Skein Lasagna Modules and Stabilization Problems

By

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Abstract

Many of the most well-known open problems in geometric topology involve the behavior of exotic smooth structures in dimension 4. One class of open problems in this field are the *Wall-type* stabilization problems, which ask about the behavior of exotic 4-manifolds under specified topological operations. For a chosen link homology theory, the associated skein lasagna module [**MWW22**] is an invariant of smooth 4-manifolds with robust gluing properties [**MWW23**]. We employ the skein lasagna module construction for Khovanov homology and study the effects of various forms of stabilization on these smooth 4-manifold invariants. Through the computations of these invariants for 4-manifolds relevant to external stabilization, we establish an isomorphism between *Rozansky-Willis homology* groups and skein lasagna modules of $\#^k(S^2 \times D^2)$ with various null-homologous links in the boundary. Furthermore, using skein-theoretic 4-manifold invariants defined in [**MWW24**], we show that skein lasagna modules are capable of distinguishing pairs of exotically knotted surfaces even after an internal stabilization.

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CHAPTER 1

Introduction

1.1. Overview

This thesis examines the application of Khovanov skein lasagna modules to the detection of exotic phenomena in dimension 4. Let X and Y be smooth 4-manifolds, then the pair (X, Y) is called an *exotic pair* if X is homeomorphic to Y, but not diffeomorphic to Y. Many of the most well-known open problems in 4-manifold topology involve the study and detection of such exotic pairs. For example, the *smooth 4-dimensional Poincaré conjecture* posits that there does not exist an exotic S^4. The *h-cobordism theorem* of Smale [**Sma07**] states that for $n \geq 5$, if $W : X_0 \to X_1$ is an $(n+1)$-dimensional cobordism between simply-connected n-manifolds X_0 and X_1 such that the inclusion maps $X_i \hookrightarrow W$ are homotopy equivalences, then W is isomorphic (in the relevant category) to the product cobordism $X_0 \times I$. Famously, for $n = 4$, this theorem is false due to the failure of the *Whitney trick* in dimension 4. However, as a consequence of the h-cobordism theorem, exotic pairs of simply-connected 4-manifolds become diffeomorphic after taking sufficiently many connect-sums with $S^2 \times S^2$ (an operation called an *external stabilization*) [**Wal64**]. One major conjecture in 4-manifold topology asks if a single external stabilization on an exotic pair of closed, simply-connected 4-manifolds is enough to produce a diffeomorphism. In other words, for this type of exotic pair (X, Y), the *(external) stabilization distance* between X and Y is 1.

Given a 4-manifold X, there are analogous questions for smoothly embedded surfaces $\Sigma_i \subset X$. For a pair of such surfaces (Σ_0, Σ_1) in X, we say that (Σ_0, Σ_1) is an *exotically knotted* pair if the surfaces are topologically isotopic in X, but not smoothly isotopic in X. For pairs of the form (X, Σ), where Σ is a smoothly embedded surface in X, taking a connect-sum of pairs of the form $(X, \Sigma)\#(S^2 \times S^2, \emptyset)$ is called an *external stabilization*, whereas a connect-sum of the form $(X, \Sigma)\#(S^4, T^2)$ is called an *internal stabilization*. Finally, if T^2 is an unknotted embedded torus, then taking the connect-sum with (S^4, T^2) is called a *standard internal stabilization*. Analogously, there are many open questions about the effects of such stabilizations on exotically knotted pairs of surfaces.

In 2019, Morrison-Walker-Wedrich [**MWW22**] provide a recipe for extending a functorial link invariant to an invariant of smooth 4-manifolds. These invariants, denoted $\mathcal{S}_0^Z(X; L)$ for an invariant Z and 4-manifold and boundary link pair (X, L), are called *skein lasagna modules*. Over the past few years, these invariants have seen many interesting applications to questions in 4-manifold topology. For example, Ren and Willis in [**RW24**] show that the Khovanov skein lasagna module is capable of distinguishing exotic pairs of 4-manifolds that have not yet been shown to be exotic through previous gauge-theoretic methods.

These skein lasagna invariants have nice gluing properties that make them well-suited for studying the above Wall-type stabilization problems. There are more general gluing maps for pairs of 4-manifolds that share a common 3-manifold, but this thesis focuses primarily on connect-sums. Given pairs (X_0, L_0) and (X_1, L_1), then over a field there is an isomorphism

$$\mathcal{S}_0^{\mathrm{Kh}}(X_0 \# X_1; L_0 \cup L_1) \xrightarrow{\cong} \mathcal{S}_0^{\mathrm{Kh}}(X_0; L_0) \otimes_{\mathbb{F}} \mathcal{S}_0^{\mathrm{Kh}}(X_1; L_1). \tag{1.1}$$

Before the results of Ren-Willis [**RW24**], there was only speculation as to whether or not these invariants could detect exotic smooth structures. Therefore, if it were the case that $\mathcal{S}_0^{\mathrm{Kh}}(S^2 \times S^2; \emptyset)$ was isomorphic to a non-trivial vector space, then Equation 1.1 along with the result of Wall would imply that the invariant $\mathcal{S}_0^{\mathrm{Kh}}$ is not sensitive to differences in smooth structure. This thesis presents an argument for the vanishing of $\mathcal{S}_0^{\mathrm{Kh}}(S^2 \times S^2; \emptyset)$ over $\mathbb{Q}$ using a handle-slide invariant object called a *Kirby color* in the Khovanov homology setting.

For exotically knotted pairs of surfaces and standard internal stabilizations in a 4-manifold, skein lasagna invariants are capable of obstructing smooth isotopy and even once stabilized smooth isotopy. This obstruction comes from extending similar obstruction results of Hayden [**Hay23**] to the 4-manifold setting using a skein lasagna module for *Bar-Natan* homology.

Results. In this thesis, we discuss properties of skein lasagna modules and prove the results of [**SZ24**] and [**Sul25**].

Outline. This thesis consists of 4 chapters and an appendix. Chapter 2 contains the background necessary for the 4-manifold invariants we employ. Chapter 3 contains much of the work carried out in [**SZ24**], which is about studying the effect of an external stabilization on skein lasagna modules. Chapter 4 contains the work discussed in [**Sul25**], which studies the application of these invariants to internal stabilization. Finally, category theory preliminaries and preliminaries about Bar-Natan categories are contained in the appendix A.

CHAPTER 2

Link Homology and Khovanov Skein Lasagna Modules

2.1. Khovanov Homology Background

In this thesis we focus on Khovanov homology and it's deformations and iterations. The famed Jones polynomial [**Jon85**] is a polynomial invariant of links that is computed by using a skein relation on a diagram of the link. In 1999, Mikhail Khovanov produced an invariant, valued in bigraded abelian groups, that *categorifies* the Jones polynomial [**Kho00**]. This means that taking the graded Euler characteristic of this invariant for some link L yields the Jones polynomial of L. In the categorified setting, the skein relation is realized by a cone of a saddle map between two crossing resolutions.

We follow the conventions of [**MWW22**] and [**MN22**], which we briefly recall and collect here in this section. Recall that a *framing* of a knot K is a choice of non-vanishing normal vector field on K. For our purposes, we work in the category of *framed*, oriented links in S^3 with framed, oriented cobordisms $\Sigma : L_0 \to L_1$ in $S^3 \times I$ with $L_i \subset S^3 \times \{i\}$.

Definition 2.1.1. *The* $\mathfrak{gl}_2$ *Khovanov-Rozansky homology (See [**KR08**]) of* L *over a ring* $\mathbb{F}$*, denoted* $\mathrm{KhR}_2(L)$*, is a bigraded vector space*

$$\mathrm{KhR}_2(L) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{KhR}_2^{i,j}(L)$$

where i *and* j *denote the* homological grading *and internal* quantum grading *respectively.*

The homology groups $\mathrm{KhR}_2^{i,j}$ are computed via an iterated mapping cone construction (or *multicone* [**Wil21**]) defined by the following cones associated to positive and negative crossings in the diagram for L.

$$\text{⤱} = h^{-1}q \asymp \to)(\qquad\qquad \text{⤲} =)(\to hq^{-1} \asymp \tag{2.1}$$

The h and q terms denote the formal shift in homological and quantum gradings respectively. In various contexts in this article, it will also be necessary for us to use the bracket notation to indicate homological and quantum shift: $[\![h]\!]$

$$[\![L]\!]_{\mathrm{BN}}[k]\{l\} = h^k q^l [\![L]\!]_{\mathrm{BN}}.$$

We will omit the Bar-Natan brackets $[\![\cdot]\!]_{\mathrm{BN}}$ in figures; the reader may assume that all diagrams represent their algebraic counterpart in the appropriate Bar-Natan category (see Appendix A). See [**Cap08**, **CMW09**, **Bla10**, **San21**, **Vog20**, **ETW18**, **BHPW23**]) for functoriality of Khovanov homology for links in $\mathbb{R}^3$. Morrison-Walker-Wedrich (Theorem 3.3 of [**MWW22**]) verify that the *sweep-around move* cobordism induces the identity map on KhR_2. Satisfying this sweep-around property upgrades the KhR_2 theory from a functorial invariant for links in $\mathbb{R}^3$ (and cobordisms in $\mathbb{R}^3 \times I$) to a functorial invariant for links in S^3 (and link cobordisms in $S^3 \times [0,1]$). In particular, let $\Sigma \subset S^3 \times [0,1]$ be a properly embedded, framed, oriented surface intersecting the boundary 3-spheres $S^3 \times \{0\}$ and $S^3 \times \{1\}$ in links L_0 and L_1 respectively. By the functoriality of KhR_2, there is a well-defined induced homogeneous linear map

$$\mathrm{KhR}_2(\Sigma) : \mathrm{KhR}_2(L_0) \to \mathrm{KhR}_2(L_1).$$

of bidegree $(0, -\chi(\Sigma))$. (This is a special case of (A.8).)

Khovanov homology conventions. Many conventions for Khovanov homology exist in the literature. We collect them here for reference. Let D_L denote a diagram for an oriented link L.

(1) Khovanov's original homology theory $\mathrm{Kh}(D_L)$, defined in [**Kho00**], uses the oriented skein relations

$$\nwarrow\!\!\!\!\!\nearrow \;=\; q\,)(\to hq^2 \asymp \qquad\qquad \nwarrow\!\!\!\!\!\nearrow \;=\; h^{-1}q^{-2} \asymp \to q^{-1})(.$$

Other reference works relevant to our discussion that use this convention include [**BN02**, **BN05**, **Lee05**, **Ras10**, **GLW18**, **CK12**]. In particular, Bar-Natan introduces an unoriented bracket

$$[\![\diagup\!\!\!\!\diagdown]\!]_{\mathrm{BN}} =)(\to hq \asymp$$

and then adds a global shift:

$$\mathrm{CKh}(D) = [\![D]\!]_{\mathrm{BN}}[-n_-]\{n_+ - 2n_-\}$$

where $n_\pm$ is the number of positive/negative ($\pm$) crossings in D_L. This construction is invariant under framing changes (i.e. Reidemeister I moves), as the global shift accounts for the writhe of the diagram.

(2) On the other hand, the literature involving Khovanov's arc algebras and other tangle-related constructions (excluding [**BN05**]) usually uses the conventions from [**Kho02**, **Kho03**], where the quantum degree is reversed. We denote this Khovanov homology convention by $\overline{\mathrm{Kh}}$. Thus

$$\overline{\mathrm{Kh}}(D_L)^{i,j} \cong \mathrm{Kh}(D_L)^{i,-j}.$$

For reference, the oriented skein relations are below.

$$\text{⤱} \;=\; q^{-1})(\to hq^{-2}\asymp \qquad\qquad \text{⤲} \;=\; h^{-1}q^{2}\asymp \to q)(.$$

This is also insensitive to changes in framing induced by Reidemeister I moves.

(3) Khovanov-Rozansky's *unframed* link invariant, defined in [**KR08**], is denoted $\mathbf{KhR}_2$ in the lasagna literature. This is related to the previous two constructions by $\mathbf{KhR}_2(L) \cong \overline{\mathrm{Kh}}(L^!)$ and $\mathbf{KhR}_2^{i,-j}(L) \cong \mathrm{Kh}^{i,-j}(L^!)$, where $L^!$ denotes the mirror of a link L. The skein relations are

$$\text{⤲} \;=\; q^{-1})(\to hq^{-2}\asymp \qquad\qquad \text{⤱} \;=\; h^{-1}q^{2}\asymp \to q)(.$$

(4) In Manolescu-Neithalath [**MN22**], the authors use a framed version of Khovanov-Rozansky's invariant, denoted KhR_2. The oriented skein relations are

Let $\overline{D}$ denote the mirror of the diagram D. Then

$$\mathrm{CKhR}_2(D) \cong \overline{\mathrm{CKh}}(D^!)\{-w(D)\} = \overline{\mathrm{CKh}}(D^!)\{w(D^!)\}$$

where $w(\cdot)$ denotes the writhe of a diagram. In some papers, such as [**Hog19**], KhR_2 is computed first using an unoriented skein relation

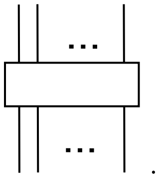
$$\llbracket \times \rrbracket_{\mathrm{KhR}} \;=\; h^{-1}q \asymp \to)(. \tag{2.2}$$

Note that this agrees with the skein relation for $\nwarrow\!\!\!\!\nearrow$; in general, if D contains n_- negative crossings, we have

$$\mathrm{CKhR}_2(D) = \llbracket D \rrbracket_{\mathrm{KhR}}[n_-]\{-n_-\}.$$

2.2. Categorified Projectors

For our computations, we use objects in $\mathrm{Kom}(\mathcal{TL}_n)$ called *categorified projectors* [**Roz14**, **CK12**]. For relevant background on the categories involved in this section, see Appendix A. Let $\mathbf{1}_n$ denote the identity braid on n-strands and let $FT_n^{\otimes m}$ denote the n-strand braid with m positive full-twists. The Jones-Wenzl (or *highest weight*) projector is given by $P_n := \operatorname{colim}_{m\to\infty} FT_n^{\otimes m}$ with degree 0 connecting maps. Throughout, we denote these projectors by boxes in tangle diagrams



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The categorified Jones-Wenzl projector is defined up to homotoy by the following properties.

Lemma 2.2.0.1. *Let P_n be a categorified Jones-Wenzl projector on n strands.*

1. *The n-strand identity braid appears in P_n only once in homological degree 0.*
2. *P_n is homologically bounded above.*
3. *P_n kills turnbacks and is idempotent: $P_n \otimes P_n \simeq P_n$.*
4. *P_n eats crossings: for any Artin generator σ_i in the braid group B_n, we have*

$$P_n \otimes \sigma_i \simeq P_n \simeq P_n \otimes \sigma_i^{-1}.$$

Note that there are no grading shifts with our conventions.

Proof. For a proof of these statements, see [**Hog19**, Theorem 3.7] and [**CK12**]. ☐

Along with categorified Jones-Wenzl projectors P_n, we will employ a related family of chain complexes in $\mathrm{Kom}(\mathcal{TL}_n)$ called *higher order projectors.*

Definition 2.2.1. *Let D be a Temperley-Lieb diagram in $\mathcal{TL}_n$. The* through-degree *of T, denoted $\tau(T)$, is the minimal integer k such that D factors into a vertical stacking $D_1 \otimes D_2$, where $D_1 \in \mathcal{TL}_k^n$ and $D_2 \in \mathcal{TL}_n^k$.*

This definition extends to complexes of Temperley-Lieb diagrams.

Definition 2.2.2. *Let τ denote the* through-degree *of a chain complex of Temperley-Lieb diagrams in* $\mathrm{Kom}(\mathcal{TL}_n)$*. In particular, for a complex $A \in \mathrm{Kom}(\mathcal{TL}_n)$, define*

$$\tau(A) := \max\{\tau(D) \mid D \textit{ a Temperley-Lieb diagram appearing in } A\}.$$

Definition 2.2.3. *[**CH15**, Definition 8.4] The k*th higher order projector *is a chain complex $P_{n,k} \in \mathrm{Kom}(\mathcal{TL}_n)$ uniquely defined by the following properties*

(1) $\tau(P_{n,k}) = k$.

(2) For each $l \in \mathbb{Z}_+$, and $a \in \mathbf{Cob}_{n,l}$, if $\tau(a) < k$, then $a \otimes P_{n,k} \simeq 0$. ($P_{n,k}$ kills complexes with sufficiently low through-degree.)

(3) There exists $C \in \mathrm{Kom}(\mathcal{TL}_n)$ with $\tau(C) < k$, and a twisted complex

$$D = \mathbf{1}_n \to C \to hP_{n,k}$$

such that $a \otimes D \simeq D \otimes \overline{a} \simeq 0$ for all $a \in \mathbf{Cob}_{n,m}$ such that $\tau(a) \leq k$.

We call the higher order projector $P_{n,0}$ of through-degree 0 *the* Rozansky projector *on n strands in* $\mathrm{Kom}(\mathcal{TL}_n)$ *(see [**Roz10**, **Wil21**]).*

Note that the higher order projector $P_{n,k}$ *factors through* P_k. Restating an observation of Cooper-Hogancamp [**CH15**, Observation 8.8], given complexes A, B in $\mathrm{Kom}(\mathcal{TL}_n)$ such that $\tau(A) \geq k$ and $\tau(B) \geq k$, there is an isomorphism $P_{n,k} \cong A \otimes P_k \otimes B$. In the decategorified setting, for some idempotents p_ϵ in the Temperley-Lieb algebra TL_n, there is a decomposition of the identity:

$$1 = \sum p_\epsilon. \tag{2.3}$$

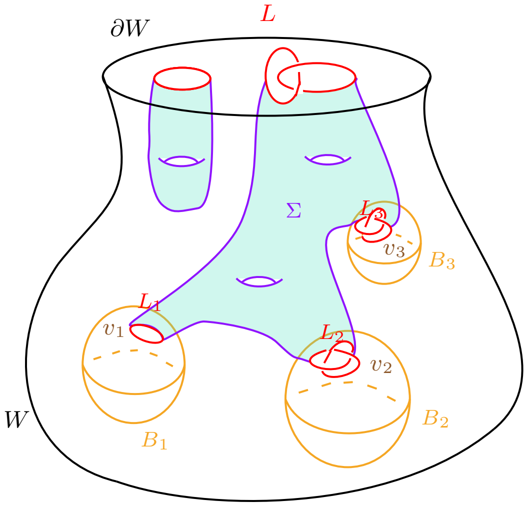
Roughly, the categorification of the equation (2.3) may be realized as the following chain homotopy equivalence.

$$\mathbf{1}_n \simeq \left(P^{\vee}_{n,n(\text{mod } 2)} \to P^{\vee}_{n,n(\text{mod } 2)+2} \to \cdots \to P^{\vee}_{n,n-2} \to P^{\vee}_{n} \right) \tag{2.4}$$

where the right-hand side has higher differentials $P^{\vee}_{n,i} \to P^{\vee}_{n,j}$ $(j > i)$ [**EH17**,**CH15**] (A *categorical diagonalization* of the identity operator). This homotopy equivalence, referred to as the *resolution of the identity* [**CH15**, Section 7, Observation 8.9], will prove to be an instrumental tool in the arguments to follow.

2.3. Skein Lasagna Modules.

Morrison-Walker-Wedrich [**MWW22**] extend the framed version of Khovanov homology for links in S^3 to an invariant $\mathcal{S}_0^2(X; L)$ of a pair $(X, L \subset \partial X)$, where X is an oriented 4-manifold, and L a link in its boundary. For a null-homologous boundary link L ($[L] = 0 \in H_1(X; \mathbb{Z})$), the invariant $\mathcal{S}_0^2$ is a triply-graded module, with trigrading (α, i, j) in $H_2^L(X) \times \mathbb{Z} \times \mathbb{Z}$. The $H_2^L(X)$ term is the $H_2(X)$-torsor, defined as $\partial^{-1}([L]) \subset H_2(X; L)$, where ∂ is the boundary map in the long exact sequence of the pair (X, L). The α-grading is called the *skein grading*, and the gradings i and j are the homological and quantum gradings inherited from KhR_2 respectively. There is an additional 0-grading that appears in the notation, this is reference to the fact that there are other invariants $\mathcal{S}_k^2$ $(k > 0)$ constructed via *blob homology* [**MW12**]. Given a 4-manifold and boundary link pair (X, L), the module $\mathcal{S}_0^2(X; L)$ are generated by *Khovanov lasagna fillings*, which are defined as follows and depicted in Figure 2.3. In this section we work primarily over $\mathbb{Q}$, but any commutative ring will do.



Definition 2.3.1. *A* lasagna filling *of* $(X, L \subset \partial W)$ *is an object consisting of the following data:* $\mathcal{F} := (\Sigma, \{(B_i, L_i, v_i)\})$*, where:*

- *A finite set of 4-balls* $\{B_i\}$*, disjointly embedded in* X*, each with a link* $L_i \subset \partial B_i$*, and a homogeneous label* $v_i \in \mathrm{KhR}_2(L_i)$*.*

- *A framed oriented surface* Σ *properly embedded in* $W \setminus \bigcup_i B_i$ *such that* $\Sigma \cap B_i = L_i$*, and* $\Sigma \cap \partial W = L$*.*

There is a well-defined bidegree for fillings $\mathcal{F}$ of a pair (X, L):

Definition 2.3.2. *The* bidegree of a lasagna filling $\mathcal{F}$ *is given by:*

$$\deg(\mathcal{F}) := \sum_i \deg(v_i) + (0, -\chi(\Sigma))$$

Furthermore, when $W = B^4$, we define $\mathrm{KhR}_2(\mathcal{F}) := \mathrm{KhR}_2(\Sigma)(\otimes_i x_i) \in \mathrm{KhR}_2(\partial W; L)$, where $\mathrm{KhR}_2(\Sigma)$ is the morphism induced by Σ of $\mathcal{F}$.

Definition 2.3.3. For a 4-manifold X and a link $L \subset \partial W$, the *skein lasagna module* of $(X; L)$ is the bigraded abelian group:

$$\mathcal{S}_0^2(X; L) := \mathbb{Q}\langle \mathcal{F} \text{ of } (X, L)\rangle / \sim$$

The relation is defined as the transitive and linear closure of the following relations.

