2))\,(2b+2)} \\
&= \frac{b^7+14b^6+82b^5+260b^4+477b^3+486b^2+216b}{b^7+14b^6+82b^5+260b^4+477b^3+502b^2+280b+64} < 1.
\end{aligned} \tag{2.2.3}$$

It follows that if $a < \frac{(b+1)(b+2)}{2}$ then $f^{(a+b+1,a-1,1)} < f^{\lambda}$ because for fixed b, Equation (2.2.3) decreases as a decreases. To see this, we examine the effect of decreasing a on $\frac{a+b+2}{a+b+3}$, $\frac{a+1}{a+b+1}$, and $\frac{a-1}{a}$ individually. Each of these factors is of the form $\frac{x}{x+d}$ for fixed $d > 0$. Notice that $g(x) = \frac{x}{x+d}$ is a strictly increasing function for $x > 0$. Therefore, each of the above factors decreases as a decreases. Thus, we have shown that for all ν such that $\lambda <_D \nu$, $f^{\lambda} > f^{(a+b,a+1)}$ and $f^{\lambda} > f^{(a+b+1,a-1,1)} > f^{\nu}$. Hence, $(a+b, a, 1)$ is maximal whenever $a \leqslant \frac{(b+1)(b+2)}{2}$.

Now in the reverse direction, if $a > \frac{(b+1)(b+2)}{2}$, then $f^{(a+b+1,a-1,1)} > f^{\lambda}$. To see this it suffices to consider $a = \frac{(b+1)(b+2)}{2} + 1$ since Equation (2.2.3) increases as a increases for the same reason as above. If $a = \frac{(b+1)(b+2)}{2} + 1$, then

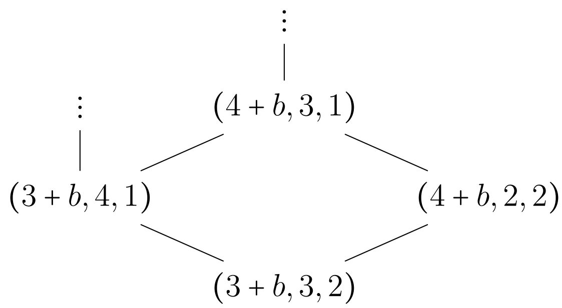
$$\frac{f^{(a+b+1,a-1,1)}}{f^{\lambda}} = \frac{b^7+14b^6+88b^5+322b^4+739b^3+1056b^2+852b+288}{b^7+14b^6+88b^5+318b^4+707b^3+964b^2+740b+240} > 1.$$

Therefore, λ is maximal if only if $a \leqslant \frac{(b+1)(b+2)}{2}$. □

PROPOSITION 2.2.10. *Let $\lambda = (a+b, a, 2)$ where $a \geqslant 3$. Then λ is maximal in the standard immersion poset if and only if $a \leqslant \frac{(b+1)(b+2)}{2}$.*

PROOF. We first prove the reverse direction by inducting on a. For our base case, let $a = 3$ and $3 \leqslant \frac{(b+1)(b+2)}{2}$. To prove that $\lambda = (3+b, 3, 2)$ is maximal, we follow a similar argument to Proposition 2.2.9. We first classify all partitions ν such that $\lambda <_D \nu$ and then show that $f^{\lambda} > f^{\nu}$ for all such ν by finding chains in the dominance order that contain maximal elements from the standard immersion poset. Our assumption that $3 \leqslant \frac{(b+1)(b+2)}{2}$ implies that $b \geqslant 1$. Hence it suffices to show that $\lambda = (3+b,3,2)$ is maximal for all $b \geqslant 1$. We consider the cases $b=1, b=2$, and $b \geqslant 3$ separately. It can be checked explicitly (for example using SageMath [The24]) that $(4,3,2)$ and $(5,3,2)$ are maximal in the standard immersion poset.

For $b \geqslant 3$, the Hasse diagram in dominance order around $\lambda = (3+b,3,2)$ looks as follows:

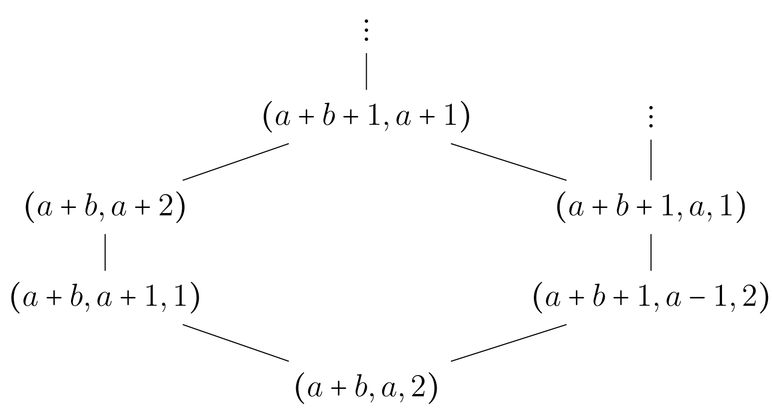


If $\lambda <_D \nu$ then $\nu = (3+b,4,1), (4+b,2,2)$, or ν is contained in some chain $\lambda <_D (3+b,4,1) <_D \nu$. By Proposition 2.2.9, $(3+b,4,1)$ is maximal in the standard immersion poset so it suffices to show that $f^\lambda > f^\nu$ for $\nu = (3+b,4,1)$ and $(4+b,2,2)$. By the hook length formula,

$$\frac{f^{(3+b,4,1)}}{f^\lambda} = \frac{4(b)(b+4)}{5(b+1)(b+3)} = \frac{4(b^2+4b)}{5(b^2+4b+3)} < 1 \qquad \text{and} \qquad \frac{f^{(4+b,2,2)}}{f^\lambda} = \frac{4(b+4)}{2(b+1)(b+6)} < 1.$$

Hence, for $b \geqslant 3$, $(3+b,3,1)$ is maximal in the standard immersion poset, so we have shown that $(3+b,3,1)$ is maximal for all $b \geqslant 1$.

Now, let $\lambda = (a+b,a,2)$ where $a \leqslant \frac{(b+1)(b+2)}{2}$ and suppose that for some $a \geqslant 3$, the partition $(c+b,c,2)$ is maximal when $c < a \leqslant \frac{(b+1)(b+2)}{2}$. Again, we show that $f^\lambda > f^\nu$ for $\lambda <_D \nu$. Observe that the Hasse diagram in dominance order around λ looks as follows:



By the Hasse diagram, if ν is a partition such that $\lambda <_D \nu$, then $\nu = (a+b, a+1, 1), (a+b, a+2), (a+b+1, a-1, 2)$, or ν is contained in some chain $\lambda <_D (a+b+1, a-1, 2) <_D \nu$. Observe that $(a+b+1, a-1, 2) = ((a-1)+(b+2), a-1, 2)$ is maximal by our inductive hypothesis since $a - 1 < a \leqslant \frac{(b+1)(b+2)}{2} \leqslant \frac{(b+3)(b+4)}{2}$. Therefore, it suffices to check that $f^\lambda > f^\nu$ for $\nu = (a+b, a+1, 1), (a+b, a+2)$, and $(a+b+1, a-1, 2)$.

For $\nu = (a+b, a+1, 1)$, we have that

$$\frac{f^{(a+b,a+1,1)}}{f^\lambda} = \frac{2b(a+b+1)(a+1)}{(a+b)(a-1)(a+2)(b+1)}. \tag{2.2.4}$$

Since

$$\frac{d}{da}\frac{f^{(a+b,a+1,1)}}{f^\lambda} = -\frac{(2b(a^4+2a^3b+4a^3+a^2b^2+5a^2b+7a^2+2ab^2+8ab+2a+3b^2+3b-2))}{((a-1)^2(a+2)^2(b+1)(a+b)^2)},$$

we have that Equation (2.2.4) decreases as a increases. Therefore, it suffices to consider $a = 3$ which we have done in our base case. Hence, $f^{(a+b,a+1,1)} < f^\lambda$.

For $\nu = (a+b, a+2)$, we have that

$$\frac{f^{(a+b,a+2)}}{f^\lambda} = \frac{2(b-1)(a+b+2)}{(a-1)(b+1)(a+b)(a+2)} \leqslant \frac{a+b+2}{(a+b)(a+2)}$$

since $a \geqslant 3$. As $(a+b)(a+2) = a^2 + 2a + ab + 2b \geqslant a + b + 2$, we have that $f^{(a+b,a+2)} < f^\lambda$.

Lastly, for $\nu = (a+b+1, a-1, 2)$, we first consider when $a = \frac{(b+1)(b+2)}{2}$. By the hook length formula, we have

$$\begin{aligned} \frac{f^{(a+b+1,a-1,2)}}{f^\lambda} &= \frac{(b+3)(a+b+1)(a-2)(a+1)}{(b+1)(a+b+3)(a-1)(a+b)} \\ &= \frac{(2b+6)((b+1)(b+2)+2b+2)((b+1)(b+2)-4)((b+1)(b+2)+2)}{((b+1)(b+2)+2b)(2b+2)((b+1)(b+2)+2b+6)((b+1)(b+2)-2)} \\ &= \frac{b^7+14b^6+78b^5+220b^4+321b^3+182b^2-80b-96}{b^7+14b^6+78b^5+220b^4+321b^3+214b^2+48b} \\ &< 1. \end{aligned} \tag{2.2.5}$$

Following a similar argument as in Proposition 2.2.9 for Equation (2.2.3), we can see that for fixed b, Equation (2.2.5) decreases as a decreases by considering $\frac{a+b+1}{a+b+3}, \frac{a-2}{a-1}$, and $\frac{a+1}{a+b}$.

Value for α	$\lambda = (\alpha, \beta)$	$\lambda = (\alpha, \beta, 1)$	$\lambda = (\alpha, \beta, 2)$
$\alpha \geqslant 2$	$(\alpha, 1)$		
$\alpha \geqslant 3$	$(\alpha, 2)$	$(\alpha, 2, 1)$	
$\alpha \geqslant 4$		$(\alpha, 3, 1)$	$(\alpha, 3, 2)$
$\alpha \geqslant 5$	$(\alpha, 3)$		
$\alpha \geqslant 6$	$(\alpha, 4)$	$(\alpha, 4, 1)$	$(\alpha, 4, 2)$
$\alpha \geqslant 7$	$(\alpha, 5)$	$(\alpha, 5, 1)$	$(\alpha, 5, 2)$
$\alpha \geqslant 8$		$(\alpha, 6, 1)$	$(\alpha, 6, 2)$
$\alpha \geqslant 9$	$(\alpha, 6)$		
$\alpha \geqslant 10$	$(\alpha, 7)$	$(\alpha, 7, 1)$	$(\alpha, 7, 2)$
$\alpha \geqslant 11$	$(\alpha, 8)$	$(\alpha, 8, 1)$	$(\alpha, 8, 2)$
$\alpha \geqslant 12$	$(\alpha, 9)$	$(\alpha, 9, 1)$	$(\alpha, 9, 2)$
$\alpha \geqslant 13$		$(\alpha, 10, 1)$	$(\alpha, 10, 2)$
$\alpha \geqslant 14$	$(\alpha, 10)$		
$\alpha \geqslant 15$	$(\alpha, 11)$	$(\alpha, 11, 1)$	$(\alpha, 11, 2)$
$\alpha \geqslant 16$	$(\alpha, 12)$	$(\alpha, 12, 1)$	$(\alpha, 12, 2)$
$\alpha \geqslant 17$	$(\alpha, 13)$	$(\alpha, 13, 1)$	$(\alpha, 13, 2)$
$\alpha \geqslant 18$	$(\alpha, 14)$	$(\alpha, 14, 1)$	$(\alpha, 14, 2)$
$\alpha \geqslant 19$		$(\alpha, 15, 1)$	$(\alpha, 15, 2)$
$\alpha \geqslant 20$	$(\alpha, 15)$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 2.1. Necessary and sufficient conditions for maximality of a partition λ.

We have thus shown that when $a \geqslant 3$, $\lambda = (a+b, a, 2)$ is maximal if $a \leqslant \frac{(b+1)(b+2)}{2}$. For the reverse direction, consider Equation (2.2.5) when $a = \frac{(b+1)(b+2)}{2} + 1$. We have that

$$\frac{f^{(a+b+1,a-1,2)}}{f^\lambda} = \frac{b^6 + 12b^5 + 60b^4 + 162b^3 + 243b^2 + 162b}{b^6 + 12b^5 + 60b^4 + 158b^3 + 219b^2 + 150b + 40} > 1.$$

Since Equation (2.2.5) increases as a increases, $\lambda \leqslant_{std} (a+b+1, a-1, 2)$ when $a > \frac{(b+1)(b+2)}{2}$.

Therefore, λ is maximal if and only if $a \leqslant \frac{(b+1)(b+2)}{2}$. □

REMARK 2.2.11. We may translate the results of Propositions 2.2.8, 2.2.9, and 2.2.10 into statements about partitions of the form (α, β), $(\alpha, \beta, 1)$, and $(\alpha, \beta, 2)$. Table 2.1 summarizes our maximality conditions.

We next classify all maximal hook shape partitions. As noted in Lemma 2.2.5, $(1^n) \lessdot_{std} (n)$ and so the single column shape is only maximal when $n = 1$. By Lemma 2.2.7, $(2, 1^b) \lessdot_{std} (2^2, 1^{b-2})$ whenever $b \geqslant 3$. Since $(2,1,1) \lessdot_{std} (3,1)$, the only maximal hook shape with arm length 2 is $(2,1)$. In the following proposition, we investigate all hook shape partitions with arm length greater than 2.

Proposition 2.2.12. *Let $\lambda = (a, 1^b)$ be a hook shape partition such that $a > 2$. Then λ is a maximal element in the standard immersion poset if and only if $b \leqslant 2$.*

Proof. When $b = 1$, the only partition that dominates $(a, 1)$ is $(a+1)$ and $f^{(a,1)} = (a+1)-1 > 1 = f^{(a+1)}$. Thus, $(a,1)$ is maximal. When $b = 2$, the only partitions that dominate $(a, 1^2)$ are $(a+2)$, $(a+1,1)$, and $(a,2)$. By the hook length formula,

$$\frac{f^{(a+1,1)}}{f^{(a,1^2)}} = \frac{2}{a} < 1 \qquad \text{and} \qquad \frac{f^{(a,2)}}{f^{(a,1^2)}} = \frac{(a+2)(a-1)}{(a+1)a} = \frac{a^2+a-2}{a^2+a} < 1.$$

Therefore, no partition dominates $(a,1^2)$ and has more standard Young tableaux, so $(a,1^2)$ is maximal.

When $b \geqslant 3$, $(a, 1^b) <_{std} (a, 2, 1^{b-2})$, by the hook length formula:

$$\frac{f^{(a,1^b)}}{f^{(a,2,1^{b-2})}} = \frac{a(a+b-1)}{(a+b)(a-1)(b-1)} \leqslant 1$$

since

$$(a+b)(a-1)(b-1) \geqslant 2(a-1)(a+b) \geqslant a(a+b-1).$$

Therefore $f^{(a,1^b)} \leqslant f^{(a,2,1^{b-2})}$ and $(a,1^b) <_D (a,2,1^{b-2})$, so we have $(a,1^b) <_{std} (a,2,1^{b-2})$ whenever $b \geqslant 3$. ☐

Proposition 2.2.13. *If λ is a maximal element in the standard immersion poset, then $\lambda_1 > \lambda_2$.*

Proof. Suppose by contradiction that $\lambda = (a^b, \lambda_{b+1}, \ldots)$ with $a > \lambda_{b+1}$ and $b \geqslant 2$. Let $\mu = (a+1, a^{b-2}, a-1, \lambda_{b+1}, \ldots)$ and denote by $\mathsf{SYT}(\lambda)$ the set of all standard Young tableaux of shape λ. The map

$$\varphi \colon \mathsf{SYT}(\lambda) \to \mathsf{SYT}(\mu),$$

where $\varphi(T)$ is the standard Young tableau obtained from T by moving the box in position (b, a) to position $(1, a+1)$, is an injection. Note that since the entry in position (b, a) is greater than the entry in $(1, a)$ from strictly increasing columns, then it follows that the newly obtained Young tableau is standard. Therefore, $\lambda <_D \mu$ and $f^\lambda \leqslant f^\mu$, which implies $\lambda \leqslant_I \mu$ and thus demonstrates that λ is not a maximal element in the standard immersion poset. □

In fact, the injection φ used in the proof of Proposition 2.2.13 remains an injection when the domain and codomain are extended to semistandard Young tableaux of content ν, for any $\nu \vdash |\lambda|$. Injection arguments between sets of semistandard Young tableaux are expanded on in Section 2.3.2. In particular, this result is extended to the immersion poset in Corollary 2.3.7.

We conclude this section with a conjecture about more general maximal elements in the standard immersion poset.

Conjecture 2.2.14. *Suppose* $\lambda = (\sum_{i=1}^{\ell} a_i, \sum_{i=1}^{\ell-1} a_i, \ldots, a_2 + a_1, a_1)$ *for* $\ell > 2$. *If*

$$\binom{a_j + 2}{2} \geqslant \sum_{i=1}^{j-1} a_i + j - 2$$

is satisfied for all $2 \leqslant j \leqslant \ell$, *then* λ *is maximal in the standard immersion poset.*

This conjecture has been verified with SageMath [**The24**] for $|\lambda| \leqslant 30$.

Remark 2.2.15. Proposition 2.2.8 addresses the case $\ell = 2$ associated to Conjecture 2.2.14. Note that for $\ell = 2$ the condition stated in Conjecture 2.2.14 reads

$$\binom{a_2 + 2}{2} \geqslant a_1, \quad \text{whereas the condition from Proposition 2.2.8 is} \quad \binom{a_2 + 2}{2} > a_1.$$

This discrepancy comes from the fact that for $\ell > 2$, there are more factors contributing to the inequality in $\frac{f^\mu}{f^\lambda} < 1$.

2.3. Immersion poset

In this section we turn to the immersion poset. In Section 2.3.1, we study basic properties of the immersion poset. In Section 2.3.2, we provide explicit injections between certain sets of semistandard Young tableaux, which are used to determine statements about maximal elements and cover relations in the immersion poset. In Sections 2.3.3 and 2.3.4, we study the immersion poset restricted to hook partitions and two column partitions, respectively. We conclude in Section 2.3.5 with conjectures about certain lower intervals in the immersion poset and prove that the conjectured intervals give Schur-positive sums of power sum symmetric functions.

2.3.1. Properties of the immersion poset. We begin by specifying the minimal element.

LEMMA 2.3.1. *The partition* (1^n) *is the unique minimal element in the immersion poset* $(\mathcal{P}(n), \leqslant_I)$.

PROOF. We have $f^{(1^n)} = 1 \leqslant f^{\lambda}$ for all $\lambda \in \mathcal{P}(n)$. Furthermore $K_{(1^n),\alpha} = 0 \leqslant K_{\lambda,\alpha}$ for all $\alpha \neq (1^n)$ and $\lambda \in \mathcal{P}(n)$. By Lemma 2.2.1 this proves the claim. □

Analogously to Lemma 2.2.5, we prove the following result.

LEMMA 2.3.2. *We have*

(1) $(1^n) \lessdot_I (n)$ *for all* n *and*

(2) $(2, 1^{n-2}) \lessdot_I (n-1, 1)$ *for all* $n \geqslant 3$.

PROOF. By Lemma 2.3.1, we have $(1^n) <_I (n)$. By Lemma 2.2.5, $(1^n) \lessdot_{std} (n)$. Since in the immersion poset there are fewer order relations than in the standard immersion poset, the first part of the lemma follows.

We have $(2, 1^{n-2}) <_I (n-1, 1)$ since

$$s_{(2,1^{n-2})} = (n-1)m_{(1^n)} + m_{(2,1^{n-2})} \text{ and } s_{(n-1,1)} = (n-1)m_{(1^n)} + (n-2)m_{(2,1^{n-2})} + \sum_{\mu \neq (1^n),(2,1^{n-2})} K_{(n-1,1),\mu} m_\mu.$$

Again, since by Lemma 2.2.5 we have $(2, 1^{n-2}) \lessdot_{std} (n-1, 1)$, the second part of the lemma follows. □

Unlike in the standard immersion poset, where λ and λ^t are always comparable as long as they are comparable in dominance order (see Remark 2.2.6), this is not always true in the immersion poset. For example $\lambda = (4, 4, 2, 1, 1)$ and λ^t are not comparable in the immersion poset since $K_{(4,4,2,1,1),(4,4,1,1,1,1)} > K_{(5,3,2,2),(4,4,1,1,1,1)}$. For hook partitions, it is however true that $\lambda <_I \lambda^t$ if $\lambda <_D \lambda^t$ (see Corollary 2.3.30).

We prove the analog of Lemma 2.2.7 in the next section using injections on semistandard Young tableaux. See Corollaries 2.3.6, 2.3.13, and 2.3.21.

2.3.2. Explicit injections. Recall from Lemma 2.2.1 that $\lambda \leqslant_I \mu$ if and only if $K_{\lambda,\nu} \leqslant K_{\mu,\nu}$ for all $\nu \in \mathcal{P}(n)$. The Kostka number $K_{\lambda,\nu}$ is the cardinality of the set of semistandard Young tableaux $\mathsf{SSYT}(\lambda,\nu)$ of shape λ and content ν. Hence we can analyze the order relations $\lambda \leqslant_I \mu$ by constructing explicit injections

$$\varphi\colon \mathsf{SSYT}(\lambda,\nu) \to \mathsf{SSYT}(\mu,\nu) \tag{2.3.1}$$

for all $\nu \in \mathcal{P}(n)$.

To this end, we present one such injection, where μ differs from λ by moving a single cell from the c-th column to the $(c+1)$-th column, and λ has a bound on the relative size of the two columns. Upon establishing this first injection, we refine it to obtain more precise bounds on the relative size of the columns. We partially characterize what elements cannot be maximal in the immersion poset, similar to those given in Section 2.2.3 for the standard immersion poset.

Let

$$\begin{aligned} \lambda &= (\lambda_1,\dots,\lambda_\alpha,c^\beta,\lambda_{\alpha+\beta+1},\dots),\\ \mu &= (\lambda_1,\dots,\lambda_\alpha,c+1,c^{\beta-2},c-1,\lambda_{\alpha+\beta+1},\dots), \end{aligned} \tag{2.3.2}$$

such that either $\alpha > 0$ and $\lambda_{\beta+\alpha+1} < c < \lambda_\alpha$, or $\alpha = 0$ and $\lambda_{\beta+\alpha+1} < c$. In particular, $\lambda_{\beta+\alpha+1}$ can be 0. We define a map

$$\varphi_0\colon \mathsf{SSYT}(\lambda,\nu) \to \mathsf{YT}(\mu,\nu),$$

where $\mathsf{YT}(\mu,\nu)$ is the set of all tableaux of shape μ and content ν, not necessarily semistandard. We will show in Proposition 2.3.5 that when $\beta \geqslant \alpha + 2$, the image of φ_0 will be contained in $\mathsf{SSYT}(\mu,\nu)$, so φ_0 will be as in (2.3.1).

For $T \in \mathsf{SSYT}(\lambda,\nu)$, we define $\varphi_0(T)$ as follows. Suppose the entries in the c-th column of T in increasing order are $x_{\beta+\alpha}, x_{\beta+\alpha-1},\dots,x_1$ and the entries in the $(c+1)$-th column of T in increasing order are $y_\alpha, y_{\alpha-1},\dots,y_1$. Let i be the smallest index such that $x_i > y_i$. If no such index exists, let $i = \alpha+1$. Then $\varphi_0(T)$ is the tableau such that the entries in the c-th column of $\varphi_0(T)$ are

$$x_{\beta+\alpha}, x_{\beta+\alpha-1},\dots,x_{i+1}, y_{i-1}, y_{i-2},\dots,y_1,$$

the entries in the $(c+1)$-th column of $\varphi_0(T)$ are

$$y_\alpha, y_{\alpha-1}, \ldots, y_i, x_i, x_{i-1}, \ldots, x_1,$$

and all other entries are the same as those in T. In other words, φ_0 moves the cell containing x_1 to the $(\alpha+1)$-th row of the $(c+1)$-th column, and swaps each x_j with y_{j-1} for all $2 \leqslant j \leqslant i$.

More concretely, the c-th and $(c+1)$-th column in T and $\varphi_0(T)$ look as follows:

(2.3.3)

T :

$x_{\beta+\alpha}$	y_α
$\vdots$	$\vdots$
$x_{\beta+i}$	y_i
$x_{\beta+i-1}$	y_{i-1}
$\vdots$	$\vdots$
$x_{\beta+1}$	y_1
x_β	
$\vdots$	
x_{i+1}	
x_i	
$\vdots$	
x_2	
x_1	

$\varphi_0(T)$:

$x_{\beta+\alpha}$	y_α
$\vdots$	$\vdots$
$x_{\beta+i}$	y_i
$x_{\beta+i-1}$	x_i
$\vdots$	$\vdots$
$x_{\beta+1}$	x_2
x_β	x_1
$\vdots$	
x_{i+1}	
y_{i-1}	
$\vdots$	
y_1	

The cells marked in green contain the entries that move from the c-th column to the $(c+1)$-th column, and the cells marked in yellow are the entries that move from $(c+1)$-th column to the c-th column. We continue to use this convention for all subsequent examples of φ_0.

Remark 2.3.3. Observe that by our choice of i, both $x_{i+1} < x_{i-1} \leqslant y_{i-1}$ and $y_i < x_i$, so the columns of $\varphi_0(T)$ are strictly increasing by construction.

Example 2.3.4. For $\lambda = (3,2,1^4)$ and $\mu = (3,2,2,1^2,0)$, we have $c = 1$, $\alpha = 2$ and $\beta = 4$. Here are some examples of the injection φ_0 on various tableaux of shape λ:

1	1	3
2	2	
3		
4		
5		
6		

$\mapsto$

1	1	3
2	2	
3	6	
4		
5		

1	1	2
2	7	
3		
4		
5		
6		

$\mapsto$

1	1	2
2	5	
3	6	
4		
7		

1	6	9
2	8	
3		
4		
5		
7		

$\mapsto$

1	4	9
2	5	
3	7	
6		
8		

Proposition 2.3.5. *Let λ and μ be as in (2.3.2) with $\beta \geqslant \alpha + 2$. Then φ_0 as defined above is an injection*

$$\varphi_0 \colon \mathsf{SSYT}(\lambda, \nu) \to \mathsf{SSYT}(\mu, \nu).$$

Proof. Let $T \in \mathsf{SSYT}(\lambda, \nu)$. Note that the content does not change under φ_0. We need to check that the c-th and $(c+1)$-th columns of $\varphi_0(T)$ are strictly increasing, and that the $(\alpha-i+2)$-th through $(\alpha+\beta-1)$-th rows of $\varphi_0(T)$ are weakly increasing, since all other entries are identical to those in T. (It may be helpful to consult (2.3.3).) The columns are strictly increasing by Remark 2.3.3.

For rows, we first consider the $(\alpha-i+2)$-th through $(\alpha+1)$-th rows. Due to the bound $\beta \geqslant \alpha+2$, in the c-th column, these rows contain $x_{\beta+i-1}, \dots, x_\beta$. In the $(c+1)$-th column, irrespective of the bound on α and β, these rows contain $x_i, \dots, x_1$. In particular, the bound $\beta \geqslant \alpha + 2$ makes it so that there is no y_j entry in these rows, so there is no "overlap" of y_j and x_k for $1 \leqslant k \leqslant i$. The rows are thus strictly increasing because $x_j > x_k$ for all $j < k$, so the x_j entries in the $(c+1)$-th column are greater than the entries to their left in the c-th column; and $x_j < x_{j-1} \leqslant y_{j-1}$ for $2 \leqslant j \leqslant i$, so the x_j entries in the $(c+1)$-th column are less than any entries to their right, originally from T. (Such entries on the right do not necessarily exist. In particular, x_1 never has any cell to its right.)

Now consider the $(\alpha+2)$-th through $(\alpha+\beta-1)$-th rows. In the c-th column, these rows contain

$$x_{\beta-1}, \dots, x_{i+1}, y_{i-1}, \dots, y_1,$$

and in the $(c+1)$-th column, these rows contain no cells. They are weakly increasing because $y_{j-1} \geqslant x_{j-1} > x_j$ for $2 \leqslant j \leqslant i$, so the y_{j-1} entries in the c-th column are greater than the entries to their left, originally from T, and they have no cells to their right.

To show injectivity, we define an explicit inverse ψ_0. Let $T' \in \varphi_0(\mathsf{SSYT}(\lambda, \nu))$. Suppose the entries in the c-th column of T' in increasing order are $x'_{\beta+\alpha-1}, x'_{\beta+\alpha-2}, \dots, x'_1$, and the entries in the $(c+1)$-th column of T' in increasing order are $y'_{\alpha+1}, y'_\alpha, \dots, y'_1$. Let i' be the smallest index such that $y'_{i'} > x'_{i'}$. This i' will be equal to the i from the definition of φ_0, because $x_i > x_{i+1}$ and $x_j \leqslant y_j$ for $1 \leqslant j \leqslant i-1$ in T.

Then $\psi_0(T')$ is the tableau of shape λ such that the entries in the c-th column of $\psi_0(T')$ are

$$x'_{\beta+\alpha-1}, x'_{\beta+\alpha-2}, \dots, x'_{\beta-1}, x'_{\beta-2}, x'_{\beta-3}, \dots, x'_{i'}, y'_{i'}, y'_{i'-1}, \dots, y'_1,$$

the entries in the $(c+1)$-th column of $\psi_0(T')$ are

$$y'_{\alpha+1}, y'_{\alpha}, \dots, y'_{i'+1}, x'_{i'-1}, x'_{i'-2}, \dots, x'_1,$$

and all other entries are the same as those in T'. In other words, ψ_0 moves the cell containing y'_1 to the $(\alpha+\beta)$-th position in the c-th column, and swaps each $y'_{j'}$ with $x'_{j'-1}$ for all $2 \leqslant j' \leqslant i'$. Concretely:

(2.3.4)

T' :

$x'_{\beta+\alpha-1}$	$y'_{\alpha+1}$
$\vdots$	$\vdots$
$x'_{\beta+i'-1}$	$y'_{i'+1}$
$x'_{\beta+i'-2}$	$y'_{i'}$
$\vdots$	$\vdots$
x'_{β}	y'_2
$x'_{\beta-1}$	y'_1
$x'_{\beta-2}$	
$\vdots$	
$x'_{i'}$	
$x'_{i'-1}$	
$\vdots$	
x'_1	

$\psi_0(T')$:

$x'_{\beta+\alpha-1}$	$y'_{\alpha+1}$
$\vdots$	$\vdots$
$x'_{\beta+i'-1}$	$y_{i'+1}$
$x'_{\beta+i'-2}$	$x'_{i'-1}$
$\vdots$	$\vdots$
x'_{β}	x'_1
$x'_{\beta-1}$	
$x'_{\beta-2}$	
$\vdots$	
$x'_{i'}$	
$y'_{i'}$	
$\vdots$	
y'_2	
y'_1	

Since $i' = i$, ψ_0 moves back exactly the entries in T' that were originally moved by φ_0 in T, so ψ_0 is the inverse of φ_0. ☐

As a corollary, the injection describes a class of cover relations in the immersion poset. As a specific example, it can partially address the two column case, which was completely addressed by Lemma 2.2.7 for the standard immersion poset.

Corollary 2.3.6. *The partitions λ and μ as in (2.3.2) with $\beta \geqslant \alpha + 2$ form a cover in the immersion poset. In particular, $\lambda = (2^\alpha, 1^\beta)$ and $\mu = (2^{\alpha+1}, 1^{\beta-2})$ form a cover.*

Proof. The partition μ covers λ in dominance order, and the injection shows that μ is greater than λ in the immersion poset, so μ must also cover λ in the immersion poset. ☐

The injection also gives a few conditions on which partitions cannot be maximal.

Corollary 2.3.7. *If* $\lambda = (a^\beta, b, \ldots)$*, where* $a > b$*, and* $\beta \geqslant 2$*, then* λ *is not maximal.*

Proof. We have $\beta \geqslant 2$ with $\alpha = 0$, so we can apply the injection. □

Corollary 2.3.8. *If* $\lambda = (a, b^\beta, c, \ldots)$*, where* $a > b > c$*, and* $\beta \geqslant 3$*, then* λ *is not maximal. In particular,* $\lambda = (a, 1^\beta)$ *is not maximal for* $a \geqslant 2$*,* $\beta \geqslant 3$*.*

Proof. We have $\beta \geqslant 3$ with $\alpha = 1$, so we can apply the injection. □

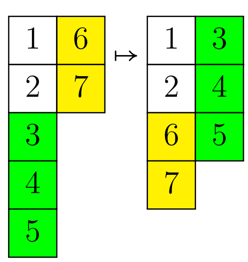
Note that Corollary 2.3.7 and Corollary 2.3.8 repeat the results from Proposition 2.2.13 and the forward direction of Proposition 2.2.12 concerning nonmaximal elements in the standard immersion poset.

Corollary 2.3.9. *If* $\lambda = (a, b, c, d)$ *is maximal in the immersion poset, then it has no more than two identical non-zero parts.*

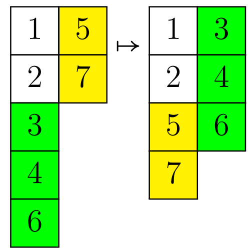
Proof. If λ has three or more identical parts, then λ is one of (a^4), (a^3, d), or (a, b^3), so we can apply the injection. □

As stated in the proof of Proposition 2.3.5, the bound $\beta \geqslant \alpha + 2$ is necessary for $\varphi_0(T)$ to be semistandard for T semistandard. When $\beta < \alpha + 2$, φ_0 can cause an "overlapping" row, where for certain $1 \leqslant j \leqslant \alpha - \beta + 2$, $y_{\beta-2+j}$ is to the left of x_j, yet $y_{\beta-2+j} > x_j$.

Example 2.3.10. For $\lambda = (2^2, 1^3)$, so $\alpha = 2$ and $\beta = 3 = \alpha + 1$, φ_0 can give:



One natural modification to restore weakly increasing rows is to swap $y_{\beta-2+j}$ and x_j whenever the problem occurs. Unfortunately, doing so on its own would not maintain injectivity. If we try to swap the 5 and 6 in the previous example, our final tableau is the same as the following tableau obtained from φ_0 with no switches:



However, if we are able to implement subsequent modifications in a way such that the resulting tableau is semistandard, yet cannot be obtained from φ_0 alone, then we can restore injectivity.

We now define the modification of our original φ_0 injection for the case when $\beta = \alpha + 1$, and $\alpha \geqslant 2$, which we call

$$\varphi_1\colon \mathsf{SSYT}(\lambda) \to \mathsf{SSYT}(\mu).$$

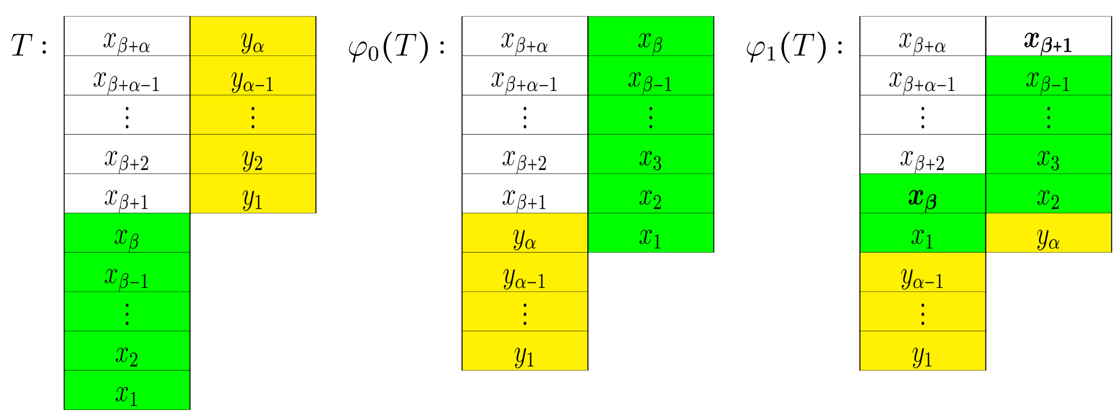
From now on, we drop the content ν as all maps in this subsection preserve the content.

Let $T \in \mathsf{SSYT}(\lambda)$. As before, suppose that the entries in the c-th column of T in increasing order are $x_{\beta+\alpha}, x_{\beta+\alpha-1}, \ldots, x_1$, and the entries in the $(c+1)$-th column of T in increasing order are $y_\alpha, y_{\alpha-1}, \ldots, y_1$. We define $\varphi_1(T)$ to be the same as $\varphi_0(T)$ if $\varphi_0(T) \in \mathsf{SSYT}(\mu)$.

If $\varphi_0(T) \notin \mathsf{SSYT}(\mu)$, then necessarily $i = \alpha + 1$ as defined for φ_0 and x_1 is to the right of $y_{\beta-1} = y_\alpha$, with $y_\alpha > x_1$. Then $\varphi_1(T)$ is the same as $\varphi_0(T)$, except we swap y_α with x_1, as well as $x_{\beta+1}$ with x_β.

Concretely, when $\varphi_1(T) \neq \varphi_0(T)$, the c-th and $(c+1)$-th columns of T, $\varphi_0(T)$, and $\varphi_1(T)$ look as follows:

(2.3.5)



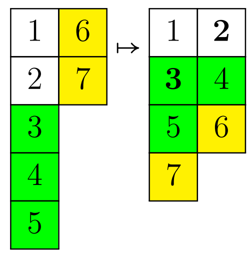
We indicate the additional swaps φ_1 adds to φ_0 using boldface on the relevant entries, $x_{\beta+1}$ and x_β. We will continue to use this convention for any subsequent modifications to φ_0.