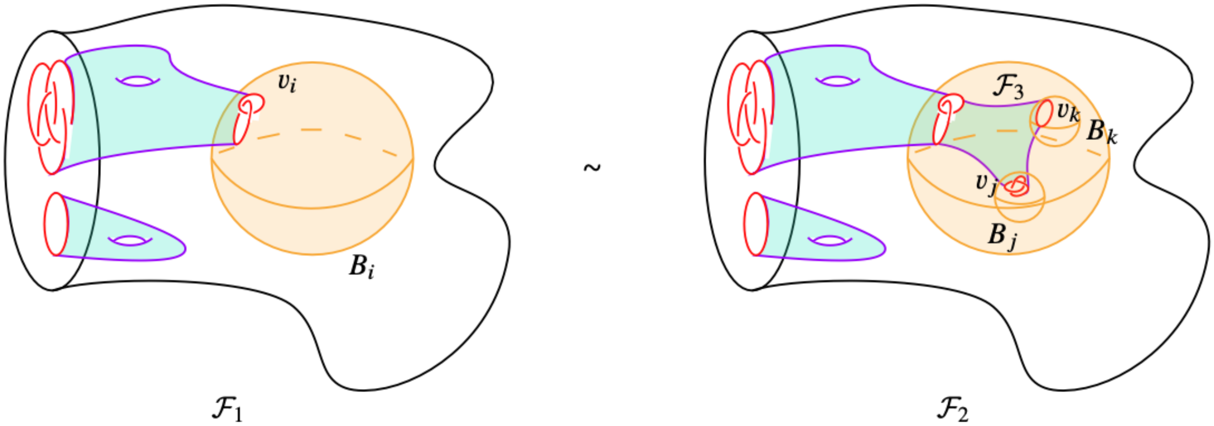


FIGURE 2.1. The second relation of Definition 2.3.3 on lasagna fillings.

- Linear combinations of lasagna fillings are multilinear in the KhR_N labels $\{v_i\}$.
- Declare two lasagna fillings $\mathcal{F}_1$ and $\mathcal{F}_2$ equivalent if $\mathcal{F}_1$ has an input ball B_i with boundary link L_i labelled v_i, and the filling $\mathcal{F}_2$ is obtain from $\mathcal{F}_1$ by inserting a lasagna filling $\mathcal{F}_3$ of (B_i, L_i) into B_i such that $v_i = \mathrm{KhR}_2(\mathcal{F}_3)$, possibly followed by an isotopy rel boundary (see Figure 2.1). This relation is called the *enclosement relation*.

These invariants can be computed from a handle decomposition of a 4-manifold, and there are formulae for 4-dimensional handles of each index [**MWW23**]. In this work, we look only at 2− and 4−handle attachment, and we only attach 2−handles to B^4. By [**MN22**, Proposition 1.6], the skein lasagna module's isomorphism class remains unchanged after the removal of a 4-ball. In particular, if X is a closed, smooth 4-manifold, there is an isomorphism $\mathcal{S}_0^2(X;\emptyset) \cong \mathcal{S}_0^2(X \setminus B^4;\emptyset)$. For 2−handles attached to the S^3 boundary of a 4-ball, there is a formula of Manolescu-Neithalath [**MN22**, Theorem 1.1] that relates the invariant of the corresponding knot trace to a particular colimit of Khovanov homology groups (or, isomorphically, skein lasagna modules of the form $\mathcal{S}_0^2(B^4;L)$). We discuss this formula now. Let $L \subset S^3$ be a framed oriented link with components $L_1, L_2, ..., L_k$, and let r^-, $r^+ \in \mathbb{Z}^k_{\geq 0}$.

DEFINITION 2.3.4. *The* (r^-, r^+)-cable of L, *denoted* $L(r^-, r^+)$, *is the framed oriented link consisting of* r_i^- *many negatively oriented parallel strands and* r_i^+ *many positively oriented parallel strands for the component* L_i. *The notion of parallel is given by pushing off along the framing of each* L_i,

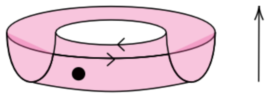


FIGURE 2.2. The dotted annulus cobordism, denoted by ⊍ throughout.

and 'positively' (resp. negatively) oriented means the orientation of the parallel strand agrees (resp. disagrees) with the original orientation of the component L_i*.*

Let ⊍ denote the dotted annulus cobordism from the empty link to two oppositely oriented parallel strands in Figure 2.2. Let e_i denote the ith unit vector. Note that ⊍ is a cobordism between cables of L;

$$⊍ : L(r^-, r^+) \to L(r^- + e_i, r^+ + e_i). \tag{2.5}$$

DEFINITION 2.3.5. *Let* $\mathfrak{S}_n$ *denote the symmetric group on* n *elements and, for* $\alpha \in \mathbb{Z}^n$*, let* α^+ *(resp.* α^-*) denote the tuple* $(\alpha_1^+, ..., \alpha_n^+)$ *where* $\alpha_i^+ := \max\{0, \alpha_i\}$ *(resp.* $\alpha_i^- := \min\{0, \alpha_i\}$*). The* cabled Khovanov homology *over* $\mathbb{Q}$ *of a framed oriented link* L *with* n *components at skein grading* $\alpha \in \mathbb{Z}^n$ *is defined as*

$$\underline{\mathrm{KhR}}_{2,\alpha}(L) = \left(\bigoplus_{r \in \mathbb{Z}^n_{>0}} \mathrm{KhR}_2(L(r - \alpha^-, r + \alpha^+))\{-2|r| - |\alpha|\} \right) / \sim$$

where $\sim$ *is the transitive and linear closure of the following identifications:*

$$\beta(b)(v) \sim v, \qquad \mathrm{KhR}_2(⊍)(v) \sim v$$

for all $b \in \mathfrak{S}_{2r_i + |\alpha_i|}$ *and for all* $v \in \mathrm{KhR}_2(L(r - \alpha^-, r + \alpha^+))$ *where*

(1) *For an element* b *of the braid group* $B_{r_i - \alpha_i^- + r_i + \alpha_i^+}$*, the map* $\beta_i(b)$ *is the automorphism induced on* $\mathrm{KhR}_2(L_i(r_i - \alpha_i^-, r_i + \alpha_i^+))$ *by the braid group action interchanging parallel strands. By [**GLW18**], this braid group action on cables factors through the symmetric group.*

(2) $\mathrm{KhR}_2(⊍)$ *denotes the morphism induced by the dotted annulus cobordism* ⊍ *(see Figure 2.2).*

We note that the undotted annulus relation is omitted from our definition as in [**MN22**, Proposition 3.8], as it is not necessary in our setting. We present an equivalent definition of $\underline{\mathrm{KhR}}_2$ tailored to 4-manifold and boundary link pairs (X, L), which we will use in this article. Let $\mathrm{Sym}(\mathrm{KhR}_2(L))$ denote the vector space $\mathrm{KhR}_2(L)$ symmetrized with respect to the braid group action in part (1) of Definition 2.3.5. If f is a linear map between vector spaces, let $\mathrm{Sym}(f)$ denote the induced map on symmetrized vector spaces.

Definition 2.3.6. *Let X be a 4-manifold with a 0-handle, k many 2-handles, and possibly a 4-handle, with 2-handles attached along a framed oriented link $L = L_1 \cup L_2 \cup \ldots \cup L_k$. Let $(I, \leq)$ be the directed set $\mathbb{Z}^k_{\geq 0}$ with the poset relation induced by the total ordering $\leq$ on $\mathbb{Z}$; this forms a poset category. The* cabling directed system *for X at skein grading α, denoted $\mathcal{B}^\alpha(X; \emptyset)$, is a functor from $\mathbb{Z}^k_{\geq 0}$ to $\mathrm{ggVect}_{\mathcal{F}}$ where*

- *for $a \in I$, $\mathcal{B}^\alpha(X; \emptyset)(a) := \mathrm{Sym}(\mathrm{KhR}_2(L_a))$, where $L_a := L(a - \alpha^-, a + \alpha^+)$ and*
- *the arrow from $\mathcal{B}^\alpha(X; \emptyset)(a)$ to $\mathcal{B}^\alpha(X; \emptyset)(a + e_i)$ is* Sym() *at the corresponding 2-handle attachment site $L_i \subset L$ as described above.*

The *cabled Khovanov homology of L* at skein grading α may then be defined as the colimit of the cabling system $\mathcal{B}^\alpha(X; \emptyset)$. The above construction for the framed oriented link corresponding to the attaching link of the link trace of L, denoted $X_f(L)$ (that is, the 4-manifold obtained by attaching 4-dimensional 2-handles to B^4 along L with framings $f = (f_1, f_2, \ldots, f_{|L|})$), is isomorphic to the skein lasagna module of the pair $(X_f(L), \emptyset)$. In particular, we have the following 2-handle formula:

Theorem 2.3.7 ([**MN22**], Theorem 1.1). *Let $X_f(L)$ denote the link trace framed n-component link L. For each $\alpha \in H_2(X; \mathbb{Z}) \cong \mathbb{Z}^n$, there is an isomorphism*

$$\Phi : \operatorname{colim}_{\mathrm{ggVect}}(\mathcal{B}^\alpha(X; \emptyset)) \xrightarrow{\cong} \mathcal{S}^2_0(X; \emptyset, \alpha). \tag{2.6}$$

We also require a relative version of the construction above for a pair $(X_f(L), K)$ with a nontrivial, null-homologous boundary link K. Specifically, let L denote the framed attaching and let K denote a link in $\partial(X_f(L))$ such that $[K] = 0 \in H_1(X_f(L))$. The authors of [**MWW23**] describe such a cabled Khovanov homology construction for this setup by considering cables of the form $L(r^-, r^+) \cup K$ as follows. Note that the braid group action defined above yields a braid group action

$\beta_i : B_{r_i^-, r_i^+} \to \mathrm{Aut}(\mathrm{KhR}_2(L(r^-, r^+) \cup K))$, and similarly for the dotted annulus map we have an induced map

$\mathrm{KhR}_2($  $) : \mathrm{KhR}_2(L(r^-, r^+) \cup K) \to \mathrm{KhR}_2(L(r^- + e_i, r^+ + e_i) \cup K).$

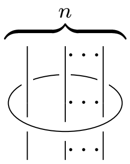
Definition 2.3.8. *Let L and K be the 2-handle attaching link and boundary link respectively, where L has k components. Then the* relative cabled Khovanov homology *is defined as*

$$\underline{\mathrm{KhR}}_{2,\alpha}(L, K) := \left(\bigoplus_{r \in \mathbb{Z}^k_{\geq 0}} \mathrm{KhR}_2(L(r - \alpha^-, r + \alpha^+) \cup K)\{-2|r| + |\alpha|\} \right) / \sim$$

where the relation $\sim$ is the relation in Definition 2.3.5. For the equivalent cabling directed systems definition, let $L_a := L(a - \alpha^-, a + \alpha^+) \cup K$ for each $a \in I$. We then obtain a new directed system $\mathcal{A}^\alpha(X_f(L); K)$ whose colimit is identically $\underline{\mathrm{KhR}}_{2,\alpha}(L, K)$.

Remark 2.3.1. *The above definition agrees with the cabled skein lasagna module construction in [**MWW23**] with $X = B^4$. that is, $\underline{\mathrm{KhR}}_{2,\alpha}(L, K) \cong \mathcal{S}^2_0(X_f(L); K, \alpha)$.*

In our approach, we instead work with tangles and cobordisms in the relevant completions of Bar-Natan's categories. We postpone closing up tangles and taking homology until after we compute homotopy colimits. The relationship between this approach and the method used in [**MN22**] is discussed in Section 3.1. We now construct our primary objects of study. Let $\mathbf{1}_n$ denote the identity braid on n strands, and let T_n denote the unoriented chain complex associated to the identity braid "wearing a belt":



Observe that $T_n^{\otimes k}$ denotes the identity braid on n strands wearing k parallel, unlinked belts. We now describe the action of the symmetric group $\mathfrak{S}_k$ on the chain complex $T_n^{\otimes k}$. There are two ribbon maps

$$\cup : \mathbf{1}_n \to T_n^{\otimes 2} \qquad \cap : T_n^{\otimes 2} \to \mathbf{1}_n.$$

The reader should note that these are shorthand symbols for cobordisms; for example, $\cup$ represents a cobordism that is topologically $\cup \times S^1$. We will sometimes also compose these maps; for example, $\cap \circ \cup$ is a torus (wrapped around the identity cobordism) and therefore represents the morphism $\mathbf{1}_n \xrightarrow{2} \mathbf{1}_n$. This follows directly from the local relations in Figure A.3. We will also use dotted ribbon cobordisms, in which case we use the symbols $\dot{\cup}$, $\dot{\cap}$, as seen in (2.5) and Figure 2.2. We write the composition $\cup \circ \cap$ as $\asymp$.

Consider the braid group action on the k belt loops in $T_n^{\otimes k}$. Let $\sigma_i \in B_k$, and let $\Sigma_i : T_n^{\otimes k} \to T_n^{\otimes k}$ denote the cobordism corresponding to the movie where the $(i+1)$st belt grows wider and moves up and around the ith belt, interchanging them; the cobordism looks like $\sigma_i \otimes S^1$ near the belts, along with n identity sheets corresponding to the n vertical strands. Let Σ_i^{-1} denote the upside-down cobordism (i.e. time-reversed movie). Grigsby-Licata-Wehrli in [**GLW18**] show that the cobordisms $\{\Sigma_i\}_{i=1}^k$ satisfy the braid relations on the nose. Furthermore, they show the braid group B_k action descends to an action by the symmetric group $\mathfrak{S}_k$, which we now describe.

Define the *swap* endomorphism $s : T_n^{\otimes 2} \to T_n^{\otimes 2}$ by

$$s = \mathrm{id} - \asymp$$

on the two belt loops. Since $(\asymp)^2 = 2\asymp$ by the local relations, we have $s^2 = \mathrm{id}$. Define s_i to be the corresponding swap endomorphism involving only the ith and $(i+1)$st belts:

$$s_i = \mathrm{id}_{i-1} \otimes s \otimes \mathrm{id}_{k-(i+1)}$$

Note that in Grigsby-Licata-Wehrli's conventions, a torus evaluates to -2; since our torus evaluates to +2, the corresponding statement of [**GLW18**, Proposition 9] is

$$\Sigma_i = s_i = \Sigma_i^{-1}. \tag{2.7}$$

Thus s_i is the morphism realizing the transposition of the ith and $(i+1)$st belts under the $\mathfrak{S}_k$ action.

In order to symmetrize the complex $P_n \otimes T_n^{\otimes k}$ under the $\mathfrak{S}_k$ action, we consider the morphism

$$e_k := \frac{1}{k!} \sum_{g \in \mathfrak{S}_k} g \quad \in \quad \mathrm{End}(T_n^{\otimes k}).$$

Definition 2.3.9. *Let $\mathcal{C}$ be a dg-category, a* homotopy idempotent $e \in \operatorname{End}(X)$ *is a closed degree 0 endomorphism of some object X such that $e^2 \sim e$. Equivalently, it is an idempotent in the homotopy category of $\mathcal{C}$. Furthermore, an object X is an* image *of a homotopy idempotent e if there exist closed degree 0 maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim e$ and $g \circ f \sim \mathrm{id}_X$. (For more details, see [**GHW22**, Section 4].)*

By standard representation theory arguments, we have the following lemma.

Lemma 2.3.9.1. *The endomorphism $e_k : T_n^{\otimes k} \to T_n^{\otimes k}$ is a homotopy idempotent, i.e. $e_k^2 \sim e_k$.*

Proof. The argument is standard and follows from the fact that left multiplication by any fixed $g \in \mathfrak{S}_k$ gives a permutation of the set $\mathfrak{S}_k$. Finally, note that we only have a homotopy equivalence between e_k^2 and e_k because the action of the generators s_i is defined only up to homotopy; the Reidemeister move equivalences are homotopy equivalences, not isomorphisms of complexes. ☐

Gorsky-Hogancamp-Wedrich in [**GHW22**] prove that the homotopy category $K(\mathcal{C})$ of a Karoubian category $\mathcal{C}$ is also Karoubian, and the images of homotopy idempotents are unique up to homotopy. By [**GW23**] Theorem A.10, bounded homotopy categories of Karoubian categories are Karoubian, so the images of homotopy idempotents are guaranteed to exist.

Definition 2.3.10. *Let $\operatorname{Sym}(T_n^{\otimes k})$ denote an image of $T_n^{\otimes k}$ under the idempotent e_k in $K(\operatorname{Kar}(\operatorname{Kom}(\mathcal{TL}_n)))$. There exist maps*

$$\begin{tikzcd} T_n^{\otimes k} \arrow[r, bend left, "p_k"] & \operatorname{Sym}(T_n^{\otimes k}) \arrow[l, bend left, "i_k"] \end{tikzcd}$$

such that

$$i_k \circ p_k \sim e_k \qquad \text{and} \qquad p_k \circ i_k \sim \mathrm{id}_{\operatorname{Sym}(T_n^{\otimes k})}. \tag{2.8}$$

For a morphism $f : T_n^{\otimes k} \to T_n^{\otimes l}$, let $\operatorname{Sym}(f) := p_l \circ f \circ i_k$ denote the induced morphism $\operatorname{Sym}(T_n^{\otimes k}) \to \operatorname{Sym}(T_n^{\otimes l})$.

We verify that the map induced by the undotted annulus is identically 0 on symmetrized $T_n^{\otimes k}$ complexes.

Lemma 2.3.10.1. *Let $k \in \mathbb{N}$. Let $\cup : T_n^{\otimes k} \to T_n^{\otimes k+2}$ be the ribbon map that introduces the last pair of belts and is the identity sheet on all other components. Then* $\mathrm{Sym}(\cup) \simeq 0$.

Proof. By (2.8),

$$\mathrm{Sym}(\cup) = p_{k+2} \circ \cup \circ i_k = p_{k+2} \circ i_{k+2} \circ p_{k+2} \circ \cup \circ i_k = p_{k+2} \circ (e_{k+2} \circ \cup) \circ i_k.$$

So, it suffices to show that $e_{k+2} \circ \cup = 0$. Note that, letting s_k denote the swap endomorphism on the kth and $(k+1)$th strands, e_{k+2} is the sum of all compositions of s_j, $j \in \{1, ..., k+1\}$. Note that $s_{k+1} \circ \cup = -\cup$, and also that the set of permutations in $\mathfrak{S}_{k+2}$ can be decomposted into pairs $(g, g \circ s_{k+1})$. We then have that any permutation composed with the cup map gives

$$g \circ \cup + g \circ s_{k+1} \circ \cup = g \circ \cup - g \circ \cup = 0.$$

Thus, $e_{k+2} \circ \cup = 0$.

□

Let $\underline{\mathcal{TL}}^{\oplus}$ denote the category $\mathrm{Ind}(K(\mathrm{Kar}(\mathrm{Kom}(\mathcal{TL}_n))))$. We are now ready to define the identity tangle with a *Kirby belt* $T_n^{\omega_\alpha}$.

Definition 2.3.11. *For $\alpha \in \mathbb{N}$, let $T_n^{\omega_\alpha} \in \underline{\mathcal{TL}}^{\oplus}$ denote the colimit of the directed system*

$$\mathcal{A}_n^\alpha := \left(\mathrm{Sym}(T_n^{\otimes \alpha}) \xrightarrow{\mathrm{Sym}(\cup)} \mathrm{Sym}(T_n^{\otimes \alpha+2}) \xrightarrow{\mathrm{Sym}(\cup)} \mathrm{Sym}(T_n^{\otimes \alpha+4}) \xrightarrow{\mathrm{Sym}(\cup)} \cdots \right).$$

Note that only the parity of α matters on the level of colimits, so there are only two Kirby-belted identity objects, corresponding to $\alpha = 0, 1$. For n vertical strands, we denote $\mathrm{colim}\,(\mathcal{A}_n^0)$ and $\mathrm{colim}\,(\mathcal{A}_n^1)$ by $T_n^{\omega_0}$ and $T_n^{\omega_1}$ respectively. We also consider the homotopy colimits of $\mathcal{A}_n^0$ and $\mathcal{A}_n^1$, described as follows.

Definition 2.3.12. *Let $T_n^{\Omega_\alpha}$ denote the homotopy colimit of the directed system $\mathcal{A}_n^\alpha$ in Definition 2.3.11. In particular, $T_n^{\Omega_\alpha}$ is the total complex* $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}_n^\alpha})$ *of the double complex $\mathcal{D}_{\mathcal{A}_n^\alpha}$ associated to $\mathcal{A}_n^\alpha$ as in Figure A.1. (see Figure 2.3 for the double complex representing $T_n^{\Omega_0}$).*

Note that $T_n^{\omega_\alpha}$ and $T_n^{\Omega_\alpha}$ are equivalent by Proposition A.0.2, so for the remainder of this work we will denote this object only by $T_n^{\Omega_\alpha}$, and label Kirby-colored components with Ω_α.

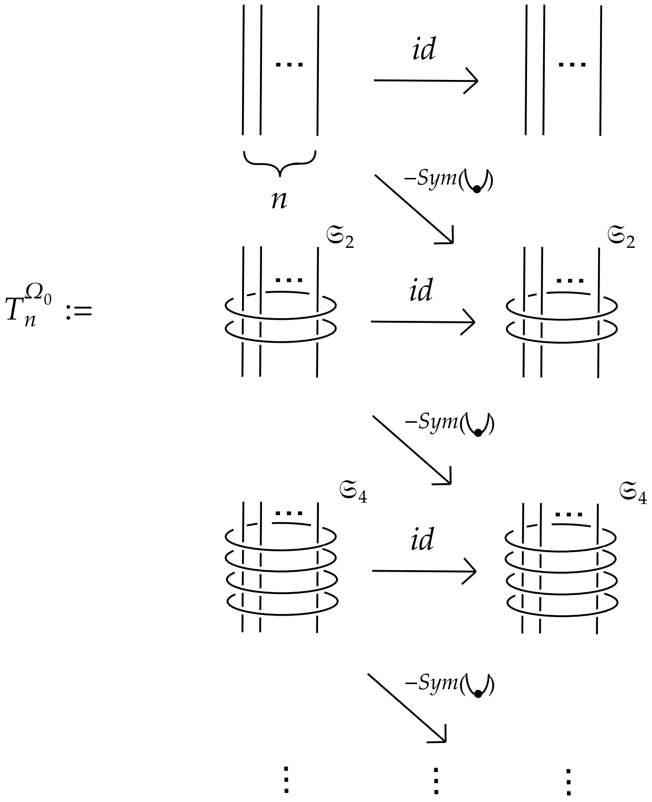


FIGURE 2.3. For $\alpha = 0$. The diagrams $(T_n^{\otimes k})^{\mathfrak{S}_k}$ denote the symmetrized complexes $\mathrm{Sym}(T_n^{\otimes k})$.

REMARK 2.3.2. *Despite using ‘Kirby-colored’ terminology, we work with homotopy colimits of tangle complexes not in the annular setting, so our construction is different from that of [**HRW22**]. The* ith Kirby object *([**HRW22**])* ω_i' *for* $i \geq 0$ *is an object in the Ind-completion of the additive closure of the Karoubi envelope of the annular Bar-Natan category represented by the colimit*

$$\omega_i' := \left(q^{-i} P_i \to q^{-(i+2)} P_{i-2} \to q^{-(i+4)} P_{i-4} \to \cdots\right)$$

where the arrows are certain dotted maps between projectors. Letting L *once again denote the* f*-framed oriented link that a 2-handle is attached to* B^4 *along, and letting* K *be a boundary link in* $\partial(X_f(L))$*, a theorem of Hogancamp-Rose-Wedrich can be stated as follows.*

THEOREM 2.3.13 ([**HRW22**], Theorem C). *Let* $(X_f(L), K)$ *denote the* 4*-manifold obtained by attaching a* 2*-handle to* B^4 *along the* n *component link* L*, with a link* K *in the boundary, and let* $\omega'_{\underline{i}}$ *denote a collection of Kirby objects where* $\underline{i} \in \{0,1\}^n$*. Decorate the* n *components of* L *with* n *Kirby objects* ω'_i *for* $i \in \{0,1\}$ *(In other words, decorate* L *with* $\omega'_{\underline{i}}$*). Then the following bigraded vector spaces are isomorphic:*

(a) The Kirby-colored Khovanov homology $\mathrm{Kh}(K \cup (L)^{\omega'_{\underline{i}}})$

(b) The relative cabled Khovanov homology of $K \cup L$ *at skein grading* $\underline{i}$*.*

(c) The $N = 2$ *skein lasagna module of* $(X_f(L), K)$ *at skein grading* $\underline{i}$*.*

The reason that items (a) and (c) are equivalent to (b) is because the relative cabled Khovanov homology of $K \cup L$ *is precisely the Manolescu-Walker-Wedrich* cabled skein lasagna *construction in [**MWW23**] for a* 4*-manifold with no* $1-$ *or* 3*-handles and an arbitrary link* K *in the boundary.*

CHAPTER 3

Rozansky-Willis Homology and External Stabilization

In this section, we compute $\mathcal{S}_0^2$ for pairs of the form $(S^2 \times D^2, L)$ and use these computations to compute $\mathcal{S}_0^2(S^2 \times S^2; \emptyset)$. We show that $\mathcal{S}_0^2(S^2 \times D^2; L)$ recovers an invariant of links in $\#^k(S^1 \times S^2)$ called *Rozansky-Willis* homology.

3.1. Skein lasagna module computations using Kirby-colored belts

Let $\{p_1, \ldots, p_n\}$ be a collection of n distinct points on S^2. In this section, we compute the skein lasagna module $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n)$, where $\widetilde{\mathbf{1}}_n$ is the geometrically essential link $\{p_1, \ldots, p_n\} \times S^1 \subset S^1 \times S^2 = \partial X$ (see Figure 3.1).

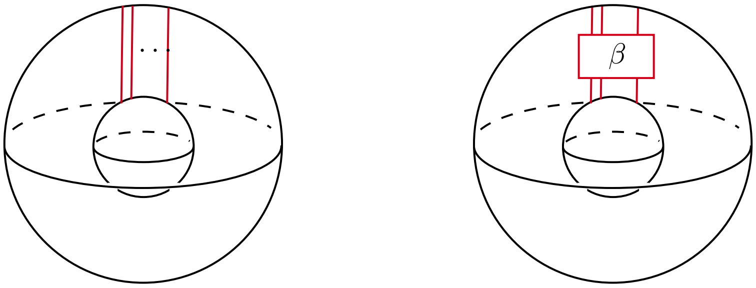
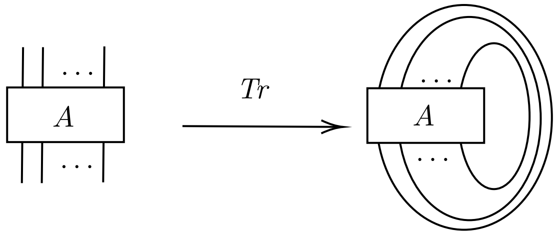


Figure 3.1. **Left:** The n component link $\widetilde{\mathbf{1}}_n$ in $S^1 \times S^2 = [0,1] \times S^2/(0,p) \sim (1,p)$. **Right:** An example of a geometrically essential link $\widetilde{\beta}$ given by a braid β in $S^1 \times S^2$.

Throughout this section, we use the symbol $\widehat{\mathbf{1}}_n$ when referring to the usual closure of the identity braid in the (thickened) plane, and use the symbol $\widetilde{\mathbf{1}}_n$ when referring to the specific link in $S^1 \times S^2$ shown on the left in Figure 3.1.

3.1.1. Equivalence of $H^*(\mathrm{Tr}(T_n^{\Omega_\alpha}))$ and $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n, \alpha)$. Here we confirm that the homology of the trace of $T_n^{\Omega_\alpha}$ is isomorphic to the skein lasagna module of $S^2 \times B^2$ with $\tilde{\mathbf{1}}_n$ in the boundary.

Definition 3.1.1. *Let* $\mathrm{Tr} : \mathcal{TL}_n \to \mathcal{BN}$ *denote the trace closure functor.*



By functoriality, we may take the tangle closure as in Definition 3.1.1 of each term in the double complex $\mathcal{D}_{\mathcal{A}_n^\alpha}$, then take the homology of each term to obtain a double complex of (bigraded) vector spaces $\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha}$ depicted in Figure 3.2.

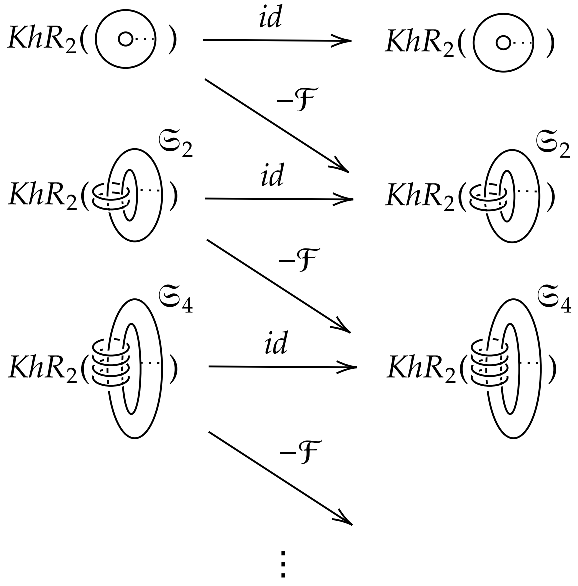


FIGURE 3.2. The double complex $\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha}$ associated to the closure of $T_n^{\Omega_0}$, where $\mathcal{F}$ denotes the morphism induced by the symmetrized dotted annulus map. The quantum degree shifts from the definition of cabled Khovanov homology have been suppressed.

The totalization of this double complex is

$$\mathrm{Tr}(T_n^{\Omega_\alpha}) := \mathrm{Tot}(\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha}) = \bigoplus_{k=0}^{\infty} \mathrm{KhR}_2(\mathrm{Sym}(\mathrm{Tr}(T_n^{\otimes 2k+\alpha}))) \xrightarrow{\mathrm{id}-\mathcal{F}} \bigoplus_{k=0}^{\infty} \mathrm{KhR}_2(\mathrm{Sym}(\mathrm{Tr}(T_n^{\otimes 2k+\alpha}))).$$

where $\mathcal{F}$ is comprised of the morphisms induced by symmetrized dotted ribbon maps.

Then, by Lemma A.0.9.3 and Corollary A.0.1, we obtain the following proposition.

Proposition 3.1.1. *Let* $U_0 \cup \widehat{\mathbf{1}}_n$ *denote the link obtained by the tangle closure of* T_n*, where the belt is a* 0*-framed unknot* U_0*. We have the following isomorphisms of vector spaces:*

$$\begin{aligned} H^*(\mathrm{Tr}(T_n^{\Omega_\alpha})) &\cong H^*(\mathrm{Tot}(\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha})) \\ &\cong \mathrm{colim}\,(\mathcal{B}^\alpha(S^2 \times B^2; \widetilde{\mathbf{1}}_n)) \\ &\cong \underline{\mathrm{KhR}}_{2,\alpha}(U_0, \widehat{\mathbf{1}}_n) \\ &\cong \mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n, \alpha) \end{aligned}$$

Proof. The first isomorphism $H^*(\mathrm{Tr}(T_n^{\Omega_\alpha})) \cong H^*(\mathrm{Tot}(\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha}))$ is an application of Lemma A.0.9.3 and Corollary A.0.1. The second and third isomorphisms follow from the observation that the homology of $\mathrm{Tot}(\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha})$ is manifestly the relative cabled Khovanov homology $\underline{\mathrm{KhR}}_{2,\alpha}(U_0, \widehat{\mathbf{1}}_n)$, which is isomorphic to $\mathrm{colim}\,(\mathcal{B}^\alpha(S^2 \times B^2; \widehat{\mathbf{1}}_n))$ by construction, and is furthermore isomorphic to $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n, \alpha)$ by Remark 2.3.1. ☐

The results presented in this thesis primarily involve the link $\widetilde{\mathbf{1}}_n$ because we consider belts wrapped around the n-strand identity braid. However, by stacking braids, we obtain results for other geometrically essential links in $\partial(S^2 \times B^2) = S^1 \times S^2$. Letting β denote the tangle complex associated to an n-strand braid, we can replace each copy of $T_n^{\otimes \alpha+2k}$ above with the tangle complex $\beta \otimes T_n^{\otimes \alpha+2k}$. Let $\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha,\beta}$ denote the double complex obtained from $\mathcal{D}_{\mathcal{A}_n^\alpha}$ with each $\mathrm{Sym}(T_n^{\otimes \alpha+2k})$ term replaced by $\mathrm{Sym}(\beta \otimes T_n^{\otimes \alpha+2k})$ and trace and homology taken term-by-term. Let $\mathrm{Tr}(\beta \otimes T_n^{\Omega_\alpha})$ denote $\mathrm{Tot}(\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha,\beta})$.

Corollary 3.1.1. *Taking the trace of* $\beta \otimes T_n^{\otimes k}$ *and taking homology, we obtain isomorphisms:*

$$\begin{aligned} H^*(\mathrm{Tr}(\beta\otimes T_n^{\Omega_\alpha})) &\cong H^*(\mathrm{Tot}(\widehat{\mathcal{D}}_{\mathcal{A}_n^\alpha,\beta})) \\ &\cong \operatorname{colim}\,(\mathcal{B}^\alpha(S^2\times B^2;\widetilde{\beta})) \\ &\cong \underline{\mathrm{KhR}}_{2,\alpha}(U_0,\widehat{\beta}) \\ &\cong \mathcal{S}_0^2(S^2\times B^2;\widetilde{\beta},\alpha) \end{aligned}$$

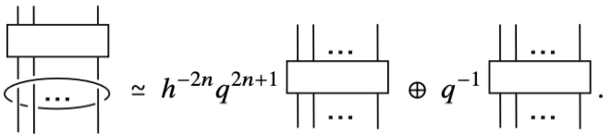
Thus, we are able to study the skein lasagna modules of pairs $(S^2\times B^2,\widetilde{\beta})$ by studying the homotopy colimits $T_n^{\Omega_\alpha}$ and $\beta\otimes T_n^{\Omega_\alpha}$.

3.2. Projectors wearing symmetrized belts

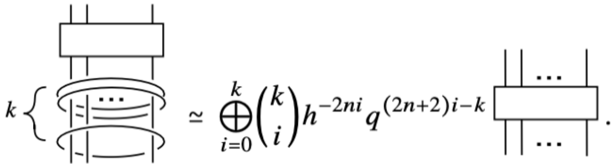
We begin the study of the homotopy colimits in the previous section by calculating $\mathrm{Sym}(P_n\otimes T_n^{\otimes k})$ for $k\geq 0$ computing $P_n\otimes T_n^{\Omega_\alpha}$ for $i=0,1$. Our first goal is to explicitly describe the symmetric part of $P_n\otimes T_n^{\otimes k}$ under the $\mathfrak{S}_k$ action permuting the k belts. Let us first consider the complex $P_n\otimes T_n$, i.e. a projector wearing one belt.

REMARK 3.2.1. *Given a tangle diagram D, we first use the unoriented skein relation (2.2) to decompose the diagram into flat tangles. We then introduce the global bigrading shift dictated by (2.1). On the other hand, the projector P_n is already defined to be an object in* $\mathrm{Kom}(\mathcal{TL}_n)$ *with absolute gradings.*

LEMMA 3.2.0.1. *[**Hog19**, Corollary 3.51] The unoriented complex $P_n\otimes T_n$ splits as*



COROLLARY 3.2.1. *For the projector P_n with k belts, we have*



Proof. Note that $P_n \otimes T_n \simeq T_n \otimes P_n$, and that P_n is idempotent, so $(P_n \otimes T_n)^{\otimes k} \simeq P_n \otimes T_n^{\otimes k}$. For the degree shifts, note that $(h^{-2n}q^{2n+1})^i(h^0q^{-1})^{k-i} = h^{-2ni}q^{(2n+2)i-k}$. $\square$

We now identify the $\mathfrak{S}_k$ action on the right-hand side of the homotopy equivalence in Corollary 3.2.1. Let V_i be a $\binom{k}{i}$-dimensional vector space at bigrading $(-2ni, (2n+2)i-k)$. Let $[k]$ denote the set of indices $\{1, 2, \ldots, k\}$. The standard basis vectors in V_i can be identified with the set of multi-indices $\{I \subset [k] \mid |I| = i\}$. Let V denote the vector space $\bigoplus_{i=0}^{k} V_i$.

The symmetric group $\mathfrak{S}_k$ acts on each V_i by permuting the elements of $[k]$. To be precise, if $\sigma \in \mathfrak{S}_k$, and $I = \{j_1, j_2, \ldots, j_i\}$, then $\sigma I = \{\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_i)\}$. Let $\mathbf{V}$ denote the $\mathfrak{S}_k$ representation $\bigoplus_{i=0}^{k} V_i$.

Let $\sigma \in \mathfrak{S}_k$ and let $\mathrm{Ext}^{i,j}(P_n)$ denote the group of bidegree (i,j) endomorphisms of P_n modulo chain homotopy. A priori, with some choice of basis for V_i, the action of σ on $P_n \otimes V_i$ is given by a $\binom{k}{i} \times \binom{k}{i}$ matrix M_σ with entries in $\mathrm{Ext}^{0,0}(P_n)$. However, by [**Hog19**, Corollary 3.35(2)] (and the universal coefficient theorem), $\mathrm{Ext}^{0,0}(P_n) \cong \mathbb{F}$, so we may view M_σ as a matrix with coefficients in $\mathbb{F}$.

Here we wish to show that by some choice of basis, M_σ is precisely the matrix representing the action of σ on V_i. To do this, we will rely on Grigsby-Licata-Wehrli's description of the $\mathfrak{S}_k$ action on the canonical generators in the Lee homology of $\mathrm{Tr}\left(T_n^{\otimes k}\right)$, so some setup is in order.

Each multi-index I determines a sign sequence ϵ_I dictating an orientation on the k belts in $T_n^{\otimes k}$. Let o_I denote the orientation on $T_n^{\otimes k}$ where the vertical strands are all oriented upwards, and the belts are oriented according to ϵ_I, where the belt at position $j \in [k]$ links negatively with the vertical strands if and only if $j \in I$.

By the naturality of the trace functor, taking an (n,n)-tangle T to $\mathrm{Tr}\,(T)$, we may instead consider the object $\mathrm{Tr}\left(P_n \otimes T_n^{\otimes k}\right)$ in $\mathrm{Kom}(\mathcal{BN})$. Since the $\mathfrak{S}_k$ acts by cobordism maps, we have that

$$\mathrm{Tr}\left(P_n \otimes T_n^{\otimes k}\right) \simeq \mathrm{Tr}\,(P_n) \otimes V.$$

We will now take the trace and apply the Lee homology functor, which will allow us to pick out a set of homology classes; by keeping track of the action of $\mathfrak{S}_k$, on this set, we will identify the representation V.

Let $\mathrm{FT}_n \in \mathrm{Kom}(\mathcal{TL}_n)$ denote the positive full twist on n strands. Recall that $P_n = \operatorname{colim}_{m\to\infty} \mathrm{FT}_n^{\otimes m}$, where each $\mathrm{FT}_n^{\otimes m} \overset{\iota}{\hookrightarrow} \mathrm{FT}_n^{\otimes m+1}$ as a subcomplex [**Roz14**]. After applying the Lee homology functor $\mathrm{Lee} : \mathrm{Kom}(\mathcal{BN}) \to \mathrm{gVect}$, by [**Lee05**] we have that $\mathrm{Lee}(\mathrm{Tr}\left(\mathrm{FT}_n^{\otimes m} \otimes T_n^{\otimes k}\right))$ is generated by the Lee canonical classes $\{\mathfrak{s}_o\}$, which are in bijection with the set of orientations on the link $\mathrm{Tr}\left(\mathrm{FT}_n^{\otimes m} \otimes T_n^{\otimes k}\right)$.

By an abuse of notation, we let o_I also denote the orientation on the $n+k$ components of $\mathrm{Tr}\left(\mathrm{FT}_n^{\otimes m} \otimes T_n^{\otimes k}\right)$ where the vertical strands are all oriented upwards, and the k belts are oriented according to I. Let $\mathfrak{s}_I^m$ denote the Lee generator corresponding to o_I. Observe that under the maps induced by the inclusion maps of subcomplexes

$$\mathrm{Lee}(\mathrm{Tr}\left(\mathrm{FT}_n^{\otimes m} \otimes T_n^{\otimes k}\right)) \xrightarrow{\iota_*} \mathrm{Lee}(\mathrm{Tr}\left(\mathrm{FT}_n^{\otimes m+1} \otimes T_n^{\otimes k}\right)),$$

the Lee class $\mathfrak{s}_I^m$ is mapped to the Lee class $\mathfrak{s}_I^{m+1}$. (This can be deduced by considering the oriented resolution of the two links, and verifying that the inclusion map ι identifies the Lee cycles s_I^m and s_I^{m+1} whose (nonzero) homology classes are $\mathfrak{s}_I^m$ and $\mathfrak{s}_I^{m+1}$, respectively.)

Hence we may define colimits $\mathfrak{s}_I := \operatorname{colim}_{m\to\infty} \mathfrak{s}_I^m$, which are (nonzero) classes in

$$\mathrm{Lee}(\mathrm{Tr}\left(P_n \otimes T_n^{\otimes k}\right)) := \operatorname{colim}_{m\to\infty} \mathrm{Lee}(\mathrm{Tr}\left(\mathrm{FT}_n^{\otimes m} \otimes T_n^{\otimes k}\right)) \cong \mathrm{Lee}(\mathrm{Tr}\left(P_n\right)) \otimes V. \tag{3.1}$$

Lemma 3.2.0.2. *The Lee homology of the trace of the projector* $\mathrm{Lee}(\mathrm{Tr}\left(P_n\right))$ *is two dimensional, generated by the Lee generators corresponding to the braidlike and antibraidlike orientations.*

Proof. The braid FT_n contains $n(n-1)$ crossings, i.e. two crossings between any two given strands.

Let $J \subset [n]$ be a multiindex of weight j. That is, in the corresponding orientation on FT_n, there are $n_\uparrow = j$ strands pointing upward (braidlike), and $n_\downarrow = n-j$ strands pointing downward (antibraidlike).

To understand the relative homological grading of $\mathfrak{s}_{o_J}$, we must understand the number of negative crossings in (FT_n, o_J).

Each $\uparrow$ strand links with other $\uparrow$ strands positively, but links with each $\downarrow$ strand once, i.e. they cross at two crossings. Since we will also consider the contribution from the other strand, we will count this as one negative crossing.

Each $\downarrow$ strand links with other $\downarrow$ strands positively, but links with each $\uparrow$ strand once, i.e. at two crossings. The contribution to negative crossings is again one.

Therefore the total number of negative crossings in (FT_n, o_J) is

$$n_\uparrow n_\downarrow + n_\downarrow n_\uparrow = 2n_\uparrow n_\downarrow = 2j(n-j).$$

The total number of negative crossings in $(\mathrm{FT}_n^{\otimes m}, o_J)$ is then $2mj(n-j)$. So, if $j \neq 0$ or n, then as the number of full twists increases ($m \to \infty$), the number of negative crossings grows without bound. On the other hand, the braidlike and antibraidlike resolutions remain at homological grading 0 and survive to the colimit. $\square$

PROPOSITION 3.2.1. *Under the chain homotopy equivalence in Corollary 3.2.1, the action of* $\mathfrak{S}_k$ *on* $P_n \otimes T_n^{\otimes k}$ *agrees with the action of* $\mathfrak{S}_k$ *on* $P_n \otimes \mathbf{V}$*, where the action of* P_n *is trivial.*

PROOF. For each fixed m, Grigsby-Licata-Wehrli show that the action of $\sigma \in \mathfrak{S}_k$ sends $\mathfrak{s}_I^m \mapsto \mathfrak{s}_{\sigma I}^m$ (see [**GLW18**, Section 7]). By functoriality of Lee homology, the $\mathfrak{S}_k$-action on $\mathrm{Lee}(\mathrm{Tr}\left(FT_n^{\otimes m} \otimes T_n^{\otimes k}\right))$ is compatible with the action on $\mathrm{Lee}(\mathrm{Tr}\left(FT_n^{\otimes m+1} \otimes T_n^{\otimes k}\right))$. Thus, in the colimit, the action of σ takes $\mathfrak{s}_I \mapsto \mathfrak{s}_{\sigma I}$.

It remains to verify that for each i, the set $\{\mathfrak{s}_I \mid |I| = i\}$ forms a basis for the $\binom{k}{i}$-dimensional vector space at homological grading $-2ni$ in $\mathrm{Lee}(\mathrm{Tr}\left(P_n \otimes T_n^{\otimes k}\right))$. Note that the homological grading is preserved as we pass from $P_n \otimes T_n^{\otimes k} \in \mathrm{Kom}(\mathcal{TL}_n)$ to $\mathrm{Tr}\left(P_n \otimes T_n^{\otimes k}\right) \in \mathrm{Kom}(\mathcal{BN})$ and further to $\mathrm{Lee}(\mathrm{Tr}\left(P_n \otimes T_n^{\otimes k}\right)) \in \mathrm{gVect}$ (with KhR_2 conventions).