Observe that for the $x_{\beta+1}$ with x_β swap to be between cells in different rows, we must have $\alpha \geqslant 2$. This property is necessary for the tableau to remain semistandard after the swap.

The intuition behind the swaps is that the swap of y_α with x_1 makes the tableau semistandard, and the swap of $x_{\beta+1}$ with x_β prevents the new tableau from being in the image of φ_0.

EXAMPLE 2.3.11. For T in Example 2.3.10, φ_1 maps:

(2.3.6)



PROPOSITION 2.3.12. *Let λ and μ be as in (2.3.2) with $\beta = \alpha + 1 \geqslant 3$. Then φ_1 as defined above is an injection*

$$\varphi_1 \colon \mathsf{SSYT}(\lambda) \to \mathsf{SSYT}(\mu).$$

PROOF. Let $T \in \mathsf{SSYT}(\lambda)$. We need to check that $\varphi_1(T)$ is semistandard. It suffices to do so for the case when $\varphi_1(T) \neq \varphi_0(T)$, where there is an overlap in $\varphi_0(T)$ consisting of a single decreasing pair of cells in a row, $y_\alpha > x_1$. In particular, $\varphi_0(T)$ would be semistandard if it were not for this single pair by the proof of Proposition 2.3.5, so it suffices to check that swapping the x_1 with y_α makes the tableau semistandard, and swapping the $x_{\beta+1}$ with x_β keeps it semistandard, by examining the changed entries.

Swapping the entries x_1 with y_α makes the $(\alpha+1)$-th row weakly increasing since $x_1 < y_\alpha$ by assumption. The c-th column remains strictly increasing since $x_{\beta+1} < x_1 < y_\alpha < y_{\alpha-1}$, and the $(c+1)$-th column remains strictly increasing since $x_2 < x_1 < y_\alpha$.

Swapping the entries x_β with $x_{\beta+1}$ keeps the relevant rows, namely the 1st row and α-th row, weakly increasing and their columns strictly increasing since $x_j > x_k$ for all $j < k$.

Specifically, the c-th column remains strictly increasing since $x_{\beta+2} < x_\beta < x_1$, and $(c+1)$-th column remains strictly increasing since $x_{\beta+1} < x_{\beta-1}$. The 1st row remains weakly increasing because $x_{\beta+1} < x_\beta$, so $x_{\beta+1}$ is also less than all entries to its right, which are originally right of x_β. The α-th row remains weakly increasing because $x_{\beta+1} < x_\beta$, so x_β is greater than all entries to its left, which are originally left of $x_{\beta+1}$.

To show injectivity, it suffices to check that the modified tableau cannot be in the image of φ_0, allowing us to define an explicit inverse ψ_1 by likewise modifying ψ_0. Namely, we must verify that $\varphi_1(T)$ is not equal to $\varphi_0(S)$ for any $S \in \mathsf{SSYT}(\lambda)$.

Indeed, consider $\varphi_0(S)$ for any $S \in \mathsf{SSYT}(\lambda)$. Let i be as in the definition of φ_0. Then the i-th entry from the bottom of the $(c+1)$-th column in $\varphi_0(S)$ must have originally been below and hence greater than the i-th entry from the bottom of the c-th column in $\varphi_0(S)$, which stays in the same place in $\varphi_0(S)$. That is, if the j-th entry from the bottom of the $(c+1)$-th column in $\varphi_1(T)$ is less than or equal to the j-th entry from the bottom of the c-th column in $\varphi_1(T)$ for all $1 \leqslant j \leqslant \alpha+1$, then $\varphi_1(T) \neq \varphi_0(S)$ for all $S \in \mathsf{SSYT}(\lambda)$. If we check these corresponding pairs of entries in $\varphi_1(T)$, we have $y_\alpha < y_1$, $x_j < y_j$ for $2 \leqslant j \leqslant \alpha - 1$, $x_{\beta-1} < x_1$, and $x_{\beta+1} < x_\beta$. Thus, we do not have a requisite pair of entries, and no S satisfies $\varphi_1(T) = \varphi_0(S)$.

We can now define our explicit inverse ψ_1. Let $T' \in \varphi_1(\mathsf{SSYT}(\lambda))$. As before, the entries in the c-th column of T' in increasing order are $x'_{\beta+\alpha-1}, x'_{\beta+\alpha-2}, \ldots, x'_1$, and the entries in the $(c+1)$-th column of T' in increasing order are $y'_{\alpha+1}, y'_\alpha, \ldots, y'_1$.

Let $\psi_1(T') = \psi_0(T')$ when $T' \in \varphi_0(\mathsf{SSYT}(\lambda))$, the domain of ψ_0. This occurs when there exists an i' such that $y'_{i'} > x'_{i'}$, so we can take the smallest such i' as in the definition of ψ_0.

If such an i' does not exist, then ψ_1 first swaps x'_β with $y'_{\alpha+1}$, and $x'_{\beta-1}$ with y'_1, undoing the modifications. Relabelling the new tableau obtained after these swaps T'', we now let $\psi_1(T') = \psi_0(T'')$.

Concretely, when ψ_1 differs from ψ_0, the c-th and $(c+1)$-th columns look as follows:

(2.3.7)

T' :

$x'_{\beta+\alpha-1}$	$\boldsymbol{y'_{\alpha+1}}$
$x'_{\beta+\alpha-2}$	y'_α
$\vdots$	$\vdots$
$\boldsymbol{x'_\beta}$	y'_2
$x'_{\beta-1}$	y'_1
$x'_{\beta-2}$	
$\vdots$	
x'_1	

T'' :

$x'_{\beta+\alpha-1}$	$\boldsymbol{x'_\beta}$
$x'_{\beta+\alpha-2}$	y'_α
$\vdots$	$\vdots$
$\boldsymbol{y'_{\alpha+1}}$	y'_2
y'_1	$x'_{\beta-1}$
$x'_{\beta-2}$	
$\vdots$	
x'_1	

$\psi_1(T')$:

$x'_{\beta+\alpha-1}$	y'_1
$x'_{\beta+\alpha-2}$	$x'_{\beta-2}$
$\vdots$	$\vdots$
$y'_{\alpha+1}$	x'_1
x'_β	
y'_α	
$\vdots$	
y'_2	
$x'_{\beta-1}$	

It is straightforward to check that for $T' = \varphi_1(T)$, ψ_1 exactly reverses all the swaps done by φ_1. □

We now obtain stronger versions of the corollaries obtained from the previous injection, in particular Corollary 2.3.6.

Corollary 2.3.13. *The partitions λ and μ as in (2.3.2) with $\beta \geqslant \alpha + 1 \geqslant 3$ form a cover in the immersion poset.*

Corollary 2.3.14. *If $\lambda = (a^2, b^\beta, c, \dots)$, where $a > b > c$, and $\beta \geqslant 3$, then λ is not maximal.*

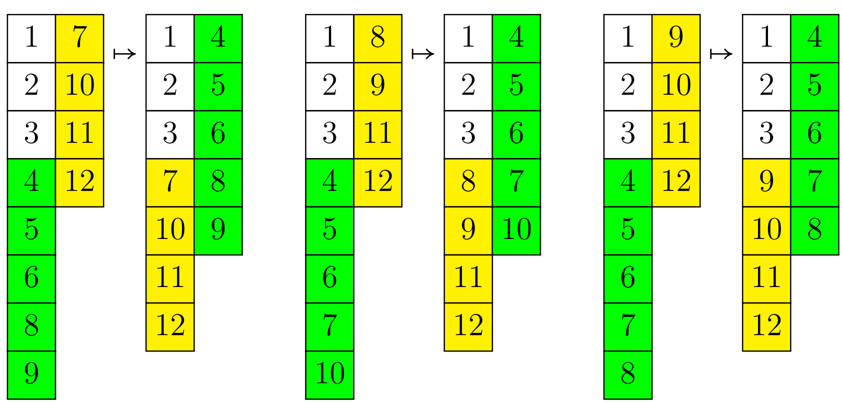
Corollary 2.3.15. *If $\lambda = (a, b, c, d, e)$ is maximal in the immersion poset, then it has no more than two identical non-zero parts.*

In order to further improve the bound for the injection, we must continue to apply modifications to resolve decreasing pairs in "overlapping" rows, and then apply further modifications to establish injectivity. However, there are now multiple cases to consider.

Firstly, any combination of the overlapping rows containing both $y_{\beta-2+j}$ and x_j can be decreasing.

Example 2.3.16. For $\lambda = (2^4, 1^4)$, so $\alpha = \beta = 4$, following φ_0 can give two overlapping rows. We have each possible combination of rows with decreasing pairs as follows:

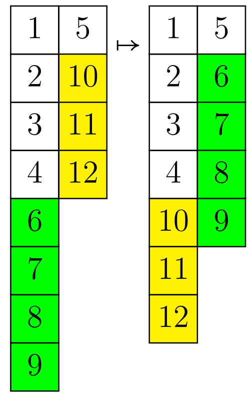
(2.3.8)



While all these previous tableaux give 2 rows of overlap, it is also possible for a tableau of the same shape to give 0 or 1 rows of overlap instead. More generally, $\varphi_0(T)$ for T of shape λ as in (2.3.2) can have anywhere between 0 and $\max\{0, \alpha - \beta + 2\}$ rows of overlap.

Example 2.3.17. For the same $\lambda = (2^4, 1^4)$ as in Example 2.3.16, φ_0 can give a single overlapping row, which contains a decreasing pair:

(2.3.9)



Hence, in our next modifications of φ_0, we must encode the information of every possible case in a way that both distinguishes the cases from φ_0 with no modifications, and distinguishes the cases from each other. To achieve this, our modifications will involve cyclically rotating certain entries in the c-th and $(c+1)$-th columns. These rotations will be analogous to the x_β with $x_{\beta+1}$ swap in φ_1, which can be thought of as a rotation of 2 elements.

We now define a second set of modifications of our original φ_0 injection for the case when $\beta = \alpha$, and $\alpha \geqslant 4$, which we call

$$\varphi_2 \colon \mathsf{SSYT}(\lambda) \to \mathsf{SSYT}(\mu).$$

Let $T \in \mathsf{SSYT}(\lambda)$. As before, suppose that the entries in the c-th column of T in increasing order are $x_{\beta+\alpha}, x_{\beta+\alpha-1}, \ldots, x_1$, and the entries in the $(c+1)$-th column of T in increasing order are $y_\alpha, y_{\alpha-1}, \ldots, y_1$. We define $\varphi_2(T)$ to be the same as $\varphi_0(T)$ if $\varphi_0(T) \in \mathsf{SSYT}(\mu)$.

If $\varphi_0(T) \notin \mathsf{SSYT}(\mu)$, then we have several cases. If there are two rows of overlap, then i as defined for φ_0 is $\alpha+1$, x_1 is to the right of $y_{\alpha-1}$, and x_2 is to the right of $y_\alpha = y_\beta$:

(2.3.10)

$T:$

$x_{\beta+\alpha}$	y_α
$x_{\beta+\alpha-1}$	$y_{\alpha-1}$
$x_{\beta+\alpha-2}$	$y_{\alpha-2}$
$\vdots$	$\vdots$
$x_{\beta+3}$	y_3
$x_{\beta+2}$	y_2
$x_{\beta+1}$	y_1
x_β	
$x_{\beta-1}$	
$\vdots$	
x_2	
x_1	

$\varphi_0(T):$

$x_{\beta+\alpha}$	$x_{\beta+1}$
$x_{\beta+\alpha-1}$	x_β
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$x_{\beta+3}$	x_4
$x_{\beta+2}$	x_3
y_α	x_2
$y_{\alpha-1}$	x_1
$y_{\alpha-2}$	
$\vdots$	
y_1	

If $y_{\alpha-1} > x_1$ and $y_\alpha \leqslant x_2$, then we swap $y_{\alpha-1}$ with x_1. We also "clockwise rotate" the entries $x_{\beta+2}$ and $x_{\beta+3}$ in the c-th column, and $x_{\beta+1}$ in the $(c+1)$-th column, as shown in our next diagram.

In our definition of φ_2, a clockwise rotation of a set of entries in the c-th and $(c+1)$-th columns moves all entries in the c-th column up one cell except the topmost entry, which moves to the topmost cell in the $(c+1)$-th column containing an entry being rotated. The rotation moves all entries in the $(c+1)$-th column down one cell except the bottommost entry, which moves to the bottommost cell in the c-th column containing an entry being rotated. As another example, a rotation of a single entry in the c-th column and a single entry in the $(c+1)$-th column is a swap of those entries. We will continue to describe all cases of φ_2 with rotations of different sets of entries.

If $y_{\alpha-1} \leqslant x_1$ and $y_\alpha > x_2$, then we swap y_α with x_2. We also clockwise rotate $x_{\beta+2}$ in the c-th column, and $x_{\beta+1}$ and x_β in the $(c+1)$-th column.

If $y_{\alpha-1} > x_1$ and $y_\alpha > x_2$, then we swap both $y_{\alpha-1}$ with x_1 and y_α with x_2. We also clockwise rotate $x_{\beta+2}$ and $x_{\beta+3}$ in the c-th column, and $x_{\beta+1}$ and x_β in the $(c+1)$-th column.

Concretely, the two row overlap cases are as follows:

(2.3.11) $\varphi_2(T)$:

$x_{\beta+\alpha}$	$\boldsymbol{x_{\beta+3}}$
$x_{\beta+\alpha-1}$	x_β
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$\boldsymbol{x_{\beta+2}}$	x_4
$\boldsymbol{x_{\beta+1}}$	x_3
y_α	x_2
x_1	$y_{\alpha-1}$
$y_{\alpha-2}$	
$\vdots$	
y_1	

$y_{\alpha-1} > x_1, \quad y_\alpha \leqslant x_2$

$x_{\beta+\alpha}$	$\boldsymbol{x_{\beta+2}}$
$x_{\beta+\alpha-1}$	$\boldsymbol{x_{\beta+1}}$
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$x_{\beta+3}$	x_4
$\boldsymbol{x_\beta}$	x_3
x_2	y_α
$y_{\alpha-1}$	x_1
$y_{\alpha-2}$	
$\vdots$	
y_1	

$y_{\alpha-1} \leqslant x_1, \quad y_\alpha > x_2$

$x_{\beta+\alpha}$	$\boldsymbol{x_{\beta+3}}$
$x_{\beta+\alpha-1}$	$\boldsymbol{x_{\beta+1}}$
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$\boldsymbol{x_{\beta+2}}$	x_4
$\boldsymbol{x_\beta}$	x_3
x_2	y_α
x_1	$y_{\alpha-1}$
$y_{\alpha-2}$	
$\vdots$	
y_1	

$y_{\alpha-1} > x_1, \quad y_\alpha > x_2$

Consider the entries involved in the clockwise rotation in each case. For the topmost entry in the c-th column to move strictly up to the $(c+1)$-th column, and the bottommost entry in the $(c+1)$-th column move strictly down to the c-th column, we must have $\alpha \geqslant 3$. This property is necessary for the tableau to remain semistandard after the rotation, which partially necessitates the $\alpha \geqslant 4$ assumption, which is analogous to the $\alpha \geqslant 2$ assumption for φ_1.

If there is one row of overlap, then i as defined for φ_0 is α, and x_1 is to the right of $y_{\alpha-1}$ with $x_1 < y_{\alpha-1}$:

(2.3.12) T :

$x_{\beta+\alpha}$	y_α
$x_{\beta+\alpha-1}$	$y_{\alpha-1}$
$x_{\beta+\alpha-2}$	$y_{\alpha-2}$
$\vdots$	$\vdots$
$x_{\beta+3}$	y_3
$x_{\beta+2}$	y_2
$x_{\beta+1}$	y_1
x_β	
$x_{\beta-1}$	
$\vdots$	
x_2	
x_1	

$\varphi_0(T)$:

$x_{\beta+\alpha}$	y_α
$x_{\beta+\alpha-1}$	x_β
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$x_{\beta+3}$	x_4
$x_{\beta+2}$	x_3
$x_{\beta+1}$	x_2
$y_{\alpha-1}$	x_1
$y_{\alpha-2}$	
$\vdots$	
y_1	

In this case, we only have the single pair of decreasing entries, $y_{\alpha-1} < x_1$, so we swap $y_{\alpha-1}$ with x_1. However, for the additional modifications after and in addition to this swap, we have different subcases.

If $y_\alpha < x_{\beta+2}$, we clockwise rotate $x_{\beta+1}$ and $x_{\beta+2}$ in the c-th column, and x_β in the $(c+1)$-th column.

If $x_{\beta+2} \leqslant y_\alpha < x_{\beta+1}$, we swap $x_{\beta+2}$ with y_α, and $x_{\beta+1}$ with x_β. Observe in particular that this subcase is two separate swaps, and not a rotation.

If $x_{\beta+1} \leqslant y_\alpha$, we clockwise rotate $x_{\beta+1}$ and $x_{\beta+2}$ in the c-th column, and y_α, x_β, and $x_{\beta-1}$ in the $(c+1)$-th column. Concretely, the one row overlap cases are as follows:

(2.3.13) $\varphi_2(T)$:

$x_{\beta+\alpha}$	y_α
$x_{\beta+\alpha-1}$	$\boldsymbol{x_{\beta+2}}$
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$x_{\beta+3}$	x_4
$\boldsymbol{x_{\beta+1}}$	x_3
$\boldsymbol{x_\beta}$	x_2
x_1	$y_{\alpha-1}$
$y_{\alpha-2}$	
$\vdots$	
y_1	

$y_\alpha < x_{\beta+2}$

$x_{\beta+\alpha}$	$\boldsymbol{x_{\beta+2}}$
$x_{\beta+\alpha-1}$	$\boldsymbol{x_{\beta+1}}$
$x_{\beta+\alpha-2}$	$x_{\beta-1}$
$\vdots$	$\vdots$
$x_{\beta+3}$	x_4
$\boldsymbol{y_\alpha}$	x_3
$\boldsymbol{x_\beta}$	x_2
x_1	$y_{\alpha-1}$
$y_{\alpha-2}$	
$\vdots$	
y_1	

$x_{\beta+2} \leqslant y_\alpha < x_{\beta+1}$

$x_{\beta+\alpha}$	$\boldsymbol{x_{\beta+2}}$
$x_{\beta+\alpha-1}$	$\boldsymbol{y_\alpha}$
$x_{\beta+\alpha-2}$	$\boldsymbol{x_\beta}$
$\vdots$	$\vdots$
$x_{\beta+3}$	x_4
$\boldsymbol{x_{\beta+1}}$	x_3
$\boldsymbol{x_{\beta-1}}$	x_2
x_1	$y_{\alpha-1}$
$y_{\alpha-2}$	
$\vdots$	
y_1	

$x_{\beta+1} \leqslant y_\alpha$

Again, consider the entries involved in the modifications in each case, either the rotations when $y_\alpha < x_{\beta+2}$ or $x_{\beta+1} \leqslant y_\alpha$, or the swaps when $x_{\beta+2} \leqslant y_\alpha < x_{\beta+1}$. For the topmost entry in the c-th column to move strictly up to the $(c+1)$-th column, and the bottommost entry in the $(c+1)$-th column to move strictly down to the c-th column, we must have $\alpha \geqslant 4$. This property is necessary for the tableau to remain semistandard after the rotation, as we will see in Lemma 2.3.19, which fully necessitates the $\alpha \geqslant 4$ assumption.

EXAMPLE 2.3.18. For $\lambda = (2^4, 1^4)$ and T from Example 2.3.16, we get all the two row overlap cases of φ_2:

(2.3.14)

$$\begin{array}{cc} 1 & 7 \\ 2 & 10 \\ 3 & 11 \\ 4 & 12 \\ 5 & \\ 6 & \\ 8 & \\ 9 & \end{array} \mapsto \begin{array}{cc} 1 & \mathbf{2} \\ \mathbf{3} & 5 \\ \mathbf{4} & 6 \\ 7 & 8 \\ 9 & 10 \\ 11 & \\ 12 & \end{array} \qquad \begin{array}{cc} 1 & 8 \\ 2 & 9 \\ 3 & 11 \\ 4 & 12 \\ 5 & \\ 6 & \\ 7 & \\ 10 & \end{array} \mapsto \begin{array}{cc} 1 & \mathbf{3} \\ 2 & \mathbf{4} \\ \mathbf{5} & 6 \\ 7 & 8 \\ 9 & 10 \\ 11 & \\ 12 & \end{array} \qquad \begin{array}{cc} 1 & 9 \\ 2 & 10 \\ 3 & 11 \\ 4 & 12 \\ 5 & \\ 6 & \\ 7 & \\ 8 & \end{array} \mapsto \begin{array}{cc} 1 & \mathbf{2} \\ \mathbf{4} & \mathbf{3} \\ \mathbf{5} & 6 \\ 7 & 9 \\ 8 & 10 \\ 11 & \\ 12 & \end{array}$$

For the same λ, we have all the one row overlap cases of φ_2 as follows, including the T from Example 2.3.17:

(2.3.15)

$$\begin{array}{cc} 1 & 3 \\ 2 & 10 \\ 4 & 11 \\ 5 & 12 \\ 6 & \\ 7 & \\ 8 & \\ 9 & \end{array} \mapsto \begin{array}{cc} 1 & 3 \\ 2 & \mathbf{4} \\ \mathbf{5} & 7 \\ \mathbf{6} & 8 \\ 9 & 10 \\ 11 & \\ 12 & \end{array} \qquad \begin{array}{cc} 1 & 4 \\ 2 & 10 \\ 3 & 11 \\ 5 & 12 \\ 6 & \\ 7 & \\ 8 & \\ 9 & \end{array} \mapsto \begin{array}{cc} 1 & \mathbf{3} \\ 2 & \mathbf{5} \\ \mathbf{4} & 7 \\ \mathbf{6} & 8 \\ 9 & 10 \\ 11 & \\ 12 & \end{array} \qquad \begin{array}{cc} 1 & 5 \\ 2 & 10 \\ 3 & 11 \\ 4 & 12 \\ 6 & \\ 7 & \\ 8 & \\ 9 & \end{array} \mapsto \begin{array}{cc} 1 & \mathbf{3} \\ 2 & \mathbf{5} \\ \mathbf{4} & \mathbf{6} \\ \mathbf{7} & 8 \\ 9 & 10 \\ 11 & \\ 12 & \end{array}$$

The proof that φ_2 is an injection of semistandard tableaux relies on the following lemma regarding the clockwise rotation.

LEMMA 2.3.19. *Suppose S is a semistandard tableau. Suppose that we perform a clockwise rotation on elements in the c-th and $(c+1)$-th columns of S to obtain S', such that the following are true:*

(1) The bottommost rotated entry in the c-th column in S is less than or equal to the topmost rotated entry in the $(c+1)$-th column in S.

(2) The entry moving from the c-th column in S to the $(c+1)$-th column in S' moves strictly upwards.

(3) The entry moving from the $(c+1)$-th column in S to the c-th column in S' moves strictly downward.

(4) The entry moving from the c-th column in S to the $(c+1)$-th column in S' is greater than the entry above it in the $(c+1)$-th column in S', if such an entry exists.

(5) The entry moving from the $(c+1)$-th column to the c-th column is less than the entry below it in the c-th column in S'.

Then S' is semistandard.

PROPOSITION 2.3.20. *Let λ and μ be as in (2.3.2) with $\beta = \alpha \geqslant 4$. Then φ_2 as defined above is an injection*

$$\varphi_2 \colon \mathsf{SSYT}(\lambda) \to \mathsf{SSYT}(\mu).$$

The proof of Proposition 2.3.20 is technical and omitted here. It follows similar ideas to the proof of Proposition 2.3.12. We can now further improve upon Corollary 2.3.6 and Corollary 2.3.13.

COROLLARY 2.3.21. *The partitions λ and μ as in (2.3.2) with $\beta \geqslant \alpha \geqslant 4$ form a cover in the immersion poset.*

We summarize the bounds on α and β needed for each map to be an injection:

Map	α	β
φ_0	$\alpha \geqslant 0$	$\beta \geqslant \alpha + 2$
φ_1	$\alpha \geqslant 2$	$\beta = \alpha + 1$
φ_2	$\alpha \geqslant 4$	$\beta = \alpha$

2.3.3. Immersion poset on hook partitions. For this section, set $\lambda^i = (i, 1^{n-i}) \vdash n$ and let $S = \{\lambda^i \mid 1 \leqslant i \leqslant n\}$ be the set of all hook partitions of size n. We study the immersion poset restricted to S.

PROPOSITION 2.3.22. *Let $1 \leqslant i \leqslant n$ and $\alpha = (\alpha_1, \dots, \alpha_k) \vdash n$ such that $\alpha \leqslant_D \lambda^i$. Then*

$$K_{\lambda^i,\alpha} = \binom{k-1}{n-i}.$$

PROOF. Since λ^i dominates α, we know that $K_{\lambda^i,\alpha} \geqslant 1$. To form a semistandard Young tableau of shape λ^i and content α, the α_1 entries 1 must be placed leftmost in the first row of λ^i. The remaining $n - i$ positions in the first column of λ^i can be filled with distinct values from the set

$\{2, 3, \ldots, k\}$. This gives $\binom{k-1}{n-i}$ choices. Once these are placed, there is only one way to fill the remainder of the first row so that the resulting tableau is semistandard. □

Recall from Lemma 2.2.1 that $\mu \leqslant_I \lambda$ if and only if $K_{\mu,\alpha} \leqslant K_{\lambda,\alpha}$ for all $\alpha \vdash n$. Hence with Proposition 2.3.22, we are now ready to describe all the relations between hook partitions $\lambda^i \in S$ in the immersion poset. To illustrate what the proposition implies, we form a matrix of values in the following way:

- The j-th column is indexed by the content α^j, where α^j is any content that has j parts.
- The i-th row is indexed by the shape λ^i.
- The (i, j) entry of this matrix is the value $T_{i,j} := K_{\lambda^i,\alpha^j} = \binom{j-1}{n-i}$ for $1 \leqslant i, j \leqslant n$.

Example 2.3.23. We give the matrix for $n = 7$:

Partition \ # of parts	1	2	3	4	5	6	7
(1^7)	0	0	0	0	0	0	1
$(2, 1^5)$	0	0	0	0	0	1	6
$(3, 1^4)$	0	0	0	0	1	5	15
$(4, 1^3)$	0	0	0	1	4	10	20
$(5, 1^2)$	0	0	1	3	6	10	15
$(6, 1)$	0	1	2	3	4	5	6
(7)	1	1	1	1	1	1	1

Remark 2.3.24. In this context, $\lambda^i \geqslant_I \lambda^j$ if and only if $T_{i,m} \geqslant T_{j,m}$ for all m. Equivalently, since $T_{j,m} = 0$ when $n - j \geqslant m$ and λ^i dominates λ^j when $i > j$, we need only show $T_{i,m} \geqslant T_{j,m}$ for all $m > n - j$ when $i > j$.

The following lemma is used to prove the structure of the immersion poset restricted to hook partitions.

Lemma 2.3.25. *Suppose* $\binom{n-1}{n-i} \geqslant \binom{n-1}{n-j}$ *and* $i > j$ *(note that this implies* $j \leqslant \frac{n}{2}$*). Then for all* $0 \leqslant p \leqslant j-1$*, we have*

$$\binom{n-1-p}{n-i} \geqslant \binom{n-1-p}{n-j}.$$

The proof follows from basic properties of binomial coefficients, and is omitted here.

Corollary 2.3.26. *If* $T_{i,n} \geqslant T_{j,n}$ *for* $i > j$*, then* $\lambda^i \geqslant_I \lambda^j$*.*

Proof. By Proposition 2.3.22, $T_{i,n-p} = K_{\lambda^i,\alpha^{n-p}} = \binom{(n-p)-1}{n-i}$. Hence if $T_{i,n} \geqslant T_{j,n}$, by Lemma 2.3.25, we also have $T_{i,n-p} \geqslant T_{j,n-p}$ for $0 \leqslant p \leqslant j-1$. By Remark 2.3.24, this implies $\lambda^i \geqslant_I \lambda^j$. □

Example 2.3.27. Take the rows corresponding to the partitions $(5,1^2)$ and $(3,1^4)$ in Example 2.3.23. Since the last column entries give $T_{5,7} = 15 \geqslant 15 = T_{3,7}$, then by Corollary 2.3.26 we also have $T_{5,7-p} \geqslant T_{3,7-p}$ for $1 \leqslant p \leqslant 2$: $T_{5,6} = 10 \geqslant 5 = T_{3,6}$, $T_{5,5} = 6 \geqslant 1 = T_{3,5}$.

We now describe the relations in the immersion poset on S depending upon whether n is even or odd.

Proposition 2.3.28. *Let* $n = 2k+1$ *be odd, then:*

(1) $\lambda^{\ell+1} \geqslant_I \lambda^\ell$ *for all* $1 \leqslant \ell \leqslant k$*.*

(2) $(\lambda^{k+1-\ell})^t = \lambda^{k+1+\ell} \geqslant_I \lambda^{k+1-\ell}$ *for all* $1 \leqslant \ell \leqslant k$*.*

(3) For any $1 < i \leqslant k+1$*,* λ^i *is incomparable to* λ^j *for all* $j > n-i+1$*.*

(4) For any $k+2 \leqslant i < n$*,* λ^i *is incomparable to* λ^j *for all* $j > i$*.*

These describe all relations in the immersion poset restricted to hook partitions S*.*

Proof. Let us first prove (1). Fix an ℓ with $1 \leqslant \ell \leqslant k$. Then by Corollary 2.3.26, $\lambda^{\ell+1} \geqslant_I \lambda^\ell$ if and only if $T_{\ell+1,n} \geqslant T_{\ell,n}$. Note that $T_{\ell+1,n} = \binom{n-1}{n-\ell-1} = \binom{n-1}{\ell}$ and $T_{\ell,n} = \binom{n-1}{n-\ell} = \binom{n-1}{\ell-1}$. Since $1 \leqslant \ell \leqslant k$, we have $\binom{n-1}{\ell} \geqslant \binom{n-1}{\ell-1}$ and the result follows.

To prove (2), note that $\lambda^{k+1+\ell} \geqslant_I \lambda^{k+1-\ell}$ if and only if $T_{k+1+\ell,n} \geqslant T_{k+1-\ell,n}$. Since $T_{k+1+\ell,n} = \binom{n-1}{k-\ell} = \binom{2k}{k-\ell} = \binom{2k}{k+\ell} = \binom{n-1}{k+\ell} = T_{k+1-\ell,n}$, the result follows.

To prove (3) we show for any $1 < i \leqslant k+1$ that λ^i is incomparable to λ^j for all $j > n-i+1$. Since λ^j dominates λ^i, we need only show there exists some α such that $K_{\lambda^i,\alpha} > K_{\lambda^j,\alpha}$. Choose $\alpha = (1^n)$. Then $K_{\lambda^i,\alpha} = \binom{n-1}{n-i} = \binom{n-1}{i-1} > \binom{n-1}{n-j} = K_{\lambda^j,\alpha}$ because $i-1 > n-j$ and $1 < i \leqslant k+1$.

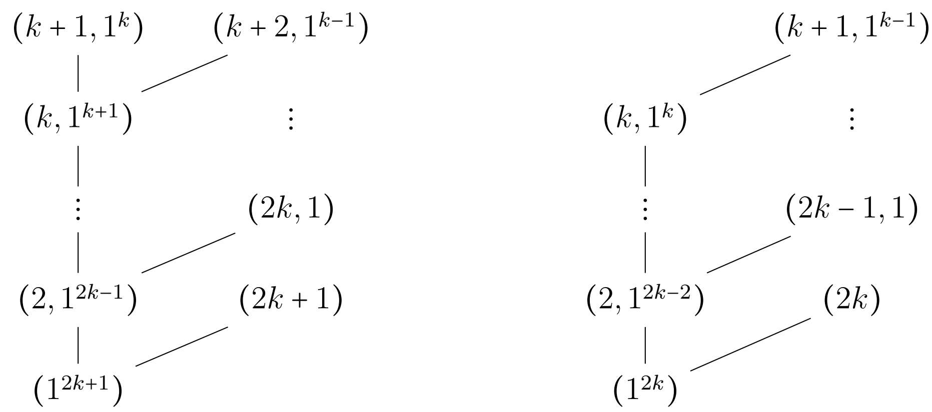


FIGURE 2.2. Immersion poset restricted to hook partitions for $n = 2k+1$ (left) and $n = 2k$ (right).

Lastly, to prove (4) we follow the same strategy as (3). Since λ^j dominates λ^i, we can let $\alpha = (1^n)$, and since $k+1 < i < j$ we get $K_{\lambda^i,\alpha} = \binom{n-1}{n-i} > \binom{n-1}{n-j} = K_{\lambda^j,\alpha}$, and the result follows. ☐

PROPOSITION 2.3.29. *Let $n = 2k$ be even, then:*

(1) $\lambda^{\ell+1} \geqslant_I \lambda^\ell$ for all $1 \leqslant \ell < k$.

(2) $(\lambda^{k-\ell})^t = \lambda^{k+1+\ell} \geqslant_I \lambda^{k-\ell}$ for all $0 \leqslant \ell \leqslant k-1$.

(3) For any $1 < i \leqslant k$, λ^i is incomparable to λ^j for all $j > n-i+1$.

(4) For any $k+1 \leqslant i < n$, λ^i is incomparable to λ^j for all $j > i$.

These describe all relations in the immersion poset restricted to hook partitions S.

The proof of the even case is similar to the odd case.

The Hasse diagram of the immersion poset restricted to hook partitions is given in Figure 2.2. Notice that item (1) in Propositions 2.3.28 and 2.3.29 proves the string of covers on the left going up each Hasse diagram, while (2) proves the covers going up the right side which are the transposes.

COROLLARY 2.3.30. *Let $\lambda \in S$ be a hook partition such that $\lambda \leqslant_D \lambda^t$. Then $\lambda \leqslant_I \lambda^t$.*

The *rank* of a poset is the length of the longest chain of elements of the poset.

COROLLARY 2.3.31. *The rank of the immersion poset $(\mathcal{P}(n), \leqslant_I)$ is at least $\lfloor n/2 \rfloor$.*

2.3.4. Immersion poset on two column partitions. Now let S be the set partitions with at most two columns, that is, $S = \{\lambda \mid \lambda_1 \leqslant 2\}$. If $n = 2k$, then for this section we define

$\lambda^j = (2^{k-j}, 1^{2j})$ for $0 \leqslant j \leqslant k$. Similarly, if $n = 2k+1$, then $\lambda^j = (2^{k-j}, 1^{2j+1})$ for $0 \leqslant j \leqslant k$. In this section, we study the immersion poset restricted to S.

REMARK 2.3.32. Note that $K_{\lambda,\mu} = 0$ if $\lambda \in S$ and $\mu \notin S$. Hence there does not exist an immersion pair $\mu \leqslant_I \lambda$ with $\lambda \in S$ and $\mu \notin S$. This implies that if λ^i is a cover for λ^j in the subposet restricted to S, then λ^i is a cover for λ^j in the immersion poset.

This remark implies that we only need to consider $K_{\lambda,\mu}$ for $\lambda, \mu \in S$ when determining the immersion relations for this subset. Recall that f^λ is the number of standard Young tableaux of shape λ.

PROPOSITION 2.3.33.

(1) Let $\lambda^j = (2^{k-j}, 1^{2j}) \vdash 2k$ for $0 \leqslant j \leqslant k$. Then $K_{\lambda^i,\lambda^j} = f^{(j+i,j-i)}$ when $i \leqslant j$ and $K_{\lambda^i,\lambda^j} = 0$ when $i > j$.

(2) Let $\lambda^j = (2^{k-j}, 1^{2j+1}) \vdash 2k+1$ for $0 \leqslant j \leqslant k$. Then $K_{\lambda^i,\lambda^j} = f^{(j+i+1,j-i)}$ when $i \leqslant j$ and $K_{\lambda^i,\lambda^j} = 0$ when $i > j$.

PROOF. For (1), if $i > j$, λ^j dominates λ^i and hence $K_{\lambda^i,\lambda^j} = 0$. If $i = j$, clearly $K_{\lambda^i,\lambda^i} = f^{(2i)} = 1$. Suppose $j > i$. Then the first $k-j$ of the $k-i$ two length rows of any tableau $T \in \mathsf{SSYT}(\lambda^i, \lambda^j)$ are fixed by the content. Hence, there is a bijection $\mathsf{SSYT}(\lambda^i, \lambda^j) \to \mathsf{SSYT}((2^{j-i}, 1^{2i}), (1^{2j}))$ by removing the first $k-j$ rows. Note that $K_{(2^{j-i},1^{2i}),(1^{2j})} = f^{(2^{j-i},1^{2i})}$, which is also equal to the number of standard tableaux of the transpose of $(2^{j-i}, 1^{2i})$. The result follows.

The proof of part (2) is similar. □

Using the hook length formula with Proposition 2.3.33, we can describe K_{λ^i,λ^j}. We present this in the form of a matrix. More explicitly, suppose $n = 2k$ or $n = 2k+1$. Then for all $0 \leqslant i, j \leqslant k$, the i-th row and j-th column entry of the matrix $T = (T_{i,j})$ is $T_{i,j} = K_{\lambda^i,\lambda^j}$. Note that the indexing starts with 0.

EXAMPLE 2.3.34. Below are the matrices in tabular form for cases $n = 14, 15$.