Since there are $\binom{k}{i}$ elements in the set, it suffices to show that they are all linearly independent. Indeed, since there are only finitely many of these classes, if there were some nonzero linear combination of the $\{\mathfrak{s}_I \mid |I| = i\}$, there would be some finite level M where for all $m \geq M$, the same relation would hold among $\{\mathfrak{s}_I^m \mid |I| = i\}$; this is impossible because the set of all $\{\mathfrak{s}_I \mid I \subset [k]\}$ forms a subset of a basis for $\mathrm{Lee}(\mathrm{Tr}\left(FT_n^{\otimes m+1} \otimes T_n^{\otimes k}\right))$.

To summarize, we have shown that the action of $\mathfrak{S}_k$ is standard on the subspace of $\mathrm{Lee}(P_n \otimes T_n^{\otimes k})$ corresponding to orientations of $P_n \otimes T_n^{\otimes k}$ where the vertical strands are oriented upward. The same holds for the set of orientations where the vertical strands are anti-braidlike; let $\bar{\mathfrak{s}}_I$ denote the Lee generator corresponding to the orientation $\bar{o}_I$, where the orientation of *all* $n+k$ strands are reversed from their orientation in o_I.

Finally, let φ denote the chain homotopy equivalence realizing (3.1). Then the images of $\{\mathfrak{s}_I\}_{I \subset [k]} \cup \{\bar{\mathfrak{s}}_I\}_{I \subset [k]}$ under φ form a basis for $\mathrm{Lee}(P_n) \otimes V$ that realizes that the action of $\sigma \in \mathfrak{S}_k$ as the standard permutation matrix on the 2^k subsets of $[k]$. Therefore $V \cong \mathbf{V}$ as $\mathfrak{S}_k$ representations. $\square$

In other words, the 2^k P_n components in Corollary 3.2.1 correspond to the 2^k subsets of $[k]$, and the $\mathfrak{S}_k$ action is the one induced by the natural action of $\mathfrak{S}_k$ on $[k]$. This action has $k+1$ orbits, indexed by the subset size $0 \leq i \leq k$. Therefore

$$\mathrm{Sym}(P_n \otimes T_n^{\otimes k}) \simeq \bigoplus_{i=0}^{k} h^{-2ni} q^{(2n+2)i-k} P_n. \tag{3.2}$$

Finally, we add orientations to the computation of the unoriented bracket (which agrees with the KhR bracket if all crossings are positive). Let T_n^+ (resp. T_n^-) denote the n-strand identity braid with a single counterclockwise (resp. clockwise) oriented belt.

3.2.1. Dual Projector with Kirby-colored belt.

For our computations, we will actually need the dual projector. Recall that diagrams in $\mathcal{TL}_n$ are drawn in the unit square $[0,1]^2$ in the xy-plane, with n endpoints each on the intervals $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. The dualizing functor $(\cdot)^\vee : \mathrm{Kom}^-(\mathcal{TL}_n) \to \mathrm{Kom}^+(\mathcal{TL}_n)$ reflects the diagrams across the line $y = 1/2$, reverses both homological and quantum degree, and is contravariant on morphisms (see more discussion following Theorem 4.12 in [**Hog20**]). Observe, for example, that the complexes for a positive and a negative crossing are dual to each other.

THEOREM 3.2.1 ([**Hog20**], Theorem 4.12). *There is a natural isomorphism*

$$\mathrm{Hom}^\bullet_{\mathrm{Kom}^-(\mathcal{TL}_n)}(A, B) \cong q^n \mathrm{Hom}^\bullet_{\mathrm{Kom}^\Pi(\mathcal{TL}_0)}\left(\emptyset, \mathrm{Tr}\left(BA^\vee\right)\right)$$

Since $\mathrm{Tr}\left(P_n \otimes P_n^\vee\right) \cong \mathrm{Tr}\left(P_n^\vee \otimes P_n\right)$ (in $\mathrm{Kom}^\Pi(\mathcal{TL}_0)$), we immediately deduce the following:

Corollary 3.2.2. *There is a (grading-preserving) isomorphism*

$$\mathrm{Ext}(P_n, P_n) \cong \mathrm{Ext}(P_n^\vee, P_n^\vee).$$

The dual projector $P_n^\vee$ satisfies the same properties as P_n (cf. Lemma 2.2.0.1): it is idempotent, eats crossings, and kills turnbacks, and the identity braid appears only at homological degree 0. Of course, the dual projector is bounded below rather than above. This difference will become important later, because it guarantees that any endomorphism of $P_n^\vee$ of negative homological degree is nilpotent.

Lemma 3.2.1.1. *Regardless of how the identity braid $\mathbf{1}_n$ is oriented, $\mathbf{1}_n \otimes T_n^+ \otimes T_n^- = T_n^+ \otimes T_n^-$ must have an equal number of positive and negative crossings. Since there are $4n$ total crossings, $n_- = n_+ = 2n$. Thus*

$$\begin{aligned} P_n \otimes T_n^+ \otimes T_n^- &= h^{n_-} q^{-n_-} P_n \otimes T_n \otimes T_n \\ &\simeq h^{-2n} q^{2n+2} P_n \oplus 2(h^0 q^0 P_n) \oplus h^{2n} q^{-2n-2} P_n. \end{aligned}$$

We now describe the dotted annulus map on $\mathrm{Sym}(P_n^\vee \otimes T_n^{\otimes k})$ with orientations. Abusing notation, denote the morphisms in $\mathrm{Kom}(\mathcal{TL}_n)$ induced by the ribbon cobordisms which wrap (resp. unwrap) two antiparallel rings around $\mathbf{1}_n$ by

$$\cup, \underset{\bullet}{\cup} : \mathrm{Tr}\left(\mathbf{1}_n\right) \to \mathrm{Tr}\left(T_n^+ \otimes T_n^-\right) \qquad \text{and} \qquad \cap, \overset{\bullet}{\cap} : \mathrm{Tr}\left(T_n^+ \otimes T_n^-\right) \to \mathrm{Tr}\left(\mathbf{1}_n\right).$$

Here, we assume $\mathbf{1}_n$ is given some orientation.

We will need to understand the symmetrized map of bidegree $(0, 2)$

$$\mathrm{Sym}(\mathrm{id}_{P_n^\vee} \otimes \underset{\bullet}{\cup}) : P_n^\vee \to h^{-2n} q^{2n+2} P_n^\vee \oplus P_n^\vee \oplus h^{2n} q^{-2n-2} P_n^\vee \tag{3.3}$$

that appear in the cabling system defining $P_n^\vee \otimes T_n^{\omega_\alpha}$.

To do this, we will rely on Hogancamp's computations for morphisms between categorified projectors, specifically the Ext groups listed below:

Lemma 3.2.1.2. *[**Hog19**, Corollary 3.35] The group $\mathrm{Ext}^{i,j}(P_n^\vee, P_n^\vee)$ of maps $h^i q^j P_n^\vee \to P_n^\vee$ mod homotopy satisfies the following.*

(1) If $k < 0$, then $\mathrm{Ext}^{k-i,i}(P_n^\vee, P_n^\vee) = 0$ *for all i.*

(2) $\mathrm{Ext}^{0-i,i}(P_n^\vee, P_n^\vee) \cong \mathbb{F}$ *when $i = 0$ and zero otherwise.*

(4) If $i \in \{2, 4, ..., 2n\}$, then $\mathrm{Ext}^{2-i,i}(P_n^\vee, P_n^\vee) \cong \mathbb{F}$ *and zero otherwise.*

PROPOSITION 3.2.2. *For $n > 0$, The homotopy colimit $P_n^\vee \otimes T_n^{\Omega_\alpha}$ for $\alpha \in \{0, 1\}$ is contractible.*

PROOF. Let $\alpha \in \{0, 1\}$ and consider the directed system

$$\mathrm{Sym}(P_n^\vee \otimes T_n^{\pm\alpha}) \to \mathrm{Sym}(P_n^\vee \otimes T_n^{\pm\alpha} \otimes (T_n^+ \otimes T_n^-)) \to \mathrm{Sym}(P_n^\vee \otimes T_n^{\pm\alpha} \otimes (T_n^{+\otimes 2} \otimes T_n^{-\otimes 2})) \to \cdots$$

denoted $P_n^\vee \otimes \mathcal{A}_n^\alpha$, where the arrows are given by $\mathrm{Sym}(\mathrm{id}_{P_n^\vee} \otimes$  $)$. By (3.2), each object each object in the directed system $P_n^\vee \otimes \mathcal{A}_n^\alpha$ is a sum of shifted projectors,
Let us study the components of the (symmetrized) dotted annulus map in (3.3) as shown in the diagram below:

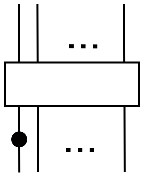
(3.4)
$$\begin{tikzcd}
 & P_n^\vee \arrow[dl, "\phi_1"'] \arrow[d, "\phi_2"] \arrow[dr, "\phi_3"] & \\
h^{-2n}q^{2n+2}P_n^\vee & P_n^\vee & h^{2n}q^{-2n-2}P_n^\vee.
\end{tikzcd}$$

(ϕ_1) The bidegree $(0, 2)$ map $\phi_1 : P_n \to h^{-2n}q^{2n+2}P_n$ corresponds to a degree-preserving map

$$h^{2n}q^{(-2n-2)+2}P_n \to P_n,$$

up to homotopy, i.e. an element of $\mathrm{Ext}^{2n,-2n}(P_n, P_n)$. This falls into Case (2) of Lemma 3.2.1.2, with $k = 0$ and $i = -2n \neq 0$. Therefore $\phi_1 \simeq 0$.

(ϕ_2) By Lemma 3.2.1.2, the space of endomorphisms of $P_n^\vee$ in bidegree $(0, 2)$, up to homotopy, is isomorphic to $\mathbb{F}$. In the same paper ([**Hog19**] Theorem 1.13), Hogancamp shows that $\mathrm{Ext}^{0,2}(P_n, P_n)$ is generated by the dotted identity map $U_1^{(n)}$, depicted below:

$U_1^{(n)} :=$ 

This is also true for $P_n^\vee$ (the dual morphism to dotted identity is still a dotted identity, but on the dual object). Thus $\phi_2 \simeq cU_1^{(n)}$ for some $c \in \mathbb{F}$. Since $U_1^{(n)}$ is nilpotent of order 2, the nilpotency order of ϕ_2 is at most 2.

(ϕ_3) Because $P_n^\vee$ is bounded below, the map ϕ_3 is nilpotent on any class in $P_n^\vee$. In other words, fixing a class x at homological degree $\mathrm{gr}_h(x)$ in the complex $P_n^\vee$, there exists some $N(x) \in \mathbb{N}$ such that for all $k > N(x)$, the map

$$\phi_3^k : P_n^\vee \to h^{2nk} q^{(-2n-2)k} P_n^\vee$$

sends $x \to 0$, simply because $h^{2nk} q^{(-2n-2)k} P_n^\vee$ has chain group 0 at homological degree $\mathrm{gr}_h(x)$.

Let x denote a class in the directed system of shifted projectors. Any sufficiently long path ρ from x through the directed system will satisfy at least one of the following:

- ρ contains an instance of ϕ_1
- ρ contains an instance of $(\phi_2)^2$
- ρ contains $> N(x)$ instances of ϕ_3.

By the discussion above, we thus have $x \sim 0$ in the colimit.

Thus, $\operatorname{colim}(P_n^\vee \otimes \mathcal{A}_n^\alpha) = P_n^\vee \otimes T_n^{\omega_\alpha} = 0$. Finally, by Proposition A.0.3, we have that $P_n^\vee \otimes T_n^{\Omega_\alpha} \simeq 0$ as desired. ☐

3.3. Homological levels with at least one odd term

In this section, we prove that the complex $T_n^{\Omega_\alpha}$ is 0 when n is odd. To do so, we use the resolution of identity (2.4) and prove that $P_{n,k}^\vee \otimes T_n^{\Omega_\alpha} \simeq 0$ for $k \leq n$ and odd. We begin by proving the following commuting property for $T_n^{\Omega_\alpha}$.

Lemma 3.3.0.1. *Let τ_i denote the Temperley-Lieb generator on the strands at positions i and $i+1$. Then $\tau_i \otimes T_n^{\Omega_\alpha} \simeq T_n^{\Omega_\alpha} \otimes \tau_i$, for $\alpha \in \{0,1\}$ (see Figure 3.3).*

Proof. To obtain the equivalence in Figure 3.3, we produce a chain homotopy equivalence map $\Phi : \tau_i \otimes T_n^{\Omega_\alpha} \to T^{\Omega_\alpha} \otimes \tau_i$. Note that, for some fixed number of belts, we have a chain homotopy

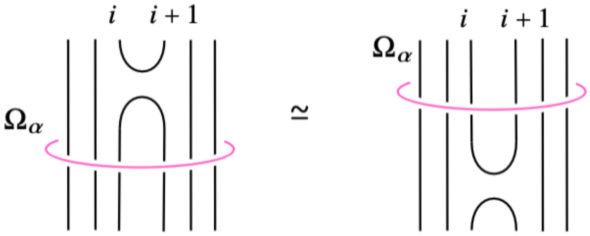
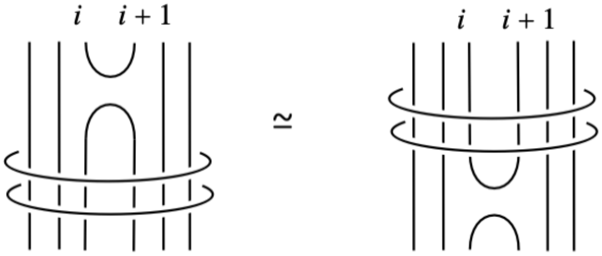
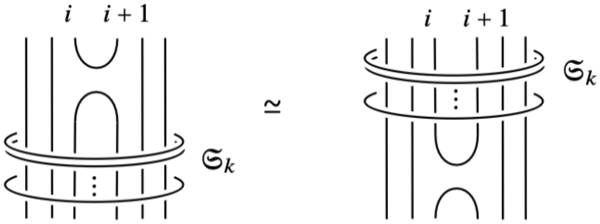


FIGURE 3.3. Commuting rule for a Kirby-colored belt and a $\mathcal{TL}_n$-generator.

equivalence given by a composition of Reidemeister II moves. The following figure is an example for $\tau_i \otimes T_n^{\otimes 2} \simeq T_n^{\otimes 2} \otimes \tau_i$.



Let $\mathcal{R}_{i,k} : \tau_i \otimes T_n^{\otimes k} \to T_n^{\otimes k} \otimes \tau_i$ denote the chain maps associated to the composition of cobordisms that slide the k belts through the diagram τ_i. Note that, by functoriality, the maps $\mathcal{R}_{i,k}$ commute with our dotted ribbon maps up to homotopy, and commute as well with the cobordisms associated to the homotopy $\mathfrak{S}_k$-action on belts. Hence, we have a homotopy equivalence:



Let $\alpha_{i,k} : \tau_i \otimes \mathrm{Sym}(T_n^{\otimes k}) \to \mathrm{Sym}(T_n^{\otimes k}) \otimes \tau_i$ be the chain map that induces the homotopy equivalence above. Then there is an induced comparison chain map $\alpha : \tau_i \otimes T_n^{\Omega_j} \to T_n^{\Omega_j} \otimes \tau_i$ that gives a homotopy equivalence of homotopy colimits by Lemma A.0.9.2(a) and A.0.9.2(c).

□

REMARK 3.3.1. *The arguments in the proof of Lemma 3.3.0.1 also hold for Temperley-Lieb diagrams with different numbers of endpoints. In particular, if τ is a chain complex associated to a planar diagram with no crossings in $\mathbf{Cob}_{n,k}$, then by an argument identical to the above, we obtain*

$$\tau \otimes T_n^{\Omega_\alpha} \simeq T_k^{\Omega_\alpha} \otimes \tau.$$

We will require the following property of a Kirby-colored belt with a cone stacked on top.

PROPOSITION 3.3.1. *Let $f : A \to B$ be a chain map in $\mathrm{Kom}(\mathcal{TL}_n)$, then there is a well-defined map $f \otimes \mathrm{id} : A \otimes T_n^{\Omega_\alpha} \to B \otimes T_n^{\Omega_\alpha}$. Furthermore, we have that*

$$\mathrm{Cone}(A \otimes T_n^{\Omega_\alpha} \xrightarrow{f\otimes \mathrm{id}} B \otimes T_n^{\Omega_\alpha}) = (\mathrm{Cone}(A \xrightarrow{f} B)) \otimes T_n^{\Omega_\alpha}.$$

See Figure 3.4.

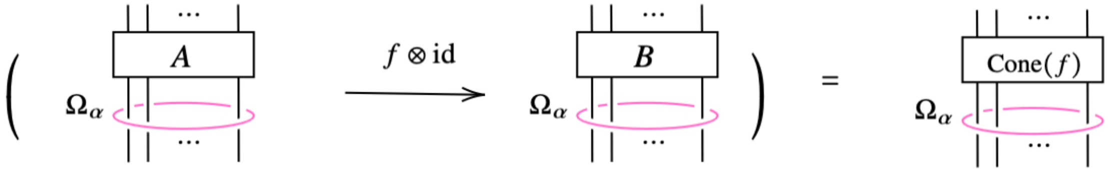


FIGURE 3.4. The cone property for a Kirby-colored belt described in Proposition 3.3.1.

PROOF. Let $\mathrm{id}^k : T_n^{\otimes k} \to T_n^{\otimes k}$ denote the identity map on $T_n^{\otimes k}$. Note first that, by cobordism invariance in $\mathrm{Kom}(\mathcal{TL}_n)$, the chain maps $f \otimes \mathrm{id}^k : A \otimes T_n^{\otimes k} \to B \otimes T_n^{\otimes k}$ commute with the dotted ribbon map and the $\mathfrak{S}_k$-action permuting the k belts. Thus, the collection of maps $\{f \otimes \mathrm{id}^k\}$ satisfies the hypothesis of Lemma A.0.9.2 for directed systems:

$$A \otimes \mathcal{A}_n^\alpha := A \otimes \mathrm{Sym}(T_n^{\otimes \alpha}) \to A \otimes \mathrm{Sym}(T_n^{\otimes \alpha+2}) \to A \otimes \mathrm{Sym}(T_n^{\otimes \alpha+4}) \to \cdots$$

$$B \otimes \mathcal{A}_n^\alpha := B \otimes \mathrm{Sym}(T_n^{\otimes \alpha}) \to B \otimes \mathrm{Sym}(T_n^{\otimes \alpha+2}) \to B \otimes \mathrm{Sym}(T_n^{\otimes \alpha+4}) \to \cdots.$$

By notational abuse, let $f \otimes \mathrm{id}$ denote the collection $\{f \otimes \mathrm{id}^k\}$, by Lemma A.0.9.2(a), we have that $f \otimes \mathrm{id}$ is a well-defined map on the homotopy colimits

$$f \otimes \mathrm{id} : \mathrm{hocolim}\,(A \otimes \mathcal{A}_n^\alpha) \to \mathrm{hocolim}\,(B \otimes \mathcal{A}_n^\alpha).$$

Let $\mathcal{C}_n^\alpha$ denote the directed system of cones $\{\mathrm{Cone}(A \otimes T_n^{\otimes \alpha+2m} \xrightarrow{f\otimes \mathrm{id}^m} B \otimes T_n^{\otimes \alpha+2m})\}_{m\in\mathbb{N}}$. By Lemma A.0.9.2(b), we also have the equality

$$\mathrm{Cone}(f \otimes \mathrm{id}) = \mathrm{hocolim}\,(\mathcal{C}_n^\alpha). \tag{3.5}$$

Since $\mathrm{Cone}(A \otimes T_n^{\otimes \alpha+2m} \xrightarrow{f\otimes \mathrm{id}^m} B \otimes T_n^{\otimes \alpha+2m}) = \mathrm{Cone}(A \xrightarrow{f} B) \otimes T_n^{\otimes m}$ by monoidality, we have that $\mathrm{hocolim}\,(\mathcal{C}^\alpha)$ is identically the chain complex $\mathrm{Cone}(A \xrightarrow{f} B) \otimes T_n^{\Omega_\alpha}$, so (3.5) becomes

$$\mathrm{Cone}(A \otimes T_n^{\Omega_\alpha} \xrightarrow{f\otimes \mathrm{id}} B \otimes T_n^{\Omega_\alpha}) = \mathrm{Cone}(A \xrightarrow{f} B) \otimes T_n^{\Omega_\alpha},$$

as desired. ☐

Since any chain complex B in $\mathrm{Kom}(\mathcal{TL}_n)$ is an iterated mapping cone of chain complexes associated to Temperley-Lieb diagrams, from Proposition 3.3.1 and Lemma 3.3.0.1 we obtain the following corollary.

COROLLARY 3.3.1. *Let B be a chain complex in $\mathrm{Kom}(\mathcal{TL}_n^k)$. Then $P_k^\vee \otimes B \otimes T_n^{\Omega_\alpha} \simeq P_k^\vee \otimes T_k^{\Omega_\alpha} \otimes B \simeq 0$ for $0 < k \leq n$. See Figure 3.5.*

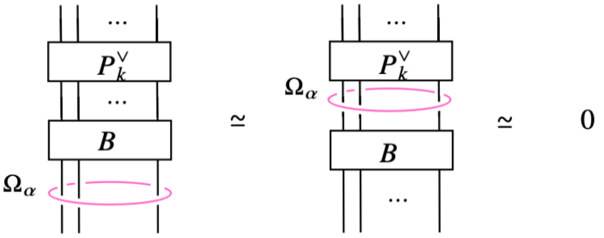


FIGURE 3.5. An illustration of Corollary 3.3.1.

PROOF. Suppose first that the complex B is given by a single $\mathcal{TL}_n^k$-diagram for a positive integer $k \leq n$. Then by Lemma 3.3.0.1 we have that $P_k^\vee \otimes B \otimes T_n^{\Omega_\alpha} \simeq P_k^\vee \otimes T_k^{\Omega_\alpha} \otimes B$. The desired equivalence then follows as $P_k^\vee \otimes T_k^{\Omega_\alpha} \simeq 0$ by Proposition 3.2.2. Next, suppose that B is an arbitrary chain complex in $\mathrm{Kom}(\mathcal{TL}_n^k)$, then $P_k^\vee \otimes B \otimes T_n^{\Omega_\alpha}$ decomposes as a multicone where each chain complex is of the form $P_k^\vee \otimes \tau \otimes T_n^{\Omega_\alpha}$ where τ is a $\mathcal{TL}_n^k$-diagram. By above, each term of $P_k^\vee \otimes B \otimes T_n^{\Omega_\alpha}$ is chain homotopy equivalent to 0 and therefore $P_k^\vee \otimes B \otimes T_n^{\Omega_\alpha} \simeq P_k^\vee \otimes T_k^{\Omega_\alpha} \otimes B \simeq 0$ as desired. □

Recalling that higher order projectors factor as $P_{n,k}^\vee = A \otimes P_k^\vee \otimes B$, Corollary 3.3.1 and Proposition 3.2.2 allows us to conclude the following.

Corollary 3.3.2. *For an integer $0 < k \leq n$, the complex $P_{n,k}^\vee \otimes T_n^{\Omega_\alpha}$ is contractible.*

We are now ready to prove the main result of this section.

Proposition 3.3.2. *If n is an odd positive integer, then $T_n^{\Omega_\alpha} \simeq 0$.*

Proof. For any n, by (2.4), we can express $T_n^{\Omega_\alpha} = \mathbf{1}_n \otimes T_n^{\Omega_\alpha}$ as

$$T_n^{\Omega_\alpha} \simeq P_{n,n(\mathrm{mod}\ 2)}^\vee \otimes T_n^{\Omega_\alpha} \to \dots \to P_{n,n-2}^\vee \otimes T_n^{\Omega_\alpha} \to P_n^\vee \otimes T_n^{\Omega_\alpha}.$$

If n is odd, then Corollary 3.3.2 implies $T_n^{\Omega_\alpha} \simeq 0$. □

Since the homology of the trace of $T_n^{\Omega_\alpha}$ is isomorphic to the skein lasagna module $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n, \alpha)$, Proposition 3.3.2 produces the following immediate corollary.

Corollary 3.3.3. *Let n be odd and let $\alpha \in H_2^{\widetilde{\mathbf{1}}_n}(S^2 \times B^2) \cong H_2(S^2 \times B^2) \cong \mathbb{Z}$. We have that $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n, \alpha) \cong 0$.*

Proof. If n is odd, then $T_n^{\Omega_\alpha} \simeq 0$, implying that $H^*(\mathrm{Tr}(T_n^{\Omega_\alpha})) \cong \mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_n, \alpha) \cong 0$ by Proposition 3.1.1. □

We apply this Corollary 3.3.3 to compute $\mathcal{S}_0^2(S^2 \times S^2; \emptyset, \underline{\alpha})$ for specific homological levels.

Theorem 3.3.1. *Let $\underline{\alpha} = (\alpha_1, \alpha_2) \in H_2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}^2$ with at least one α_1 or α_2 odd, we have that $\mathcal{S}_0^2(S^2 \times S^2; \emptyset, \underline{\alpha}) \cong 0$.*

Proof. Recall that a Kirby diagram of $S^2 \times S^2$ is the Hopf link $L = L_1 \cup L_2$ with 0-framing on both components. Let $I = \mathbb{Z}_{\geq 0}$ and $J = \mathbb{Z}_{\geq 0}$, both equipped with the usual poset relation. The cabling directed system of $(S^2 \times S^2, \emptyset)$ at homological level $\underline{\alpha}$, denoted $\mathcal{B}^{\underline{\alpha}}$, lies over the indexing set $I \times J$. Let $D^{\underline{\alpha}}(0,0)$ denote the cabling of the Hopf link corresponding to $\underline{\alpha}$ and associated to the index $(0,0)$ (that is, the cable of the Hopf link with $|\alpha_i|$ parallel strands for the ith component L_i, oriented according to the sign of α_i. See the left-most diagram in Figure 3.6).

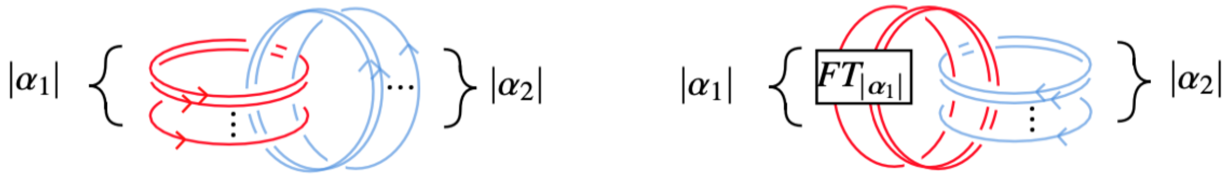


FIGURE 3.6. **Left:** The diagram $D^{\underline{\alpha}}(0,0)$ for $\underline{\alpha} = (\alpha_1, \alpha_2)$, representing the $(0,0)$ object in the cabling directed system of $S^2 \times S^2$. **Right:** The diagram $D^{(\alpha_1,\alpha_2)}(0,0)$ representing the $(0,0)$ object in the cabling directed system of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

Let $D^{\underline{\alpha}}(i,j)$ be the link diagram obtained from $D^{\underline{\alpha}}(0,0)$ by adding $2i$ parallel strands to the cable of L_1, with i positively oriented and i negatively oriented, and adding $2j$ parallel strands to the cable of L_2, with j positively oriented and j negatively oriented. By Definition (2.3.6), the (i,j)th object of $\mathcal{B}^{\underline{\alpha}}$ is $\mathrm{KhR}_2(\mathrm{Sym}(D^{\underline{\alpha}}(i,j)))$ where $\mathrm{Sym}(D^{\underline{\alpha}}(i,j))$ is the complex symmetrized under the $\mathfrak{S}_{|\alpha_1|+2i} \times \mathfrak{S}_{|\alpha_2|+2j}$ action on parallel strands. The morphisms $(i,j) \to (i+2,j)$ (respectively, $(i,j) \to (i,j+2)$) are the symmetrized dotted ribbon maps associated to cables of L_1 (respectively, L_2).

Since $\mathcal{S}_0^2(S^2 \times S^2; \emptyset, \underline{\alpha}) \cong \operatorname{colim}_{I \times J}(\mathcal{B}^{\underline{\alpha}}) \cong \operatorname{colim}_I \operatorname{colim}_J(\mathcal{B}^{\underline{\alpha}})$, we can compute the skein lasagna module of $(S^2 \times S^2, \emptyset)$ by computing the colimits of the directed systems given by a fixed $i \in I$ (or fixed $j \in J$). Without loss of generality, suppose that α_1 is an odd integer, and fix an $i \in I$. The corresponding cabling directed system is of the form

$$\cdots \to \mathrm{KhR}_2(\mathrm{Sym}(D^{\underline{\alpha}}(|\alpha_1| + 2i, |\alpha_{2|} + 2j))) \to \mathrm{KhR}_2(\mathrm{Sym}(D^{\underline{\alpha}}(|\alpha_1| + 2i, |\alpha_2| + 2(j+1)))) \to \cdots.$$

Observe that the colimits of these directed systems are precisely $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_{|\alpha_1|+2i}, \alpha_2)$ with the strands of $\widetilde{\mathbf{1}}_{|\alpha_1|+2i}$ oriented. In particular, we have that

$$\operatorname{colim}_{I \times J}(\mathcal{B}^{\underline{\alpha}}) = \operatorname{colim}_I \operatorname{colim}_J(\mathcal{B}^{\underline{\alpha}}) \cong \operatorname{colim}_I(\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_{|\alpha_1|+2i}, \alpha_2)).$$

As α_1 is odd, $|\alpha_1| + 2i$ is odd for all i, so $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_{|\alpha_1|+2i}, \alpha_2) \cong 0$ for all i by Corollary 3.3.3. Therefore, $\operatorname{colim}_{I\times J}(\mathcal{B}^{\underline{\alpha}}) \cong 0$, as desired. □

There is a corresponding result for $S^2 \tilde{\times} S^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. Recall that a Kirby diagram representing $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is a Hopf link $L = L_1 \cup L_2$ where L_1 has $+1$ framing and L_2 has 0-framing. Although the cabling directed system corresponding to $(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \emptyset)$ does not admit the same symmetry of indexing sets, we have the following.

Corollary 3.3.4. *Let $L = L_1 \cup L_2$ be the framed oriented Hopf link in the Kirby diagram of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, and let α_1 (respectively α_2) represent the generator of $H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2; \mathbb{Z})$ corresponding to the $(+1)$-framed component L_1 (respectively, the 0-framed component L_2). Then, if α_1 is odd, we have*

$$\mathcal{S}_0^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2; \emptyset, (\alpha_1, \alpha_2)) \cong 0.$$

Proof. Let FT_n denote the full-twist tangle on n strands. The skein lasagna module of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ at level $(\alpha_1, \alpha_2) \in H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2; \mathbb{Z})$ is isomorphic to the colimit of the cabling directed system of L. Denote this cabling directed system by $\widetilde{\mathcal{B}}^{(\alpha_1,\alpha_2)}$. Define indexing sets I and J as in the proof of Theorem 3.3.1 and observe that, unlike the case for $S^2 \times S^2$, cables of the component L_1 are $T(n,n)$ torus links. Therefore, by fixing $i \in I$, the colimit of the corresponding directed system is instead isomorphic to $H^*(\mathrm{Tr}(\mathrm{FT}_{|\alpha_1|+2i} \otimes T^{\Omega_{\alpha_2}}_{|\alpha_1|+2i})) \cong \mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathrm{FT}}_{|\alpha_1|+2i}, \alpha_2)$ by Corollary 3.1.1. However, since α_1 is odd and therefore $|\alpha_1| + 2i$ is odd for all i, we have that $T^{\Omega_{\alpha_2}}_{|\alpha_1|+2i} \simeq 0$, and therefore $H^*(\mathrm{Tr}(\mathrm{FT}_{|\alpha_1|+2i} \otimes T^{\Omega_{\alpha_2}}_{|\alpha_1|+2i})) \cong 0$ for all i. It follows that

$$\begin{aligned}
\mathcal{S}_0^2(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2; \emptyset, (\alpha_1, \alpha_2)) &\cong \operatorname{colim}_{I\times J}(\widetilde{\mathcal{B}}^{(\alpha_1,\alpha_2)}) \\
&\cong \operatorname{colim}_{I}(\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathrm{FT}}_{|\alpha_1|+2i}, \alpha_2)) \\
&\cong 0.
\end{aligned}$$

□

Theorem 3.3.1 and Corollary 3.3.4 provide a partial picture of the skein lasagna modules of $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$. To complete this picture, we now comment on the case where the homological levels have only even values.

3.4. Even homological levels

If the number of strands n is even, by the resolution of the n strand identity braid, by the argument used in the proof of Proposition 3.3.2, we have instead that $T_n^{\Omega_\alpha} \simeq P_{n,0}^\vee \otimes T_n^{\Omega_\alpha}$. The higher order projector $P_{n,0}^\vee$ has through-degree 0. Cobordisms between tangles with through-degree 0 have a certain splitting property.

Definition 3.4.1. *Let $T = T_0 \cup T_1$ be a split tangle (so the connected components T_i may each be placed in a 3-ball B_i^3 such that $B_0^3 \cap B_1^3 = \emptyset$). A cobordism between split tangles $C : T \to T'$ is a* split cobordism *if it can be written as $C = C_0 \cup C_1$, where each $C_i : T_i \to T_i'$ is a tangle cobordism entirely contained in $B_i \times [0,1]$.*

By neck-cutting (see Appendix A), cobordism maps in a Bar-Natan cobordism category between through-degree 0 tangles can be reinterpreted as a sum of split cobordism maps. Hence, the differentials of a chain complex of through-degree 0 tangles can be realized as linear combinations of split cobordism maps. With this in mind, we prove the following sliding-off property for the Kirby-colored belt on through-degree 0 chain complexes.

Theorem 3.4.2. *Let A be a chain complex in $\mathrm{Kom}(\mathcal{TL}_n)$ of through-degree 0, let U^k denote the k component unlink, and let U^{Ω_α} denote the Kirby colored 0-framed unknot. Then there is a chain homotopy equivalence $A \otimes T_n^{\Omega_\alpha} \simeq A \sqcup U^{\Omega_\alpha}$ (using the notation from Remark A.1.1).*

Proof. We begin by showing that the chain complex $A \otimes T_n^{\otimes k}$ is chain homotopy equivalent to the complex $A \sqcup U^k$ for each k. By assumption, each chain group of A is given by a direct sums of shifted flat tangles of through-degree 0, denoted A_i. Furthermore, the differentials of A are matrices of linear combinations of chain maps induced by split cobordisms. Let $A_i = \bigoplus_j q^{k_j} \tau_j^i$, where each τ_j^i is a through-degree 0 tangle as above. Observe that there exist natural cobordism maps $\Sigma_j^i : \tau_j^i \otimes T_n^{\otimes k} \to \tau_j^i \sqcup U^k$ given simply by a composition of isotopies that slide the belts off of τ_j^i (see Figure 3.7). These Σ_j^i maps are given by compositions of Reidemeister II moves and are therefore homotopy equivalence maps $\tau^i_j \otimes T_n^{\otimes k} \simeq \tau^i_j \sqcup U^k$. Then, letting $\Sigma^i := \bigoplus_j \Sigma^i_j$, these maps are also chain homotopy equivalence maps $A_i \otimes T_n^{\otimes k} \simeq A_i \sqcup U^k$.

By Proposition A.0.1, we have

$$A \otimes T_n^{\otimes k} \simeq \operatorname{Tot}(\{A_i \sqcup U^k, g_{i,j}\}, \qquad g_{i,i+1} = \Sigma^{i+1} \circ (\partial_i \otimes \mathrm{id}) \circ (\Sigma^i)^{-1}$$

where $\partial_i : A_i \to A_{i+1}$ is a differential of A and $g_{i,j}$ are morphisms of homological degree $j - i - 1$ satisfying Equation (A.1) of Definition A.0.6 (see Figure 3.8). Note that if $j - i > 1$ for $g_{i,j} : A_i \sqcup U^k \to A_j \sqcup U^k$, then $g_{i,j}$ is the zero map, as $A_j \sqcup U^k$ is itself a chain complex supported only in homological degree 0 (it is a direct sum of flat tangles). Thus, the twisted complex $\{A_i \sqcup U^k, g_{i,j}\}$ is an actual chain complex. Furthermore, by functoriality, $g_{i,i+1}$ and $\partial_i \sqcup \mathrm{id}$ are homotopic maps, since the cobordisms they represent are isotopic. In other words, the following diagram homotopy commutes:

$$\begin{tikzcd}
A_i \otimes T_n^{\otimes k} \arrow[r, "\partial_i \otimes \mathrm{id}"] \arrow[d, "\Sigma^i"'] & A_{i+1} \otimes T_n^{\otimes k} \arrow[d, "\Sigma^{i+1}"] \\
A_i \sqcup U^k \arrow[r, "\partial_i \sqcup \mathrm{id}"] & A_{i+1} \sqcup U^k
\end{tikzcd}$$

Hence $A \sqcup U^k = (\bigoplus_i A_i \sqcup U_k, \bigoplus_i \partial_i \sqcup \mathrm{id})$ is chain homotopy equivalent to the complex $A \otimes T_n^{\otimes k}$. Now let $\Sigma^{(k)} : A \otimes T_n^{\otimes k} \to A \sqcup U^k$ denote the chain homotopy equivalence map provided by Proposition A.0.1. Then, the cobordism maps $\Sigma^{(k)}$ commute with the symmetrizing cobordisms and dotted cup cobordisms. We may then define the following directed systems:

$$A \otimes \mathcal{A}_n^\alpha := A \otimes \mathrm{Sym}(T_n^{\otimes|\alpha|}) \xrightarrow{\mathrm{id} \otimes \mathrm{Sym}(\underset{\bullet}{\cup})} A \otimes \mathrm{Sym}(T_n^{\otimes(|\alpha|+2)}) \xrightarrow{\mathrm{id} \otimes \mathrm{Sym}(\underset{\bullet}{\cup})} A \otimes \mathrm{Sym}(T_n^{\otimes|\alpha|+4}) \to \cdots$$

$$A \sqcup \mathcal{A}_n^\alpha := A \sqcup \mathrm{Sym}(T_n^{\otimes|\alpha|}) \xrightarrow{\mathrm{id} \sqcup \mathrm{Sym}(\underset{\bullet}{\cup})} A \sqcup \mathrm{Sym}(T_n^{\otimes(|\alpha|+2)}) \xrightarrow{\mathrm{id} \sqcup \mathrm{Sym}(\underset{\bullet}{\cup})} A \sqcup \mathrm{Sym}(T_n^{\otimes|\alpha|+4}) \to \cdots$$

Observe also that $A \otimes T_n^{\Omega_\alpha} = \operatorname{hocolim}(A \otimes \mathcal{A}_n^\alpha)$ and $A \sqcup U^{\Omega_\alpha} = \operatorname{hocolim}(A \sqcup \mathcal{A}_n^\alpha)$. Since we have homotopy equivalence maps $\Sigma^{(m)}$ between each object in our directed systems, by Lemma

A.0.9.2(a), there is a well-defined chain map $\Sigma : A \otimes T_n^{\Omega_\alpha} \to A \sqcup U^{\Omega_\alpha}$ on homotopy colimits. Furthermore, by Lemma A.0.9.2(c), we immediately have that $A \otimes T_n^{\Omega_\alpha} \simeq A \sqcup U^{\Omega_\alpha}$, as desired. $\square$

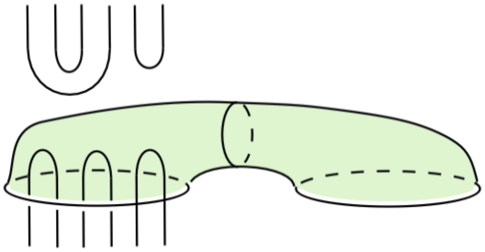


Figure 3.7. Belt slide-off cobordism from Reidemeister II moves. The split cobordisms $A_i \to A_j$ between split tangles do not intersect the shaded surface.

A Kirby diagram of $S^2 \times B^2$ is the 0-framed unknot. In [**MN22**], the $N = 2$ skein lasagna module of $(S^2 \times B^2, \emptyset)$, equivalent to the cabled Khovanov homology of the 0-framed unknot, was shown to be isomorphic to $\mathbb{F}[A_0, A_0^{-1}, A_1]$ for formal variables A_0 and A_1 in q-degrees 0 and -2 respectively. At homological level α, the skein lasagna module $\mathcal{S}_0^2(S^2 \times B^2; \emptyset, \alpha)$ is isomorphic to the subgroup of $\mathbb{F}[A_0, A_0^{-1}, A_1]$ generated by homogeneous polynomials of degree α. Denote this subgroup by $\mathbb{F}_{|\alpha|}[A_0, A_0^{-1}, A_1]$.

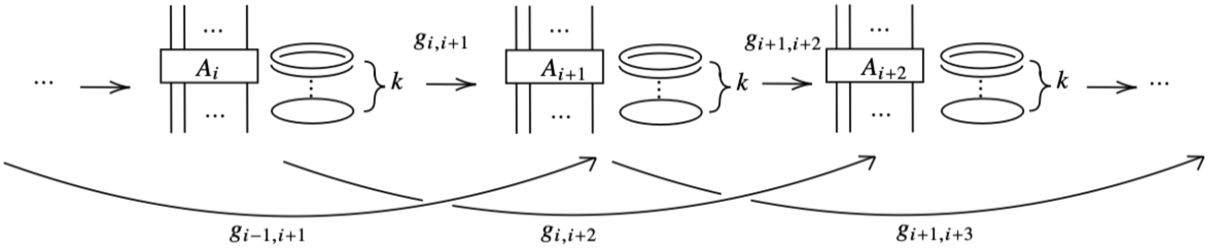


Figure 3.8. The twisted complex $\{A_i \sqcup T_0^{\otimes k}, g_{i,j}\}$

Theorem 3.4.2 then has the immediate corollary for pairs $(S^2 \times B^2, \widetilde{\mathbf{1}}_n)$ for an even integer n.

Corollary 3.4.1. *Let* $\alpha \in H_2(S^2 \times B^2; \mathbb{Z}) \cong \mathbb{Z}$ *and let* $k \in \mathbb{N}$. *Then there is an isomorphism* $\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_{2k}, \alpha) \cong H^*(\mathrm{Tr}(P_{2k,0}^\vee)) \otimes \mathbb{F}_{|\alpha|}[A_0, A_0^{-1}, A_1]$.

Proof. The skein lasagna module of the pair $(S^2 \times B^2, \widetilde{\mathbf{1}}_{2k})$ is isomorphic to $H^*(\mathrm{Tr}(T_{2k}^{\Omega_\alpha}))$ by Corollary 3.1.1. After tensoring $T_{2k}^{\Omega_\alpha}$ with the resolution of the identity $\mathbf{1}_{2k}$, we obtain a chain homotopy equivalence

$$T_{2k}^{\Omega_\alpha} \simeq P_{2k,0}^\vee \otimes T_{2k}^{\Omega_\alpha}.$$

However, since $P_{2k,0}^\vee$ is a through-degree 0 complex in $\mathrm{Kom}(\mathcal{TL}_n)$, by Theorem 3.4.2, we have that $T_{2k}^{\Omega_\alpha} \simeq P_{2k,0}^\vee \sqcup U^{\Omega_\alpha}$. Note that $\mathrm{Tr}(P_{2k,0}^\vee \sqcup U^{\Omega_\alpha}) = \mathrm{Tr}(P_{2k,0}^\vee) \sqcup U^{\Omega_\alpha}$, implying

$$\begin{aligned}
\mathcal{S}_0^2(S^2 \times B^2; \widetilde{\mathbf{1}}_{2k}, \alpha) &\cong H^*(\mathrm{Tr}(P_{2k,0}^\vee \otimes T_{2k}^{\Omega_\alpha})) \\
&\cong H^*(\mathrm{Tr}(P_{2k,0}^\vee) \sqcup U^{\Omega_\alpha}) \\
&\cong H^*(\mathrm{Tr}(P_{2k,0}^\vee)) \otimes \mathbb{F}_{|\alpha|}[A_0, A_0^{-1}, A_1].
\end{aligned}$$

proving the claim. □

We can similarly extend the result of Corollary 3.4.1 to the pair $(S^2 \times S^2, \emptyset)$, and may now complete the proof of Corollary 3.3.1.