The case when $n = 14$:

Shape \ Content	(2^7)	$(2^6, 1^2)$	$(2^5, 1^4)$	$(2^4, 1^6)$	$(2^3, 1^8)$	$(2^2, 1^{10})$	$(2, 1^{12})$	(1^{14})
(2^7)	1	1	2	5	14	42	132	429
$(2^6, 1^2)$	0	1	3	9	28	90	297	1001
$(2^5, 1^4)$	0	0	1	5	20	75	275	1001
$(2^4, 1^6)$	0	0	0	1	7	35	154	637
$(2^3, 1^8)$	0	0	0	0	1	9	54	273
$(2^2, 1^{10})$	0	0	0	0	0	1	11	77
$(2, 1^{12})$	0	0	0	0	0	0	1	13
(1^{14})	0	0	0	0	0	0	0	1

The case when $n = 15$:

Shape \ Content	$(2^7, 1)$	$(2^6, 1^3)$	$(2^5, 1^5)$	$(2^4, 1^7)$	$(2^3, 1^9)$	$(2^2, 1^{11})$	$(2, 1^{13})$	(1^{15})
$(2^7, 1)$	1	2	5	14	42	132	429	1430
$(2^6, 1^3)$	0	1	4	14	48	165	572	2002
$(2^5, 1^5)$	0	0	1	6	27	110	429	1638
$(2^4, 1^7)$	0	0	0	1	8	44	208	910
$(2^3, 1^9)$	0	0	0	0	1	10	65	350
$(2^2, 1^{11})$	0	0	0	0	0	1	12	90
$(2, 1^{13})$	0	0	0	0	0	0	1	14
(1^{15})	0	0	0	0	0	0	0	1

Since the columns and rows are decreasing in dominance order, for any $i < j$ we have $\lambda^i \geqslant_I \lambda^j$ if $T_{i,m} \geqslant T_{j,m}$ for all $0 \leqslant m \leqslant k$. In the following lemma, we prove some properties of the matrix T that will show that this statement is equivalent to only comparing values in the last column of the matrix. That is, if $i < j$ and $T_{i,k} \geqslant T_{j,k}$, then $\lambda^i \geqslant_I \lambda^j$. The reader can verify this in Example 2.3.34.

LEMMA 2.3.35. *The matrix $(T_{i,j}) = (K_{\lambda^i,\lambda^j})$ defined above with $0 \leqslant i, j \leqslant k$ has the following properties:*

(1) The entries weakly increase within each row.

(2) The entries within each column are unimodal.

(3) The rate of change of entries within a row increases as the row number increases. In particular, for any fixed i and j with $0 \leqslant i < j \leqslant k$, we have for all $j \leqslant r < k$:

$$\frac{T_{i,r+1}}{T_{i,r}} < \frac{T_{j,r+1}}{T_{j,r}}.$$

(4) For any fixed i and j with $i < j$, if $T_{i,k} \geqslant T_{j,k}$, then $T_{i,m} \geqslant T_{j,m}$ for all $0 \leqslant m \leqslant k$.

PROOF. We begin by proving (1). Let $n = 2k$ be even. Then for a fixed row i, given any $i \leqslant j < k$, we need to show that $T_{i,j+1} \geqslant T_{i,j}$. Using Proposition 2.3.33, we have:

$$\frac{T_{i,j+1}}{T_{i,j}} = \frac{f^{(j+1+i,j+1-i)}}{f^{(j+i,j-i)}} = \frac{(2j+2)(2j+1)}{(j+i+2)(j+1-i)} \geqslant 1$$

because $j \geqslant i$ implies

$$2j+2 \geqslant j+i+2 \quad \text{and} \quad 2j+1 \geqslant j+1-i.$$

Now let $n = 2k+1$ be odd. Using the same strategy as the even case we have:

$$\frac{T_{i,j+1}}{T_{i,j}} = \frac{f^{(j+2+i,j+1-i)}}{f^{(j+i+1,j-i)}} = \frac{(2j+3)(2j+2)}{(j+i+3)(j+1-i)} \geqslant 1$$

because $j \geqslant i$ implies

$$2j+3 \geqslant j+i+3 \quad \text{and} \quad 2j+2 \geqslant j+1-i.$$

Next we prove statement (2). Let $n = 2k$ be even. Since statement (2) holds trivially if there is only one non-zero entry in the column, we focus on columns with more than one non-zero entry. Fix a $2 \leqslant j \leqslant k$. To determine when the column is increasing and decreasing we consider the fraction:

$$\frac{T_{i+1,j}}{T_{i,j}} = \frac{f^{(j+i+1,j-i-1)}}{f^{(j+i,j-i)}} = \frac{\frac{(2j)!(2i+3)}{(j+i+2)!(j-i-1)!}}{\frac{(2j)!(2i+1)}{(j+i+1)!(j-i)!}} = \frac{(2i+3)(j-i)}{(2i+1)(j+i+2)}.$$

Analyzing the following inequalities gives:

$$\begin{aligned} \frac{T_{i+1,j}}{T_{i,j}} > 1 &\iff 2i^2+4i+1 < j, \\ \frac{T_{i+1,j}}{T_{i,j}} = 1 &\iff 2i^2+4i+1 = j, \\ \frac{T_{i+1,j}}{T_{i,j}} < 1 &\iff 2i^2+4i+1 > j. \end{aligned} \tag{2.3.16}$$

Thus, for values of i such that $2i^2+4i+1 < j$ the column entries are increasing, and when the values of i satisfy $2i^2+4i+1 > j$ the column entries are decreasing. This proves (2) for the even case.

Now let $n = 2k+1$ be odd. Fix a $2 \leqslant j \leqslant k$. Similar to the even case we have:

$$\frac{T_{i+1,j}}{T_{i,j}} = \frac{f^{(j+i+2,j-i-1)}}{f^{(j+i+1,j-i)}} = \frac{\frac{(2j+1)!(2i+4)}{(j+i+3)!(j-i-1)!}}{\frac{(2j+1)!(2i+2)}{(j+i+2)!(j-i)!}} = \frac{(2i+4)(j-i)}{(2i+2)(j+i+3)}.$$

Analyzing the following inequalities gives:

$$\begin{aligned} \frac{T_{i+1,j}}{T_{i,j}} > 1 &\iff 2i^2+6i+3 < j, \\ \frac{T_{i+1,j}}{T_{i,j}} = 1 &\iff 2i^2+6i+3 = j, \\ \frac{T_{i+1,j}}{T_{i,j}} < 1 &\iff 2i^2+6i+3 > j. \end{aligned} \tag{2.3.17}$$

Again, we notice that for values of i such that $2i^2+6i+3 < j$ the column entries are increasing, and when the values of i satisfy $2i^2+6i+3 > j$ the column entries are decreasing. This concludes the proof of (2).

To prove statement (3), we use Proposition 2.3.33 and the hook length formula to get the following equivalences:

$$\frac{T_{i,r+1}}{T_{i,r}} < \frac{T_{j,r+1}}{T_{j,r}} \iff \frac{K_{\lambda^i,\lambda^{r+1}}}{K_{\lambda^i,\lambda^r}} < \frac{K_{\lambda^j,\lambda^{r+1}}}{K_{\lambda^j,\lambda^r}} \iff \frac{(r+j+2)(r+1-j)}{(r+i+2)(r+1-i)} < 1 \iff i^2+i < j^2+j.$$

The last inequality is always true since $0 \leqslant i < j$, thus proving (3).

To prove (4), fix i and j with $i < j$ where $T_{i,k} \geqslant T_{j,k}$. Then by statement (3) it directly follows that $T_{i,m} \geqslant T_{j,m}$ for all $j \leqslant m \leqslant k$. Because $T_{j,m} = 0$ for all $0 \leqslant m < j$, it trivially follows that $T_{i,m} \geqslant T_{j,m}$ for these values of m, this finishes the proof of (4). □

The beauty of Lemma 2.3.35, in particular statement (4), is that we can now reduce much of the work in determining the immersion relations between partitions in S to just comparing the numbers of standard Young tableaux, as is done in the next proposition.

Proposition 2.3.36. *For $n = 2k$ even or $n = 2k + 1$ odd, the last (k-th) column of T can be used to completely determine relations in the immersion poset restricted to the subset S. In particular:*

(1) $\lambda^i \geqslant_I \lambda^j$ if and only if $i < j$ and $T_{i,k} \geqslant T_{j,k}$,

(2) For $i < j$, λ^i and λ^j are incomparable if and only if $T_{j,k} > T_{i,k}$.

Proof. To prove (1), by definition $\lambda^i \geqslant_I \lambda^j$ if and only if $\lambda^i >_D \lambda^j$ and $T_{i,\ell} \geqslant T_{j,\ell}$ for all $0 \leqslant \ell \leqslant k$. But λ^i dominates λ^j if and only if $i < j$, and by (4) of Lemma 2.3.35, $T_{i,\ell} \geqslant T_{j,\ell}$ for all $0 \leqslant \ell \leqslant k$ if and only if $T_{i,k} \geqslant T_{j,k}$ (when $i < j$).

To prove (2), let $i < j$. If λ^i and λ^j are incomparable, then there exists some ℓ such that $T_{j,\ell} > T_{i,\ell}$. By (3) of Lemma 2.3.35, we have:

$$\frac{T_{i,r+1}}{T_{i,r}} \leqslant \frac{T_{j,r+1}}{T_{j,r}}$$

for all $\ell \leqslant r < k$, which guarantees that $T_{j,k} > T_{i,k}$. □

As a consequence we obtain the following immediate corollary.

Corollary 2.3.37. *The cover relations for the immersion poset of the set S are the exact same as those in the standard immersion poset.*

We can now explain the cover relations of the immersion poset restricted to the set S.

Proposition 2.3.38. *Let $n = 2k$ be even or $n = 2k + 1$ be odd, then:*

(1) $\lambda^i >_I \lambda^{i+1}$ when $2i^2 + 4i + 2 > k$ for n even and $2i^2 + 6i + 4 > k$ for n odd. This also coincides with Lemma 2.2.7, taking $a = k - i - 1$ and $b = 2i + 2$ (n even) or $b = 2i + 3$ (n odd).

(2) λ^i *and* λ^j *are incomparable in the immersion poset for all* $0 \leqslant i, j \leqslant i_{max}$ *with* $i \neq j$ *and* i_{max} *being the largest* i *value not satisfying (1).*

(3) Fix i *with* $0 \leqslant i \leqslant i_{max}$ *and let* $m > i_{max} - i$ *be smallest such that* $T_{i,k} \geqslant T_{i+m,k}$*. Then* $\lambda^i \geqslant_I \lambda^{i+m}$*.*

Proof. If $T_{i,k} \geqslant T_{i+1,k}$, then by Proposition 2.3.36 and Remark 2.3.32, we have that $\lambda^i \geqslant_I \lambda^{i+1}$ is a cover. We determine the values for i such that $T_{i,k} \geqslant T_{i+1,k}$ by using the middle equation and bottom inequality of (2.3.16) (for n even) and (2.3.17) (for n odd), where we replace j with k. Specifically, for n even:

$$2i^2 + 4i + 1 \geqslant k \quad \Longrightarrow \quad 2i^2 + 4i + 2 > k,$$

and for n odd:

$$2i^2 + 6i + 3 \geqslant k \quad \Longrightarrow \quad 2i^2 + 6i + 4 > k.$$

To prove (2), notice that since i_{max} is the number of the row containing the first maximum, by the increasing nature of the column up to the maximum value given by (2) of Lemma 2.3.35, then for any $0 \leqslant i < j \leqslant i_{max}$ we have $T_{i,k} < T_{j,k}$. Hence by Proposition 2.3.36 (2), λ^i and λ^j are incomparable.

To prove (3) notice that by Proposition 2.3.36 statement (1), since m is the smallest value it must be a cover. ☐

Example 2.3.39. Suppose $n = 14$, so that $k = 7$. By (1) of Proposition 2.3.38, the inequality holds for $1 \leqslant i \leqslant k = 7$ so we obtain:

$$\lambda^7 \lessdot_I \lambda^6 \lessdot_I \lambda^5 \lessdot_I \lambda^4 \lessdot_I \lambda^3 \lessdot_I \lambda^2 \lessdot_I \lambda^1.$$

Applying (3) of Proposition 2.3.38 to λ^0, with $i = 0$ we find that $m = 4$:

$$T_{0,7} = 429 \geqslant 273 = T_{4,7}.$$

Notice that $m = 3$ does not satisfy the inequality:

$$T_{0,7} = 429 \ngeqslant 637 = T_{3,7}.$$

So our final cover relation for the poset is $\lambda^4 \lessdot_I \lambda^0$.

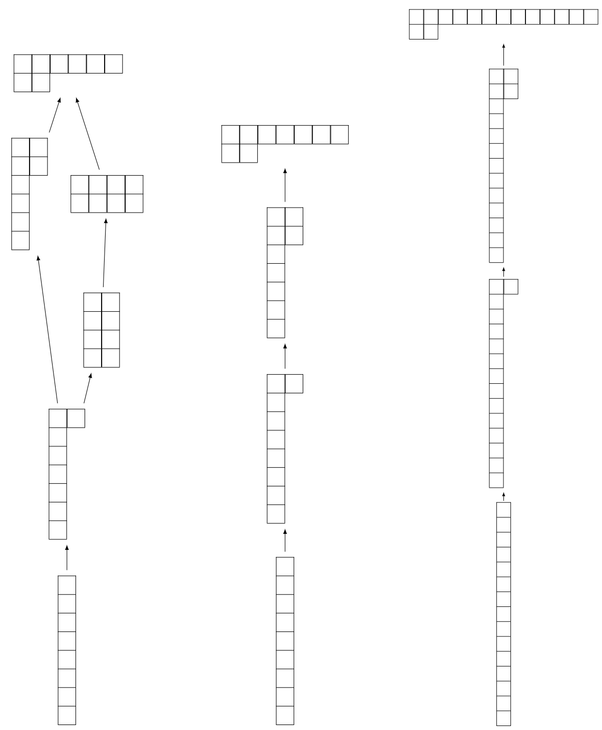


FIGURE 2.3. Subposet of the immersion poset only containing partitions in $A_{(n-2,2)}$ for $n = 8$ (left), $n = 9$ (middle), and $n = 15$ (right).

2.3.5. Lower intervals and Schur-positivity of interval power sums. In this section, we make conjectures about certain lower intervals $A_\mu := \{\lambda \,|\, (1^n) \leqslant_I \lambda \leqslant_I \mu\}$ in the immersion poset. Determining intervals will

(1) enhance our understanding of the immersion of polynomial representations for $GL_N(\mathbb{C})$ and

(2) allow us to investigate when p_{A_μ} of Equation (2.1.1) is Schur-positive, as asked in Question 2.1.3. We call p_{A_μ} an interval power sum. It also helps towards constructing a natural corresponding representation of the symmetric group.

In this section, we prove that p_{A_μ} is Schur-positive for the conjectured intervals.

CONJECTURE 2.3.40. *For $n = 5$ and $n \geqslant 9$, the interval $A_{(n-2,2)} = \{\lambda \,|\, (1^n) \leqslant_I \lambda \leqslant_I (n-2,2)\}$ is exactly*

$$(1^n) \lessdot_I (2, 1^{n-2}) \lessdot_I (2, 2, 1^{n-4}) \lessdot_I (n-2, 2).$$

REMARK 2.3.41. The first two covers are consequences of Proposition 2.3.38 (1). The map

$$\mathsf{SSYT}((2,2,1^{n-4}),\nu) \longrightarrow \mathsf{SSYT}((n-2,2),\nu),$$

which is the transpose if $\nu_1 = 1$ and which moves the boxes in positions $(3,1),\dots,(n-2,1)$ to positions $(1,3),\dots,(1,n-2)$ if $\nu_1 = 2$ is an injection. This shows that $(2,2,1^{n-4}) <_I (n-2,2)$. Therefore, we have

$$\{(1^n),(2,1^{n-2}),(2,2,1^{n-4}),(n-2,2)\} \subseteq A_{(n-2,2)}.$$

However, we have not proven the cover relation $(2,2,1^{n-4}) \lessdot_I (n-2,2)$. One strategy to show the reverse containment is to argue that for all partitions λ such that λ and λ^t are not included in the above list, we have $f^\lambda > \frac{n(n-3)}{2} = f^{(n-2,2)}$. This would prove that $\lambda \nless_I (n-2,2)$, hence $\lambda \notin A_{(n-2,2)}$. We have confirmed the conjecture up to $n = 18$. See Figure 2.3.

Proposition 2.3.42.

(1) For $n < 7$ and $n = 8$, $p_{A_{(n-2,2)}}$ is Schur-positive.

(2) For $n \geqslant 9$, $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(n-2,2)}$ is Schur-positive.

Proof. Part (1) can be checked explicitly by SageMath. For part (2), let

$$p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(n-2,2)} = \sum_{\lambda\vdash n} c_\lambda s_\lambda. \tag{2.3.18}$$

We prove that $c_\lambda \geqslant 0$ for all $\lambda \vdash n$ by proving that all partial sums $p_{(1^n)}$, $p_{(1^n)} + p_{(2,1^{n-2})}$, $p_{(1^n)}+p_{(2,1^{n-2})}+p_{(2,2,1^{n-4})}$, $p_{(1^n)}+p_{(2,1^{n-2})}+p_{(2,2,1^{n-4})}+p_{(n-2,2)}$ are Schur-positive. We employ the combinatorial Murnaghan–Nakayama rule involving ribbon tableaux (see for example [**Sta99**, Chapter 7.17])

$$p_\mu = \sum_{\lambda\vdash n} \chi^\lambda(\mu) s_\lambda \qquad \text{where} \qquad \chi^\lambda(\mu) = \sum_{T\in \mathsf{R}(\lambda,\mu)} (-1)^{\mathrm{ht}(T)}$$

and $\mathsf{R}(\lambda,\mu)$ is the set of all ribbon tableaux of shape λ and type μ and $\mathrm{ht}(T)$ is equal to the sum of the heights of all ribbons in T. We show that each subset of ribbon tableaux that contributes a negative term to c_λ is in bijection with a distinct subset of ribbon tableaux that contributes a positive number to c_λ, ensuring that $c_\lambda \geqslant 0$. We examine each partial sum of power sum symmetric functions, and demonstrate Schur-positivity at each step through these bijections.

(1) It is well-known that $p_{(1^n)} = \sum_{\lambda\vdash n} f^\lambda s_\lambda$ since $\mathsf{R}(\lambda,(1^n))$ is the set of all standard Young tableaux of shape λ.

(2) For $T \in \mathsf{R}(\lambda, (2, 1^{n-2}))$, T has either a horizontal or a vertical 2-ribbon and the remaining are single box ribbons. If T has a horizontal 2-ribbon, then $\mathrm{ht}(T) = 0$ and T contributes $+1$ to $\chi^\lambda((2, 1^{n-2}))$. There are $f^{\lambda/(2)}$ such ribbon tableaux in $\mathsf{R}(\lambda, (2, 1^{n-2}))$, where $f^{\lambda/\mu}$ is the cardinality of $\mathsf{SYT}(\lambda/\mu)$, the set of standard Young tableaux of skew shape λ/μ. If T has a vertical 2-ribbon, then $\mathrm{ht}(T) = 1$ and T contributes -1 to $\chi^\lambda((2, 1^{n-2}))$. There are $f^{\lambda/(1,1)}$ such ribbon tableaux in $R(\lambda, (2, 1^{n-2}))$. Therefore, the coefficient of s_λ in $p_{(2,1^{n-2})}$ is $f^{\lambda/(2)} - f^{\lambda/(1,1)}$. If $(1, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(1,1)}$. The natural bijection

$$\mathsf{SYT}(\lambda/(1,1)) \to \{T \in \mathsf{SYT}(\lambda) \mid T_{1,1} = 1 \text{ and } T_{2,1} = 2\}$$

demonstrates that $f^\lambda - f^{\lambda/(1,1)} \geqslant 0$. Hence $p_{(1^n)} + p_{(2,1^{n-2})}$ is Schur-positive.

(3) For any $T \in \mathsf{R}(\lambda, (2, 2, 1^{n-4}))$, there are six possible ways to arrange two 2-ribbons.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 2 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$\mathrm{ht}(T) = 0 \quad \mathrm{ht}(T) = 2 \quad \mathrm{ht}(T) = 0 \quad \mathrm{ht}(T) = 2 \quad \mathrm{ht}(T) = 1 \quad \mathrm{ht}(T) = 1$$

The remaining $n - 4$ ribbons in T are single boxes. Therefore, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ is

$$2f^{\lambda/(2,2)} + f^{\lambda/(4)} + f^{\lambda/(1^4)} - f^{\lambda/(3,1)} - f^{\lambda/(2,1,1)}.$$

If $(3, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(3,1)}$. Consider the bijection

$$\mathsf{SYT}(\lambda/(3,1)) \to \{T \in \mathsf{SYT}(\lambda/(2)) \mid T_{1,3} = 1, \text{ and } T_{2,1} = 2\}.$$

If $(2, 1, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(2,1,1)}$. Consider the bijection

$$\mathsf{SYT}(\lambda/(2,1,1)) \to \{T \in \mathsf{SYT}(\lambda/(2)) \mid T_{2,1} = 1 \text{ and } T_{3,1} = 2\}.$$

Hence $f^{\lambda/(2)}$ from (2) and the terms $-f^{\lambda/(3,1)} - f^{\lambda/(2,1,1)}$ from (3) satisfy $f^{\lambda/(2)} - f^{\lambda/(3,1)} - f^{\lambda/(2,1,1)} \geqslant 0$. So far, we have shown that $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})}$ is Schur-positive.

(4) For $T \in \mathsf{R}(\lambda, (n-2,2))$, the possible ways of arranging one $(n-2)$-ribbon and one 2-ribbon in T are the following. Note that 1^0 appearing in λ means that there are no parts of size 1 in λ.

$\lambda^{(1)} = (a, 1^b)$, $0 \leqslant b \leqslant n-3$:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$\mathrm{ht}(T) = b$

$\lambda^{(2)} = (a, 1^b)$, $2 \leqslant b \leqslant n-1$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline 1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$\mathrm{ht}(T) = b-2+1$

$\lambda^{(3)} = (a, 3, 1^b)$, $0 \leqslant b \leqslant n-6$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline 1 & 2 & 2 \\ \cline{1-3} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$\mathrm{ht}(T) = b+1$

$\lambda^{(4)} = (a, 2, 2, 1^b)$, $0 \leqslant b \leqslant n-6$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline 1 & 2 \\ \cline{1-2} 1 & 2 \\ \cline{1-2} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$\mathrm{ht}(T) = b+2+1$

$\lambda^{(5)} = (2, 2, 1^{n-4})$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$\mathrm{ht}(T) = n-3+1$

$\lambda^{(6)} = (n-2, 2)$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

$\mathrm{ht}(T) = 0$

If $\lambda = (a, 1^b)$ with $2 \leqslant b \leqslant n-3$, then cases $\lambda^{(1)}$ and $\lambda^{(2)}$ both apply. Since $\mathrm{ht}(\lambda^{(1)})$ and $\mathrm{ht}(\lambda^{(2)})$ have opposite parity, $\chi^{(a,1^b)}((n-2,2)) = 0$. For $\lambda = (2, 1^{n-2})$, only $\lambda^{(2)}$ applies and $\chi^{(2,1^{n-2})}((n-2,2)) = (-1)^{n-3}$. For $\lambda = (1^n)$, only $\lambda^{(2)}$ applies and $\chi^{(1^n)}((n-2,2)) = (-1)^{n-1}$. Since (1^4) is contained in both $\lambda = (2,1^{n-2}), (1^n)$, c_λ also includes $f^{\lambda/(1^4)} \geqslant 1$. For $\lambda = (n-1,1)$, only $\lambda^{(1)}$ applies and $\chi^{(n-1,1)}((n-2,2)) = -1$. In this case, $(4) \subseteq \lambda$, and thus c_λ also includes $f^{\lambda/(4)} \geqslant 1$. For $\lambda = (n)$, the height of any ribbon tableaux is 0, so there are no negatives to worry about.

If λ is of the form $\lambda^{(3)}$, $\lambda^{(4)}$, or $\lambda^{(5)}$, then it is possible that $c_{\lambda^{(i)}}$ includes -1 from the unique $T \in \mathsf{R}(\lambda^{(i)}, (n-2,2))$ for $i = 3,4,5$. In any of these disjoint cases, $(2,2) \subseteq \lambda^{(i)}$, which means $c_{\lambda^{(i)}}$ also includes $f^{\lambda^{(i)}/(2,2)} \geqslant 1$. Hence $f^{\lambda/(1^4)}$, $f^{\lambda/(4)}$, and $f^{\lambda/(2,2)}$ from (3) and $\chi^\lambda((n-2,2))$ from (4) satisfy $f^{\lambda/(1^4)} + f^{\lambda/(4)} + f^{\lambda/(2,2)} + \chi^\lambda((n-2,2)) \geqslant 0$. We have shown that $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(n-2,2)}$ is Schur-positive. □

Conjecture 2.3.43. *For $n \geqslant 9$, the interval $A_{(n-2,1,1)} = \{\lambda \mid (1^n) \leqslant_I \lambda \leqslant_I (n-2,1,1)\}$ is exactly*

$$(1^n) \lessdot_I (2,1^{n-2}) \lessdot_I (2,2,1^{n-4}) \lessdot_I (3,1^{n-3}) \lessdot_I (n-2,1,1).$$

Remark 2.3.44. The first two covers are consequences of Corollary 2.3.6. By Proposition 2.3.5,

$$\varphi_0 \colon \mathsf{SSYT}((2,2,1^{n-4}), \nu) \to \mathsf{SSYT}((3,1^{n-3}), \nu)$$

is an injection (with $\alpha = 0, \beta = 2$). Since $(2,2,1^{n-4}) <_D (3,1^{n-3})$, this implies $(2,2,1^{n-4}) \lessdot_I (3,1^{n-3})$. By Corollary 2.3.30, we know $(3,1^{n-3}) <_I (n-2,1,1)$ because $(3,1^{n-3}) <_D (3,1^{n-3})^t = (n-2,1,1)$. This implies

$$\{(1^n),(2,1^{n-2}),(2,2,1^{n-4}),(3,1^{n-3}),(n-2,1,1)\} \subseteq A_{(n-2,1,1)}.$$

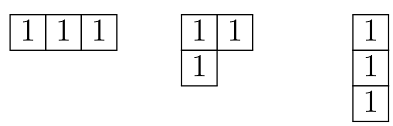
However, we have not proven the cover relation $(3,1^{n-3}) \lessdot_I (n-2,1,1)$. To show that $A_{(n-2,1,1)}$ is contained in the above set, one could show that for all partitions λ such that λ and λ^t are not included in the above list, we have $f^\lambda > \frac{(n-1)(n-2)}{2} = f^{(n-2,1,1)}$. This would prove that $\lambda \notin A_{(n-2,1,1)}$. We have confirmed the conjecture up to $n = 18$.

Proposition 2.3.45.

(1) For $n < 9$, $p_{A_{(n-2,1,1)}}$ is Schur-positive.

(2) For $n \geqslant 9$, $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(3,1^{n-3})} + p_{(n-2,1,1)}$ is Schur-positive.

Proof. Part (1) can be checked explicitly using SageMath. For part (2), as shown in the proof of Proposition 2.3.42, $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})}$ is Schur-positive. We next show $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(3,1^{n-3})}$ is Schur-positive. For $T \in \mathsf{R}(\lambda,(3,1^{n-3}))$, there are three possible ways of arranging one 3-ribbon in T.



$\mathrm{ht}(T) = 0 \quad \mathrm{ht}(T) = 1 \quad \mathrm{ht}(T) = 2$

Therefore, the coefficient of s_λ in $p_{(3,1^{n-3})}$ is $f^{\lambda/(3)} - f^{\lambda/(2,1)} + f^{\lambda/(1,1,1)}$.

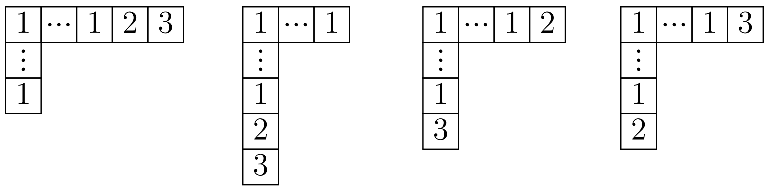
If $(2,1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(2,1)}$. Consider the bijection

$$\mathsf{SYT}(\lambda/(2,1)) \to \{T \in \mathsf{SYT}(\lambda) \mid T_{1,1} = 1, T_{1,2} = 2, \text{ and } T_{2,1} = 3\}.$$

Note, the above subset of standard Young tableaux of shape λ in the term f^λ in $p_{(1^n)}$ was not used in the bijections in the proof of Proposition 2.3.42. This shows that $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(3,1^{n-3})}$ is Schur-positive.

We now examine the Schur expansion of $p_{(n-2,1,1)}$. There are a few specific shapes λ where $\mathsf{R}(\lambda,(n-2,1,1))$ is nonempty. Note that for a ribbon tableau T of type $(n-2,1,1)$, $\mathrm{ht}(T) = \mathrm{ht}(R_1)$, where R_1 is the $(n-2)$-ribbon of 1's in T. In Case 1, we examine all hook shapes $\lambda = (a, 1^b)$. In Case 2, we examine all shapes $\lambda = (a, 2, 1^b)$. In Case 3, we examine the remaining two shapes $(a, 3, 1^b)$ and $(a, 2, 2, 1^b)$.

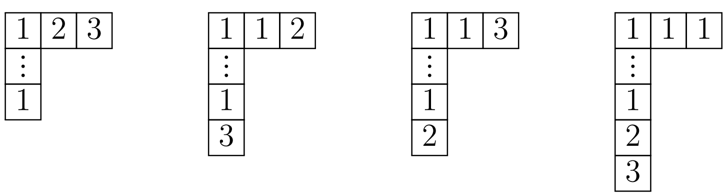
Case 1a: $\lambda = (a, 1^b)$ with $4 \leqslant a \leqslant n-3$ and $3 \leqslant b \leqslant n-4$. These conditions on a, b require that the $(n-2)$-ribbon forms a hook with nontrivial arm and nontrivial leg.



$\mathrm{ht}(T) = b$ $\quad$ $\mathrm{ht}(T) = b-2$ $\quad$ $\mathrm{ht}(T) = b-1$ $\quad$ $\mathrm{ht}(T) = b-1$

Since $b, b-2$ have opposite parity to $b-1$, $\chi^\lambda((n-2,1,1)) = 0$ for $\lambda = (a, 1^b)$ with $4 \leqslant a \leqslant n-3$ and $3 \leqslant b \leqslant n-4$.

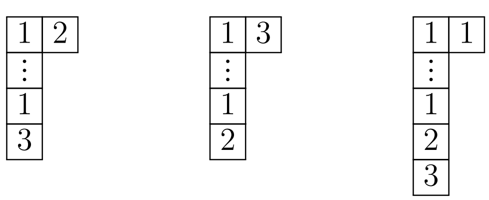
Case 1b: $\lambda = (3, 1^{n-3})$.



$\mathrm{ht}(T) = n-3$ $\quad$ $\mathrm{ht}(T) = n-4$ $\quad$ $\mathrm{ht}(T) = n-4$ $\quad$ $\mathrm{ht}(T) = n-5$

Since $n-3, n-5$ have opposite parity to $n-4$, $\chi^\lambda((n-2,1,1)) = 0$ for $\lambda = (3, 1^{n-3})$.

Case 1c: $\lambda = (2, 1^{n-2})$.



$\mathrm{ht}(T) = n-3$ $\quad$ $\mathrm{ht}(T) = n-3$ $\quad$ $\mathrm{ht}(T) = n-4$

Since $n-3$ and $n-4$ have opposite parity, $\chi^\lambda((n-2,1,1)) = (-1)^{n-3}$ for $\lambda = (2,1^{n-2})$. Since $n \geqslant 5$, $(1^4) \subseteq \lambda$ and $f^{\lambda/(1^4)} \geqslant 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative.

Case 1d: $\lambda = (1^n)$. The unique ribbon tableau $T \in \mathsf{R}((1^n),(n-2,1,1))$ has height $n-3$, hence $\chi^\lambda((n-2,1,1)) = (-1)^{n-3}$ for $\lambda = (1^n)$. Since $n \geqslant 5$, $(1^4) \subseteq \lambda$ and $f^{\lambda/(1^4)} \geqslant 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative.

Case 1e: $\lambda = (n-2,1,1)$.

$$\begin{array}{|c|c|c|}\hline 1 & \cdots & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 2 \\ \hline 1 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 3 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & \cdots & 1 & 2 & 3 \\ \hline 1 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$$\mathrm{ht}(T) = 0 \quad \mathrm{ht}(T) = 1 \quad \mathrm{ht}(T) = 1 \quad \mathrm{ht}(T) = 2$$

Thus, $\chi^\lambda((n-2,1,1)) = 0$ for $\lambda = (n-2,1,1)$.

Case 1f: $\lambda = (n-1,1)$.

$$\begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & \cdots & 1 & 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\mathrm{ht}(T) = 0 \quad \mathrm{ht}(T) = 0 \quad \mathrm{ht}(T) = 1$$

Thus, $\chi^\lambda((n-2,1,1)) = 1$ for $\lambda = (n-1,1)$.

Case 1g: $\lambda = (n)$. The unique ribbon tableau $T \in \mathsf{R}((n),(n-2,1,1))$ has height 0, hence $\chi^\lambda((n-2,1,1)) = 1$ for $\lambda = (n)$.

Case 2a: $\lambda = (a,2,1^b)$ with $3 \leqslant a \leqslant n-3$ and $1 \leqslant b \leqslant n-5$.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 3 \\ \hline 1 & 2 \\ \cline{1-2} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 2 \\ \hline 1 & 3 \\ \cline{1-2} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline 1 & 2 \\ \cline{1-2} \vdots \\ \cline{1-1} 1 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline 1 & 3 \\ \cline{1-2} \vdots \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$\mathrm{ht}(T) = b+1 \quad \mathrm{ht}(T) = b+1 \quad \mathrm{ht}(T) = b \quad \mathrm{ht}(T) = b$$

Since b and $b+1$ have opposite parity, $\chi^\lambda((n-2,1,1)) = 0$ for $\lambda = (a,2,1^b)$ with $3 \leqslant a \leqslant n-3$ and $1 \leqslant b \leqslant n-5$.

Case 2b: $\lambda = (n-2,2)$.

$$\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 2 & 3 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & \cdots & 1 & 3 \\ 1 & 2 & & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & \cdots & 1 & 2 \\ 1 & 3 & & & \end{array}$$

$$\mathrm{ht}(T) = 0 \qquad \mathrm{ht}(T) = 1 \qquad \mathrm{ht}(T) = 1$$

Thus, $\chi^\lambda((n-2,1,1)) = -1$ for $\lambda = (n-2,2)$. Since $(2,2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geqslant 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this negative.

Case 2c: $\lambda = (2,2,1^{n-4})$.

$$\begin{array}{cc} 1 & 2 \\ 1 & 3 \\ \vdots & \\ 1 & \\ 1 & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 1 & 3 \\ \vdots & \\ 1 & \\ 2 & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 1 & 2 \\ \vdots & \\ 1 & \\ 3 & \end{array}$$

$$\mathrm{ht}(T) = n-3 \quad \mathrm{ht}(T) = n-4 \quad \mathrm{ht}(T) = n-4$$

Since $n-3$ and $n-4$ have opposite parity, $\chi^\lambda((n-2,1,1)) = (-1)^{n-4}$ for $\lambda = (2,2,1^{n-4})$. Since $(2,2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geqslant 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative.

Case 3a: $\lambda = (n-(3+b),3,1^b)$ for $0 \leqslant b \leqslant n-6$. The unique ribbon tableau $T \in \mathsf{R}((n-(3+b),3,1^b),(n-2,1,1))$ has height $b+1$, hence $\chi^\lambda((n-2,1,1)) = (-1)^{b+1}$ for $\lambda = (n-(3+b),3,1^b)$. Since $(2,2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geqslant 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative.

Case 3b: $\lambda = (n-(4+b),2,2,1^b)$ for $0 \leqslant b \leqslant n-6$. The unique ribbon tableau $T \in \mathsf{R}((n-(4+b),2,2,1^b),(n-2,1,1))$ has height $b+2$, hence $\chi^\lambda((n-2,1,1)) = (-1)^{b+2}$ for $\lambda = (n-(4+b),2,2,1^b)$. Since $(2,2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geqslant 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative. □

2.4. Discussion

In this thesis, we studied various properties of the immersion and standard immersion poset, which are tightly linked to finite-dimensional irreducible polynomial representations of $GL_N(\mathbb{C})$ through their immersion pairs.

There are still many open questions to pursue in this line of research. It would be interesting to characterize all maximal elements in the immersion and standard immersion poset. In particular, a proof of Conjecture 2.2.14 seems in reach with the methods developed in this thesis. In Corollary 2.3.30, we showed that for hook shapes λ and λ^t form an immersion pair. The same seems true for two column partitions. It would be interesting to classify when λ and its transpose form an immersion pair. In Corollary 2.3.31, we showed that the rank of the immersion poset is at least $\lfloor n/2 \rfloor$. It would be desirable to find better bounds for the rank.

Furthermore, it would be interesting to classify all intervals and chains in the immersion poset, in particular to obtain proofs of Conjectures 2.3.40 and 2.3.43. In view of the results of Section 2.3.5, the following question is natural.

Question 2.4.1. Which intervals $A_\mu := \{\lambda \,|\, (1^n) \leqslant_I \lambda \leqslant_I \mu\}$ in the immersion poset give rise to Schur-positivity of p_{A_μ}?

Sundaram conjectured that all intervals $[(1^n), \mu]$ in reverse lexicographic order make (2.1.1) Schur-positive [**Sun18**, Conjecture 1], and has proven the conjecture for certain intervals [**Sun19**]. When $n \geqslant 5$, it appears that the immersion poset always contains some interval(s) which do not give rise to Schur-positivity. For example, $p_{A_{(n-1,1)}} = p_{(1^n)} + p_{(2,1^{n-2})} + p_{(n-1,1)}$ contains $-s_{(1^n)}$ when n is odd. This observation shows that the analog of [**Sun18**, Conjecture 1] is false for the immersion poset order. However, it does seem true that a large percentage of intervals A_μ in the immersion poset yield Schur-positivity. Using SageMath [**The24**], we observe that when $6 \leqslant n \leqslant 9$ at least 91% of the intervals in the immersion poset make (2.1.1) Schur-positive. When $n = 10, 11$ the percentage of Schur-positive intervals drops to at least 81%, and when $n = 18$, the percentage is approximately 73.5%.