Proof of Corollary 3.3.1. By Theorem 3.3.1, it remains to show that the skein lasagna module of $S^2 \times S^2$ vanishes for $(\alpha_1, \alpha_2) \in H_2(S^2 \times S^2)$ where both entries are even. Let $\dot{\cup}^*$ denote the morphisms on colimits induced by $\dot{\cup}$ and let α_1 and α_2 be even integers. The skein lasagna module $\mathcal{S}_0^2(S^2 \times S^2; \emptyset, (\alpha_1, \alpha_2))$ is isomorphic to $\mathrm{colim}\,(\mathcal{V}) \otimes \mathbb{F}_{|\alpha_2|}[A_0, A_0^{-1}, A_1]$, where $\mathcal{V}$ is the directed system

$$\mathcal{V} := H^*(\mathrm{Tr}(P_{|\alpha_1|,0}^\vee)) \xrightarrow{\dot{\cup}^*} H^*(\mathrm{Tr}(P_{|\alpha_1|+2,0}^\vee)) \xrightarrow{\dot{\cup}^*} H^*(\mathrm{Tr}(P_{|\alpha_1|+4,0}^\vee)) \xrightarrow{\dot{\cup}^*} \cdots$$

However, if α_2 is taken to be odd instead, we have that $\mathrm{colim}\,(\mathcal{V}) \otimes \mathbb{F}_{|\alpha_2|}[A_0, A_0^{-1}, A_1] \cong 0$, implying that $\mathrm{colim}\,(\mathcal{V}) = 0$. Therefore, $\mathcal{S}_0^2(S^2 \times S^2; \emptyset, (\alpha_1, \alpha_2)) \cong 0$ for all $(\alpha_1, \alpha_2) \in H_2(S^2 \times S^2)$.

□

3.4.1. Relation to Rozansky-Willis homology.

Given a link $L \subset \#^k(S^1 \times S^2)$, Rozansky [**Roz10**] (For a single copy of $S^1 \times S^2$) and Willis [**Wil21**] (generalized to connect-sums $\#^k(S^1 \times S^2)$) define an invariant $H_{RW}^{*,*}(\#^k(S^1 \times S^2); L)$ of L that recovers Khovanov homology for $k = 0$. Depicting each $S^1 \times S^2$ connect-summand as a bracketed 0-framed unknot, we say that a link

$L \subset \#^k(S^1 \times S^2)$ is in *standard position* if it admits a diagram where the $S^1 \times S^2$ connect-summands may be lined up, and L locally looks like Figure 3.9 near each surgery region.

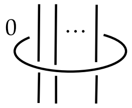


Figure 3.9. A local picture of a diagram of a standard position link in $S^1 \times S^2$
.

This invariant is defined as the homology of the Khovanov complex obtained by replacing each $S^1 \times S^2$ connect-summand $(S^1 \times S^2)_i$ with an n_i-strand Rozansky projector $P^{\vee}_{n_i,0}$ when L transversely intersects the attaching sphere in n_i points, see Figure 3.10.

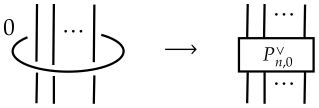


Figure 3.10. Replacing $S^1 \times S^2$ summands with Rozansky projectors.

We note first that by working with Kirby-colored belts on tangles in Chapter 3, the results are easily extended to pairs of the form $(\natural^k(S^2 \times D^2); L)$. The arguments presented in Chapters 2 and 3 about the resolution of identity directly imply the following theorem.

Theorem 3.4.3. *Let L be an admissible link in the boundary $\#^k(S^1 \times S^2) = \partial(\natural^k(S^2 \times D^2))$, then there is a homogeneous isomorphism of $\mathbb{Q}$-vector spaces*

$$\mathcal{S}^2_0(\natural^k(S^2 \times D^2); L) \xrightarrow{\cong} H^{*,*}_{RW}(\#^k(S^1 \times S^2), L) \otimes \mathcal{S}^2_0(\natural^k(S^2 \times D^2); \emptyset)$$

where $H^{*,*}_{RW}$ denotes (an appropriately normalized) Rozansky-Willis homology of $L \subset \#^k(S^1 \times S^2)$.

CHAPTER 4

Bar-Natan Skein Lasagna Modules and Strong Internal Stabilization

In this section, we discuss some applications of importing *Bar-Natan* homology (see [**BN05**]) to the smooth 4-manifold setting by the skein lasagna construction. We study the naturally arising $\mathbb{F}_2[H]$-module structure of this invariant, which we denote by $\mathcal{S}_0^{BN}(X;L)$ for a 4-manifold and boundary link pair (X,L). We interpret the effect of multiplication by H as a 3-dimensional 1-handle surgery in X corresponding to attaching a handle to the skein surfaces representing elements of $\mathcal{S}_0^{\mathrm{BN}}(X;L)$. Using this interpretation, we study the H-torsion elements in this Bar-Natan lasagna setting, and use a notion of *H-torsion order* for $\mathcal{S}_0^{BN}(X;L)$ to extend internal stabilization results of Hayden [**Hay23**]. Using results about a more general form of this invariant, defined and explored in [**MWW24**], we are able to produce examples of exotically knotted pairs of surfaces in 4-manifolds other than B^4, that remain exotic after a single *internal stabilization*.

4.1. Knotted surfaces and internal stabilizations

We begin by establishing our definition of exotically knotted pairs of surfaces and internal stabilization, as well as the notation and conventions for the link homology tools that we use throughout this section.

We are most interested in exotic phenomena in dimensions 2 and 4, specifically, phenomena involving exotically knotted pairs of surfaces in 4-manifolds. We recall that a pair of smooth 4-manifolds is called an *exotic pair* if they are homeomorphic but not diffeomorphic. We recall also that there exists a natural number k such that $X\#^k(S^2\times S^2)$ and $Y\#^k(S^2\times S^2)$ are diffeomorphic (see [**Wal64**]). We refer to the operation corresponding to taking a connect-sum with a copy of $S^2\times S^2$ as an *external stabilization* of the original 4-manifold. We turn now to surfaces in 4-manifolds. All manifolds (resp. submanifolds) are assumed to be smooth (resp. smoothly embedded) throughout.

Definition 4.1.1. *Let* Σ_1 *and* Σ_2 *be homologous surfaces with equal genus in some* 4*-manifold* X*. Then* (Σ_1, Σ_2) *form an* exotically knotted pair *of surfaces in* X *if they are topologically isotopic (rel boundary if the surfaces have boundary), but not smoothly isotopic (rel boundary) in* X*.*

For pairs of exotically knotted surfaces, there may exist topological operations that dissolve their exotic relationship.

Definition 4.1.2. *Let* Σ *be a smoothly embedded surface in a 4-manifold* X*. Attaching a 1-handle to* Σ *in* X *while preserving orientability is called a* weak internal stabilization *of* Σ*. If the surgery corresponds to taking a connect-sum of pairs* $(X, \Sigma)\#(S^4, T^2)$ *where* T^2 *is an unknotted torus, then the operation is called a* standard internal stabilization *of* Σ*. Throughout, we work entirely with standard internal stabilizations, simply referring to them as* internal stabilizations.

For certain pairs of exotically knotted surfaces, the resulting surfaces after the application of sufficiently many internal stabilizations are smoothly isotopic rel boundary.

Theorem 4.1.3 ([**BS16**], Theorem 1). *Let* (Σ_1, Σ_2) *be a pair of exotically knotted surfaces in* X*, then* Σ_1 *and* Σ_2 *become smoothly isotopic after some number* $n \geq 0$ *of weak internal stabilizations. Furthermore, if the inclusion map* $i : \partial(\nu\Sigma_i) \hookrightarrow X \setminus \Sigma_i$ *induces a surjection on* π_1*, then the result holds for standard internal stabilizations.*

Note that the integer n establishes a notion of *stabilization distance* between exotically knotted pairs of surfaces. The above Theorem 4.1.3 of Baykur and Sunukjian states that for exotically knotted pairs of surfaces (Σ_1, Σ_2) satisfying the surjectivity on π_1 condition, there exists an integer $n > 0$ such that the resulting surfaces $\Sigma_1\#^nT^2$ and $\Sigma_2\#^nT^2$ are smoothly isotopic. We remark also that exotically knotted surfaces also become smoothly isotopic after sufficiently many external stabilizations to their ambient 4-manifold.

Definition 4.1.4. *Let* (Σ_1, Σ_2) *be an exotically knotted pair of surfaces in* (B^4, L) *for some possibly empty link* L*. Define*

$$d(\Sigma_1, \Sigma_2) = \min \{k \mid \Sigma_1\#^kT^2 \textit{ and } \Sigma_2\#^kT^2 \textit{ are smoothly isotopic}\}$$

to be the (strong) internal stabilization distance *between* Σ_1 *and* Σ_2*.*

Baykur and Sunukjian show, for many families of pairs of exotically knotted surfaces, that $d(\Sigma_1, \Sigma_2) = 1$. However, a surface analogue of the external stabilization conjecture remains open and conjectures the following: If (Σ_1,Σ_2) is an exotically knotted pair of closed surfaces in B^4 satisfying the conditions in Theorem 4.1.3, then $d(\Sigma_1, \Sigma_2) = 1$.

Although this question is very much open for closed surfaces, it is generally not true for surfaces with boundary in the 4-ball. For example, in [**Hay23**], Hayden constructs infinitely many exotically knotted pairs of surfaces with boundary, with arbitrarily high genus, such that one internal stabilization is not enough. Also, Guth in [**Gut22**] produces yet another infinite family of exotically knotted pairs of relative surfaces in the 4-ball with arbitrarily large internal stabilization distances. We remark also on the existence of closed exotically knotted pairs in 4-manifolds with boundary other than the 4-ball produced in [**HKM23**]. The result of Hayden is the most relevant to this work, so we record it now.

THEOREM 4.1.5 ([**Hay23**], Theorem A). *For any $g \geq 1$, there exist exotically knotted pairs of surfaces with boundary of genus g in B^4 that remain exotic after one internal stabilization.*

This theorem is proven by constructing pairs of surfaces that induce different maps on Bar-Natan homology. In particular, Hayden produces a knot K_H (see Figure 4.1), that bounds a pair of disks that are neither topologically nor smoothly isotopic, these disks yield a pair of genus-1 exotically knotted surfaces after attaching bands. Let these genus 1 surfaces be denoted F_1 and F_1' respectively, and let $K_1 = \partial F_1 = \partial F_1'$ denote their boundary. We may then connect-sum F_1 and F_1' with the fiber surface of $T_{2,2g-1}$ and we let the resulting exotically knotted pair be denoted (F_g, F_g') with boundary denoted K_g (see Figure 4.1). We recall briefly the method used to prove Theorem 4.1.5 using Bar-Natan homology.

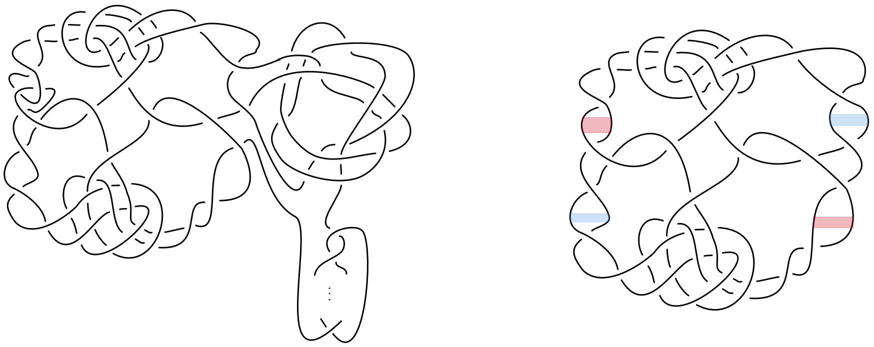


FIGURE 4.1. **Left:** A diagram of the knot $K_g = \partial F_g = \partial F'_g$. **Right:** The knot K_H with disks D and D'. Both diagrams are found in [**Hay23**].

REMARK 4.1.1 ([**Hay23**], Propositions 4.1 and 4.2). *Let $-D$ and $-D'$ denote the time-reversed (punctured) disks bounded by the knot K_H. These disks, taken as cobordisms $-D, -D' : U \to K_H$, induce distinct maps on reduced Bar-Natan homology with $\mathbb{F}_2[H]$ coefficients. Furthermore, these maps remain distinct after multiplication by H, implying that the disks D and D' remain exotic after a single internal stabilization.*

4.2. Bar-Natan/equivariant link homology

To establish preliminaries and conventions, as well as state Theorem 4.1.5 and Remark 4.1.1 in a more explicit way, we briefly review Bar-Natan's link homology theory, defined originally in [**BN05**]. See Appendix A A for more details on Bar-Natan categories.

DEFINITION 4.2.1. *The* $U(2)$-equivariant Frobenius pair *is the pair (R, A) where R is the ground ring $\mathbb{F}_2[E_1, E_2]$, and A is the Frobenius algebra*

$$A := \mathrm{KhR}_{U(2)}(unknot) = \frac{R[X]}{(X^2 - E_1X + E_2)}$$

where E_1 and E_2 are classes of degree 2 and 4 respectively. We remark that the classes E_1 and E_2 can be identified with the first and second elementary symmetric polynomials in two variables $\{r_1, r_2\}$, each of degree 2.

By letting H denote $E_1(r_1, r_2)$ with $r_2 = 0$, which implies that $E_2 = 0$, we obtain the Frobenius pair for Bar-Natan homology

$$R^{\mathrm{BN}} = \mathbb{F}_2[H], \quad A^{\mathrm{BN}} = \frac{R^{\mathrm{BN}}[X]}{(X^2 - HX)}.$$

Note that in these conventions, multiplication by H is a bi-degree $(0,2)$ map. The link homology theory associated to the TQFT corresponding to the pair $(R^{\mathrm{BN}}, A^{\mathrm{BN}})$ will be denoted $BN(\_)$ throughout; we refer to this theory as *Bar-Natan homology*. The local relations with dots and some consequences for Bar-Natan homology are given in Figure 4.2.

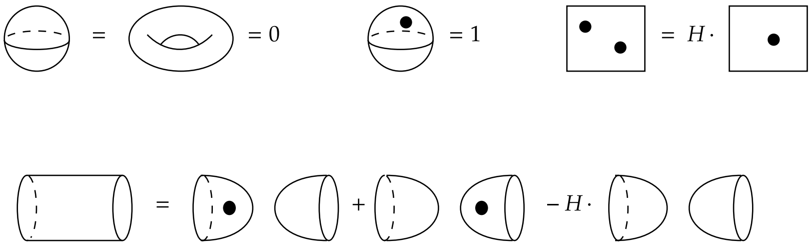


FIGURE 4.2. **Top:** The sphere, torus, dotted sphere, and H-trading relation over $\mathbb{F}_2$ coefficients. **Bottom:** the neck-cutting relation.

Using the neck-cutting relation in Figure 4.2, we can re-interpret the internal stabilization operation as in Figure 4.3. We note that connect-summing with T^2 is equivalent to the creation of a dotted torus, disjoint from the original surface.

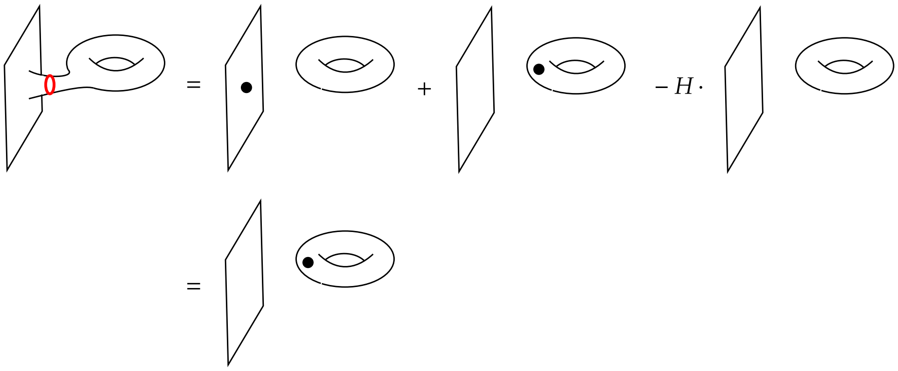


FIGURE 4.3. The effect of neck-cutting along the small red circle.

It will be necessary to discuss the reduced form of this theory, which we denote by $\widetilde{\mathrm{BN}}(\_)$ in general, and $\widetilde{\mathrm{BN}}_y(\_)$, $y \in \{1, x\}$ for a specific version of reduced Bar-Natan homology. For references, see [**KWZ19**, **Wig16**, **AZ22**].

Definition 4.2.2. *Let (L, p) be a link with a basepoint p. Let f_x denote the chain map induced by the cobordism corresponding to the creation and merging of an x-labeled circle onto the component of L containing p. We define the* reduced Bar-Natan complex $\widetilde{CBN}_x(L, p)$ *as the image $im(f_x) \subset CBN(L)$. We define $\widetilde{CBN}_1(L, p)$ to be the quotient complex $CBN(L)/\widetilde{CBN}_x(L)$. The homology of these complexes are isomorphic up to a shift in q-grading, and we denote the isomorphism type by $\widetilde{\mathrm{BN}}(L, p)$.*

Note that, due to the basepoint requirement, the reduced Bar-Natan homology theory is functorial for non-empty links and requires the introduction of punctures for cobordisms of the form $L \to \emptyset$ or $\emptyset \to L$. Note that a 1-handle surgery corresponding to an internal stabilization on a link cobordism $\Sigma : L_0 \to L_1$ has the effect of multiplication by $2X - H$, or simply H when working with $\mathbb{F}_2$ coefficients. In the following remark we record the more precise statement of Hayden. As per Remark 4.1.1, we have that $\widetilde{\mathrm{BN}}(-D) \neq \widetilde{\mathrm{BN}}(-D')$. Furthermore, we have that $H \cdot \widetilde{\mathrm{BN}}(-D) \neq H \cdot \widetilde{\mathrm{BN}}(-D')$, implying that $-D$ and $-D'$ do not become smoothly isotopic after a single internal stabilization.

Remark 4.2.1. *Hayden proves this result by studying the element $\widetilde{\delta}_0 := \widetilde{\mathrm{BN}}(-D)(1) - \widetilde{\mathrm{BN}}(-D')(1) \in \widetilde{\mathrm{BN}}(K_H)$. They show that $\widetilde{\delta}_0$ is non-zero and has H-torsion order greater than 1. We now remark that a corresponding element in the unreduced theory also satisfies this property. This is required as we do not work in the reduced setting later on. To see this, we use a theorem of Wigderson [**Wig16**].*

Theorem 4.2.3 ([**Wig16**], Theorem 3). *There is a natural $\mathbb{F}_2[H]$-module isomorphism*

$$\Phi_K : H_*(CBN(K)) \xrightarrow{\cong} H_*(\widetilde{CBN}_x(K)) \oplus H_*(\widetilde{CBN}_1(K)).$$

Furthermore, there is an isomorphism $f_K : H_*(\widetilde{CBN}_x(K)) \xrightarrow{\cong} H_*(q^{-2}h^0\widetilde{CBN}_1(K))$.

Equipped with this splitting isomorphism, we now have the following lemma.

Lemma 4.2.3.1. *Let* $\widetilde{\delta}_0 \in \widetilde{\mathrm{BN}}_x(K_H)$ *be the* H*-torsion element defined in Remark 4.2.1 and let* δ_0 *denote the element* $\Phi^{-1}_{K_H}((\widetilde{\delta}_0, 0)) \in BN(K_H)$*. The element* δ_0 *is non-trivial with the same* H*-torsion order as* $\widetilde{\delta}_0$*.*

Proof. The result follows immediately from the definitions. □

Definition 4.2.4. *(Inverting the discriminant) We now recall the definition of the* localized *version of Bar-Natan homology. Let* D *denote the discriminant of the polynomial* $X^2 - E_1X + E_2$*, then define a Frobenius pair by* $R_D := R[D^{-1}]$ *and* $A_D = A \otimes_R R_D$*. The TQFT associated to this Frobenius pair defines the localized* $U(2)$*-equivariant homology. In our case, after setting* $E_1 = H$ *and* $E_2 = 0$*, we have that the discriminant is* H^2 *and we obtain an equivalent theory if we let* $D = H$*. Finally, after inverting the discriminant as above, we let* $BN_{H=1}$ *denote the link homology theory obtained from setting* $H = 1$ *in the localized Bar-Natan theory. We remark that the link homology theory associated to this pair* $(R^{\mathrm{BN}}_{H^{-1}}, A^{\mathrm{BN}}_{H^{-1}})$ *with* $H = 1$ *is equivalent to the Lee homology theory [**Lee05**, **KR22**].*

We now claim that the Proposition of Hayden in Remark 4.1.1 can be interpreted and subsequently extended by passing to the skein lasagna module setting.

Definition 4.2.5. *Let* Z *denote a TQFT for links in* $\mathbb{S}^3$ *that is (lax) monoidal under disjoint union, and let* (X, L) *denote a 4-manifold and boundary link pair. A* skein surface *of* (X, L) *consists of the topological data* $(S, \{B_i\}_{i\in I}, \{L_i\}_{i\in I})$*, where* S *is a properly embedded, oriented, framed surface such that* $\partial S \cap \partial X = L$*, where* $\{B_i\}_{i\in I}$ *is a finite collection of disjoint embedded* 4*-balls with their interiors deleted from int(*X*), such that* ∂S *intersects the boundary of each* B_i *in the link* L_i*. Skein surfaces are upgraded to* Z-lasagna fillings *when the input links* $\{L_i\}_{i\in I}$ *are labeled by elements of* $Z(L_i)$ *for each* L_i*. For a skein surface* S *with label* v *we use the notation* (S, v) *to denote a corresponding* Z*-lasagna filling.*

Recall that the *skein lasagna module* is $\mathcal{S}^Z_0(X; L) := \mathbb{F}\langle Z\text{-lasagna fillings of (X,L)}\rangle / \sim$ where $\sim$ is the enclosement relation as discussed in Chapter 2, Definition 2.3.3. When say two lasagna fillings (S, v) and (S', v') are *skein equivalent* if $(S, v) \sim (S', v')$.

Throughout this section, it is necessary to distinguish between skein surfaces, Z-lasagna fillings, and skein equivalence classes of Z-lasagna fillings. For a lasagna filling (S, v), we use the notation $[(S, v)]$ to denote the corresponding element in $\mathcal{S}_0^Z(X; L)$; if the label is not relevant, we will drop it from the notation and write $[S]$ instead. When context is clear, we refer to lasagna elements $[S]$ simply as fillings.

With this recipe in hand, one may define a smooth 4-manifold invariant $\mathcal{S}_0^{\mathrm{BN}}$ by importing the Bar-Natan TQFT given by $(R^{\mathrm{BN}}, A^{\mathrm{BN}})$. This invariant is not strictly brand new, as the authors in [**MWW24**] construct and discuss such an invariant for the more general $GL(2)$-equivariant version of $\mathfrak{gl}_2$ link homology. The invariant we are interested in is obtained from the construction in [**MWW24**] by setting $N = 2$, the ground ring $R = \mathbb{F}_2$, and the parameters E_1 and E_2 to H and 0 respectively. Since $\mathcal{S}_0^{\mathrm{BN}}$ is a specialization of the $GL(2)$-equivariant version, it enjoys some analogous features. We will use the following natural isomorphism frequently throughout this chapter.

Lemma 4.2.5.1. *There is a natural isomorphism*

$$\mathcal{S}_0^{BN}(B^4; L) \xrightarrow{\cong} BN(L).$$

We will make use of this isomorphism when we extend the ideas of Hayden later on.

Proof. This is a feature of the skein lasagna module construction for any TQFT. In particular, for a pair (B^4, L), the isomorphism is given by mapping $x \in Z(L)$ to the equivalence class of Z-lasagna fillings represented by the filling with skein surface consisting of a "large" input 4-ball and skein surface $L \times I$, with label x. $\square$

We now wish to specify a special class of surfaces in (X, L) for Bar-Natan lasagna purposes.

Definition 4.2.6. *A surface $S \subset (X, L)$ is called* homologically diverse *if no non-trivial union of its closed components is null-homologous in X.*

Simple examples of such surfaces include surfaces with boundary with no closed components, or closed homologically essential surfaces with a single closed component. The authors of [**MWW24**] prove that homologically diverse surfaces correspond to non-trivial free generators of the $GL(2)$-equivariant skein lasagna module. In particular, the authors show that a homologically diverse surface $S \subset (X, L)$ defines a class in tri-degree $([S], \deg_t(S), \deg_q(S))$, where $[S] \in H_2^L(X)$, that is non-trivial modulo torsion in the $GL(2)$-equivariant skein lasagna module. We will return to these surfaces and their significance after establishing some properties of the invariant $\mathcal{S}_0^{\mathrm{BN}}$. We now interpret the action of multiplication by H on Bar-Natan lasagna modules. The invariant $\mathcal{S}_0^{\mathrm{BN}}$ inherits an $\mathbb{F}_2[H]$-module structure from Bar-Natan homology. We begin studying the $\mathbb{F}_2[H]$-module structure by noting that the cabling directed system perspective of Manolescu-Neithalath [**MN22**] applies to $\mathcal{S}_0^{\mathrm{BN}}$.

Lemma 4.2.6.1. *Let (X, L) be a 4-manifold and boundary link pair, where X is a 2-handlebody with Kirby diagram given by the framed oriented link K. Then there is an isomorphism*

$$\operatorname{colim}_{r \in \mathbb{Z}_{\geq 0}} BN(K(r+\alpha^+, r-\alpha^-) \cup L)^{B_{r+\alpha^+, r-\alpha^-}}\{2r - |\alpha|\} \xrightarrow{\cong} \mathcal{S}_0^{\mathrm{BN}}(X; L, \alpha)$$

where $BN(K(r+\alpha^+, r-\alpha^-) \cup L)^{B_{r+\alpha^+, r-\alpha^-}}\{2r - |\alpha|\}$ denotes the Bar-Natan homology group of the cable $K(r+\alpha^+, r-\alpha^-)$, symmetrized with respect to the braid group action permuting the components of $K(r+\alpha^+, r-\alpha^-)$. The colimit is taken along annuli with (possibly zero) dots (or, equivalently, H^k times a dotted annulus for $k \in \mathbb{Z}_{\geq 0}$ as per the H-trading relation (Figure 4.2)).

Proof. The proof of this lemma follows identically from the arguments presented in [**MN22**] and [**MWW23**], with Bar-Natan homology taking the place of KhR_2. ☐

Remark 4.2.2. *Over $\mathbb{F}_2$, this colimit perspective may prove difficult to use for any direct computations, with difficulty largely attributed to the appearance of braid group representations of modules with $\mathbb{F}_2$ coefficients. In particular, many of the arguments presented in Chapters 2 and 3 may not apply here due to difficulties involving symmetric group representations over $\mathbb{F}_2$. There is no issue with choosing coefficients in a field more amicable to the symmetrization maps in the previous chapters, such as $\mathbb{Q}$, but this may complicate the topological interpretation of the H-action.*

4.2.1. Lasagna H-action for (X, L).

We work over $\mathbb{F}_2$ for the remainder of this chapter. For a 4-manifold and boundary link pair (X, L) and for a cobordism S from L to itself in $\partial X \times I$, we adopt the notation $\mathcal{S}_0^Z(\partial X \times I; S) : \mathcal{S}_0^Z(X; L) \to \mathcal{S}_0^Z(X; L)$ to denote the map defined by gluing the surface S to fillings of (X, L) along L.

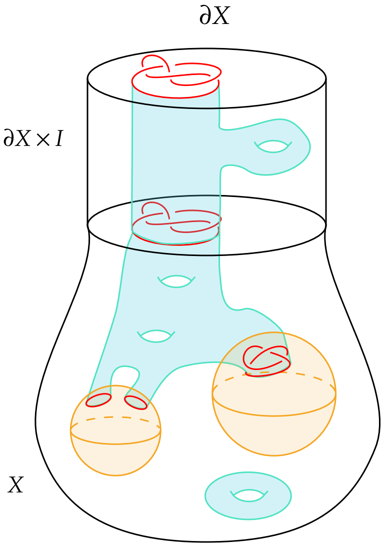


FIGURE 4.4. The H-action map for $L \neq \emptyset$ as in Definition 4.2.7.

DEFINITION 4.2.7. *Given a 4-manifold and boundary link pair (X, L) where L is not the empty link, we can interpret multiplication by H as the following endomorphism on $\mathcal{S}_0^{\mathrm{BN}}(X; L)$*

$$H \cdot \_ := \mathcal{S}_0^{\mathrm{BN}}(\partial X \times I; T^2 \#(L \times I)) : \mathcal{S}_0^{\mathrm{BN}}(X; L) \to \mathcal{S}_0^{\mathrm{BN}}(X; L).$$

Depicted in Figure 4.4, this H-action on $\mathcal{S}_0^{\mathrm{BN}}(X; L)$ has the topological interpretation of attaching an unknotted handle to a connected component of the skein surface of a Bar-Natan lasagna filling. Alternatively, if $L = \emptyset$, we realize the H-action as the creation of a local unknotted dotted torus in int(X) by a neck-cutting on the skein surface (see Figure 4.3), followed by an isotopy moving the dotted torus to the boundary of X. If X is closed, we can remove an embedded B^4 to create such a boundary with no effect on $\mathcal{S}_0^{\mathrm{BN}}$. Letting $T^2_\bullet$ denote the dotted torus, we may define

$$H \cdot \_ := \mathcal{S}_0^{\mathrm{BN}}(\partial X \times I; T^2_\bullet) : \mathcal{S}_0^{\mathrm{BN}}(X; \emptyset) \to \mathcal{S}_0^{\mathrm{BN}}(X; \emptyset)$$

with X replaced by $X \setminus$ int(B^4) above in the case where $\partial X = \emptyset$.

With the H-action established, we define an analogue of H-torsion order.

Definition 4.2.8. *The* lasagna H-torsion order *of a Bar-Natan lasagna element* $\mathcal{F} \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$ *is*

$$ord_H^{(X,L)}(\mathcal{F}) = \min\{k \in \mathbb{Z}_{\geq 0} \mid H^k \cdot \mathcal{F} = 0\}.$$

When $X = B^4$, the lasagna H-torsion order and the usual H-torsion order of an element in Bar-Natan homology coincide under the identification from Lemma 4.2.5.1. Hence, this definition is an extension of the usual notion of Bar-Natan homology H-torsion order (see [**Ali19**, **Guj20**] for more details and applications of this invariant for surfaces in B^4) for links in S^3 to surfaces in pairs (X, L).

We complete this section by relating the invariant $\mathcal{S}_0^{\mathrm{BN}}$ to the KhR_2 version. Recall that we recover the original Khovanov homology construction by setting $H = 0$ in the Bar-Natan theory.

Lemma 4.2.8.1. *There exists a linear map*

$$F : \frac{\mathbb{F}_2[H]}{H} \otimes \mathcal{S}_0^{\mathrm{BN}}(X; L) \to \mathcal{S}_0^2(X; L; \mathbb{F}_2)$$

obtained by applying a natural transformation $\frac{\mathbb{F}_2[H]}{H} \otimes BN(\_) \Rightarrow \mathrm{KhR}_2(\_)$ *to the labels on input links. The map* F *is* $H_2^L(X) \times \mathbb{Z}_t$*-bigrading preserving.*

Proof. Bar-Natan homology and KhR_2 are related by tensoring with the module $\frac{\mathbb{F}_2[H]}{H}$ on the chain complex level. The universal coefficient theorem yields the corresponding natural transformation $\frac{\mathbb{F}_2[H]}{H} \otimes BN(\_) \Rightarrow \mathrm{KhR}_2(\_)$. This natural transformation respects the lasagna skein relation and therefore induces a homomorphism of skein modules. □

This lemma is an instance of Lemma 3.17 and Proposition 3.18 in [**MWW24**].

4.2.2. A connect-sum gluing map for Bar-Natan skein lasagna. In order to extend exotically knotted pairs of surfaces in the 4-ball to other smooth 4-manifolds, it is necessary to understand the behavior of $\mathcal{S}_0^{\mathrm{BN}}$ on connect-sums of 4-manifolds. Since we do not work over a field, this invariant does not admit a connect-sum formula in the same manner as $\mathcal{S}_0^2$ over field coefficients. Instead, we opt to study properties of the gluing map corresponding to the connect-sum operation. Letting $X = X_0 \cup_Y X_1$ where Y is a properly embedded separating 3-manifold, there is a map $\phi : \mathcal{S}_0^{\mathrm{BN}}(X_0 \sqcup X_1; L_0 \sqcup L_1) \to \mathcal{S}_0^{\mathrm{BN}}(X; L_0 \sqcup L_1)$ corresponding to gluing X_0 to X_1 along Y (for simplicity we assumed boundary links $L_i \subset \partial X_i$ do not intersect Y, although this is allowed in general, see [**MWW23**]). For connect-sums of 4-manifolds, the 3-manifold Y is S^3, and we have the corresponding map

$$\phi_\# : \bigoplus_{L \subset S^3} \mathcal{S}_0^{\mathrm{BN}}(X_0 \sqcup X_1; L_0 \sqcup L_1 \sqcup L) \to \mathcal{S}_0^{\mathrm{BN}}(X_0 \# X_1; L_0 \sqcup L_1).$$

The map $\phi_\#$ is the surjective homomorphism given by taking the connect-sum of X_0 and X_1 with direct sum taken over links in the boundary of the connect-sum B^4. Note also that there exists an injective homomorphism

$$\iota : \mathcal{S}_0^{\mathrm{BN}}(X_0; L_0) \otimes \mathcal{S}_0^{\mathrm{BN}}(X_1; L_1) \hookrightarrow \mathcal{S}_0^{\mathrm{BN}}(X_0 \sqcup X_1; L_0 \sqcup L_1)$$

given by applying the natural injective map $\mu : BN(K) \otimes BN(K') \to BN(K \sqcup K')$ to the input link labels.

Definition 4.2.9. *Let (X_0, L_0) and (X_1, L_1) denote 4-manifold and boundary link pairs with skein surfaces S_0 and S_1 respectively. For each S_i, we let K_i denote the input link after merging the input balls into a single B^4. Note that merging input B^4s yields a well-defined class. We let $[(S_0 \sqcup S_1, v)]$ denote the skein equivalence class of the filling given by $(S_0 \sqcup S_1, v)$. We define the* half torsion-free submodule $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$ *of $\mathcal{S}_0^{\mathrm{BN}}(X_0 \# X_1; L_0 \sqcup L_1)$ as the submodule*

$$\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1) := \mathbb{F}_2[H]\Big\langle [(S_0 \sqcup S_1, v)] \mid Tor_1^{\mathbb{F}_2[H]}(BN(K_0), BN(K_1)) = 0 \Big\rangle.$$

In other words, $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$ is the submodule generated by elements in $\mathcal{S}_0^{\mathrm{BN}}(X_0 \# X_1; L_0 \sqcup L_1)$ represented by a disjoint union of fillings whose Bar-Natan homology groups of their respective input links have pairwise vanishing torsion. We work with $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$ to avoid the issues with torsion later on.

Proposition 4.2.1. *The composition of maps $(\phi_\# \circ \iota)$ is invertible on $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$. In particular, suppose that $[S_0 \sqcup S_1] \in \mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$, then*

$$ord_H^{(X_0 \# X_1, L_0 \sqcup L_1)}([S_0 \sqcup S_1]) = ord_H^{(X_0 \sqcup X_1, L_0 \sqcup L_1)}(\iota([S_0] \otimes [S_1])).$$

Proof. We consider the composition of maps

$$\phi_{\#} \circ \iota : \mathcal{S}_0^{\mathrm{BN}}(X_0; L_0) \otimes \mathcal{S}_0^{\mathrm{BN}}(X_1; L_1) \to \mathcal{S}_0^{\mathrm{BN}}(X_0 \# X_1; L_0 \sqcup L_1).$$

We are done if we are able to define an inverse map $(\phi_{\#} \circ \iota)^{-1}$ with the desired image when restricted to $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$. Let $[(S_0 \sqcup S_1, v)]$ be a generator of $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$. Then, letting K_0 and K_1 denote the input links of S_0 and S_1 respectively, we have that $v \in BN(K_0 \sqcup K_1)$. Since $BN(K_0)$ and $BN(K_1)$ satisfy the pairwise vanishing torsion condition, we have the natural isomorphism $\mu^{-1} : BN(K_0 \sqcup K_1) \xrightarrow{\cong} BN(K_0) \otimes BN(K_1)$. Note that the preimage of a generator $[(S_0 \sqcup S_1, v)]$ under the map $\phi_{\#}$ is the element in $\mathcal{S}_0^{\mathrm{BN}}(X_0 \sqcup X_1; L_0 \sqcup L_1)$ represented by the filling $(S_0 \sqcup S_1, v)$. Letting $\mu^{-1}(v) = y \otimes z$, we may then apply the map μ^{-1} to the input link labels to obtain the element $[(S_0, y)] \otimes [(S_1, z)] \in \mathcal{S}_0^{\mathrm{BN}}(X_0; L_0) \otimes \mathcal{S}_0^{\mathrm{BN}}(X_1; L_1)$. We note that $\phi_{\#}^{-1}([(S_0 \sqcup S_1, v)])$ is $\iota([(S_0, y)] \otimes [(S_1, z)])$, hence, the the lasagna H torsion orders of $[(S_0, y)] \otimes [(S_1, z)]$ and $[(S_0 \sqcup S_1, v)] \in \mathcal{S}_0^{\mathrm{BN}}(X_0 \# X_1; L_0 \sqcup L_1)$ coincide. □

Remark 4.2.3. *The submodule $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$ contains elements represented by surfaces connected through the 4-manifold connect-sum region by cobordisms $L \times I$ for some link L provided that $Tor_1^{\mathbb{F}_2[H]}(BN(K_0 \sqcup L), BN(K_1 \sqcup \overline{L}))$ vanishes. Letting $[S]$ denote the element corresponding to the connected surface and letting $\sum_i([S_{0,i}] \otimes [S_{1,i}])$ denote the result of neck-cutting along $L \times I$, we have that*

$$ord_H^{(X_0 \# X_1, L_0 \sqcup L_1)}([S]) = ord_H^{(X_0 \sqcup X_1, L_0 \sqcup L_1)}(\sum_i \iota([S_{0,i}] \otimes [S_{1,i}])).$$

4.2.3. Deformations and decompositions into relative second homology.

In this section, we relate results of Ren–Willis [**RW24**] and Morrison–Walker–Wedrich [**MWW24**]. We then describe the structure of the free part of $\mathcal{S}_0^{\mathrm{BN}}(X; L)$ using a deformation of the invariant. Throughout the remainder of this work, we let $H_2(X)^L$ denote the $H_2(X, L)$-torsor given by the preimage $\partial^{-1}([L])$, where ∂ is the connecting homomorphism in the homology long exact sequence of the pair (X, L).

Definition 4.2.10. *Let L be a link in S^3 and let $H(L)$ denote the bigraded rank one free abelian group $\mathbb{F}_2$ concentrated in bidegree $(-lk(L), lk(L))$, generated by preimage $\partial^{-1}([L])$ in $H_2(B^4, L)$.*

Remark 4.2.4. *The invariant $H(L)$ is functorial for links in S^3, and is naturally isomorphic to the* $\mathfrak{gl}_1$ *link homology theory* KhR_1 *as bigraded link homology theories. Thus, we let $\mathcal{S}_0^1(X;L)$ denote the homological skein module constructed from the $H(\_)$ link homology theory. We note that any surface S representing an element $\alpha \in H_2(X)^L$ yields an element $[S] \in \mathcal{S}_0^1(X;L,\alpha)$.*

Definition 4.2.11. *Let $\Sigma = \{\lambda_1, \lambda_2\}$ be a set of deformation parameters in $\mathbb{F}_2$. We let $\mathcal{S}_0^\Sigma(X;L)$ denote the skein lasagna module built from the link homology theory obtained by replacing E_1 and E_2 in Definition 4.2.1 with $E_1(\lambda_1,\lambda_2)$ and $E_2(\lambda_1,\lambda_2)$.*

It is possible to characterize $\mathcal{S}_0^\Sigma(X;L)$ in terms of $\mathcal{S}_0^1(X;L)$ modules.

Corollary 4.2.1 (Corollary 3.14 – [**MWW24**]). *Let $\Sigma = \{1,0\}$, and let $C(L,\Sigma)$ denote the set of* colorings *of the components of L by elements of Σ. Let $L_{c^{-1}(i)}$ denote the sublink of L with color i colored by c. There is an isomorphism of $H_2(X)^L \times \mathbb{Z}_q \times \mathbb{Z}_h$-graded $\mathbb{F}_2$-vector spaces*

$$\mathcal{S}_0^\Sigma(X;L) \cong \bigoplus_{c\in C(L,\Sigma)} \mathcal{S}_0^1(X;L_{c^{-1}(1)},\mathbb{F}_2) \otimes \mathcal{S}_0^1(X;L_{c^{-1}(0)},\mathbb{F}_2) \tag{4.1}$$

Remark 4.2.5. *Let S be an oriented surface in (X,L) with no closed components. For a coloring $c \in C(L,\Sigma)$, the element corresponding to $[S] \in \mathcal{S}_0^\Sigma(X;L)$ under the isomorphism in (4.1) is the tensor product $[S_{c^{-1}(1)}] \otimes [S_{c^{-1}(0)}]$, where $[S_{c^{-1}(\lambda_i)}]$ denotes the relative second homology class given by the subsurface $S_{c^{-1}(\lambda_i)}$ whose boundary is the sublink colored λ_i. We can relax the conditions on S to include closed components as long as S remains homologically diverse.*

We then adopt the notation $H_2^{L,\times 2}(X)$ from [**RW24**] to denote the set of relative *double classes* of the pair (X,L). Defined as

$$H_2^{L,\times 2}(X) := \{(\alpha_-,\alpha_+) \in H_2(X;L)^2 \mid \partial\alpha_\pm = \sum_i \epsilon_{i,\pm}[L_i],\ \epsilon_{i,\pm} \in \{0,1\}, \epsilon_{i,+} + \epsilon_{i,-} = 1\}.$$

We also require the notion of a *skein category*, defined as follows.

Definition 4.2.12. *Let (X,L) be a 4-manifold and boundary link pair. The corresponding* skein category *$\mathcal{C}(X;L)$ has skein surfaces of (X,L) as its objects, and, for any Σ_0, $\Sigma_1 \in Ob(\mathcal{C}(X;L))$*

with input data $\{B_{0,i}, L_{0,i}\}_{i\in I}$ *and* $\{B_{1,j}, L_{1,j}\}_{j\in J}$ *respectively, the morphisms are*

$$Hom(\Sigma_0, \Sigma_1) = \{([S] : \Sigma_0 \to \Sigma_1, f)\}$$

where $[S]$ *is the isotopy class of a surface* S *such that* $S|_{\partial B_{0,i}} = \Sigma_0|_{\partial B_{0,i}}$ *and* $S|_{\partial B_{1,j}} = \Sigma_1|_{\partial B_{1,j}}$*, and* f *is an isotopy class of isotopies* $S \cup \Sigma_1$ *to* Σ_0*.*

For more details, see [**RW24**] Section 2. We now study a *localized* version of the invariant $\mathcal{S}_0^{\mathrm{BN}}(X;L)$ with $H = 1$. Let $BN_{H=1}$ denote the link homology theory obtained by setting $H = 1$ in the Bar-Natan Frobenius algebra A^{BN} as in Definition 4.2.4. Equivalently, $BN_{H=1}$ is the link homology theory obtained by deforming the $U(2)$-equivariant Frobenius pair by $\Sigma = \{1, 0\}$. Note that the skein lasagna module constructed from $BN_{H=1}$ is exactly the deformation $\mathcal{S}_0^{\Sigma}(X;L)$ for $\Sigma = \{1, 0\}$, and therefore satisfies analogous properties. Our goal is to prove the following.

PROPOSITION 4.2.2. *The rank of the free part of* $\mathcal{S}_0^{\mathrm{BN}}(X;L)$ *is bounded below by the dimension of* $\mathcal{S}_0^{BN_{H=1}}(X;L)$*. Furthermore, there is an isomorphism*

$$\Phi : \mathcal{S}_0^{BN_{H=1}}(X;L) \xrightarrow{\cong} \mathbb{F}_2^{H_2^{L,\times 2}(X)}$$

and there exists a basis for $\mathcal{S}_0^{BN_{H=1}}(X;L)$ *of canonical generators that correspond to elements of* $H_2^{L,\times 2}(X)$*.*

This proposition follows from the isomorphism in (4.1), but for completeness, we reprove this result using the terminology in [**RW24**]. Recall that a *double skein* is an oriented surface $\Sigma \subset (X, L)$ and a partition $\Sigma = \Sigma_+ \cup \Sigma_-$ of the surface given by a second orientation $\mathfrak{D}$. If the original orientation agrees (resp. disagrees) with $\mathfrak{D}$ on certain components, we denote this subsurface by Σ_+ (resp. Σ_-).