We conclude with some probabilistic and asymptotic questions.

Question 2.4.2. For randomly chosen partitions $\lambda <_D \mu$, what is the probability that $\lambda \leqslant_I \mu$?

For a partition λ of any size, consider the padded partition $\lambda[N] := (N - |\lambda|, \lambda_1, \lambda_2, \dots)$ of size N, where $N \geqslant |\lambda|$. For any two partitions $\lambda \leqslant_D \mu$ (of any size), what can we say about $\lambda[N] \leqslant_I \mu[N]$ for $N \gg 1$? Furthermore, it would be interesting to study the asymptotical behaviors of the (standard) immersion poset.

2.5. Maximal Element Conjecture

For this section, let λ be a partition of n with the form of $\lambda = (\sum_{i=1}^{k} a_i, \sum_{i=1}^{k-1} a_i, \ldots, a_2 + a_1, a_1)$. To suppress some of the notation, we will let $s_i^j = \sum_{l=i}^{j} a_l$, so that $\lambda = (s_1^k, s_1^{k-1}, \ldots, s_1^2, s_1^1)$. In this section, we begin with (2.5.1) which relates the number of standard Young tableau of two particular shapes. This allows us to prove a few propositions that serve as framework for the end of the section, where in Proposition 2.5.11 we prove the maximality Conjecture 2.5.2 for the case when $k = 3$.

Definition 2.5.1. We say that $\lambda = (s_1^k, s_1^{k-1}, \ldots, s_1^2, s_1^1)$ satisfies the *maximality inequalities* if for all $1 \leqslant p \leqslant k-1$ we have:

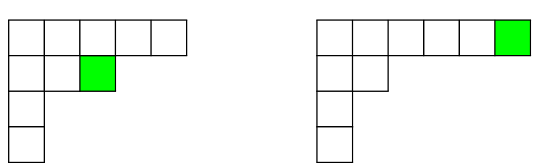
$$\binom{a_{p+1}+2}{2} \geqslant \left(\sum_{i=1}^{p} a_i\right) + p - 1,$$

We now restate the maximality conjecture for partitions of the immersion poset.

Conjecture 2.5.2. *If $\lambda \vdash n \geqslant 5$ satisfies the maximality inequalities, then λ is a maximal element in the standard immersion poset and consequently the immersion poset as well.*

Definition 2.5.3. A partition λ can be thought of in terms of Ferrers diagrams. We let the number of boxes in the ith row of the Ferrers diagram corresponding to λ equal the number in the ith part of the partition λ. Let $p, m \in \mathbb{Z}_{>0}$ and recall λ has k parts, or k rows in its corresponding Ferrers diagram. Then we define the new partition $\lambda_{(p,m)}$ as follows. The corresponding Ferrers diagram for $\lambda_{(p,m)}$ is the Ferrers diagram obtained by taking λ's corresponding Ferrers diagram and moving m boxes from the $(k-p+1)$th row to the $(k-p)$th row. As a partition, this means $\lambda_{(p,m)} = (s_1^k, s_1^{k-1}, \ldots, s_1^{p+2}, s_1^{p+1} + m, s_1^p - m, s_1^{p-1}, \ldots, s_1^2, s_1^1)$, assuming $\lambda_{(p,m)}$ is also a partition of n, which we require.

Example 2.5.4. Let $\lambda = (5,3,1,1)$, Then $\lambda_{(3,1)} = (6,2,1,1)$. Below we have the Ferrers diagrams of λ (on the left) and $\lambda_{(3,1)}$ (on the right), where we have highlighted the box being moved in green.



We refer the reader back to Equation (2.2.2) for the hook length formula. To calculate f^λ, the number of standard Young tableau of shape λ, we fill the Ferrers diagram boxes with values equal to the number of boxes in the row right of it plus the number of boxes in the column below it plus one, which we call the hook length. For example with $\lambda = (5,3,1,1)$ and $\lambda_{(3,1)} = (6,2,1,1)$ we have

8	5	4	2	1
5	2	1		
2				
1				

9	6	4	3	2	1
4	1				
2					
1					

.

We refer to the hook length entry of the ith row and jth column as $h_{i,j}$. Then by the hook length formula from (2.2.2) we have the following equation for $\lambda = (5,3,1,1)$ and $\lambda_{(3,1)} = (6,2,1,1)$, and the ratio $\frac{f^{(5,3,1,1)}}{f^{(6,2,1,1)}}$

$$f^{(5,3,1,1)} = \frac{10!}{8\cdot 5\cdot 4\cdot 2\cdot 1\cdot 5\cdot 2\cdot 1\cdot 2\cdot 1}$$

,

$$f^{(6,2,1,1)} = \frac{10!}{9\cdot 6\cdot 4\cdot 3\cdot 2\cdot 1\cdot 4\cdot 1\cdot 2\cdot 1}$$

,

$$\frac{f^{(5,3,1,1)}}{f^{(6,2,1,1)}} = \frac{81}{50}$$

Proposition 2.5.5 will provide us with a formula for $\frac{f^\lambda}{f^{\lambda_{(p,m)}}}$. This formula will be incredibly useful in proving many of the propositions that follow. One should write out the hook lengths for $\lambda = (s_1^k, s_1^{k-1}, \ldots, s_1^2, s_1^1)$ and $\lambda_{(p,m)} = (s_1^k, s_1^{k-1}, \ldots, s_1^{p+2}, s_1^{p+1} + m, s_1^p - m, s_1^{p-1}, \ldots s_1^2, s_1^1)$ to be able to understand (2.5.1) in Proposition 2.5.5.

Proposition 2.5.5. *Let* $\lambda = (s_1^k, s_1^{k-1}, \ldots, s_1^2, s_1^1)$ *and* $\lambda_{(p,m)} = (s_1^k, s_1^{k-1}, \ldots, s_1^{p+2}, s_1^{p+1} + m, s_1^p - m, s_1^{p-1}, \ldots s_1^2, s_1^1)$ *as defined above where p and m are positive integers such that $\lambda_{(p,m)}$ is a partition.*

Then when $a_1, a_p, a_{p+2} \geqslant 1$ and $a_i \geqslant 0$ for all other $i \neq 1, p, p+2$, we have the following:

$$\frac{f^\lambda}{f^{\lambda_{(p,m)}}} = \frac{\left(\prod_{j=p+1}^{p+m}(s_1^{p+1}+j)\right)\left(s_{p+1}^{p+1}+1\right)}{\left(\prod_{j=p-m}^{p-1}(s_1^{p}+j)\right)\left(s_{p+1}^{p+1}+2m+1\right)} P_{\lambda,p,m} Q_{\lambda,p,m}, \tag{2.5.1}$$

where

$$P_{\lambda,p,m} = \begin{cases} \prod_{j=2}^{p} \frac{\left(s_j^{p+1}+p+2-j\right)\left(s_j^{p}+p+1-j\right)}{\left(s_j^{p+1}+p+m+2-j\right)\left(s_j^{p}+p+1-m-j\right)} & \text{if } 2 \leqslant p \leqslant k-1 \\ 1 & \text{if } p = 1 \end{cases}$$

and

$$Q_{\lambda,p,m} = \begin{cases} \prod_{j=p+2}^{k} \frac{\left(s_{p+1}^{j}+j-p\right)\left(s_{p+1}^{j}+j-p-1\right)}{\left(s_{p+1}^{j}+j-p+m\right)\left(s_{p+1}^{j}+j-p-1-m\right)} & 1 \leqslant p \leqslant k-2 \\ 1 & \text{if } p = k-1 \end{cases}.$$

The proof of the proposition is a straightforward calculation using the hook lengths formula.

EXAMPLE 2.5.6. In Example 2.5.4, we calculated f^λ and $f^{\lambda_{(3,1)}}$ for $\lambda = (5,3,1,1)$ and $\lambda_{(3,1)} = (6,2,1,1)$. We can now verify (2.5.1) for these partitions.

$$P_{\lambda,3,1} = \prod_{j=2}^{3} \frac{\left(s_j^4+5-j\right)\left(s_j^3+4-j\right)}{\left(s_j^4+6-j\right)\left(s_j^3+3-j\right)} = \frac{(7)(4)}{(8)(3)}\frac{(6)(3)}{(7)(2)}$$

$$Q_{\lambda,3,1} = 1$$

$$\begin{aligned}\frac{f^\lambda}{f^{\lambda_{(3,1)}}} &= \frac{\left(\prod_{j=4}^{4}(s_1^4+j)\right)\left(s_4^4+1\right)}{\left(\prod_{j=2}^{2}(s_1^3+j)\right)\left(s_4^4+3\right)} P_{\lambda,3,1}Q_{\lambda,3,1} \\ &= \frac{(9)(3)}{(5)(5)}\frac{(7)(4)}{(8)(3)}\frac{(6)(3)}{(7)(2)} \\ &= \frac{81}{50}\end{aligned}$$

PROPOSITION 2.5.7. *Let $k \geqslant 3$, $1 \leqslant p \leqslant k-1$ and $\lambda \vdash n$ with $\lambda = (s_1^k, s_1^{k-1}, \ldots, s_1^2, s_1^1)$. If λ satisfies the maximality inequalities, then $f^\lambda > f^{\lambda_{(p,1)}}$ when $\lambda_{(p,1)} \vdash n$.*

PROOF. Let $y > x > 0$. Then every fraction in the products of $P_{\lambda,p,1}$ and $Q_{\lambda,p,1}$ is of the form:

$$\frac{(y+2)(x+1)}{(y+3)(x)} > 1$$

Thus each of these is strictly greater than 1 and since $k \geqslant 3$ there is at least one of these fractions to guarantee that $P_{\lambda,p,1}Q_{\lambda,p,1} > 1$. Then, since:

$$\frac{\Big((\sum_{i=1}^{p+1} a_i) + p + 1\Big)\Big((a_{p+1}) + 1\Big)}{\Big((\sum_{i=1}^{p} a_i) + p - 1\Big)\Big((a_{p+1}) + 3\Big)} \geqslant 1 \qquad \Longleftrightarrow \qquad \binom{a_{p+1}+2}{2} \geqslant \Big(\sum_{i=1}^{p} a_i\Big) + p - 1,$$

the result follows. ☐

PROPOSITION 2.5.8. *Let $k \geqslant 3$, $m \geqslant 1$ and $\lambda \vdash n$ with $\lambda = (s_1^k, s_1^{k-1}, \dots, s_1^2, s_1^1)$. If*

$$\binom{a_{p+1}+2}{2} \geqslant \Big(\sum_{i=1}^{p} a_i\Big) + p - 1,$$

for all $1 \leqslant p \leqslant k-1$, then $f^\lambda > f^{\lambda_{(p,m)}}$ when $\lambda_{(p,m)} \vdash n$.

PROOF. We proceed by induction on m. When $m = 1$, Proposition 2.5.7 proves the base case. Now suppose the proposition holds for $m-1$ and $\lambda_{(p,m)} \vdash n$, we will show it holds for m. To show $f^\lambda > f^{\lambda_{(p,m)}}$, we will show $\frac{f^\lambda}{f^{\lambda_{(p,m)}}} > \frac{f^\lambda}{f^{\lambda_{(p,m-1)}}}$, which is equivalent to $f^{\lambda_{(p,m-1)}} > f^{\lambda_{(p,m)}}$, then by the inductive hypothesis the result will follow.

First notice that $P_{\lambda,p,m-1} > P_{\lambda,p,m}$ and $Q_{\lambda,p,m-1} > Q_{\lambda,p,m}$ follows immediately. But,

$$\frac{\Big(\prod_{j=p+1}^{p+m}(s_1^{p+1}+j)\Big)\Big(s_{p+1}^{p+1}+1\Big)}{\Big(\prod_{j=p-m}^{p-1}(s_1^{p}+j)\Big)\Big(s_{p+1}^{p+1}+2m+1\Big)} \geqslant \frac{\Big(\prod_{j=p+1}^{p+(m-1)}(s_1^{p+1}+j)\Big)\Big(s_{p+1}^{p+1}+1\Big)}{\Big(\prod_{j=p-(m-1)}^{p-1}(s_1^{p}+j)\Big)\Big(s_{p+1}^{p+1}+2(m-1)+1\Big)}$$

which simplifies to

$$(s_1^{p+1}+p+m)(s_{p+1}^{p+1}+2m-1) \geqslant (s_1^{p+1}+p-m)(s_{p+1}^{p+1}+2m+1)$$

and again simplifies to

$$\binom{a_{p+1}+2m}{2} + m - 1 \geqslant s_1^p + p - 1$$

and since $m \geq 1$ we have

$$\binom{a_{p+1}+2m}{2} + m - 1 \geqslant \binom{a_{p+1}+2}{2}$$

and finally

$$\binom{a_{p+1}+2}{2} \geqslant s_1^p + p - 1$$

is assumed to be true, completing the proof. ☐

Proposition 2.5.9. *Let* $\lambda = (s_1^k, s_1^{k-1}, \dots, s_1^2, s_1^1) \vdash n$. *If* λ *satisfies the maximality inequalities, then* $\lambda' = (s_1^k + 1, s_1^{k-1}, \dots, s_1^2, s_1^1 - 1)$ *also satisfies the maximality inequalities. In particular, as long as* $k > 1$ *(which implies* $s_1^1 > 0$*),* λ' *always exists.*

The proof is straightforward, as only the first and last inequalities are different. This simple result will be the critical piece for our first major result.

Proposition 2.5.10. *If* $\lambda = (s_1^3, s_1^2, s_1^1) \vdash n \geqslant 5$ *satisfies the maximality inequalities, then* $f^\lambda > f^{\lambda'}$, *where* $\lambda' = (s_1^3 + 1, s_1^2, s_1^1 - 1)$.

Proof. The reader can verify that:

$$\frac{f^\lambda}{f^{\lambda'}} = \frac{(s_1^3 + 3)(s_2^3 + 2)(s_2^2 + 1)(s_3^3 + 1)}{(s_1^1)(s_2^3 + 4)(s_2^2 + 2)(s_3^3 + 2)}$$

From this point on, we will assume $a_1 \geqslant 1$ (required for λ to have three parts), $a_2 \geqslant 1$ and $a_3 \geqslant 2$. The only $\lambda \vdash n \geqslant 5$ with three parts where these inequalities are not satisfied, but the maximality inequalities are satisfied is $(3,2,1)$ and $(3,1,1)$ and the reader can verify that if λ is one of these two, then $f^\lambda > f^{\lambda'}$. Now recall the two maximality inequalities for when $k = 3$:

$$\binom{a_3 + 2}{2} \geqslant \sum_{i=1}^{2} a_i + 1 \qquad \Longleftrightarrow \qquad a_3^2 + 3a_3 \geqslant 2a_2 + 2a_1$$

$$\binom{a_2 + 2}{2} \geqslant \sum_{i=1}^{1} a_i \qquad \Longleftrightarrow \qquad a_2^2 + 3a_2 + 2 \geqslant 2a_1$$

Now with a little rewriting we have:

$$f^\lambda > f^{\lambda'} \iff (s_1^3+3)(s_2^3+2)(s_2^2+1)(s_3^3+1) - (s_1^1)(s_2^3+4)(s_2^2+2)(s_3^3+2) > 0$$

$$\iff a_2a_3^3 + a_2^3a_3 + 2a_2^2a_3^2 + a_3^3 + 8a_2a_3^2 + 8a_2^2a_3 + a_2^3 + 6a_3^2 + 6a_2^2 + 18a_2a_3 + 11a_2 + 11a_3 + 6$$

$$- a_1a_3^2 - a_1a_2^2 - 4a_1a_2a_3 - 9a_1a_2 - 9a_1a_3 - 14a_1 > 0$$

$$\iff (\frac{1}{2}a_2^2a_3^2 + \frac{3}{2}a_2^2a_3 - a_2^3 - a_1a_2^2)$$

$$+ (a_2^3a_3 + 3a_2^2a_3 + 2a_2a_3 - 2a_1a_2a_3)$$

$$+ (a_2a_3^3 + 3a_2a_3^2 - 2a_2^2a_3 - 2a_1a_2a_3)$$

$$+ (\frac{1}{2}a_2^2a_3^2 + \frac{3}{2}a_2a_3^2 + a_3^2 - a_1a_3^2)$$

$$+ (a_3^3 + 3a_3^2 - 2a_2a_3 - 2a_1a_3)$$

$$+ (\frac{7}{2}a_2^2a_3 + \frac{21}{2}a_2a_3 + 7a_3 - 7a_1a_3)$$

$$+ (2a_2a_3^2 + 6a_2a_3 - 4a_2^2 - 4a_1a_2)$$

$$+ (3a_2^2 + 9a_2 + 6 - 6a_1)$$

$$+ (a_3^2 + 3a_3 - 2a_1 - 2a_2)$$

$$+ (2a_2^3 + 6a_2^2 + 4a_2 - 4a_1a_2)$$

$$+ (\frac{1}{2}a_2a_3^2 + \frac{3}{2}a_2a_3 - a_1a_2 - a_2^2)$$

$$+ (a_2^2a_3^2 + 2a_2^2a_3 + 2a_2^2 + a_2a_3^2 + a_3^2 + a_3 - 6a_1) > 0$$

There are eleven parentheses in the above inequality. We will show that for each parentheses, the terms being added are at least as big as those being subtracted by using the maximality inequalities.

$$\frac{1}{2}a_2^2a_3^2 + \frac{3}{2}a_2^2a_3 = \frac{1}{2}a_2^2(a_3^2 + 3a_2) \geqslant a_1a_2^2 + a_2^3$$

$$a_2^3a_3 + 3a_2^2a_3 + 2a_2a_3 = a_2a_3(a_2^2 + 3a_2 + 2) \geqslant 2a_1a_2a_3$$

$$a_2a_3^3 + 3a_2a_3^2 = a_2a_3(a_3^2 + 3a_3) \geqslant 2a_2^2a_3 + 2a_1a_2a_3$$

$$\frac{1}{2}a_2^2a_3^2 + \frac{3}{2}a_2a_3^2 + a_3^2 = \frac{1}{2}a_3^2(a_2^2 + 3a_2 + 2) \geqslant a_1a_3^2$$

$$a_3^3 + 3a_3^2 = a_3(a_3^2 + 3a_3) \geqslant 2a_1a_3 + 2a_2a_3$$

$$\frac{7}{2}a_2^2a_3 + \frac{21}{2}a_2a_3 + 7a_3 = \frac{7}{2}a_3(a_2^2 + 3a_2 + 2) \geqslant 7a_1a_3$$

$$2a_2a_3^2 + 6a_2a_3 = 2a_2(a_3^2 + 3a_3) \geqslant 4a_1a_2 + 4a_2^2$$

$$3a_2^2 + 9a_2 + 6 = 3(a_2^2 + 3a_2 + 2) \geqslant 6a_1$$

$$a_3^2 + 3a_3 \geqslant 2a_1 + 2a_2$$

$$2a_2^3 + 6a_2^2 + 4a_2 = 2a_2(a_2^2 + 3a_2 + 2) \geqslant 4a_1a_2$$

$$\frac{1}{2}a_2a_3^2 + \frac{3}{2}a_2a_3 = \frac{1}{2}a_2(a_3^2 + 3a_3) \geqslant a_1a_2 + a_2^2$$

$$a_2^2a_3^2 + 2a_2^2a_3 + 2a_2^2 + a_2a_3^2 + a_3^2 + a_3 \geqslant 10a_2^2 + 4a_2 + 6 > 3a_2^2 + 9a_2 + 6 \geqslant 6a_1$$

All but the last inequality follows directly from the maximality inequalities we are assuming. The last inequality follows because $a_3 \geqslant 2$, and when $a_2 \geqslant 1$ we have $a_2^2 \geqslant a_2$. Notice the strictness of the last inequality guarantees the strictness of the whole sum and the result follows. □

PROPOSITION 2.5.11. *Let* $\lambda = (s_1^3, s_1^2, s_1^1) \vdash n \geqslant 5$. *If* λ *satisfies the maximality inequalities, then* λ *is maximal in the immersion poset.*

PROOF. We proceed with a proof by induction on the number of steps away from (n) that λ is in dominance order. The base case is trivial; there are no inequalities for (n) to satisfy, and it is indeed maximal in the immersion poset. Now assume every partition less than m steps away from (n) that satisfies the maximality inequalities is also a maximal element in the immersion poset. Let λ satisfy the maximality inequalities and be m steps away from (n) in dominance order.

If we think about the partition as a tableau, then all partitions that dominate λ are obtained by either (1) moving only boxes from the third row to the second row, (2) moving only boxes from the second row to the first row, or (3) some combination of both (this includes the case of only moving boxes from the third row to the first). We need to show that if μ is a partition that dominates λ, then $f^\lambda > f^\mu$.

For μ a partition in case (1) or (2), $f^\lambda > f^\mu$ by Proposition 2.5.8. Let λ' be defined as it is in Proposition 2.5.9, then every μ in case (3) dominates λ'. Since λ' is less than m moves away from (n), by the induction hypothesis we have $f^{\lambda'} > f^\mu$. Finally, by Proposition 2.5.10, $f^\lambda > f^{\lambda'}$ and the result follows. $\square$

CHAPTER 3

A crystal analysis of claw-free graphs

This chapter is based on work in collaboration with Evuilynn Nguyen and Anne Schilling.

3.1. Background and definitions

DEFINITION 3.1.1. Let $G = (V, E)$ be a graph with $|V| = n$. Given $S \subseteq \mathbb{N}$, a *proper S-coloring of G* is a function $\kappa: V \to S$ such that $\kappa(i) \neq \kappa(j)$ when $(i,j) \in E$. In [**Sta95**], Stanley defined the *chromatic symmetric function of G* as

$$X_G(\mathbf{x}) := \sum_{\kappa \in \mathcal{K}(G)} x_{\kappa(1)} \cdots x_{\kappa(n)}, \tag{3.1.1}$$

where $\mathcal{K}(G)$ is the set of all proper $\mathbb{N}$-colorings of G. It is easy to see that $X_G(\mathbf{x})$ lies in the ring of symmetric functions $\Lambda_{\mathbb{Z}}$ in $x_1, x_2, \ldots$ with integer coefficients.

In [**Sta95**], Stanley conjectured the chromatic symmetric functions of incomparability graphs of $(3+1)$-free posets are e-positive where e refers to the elementary symmetric functions, known as the Stanley-Stembridge conjecture. A weaker result stating that $X_G(\mathbf{x})$ is Schur-positive when G is as in the conjecture was proven by Gasharov [**Gas96**].

Motivated by this conjecture, Shareshian and Wachs define a refinement of the chromatic symmetric functions called chromatic quasisymmetric functions [**SW16**].

DEFINITION 3.1.2. The *chromatic quasisymmetric function* of a graph G is

$$X_G(\mathbf{x}, t) := \sum_{\kappa \in \mathcal{K}(G)} x_{\kappa(1)} \cdots x_{\kappa(n)} t^{\operatorname{asc}(\kappa)}$$

where $\operatorname{asc}(\kappa) = |\{(i,j) \in E : i < j \text{ and } \kappa(i) < \kappa(j)\}|$. We note that $X_G(\mathbf{x}, 1) = X_G(\mathbf{x})$, so it follows that $X_G(\mathbf{x}, 1)$ is Schur positive when G is the incomparability graph of a $(3+1)$-free poset.

Using a proof similar to Gasharov's [**Gas96**], Shareshian and Wachs prove Schur positivity of $X_G(\mathbf{x}, t)$ when G is the incomparability graph of a $(2+2)$ and $(3+1)$-free poset [**SW16**].

In [SW16], Shareshian and Wachs also show that for unit interval graphs, $X_G(\mathbf{x},t) \in \Lambda_{\mathbb{Z}}[t]$, that is, $X_G(\mathbf{x},t)$ is a polynomial in t with coefficients in $\Lambda_{\mathbb{Z}}$. Although this is a much smaller class of posets, Guay-Paquet showed that proving e-positivity of $(3+1)$-free posets was equivalent to proving e-positivity of unit interval graphs, thereby simplifying the Stanley-Stembridge conjecture [GP13]. As of October 2024, the Stanley-Stembridge conjecture has been proven by Tatsuyuki Hikita [Hik24].

Now, Stanley [Sta98] conjectured that the chromatic symmetric function of all claw-free graphs G is Schur positive.

CONJECTURE 3.1.3 ([Sta98]). *The chromatic symmetric function of a claw-free graph G is Schur-positive.*

Special cases of this conjecture were considered in [Gas96, WW20, Ehr22, SvW24]. In particular, Gasharov [Gas96] proved that claw-free incomparability graphs are Schur positive. Ehrhard [Ehr22] reproved Gasharov's results by defining a crystal structure on P-arrays.

Kashiwara crystals, introduced by Kashiwara in [Kas90], have been a common tool to prove Schur positivity results. In [Ste03], Stembridge showed that when a crystal satisfies certain local axioms, then the character of the crystal corresponds to the character of a certain representation, providing a connection to representation theory and providing an immediate proof of Schur positivity.

In this chapter, we begin in Section 3.2 by defining crystal operators on claw-free graphs and end the section by showing the crystal structure of these crystal operators is isomorphic to that of Ehrhard's [Ehr22] crystal structure when restricted to claw-free incomparability graphs. In Section 3.3 we review the definition of a Stembridge crystal and show our crystal operators don't satisfy the local axioms for the crystal to be Stembridge. In Section 3.4, we show that when restricted to unit interval graphs which do not contain an induced subgraph isomorphic to P_4 (the path graph of length 4), then our operators do satisfy the Stembridge axioms. The proof is technical and long, suggesting that for claw-free graphs, a different approach to proving Schur positivity using the crystal structure would be more suitable. So, we end with Section 3.5, where we discuss how to extend Schur positivity results using crystals beyond claw-free incomparability graphs.

3.2. Crystal Operators

In Section 3.2.1, we will define the crystal operators that will act on the colorings of claw-free graphs (see Definition 1.2.8) and we will show that these operators do satisfy the crystal axioms defined in Definition 3.2.5. In Section 3.2.2, we will show that the crystal structure generated by these operators is isomorphic to Ehrhard's crystal structure in [**Ehr22**] when restricted to incomparability graphs of $(3+1)$-free posets.

3.2.1. Crystal operators on claw-free graphs.

Definition 3.2.1. We define the *simple roots*, α_i ($i \in \{1, 2, \ldots, k-1\}$), of a type A_{k-1} ($GL(r)$ version) root system in $\mathbb{R}^k$ to be $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, where $\mathbf{e}_i$ is the ith standard basis vector. The *simple coroots*, $\alpha_i^\vee$ ($i \in \{1, 2, \ldots, k-1\}$), are defined to be:

$$\alpha_i^\vee = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$$

The weight lattice for the $GL(r)$ version is $\mathbb{Z}^k$. Let V be the quotient space of $\mathbb{R}^k$ by the subspace spanned by the diagonal vector $\sum_{i=1}^k \mathbf{e}_i$. Then the type A_{k-1} ($SL(k)$ version) root system has for simple roots, the images of the simple roots of the $GL(k)$ version under the quotient map. The weight lattice Λ is the image of $\mathbb{Z}^k$ in the quotient space.

Notice that since $\langle \cdot, \cdot \rangle$ is the usual dot product, $\alpha_i = \alpha_i^\vee$ for a type A root system.

We note that the weight lattice for the $SL(k)$ version is semi-simple, whereas the $GL(k)$ version is not. Since we want it to be semi-simple, we need to use the $SL(k)$ version. Since the root systems are in bijection through the quotient map, and the $GL(k)$ version is easier to work with, we will use this throughout the rest of the chapter, with the understanding that one can easily use the quotient map to identify the $SL(k)$ version.

Definition 3.2.2. Let A_{k-1} be the root system, $I = \{1, \ldots, k-1\}$ be the index set, and $\mathbb{Z}^k$ be the weight lattice. A *finite type crystal* of type A_{k-1} is a nonempty set $\mathcal{C}$ together with the maps:

$$e_i, f_i : \mathcal{C} \to \mathcal{C} \sqcup \{0\}$$

$$\varepsilon_i, \varphi_i : \mathcal{C} \to \mathbb{Z}$$

$$\mathrm{wt} : \mathcal{C} \to \Lambda$$

satisfying the following conditions:

A1: If $x, y \in \mathcal{C}$, then $e_i(x) = y$ if and only if $f_i(y) = x$. If so, then we assume:

- $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$
- $\varepsilon_i(y) = \varepsilon_i(x) - 1$
- $\varphi_i(y) = \varphi_i(x) + 1$

A2: We must have:

$$\varphi_i(x) - \varepsilon_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$$

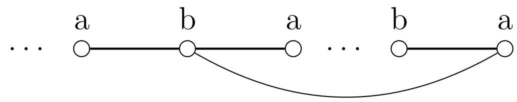
for all $x \in \mathcal{C}$ and $i \in I$. e_i and f_i will be referred to as our *crystal operators*.

More specifically, the elements of $\mathcal{C}$ will be the proper colorings of some claw-free graph that uses only the colors $1, 2, \ldots, k$. For a more extensive treatment of crystals of various root systems see chapter 2 of [**BS17**].

Definition 3.2.3. Let x be a proper coloring. We define the *induced i-coloring* of x to be the coloring on the induced subgraph containing only the vertices colored i or $i+1$ and the edges between these vertices, however we will also for brevity, sometimes refer to the induced subgraph as the induced i-coloring of x as well, and the context should make this difference clear. Similarly, we define the *induced $i, i+1$-coloring* of x to be the same as before, except that we keep the vertices colored i, $i+1$, and $i+2$.

Proposition 3.2.4. *Let G be a claw-free graph, I the index set with $i, i+1 \in I$ and let x be a proper coloring of G. Then each connected component of the induced i-coloring of x must be a subgraph isomorphic to the path graph P_k or the cycle graph C_k (see Definition 1.2.9) for some k. In addition, C_k is only possible if the component has an even number of vertices. If G is a unit interval graph, then C_k is not possible.*

Proof. The reader can check by hand that if there are 3 or less vertices, the component must be isomorphic to P_1, P_2 or P_3. Now suppose our component consists of four or more vertices. We can reconstruct this component by starting with any vertex in the component and then repetitively adding vertices that share an edge with at least one of the vertices we have already added, and add any edges this new vertex shares with any of the vertices we already have. We now prove the result through induction. Suppose we have already added k vertices ($k \geq 4$) and the component is isomorphic to P_k. If we add a vertex and edge to one of the end vertices and this is all that is added, we get P_{k+1} and we are done. If we add an additional edge to a vertex not on one of the ends, we have the following look.



Here a is the color i or $i+1$, and b is the other color that a is not. Notice that if this b colored vertex in the middle (that the newly added a colored vertex on the end shares an edge with) must share an edge with three a colored vertices. If we delete all other vertices, we get the claw; hence this is not possible. So, the only other possibility is that it could share an edge with the other end vertex. But this is only possible if that vertex is colored differently, and hence the number of vertices is even, giving us C_{k+1}.

Now suppose the newly added vertex does not share an edge with a vertex on the end. If it shares an edge with a vertex in the middle, we get a claw again by the same logic as before. So, if we start with P_k, the only possibilities are that we get P_{k+1} or C_{k+1}, and in the case of C_{k+1}, $k+1$ must be even.

Now suppose that the component is isomorphic to C_k and k is even. If we were to try to add a new vertex colored a (again a is i or $i+1$ and b will represent the other color) to this, this new vertex must share an edge with some vertex colored b. But that b colored vertex already has edges with two other a colored vertices. Deleting everything except these four vertices will again give us the claw. Hence, the component must be C_k and the process terminates.

If G is a unit interval graph, it should be clear from Definition 1.2.8 that C_k is not possible. □

We will now define the crystal operators f_i and e_i that will act upon the proper colorings of a claw-free graph G.

Definition 3.2.5. Let G be a claw-free graph with vertex set V, with $|V| = n$ and whose vertices are labeled $1, \ldots, n$. Now, let x be a proper coloring of G. If we act on x with f_i or e_i, then we only consider the induced i-coloring of x. At this point, by Proposition 3.2.4, the induced subgraph can contain only connected components where the colors alternate between i and $i+1$

within each component and are isomorphic to P_k or C_k for some k. We will think of the *position* of a component as the value of the vertex that is the largest in magnitude in the component. We then order the components based upon the position of the component in increasing order. If the component consists entirely of a path of alternating i and $i+1$'s of even length, we disregard these, but often refer to this as an *even i-bracket*. When the graph is a unit interval graph, we may wish to refer to an even i-bracket more specifically based upon the color of the smallest-valued vertex. For example, an *i starting even i-bracket* or an *$i+1$ starting even i-bracket* depending upon whether it begins with an i or $i+1$ colored vertex. If the component is of odd length, and if the ends of the path graph are colored i, we label this a *right i-bracket*; if the ends of the path graph are colored $i+1$, we label this a *left i-bracket*. At this point, we pair any left i-bracket with a right i-bracket that is to the right of it and contains no unpaired left and right i-brackets in between them. Once all possible pairings have been made, the f_i operator will act on the rightmost unpaired right i-bracket, while the e_i operator will act on the leftmost unpaired left i-bracket. If there is no such unpaired bracket for the operator, then it sends x to 0. When the operator acts on a bracket, it simply changes i's to $i+1$'s and vice versa. Figure 3.1 shows an example of this process. For all our pictures involving colorings of graphs, the colors will be located directly above the vertices. Occasionally the vertices will be represented by their numbers, but it should still be easy to tell what numbers represent vertices because there will be edges going between them.

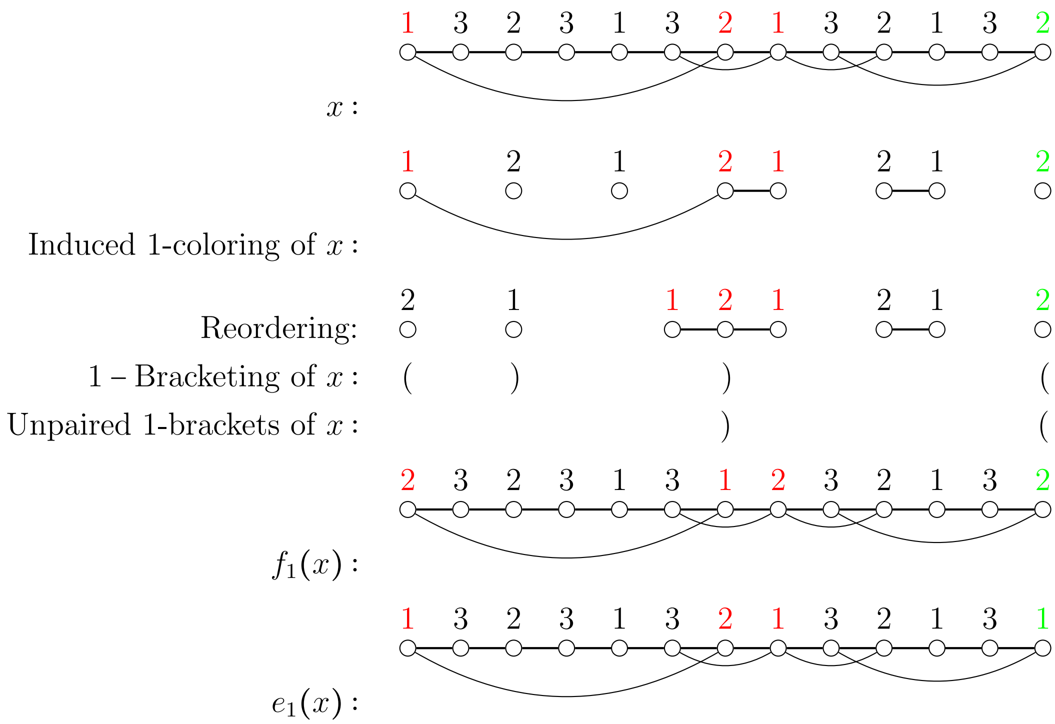


FIGURE 3.1. Notice that f_1 acts on the component whose colors are labeled in red, while e_1 acts on the component whose colors are labeled in green.

REMARK 3.2.6. When we want to refer to a coloring x without drawing the coloring out, we will simply write it in vector form, where the ith position of the vector contains the color of vertex i in the coloring x. For instance, in Figure 3.1, $x = [1, 3, 2, 3, 1, 3, 2, 1, 3, 2, 1, 3, 2]$.

REMARK 3.2.7. We note that when an f_i operator is applied to a coloring x, and $f_i x \neq 0$, then the rightmost unpaired right i-bracket that f_i acts on in x, becomes a left i-bracket. In particular, it becomes the leftmost unpaired left i-bracket of $f_i x$. Similarly, when we apply e_i to $f_i x$, the leftmost unpaired left i-bracket it acts upon becomes a right i-bracket–the rightmost unpaired right i-bracket we had in x. That is $e_i f_i x = x$ when $f_i x \neq 0$. Similarly $f_i e_i x = x$ when $e_i x \neq 0$. So, the e_i and f_i operators are partial inverses in the sense that they are inverses provided that applying the operator does not send the coloring to 0.

DEFINITION 3.2.8. We define the *string lengths* to be $\varphi_i(x) = \max\{k \in \mathbf{Z}_{\geq 0} | f_i^k \neq 0\}$ and $\varepsilon_i(x) = \max\{k \in \mathbf{Z}_{\geq 0} | e_i^k \neq 0\}$. Our string lengths will only take on finite non-negative integers. As a result, we say our crystal is of *finite type*. Any crystal whose string lengths are defined in the exact way that we have is said to be *seminormal*.

Definition 3.2.9. For a type A_{k-1} crystal of a claw-free graph, we define the weight of coloring x, $\mathrm{wt}(x)$, to be the vector of length k whose ith position contains the number of vertices of x colored i. For example, in type A_4, the weight of x in Example 3.1 is $\mathrm{wt}(x) = \langle 4, 4, 5, 0, 0 \rangle$. We define a *highest weight* to be a weight of a coloring x where $e_i x = 0$ for all $i \in I$.

Definition 3.2.10. Let $\mathcal{C}$ be a crystal. We define the *character* of $\mathcal{C}$ to be $\chi_{\mathcal{C}}(x) = \sum_{b \in \mathcal{C}} x^{wt(b)}$.

We will now show that our operators satisfy the crystal axioms **A1** and **A2**.

Proposition 3.2.11. *The crystal operators e_i and f_i we defined previously satisfy the crystal axioms **A1** and **A2**.*

Proof. Let x be the coloring of a claw-free graph. We first note that the vertices in the components of the induced i-coloring of x are fixed when applying e_i or f_i multiple times (assuming that it does not get sent to 0). This means that the positions of the components remain fixed when applying e_i or f_i, so that our crystal operators are well defined. By Remark 3.2.7, it clearly follows that if $x, y \in \mathcal{C}$, then $e_i(x) = y$ if and only if $f_i(y) = x$.

Let $y = e_i(x)$. When we apply e_i to x, we gain an i-colored vertex and lose an $i+1$-colored vertex. This is the same as saying $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$.

From Definition 3.2.8 and Definition 3.2.5, $\varepsilon_i(x)$ is equal to the number of unpaired left i-brackets in x and $\varphi_i(x)$ is equal to the number of unpaired right i-brackets in x. Since applying f_i to x decreases the number of unpaired right i-brackets by one and increases the number of unpaired left i-brackets by one, it follows that $\varepsilon_i(y) = \varepsilon_i(x) - 1$ and $\varphi_i(y) = \varphi_i(x) + 1$.

We now prove **A2** holds. Because $\alpha_i = \alpha_i^\vee$ (see Definition 3.2.1), it follows that the right hand side of the equation is equal to the number of vertices colored i minus the number of vertices colored $i+1$. On the left hand side of the equation, consider the i-bracketing. Every vertex colored i or $i+1$ participates in an even i-bracket, left i-bracket, or a right i-bracket. Since the number of vertices colored i minus the number of vertices colored $i+1$ is equal to -1 for a left i-bracket, 0 for an even i-bracket, and 1 for a right i-bracket, then the number of vertices colored i minus the number of vertices colored $i+1$ can be rephrased as the number of right i-brackets minus the number of left i-brackets. Additionally, since paired left and right i-brackets will cancel each other out, we can also rephrase it to be the number of unpaired right i-brackets minus the number of unpaired left i-brackets. Since $\varphi_i(x)$ refers to the number of unpaired right i-brackets in x and $\varepsilon_i(x)$ refers to the number of unpaired left i-brackets in x, the result follows. □

As noted in Section 3.1, in [**SW16**], Shareshian and Wachs showed that for unit interval graphs G, the chromatic quasisymmetric function of G has symmetric functions in x for each coefficient of powers of t, the ascent statistic. That is, it should be possible to have a crystal structure that preserves the number of ascents in each connected component of the crystal. We now show that our crystal operators are ascent-preserving on unit interval graphs.

PROPOSITION 3.2.12. *The crystal operators e_i and f_i on the set of colorings of a unit interval graph G are ascent preserving.*

PROOF. When f_i or e_i acts on a coloring x, it is only changing i's to $i+1$'s and vice versa. If we consider an edge between a vertex A colored i or $i+1$ in the i-bracket that f_i or e_i acts on, and a vertex B not in the i-bracket, we first note that the vertex B must be colored something other than i or $i+1$, otherwise it would be part of the i-bracket. Then it must be the case that changing the color of vertex A to i or $i+1$ does not change whether or not the edge between A and B is an ascent. So we need only consider whether the number of ascents within the i-bracket that e_i or f_i acts on changes. But regardless of whether it is a left i-bracket or a right i-bracket, an odd length i-bracket of length $2k+1$ for $k \in \mathbb{Z}_{\geq 0}$, has exactly k ascents within the i-bracket. This shows that e_i and f_i are ascent preserving. □

DEFINITION 3.2.13. If $\mathcal{C}$ is a crystal and we want to visualize it, we can do so by creating a directed graph called a *crystal graph of $\mathcal{C}$*. The crystal graph of $\mathcal{C}$ has vertices in $\mathcal{C}$ and edges labeled by some $i \in I$, where if $f_i x = y$ for some $x, y \in \mathcal{C}$, then we draw an edge labeled i from x to y.

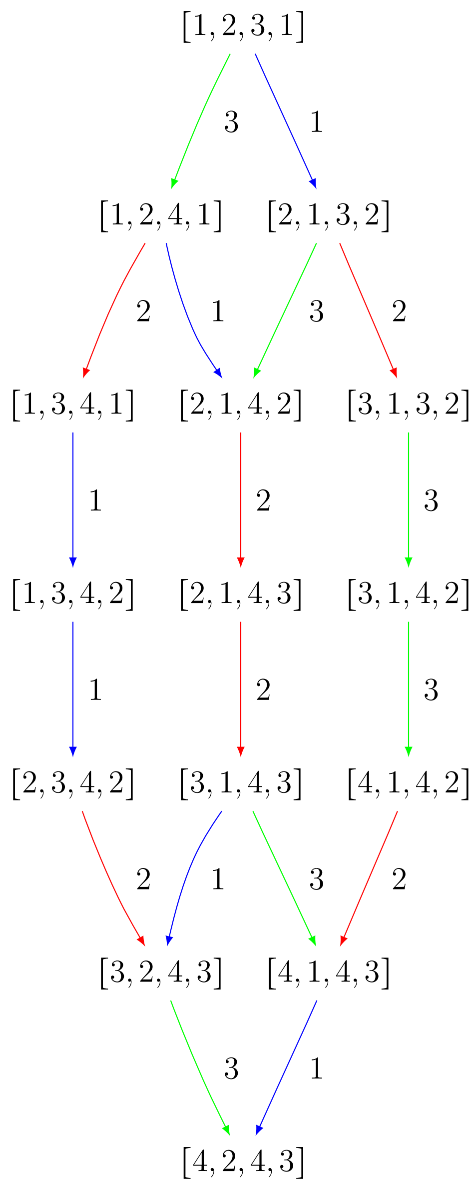
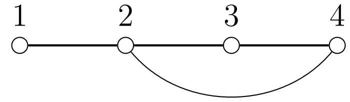


FIGURE 3.2. The crystal of highest weight $[1, 2, 3, 1]$ using 4 colors for the graph G in Example 3.2.14

EXAMPLE 3.2.14. Let G be the graph shown below.



Then the component with highest weight color $[1, 2, 3, 1]$ of the crystal graph of G using $k = 4$ colors is shown in Figure 3.2.

3.2.2. P-array bijection. In this section, we will define the crystal structure on P-arrays that Ehrhard [**Ehr22**] introduced and show that our crystal, when restricted to (3+1)-free incomparability graphs, is equivalent to that of Ehrhard's.

Definition 3.2.15. Let $\mathcal{B}$ and $\mathcal{C}$ be two crystals associated to the root system Φ and index set I. A *crystal morphism* is a map $\psi : \mathcal{B} \to \mathcal{C} \sqcup \{0\}$ such that

(1) if $b \in \mathcal{B}$ and $\psi(b) \in \mathcal{C}$, then
   (a) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$,
   (b) $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ for all $i \in I$, and
   (c) $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $i \in I$;
(2) if $b, e_i b \in \mathcal{B}$ such that $\psi(b), \psi(e_i b) \in \mathcal{C}$, then we have $\psi(e_i b) = e_i \psi(b)$;
(3) if $b, f_i b \in \mathcal{B}$ such that $\psi(b), \psi(f_i b) \in \mathcal{C}$, then we have $\psi(f_i b) = f_i \psi(b)$.

A crystal morphism $\psi : \mathcal{B} \to \mathcal{C} \sqcup \{0\}$ is called a *crystal isomorphism* if the induced map $\psi : \mathcal{B} \sqcup \{0\} \to \mathcal{C} \sqcup \{0\}$ with $\psi(0) = 0$ is a bijection.

In this section, we will refer to three pairs of crystal operators. The first pair is the one we have defined already and we will refer to these as f_i^C and e_i^C (the C refers to the fact that these are crystal operators on claw-free graphs). The second set of operators, which we refer to as f_i^R and e_i^R, is isomorphic to the first and is only introduced because it will make the bijection with Ehrhard's operators easier to establish. We introduce these operators now.

Definition 3.2.16. Let x be the coloring of a claw-free graph G. We define the crystal operators f_i^R and e_i^R, similar to those in Definition 3.2.5. The differences are that these operators will label brackets in the exact opposite way, i.e. left brackets will be right brackets and vice versa, and the position of the bracket will be based upon the vertex number that is least in value in the bracket. f_i^R will now act on the leftmost unpaired left i-bracket in the induced i coloring of x, and e_i^R will now act on the rightmost unpaired right i-bracket in the induced i coloring of x.

Proposition 3.2.17. *The crystal operators e_i^R and f_i^R defined on claw-free graphs satisfy the crystal axioms.*

We omit the proof as it is essentially the same as in Proposition 3.2.11.

Proposition 3.2.18. *Let G be a claw-free graph G and let $\mathcal{C}^C$ be the crystal defined by the operators e_i^C and f_i^C on the colorings of G. Let G^R be the reverse graph of G (the graph that relabels vertices so that vertex i is now relabeled vertex $n-i+1$ for each i). Now let $\mathcal{C}^R$ be the crystal defined by the operators e_i^R and f_i^R on the colorings of G^R. Then $\mathcal{C}^C$ and $\mathcal{C}^R$ are isomorphic as crystals.*

We omit this proof as it is straightforward to verify this map is bijective, weight preserving, and also preserves the bracketing.

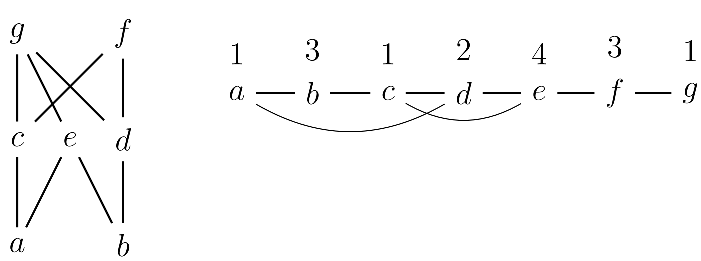
Now we will review Erhard's crystal structure in [**Ehr22**] with examples.

Definition 3.2.19. Let P be a finite poset. A *P-array* is an indexing $\{A_{i,j}\}$ of the elements of P such that if $A_{i,j}$ is defined with $j > 1$, then $A_{i,j-1}$ is also defined and $A_{i,j-1} <_P A_{i,j}$. Notice that this implies that the elements of P in row i must form a chain in the poset. Let $\mathcal{A}_P$ denote the set of P-arrays.

Now a P-array can be thought of as a way of encoding a coloring of $\mathrm{inc}(P)$.

Definition 3.2.20. Fix a poset P, let $G = \mathrm{inc}(P)$ be its incomparability graph, and let A be any P-array. We will denote by x_A a coloring of G that corresponds to A in the following way. If $v \in P$ is in the ith row of A, then vertex v of G is colored i in the coloring x_A. Let $\mathcal{X}_P$ denote the set of all proper colorings of the graph G using only natural numbers for colors. When P is $(3+1)$-free, we will define the map $\psi : \mathcal{X}_G \to \mathcal{A}_P$ by $\psi(x_A) = A$.

Example 3.2.21. On the left is a poset P. In the middle, we have a coloring of $\mathrm{inc}(P)$. On the right is a P-array A corresponding to the coloring of $\mathrm{inc}(P)$.



a	c	g
d		
b	f	
e		

Proposition 3.2.22. *The map ψ in Definition 3.2.20 is a well defined bijection.*

Proof. Let x be a coloring of G. If vertex v of G is colored i, we place it in the ith row of the P-array. Since any two vertices colored i in G can not share an edge, this implies all vertices placed in row i of the corresponding P-array $\psi(x)$ must be comparable, so we order all vertices in the P-array in ascending order as determined by relations in P. This ordering is unique, hence we get a unique P-array, so that ψ is well defined.

The inverse map $\psi^{-1} : \mathcal{A}_P \to \mathcal{X}_G$ is given by $\psi^{-1}(A) = x_A$. Here, if $v \in P$ is in row i of A, then we color vertex v of G with i. Since we have specified that the colorings are of the natural numbers only, then the number of the row coincides, so that we have a bijection. □

We define the *weight* of a P-array A, denoted $\mathrm{wt}(A)$, to be the weak composition whose ith part is the number of elements in row i. Notice that the weight of A and the weight of x_A coincide.

Proposition 3.2.23. *Let P be a finite $(3+1)$-free poset. Then for each $x_A \in \mathcal{X}_P$ and each $A \in \mathcal{A}_P$ we have $wt(\psi(x_A)) = wt(A)$.*

So we can think of a poset as actually just a graph, namely $G = \mathrm{inc}(P)$ and the P-arrays can actually just be thought of as colorings of G. Making this association early will help the reader understand the crystal isomorphism that we will describe later between the crystal operators acting on colorings of (3+1)-free incomparabilty graphs and Ehrhard's crystal operators acting on P-arrays of (3+1)-free posets. To this end, we now describe Ehrhard's crystal operators. To avoid being repetitive for the rest of this section, we will always take P as a finite (3+1)-free poset.

Definition 3.2.24. [**Ehr22**] Let A be a P-array with $r \geq 1$. Let the chains $a_1 <_P \cdots <_P a_m$ and $b_1 <_P \cdots <_P b_n$ be the elements of rows r and $r+1$ of A respectively. Let $C : \{a_1, \ldots, a_m, b_1, \ldots, b_n\} \to \mathbb{Z}_+$ be the function inductively defined so that $C(b_k) = k$ for each $1 \leq k \leq n$, and

$$C(a_k) = \max(\{C(b_i) | b_i <_P a_k\} \cup \{C(a_I) | 1 \leq i < k\}) + 1.$$

Then we define the r *pre-alignment* of A to be the map

$$\{a_1, \ldots, a_m, b_1, \ldots, b_n\} \to \{r, r+1\} \times \mathbb{Z}_+$$

such that each a_k maps to $(r, C(a_k))$, and each b_k to $(r+1, C(b_k))$.

Definition 3.2.25. [Ehr22] Let A be a P-array with $r \geq 1$. Let the chains $a_1 <_P \cdots <_P a_m$ and $b_1 <_P \cdots <_P b_n$ be the elements of rows r and $r+1$ of A respectively. Let ϕ_0 be the r pre-alignment. We construct the r *alignment* of A as follows.

Suppose we have some $\phi_k : \{a_1, \dots, a_m, b_1, \dots, b_n\} \to \{r, r+1\} \times \mathbb{Z}_+$. Select the rightmost element x mapped to some (i, c) such that column $c+1$ of ϕ_k is nonempty and contains no $y >_P x$, if such an x exists. Then we define

$$\phi_{k+1} : \{a_1, \dots, a_m, b_1, \dots, b_n\} \to \{r, r+1\} \times \mathbb{Z}_+$$

so that $\phi_{k+1}(x) = (i, c+1)$ and ϕ_{k+1} coincides with ϕ_k elsewhere. If no such x exists, then the r alignment of A is defined to be ϕ_k.

Definition 3.2.26. [Ehr22] The *P-array crystal lowering operator* $f_r^P : \mathcal{A}_P \to \mathcal{A}_P \cup \{0\}$ acts on $A \in \mathcal{A}_P$ as follows. Let $a_1 <_P \cdots <_P a_m$ and $b_1 <_P \cdots <_P b_n$ be the entries of rows r and $r+1$ of A respectively.

- If every column of the r alignment with an entry in row r also contains an entry in row $r+1$, then define $f_r(A) = 0$.
- Otherwise, let p be minimal such that a_p does not share a column with an element in row $r+1$ in the r alignment. Let $t \geq 0$ be minimal such that there is $b_i >_P a_{p+t}$ one column right of a_{p+t} or there is no b_i one column right of a_{p+t}. Then we move $a_p, \dots, a_{p+t}$ to row $r+1$, and any b_i that shares a column with one of these entries to row r.

In this section, we will refer to the set of elements $a_p, \dots, a_{p+t}$ as well as any b_i that shares a column with one of these entries as the *leftmost unpaired left r-bracket of A* to help facilitate the correspondence between these crystal operators and the ones defined in Definition 3.2.16. If p is not necessarily chosen to be minimal, then we refer to this set of entries as an *unpaired left r-bracket of A*.

The *P-array crystal raising operator* $e_r^P : \mathcal{A}_P \to \mathcal{A}_P \cup \{0\}$ acts on $A \in \mathcal{A}_P$ as follows. Let $a_1 <_P \cdots <_P a_m$ and $b_1 <_P \cdots <_P b_n$ be the entries of rows r and $r+1$ of A respectively.

- If every column of the r alignment with an entry in row $r+1$ also contains an entry in row r, then define $e_r(A) = 0$.

- Otherwise, let p be maximal such that b_p does not share a column with an element in row r in the r alignment. Let $t \geq 0$ be minimal such that there is $a_i >_P b_{p+t}$ one column right of b_{p+t}, or there is no a_i one column right of b_{p+t}. Then we move $b_p, \ldots, b_{p+t}$ to row r, and any a_i that shares a column with one of these entries to row $r+1$.

We will refer to the set of elements $b_p, \ldots, b_{p+t}$ as well as any a_i that shares a column with one of these entries as the *rightmost unpaired right r-bracket of A*. Again if p is not necessarily chosen to be maximal, then we refer to this set of entries as an *unpaired right r-bracket of A*.

Figure 3.3 shows an example of the 1 pre-alignment of a P-array A as well as the 1 alignment of A, $f_1^P A$, $e_1^P A$, and the associated coloring x_A.

Our goal now will be to show that there is a crystal isomorphism between the crystal structure $\mathcal{C}^P$ on P-arrays given by the crystal operators f_i^P and e_i^P for some finite $(3+1)$-free poset P and the crystal $\mathcal{C}^R$ on the colorings of the graph $G = \text{inc}(P)$ (with a particular relabeling that we will soon describe) given by the crystal operators f_i^R and e_i^R.

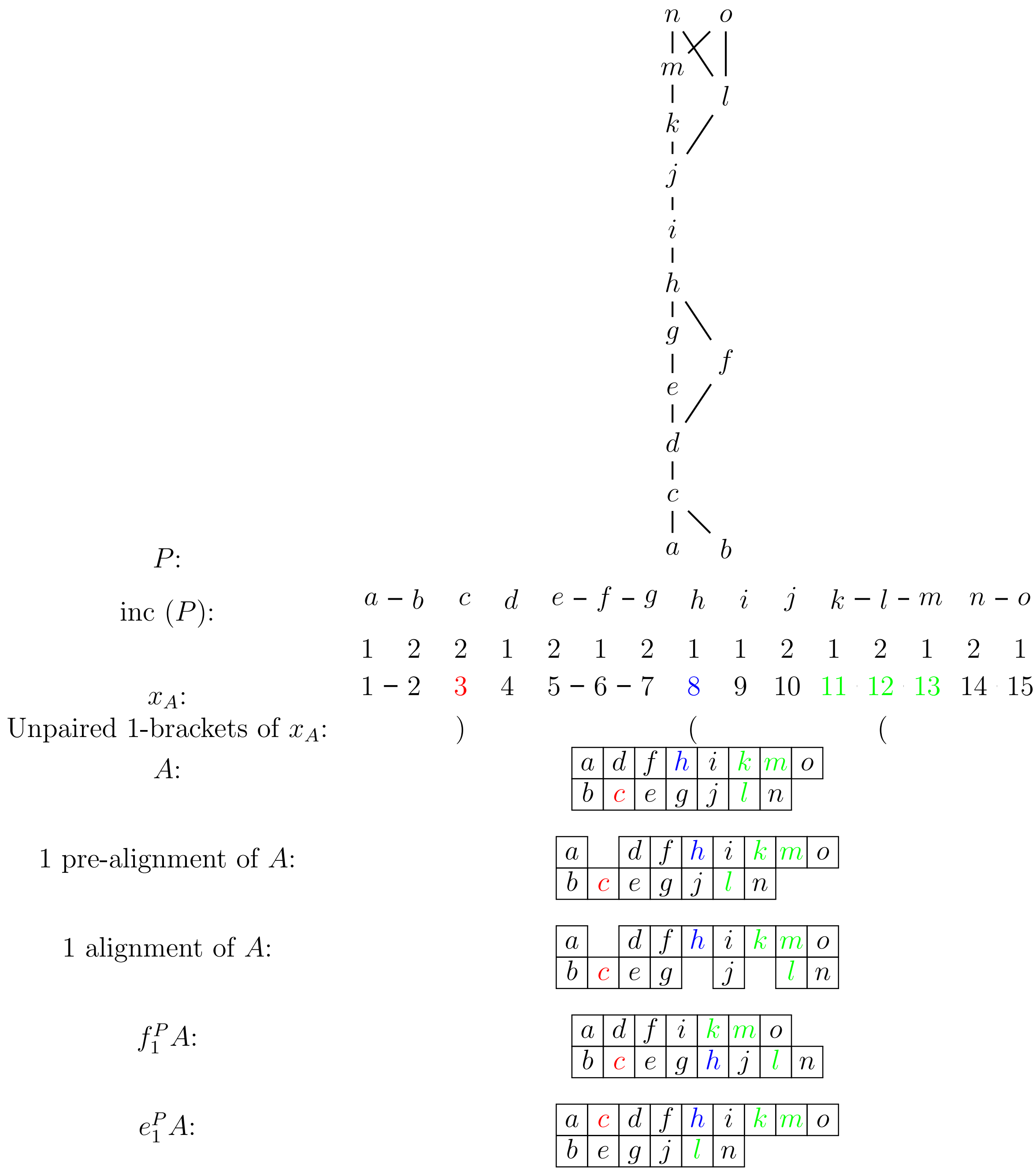


FIGURE 3.3. For a poset P, an example of a P-array A and corresponding coloring x_A associated through the map ψ. Notice how the unpaired 1-brackets of x_A can be used to exactly determine the 1 alignment of A.

REMARK 3.2.27. Let P be a $(3+1)$-free poset with n elements and let $G = \text{inc}(P)$ be its incomparability graph. In order for e_i^R and f_i^R to act on the colorings of G, we need to relabel the vertices of G 1 through n. We do so in the following way. For any $a, b \in P$, let $n_a, n_b \in [n]$ be the number labels of a and b in the relabeling of G. We need only require that if $a <_P b$, then we need $n_a < n_b$. One might wonder why this is the only condition required. For the crystal structure $\mathcal{C}^R$, when we consider an induced i-coloring of G, the components are disjoint. The only thing that a labeling could change in the way the crystal operators act is the position of a left or right bracket in the bracketing phase. However, if we compare two components, every vertex in one component can not share an edge with any vertex in the other component. This implies that every vertex in a component is comparable to every vertex in another component (recall two vertices share an edge if and only if they are incomparable in the poset). Requiring that if $a <_P b$, then $n_a < n_b$ is all we need for the bracketing to be well defined. This means that when we are confined to claw-free incomparability graphs, the position of the bracket could be determined by any vertex in the bracket, and we would still end up with the exact same bracketing. We summarize this in Lemma 3.2.28.

LEMMA 3.2.28. *In the relabeling of $G = inc(P)$, any vertices n_a, n_b, with $n_a < n_b$ where vertices n_a and n_b appear in different components of the induced i-coloring of x_A, then we must have $a <_P b$.*

Next we show that unpaired left and right brackets of $\mathcal{C}^R$ and $\mathcal{C}^P$ coincide through correspondence of x_A with A.

PROPOSITION 3.2.29. *Let $A \in \mathcal{A}_P$, $r \in I$, and let x_A be the corresponding coloring of $G = inc(P)$ with relabeling. Now suppose $e_r^R x_A \neq 0$. If an unpaired right r-bracket of the induced r-coloring of x_A is the $2k+1$ vertices $n_{a_i}, n_{a_{i+1}}, \ldots n_{a_{i+k-1}}, n_{b_j}, n_{b_{j+1}}, \ldots, n_{b_{j+k}}$, where the a's are colored r and the b's are colored $r+1$, then $e_r^P A \neq 0$ and the elements of A corresponding to these vertices, $a_i, a_{i+1}, \ldots a_{i+k-1}, b_j, b_{j+1}, \ldots, b_{j+k}$, are exactly the elements of an unpaired right r-bracket of A.*

PROOF. First, suppose $i > j$. Then it follows from Lemma 3.2.28 that there are more r colored vertices than $r+1$ colored vertices in components left of the right r-bracket in the induced r-coloring of x_A, so that there must be an unpaired left r-bracket left of the unpaired right r-bracket, a contradiction. Hence $i \leq j$.

We have two scenarios. If a_{i-1} exists, let's first assume that a_{i-1} is in a column left of column j in the r prealignment. Then, the r prealignment of A for columns j through $j+k$ has the following look.

a_i	$\cdots$	a_{i+k-1}	
b_j	$\cdots$	b_{j+k-1}	b_{j+k}

This follows because by Lemma 3.2.28, if b_{j-1} exists, we know $a_i >_P b_{j-1}$ and if a_{i+k} exists, we must have $a_{i+k} >_P b_{j+k}$. We also have that a_{i+m} is incomparable to b_{j+m} and b_{j+1+m} for all $0 \le m < k$, due to the shared edges in the graph. This explains why $a_i, \ldots, a_{i+k-1}$ appear in columns $j+1$ through $j+k$ of the r prealignment of A. Next, because of the incomparability we just mentioned and the fact that $a_{i+m} <_P b_{j+m+2}$ for all $0 \le m \le k-1$ (not including $k-1$ if b_{j+k+1} doesn't exist), the r alignment of A for columns j through $j+k$ has the following look.

	a_i	$\cdots$	a_{i+k-1}
b_j	b_{j+1}	$\cdots$	b_{j+k}

Hence, they form an unpaired right r-bracket of A. Now suppose that a_{i-1} is not in a column left of column j in the r prealignment. Then if we look left of entry a_{i-1}, there must be at least one empty column entry in row r. Choose the first that occurs when scanning from the a_{i-1} entry leftward. Suppose this occurs in column l, with $a_{i'}$ being the entry in row r in column $l+1$. Then we have the following look at that area.

	$a_{i'}$
b_l	b_{l+1}

Now we want to be able to say that every entry in a column left of and including column l in the r prealignment is in a component (of the induced r coloring of x_A) left of any component that contains an entry in a column right of column l in the r prealignment. By Lemma 3.2.28, we need only prove that $a_{i'} >_P a_q$ and $b_{l+1} >_P a_q$ for all $1 \le q < i'$ and $a_{i'} >_P b_t$ and $b_{l+1} >_P b_t$ for all $1 \le t \le l$. Since $a_{i'} >_P b_l$ and $a_{i'} >_P a_{i'-1}$ (assuming $a_{i'-1}$ exists), then it clearly follows that $a_{i'} >_P a_q$ for all $1 \le q < i'$ and $a_{i'} >_P b_t$ for all $1 \le t \le l$. It is also clear that $b_{l+1} >_P b_t$ for all $1 \le t \le l$. So we need only prove that if $a_{i'-1}$ exists, then $b_{l+1} >_P a_{i'-1}$ so that we will have $b_{l+1} >_P a_q$ for all $1 \le q < i'$. If $a_{i'-1}$ exists, then b_{l-1} must also exist. Now row r column l being open in the r prealignment means that $a_{i'-1} \not>_P b_{l-1}$, and hence $a_{i'-1} \not>_P b_{l+1}$ but if $a_{i'-1} \not<_P b_{l+1}$, the only other possibility would be that it is incomparable to b_{l-1}, b_l and b_{l+1}. Since these three form a chain, this is a contradiction since P is $(3+1)$-free.

Now this means that in the components of the induced r coloring of x_A that are right of the components that contain $a_{i'-1}$ (if it exists) and b_l and left of the unpaired right r-bracket that contains $n_{a_i}, n_{a_{i+1}}, \dots n_{a_{i+k-1}}, n_{b_j}, n_{b_{j+1}}, \dots, n_{b_{j+k}}$, we must have $i - i'$ vertices colored r and $j - l - 1$ vertices colored $r+1$. But by the fact that a_{i-1} is not in a column left of column j and the choice of $a_{i'}$, it follows that $i - i' \geq j - l - 1$. If $i - i' > j - l - 1$, then there must be an unpaired left r-bracket that is left of our unpaired right r-bracket, a contradiction. So it must be the case that $i - i' = j - l - 1$. Then the r alignment of A for columns j through $j+k$ has the following look.

a_{i-1}	a_i	$\cdots$	a_{i+k-1}
b_j	b_{j+1}	$\cdots$	b_{j+k}

Now it follows that $a_{i-1} >_P a_q$ and $b_j >_P a_q$ for all $1 \leq q < i-1$ and $a_{i-1} >_P b_t$ and $b_j >_P b_t$ for all $1 \leq t < j$. So all entries in columns left of j are in components of the induced r coloring of x_A that are left of components that contain any of the elements shown in columns j through $j+k$ of the r prealignment. But we know that $a_{i-1} <_P a_i$ and $a_{i-1} <_P b_j$ because it must be in a component left of our unpaired left r-bracket in the induced r coloring of x_A. This implies $n_{a_{i-1}}$ is a left r-bracket immediately left of our unpaired right r-bracket of x_A, a contradiction. □

PROPOSITION 3.2.30. *Let $A \in \mathcal{A}_P$, $r \in I$, and let x_A be the corresponding coloring of $G = inc(P)$ with relabeling. Now suppose $f_r^R x_A \neq 0$. If an unpaired left r-bracket of the induced r coloring of x_A is the $2k+1$ vertices $n_{a_i}, n_{a_{i+1}}, \dots n_{a_{i+k}}, n_{b_j}, n_{b_{j+1}}, \dots, n_{b_{j+k-1}}$, where the a's are colored i and the b's are colored $i+1$, then $f_r^P A \neq 0$ and the elements of A corresponding to these vertices, $a_i, a_{i+1}, \dots a_{i+k}, b_j, b_{j+1}, \dots, b_{j+k-1}$, are exactly the elements of an unpaired left r-bracket of A.*

PROOF. Suppose $i \leq j$ and element a_{i-1} appears in a column left of column j in the r prealignment of A. Then by the same arguments made in Proposition 3.2.29, we have that the r pre alignment of A has the following look for columns j through $j+k$.

a_i	$\cdots$	a_{i+k-1}	a_{i+k}
b_j	$\cdots$	b_{j+k-1}	b_{j+k}

Note that b_{j+k} may not exist, but if it does, a_{i+k+1} must exist and we must not have $a_{i+k+1} >_P b_{j+k}$. If b_{j+k} exists and there is no a_{i+k+1}, then there are more $r+1$ colored vertices in x_A after this unpaired left r-bracket of x_A, meaning that there is an unpaired right r-bracket right of it, a contradiction. Similarly, if b_{j+k} exists but $a_{i+k+1} >_P b_{j+k}$, then $n_{b_{j+k}}$ is an $r+1$ colored vertex in a component by itself in the induced r coloring of x_A that is immediately right of our unpaired left r-bracket of x_A, a contradiction. Hence the component that is immediately right of our unpaired left r-bracket of x_A must be an even bracket or another left r-bracket of x_A, so that in the r alignment phase of A, we are guaranteed to have at some point b_{j+k} move a column right so we have the following look.

a_i	$\cdots$	a_{i+k-1}	a_{i+k}
b_j	$\cdots$	b_{j+k-1}	

Now because the vertices $n_{a_i}, n_{a_{i+1}}, \ldots n_{a_{i+k}}, n_{b_j}, n_{b_{j+1}}, \ldots, n_{b_{j+k-1}}$ are in an unpaired left r-bracket of x_A, then it follows that b_{j+m} is incomparable to a_{i+m+1} for $0 \le m < k$. Hence it follows that after the r alignment phase we have the following look.

a_i	$\cdots$	a_{i+k-1}	a_{i+k}
	b_j	$\cdots$	b_{j+k-1}

So that it is easy to verify that these elements form an unpaired left r-bracket of A.

Now suppose that a_{i-1} is not in a column left of column j in the r prealignment. Then if we look left of entry a_{i-1}, there must be at least one empty column entry in row r. Choose the first that occurs when scanning from the a_{i-1} entry leftward. Suppose this occurs in column l, with $a_{i'}$ being the entry in row r in column $l+1$. Then we have the following look at that area.

	$a_{i'}$
b_l	b_{l+1}

Now we must have $l < j$ because this state of the r prealignment implies $a_{i'} >_P b_l$ and $i' < i$ and we know $b_{j+m} >_P a_q$ for any $q < i$ and $0 \le m < k$. Now this means that b_{j+m} can move up to the column that contains a_{i+m+1} in the r alignment phase as long as the b_q with $q \ge j+k$ can move past the column containing a_{i+k}. Suppose this isn't possible. We know that $b_q >_P a_{i+m}$ for $q \ge j+k$ and $0 \le m \le k$, so if it is not possible it is because there is either an empty column in row r after some element a_{i+k+p} or there exists some elements $a_{i+k+p+1} >_P b_{j+k+q}$ with $0 \le q < p$. So that we have the following look for the a_{i+k} column and those until the empty space or the existence of $a_{i+k+p+1}$ as defined.

a_{i+k}	$\cdots$	a_{i+k+p}	$a_{i+k+p+1}$
b_{j+k+q}	$\cdots$	$b_{j+k+q+p}$	

Now all elements in columns right of the a_{i+k+p} element correspond to vertices contained in components strictly right of the component that contains our unpaired left r-bracket of x_A. This also means that the vertices corresponding to the elements $\{a_{i+k+1}, \ldots, a_{i+k+p}, b_{j+k}, \cdots, b_{j+k+q+p}\}$ occur in components immediately right of the component that contains our unpaired left r-bracket of x_A. But this means we have more $r+1$ colored vertices than r colored vertices in this set of components, so that there must be at least one unpaired right r-bracket following our unpaired left r-bracket of x_A, a contradiction. So it must be the case that we can slide all b_{j+m} with $0 \le m < k$ into the desired columns, so that the unpaired left r-bracket of x_A does correspond again to an unpaired left r-bracket of A.

Now suppose $i > j$. If we repeat the argument noting that we may not have an empty column left of a_i in row r (but this doesn't stop the proof from working) we again get that the unpaired left r-brackets correspond. □

Before we can do the next proposition, we will need the following new definition.

Definition 3.2.31. Define a *maximal set of paired r-brackets* to be all of the vertices of x_A in any set of components where the set includes every component in the induced r-coloring of x_A in between two consecutive unpaired r-brackets, or every component that is left of the first unpaired r-bracket, or every component that is right of the last unpaired r-bracket.

EXAMPLE 3.2.32. In Figure 3.3, there are four maximal sets. Vertices 1 and 2 form the first set, then the second is vertices 4, 5, 6, and 7, the third set is vertices 9 and 10, and the fourth set is vertices 14 and 15.

LEMMA 3.2.33. *Let $n_{a_i}, n_{a_{i+1}}, \ldots, n_{a_{i+k}}$ be the vertices colored r and let $n_{b_j}, n_{b_{j+1}}, \ldots, n_{b_{j+k}}$ be the vertices colored $r+1$ in a maximal set of paired r-brackets for some coloring x_A. Then the maximal set has the following property*

$$a_{i+m} \not>_P b_{j+m} \text{ for all } 0 \leq m \leq k.$$

PROOF. By the definition of maximal set, if we have a pair of left and right r-brackets, it must be the case that the left r-bracket must occur before the right r-bracket it is paired with. Hence if $a_{i+m} >_P b_{j+m}$ for some m, then we know vertex $n_{a_{i+m}}$ occurs in a component that is right of $n_{b_{j+m}}$. But this implies that in the maximal set we have more $r+1$ colored vertices than r colored vertices left of the $n_{a_{i+m}}$ vertex's component (call this component Y) within the maximal set. And that would imply we have at least one more right r-bracket then left r-bracket that is left of component Y within the maximal set, a contradiction. □

PROPOSITION 3.2.34. *The corresponding elements of A in any maximal set of paired r-brackets for a coloring x_A must appear as a block of consecutive fully filled columns in the r alignment of A. That is, if the elements are $a_i, \ldots, a_{i+k}, b_j, \ldots, b_{j+k}$, then we must have the corresponding look in the r alignment of A.*

a_i	$\cdots$	a_{i+k}
b_j	$\cdots$	b_{j+k}

PROOF. Suppose $i \leq j$. Then because $a_i >_P b_{j-1}$ (if b_{j-1} exists), then it is clear that a_i is in a column whose value is at least j in the r prealignment. By Lemma 3.2.33, we know that $a_{i+m} \not>_P b_{j+m}$ for all $0 \leq m \leq k$. Hence the corresponding look shown above is exactly the look of columns j through $j+k$ in the r prealignment of A. Lastly, since $b_{j+k} <_P a_{i+k+1}, b_{j+k+1}$ should either exist, then it follows no b_{j+m} can move rightward in the r alignment phase for any $0 \leq m \leq k$.

Now suppose $i > j$. Now since any vertex n_{a_q} with $q < i$ is in a component left of every component in the maximal set, it follows that $a_q <_P b_{j+m}$ for all $0 \le m \le k$. Again recall, $b_{j+k} <_P a_{i+k+1}, b_{j+k+1}$ should either exist, so in the r alignment phase b_{j+k} can only move at most to the column that contains a_{i+k}. Lemma 3.2.33 implies that $a_q \not>_P b_{j+m}$ for any $0 \le m \le k$ and $1 \le q \le i+m$. Hence we know that b_{j+k} can move to the column with a_{i+k} and no further. And because b_{j+k-1} can move to the column of a_{i+k-1}, but no further because b_{j+k} is in the next column, the process repeats so that we get the r alignment of A as pictured.

There is only two things that could stop this process. The first is if there was no element in row r in a column right of a b_{j+m} at some point in the r alignment phase before it achieved its position in a column with a_{i+m}. This is a contradiction though. The only way an a_q can have an empty column before it after the r prealignment, is if a_q was greater in the poset than the element in row $r+1$ that was in a column before it, but $b_{j+m} >_P a_q$ for all $q < i$ so this couldn't have happened. And because $a_i, \ldots, a_{i+k}$ can't move in the r alignment phase ($a_{i+k} <_P a_{i+k+1}, b_{j+k+1}$ should either exist) we can't have any empty columns before a_i be created in the r alignment phase that would prevent any of the b_{j+m}'s from moving.

The second case is that there is an empty column in row r or $r+1$ that occurs after a_{i+k} so that we can't move entries in the alignment phase to get the desired picture. Then after the alignment phase, consider the new block of columns that aren't participating in an unpaired left or right r-bracket of A, where $q, l \ge 1$ and $p \ge 0$.

a_{i-p-q}	$\cdots$	a_{i-q}	$\cdots$	a_i	$\cdots$	a_{i+k}	$\cdots$	a_{i+k+l}
b_{j-p}	$\cdots$	b_j	$\cdots$	b_{j+q}	$\cdots$	b_{j+q+k}	$\cdots$	$b_{j+q+k+l}$

Since we choose the set to be maximal, this implies there are some vertices of the set

$$\{n_{a_{i-p-q}}, \ldots, n_{a_{i-1}}, n_{a_{i+k+1}}, \ldots, n_{a_{i+k+l}}, n_{b_{j-p}}, \ldots, n_{b_{j-1}}, n_{b_{j+k+1}}, \ldots, n_{b_{j+q+k+l}}\}$$

That are in unpaired left or right i-brackets in x_A and by Proposition 3.2.29 and Proposition 3.2.30, we must have also as unpaired left or right i-brackets in A, but this is not the case, a contradiction. ∎

Proposition 3.2.35. *Let P be a finite $(3+1)$-free poset and let $G = inc(P)$ be its incomparability graph, with relabeling according to what is specified in Remark 3.2.27. Let $A \in \mathcal{A}_P$ be a P-array, let x_A be the coloring of G corresponding to A and let I be the index set of $\mathcal{C}^P$ and $\mathcal{C}^R$. Then we have*

(1) $\varepsilon_i^R(x_A) = \varepsilon_i^P(A)$ for all $i \in I$,

(2) $\varphi_i^R(x_A) = \varphi_i^P(A)$ for all $i \in I$,

(3) If $f_i^R x_A \neq 0$, let $B = f_i^P A$. Then $f_i^R x_A = x_B$, and

(4) If $e_i^R x_A \neq 0$, let $B = e_i^P A$. Then $e_i^R x_A = x_B$.

Proof. By Proposition 3.2.29, for every unpaired right i-bracket of x_A, we have a corresponding unpaired right i-bracket of A. By Proposition 3.2.30, for every unpaired left i-bracket of x_A, we have a corresponding unpaired left i-bracket of A. Lastly, by Proposition 3.2.34, we know that since every vertex of the induced i coloring of x_A is contained in an unpaired left i-bracket, an unpaired right i-bracket, or a maximal set of paired i-brackets, then all of the elements of rows i and $i+1$ of the i alignment of A are accounted for and we can definitively say the number of unpaired left i-brackets of A and x_A are equal and the number of unpaired right i-brackets of A and x_A are equal. So it follows that $\varepsilon_i^R(x_A) = \varepsilon_i^P(A)$ and $\varphi_i^R(x_A) = \varphi_i^P(A)$ for all $i \in I$.

Let $f_i^R x_A \neq 0$, and let $B = f_i^P A$. Then by Proposition 3.2.30 and (2), since the unpaired left i-brackets of A and x_A coincide, we have $f_i^R x_A = x_B$. Now let $e_i^R x_A \neq 0$, and let $B = e_i^P A$. Then by Proposition 3.2.29 and (1), since the unpaired right i-brackets of A and x_A coincide, we have $e_i^R x_A = x_B$. □

Theorem 3.2.36. *$\mathcal{C}^P$ and $\mathcal{C}^R$ are isomorphic as crystals.*

Proof. Let P be a finite $(3+1)$-free poset and let $G = \text{inc}(P)$ be its incomparability graph, with relabeling according to what is specified in Remark 3.2.27. Let $A \in \mathcal{A}_P$ be a P-array and let x_A be the coloring of G corresponding to A. Let ψ be the bijective map from Definition 3.2.19. We need only show that ψ satisfies the criteria of Definition 3.2.15.

By Proposition 3.2.23, we know that ψ is weight preserving. By Proposition 3.2.35 (1) and (2), we know that ψ preserves the string lengths.

Suppose $e_i^R x_A \neq 0$. Then let $x_B = e_i^R x_A$. By Proposition 3.2.35 (4), we have

$$\psi(e_i^R x_A) = \psi(x_B) = B = e_i^P A = e_i^P \psi(x_A)$$

Similarly, suppose $f_i^R x_A \neq 0$. Then let $B = f_i^P A$. By Proposition 3.2.35 (3), we have

$$\psi(f_i^R x_A) = \psi(x_B) = B = f_i^P A = f_i^P \psi(x_A)$$

This shows that the bijective map ψ satisfies the criteria of Definition 3.2.15, hence $\mathcal{C}^P$ and $\mathcal{C}^R$ are isomorphic as crystals. □

Since Proposition 3.2.18 shows $\mathcal{C}^R$ and $\mathcal{C}^C$ are isomorphic as crystals, we immediately have the following corollary.

COROLLARY 3.2.37. *$\mathcal{C}^P$ and $\mathcal{C}^C$ are isomorphic as crystals.*

3.3. Stembridge Crystals

In this section, we will review the Stembridge axioms. We will then show that the crystal operators defined in Section 3.2.1 do not satisfy all of the Stembridge axioms.

DEFINITION 3.3.1. A *simply laced* root system is a root system where all the roots have the same length, hence a type A_{k-1} root system is a simply laced root system. A finite type, seminormal crystal $\mathcal{C}$ with a simply laced root system that satisfies the Stembridge axioms is a *Stembridge crystal*.

Before we give the definition of the Stembridge axioms, we will provide some motivation for why we want a crystal with a simply laced root system to satisfy these axioms. When the crystal satisfies the Stembridge axioms we are guaranteed that each connected component in the crystal has a unique highest weight element. Moreover, any two crystals whose highest weight elements have the same weight are isomorphic. Furthermore, the character of a Stembridge crystal of weight λ will coincide with the character of an irreducible representation with the same highest weight λ. We state this formally in the following proposition:

THEOREM 3.3.2. *The following are true:*

(1) If $\mathcal{C}$ is a connected Stembridge crystal, then $\mathcal{C}$ has a unique highest weight element.

(2) Let $\mathcal{C}$ and $\mathcal{C}'$ be connected Stembridge crystals with $u \in \mathcal{C}$ and $u' \in \mathcal{C}'$ being highest weight vectors. If $wt(u) = wt(u')$, then $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic.

(3) The character of a connected Stembridge crystal with a unique highest λ equals the character of the irreducible representation with highest weight λ.

(4) The character of a Stembridge crystal is Schur positive.

To read more about, or see a proof of Theorem 3.3.2 please refer to Theorems 4.12, 4.13 and Corollary 13.9 of [**BS17**]. Statement (4) of Theorem 3.3.2 is a direct consequence of (3).

DEFINITION 3.3.3. The *Stembridge axioms*, as mentioned in Definition 3.3.1, are the local conditions a crystal needs to satisfy to be a Stembridge crystal. There are four conditions for the e_i crystal operators: **S0**, **S1**, **S2**, and **S3**; and four conditions for the f_i crystal operators: **S0'**, **S1'**, **S2'**, and **S3'**. We list the axioms now:

S0: If $e_i(x) = 0$, then $\varepsilon_i(x) = 0$.

S0': If $f_i(x) = 0$, then $\varphi_i(x) = 0$.

S1: Assume $i, j \in I$ and $i \neq j$. If $x, y \in \mathcal{C}$ and $y = e_i x$, then $\varepsilon_j(y)$ equals either $\varepsilon_j(x)$ or $\varepsilon_j(x) + 1$. The second case is only possible when α_i and α_j are not orthogonal.

S1': Assume $i, j \in I$ and $i \neq j$. If $x, y \in \mathcal{C}$ and $y = f_i x$, then $\varphi_j(y)$ equals either $\varphi_j(x)$ or $\varphi_j(x) + 1$. The second case is only possible when α_i and α_j are not orthogonal.

S2: Assume $i, j \in I$ and $i \neq j$. If $x \in \mathcal{C}$ with $\varepsilon_i(x) > 0$ and $\varepsilon_j(e_i x) = \varepsilon_j(x) > 0$, then $e_i e_j x = e_j e_i x$ and $\varphi_i(e_j x) = \varphi_i(x)$.

S2': Assume $i, j \in I$ and $i \neq j$. If $x \in \mathcal{C}$ with $\varphi_i(x) > 0$ and $\varphi_j(f_i x) = \varphi_j(x) > 0$, then $f_i f_j x = f_j f_i x$ and $\varepsilon_i(f_j x) = \varepsilon_i(x)$.

S3: Assume $i, j \in I$ and $i \neq j$. If $x \in \mathcal{C}$ with $\varepsilon_j(e_i x) = \varepsilon_j(x) + 1 > 1$ and $\varepsilon_i(e_j x) = \varepsilon_i(x) + 1 > 1$, then $e_i e_j^2 e_i x = e_j e_i^2 e_j x \neq 0$ and we also have $\varphi_i(e_j x) = \varphi_i(e_j^2 e_i x)$ and $\varphi_j(e_i x) = \varphi_j(e_i^2 e_j x)$.

S3': Assume $i, j \in I$ and $i \neq j$. If $x \in \mathcal{C}$ with $\varphi_j(f_i x) = \varphi_j(x) + 1 > 1$ and $\varphi_i(f_j x) = \varphi_i(x) + 1 > 1$, then $f_i f_j^2 f_i x = f_j f_i^2 f_j x \neq 0$ and we also have $\varepsilon_i(f_j x) = \varepsilon_i(f_j^2 f_i x)$ and $\varepsilon_j(f_i x) = \varepsilon_j(f_i^2 f_j x)$.

Remark 3.3.4. Recall the definiton of a seminormal crystal in Definition 3.2.8. Since our crystal is seminormal, we are guaranteed to satisfy Stembridge axioms **S0** and **S0'**. We will show in the next section that **S1** and **S1'** are also satisfied for all unit interval graphs.

Now that we are formally introduced to the Stembridge axioms, we will now show that the Type A crystal operators defined in Section 3.2.1 do not satisfy all of the axioms. Specifically, **S2**, **S3**, **S2'**, and **S3'** are not satisfied. We will show now an example that is the first occurence of a Stembridge axiom violation, more specifically a violation of **S2'**.

Example 3.3.5. Consider the coloring $x = [1,2,1,3]$ of the graph $G = P_4$. Then $f_1 x = [2,1,2,3]$, $f_2 x = [1,3,1,3]$, and we have the following string lengths: $\varphi_2(x) = \varphi_2(f_1 x) = 1$. Since $\varphi_2(x) = 1 > 0$, and $\varphi_2(f_1 x) = \varphi_2(x) = 1 > 0$, then we should have that $f_1 f_2 x = f_2 f_1 x$ in order to satisfy **S2'**. However, $f_1 f_2 x = [1,3,2,3]$ and $f_2 f_1 x = [3,1,2,3]$, hence the crystal operators don't satisfy all of the Stembridge axioms. Figure 3.4 shows the connected component of the crystal graph where this takes place.

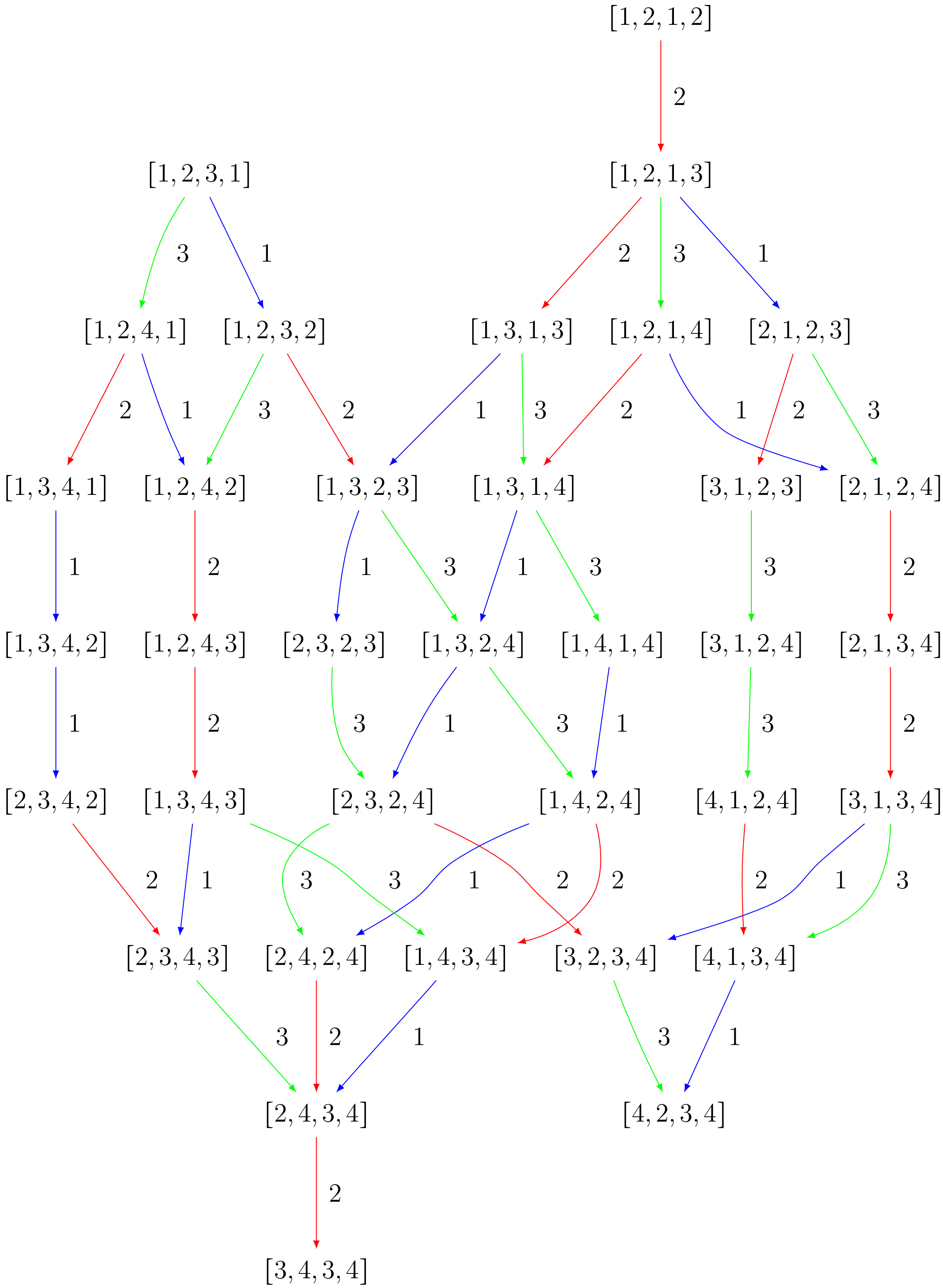


FIGURE 3.4. One connected component of the crystal graph associated to the graph P_4 using 4 colors, which is not a Stembridge crystal

3.4. Graphs of $\mathcal{G}_4$ are Stembridge crystals

In this section, we focus on unit interval graphs. Recall that a graph $G \in \mathcal{G}_4$ is a unit interval graph that does not contain an induced subgraph isomorphic to P_4 (Definition 1.2.9). For any graph $G \in \mathcal{G}_4$, let $\mathcal{C}$ be the crystal structure of the colorings of G using k colors. Then we have the following theorem.

THEOREM 3.4.1. *The crystal $\mathcal{C}$ forms a type A_{k-1} Stembridge crystal structure using the operators e_i and f_i with $i \in I = \{1, \ldots, k-1\}$.*

We have already shown $\mathcal{C}$ satisfies the crystal axioms, we need only show it satisfies the Stembridge Axioms. Remark 3.3.4 shows **S0** and **S0'** are already satisfied. In Section 3.4.1, we will show $\mathcal{C}$ satisfies **S1** and **S1'**. In Section 3.4.2, we will describe six cases for when adjacent operators act on a coloring x that will help us prove the remaining axioms. In Section 3.4.3, we will show $\mathcal{C}$ satisfies **S2** and **S2'** and in Section 3.4.4, we will show $\mathcal{C}$ satisfies **S3** and **S3'**.

3.4.1. Stembridge Axiom 1.

In this section we will prove that the crystal operators e_i and f_i do satisfy **S1** and **S1'** for all unit interval graphs. To this end, we begin with the following proposition:

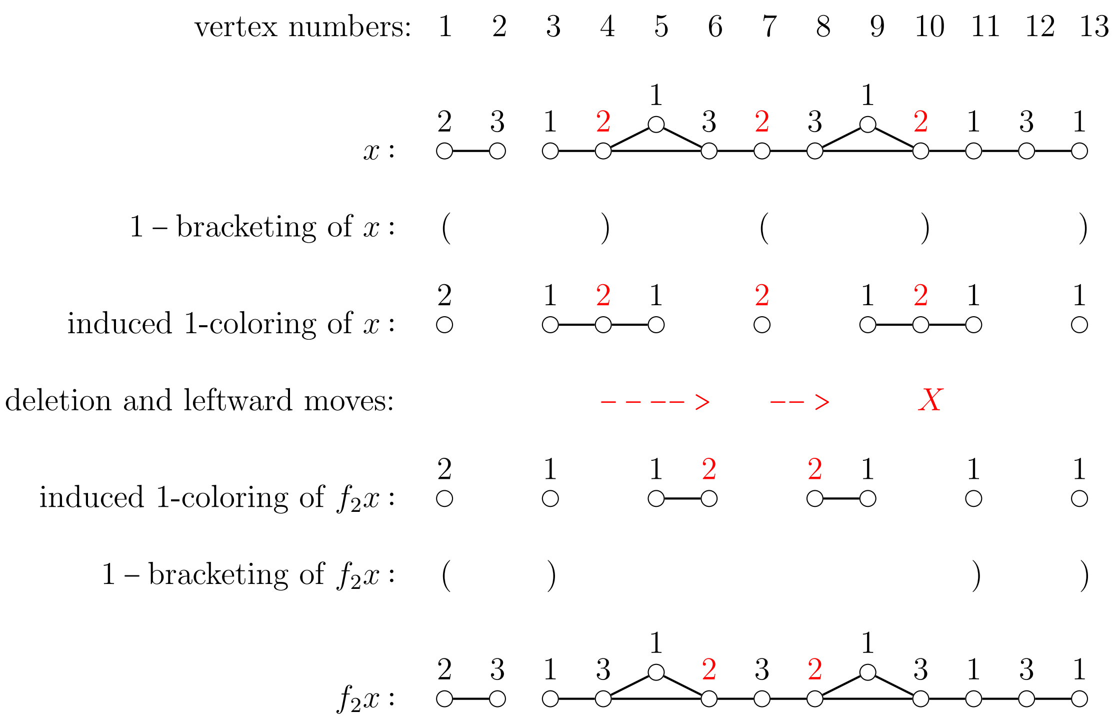
PROPOSITION 3.4.2. *Let x be a coloring and $i, j \in I$ with $j \neq i-1, i$ or $i+1$. If $f_j x \neq 0$, then $\varphi_i x = \varphi_i(f_j x)$. Also, if $e_j x \neq 0$, then $\varepsilon_i x = \varepsilon_i(e_j x)$.*

PROOF. The crystal operators f_i and e_i can only change colors on vertices if they are colored i or $i+1$, since the string lengths represent how many times the operators can be applied, and f_j and e_j cannot change the i-induced coloring of x, the result follows immediately. □

Let x be a coloring of a unit interval graph. In order to prove **S1'** completely, we will need a clear way to think about how f_i acting on x can affect the $i+1$-bracketing and similarly, how f_{i+1} acting on x can affect the i-bracketing. Similarly, to prove **S1**, we will need to consider how e_i acting on x can affect the $i+1$-bracketing and how e_{i+1} acting on x can affect the i-bracketing.

In this section, we will show how f_{i+1} applied to a coloring x can affect the i-bracketing. Since the process is similar for the other three, we will omit it. Applying f_{i+1} to x will change a sequence of alternating $i+1$ and $i+2$'s that begins and ends with an $i+1$ into an alternating sequence that begins and ends with $i+2$. The i-coloring of $f_{i+1}x$ can be thought of as being obtained by performing a sequence of steps to the induced i-coloring of x. The sequence of steps will consist of a deletion of an $i+1$, followed by sequence of 'rightward moves' of $i+1$'s. The sequence begins with the last $i+1$ in the right $i+1$-bracket being deleted. Then the remaining steps consist of moving each of the other $i+1$'s that remain in the right $i+1$-bracket rightward. To determine the number of $i+1$'s that will be moved rightward, we take the length of the right $i+1$-bracket, subtract one, and divide by two.

Example 3.4.3. Let's clarify what we've said so far with an example where $i=1$.



The alternating sequence of 2's and 3's begins on vertex 4 and ends on vertex 10 in the coloring x where the 2's in this sequence are colored red. Compare the induced 1-colorings of x and f_2x. Following the procedure we discussed above we first begin with the deletion of the coloring of 2 on vertex 10 (denoted by the X). Then, since the alternating sequence of 2's and 3's is length 5, we expect 2 rightward moves of 2's in the induced 1-coloring of x. This is the rightward move of the color 2 on vertex 7 to vertex 8 (denoted by the $-->$), followed by the rightward move of the color 2 on vertex 4 to vertex 6 (denoted by the $---->$). Notice that applying these three moves to the induced 1-coloring of x gives us the induced 1-coloring f_2x.

Definition 3.4.4. For the rest of this section, we will call the 'movement' of an $i+1$ in the induced i-coloring of x rightward to its new position in the induced i-coloring of $f_{i+1}x$ a *rightward move*. It will be important to remember that this 'movement' is from a vertex that was colored $i+1$ in x to a vertex that is colored $i+2$ in x, and these vertices share an edge because they are part of a right $i+1$-bracket of x.

Definition 3.4.5. Let x be a coloring. We say that there is *no net change in the i-bracketing* of $f_{i+1}x$ if the number of unpaired left i-brackets of x and $f_{i+1}x$ are equal and the number of unpaired right i-brackets of x and $f_{i+1}x$ are equal. If there is a change, we will specify what left or right i-brackets we gained or lost.

Remark 3.4.6. It should be noted, that in the following propositions we will know that we gained or lost brackets when applying the crystal operators to colorings. However specifying exactly what unpaired brackets we gained or lost is left to the end of this section. For example, if we gain a right i-bracket, it is possible that there are no unpaired left i-brackets left of it, so that we gain an unpaired right i-bracket. However, it is also possible this bracket pairs up with a previously unpaired left i-bracket left of it and we end up losing an unpaired left i-bracket.

We now show that the induced i-coloring obtained after each one of these steps will always either keep the string length the same ($\varphi_i(f_{i+1}x) = \varphi_i(x)$), or increase it by one ($\varphi_i(f_{i+1}x) = \varphi_i(x)+1$). Since the i-bracketing of $f_{i+1}x$ is equivalent to a deletion and a sequence of rightward moves in the i-bracketing of x as described previously, we will simply track what can happen to the i-bracketing after each of these steps.

We begin by considering what happens to the i-bracketing after a deletion of an $i+1$.

Proposition 3.4.7. *The net change of i-bracketing for a coloring x after the deletion of an $i+1$ (as described above as the last $i+1$ in a right $i+1$-bracket) is:*

(1) a deletion of a left i-bracket, or

(2) the creation of an additional right i-bracket.

Proof. The deleted $i+1$ is either in a left i-bracket, an even i-bracket or a right i-bracket. If it is in a right i-bracket, then its deletion results in a right i-bracket that is left of its position and a right i-bracket that is right of its position, the net change in the i-bracketing here is the creation of a right i-bracket. If the $i+1$ is in an i starting even i-bracket, then after deletion we obtain a right i-bracket that is left of its position and an i starting even i-bracket that is right of its position (assuming it didn't end with the $i+1$ we deleted); if it is an $i+1$ starting even i-bracket, then after deletion we obtain a right i-bracket that is right of its position and an $i+1$ starting even i-bracket that is left of its position (assuming it didn't start with the $i+1$ we deleted). The net change for a deletion within an even i-bracket is the creation of a right i-bracket. If the $i+1$ is in a left i-bracket, then after deletion we get an $i+1$ starting even i-bracket on the left (assuming the $i+1$ deleted wasn't at the beginning of the bracket) and an i starting even i-bracket on the right (assuming the $i+1$ deleted wasn't at the end of the bracket). The net change is the deletion of a left i-bracket. Since these are the only possibilities, the proof is completed. □

Now let's consider the possibilities of rightward moves. We first note that because we began with the deletion of the $i+1$ in the right $i+1$-bracket that the f_{i+1} is acting upon, after each rightward move the induced i-coloring is a proper coloring as long as we start with the rightmost rightward move and go from right to left. The situation for rightward moves is more complicated than just a simple deletion and we need to know what possibilities there are for the look of the induced i-coloring of x before and after a rightward move. The following proposition will help make this clearer.

PROPOSITION 3.4.8. *Suppose we are performing a rightward move in the induced i-coloring of x. Then the $i+1$ can't move past two vertices colored i. And if the $i+1$ is moved from a vertex left of a vertex colored i, to a vertex that is right of that same vertex colored i, then the following are true:*

(1) it must share an edge with that vertex colored i before and after the rightward move;

(2) the $i+1$ must be the last $i+1$ of the i-bracket it is in;

(3) the i colored vertex and the vertex colored $i+1$ after the rightward move must no longer be connected to the i-bracket it was in before the rightward move was performed; and the new bracket they are in begins with the i colored vertex.

Proof. First we prove the $i+1$ can't move past two vertices colored i. Since the movement of the $i+1$ is to and from vertices that share an edge, we know that for a unit interval graph (Definition 1.2.8) every vertex in between these two vertices must also share an edge. If there was more than one i colored vertex in between them, then it would not have been a proper coloring.

For (1), again, this follows because it is a unit interval graph. Every vertex in between the start and end vertex of the move must have an edge with both the start and end vertices because the start and end vertices of a rightward move share an edge.

For (2), an i-bracket consists of only i's and $i+1$'s. The only colored vertex that could have been right of the i colored vertex (that the $i+1$ is moving right of) and share an edge with it, is an $i+1$ colored vertex. However, since we know the movement of the $i+1$ must create a proper coloring, if the $i+1$ being moved is moved right of the i colored vertex, by properties of unit interval graphs, its new vertex must also share an edge with that $i+1$ colored vertex that is already on the right. This creates a contradiction, hence the i colored vertex must be at the end of the i-bracket and the $i+1$ must be the last of its color in the i-bracket.

For (3), since an i-bracket consists of only i's and $i+1$'s, an $i+1$ can only share an edge on its left with an i in the induced i-coloring. Let's suppose the $i+1$ has a vertex on its left (before the rightward move) colored i, then we shall call this vertex A. Then the i colored vertex that is right of the $i+1$ that will be moved can't share an edge with vertex A, and by extension of the property of unit interval graphs no vertex right of the i can share an edge with vertex A as well. So, it is clear the new bracket they are in must begin with the i colored vertex. □

Definition 3.4.9. In a rightward move, if the $i+1$ moves to a vertex that shares an edge with an i colored vertex and the $i+1$'s original vertex didn't share an edge with, we say the $i+1$ *acquires* an edge.

From Proposition 3.4.8, we can now say that there are four possible cases for the induced i-coloring of x when performing a rightward move:

(1) The $i+1$ doesn't move past an i and doesn't acquire a new edge on its right.
(2) The $i+1$ doesn't move past an i, does acquire a new edge on its right, and maintains the edge on its left.

(3) The $i+1$ doesn't move past an i, does acquire a new edge on its right, and loses the edge on its left.

(4) The $i+1$ does move past an i.

We now want to determine what the net change is in the i-bracketing of each of these four cases. Case (1) is covered in Proposition 3.4.10. Case (2) is covered in Proposition 3.4.11. Case (3) is covered in Proposition 3.4.12. Case (4) is covered in Proposition 3.4.13.

PROPOSITION 3.4.10. *Suppose we are performing a rightward move. If the $i+1$ is not moved past an i colored vertex, and does not acquire an edge in the i-induced coloring of x, then we have the following situations:*

(1) If it was in a left i-bracket, there is no net change in the i-bracketing.

(2) If it was in a right i-bracket, there is no net change in the i-bracketing.

(3) If it was in an $i+1$ starting even i-bracket, there is no net change in the i-bracketing.

(4) If it was in an i starting even i-bracket, there is either no net change, or the change in the i-bracketing is the even i-bracket gets replaced by a right i-bracket immediately followed by a left i-bracket on its right.

PROOF. For all four situations, if there is no loss of edges in moving from the old vertex to the new, clearly there is no change in the i-bracketing. So for each of the situations we will now assume that in moving the $i+1$ to the right, we lose an edge with the i colored vertex it had on its left.

For (1), when we lose the edge we end up with an $i+1$ starting even i-bracket, immediately followed by a left i-bracket. So we have no net change in the i-bracketing.

For (2), when we lose the edge we end up with a right i-bracket, immediately followed by an $i+1$ starting even i-bracket. So we have no net change in the i-bracketing.

For (3), when we lose the edge, we end up with two $i+1$ starting even i-brackets. So we have no net change in the i-bracketing.

For (4), when we lose the edge, we end up with a right i-bracket immediately followed by a left i-bracket. ∎

Proposition 3.4.11. *Suppose we are performing a rightward move. If the $i+1$ is not moved past a vertex colored i, acquires an edge with another i-bracket on its right, and maintains the edge it shared with a vertex colored i on its left (assuming there was such an i), then we have the following situations:*

(1) If it was in an i starting even i-bracket and it joined on its right an i starting even i-bracket, then there is no net change in the i-bracketing.
(2) If it was in a left i-bracket and it joined on its right an i starting even i-bracket, then there is no net change in the i-bracketing.
(3) If it was in an i starting even i-bracket and it joined on its right a right i-bracket, then there is no net change in the i-bracketing.
(4) If it was in a left i-bracket and it joined on its right a right i-bracket, then there is no net change in the i-bracketing.

Proof. For (1), when the two i starting even i-brackets combine, we get one i starting even i-bracket. Hence, no net change in the i-bracketing.

For (2), when the left i-bracket combines with the i starting even i-bracket, we get one left i-bracket. Hence, no net change in the i-bracketing.

For (3), when the i starting even i-bracket combines with the right i-bracket we get one right i-bracket. Hence, no net change in the i-bracketing.

For (4), when the left i-bracket combines with the right i-bracket we get one $i+1$ starting even i-bracket. Since before the rightward move the right and left i-brackets must have been paired, we have no net change in the i-bracketing. □

Proposition 3.4.12. *Suppose we are performing a rightward move. If the $i+1$ is not moved past an i, acquires an edge on the right, and loses the edge it shared with the i colored vertex it had on its left (assuming there was such an i), then we have the following situations:*

(1) If it was in an i starting even i-bracket and it joined on its right an i starting even i-bracket, then the i starting even i-brackets get replaced by a right i-bracket followed by a left i-bracket.

(2) If it was in a left i-bracket and it joined on its right an i starting even i-bracket, then there is no net change in the i-bracketing.

(3) If it was in an i starting even i-bracket and it joined on its right a right i-bracket, then there is no net change in the i-bracketing.

(4) If it was in a left i-bracket and it joined on its right a right i-bracket, then there is no net change in the i-bracketing.

PROOF. For (1), removing the $i+1$ from the i starting even i-bracket on the left creates a right i-bracket. Adding the $i+1$ to the start of an i starting even i-bracket will create a left i-bracket on the right.

For (2), removing the $i+1$ from the left i-bracket on the left creates an i starting even i-bracket. Adding the $i+1$ to the beginning of an i starting even i-bracket will create a left i-bracket on the right. Hence, there is no net change in the i-bracketing.

For (3), removing the $i+1$ from the i starting even i-bracket on the left creates a right i-bracket. Adding the $i+1$ to the beginning of a right i-bracket will create an $i+1$ starting even i-bracket on the right. Hence, there is no net change in the i-bracketing.

For (4), removing the $i+1$ from the left i-bracket on the left creates an $i+1$ starting even i-bracket. Adding the $i+1$ to the beginning of a right i-bracket will create an $i+1$ starting even i-bracket on the right. Since the right and left i-brackets before the rightward move would have been paired, there is no net change in the i-bracketing. □

PROPOSITION 3.4.13. *Suppose we are performing a rightward move. If the $i+1$ moves past an i, then there is no net change in the i-bracketing.*

PROOF. By Proposition 3.4.8, we know that whatever i-bracket the $i+1$ and i were in, they must be at the end of the i-bracket and after the rightward move they must no longer be in the i-bracket. Suppose the i-bracket they were in contained more than just two colors, then cutting off an even amount of colors from the right of the bracket doesn't change what the bracket is (i.e., if it was a left i-bracket, it still is) so cutting the $i+1$ and i from the i-bracket on the left doesn't change the i-bracketing. Similarly, if we add an even number of alternating i's and $i+1$'s to an i-bracket, it will also not change what the i-bracket is. If it was not connected to anything on the left to begin with, or doesn't connect with an i-bracket on the right after the rightward move, it doesn't matter since it is just an even i-bracket, hence there is no net change in the i-bracketing. □

PROPOSITION 3.4.14. *Let x be a proper coloring of a unit interval graph, $i, i+1 \in I$.*

(1) *If $f_{i+1}x \neq 0$, then we have two possibilities:*
   (a) $\varphi_i(f_{i+1}x) = \varphi_i(x)$ *and* $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x) - 1$ *or,*
   (b) $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1$ *and* $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x)$
(2) *If $f_i x \neq 0$, then we have two possibilities:*
   (a) $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)$ *and* $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x) - 1$ *or,*
   (b) $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$ *and* $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x)$
(3) *If $e_{i+1}x \neq 0$. Then we have two possibilities:*
   (a) $\varepsilon_i(e_{i+1}x) = \varepsilon_i(x)$ *and* $\varphi_i(e_{i+1}x) = \varphi_i(x) - 1$ *or,*
   (b) $\varepsilon_i(e_{i+1}x) = \varepsilon_i(x) + 1$ *and* $\varphi_i(e_{i+1}x) = \varphi_i(x)$
(4) *If $e_i x \neq 0$. Then we have two possibilities:*
   (a) $\varepsilon_{i+1}(e_i x) = \varepsilon_{i+1}(x)$ *and* $\varphi_{i+1}(e_i x) = \varphi_{i+1}(x) - 1$ *or,*
   (b) $\varepsilon_{i+1}(e_i x) = \varepsilon_{i+1}(x) + 1$ *and* $\varphi_{i+1}(e_i x) = \varphi_{i+1}(x)$

PROOF. We will prove (1). A similar process can be used to prove the other three cases, but we omit that here for brevity.

Let x be a proper coloring of a unit interval graph, $i, i+1 \in I$, and $f_{i+1}x \neq 0$. First, recall that we can view f_{i+1} being applied to x as a sequence of steps where we begin with the deletion of an $i+1$ in the induced i-coloring of x, and then perform a sequence of rightward moves that shift the positions of certain $i+1$'s rightward in the induced i-coloring of x and after this has been completed we obtain the induced i-coloring of $f_{i+1}x$.

Now, by Proposition 3.4.7, we know after the deletion we either gain a right i-bracket or lose a left i-bracket.

After the deletion, if we gained a right i-bracket, then it increases the string length φ_i by one, or keeps it the same if there are unpaired left i-brackets left of it. If we lost a left i-bracket, then it increases the string length φ_i by one if it was paired with a right i-bracket that was right of it and there are no unpaired left i-brackets left of that right i-bracket, or keeps the string length the same otherwise. So if we terminated here, we are done, the string length φ_i either remained the same or increased by one after applying f_{i+1} to x.

By Propositions 3.4.10, 3.4.11, 3.4.12 and 3.4.13, we know that for each rightward move there is either no net change in the i-bracketing, or we create a right i-bracket immediately followed by a left i-bracket.

Now suppose we have already performed the deletion and a sequence of rightward moves and the number of unpaired right i-brackets has remained the same or increased by one. Now we perform another rightward move. If there was no net change in the i-bracketing, well of course we maintain the number of unpaired right i-brackets we had prior to the move. If instead we created a right i-bracket followed by a left i-bracket, then we have one of two possibilities. Either we have an unpaired right i-bracket right of it, or we don't. If we do, there is no net change in the i-bracketing, since the left i-bracket would pair with it and then we are left with a right i-bracket that must also be unpaired. If there was no unpaired right i-brackets right of them, then the deletion or any rightward moves which occurred before this one, could not have increased the number of unpaired right i-brackets (after all since they are to the right, they would have needed to create an unpaired right i-brackets right of it), hence performing the rightward move would either increase the number of right i-brackets by one if there are no unpaired left i-brackets left of the right i-bracket that was just created, or keep it the same if there are unpaired left i-brackets.

This shows that $\varphi_i(f_{i+1}x) = \varphi_i(x)$ or $\varphi_i(x) + 1$.

We now prove (a). Assume $\varphi_i(f_{i+1}x) = \varphi_i(x)$. We need to show the number of unpaired left i-brackets decreases by 1 when f_{i+1} is applied to x. Once again we think of f_{i+1} applied to x as a deletion, followed by a sequence of rightward moves. We will induct on the number of rightward moves. When we perform the deletion, we can not not gain an unpaired right i-bracket. So if the deletion created a new right i-bracket, it must have paired up with a left i-bracket that was unpaired, or if we lost a left i-bracket, it must have been unpaired, or was paired with a right i-bracket that again paired up. Either way, we lost an unpaired left i-bracket.

Now suppose we have performed a deletion and a sequence of rightward moves and the number of unpaired left i-brackets is one less than it was in x. When we perform another rightward move, if we have no net change in the i-bracketing, then we are done. Suppose we gain a right i-bracket, followed by a left i-bracket. Since $\varphi_i(f_{i+1}x) = \varphi_i(x)$, the right i-bracket must pair up and decrease the number of unpaired left i-brackets by one. However the left i-bracket must lead to the addition of one more unpaired left i-bracket because there are no unpaired right i-brackets. Hence the number of unpaired left i-brackets remains one less than it was in x. Hence when $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)$, it follows we must also have $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x) - 1$.

The proof of (b) is similar to (a). □

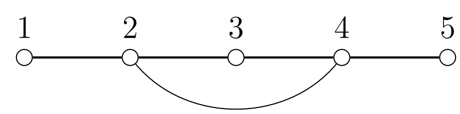
Now we are ready to conclude the section:

PROPOSITION 3.4.15. *The crystal operators e_i and f_i defined on unit interval graphs satisfy the Stembridge axioms **SA1** and **SA1'**.*

PROOF. By Proposition 3.4.2, we know that when the operators are not adjacent, the first Stembridge axioms hold. By Proposition 3.4.14, the first Stembridge axioms hold when the operators are adjacent. □

3.4.2. The Six Cases. We begin this section with a definition:

DEFINITION 3.4.16. We define the *Bull graph* to be the graph with vertex set $V = \{1, 2, 3, 4, 5\}$ and whose edge set is $E = \{12, 23, 24, 34, 45\}$, the graph of the Bull is below. Notice that deleting vertex 3 shows that the Bull graph contains a subgraph isomorphic to P_4.



In this section, we will describe the six cases that can occur when f_i and f_{i+1} (or e_i and e_{i+1}) act on a coloring x. We will prove that certain cases only occur when the graph contains an induced subgraph isomorphic to the Bull. This is important because $\mathcal{G}_4$ does not contain graphs which contain an induced subgraph isomorphic to the Bull.

DEFINITION 3.4.17. Let A be an i-bracket, and let B be a j-bracket for some $i, j \in I$. Then we say A is *strictly left of* B if the last vertex of A is left of the first vertex of B and there is no edge between these two vertices. We define A to be *strictly right of* B in a similar fashion.

The main issue with the crystal operators as defined, is that when a right i-bracket overlaps with a right $(i+1)$-bracket, there can be different scenarios depending upon whether the f_i or the f_{i+1} operator acts first. Therefore, we have broken this down into six cases which describe the overlap of where the operators act and their relative positions. This will help us tremendously with proving Stembridge axioms. We will refer to each of these cases by their number as seen in Figure 3.5.

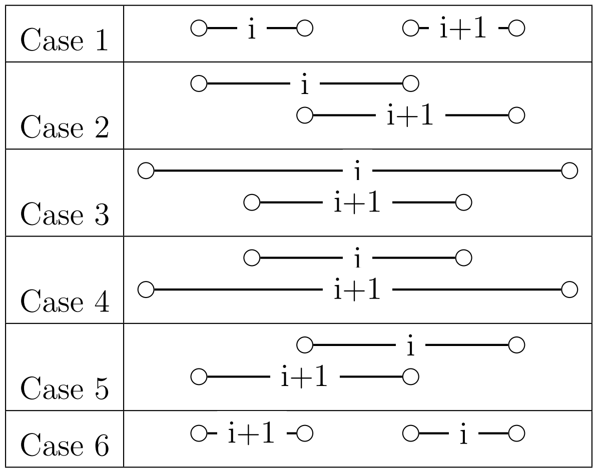
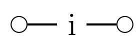


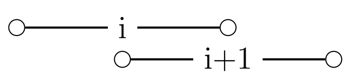
FIGURE 3.5. The six cases of where the f_i and f_{i+1} operators will act on a coloring.

When the reader sees



we mean that this is the location of the right i-bracket that f_i will act on in the coloring. So for Case 1, since we have

it means that the right i-bracket that f_i will act on is strictly left of the right $i+1$-bracket that f_{i+1} will act upon in the coloring. And for Case 2, since we have



it means that the the right i-bracket begins before the right $i+1$-bracket does, and the right i-bracket ends after the right $i+1$-bracket begins, but before the right $i+1$-bracket ends.

REMARK 3.4.18. The six cases also apply to the e_i operators in the exact same fashion, so for Case 1, it means that the left i-bracket that e_i will act on is strictly left of the left $i+1$-bracket that e_{i+1} will act upon in the coloring.

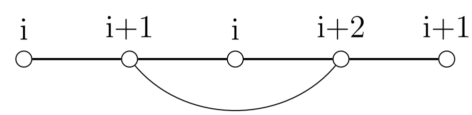
Definition 3.4.19. To reference these cases easily, we denote which operator before the Case, for instance if we want to refer to the f_i operator version of Case 3, we will say *f-Case 3* and similarly if we want to refer to the e_i operator version of Case 3, we will say *e-Case 3*. It should be noted that what i value the case is referring to should be clear from the context.

Cases 2-5 are where the brackets overlap and these will be the cases that give us the most trouble. However, as noted previously, many of these cases involve the Bull graph, and we will now prove which cases do.

Remark 3.4.20. In the following proofs we will talk about deleting vertices to create an induced subgraph, to be clear when we say delete a vertex of the graph, we also mean delete any edge that was attached to that vertex so that we get an induced subgraph.

Proposition 3.4.21. *If f-Case 2 applies to a coloring x of a unit interval graph G, then the graph G must contain an induced subgraph isomorphic to the Bull graph.*

Proof. We claim the graph G must contain an induced subgraph with the following induced coloring of x:



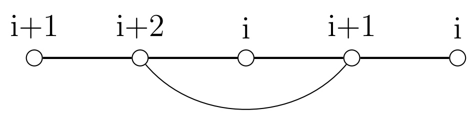
To achieve this, begin by deleting every vertex not contained in the f-Case 2 right i-bracket and right $i+1$-bracket. Then in the right i-bracket, delete all but the last three vertices. The $i+1$ colored vertex in the right i-bracket must be part of the overlap of the right brackets, and hence the $i+1$ colored vertex must also be a part of the right $i+1$-bracket. Now the right $i+1$-bracket must continue past the right i-bracket because this is f-Case 2, so there must be at least two more vertices colored $i+2$ and $i+1$ that are right of this $i+1$ colored vertex. Now delete all vertices in the right $i+1$-bracket that are not these three vertices. This explains all but one part of the picture, the ordering of the i colored vertex that is left of the $i+2$ colored vertex. But this is because we have specified that this is the last i colored vertex in the right i-bracket. Since the two vertices on the end of the right $i+1$-bracket must be connected by an edge, if the i colored vertex were right of the $i+2$ colored vertex, it would be forced to share an edge with the $i+1$ colored vertex (because of the properties of unit interval graphs), hence contradicting the fact that it is the last vertex in the right i-bracket.

Since this subgraph is isomorphic to the Bull graph, we have the result. □

The proofs of the next three propositions are similar to Proposition 3.4.21 and we omit many of the details for brevity.

PROPOSITION 3.4.22. *If f-Case 5 applies to a coloring x of a unit interval graph G, then the graph G must contain an induced subgraph isomorphic to the Bull graph.*

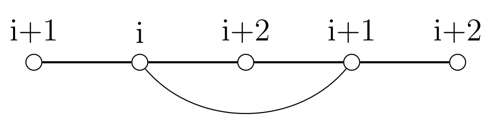
PROOF. Through a process similar to Proposition 3.4.21, it can be shown that the graph G must contain an induced subgraph whose look and corresponding induced coloring of x is the following:



Since this subgraph is isomorphic to the Bull graph, the result follows. □

PROPOSITION 3.4.23. *If e-Case 2 applies to a coloring x of a unit interval graph G, then the graph G must contain an induced subgraph isomorphic to the Bull graph.*

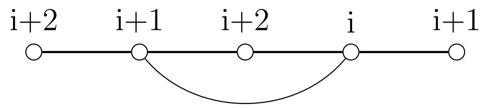
PROOF. Through a process similar to Proposition 3.4.21, it can be shown that the graph G must contain an induced subgraph whose look and corresponding induced coloring of x is the following:



Since this subgraph is isomorphic to the Bull graph, the result follows. □

PROPOSITION 3.4.24. *If e-Case 5 applies to a coloring x of a unit interval graph G, then the graph G must contain an induced subgraph isomorphic to the Bull graph.*

PROOF. Through a process similar to Proposition 3.4.21, it can be shown that the graph G must contain an induced subgraph whose look and corresponding induced coloring of x is the following:

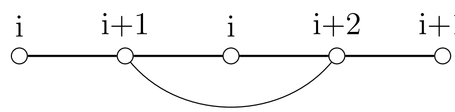


Since this subgraph is isomorphic to the Bull graph, the result follows. □

We now cover the last two cases: f-Case 4 and e-Case 3.

Proposition 3.4.25. *If f-Case 4 applies to a coloring x of a unit interval graph G, then the graph G must contain an induced subgraph isomorphic to the Bull graph.*

Proof. We first note that the right i-bracket must have a length of at least 3. If the length was only one, since it is contained in a right $i+1$-bracket, the vertex colored i is forced to share an edge with a vertex colored $i+1$, a contradiction. We now claim the graph G must contain an induced subgraph whose look and corresponding induced coloring of x is the following:



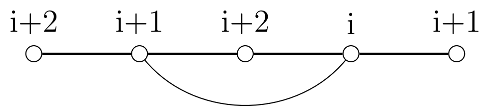
The proof will now follow a similar reasoning as in the proof of Proposition 3.4.21. To obtain the induced subgraph with corresponding induced coloring of x shown above, we first delete every vertex not in the right i-bracket and right $i+1$-bracket that the f-Case 4 refers to. Then, we delete every vertex in the right i-bracket except for the last three vertices. The vertex colored $i+1$ in these last three vertices must also be in the right $i+1$-bracket, we delete every vertex in the right $i+1$-bracket except for that vertex and the two vertices immediately right of it. These vertices must exist because we are in f-Case 4, and a portion of the right $i+1$-bracket must extend past the right i-bracket. These five vertices are the ones shown above. We need only explain why the middle i colored vertex is left of the $i+2$ colored vertex. This is because if it were right of the $i+2$ colored vertex, by properties of the unit interval graphs, it must share an edge with the $i+1$ colored vertex that is on the right end in the picture, a contradiction because the right i-bracket wouldn't end with the i colored vertex in the middle.

Since this subgraph is isomorphic to the Bull graph, the result follows. □

The next proposition will follow a similar proof to Proposition 3.4.25, so we omit many of the details.

Proposition 3.4.26. *If e-Case 3 applies to a coloring x of a unit interval graph G, then the graph G must contain an induced subgraph isomorphic to the Bull graph.*

Proof. We first note that the right $i+1$-bracket must have a length of at least 3. If the length was only one, since it is contained in a right i-bracket, the vertex colored $i+2$, by properties of unit interval graphs, is forced to share an edge with a vertex colored $i+1$, a contradiction. By following the same proof idea in Proposition 3.4.25, it can be shown the graph G must contain an induced subgraph whose look and corresponding induced coloring of x is the following:



Since this subgraph is isomorphic to the Bull graph, we have the result. □

Notice that the Bull graph contains an induced subgraph isomorphic to P_4. This leads us to the main result of the section.

Proposition 3.4.27. *Suppose our operators are adjacent, meaning for some i we have $i, i+1 \in I$. To prove Stembridge axioms **S2'** and **S3'** for $\mathcal{G}_4$, it suffices to show the axioms hold for f-Cases 1, 3, and 6. To prove Stembridge axioms **S2** and **S3** for $\mathcal{G}_4$, it suffices to show the axioms hold for e-Cases 1, 4, and 6.*

The proof of the f-Cases of the proposition follows directly from Proposition 3.4.21, Proposition 3.4.25, and Proposition 3.4.22. The proof of the e-Cases of the proposition follows directly from Proposition 3.4.23, Proposition 3.4.26, and Proposition 3.4.24.

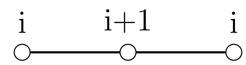
We end this section by showing that when restricted to graphs $G \in \mathcal{G}_4$, there is only one way for f-Case 3 and e-Case 4 to occur, as the next proposition will show:

Proposition 3.4.28. *Let $G \in \mathcal{G}_4$ and let $x \in G$, if f-Case 3 applies to x, then the right i-bracket must have length 3 and the right $i+1$-bracket contained within it must have length 1. Additionally, it must be the case that $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1$ and $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$. This means that **S2'** will never apply to this situation, only **S3'**.*

Proof. First, if the length was greater than 3, then we would have to contain a subgraph isomorphic to P_4. To see this, simply delete all vertices except the first 4 in the right i-bracket.

Second, since the right $i+1$-bracket must be of odd length and contained within the right i-bracket, if it were of length 3 or more, the right i-bracket would contain at least two vertices colored $i+1$, forcing its length to be greater than 3, a contradiction.

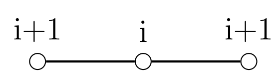
Now, we must have the following look in our induced $i, i+1$-coloring of x:



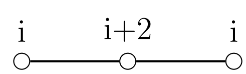
To complete this part of the proof we need to show there are no other vertices sharing edges with these three that are colored i, $i+1$, or $i+2$. The fact that we can have no more vertices colored i or $i+1$ sharing an edge with any of these vertices follows because if they did exist, it would alter the length of the right i-bracket, which we can't do. So let's now consider if it is possible to have a vertex colored $i+2$ sharing an edge with one of these vertices. First, it cannot share an edge with the vertex colored $i+1$, or it would affect the length of the right $i+1$-bracket. Suppose it shared an edge with one of the vertices colored i, then because it can not also share an edge with the $i+1$ colored vertex, we would have an induced subgraph isomorphic to P_4. Hence in the induced $i, i+1$-coloring of x, this is the entire connected component, we have no additional vertices sharing edges that have colors i, $i+1$, or $i+2$.

Then, after applying f_i and f_{i+1} to x, this section becomes:

After applying f_i :



After applying f_{i+1} :



After applying f_i, we gain a right $i+1$-bracket, and because the original right $i+1$-bracket was an unpaired bracket, we must have both of these be unpaired as well, so that $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)+1$. Similarly, after applying f_{i+1}, we gain a right i-bracket, and because the original right i-bracket was an unpaired bracket, we must have both of these be unpaired as well, so that $\varphi_i(f_{i+1}x) = \varphi_i(x)+1$.

Since this is the only possible look of f-Case 3 for graphs in $\mathcal{G}_4$, we can never get a situation where $\varphi_i(f_{i+1}x) = \varphi_i(x)$ or $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)$ when f-Case 3 applies to x. Hence **S2'** will never apply to this situation, only **S3'**. □

Proposition 3.4.29. *Let $G \in \mathcal{G}_4$ and let $x \in G$, if e-Case 4 applies to x, then the left $i+1$-bracket must have length 3 and the left i-bracket contained within it must have length 1. Additionally, it must be the case that $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x) + 1$ and $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x) + 1$. This means that **S2** will never apply to this situation, only **S3**.*

The proof of this is similar and is omitted.

3.4.3. Stembridge Axiom 2. In this section, we will show Stembridge axioms **S2** and **S2'** are both satisfied for the crystal operators e_i and f_i for colorings of a unit interval graph $G \in \mathcal{G}_4$.

Definition 3.4.30. Suppose x is the coloring of a unit interval graph G and $f_i x \neq 0$ and $f_j x \neq 0$. Let R be a right i-bracket of x. If R is exactly the same before and after f_j is applied to x, we say that R is *unaltered* by the action of f_j on x. Similarly, we can use unaltered when talking about left i-brackets that are not changed by applying e_j to a coloring. Now, if the rightmost unpaired right i-bracket of x is unaltered by the action of f_j on x and is still the rightmost right i-bracket of $f_j x$, then we say that f_i acts *independently* of f_j on x.

Remark 3.4.31. It should be clear from the definition that if f_i acts independently of f_j on x and f_j acts independently of f_i on x, then it must be the case that $f_i f_j x = f_j f_i x$.

Proposition 3.4.32. *Let x be a coloring of a unit interval graph. Let $i, j \in I$ with $j \neq i-1, i$, or $i+1$ and both $f_i x \neq 0$ and $f_j x \neq 0$. Then **S2'** holds.*

Proof. Because $j \neq i-1, i$, or $i+1$, it directly follows that f_i and f_j act independently of each other since f_j can't affect the induced i-coloring of x and vice versa. So it follows that $f_i f_j x = f_j f_i x$. Additionally, it also follows that $\varphi_j(f_i x) = \varphi_j(x) > 0$ and $\varepsilon_i(f_j x) = \varepsilon_i(x)$ for the same reason, since f_j can't affect the induced i-coloring of x and vice versa. □

Proposition 3.4.33. *Let x be a coloring of a unit interval graph. Let $i, j \in I$ with $j \neq i-1, i$, or $i+1$ and both $e_i x \neq 0$ and $e_j x \neq 0$. Then **S2** holds.*

The proof of Proposition 3.4.33 is similar to Proposition 3.4.32 and is omitted. Now we need only focus on the case where the operators are adjacent to finish proving the second Stembridge axioms. We consider f-Case 1 and f-Case 6 in this next proposition.

Proposition 3.4.34. *Let x be a proper coloring of a unit interval graph G. Then we have the following:*

(1) Suppose that $f_{i+1}x \neq 0$ and $f_i x \neq 0$. If f-Case 1 applies, then f_{i+1} acts independently of f_i in x.

(2) Suppose that $f_{i+1}x \neq 0$, $f_i x \neq 0$, and for operators f_i and f_{i+1}, f-Case 1 applies. Then $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$.

(3) Suppose that $f_{i+1}x \neq 0$ and $f_i x \neq 0$. If f-Case 6 applies, then f_i acts independently of f_{i+1} in x.

(4) Suppose that $f_{i+1}x \neq 0$, $f_i x \neq 0$, and for operators f_i and f_{i+1}, f-Case 6 applies. Then $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1$.

Proof. To prove (1), since f-Case 1 applies, the rightmost unpaired right $i+1$-bracket is unaltered by f_i applied to x. We need only show that it is still the rightmost unpaired right $i+1$-bracket in $f_i x$. We first note that we can't have any newly created right $i+1$-brackets right of it (because everything is unaltered right of it). Secondly, the coloring x either gains a right $i+1$-bracket or loses a left $i+1$-bracket. Since the creation of a right $i+1$-bracket must be strictly left of the rightmost right $i+1$-bracket of x, this would not change where f_{i+1} acts. And since we are left of the rightmost unpaired right $i+1$-bracket, it must be the case that a loss of a left $i+1$-bracket unpairs a right $i+1$-bracket that must be left of the rightmost right $i+1$-bracket of x, finishing the proof of (1).

We now prove (2). Applying f_i either creates a right $i+1$-bracket, or destroys a left $i+1$-bracket. This must occur left of the rightmost right $i+1$-bracket of x, which is still the rightmost right $i+1$-bracket of $f_i x$ by (2). We have two scenarios. If we gained a right $i+1$-bracket, then clearly $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$. If we lost a left $i+1$-bracket, then since we are left of the rightmost right $i+1$-bracket, it must have been paired and this would mean we gain an unpaired right $i+1$-bracket, so we have $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$.

The proofs of (3) and (4) are similar to (1) and (2). □

We now show **S2'** holds for f-Cases 1 and 6.

Proposition 3.4.35. ***S2'*** *is satisfied by a coloring x where f-Case 1 applies. Meaning, if $i, i+1 \in I$, $\varphi_{i+1}(x) > 0$, and $\varphi_i(f_{i+1}x) = \varphi_i(x) > 0$, then:*

(1) $f_i f_{i+1} x = f_{i+1} f_i x$, and

(2) $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x)$.

Proof. From Proposition 3.4.34 (2), we have $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$. Hence, it must be the case that $\varphi_i(f_{i+1}x) = \varphi_i(x)$ for **S2'** to apply to the coloring x.

We first prove (1). Let $\varphi_i(f_{i+1}x) = \varphi_i(x)$. Then the unpaired right i-brackets of x are unaltered by applying f_{i+1} because it acts strictly right of them. And because $\varphi_i(f_{i+1}x) = \varphi_i(x)$, we couldn't have gained an unpaired right i-bracket, so that the rightmost right i-bracket of x remains such for $f_{i+1}x$. Thus, f_i acts independently of f_{i+1} in x. By Proposition 3.4.34 (1), we know f_{i+1} acts independently of f_i in x. Hence $f_i f_{i+1} x = f_{i+1} f_i x$.

Now we prove (2). By Proposition 3.4.14 (1), since $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1$, it must be the case that $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x)$. □

Proposition 3.4.36. ***S2'*** *is satisfied by a coloring x where f-Case 6 applies. Meaning, if $i, i+1 \in I$, $\varphi_i(x) > 0$, and $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) > 0$, then:*

(1) $f_i f_{i+1} x = f_{i+1} f_i x$, and

(2) $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x)$.

Proof. From Proposition 3.4.34 (4), we have $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1$. Hence, it must be the case that $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)$ for **S2'** to apply to the coloring x.

We first prove (1). Let $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)$. Then the unpaired right $i+1$-brackets of x are unaltered by applying f_i because it acts strictly right of them. And because $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x)$, we couldn't have gained an unpaired right $i+1$-bracket, so that the rightmost right $i+1$-bracket of x remains such for $f_i x$. Thus, f_{i+1} acts independently of f_i in x. By Proposition 3.4.34 (3), we know f_i acts independently of f_{i+1} in x. Hence $f_i f_{i+1} x = f_{i+1} f_i x$.

Toprove (2), by Proposition 3.4.14, since $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1$, it must be the case that $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x)$.

□

Using a similar strategy we could prove the following propositions for **S2**.

Proposition 3.4.37. ***S2*** *is satisfied by a coloring x where e-Case 1 applies. Meaning, if $i, i+1 \in I$, $\varepsilon_i(x) > 0$, and $\varepsilon_{i+1}(e_i x) = \varepsilon_{i+1}(x) > 0$, then:*

(1) $e_i e_{i+1} x = e_{i+1} e_i x$, and

(2) $\varphi_i(e_{i+1} x) = \varphi_i(x)$.

Proposition 3.4.38. ***S2*** *is satisfied by a coloring x where e-Case 6 applies. Meaning, if $i, i+1 \in I$, $\varepsilon_i(x) > 0$, and $\varepsilon_i(e_{i+1} x) = \varepsilon_i(x) > 0$, then:*

(1) $e_i e_{i+1} x = e_{i+1} e_i x$, and

(2) $\varphi_{i+1}(e_i x) = \varphi_{i+1}(x)$.

Proposition 3.4.39. *Let x be the coloring of a unit interval graph $G \in \mathcal{G}_4$. Then the for the crystal operators e_i and f_i,* ***S2*** *and* ***S2'*** *are satisfied.*

Proof. Let $i, j \in I$ and $i \neq j$. If $\varphi_i(x) > 0$ and $\varphi_j(f_i x) = \varphi_j(x) > 0$, then we need to show $f_i f_j x = f_j f_i x$ and $\varepsilon_i(f_j x) = \varepsilon_i(x)$.

If $j \neq i-1$ or $i+1$, then by Proposition 3.4.32 **S2'** holds.

If $j = i-1$ or $i+1$, then we have the six Cases to consider for the adjacent operators. Propositions 3.4.21, 3.4.22, and 3.4.25 show that f-Cases 2, 4, and 5 can never occur for a coloring x of a graph $G \in \mathcal{G}_4$. Proposition 3.4.28 shows that **S2'** will never apply to f-Case 3 for a coloring x of a graph $G \in \mathcal{G}_4$. Proposition 3.4.35 shows **S2'** holds for f-Case 1 and Proposition 3.4.36 shows **S2'** holds for f-Case 6.

Since this exhausts all possibilities, this shows the crystal operators satisfy **S2'**.

Again, let $i, j \in I$ and $i \neq j$. If $\varepsilon_i(x) > 0$ and $\varepsilon_j(e_i x) = \varepsilon_j(x) > 0$, then we need to show $e_i e_j x = e_j e_i x$ and $\varphi_i(e_j x) = \varphi_i(x)$.

If $j \neq i-1$ or $i+1$, then by Proposition 3.4.33 **S2** holds.

If $j = i-1$ or $i+1$, then we have the six Cases to consider for the adjacent operators. Propositions 3.4.23, 3.4.24, and 3.4.26 show that e-Cases 2, 3, and 5 can never occur for a coloring x of a graph $G \in \mathcal{G}_4$. Proposition 3.4.29 shows that **S2** will never apply to e-Case 4 for a coloring x of a graph $G \in \mathcal{G}_4$. Proposition 3.4.37 shows **S2** holds for e-Case 1 and Proposition 3.4.38 shows **S2** holds for e-Case 6.

Since this exhausts all possibilities, this shows the crystal operators satisfy **S2**. ☐

3.4.4. Stembridge Axiom 3. In this section we will prove that the crystal operators satisfy **S3** and **S3'** when restricted to colorings of unit interval graphs $G \in \mathcal{G}_4$. We begin with the case of when the operators are not adjacent.

PROPOSITION 3.4.40. *Let x be the coloring of a unit interval graph and $i, j \in I$. Suppose $j \neq i-1$, i, or $i+1$. If $f_i x \neq 0$ and $f_j x \neq 0$, then $\varphi_i(x) = \varphi_i(f_j x)$ and $\varphi_j(x) = \varphi_j(f_i x)$. If $e_i x \neq 0$ and $e_j x \neq 0$, then $\varepsilon_i(x) = \varepsilon_i(e_j x)$ and $\varepsilon_j(x) = \varepsilon_j(e_i x)$. In other words,* ***S3*** *and* ***S3'*** *will never apply to these cases.*

PROOF. Let $f_i x \neq 0$ and $f_j x \neq 0$. The colors i and $i+1$ are not present in the induced j-coloring of x and the colors j and $j+1$ are not present in the induced i-coloring of x, hence string lengths are unaffected: $\varphi_i(x) = \varphi_i(f_j x)$ and $\varphi_j(x) = \varphi_j(f_i x)$.

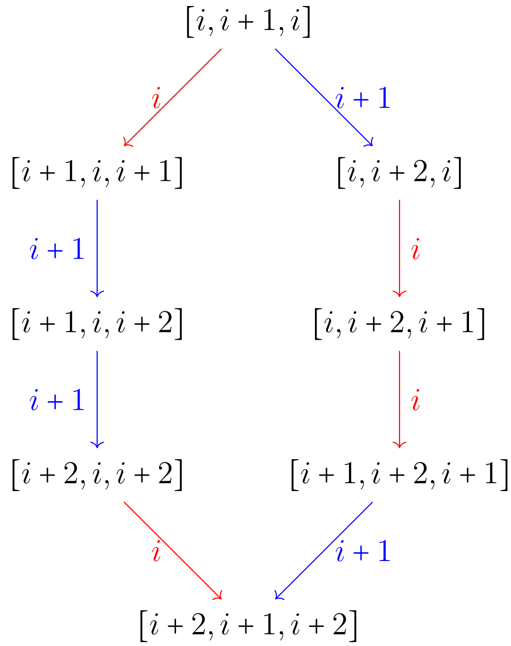
Similarly, let $e_i x \neq 0$ and $e_j x \neq 0$. For the same reason we have $\varepsilon_i(x) = \varepsilon_i(e_j x)$ and $\varepsilon_j(x) = \varepsilon_j(e_i x)$.

The last statement follows because Stembridge axiom 3 applies only when the string length of both of the operators increases by one. □

We now delve into the cases where the crystal operators are adjacent. Recall from the six cases section that we need only consider f-Cases 1, 3, and 6 and e-Cases 1, 4, and 6. We begin with f-Case 3 and e-Case 4.

PROPOSITION 3.4.41. *Let x be the coloring of a unit interval graph $G \in \mathcal{G}_4$. Let $i, i+1 \in I$ and suppose f-Case 3 applies. Then the crystal operators obey* ***S3'****.*

PROOF. By Proposition 3.4.28, we know that there is only one situation for which f-Case 3 can apply to such a coloring when restricted to graphs in $\mathcal{G}_4$, and this component in the induced $i, i+1$-coloring of x contains only three vertices colored i, $i+1$, i, in that order. Here we show the directed graph for the component of the induced $i, i+1$-coloring of x that f_i and f_{i+1} act on.



Let's label this component A. Now we will explain why A is the only component of the induced $i, i+1$-coloring of x that needs to be tracked to determine what happens to the coloring of x. First, because it is f-Case 3, we know that the first paths of the directed graph f_i, headed down and to the left, and f_{i+1}, headed down and to the right, must act on A exactly as detailed.

Now let's consider the right path. Since A is the rightmost unpaired right i-bracket in x, it must be the case that when f_{i+1} acts and changes the $i+1$ to an $i+2$, the two i colored vertices in the A component must now be the two rightmost unpaired right i-brackets of the new coloring. This explains why the next two f_i's act where they do in $f_{i+1}x$ and $f_i f_{i+1} x$. Now, since there was nothing else in the A component of x and f_{i+1} would have acted on the $i+1$ in the A component of x, it must also be the case that in $f_i^2 f_{i+1} x$, that f_{i+1} must act on the length 3 right $i+1$-bracket that is in component A. This explains the path down the right.

For the path down the left, we have a similar reasoning. Since the $i+1$ colored vertex in A is the rightmost unpaired right $i+1$-bracket in x, it must be the case that when f_i acts and changes A to the $i+1$, i, $i+1$ sequence, the two $i+1$ colored vertices in the A component must now be the two rightmost unpaired right $i+1$-brackets of the new coloring. This explains why the next two f_{i+1}'s act where they do in $f_i x$ and $f_{i+1} f_i x$. Now since there was nothing else in the A component of x, and f_i would have acted on the A component of x, it must also be the case that in $f_{i+1}^2 f_i x$, that f_i must act on the i colored vertex that is in component A. This explains the path down the left.

This shows that $f_i f_{i+1}^2 f_i x = f_{i+1} f_i^2 f_{i+1} x \neq 0$. Now we need to show $\varepsilon_i(f_{i+1}x) = \varepsilon_i(f_{i+1}^2 f_i x)$ and $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(f_i^2 f_{i+1} x)$.

Component A of x, $f_{i+1}x$, and $f_{i+1}^2 f_i x$ does not contain any left i-brackets, and their right i-brackets must all be unpaired. Therefore, $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x) = \varepsilon_i(f_{i+1}^2 f_i x)$. Similarly, component A of x, $f_i x$, and $f_i^2 f_{i+1} x$ does not contain any left $i+1$-brackets, and their right $i+1$-brackets must all be unpaired. Hence, $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x) = \varepsilon_{i+1}(f_i^2 f_{i+1} x)$. This shows **S3'** is satisfied. $\square$

PROPOSITION 3.4.42. *Let x be the coloring of a unit interval graph $G \in \mathcal{G}_4$. Let $i, i+1 \in I$ and suppose e-Case 4 applies. Then the crystal operators obey **S3**.*

The proof of this is similar and we omit it.

In order to prove **S3'** for f-Case 1 or 6, we will need to prove that for a coloring x where one of these cases applies, $f_i f_{i+1}^2 f_i x = f_{i+1} f_i^2 f_{i+1} x$. To structure our way of thinking about this scenario, we introduce some new language.

DEFINITION 3.4.43. Let x be a coloring where f-Case 1 and **S3'** both apply. Define $(i)_1$ to be the rightmost unpaired right i-bracket of x and define $(i+1)_1$ to be the rightmost unpaired right $i+1$-bracket of x. Then define $(i)_2$ to be the rightmost unpaired right i-bracket of $f_{i+1}x$ and define $(i+1)_2$ to be the second rightmost unpaired right $i+1$-bracket of $f_i x$.

By Proposition 3.4.45, we will have the following look of x and $f_{i+1} f_i x$:

$$\text{Coloring of } x: \quad \cdots \quad (i)_1 \quad \cdots \quad (i+1)_1 \quad \cdots$$

$$\text{Coloring of } f_{i+1} f_i x: \quad \cdots \quad (i+1)_2 \quad \cdots \quad (i)_2 \quad \cdots$$

LEMMA 3.4.44. *Let x be the coloring of a unit interval graph $G \in \mathcal{G}_4$ and let $i, i+1 \in I$. Then we have the following:*

(1) Let A be a right $i+1$-bracket that is strictly left of the rightmost unpaired right i-bracket of x. Then A is unaltered by the action of f_i on x.

(2) Let A be a right $i+1$-bracket that is strictly right of the rightmost unpaired right i-bracket of x. Then A is unaltered by the action of f_i on x.

(3) Let A be a right i-bracket that is strictly left of the rightmost unpaired right $i+1$-bracket of x. Then A is unaltered by the action of f_{i+1} on x.

(4) Let A be a right i-bracket that is strictly right of the rightmost unpaired right $i+1$-bracket of x. Then A is unaltered by the action of f_{i+1} on x.

Proof. For (1), applying f_i to x can only modify where $i+1$'s occur in the induced $i+1$-coloring of x, so we can't change the $i+2$'s in A. Since A is strictly left of B, it is not possible to change the $i+1$'s in A either. Nor can we add to the length of this bracket, as it would require an $i+2$ color on the end of A, a contradiction since it is a right $i+1$-bracket. Hence A is unaltered by the action of f_i on x. A similar reasoning proves (2), (3), and (4). □

Proposition 3.4.45. *Let x be the coloring of a unit interval graph $G \in \mathcal{G}_4$. Let $i, i+1 \in I$ and suppose f-Case 1 and **S3'** apply to x for i. Then we have the following:*

(1) $(i+1)_1$ is unaltered by the action of f_i on x.

(2) $(i)_1$ is unaltered by the action of f_{i+1} on x.

(3) $(i+1)_1$ is still the rightmost unpaired right $i+1$-bracket of $f_i x$.

(4) $(i)_1$ is the second rightmost unpaired right i-bracket of $f_{i+1}x$.

(5) If $(i)_2$ was altered by applying f_{i+1} to x, then we must have at least one vertex that shared an edge with $(i+1)_1$.

(6) If $(i+1)_2$ was altered by applying f_i to x, then we must have at least one shared vertex with $(i)_1$.

(7) $(i+1)_2$ is strictly left of $(i)_2$.

(8) $(i)_2$ is unaltered by the action of f_{i+1} on $f_{i+1}f_i x$.

Proof. (1) and (2) follow directly from Lemma 3.4.44.

To prove (3), by (1) we know that $(i+1)_1$ is unaltered by the action of f_i on x. By Proposition 3.4.14, we know that we either gained a right $i+1$-bracket or lost a left $i+1$-bracket that is left of $(i+1)_1$. If we gained a right $i+1$-bracket, that is left, then it is clear $(i+1)_1$ is still the rightmost unpaired right $i+1$-bracket. If we lost a left $i+1$-bracket, then it must have been paired with a right $i+1$-bracket that is left of $(i+1)_1$, hence $(i+1)_1$ is still the rightmost unpaired right $i+1$-bracket.

To prove (4), we first note that by (2), $(i)_1$ is unaltered by the action of f_{i+1} on x. By a similar rationale, every i-bracket of x that is left of $(i)_1$ is unaltered by the action of f_{i+1} on x. Hence, the number of unpaired right i-brackets left of $(i)_1$ remains the same as in x. Since **S3'** applies, there must be one unpaired right i-bracket that is right of it.

For (5) to apply, some part of $(i)_2$ must be changed when going from x to $f_{i+1}x$. Since applying f_{i+1} only changes the amount or locations of $i+1$ colors in the induced i-colorings of x and $f_{i+1}x$. Then $(i)_2$ either contains an $i+1$ color that was moved into the component, within the component, or out of the component. In any of these three cases, it must be true that some vertex of $(i)_2$ shared an edge with $(i+1)_1$.

For (6) to apply, some part of $(i+1)_2$ must be changed when going from x to f_ix. Since applying f_i only changes the number and locations of $i+1$ colors in the induced $i+1$-colorings of x and $f_{i+1}x$, then $(i+1)_2$ either contains an $i+1$ color that was moved into the component or within the component (here, we can't have lost an $i+1$ or it wouldn't be a right $i+1$-bracket). In both of these scenarios, one of the $i+1$ colored vertices is moved to a vertex shared by both $(i)_1$ and the $(i+1)_2$.

For (7), $(i+1)_2$ was either altered or unaltered, when applying f_i to x and we break our arguments into these two cases.

Let's first suppose $(i+1)_2$ is altered. From Proposition 3.4.27, we know that there is only three possibilities for $(i+1)_2$ and $(i)_2$, f-Case 1, 3, or 6. Let's consider f-Case 3. By (6), the $i+1$ colored vertex in the $i, i+1, i$ component must have been an i changed to an $i+1$ in the rightmost right i-bracket of x. But then it must be the case that the i colored vertices in the component also had to be $i+1$ colored vertices in x, and this implies the length of $(i)_1$ was greater than 3 in length, violating Proposition 3.4.28. If we consider f-Case 1, because of (5) and (6), there is no way for this to occur if both the right brackets were altered. If $(i)_2$ was unaltered, then it must be strictly right of $(i+1)_1$, which is strictly right of $(i+1)_2$. Hence only f-Case 6 is possible if $(i+1)_2$ is altered.

Now, let's consider if $(i+1)_2$ was unaltered. It must still be strictly left of $(i+1)_1$. If $(i)_2$ is unaltered, then it must be strictly right of $(i+1)_1$, so f-Case 6 applies. If $(i)_2$ is altered, then by (5), it is at most an edge away from the vertices of $(i+1)_1$, so f-Case 1 is not possible. If we consider f-Case 3, we are assuming the $i+1$ colored vertex in the $i, i+1, i$ component of the induced $i, i+1$-coloring must be unaltered as a right $i+1$-bracket, and by (5), the right i colored vertex must have shared an edge with $(i+1)_1$. This vertex can't share an edge with the $i+1$, or else it would have been in the right $i+1$-bracket. But this can't happen since this would have given us an induced subgraph isomorphic to P_4. This means again that only f-Case 6 is possible, proving (7).

By (7), f-Case 6 applies to $f_{i+1}f_i x$, hence applying Lemma 3.4.44 proves (8). ☐

Definition 3.4.46. Similar to Definition 3.4.43, let x be a coloring where f-Case 6 and **S3'** both apply. Define $(i)_1$ to be the rightmost unpaired right i-bracket of x and define $(i+1)_1$ to be the rightmost unpaired right $i+1$-bracket of x. Then define $(i+1)_2$ to be the rightmost unpaired right $i+1$-bracket of $f_i x$ and define $(i)_2$ to be the second rightmost unpaired right i-bracket of $f_{i+1}x$.

By Proposition 3.4.47, we will have the following look of x and $f_i f_{i+1} x$:

$$\text{Coloring of } x: \quad \cdots \quad (i+1)_1 \quad \cdots \quad (i)_1 \quad \cdots$$
$$\text{Coloring of } f_i f_{i+1} x: \quad \cdots \quad (i)_2 \quad \cdots \quad (i+1)_2 \quad \cdots$$

Again, this will help to visualize some of the results in the next proposition.

Proposition 3.4.47. *Let x be the coloring of a unit interval graph $G \in \mathcal{G}_4$. Let $i, i+1 \in I$ and suppose f-Case 6 and **S3'** apply to x for i. Then we have the following:*

1. *$(i)_1$ is unaltered by the action of f_{i+1} on x.*
2. *$(i+1)_1$ is unaltered by the action of f_i on x.*
3. *$(i)_1$ is still the rightmost unpaired right i-bracket of $f_{i+1}x$.*
4. *$(i+1)_1$ is the second rightmost unpaired right $i+1$-bracket of $f_i x$.*

(5) If $(i+1)_2$ was altered by applying f_i to x, then we must have at least one shared vertex with $(i)_1$.

(6) If $(i)_2$ was altered by applying f_{i+1} to x, then we must have at least one vertex that shared an edge with $(i+1)_1$.

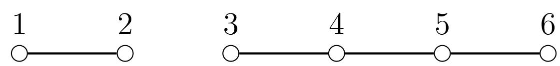
(7) $(i)_2$ is strictly left of $(i+1)_2$.

(8) $(i+1)_2$ is unaltered by the action of f_i on $f_i f_{i+1} x$.

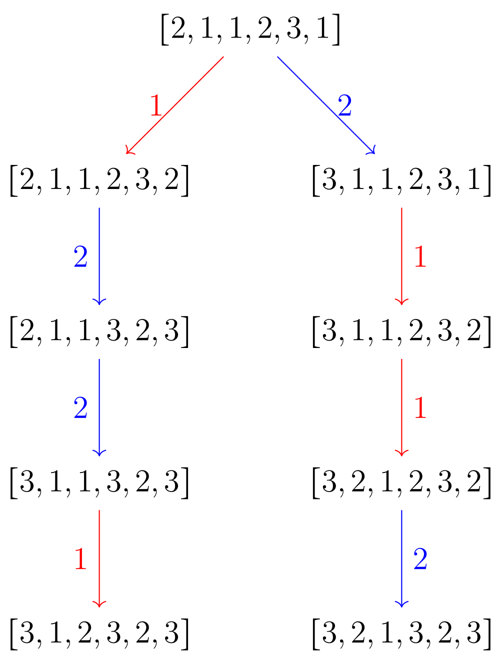
The proof of this is similar and is omitted.

Now, Proposition 3.4.45 and Proposition 3.4.47 provide most of the structure we need to prove **S3'**. However, we still need to prove that there is certain situations that can't happen. Examples of these situations are given in Example 3.4.48 and Example 3.4.50.

Example 3.4.48. Let G be the following graph:



Let $x = [2,1,1,2,3,1]$ be a coloring of this graph. Then consider the following crystal digraph:

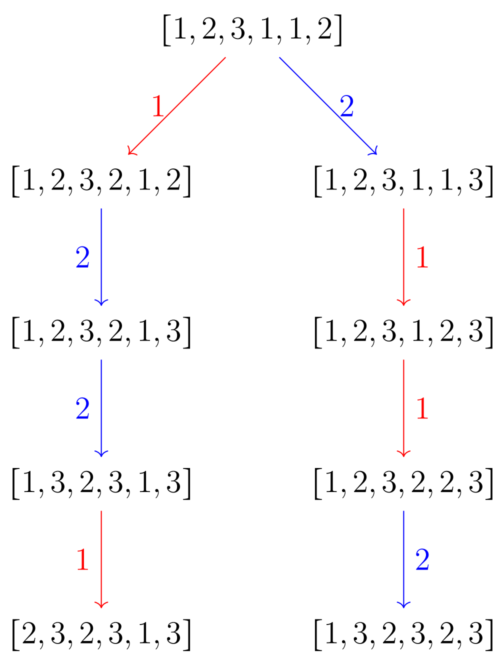


Notice that **S3'** applies to x, but $f_1 f_2^2 f_1 x \neq f_2 f_1^2 f_2 x$.

Remark 3.4.49. How did this occur? Suppose we have f-Case 6 for a coloring x, like we do in Example 3.4.48. When we consider the path $f_{i+1} f_i^2 f_{i+1}$ takes when applied to x, $(i+1)_1$ is acted on first, then $(i)_1$, then $(i)_2$, and lastly $(i+1)_2$. If we consider the other path, $f_i f_{i+1}^2 f_i$, first we act on $(i)_1$, then on $(i+1)_2$, and then on $(i+1)_1$. But when we act with f_{i+1} on $f_i x$, if by acting on $(i+1)_2$ we have $\varphi(f_{i+1} f_i x) = \varphi(f_i x) + 1$, then by Proposition 3.4.47 (4), we must gain a right i-bracket that is strictly right of $(i)_2$ (call this $(i)_3$), meaning this second path would act on $(i)_3$ instead of $(i)_2$, thereby violating **S3'**. A similar situation occurs for f-Case 1.

Example 3.4.50. Let G be the following graph:

Let $x = [1,2,3,1,1,2]$ be a coloring of this graph. Then consider the following crystal digraph:



Notice that **S3'** applies to x, but $f_1 f_2^2 f_1 x \neq f_2 f_1^2 f_2 x$.

Remark 3.4.51. Again we ask, how did this occur? Suppose we have f-Case 1 for a coloring x, like we do in Example 3.4.50. When we consider the path $f_{i+1} f_i^2 f_{i+1}$ takes when applied to x, $(i+1)_1$ is acted on first, then $(i)_2$, then $(i)_1$, and lastly $(i+1)_2$. If we consider the other path, $f_i f_{i+1}^2 f_i$, first we act on $(i)_1$, then on $(i+1)_1$, and then on $(i+1)_2$. Now $(i)_2$ is a right i-bracket that is paired up with the left i-bracket that $(i)_1$ became. If when f_{i+1} acts on $f_{i+1} f_i x$ it does not destroy a paired left i-bracket so that $(i)_2$ becomes unpaired, but instead creates a right i-bracket that is left of the left i-bracket that $(i)_1$ now is, then this second path doesn't act on $(i)_2$, but this new right i-bracket, thereby violating **S3'**. A similar thing can happen for f-Case 6.

This next proposition will show that when restricted to graphs $G \in \mathcal{G}_4$, a situation like what occurred in Example 3.4.48 will never occur.

PROPOSITION 3.4.52. *Let x be the coloring of a graph $G \in \mathcal{G}_4$ where f-Case 1 applies to x for an index $i \in I$. Then $\varphi_{i+1}(f_i f_{i+1} x) = \varphi_{i+1}(f_{i+1} x)$, meaning $(i+1)_3$ can not occur. Similarly, if f-Case 6 applies to the coloring x, then $\varphi_i(f_i^2 f_{i+1} x) = \varphi_i(f_i f_{i+1} x)$, meaning $(i)_3$ can not occur.*

PROOF. We will prove that for f-Case 1, $\varphi_{i+1}(f_i f_{i+1} x) = \varphi_{i+1}(f_{i+1} x)$ and because the proof of f-Case 6 is similar to the proof in Proposition 3.4.53, we will omit it.

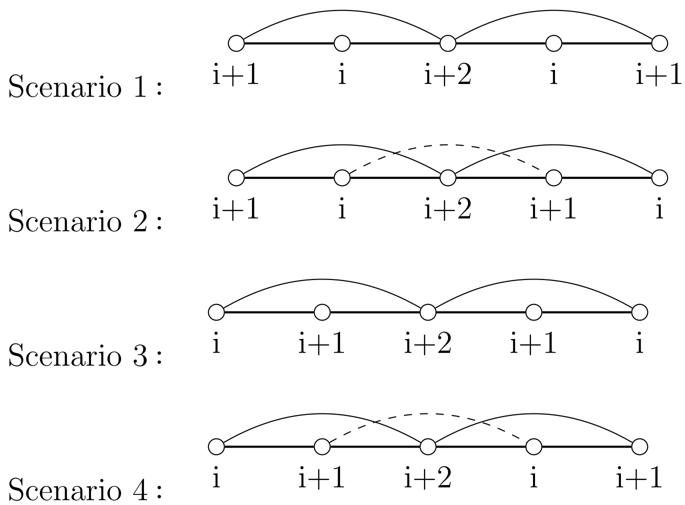
Suppose f-Case 1 applies to a coloring x of a graph $G \in \mathcal{G}_4$. We will use the terminology given in Definition 3.4.43. First, note that by Proposition 3.4.45 the second rightmost unpaired right $i+1$-bracket of $f_i x$ that we call $(i+1)_2$ is strictly left of $(i+1)_1$, which will be important later.

Now suppose $\varphi_{i+1}(f_i f_{i+1} x) = \varphi_{i+1}(f_{i+1} x) + 1$. Applying f_{i+1} to $(i+1)_1$ would have turned $(i+1)_1$ into the leftmost unpaired left $i+1$-bracket in $f_{i+1} x$, denote this as a left bracket by $(i+1)_1^L$. Applying f_i to $f_{i+1} x$ means that f_i acts on $(i)_2$ and this results in the creation of $(i+1)_3$. We now consider two cases depending upon whether $(i)_2$ was altered or unaltered by applying f_{i+1} to x.

Suppose $(i)_2$ was unaltered by applying f_{i+1} to x. Then we know that $(i)_2$ is strictly right of $(i+1)_1^L$. And when we apply f_i and act on $(i)_2$, the net change in the $i+1$-bracketing is that we either lose a left $i+1$-bracket or gain a right $i+1$-bracket. If we lose a left $i+1$-bracket and it wasn't paired, then we don't gain an unpaired right $i+1$-bracket and $(i+1)_3$ couldn't exist. If we lose a left $i+1$-bracket and it was paired, then it unpairs a right $i+1$-bracket that must then pair up with $(i+1)_1^L$. If we gain a right $i+1$-bracket, then $(i+1)_1^L$ would ensure that it would have to be a paired bracket and again $(i+1)_3$ couldn't exist.

Suppose $(i)_2$ was altered by applying f_{i+1} to x. Then, any right bracket must be of length 3 or less since otherwise we would have an induced subgraph isomorphic to P_4. Now suppose $(i+1)_1$ and $(i)_2$ have length 3. By Proposition 3.4.45 (5), we know that at least one vertex of $(i)_2$ shares an edge with $(i+1)_1$. Suppose one of the $i+1$ colored vertices of $(i+1)_1$ shares an edge with one of the i colored vertices of $(i)_2$. If this was the only edge, we contain a subgraph isomorphic to P_4. Now the $i+1$ colored vertex of $(i+1)_1$ can't share another edge with $(i)_2$ and still be a proper coloring. The only other vertex the i colored vertex of $(i)_2$ can share an edge with is the central

$i+2$ colored vertex of $(i+1)_1$, but even if this edge exists, we still contain a subgraph isomorphic to P_4. So then it must be the case that $(i)_2$ and $(i+1)_1$ share at least one vertex. If this is the case, after f_{i+1} is applied to x, the $i+2$ colored vertex must become the central $i+1$ colored vertex of $(i)_2$. This gives us four scenarios that come from graphs in $\mathcal{G}_4$.

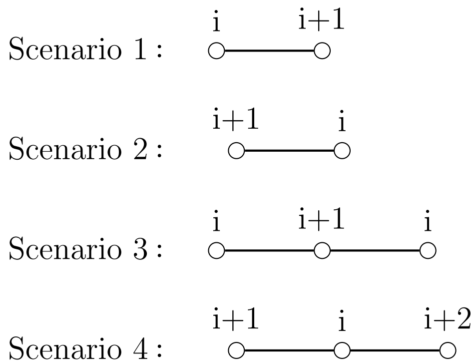


Here, the dashed edges may or may not be present. In all four scenarios, we begin with a right $i+1$-bracket $(i+1)_1$. After f_{i+1} is applied to x, this becomes a left $i+1$-bracket. And after f_i is applied to $f_{i+1}x$ and acts on $(i)_2$, the left $i+1$-bracket is lost and replaced with even $i+1$ brackets. Hence in all scenarios $(i+1)_3$ is not created and we have $\varphi_{i+1}(f_i f_{i+1} x) = \varphi_{i+1}(f_{i+1}x)$.

Now suppose $(i)_2$ has length 1 and $(i+1)_1$ has length 3. Then $(i)_2$ is just an i colored vertex. By Proposition 3.4.45 (5), the i colored vertex shares an edge with some vertex of $(i+1)_1$ in x. Suppose the i colored vertex shares an edge with an $i+1$ and an $i+2$ colored vertex, then it wouldn't be a right i-bracket after $(i+1)_1$ is acted upon by f_{i+1}. The only other possibility is that it shares an edge with an $i+1$ colored vertex only, but then we have an induced subgraph isomorphic to P_4, a contradiction.

Now suppose $(i+1)_1$ was length 1, which means it is just an $i+1$ colored vertex, and $(i)_2$ was length 3. Again, by Proposition 3.4.45 (5) the $i+1$ colored vertex of $(i+1)_1$ must share an edge with $(i)_2$ in x, and this means it must only share an edge with an i colored vertex on the end. But then we would have an induced subgraph isomorphic to P_4.

If $(i)_2$ and $(i+1)_1$ were both length 1, then for $(i)_2$ to be altered, we must have the i and $i+1$ in $(i)_2$ and $(i+1)_1$ respectively, be connected by an edge in the same component of the induced $i, i+1$-coloring of x. Then there are exactly four scenarios for this component in the induced $i, i+1$-coloring of x when restricted to graphs $G \in \mathcal{G}_4$.



These are the only four scenarios because the i can only be connected to another vertex colored $i+2$ and the $i+1$ can only be connected to another vertex colored i. Moreover the $i+2$ must be right of the $i+1$ or else it would pair up with it in the $i+1$-bracketing. Adding any other vertices would create an induced subgraph isomorphic to P_4.

In all four scenarios, after $(i+1)_1$ is acted on that $i+1$ becomes an $i+2$. After $(i)_2$ is acted on, that i becomes an $i+1$. The i becoming an $i+1$ destroys the left $i+1$-bracket that $i+2$ was (or in the case of scenario 4 the two $i+2$'s were two left $i+1$-brackets that become one with the $i+1$ between them), but this can't create an unpaired right $i+1$-bracket, because if there was it would have been unpaired before $(i+1)_1$ was acted on, violating the fact that $(i+1)_1$ was the rightmost right $i+1$-bracket. □

And now the next proposition will address the issue that occurred in Example 3.4.50, again showing that this situation cannot occur when we restrict ourselves to graphs $G \in \mathcal{G}_4$.

PROPOSITION 3.4.53. *Let x be the coloring of a graph $G \in \mathcal{G}_4$ where f-Case 1 applies to x for an index $i \in I$. Then:*

(1) $\varphi_i(f_{i+1}^2 f_i x) = \varphi_i(f_{i+1} f_i x) + 1$ and

(2) $(i)_2$ is the rightmost unpaired right i-bracket of $f_{i+1}^2 f_i x$.

Similarly, if f-Case 6 applies to the coloring x for i, then:

(1) $\varphi_{i+1}(f_i^2 f_{i+1}x) = \varphi_{i+1}(f_i f_{i+1}x) + 1$ *and*

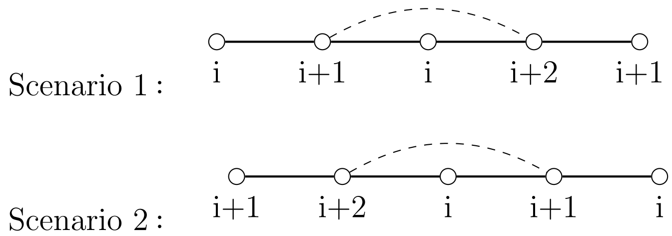
(2) $(i+1)_2$ *is the rightmost unpaired right* $i+1$*-bracket of* $f_i^2 f_{i+1}x$*.*

Proof. We prove (1) and (2) for f-Case 1 and f-Case 6 (1) follows from a similar reasoning, while the proof of (2) is similar to that of the proof of Proposition 3.4.52.

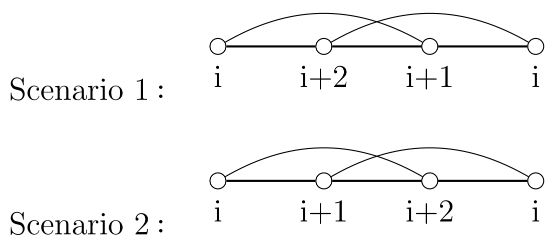
First, consider $f_{i+1}f_ix$. At this point we have $(i+1)_2$ strictly left of $(i)_2$ and since $(i)_1$ has already been acted upon, it is a left i-bracket (denote this by $(i)_1^L$) that is strictly left of $(i)_2$ and hence paired with it. Now the leftmost unpaired left i-bracket must be right of $(i)_2$ because $(i)_2$ is the rightmost right i-bracket of $f_{i+1}x$ and in $f_{i+1}f_ix$ the only difference is that we have $(i)_1$ as a left i-bracket, not a right i-bracket. Now (1) follows directly because when we act on $(i+1)_2$ we are strictly left of the leftmost left i-bracket, and regardless of whether this creates a right i-bracket, or we lose a left i-bracket, both result in the creation of an unpaired right i-bracket. Now we proceed to prove (2) with two cases, depending upon whether $(i+1)_2$ was altered or not. We note that if acting with f_{i+1} on $f_{i+1}f_ix$ unpairs $(i)_2$, it would make $(i)_2$ the rightmost right i-bracket of $f_{i+1}^2 f_ix$, so most of the proofs will focus on proving that $(i)_2$ becomes unpaired.

Suppose $(i+1)_2$ is unaltered by f_i acting on x. Then it must be strictly right of $(i)_1^L$ and since we are strictly left of $(i)_2$, it must be the case that creating a right i-bracket must result in $(i)_1^L$ being paired with a right i-bracket that is left of $(i)_2$, hence unpairing $(i)_2$. Similarly, if a left i-bracket was lost, it must be the case that the right i-bracket it was paired with is left of $(i)_2$ (because otherwise $(i)_2$ would be paired with it) and would pair with $(i)_1^L$, hence unpairing $(i)_2$ again.

Now suppose $(i+1)_2$ is altered by f_i acting on x. We know that when restricted to graphs in $\mathcal{G}_4$, the length of right brackets must be 1 or 3. Let's first consider the case where the lengths of $(i+1)_2$ and $(i)_1$ are both 3. By Proposition 3.4.45 (6), we know that $(i+1)_2$ and $(i)_1$ must share a vertex. Since the shared vertex must be an $i+1$ colored vertex of $(i+1)_2$, we can either share 1 or 2 vertices. If we share only one vertex, we must have had one of two scenarios for these vertices in x, where the dashed edge is optional.

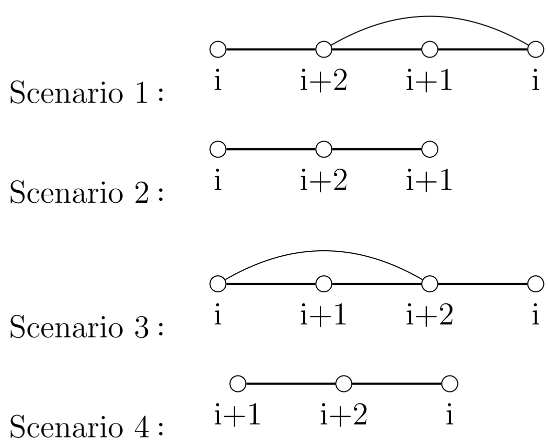


However, all of these would imply that the graph contains an induced subgraph isomorphic to P_4, so these cannot occur. Now suppose $(i+1)_2$ and $(i)_1$ share two vertices. Then we again have two scenarios:



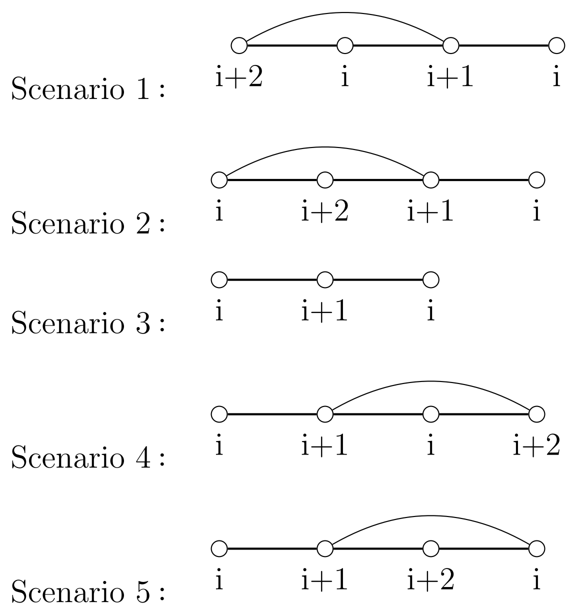
In both of these scenarios, after we apply f_i to $(i)_1$ and f_{i+1} to $(i+1)_2$, we lose $(i)_1^L$ as a left i-bracket, so that $(i)_2$ would unpair as desired and become the rightmost right i-bracket of x.

Now suppose $(i)_1$ has length 1 and $(i+1)_2$ has length 3. Again, $(i)_1$ and $(i+1)_2$ must share a vertex, in this case this is the i colored vertex that $(i)_1$ is in x, and becomes an $i+1$ colored vertex in $f_i x$ for $(i+1)_2$. We have the following 4 scenarios when restricted to graphs $\mathcal{G}_4$.



The i colored vertex on the right in scenario 1 and on the left in scenario 3 is not in $(i)_1$ or $(i+1)_2$, but could still be in the same component of the induced $i, i+1$-coloring of x. In all four of these scenarios we again have that after f_i is applied to $(i)_1$ and f_{i+1} is applied to $(i+1)_2$, we lose $(i)_1^L$ as a left i-bracket so that $(i)_2$ becomes an unpaired right i-bracket.

Now suppose $(i)_1$ has length 3 and $(i+1)_2$ has length 1. Again, $(i)_1$ and $(i+1)_2$ must share a vertex, in this case this is one of the i colored vertices of $(i)_1$ in x that becomes an $i+1$ colored vertex in $f_i x$ for $(i+1)_2$. We have the following 4 scenarios when restricted to graphs $\mathcal{G}_4$.



The $i+2$ colored vertex in scenarios 1, 2, 4, and 5 is not in $(i)_1$ or $(i+1)_2$, but can be present in the same component of the induced $i, i+1$-coloring of x. Again in all five scenarios we have that after f_i is applied to $(i)_1$ and f_{i+1} is applied to $(i+1)_2$, we lose $(i)_1^L$ as a left i-bracket so that $(i)_2$ becomes an unpaired right i-bracket.

The last case is when both $(i)_1$ and $(i+1)_2$ have length 1. There is only one possibility for this. $(i)_1$ is an i colored vertex. After f_i acts on it, it becomes an $i+1$ colored vertex. This vertex is both $(i+1)_2$ and $(i)_1^L$. When f_{i+1} acts on it, we lose $(i)_1^L$ as a left i-bracket, which unpairs $(i)_2$.

This exhausts all cases, so it must be the case that $(i)_2$ unpairs and becomes the rightmost unpaired right i-bracket in $f_{i+1}^2 f_i x$. □

Proposition 3.4.54. *Let x be the coloring of a graph $G \in \mathcal{G}_4$. Assume $i, i+1 \in I$ and f-Case 1 or f-Case 6 applies to x for i. If $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1 > 1$ and $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1 > 1$, then:*

(1) $f_i f_{i+1}^2 f_i x = f_{i+1} f_i^2 f_{i+1} x \neq 0$;

(2) $\varepsilon_i(f_{i+1}x) = \varepsilon_i(f_{i+1}^2 f_i x)$ and $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(f_i^2 f_{i+1} x)$.

Proof. We begin by proving (1) for f-Case 1. We will again be using the language of Definition 3.4.43. To prove this we will simply show that both paths act on exactly $(i)_1$, $(i+1)_1$, $(i)_2$, $(i+1)_2$, and nothing else.

Let's first consider the path $f_{i+1} f_i^2 f_{i+1}$. When f_{i+1} acts on x, it acts on $(i+1)_1$ by definition. Next, when f_i acts on $f_{i+1}x$, it acts on $(i)_2$, again by definition. By Proposition 3.4.45 (4), we know that f_i acting on $f_i f_{i+1} x$ will act on $(i)_1$. Now we know acting on $(i)_1$ creates $(i+1)_2$, and this is strictly left of $(i)_2$ by Proposition 3.4.45 (7). By Proposition 3.4.45 (8), we know that $(i)_2$ can't alter $(i+1)_2$, and by Proposition 3.4.52, it follows that $(i+1)_2$ is the rightmost right $i+1$-bracket of $f_i^2 f_{i+1} x$. Hence it is the case that f_{i+1} acting on $f_i^2 f_{i+1} x$ acts on $(i+1)_2$.

Now let's consider the path $f_i f_{i+1}^2 f_i$. When f_i acts on x, it acts on $(i)_1$ by definition. Next, when f_{i+1} acts on $f_i x$, it acts on $(i+1)_1$ because by Proposition 3.4.45 (3), we know it is still the rightmost unpaired right $i+1$-bracket of $f_i x$ and it is unaltered by f_i acting on x by Proposition 3.4.45 (1). When f_{i+1} acts on $f_{i+1} f_i x$, it acts on $(i+1)_2$ by definition. And lastly, by Proposition 3.4.45 (8), we know that $(i)_2$ is unaltered by f_{i+1} acting on $f_{i+1} f_i x$ and by Proposition 3.4.53, we know it is the rightmost right i-bracket of $f_{i+1}^2 f_i x$, so that f_i acts on $(i)_2$ when it acts on $f_{i+1}^2 f_i x$. Hence $f_i f_{i+1}^2 f_i x = f_{i+1} f_i^2 f_{i+1} x$, and this clearly isn't 0.

Now we prove (2). Let's first begin by proving $\varepsilon_i(f_{i+1}x) = \varepsilon_i(f_{i+1}^2 f_i x)$. Since we know that $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1 > 1$, by Proposition 3.4.14, this implies $\varepsilon_i(f_{i+1}x) = \varepsilon_i(x)$. Now, it is clear that $\varepsilon_i(f_i x) = \varepsilon_i(x) + 1$. When f_{i+1} is applied to $f_i x$, it acts on $(i+1)_1$ and we know that we gain a right i-bracket, but since $(i)_1$ was acted upon and is now an unpaired left i-bracket strictly left of it, these must be paired. Hence, $\varepsilon_i(f_{i+1} f_i x) = \varepsilon_i(x)$. Lastly, by Proposition 3.4.53, we know that when we apply f_{i+1} again to act on $(i+1)_2$, we know it must increase the number of unpaired right i-brackets by one, meaning $\phi_i(f_{i+1}^2 f_i x) = \phi_i(f_{i+1} f_i x) + 1$, which again by Proposition 3.4.14 implies that $\varepsilon_i(f_{i+1}^2 f_i x) = \varepsilon_i(f_{i+1} f_i x)$. Hence $\varepsilon_i(f_{i+1}^2 f_i x) = \varepsilon_i(x) = \varepsilon_i(f_{i+1} x)$.

Now let's prove $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(f_i^2 f_{i+1} x)$. Since we know that $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1 > 1$, by Proposition 3.4.14, this implies $\varepsilon_{i+1}(f_i x) = \varepsilon_{i+1}(x)$. Now it is clear that $\varepsilon_{i+1}(f_{i+1}x) = \varepsilon_{i+1}(x) + 1$. By Proposition 3.4.52, we know that when we apply f_i to $f_{i+1}x$, it will act on $(i)_2$, and we know it doesn't increase the number of unpaired right $i+1$-brackets, meaning $\varphi_{i+1}(f_i f_{i+1} x) = \varphi_{i+1}(f_{i+1}x)$, and again by Proposition 3.4.14, this implies $\varepsilon_{i+1}(f_i f_{i+1} x) = \varepsilon_{i+1}(f_{i+1}x) - 1 = \varepsilon_{i+1}(x)$. Lastly, when f_i acts again, this time on $(i)_1$, we know we gain an unpaired right $i+1$-bracket, so $\varphi_{i+1}(f_i^2 f_{i+1} x) = \varphi_{i+1}(f_i f_{i+1} x) + 1$. By applying Proposition 3.4.14 again we have $\varepsilon_{i+1}(f_i^2 f_{i+1} x) = \varepsilon_{i+1}(f_i f_{i+1} x) = \varepsilon_{i+1}(x) = \varepsilon_{i+1}(f_i x)$. ☐

A similar strategy can be used to prove the e version of Proposition 3.4.54, and we state this now without proof.

PROPOSITION 3.4.55. *Let x be the coloring of a graph $G \in \mathcal{G}_4$. Assume $i, i+1 \in I$ and e-Case 1 or e-Case 6 applies to x for i. If $\varepsilon_{i+1}(e_i x) = \varepsilon_{i+1}(x) + 1 > 1$ and $\varepsilon_i(e_{i+1}x) = \varepsilon_i(x) + 1 > 1$, then:*

(1) $e_i e_{i+1}^2 e_i x = e_{i+1} e_i^2 e_{i+1} x \neq 0$;

(2) $\varphi_i(e_{i+1}x) = \varphi_i(e_{i+1}^2 e_i x)$ and $\varphi_{i+1}(e_i x) = \varphi_{i+1}(e_i^2 e_{i+1} x)$.

We are now ready to state the final result of this section.

PROPOSITION 3.4.56. *Let x be the coloring of a graph $G \in \mathcal{G}_4$. Then the crystal operators satisfy the Stembridge axioms **S3** and **S3'**.*

PROOF. Let $i, j \in I$ and $i \neq j$. If $j \neq i-1$ or $i+1$, then by Proposition 3.4.40, **S3** and **S3'** never apply to these cases, so we now assume we have adjacent operators for the rest of the proof.

Suppose **S3'** applies to x, meaning for $i, i+1 \in I$ we have $\varphi_{i+1}(f_i x) = \varphi_{i+1}(x) + 1 > 1$ and $\varphi_i(f_{i+1}x) = \varphi_i(x) + 1 > 1$. Then there are three possible situations for these operators, f-Case 1, 3 or 6. Proposition 3.4.41 shows that **S3'** holds for f-Case 3, and Proposition 3.4.54 shows that it holds for f-Cases 1 and 6.

Suppose **S3** applies to x, meaning for $i, i+1 \in I$ we have $\varepsilon_{i+1}(e_i x) = \varepsilon_{i+1}(x) + 1 > 1$ and $\varepsilon_i(e_{i+1}x) = \varepsilon_i(x) + 1 > 1$. Then there are three possible situations for these operators, e-Case 1, 4 or 6. Proposition 3.4.42 shows that **S3** holds for e-Case 4, and Proposition 3.4.55 shows that it holds for e-Cases 1 and 6. ☐

3.5. Schur positivity using crystal structure

In Section 3.4, we showed that colorings of unit interval graphs $G \in \mathcal{G}_4$ create a Stembridge crystal structure, and by Theorem 3.3.2, this is a way to prove Schur positivity of these graphs. However, producing crystal operators that give a Stembridge crystal structure for a larger set of graphs than $\mathcal{G}_4$ appears to be a significantly challenging task. And proving that these new operators satisfy the Stembridge axioms would also be significantly challenging. So, it does not seem that seeking Stembridge crystal operators is desirable for claw-free graphs more generally.

The advantages of our crystal operators to Ehrhard's [**Ehr22**], is that our operators can be applied directly to colorings of graphs, seem easier to apply and understand, and can be applied to the set of all claw-free graphs, rather than just claw-free incomparability graphs. Given that Ehrhard proved Schur positivity using the crystal structure for claw-free incomparability graphs and our operators are isomorphic on this set of graphs, ours by extension can use the exact same proof technique to prove Schur positivity. So it is possible to try to extend Schur positivity to claw-free graphs which are not incomparability graphs using our crystal structure and a Schur positivity proof similar to one used by Ehrhard in Theorem 6.2 of [**Ehr22**]. This is what we are currently pursuing.

Our current idea is to consider types of small graphs which are claw-free but not incomparability graphs such as cycle graphs of length greater than 4, and prove Schur positivity of these graphs using our crystal structure and find ways of attaching claw-free incomparability graphs to these graphs such that the proof techniques still work to produce greater and greater subsets of claw-free graphs.

In general, if the crystal operators were to give us the correct number of highest weight colorings, then the Schur positivity proof would be straightforward. However, it seems for most (if not all) claw-free graphs which are not incomparability graphs, we do not get the correct number of highest weight colorings. So we need to either modify the operators until we do, or modify the proof to account for this difference.

Despite the difficulty, this process at least seems like it could be used to extend the Schur positivity results to rather large sets of claw-free graphs in the future and we are currently considering crystal operator modifications for claw-free graphs that will make this possible.

Bibliography

[BS17] Daniel Bump and Anne Schilling. *Crystal Bases: Representations And Combinatorics*. World Scientific Publishing Company Pte. Limited, 2017.

[Ehr22] Henry Ehrhard. A crystal analysis of p-arrays. preprint, arXiv:2205.01834, 2022.

[Gas96] Vesselin Gasharov. Incomparability graphs of $(3+1)$-free posets are s-positive. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 193–197, 1996.

[GP13] Mathieu Guay-Paquet. A modular relation for the chromatic symmetric functions of $(3+1)$-free posets. preprint, arXiv:1306.2400, 2013.

[GV89] Ira M. Gessel and Gérard Viennot. Determinants, paths, and plane partitions. *Preprint*, 1989.

[Hik24] Tatsuyuki Hikita. A proof of the Stanley–Stembridge conjecture. preprint, arXiv:2410.12758, 2024.

[JKN+25] Lisa Johnston, David Kenepp, Evuilynn Nguyen, Digjoy Paul, Anne Schilling, Mary Claire Simone, and Regina Zhou. The immersion poset on partitions. *Journal of Algebraic Combinatorics*, 61(2), February 2025.

[Kas90] Masaki Kashiwara. Crystalizing the q-analogue of universal enveloping algebras. *Comm. Math. Phys.*, 133(2):249–260, 1990.

[PPS24a] Amrutha P, Amritanshu Prasad, and Velmurugan S. Cyclic characters of alternating groups. preprint, arXiv:2403.05109, 2024.

[PPS24b] Amrutha P., Amritanshu Prasad, and Velmurugan S. Existence of elementwise invariant vectors in representations of symmetric groups. *Algebraic Combinatorics*, 7(4):915–929, 2024.

[PR22] Dipendra Prasad and Ravi Raghunathan. Relations between cusp forms sharing Hecke eigenvalues. *Represent. Theory*, 26:1063–1079, 2022.

[Sag91] Bruce E. Sagan. *The symmetric group - representations, combinatorial algorithms, and symmetric functions*. Wadsworth and Brooks / Cole mathematics series. Wadsworth, 1991.

[Sch07] J. Schur. Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.*, 132:85–137, 1907.

[Sol61] Louis Solomon. On the sum of the elements in the character table of a finite group. *Proc. Amer. Math. Soc.*, 12:962–963, 1961.

[Sta95] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.

[Sta98] Richard P. Stanley. Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications, & notable open problems. volume 193, pages 267–286. 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ste03] J. R. Stembridge. A local characterization of simply-laced crystals. *Transactions of the American Mathematical Society*, 355(12):5095–5123, 2003.

[Sun18] Sheila Sundaram. The conjugacy action of S_n and modules induced from centralisers. *J. Algebraic Combin.*, 48(2):179–225, 2018.

[Sun19] Sheila Sundaram. On a positivity conjecture in the character table of S_n. *Electron. J. Combin.*, 26(1):Paper No. 1.1, 43, 2019.

[SvW24] Ethan Shelburne and Stephanie van Willigenburg. Schur-positivity for generalized nets. preprint, arXiv:2409.00943, 2024.

[SW16] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions. *Adv. Math.*, 295:497–551, 2016.

[The24] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 10.3)*, 2024. https://www.sagemath.org.

[Wey39] Hermann Weyl. *The Classical Groups. Their Invariants and Representations.* Princeton University Press, Princeton, NJ, 1939.

[WW20] David G. L. Wang and Monica M. Y. Wang. A combinatorial formula for the Schur coefficients of chromatic symmetric functions. *Discrete Appl. Math.*, 285:621–630, 2020.