DEFINITION 4.2.13. *A* canonical Bar-Natan lasagna filling*, denoted* $x(\Sigma_+, \Sigma_-)$*, is a filling defined by labeling each boundary link* L_i *with the canonical generator* $x_{\mathfrak{D}}|_{L_i}$*, and we label the components of* Σ_+ *with* $\boldsymbol{a}$ *and* Σ_- *with* $\boldsymbol{b}$*, where* $\boldsymbol{a} = x - 1$ *and* $\boldsymbol{b} = x$*. We note that, unlike the skein lasagna module construction for Lee homology, we do not require any re-scaling with this theory, as the discriminant for this localized theory is* 1*.*

LEMMA 4.2.13.1. *The following items are analogues of results from [**RW24**].*

(1) *Let $\Sigma \subset (X, L)$ be a skein surface, then every lasagna filling with surface Σ in the free part of $\mathcal{S}_0^{BN_{H=1}}(X; L)$ is skein equivalent to a linear combination of canonical Bar-Natan lasagna fillings.*

(2) *Let $S : \Sigma \to \Sigma'$ be a morphism in the category of skeins $\mathcal{C}(X; L)$ that respects double skein structures, then the map induced by S in $\mathcal{S}_0^{BN_{H=1}}(X; L)$ maps $x(\Sigma_+, \Sigma_-)$ to $x(\Sigma'_+, \Sigma'_-)$.*

PROOF. The argument in [**RW24**] for the Lee lasagna module adapts immediately to this setting for part (1). For part (2), we make the following observation about the localized theory. Let $S : L \to L'$ be a framed cobordism and let $\mathfrak{s}_o$ be a canonical generator corresponding to the orientation o of L. Then

$$BN_{H=1}(S)(\mathfrak{s}_o) = \sum_{\mathfrak{D}|_L = o} \mathfrak{s}_{\mathfrak{D}|_{L'}}$$

where the sum runs over all orientations of S that are compatible with the orientation o on L. This follows from [**Ras04**] using a modification of the Frobenius algebra and $\mathbb{F}_2$ coefficients. As a consequence of this, on the tensor product of generators of the boundary links of a skein surface Σ, we have

$$BN_{H=1}(S)(\otimes_i \mathfrak{s}_{\mathfrak{D}|_{L_i}}) = \otimes_j \mathfrak{s}_{\mathfrak{D}|_{L'_j}} + Y$$

where Y consists of terms with incompatible orientations. Thus, the map induced by S on $x(\Sigma_+, \Sigma_-)$ as a morphism in $\mathcal{C}(X; L)$ maps $x(\Sigma_+, \Sigma_-) \mapsto x(\Sigma'_+, \Sigma'_-)$. □

PROOF. (*of Proposition 4.2.2*) As in [**RW24**], define a map $\epsilon : \mathcal{S}_0^{BN_{H=1}}(X; L) \to \mathbb{F}_2^{H_2^{L,\times 2}(X)}$ by $x(\Sigma_+, \Sigma_-) \mapsto e_{([\Sigma_+],[\Sigma_-])}$, the generator of the $([\Sigma_+], [\Sigma_-])$-th coordinate of $\mathbb{F}_2^{H_2^{L,\times 2}(X)}$. Lemma 4.2.13.1 implies that ϵ is well-defined, and the remainder of the argument presented in [**RW24**] occurs entirely in the skein category before the application of $BN_{H=1}$, so the result follows. □

REMARK 4.2.6. *One may expect these similarities between $\mathcal{S}_0^{Lee}$ and $\mathcal{S}_0^{BN_{H=1}}$, as the localized theory is equivalent to Lee homology [**KR22**].*

We note that homologically diverse surfaces are non-trivial modulo H-torsion in $\mathcal{S}_0^{\mathrm{BN}}(X; L)$ with $\mathbb{F}_2$ coefficients.

Corollary 4.2.2. *Let $S \subset (X, L)$ be a homologically diverse surface, then the element $[S] \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$ is non-trivial modulo H-torsion.*

Proof. As in the proof of the analogous result in [**MWW24**], homologically diverse surfaces represent non-trivial generators of $\mathbb{F}_2^{H_2^{L,\times 2}(X)}$ which is isomorphic to $\mathcal{S}_0^{BN_{H=1}}(X; L)$ by Proposition 4.2.2. The result then follows. $\square$

4.3. Extending the exotically knotted pairs of surfaces in the 4-ball to other 4-manifolds

We now explain the skein lasagna module interpretation of the results of Hayden in [**Hay23**]. For each disk D and D' with common boundary K_H, we define lasagna fillings represented by punctured, time-reversed disks $[-\overset{\circ}{D}]$ and $[-\overset{\circ}{D}']$, with a single input ball with an unknot on the boundary equipped with label $\Phi_U^{-1}((1, 0))$. The propositions in Remark 4.1.1, Lemma 4.2.5.1, and Lemma 4.2.3.1 then imply that these fillings are not equal in $\mathcal{S}_0^{\mathrm{BN}}(B^4; K_H)$.

Occasionally, we will start with an embedded surface $S \subset (X, L)$, then discuss the filling $[S]$ *corresponding* to S. By this, we mean to take the surface S, with the possible addition of some framing-changing input balls, as the lasagna filling representing $[S]$.

Definition 4.3.1. *Letting these fillings be denoted $\mathcal{F}_0$ and $\mathcal{F}_0'$ respectively, we define the Bar-Natan lasagna* distinguishing element $\delta_0^L = \mathcal{F}_0 - \mathcal{F}_0'$. *This is precisely the element $\delta_0 \in BN(K_H)$ defined in Lemma 4.2.3.1, reinterpreted as an element in the Bar-Natan skein lasagna module. Similarly, for the exotic genus g surfaces F_g and F_g' in [**Hay23**], we let $[F_g]$ and $[F_g']$ denote the corresponding Bar-Natan fillings in (B^4, K_H), and define the distinguishing element $\delta_g^L := [F_g] - [F_g']$.*

Remark 4.3.1. *As stated in Lemma 4.2.5.1, when the 4-manifold and boundary link pair is (B^4, L), skein lasagna modules recover the TQFT from which they were constructed, allowing for the following restatement in terms of internal stabilization distance.*

Corollary 4.3.1. *Let δ_0^L and δ_g^L be defined as above, then for all $g \geq 0$, we have that $\mathrm{ord}_H^{(B^4, K_g)}(\delta_g^L) > 1$.*

Remark 4.3.2. *Unfortunately, the Bar-Natan homology package, unlike link Floer homology (in reference to the results of Guth in [**Gut22**]), does not immediately produce exotically knotted pairs* *of surfaces with arbitrarily large internal stabilization distance in B^4. This leads to some difficulty in producing "one is not enough" results in other 4-manifolds through a direct application of the gluing map in 4.2.1. However, this difficulty can be overcome by restricting to certain types of homologically diverse surfaces.*

Question 4.3.2. *Can a version of the Floer lasagna module construction (see [**Che22**]) obstruct smooth isotopy between a pair of knotted surfaces strongly stabilized more than once? In other words, can the results of Guth in [**Gut22**] be extended in a similar way?*

Definition 4.3.3. *We say that a surface $S \subset (X, L)$ has* local genus *g if there exists a 4-ball B embedded in int(X) such that $S \cap B^4$ is a punctured oriented surface of genus g.*

Definition 4.3.4. *For a homologically diverse surface $S \subset (X, L)$, a corresponding Bar-Natan lasagna filling is* primitive *if represents a a non-trivial free generator $[S] \in \mathcal{S}_0^{\mathrm{BN}}(X; L, \alpha)$ and H does not divide $[S]$.*

Lemma 4.3.4.1. *For homologically diverse surface $S \subset (X, L)$, a filling $[S]$ is primitive if and only if S has no local genera and $[S]$ is not equivalent in $\mathcal{S}_0^{\mathrm{BN}}(X; L)$ to a sum of elements given by fillings each with surfaces having local genera.*

Proof. Suppose first that $S \subset (X, L)$ has local g-genera and let B denote the corresponding 4-ball. We can isotope the subsurface $S \cap B$ into a collar neighborhood of ∂X. This isotopy realizes S as the image of the surface $S' = S \setminus (S \cap B) \cup_U D^2$ (the surface obtained by removing the genus g subsurface and replacing it with a disk) under g many H-maps, implying that $[S]$ is divisible by H. Suppose, alternatively, that the lasagna filling is not primitive, then $[S] = H^k \cdot v$ for some $v \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$, $k \in \mathbb{N}$. It is sufficient to let $k = 1$. Recall that, over $\mathbb{F}_2$ coefficients, an H factor is interchangeable with a T^2 connect-summand. The presence of this H factor implies that, for some input ball and link pair (B_i, L_i) in $[S]$, the lasagna filling $[S]$ is equivalent to a filling $[S']$ obtained by replacing (B_i, L_i) with a slightly smaller $B_i' \subset B_i$ such that $S' \cap (B_i \setminus B_i') = (L_i \times I) \# T^2$. Thus, we can choose yet another small input ball B_i'' in the $S^3 \times I$ region between B_i and B_i' such that $B_i'' \cap S' = T^2 \setminus pt$. Therefore, S is equivalent to a filling S' with local genera.

The case where $L = \emptyset$ is similar. If S has local genera, then we may neck-cut (see Figure 4.3) to produce a dotted torus embedded in $\mathrm{int}(X)$. After removing the 4-ball, we may isotope the dotted torus to the newly created S^3 boundary.

Alternatively, if $S \subset (X, L)$ has no local genera but is skein equivalent to a sum of fillings with surfaces with local genera, then we may apply the argument above on each summand. □

We now state the main result allowing for the extension of non-stabilizing exotic knotted pairs of surfaces to other 4-manifolds.

Proposition 4.3.1. *Let $S \subset (X, L)$ be a homologically diverse surface with a primitive filling $[S] \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$, then for any $g \geq 1$, the surfaces $F_g \sqcup S$ and $F'_g \sqcup S$ form an exotically knotted pair in $(X \setminus \mathit{int}(B^4), K_g \sqcup L)$, and remain exotic after a single internal stabilization.*

Proof. This follows directly from the definition of primitive fillings and Proposition 4.2.1. For simplicity, we present the proof in the case where the boundary link of X is the empty link; the proof is identical in the other case.

Let S be a homologically diverse surface in $(X, \emptyset)$, and let $[S]$ be a corresponding primitive filling. Then the element $[S]$ is a free generator with no H factors in $\mathcal{S}_0^{\mathrm{BN}}(X)$. By Corollary 4.3.1, we have that δ_g^L is non-trivial and H-torsion in $\mathcal{S}_0^{\mathrm{BN}}(B^4; K_g)$. Then, by Proposition 4.2.1, we have that the lasagna H-torsion order of the element given by $[F_g \sqcup S] - [F'_g \sqcup S]$ is equal to $\mathrm{ord}_H^{(X \setminus \mathrm{int}(B^4), K_g \sqcup L)}(\iota(\delta_g^L \otimes [S]))$. The element $\iota(\delta_g^L \otimes [S])$ is equal to $\delta_g^L \otimes [S]$ in the isomorphic copy of $\mathcal{S}_0^{\mathrm{BN}}(B^4; K_g) \otimes \mathcal{S}_0^{\mathrm{BN}}(X; L)$ in $\mathcal{S}_0^{\mathrm{BN}}(B^4 \sqcup X; K_g \sqcup L)$, implying $[F_g \sqcup S] - [F'_g \sqcup S]$ is non-trivial with the same lasagna H-torsion order as δ_g^L. □

Before we provide any explicit examples, we need a method to determine if the filling represented by a homologically diverse surface is indeed primitive.

Lemma 4.3.4.2. *Recall the linear map $F : \frac{\mathbb{F}_2[H]}{H} \otimes \mathcal{S}_0^{\mathrm{BN}}(X; L) \to \mathcal{S}_0^2(X; L)$ from Lemma 4.2.8.1. For a homologically diverse surface $S \subset (X, L)$, if the image of it's corresponding filling $F([S]) \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$ is non-zero, then $[S] \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$ is primitive.*

Proof. Given a filling $\mathcal{F} \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$, the image $F(\mathcal{F}) \in \mathcal{S}_0^2(X; L)$ is a filling consisting of the same topological data as $\mathcal{F}$, with input link labels obtained by setting $H = 0$ in the labels of

$\mathcal{F}$. If a surface $S \subset (X, L)$ has local genus (or is equivalent in $\mathcal{S}_0^{\mathrm{BN}}(X; L)$ to a sum of fillings each with local genus), then the corresponding filling is of the form $H \cdot v$ for some $v \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$ and therefore $F([S]) = 0 \in \mathcal{S}_0^2(X; L)$. $\square$

We remark now on the steps necessary to produce examples of exotically knotted pairs of surfaces as in Proposition 4.3.1. Suppose that $S \subset X$ is an embedded surface in 4-manifold X, and suppose that S is homologically diverse. To verify that S admits a primitive filling in $\mathcal{S}_0^{\mathrm{BN}}(X; L)$, we must begin with a corresponding filling $[S]_{\mathrm{KhR}_2}$ in $\mathcal{S}_0^2(X; L; \mathbb{F}_2)$. We then must check that $[S]_{\mathrm{KhR}_2} \neq 0$ through some pre-existing non-vanishing criterion and that $F^{-1}([S]_{\mathrm{KhR}_2})$ is non-empty. If these criteria are met, we may obtain a non-zero element $1 \otimes [S]$ in $F^{-1}([S]_{\mathrm{KhR}_2})$, implying the filling $[S] \in \mathcal{S}_0^{\mathrm{BN}}(X; L)$ is primitive. We may then consider the BN lasagna filling represented by the surfaces $F_g \sqcup S$ and $F_g' \sqcup S$ in $X \setminus \mathrm{int}(B^4)$, where the common boundary K_g is a boundary link on ∂B^4. Proposition 4.2.1 allows us to identify the fillings $[F_g \sqcup S]$ and $[F_g' \sqcup S]$ with tensor products $[F_g] \otimes [S]$ and $[F_g'] \otimes [S]$ respectively, implying that $\delta_g^L \otimes [S] = [F_g] \otimes [S] - [F_g'] \otimes [S] \neq 0$ and has lasagna H-torsion order equal to that of δ_g^L.

Remark 4.3.3. *An explicit non-vanishing criterion for $\mathcal{S}_0^2(X; \emptyset; \mathbb{F}_2)$ is not known, only interpretations of other non-vanishing results using $\mathbb{F}_2$ coefficients. If such a criterion existed, it would still be necessary to describe the skein surfaces that represent non-zero Bar-Natan lasagna fillings in order to produce explicit examples. We expect that this is possible for D^2-bundles over S^2 with nonpositive Euler number. Fixing some $n \leq 0$, let $D(n)$ denote the corresponding disk bundle and let $S \subset D(n)$ denote the sphere with self-intersection $[S] \cdot [S] = -n$. Next, let $f_- \in BN(U)$ denote the label of the -1-framed input unknot on an input ball in $D(n)$. In other words, f_- is the image of the birth cobordism composed with the corresponding Reidemeister 1 map. We then construct the skein surface S' and filling $[S']$ as*

$$[S'] := (S \setminus \bigsqcup_{i=1}^{n} D_i^2, f_- \otimes ... \otimes f_-).$$

This new skein surface S' is given by removing n disks from S, and adding input balls, with -1-framed boundary unknots labeled by f_-.

Remark 4.3.4. *As a consequence of Proposition 1.15 in [**RW24**] and Theorem 2.1(1) [**Ren24**], we have the following results for the pair* $(\overline{\mathbb{C}P^2}, \emptyset)$ *(the* D^2*-bundle over* S^2 *with Euler number -1) over field coefficients:*

$$\mathcal{S}_{0,-2p^2,2p^2+2p}(\overline{\mathbb{C}P^2};\emptyset;0) \cong \mathbb{F},\ (p \geq 0)$$

$$\mathcal{S}_{0,-2p^2+2p,2p^2-1}(\overline{\mathbb{C}P^2};\emptyset;1) \cong \mathbb{F},\ (p \geq 1)$$

In particular, the empty skein and the sphere of self intersection -1 (with a single input ball with a -1-framed input unknot and label f_-*) represent non-trivial elements in* $\mathcal{S}_0^2(\overline{\mathbb{C}P^2};\emptyset)$ *(see [**RW24**] Section 6). We now proceed with the construction of exotic surfaces in* $\overline{\mathbb{C}P^2}$ *using the exotic knotted pairs of Hayden.*

Example 4.3.5. *Let* $X = \overline{\mathbb{C}P^2}$ *and let* $\overline{\mathbb{C}P^1}$ *denote the sphere of -1 self-intersection. Note that* $\overline{\mathbb{C}P^1}$ *is homologically diverse in* $(\overline{\mathbb{C}P^2}, \emptyset)$*, as it represents a generator of* $H_2(\overline{\mathbb{C}P^2}) \cong \mathbb{Z}$*. Let* $[\overline{\mathbb{C}P^1}]_{\mathrm{BN}} \in \mathcal{S}_0^{\mathrm{BN}}(\overline{\mathbb{C}P^2};\emptyset)$ *and* $[\overline{\mathbb{C}P^1}]_{\mathrm{KhR}_2} \in \mathcal{S}_0^2(\overline{\mathbb{C}P^2};\emptyset;\mathbb{F}_2)$ *denote the fillings obtained from* $\overline{\mathbb{C}P^1}$ *as in Remark 4.3.3 in their respective lasagna modules. Let* f_-^{BN} *and* $f_-^{\mathrm{KhR}_2}$ *denote the unknot labels for the skein surfaces of* $[\overline{\mathbb{C}P^1}]_{\mathrm{BN}}$ *and* $[\overline{\mathbb{C}P^1}]_{\mathrm{KhR}_2}$ *respectively. The natural map* $\mu : \frac{\mathbb{F}_2[H]}{H} \otimes BN(\_) \to \mathrm{KhR}_2(\_)$ *is given by realizing* KhR_2 *as the homology of the Bar-Natan complex tensored with* $\frac{\mathbb{F}_2[H]}{H}$ *then applying the universal coefficient theorem. Note that, for the homologies of the -1-framed unknot,* $\mu(f_-^{\mathrm{BN}}) = f_-^{\mathrm{KhR}_2}$*. Hence, the filling* $[\overline{\mathbb{C}P^1}]_{\mathrm{BN}}$ *is mapped to* $[\overline{\mathbb{C}P^1}]_{\mathrm{KhR}_2}$ *by the composition of maps*

$$\mathcal{S}_0^{\mathrm{BN}}(\overline{\mathbb{C}P^2};\emptyset,1) \xrightarrow{1\otimes} \frac{\mathbb{F}_2[H]}{H} \otimes \mathcal{S}_0^{\mathrm{BN}}(\overline{\mathbb{C}P^2};\emptyset,1) \xrightarrow{F} \mathcal{S}_0^2(\overline{\mathbb{C}P^2};\emptyset,1) \tag{4.2}$$

where F *is the linear map from Lemma 4.2.8.1. Since the filling* $[\overline{\mathbb{C}P^1}]_{\mathrm{KhR}_2}$ *is a generator of* $\mathcal{S}_0^2(\overline{\mathbb{C}P^2};\emptyset)$*, we have that,* $1 \otimes [\overline{\mathbb{C}P^1}]_{\mathrm{BN}} \in \frac{\mathbb{F}_2[H]}{H} \otimes \mathcal{S}_0^{\mathrm{BN}}(\overline{\mathbb{C}P^2};\emptyset)$ *is non-zero, as its image under the linear map from Lemma 4.2.8.1 is the* KhR_2 *filling with an identical skein surface and label given by* $f_-^{\mathrm{KhR}_2}$*. This implies, by Lemma 4.3.4.2, that* $[\overline{\mathbb{C}P^1}]$ *is a primitive filling in* $\mathcal{S}_0^{\mathrm{BN}}(\overline{\mathbb{C}P^2};\emptyset)$*.*
We may now apply Proposition 4.2.1 and Proposition 4.3.1 for the torsion element δ_g^L*, we have that*

$$ord_H^{(B^4 \sqcup \overline{\mathbb{C}P^2}, K_g)}(\iota(\delta_g^L \otimes [S])) = ord_H^{(\overline{\mathbb{C}P^2} \setminus int(B^4); K_g)}([F_g \sqcup \overline{\mathbb{C}P^1}] - [F_g' \sqcup \overline{\mathbb{C}P^1}]) = ord_H(\delta_g^L) > 1$$

This directly implies that the surfaces $F_g \sqcup \overline{\mathbb{C}P^1}$ *and* $F_g' \sqcup \overline{\mathbb{C}P^1}$ *form an exotically knotted pair in* $(\overline{\mathbb{C}P^2} \setminus \mathit{int}(B^4); K_g)$ *that do not become smoothly isotopic after a single internal stabilization.*

Similar arguments may hold for $S^2 \times D^2$ and D^2-bundles over S^2 using the non-vanishing results of Manolescu-Neithalath [**MN22**] and Ren-Willis [**RW24**].

Note that for a pair of 4-manifold and boundary link pairs (X_0, L_0) and (X_1, L_1), and skein surfaces S_0 and S_1 respectively with K_0 and K_1 as respective input links, that $\mathrm{Tor}_1^{\mathbb{F}_2[H]}(BN(K_0), BN(K_1)) = 0$ implies that a filling $[S_0 \# S_1]$ is an element of $\mathcal{R}(X_0 \# X_1; L_0 \sqcup L_1)$ as neck-cutting the connect-sum region only produces an additional unknot. Letting $S^\bullet$ denote the surface S with a dot, we have that Proposition 4.3.1 states

$$\begin{aligned}\mathrm{ord}_H^{(X_0\#X_1, L_0\sqcup L_1)}([S_0\#S_1]) &= \mathrm{ord}_H^{(X_0\sqcup X_1, L_0\sqcup L_1)}([S_0^\bullet \sqcup S_1] + [S_0 \sqcup S_1^\bullet] + H[S_0 \sqcup S_1]) \\ &= \mathrm{ord}_H([S_0^\bullet] \otimes [S_1] + [S_0] \otimes [S_1^\bullet] + H([S_0] \otimes [S_1]))\end{aligned}$$

We may conclude then that, if K is a knot that bounds exotic surfaces in B^4 that induce different maps on Bar-Natan homology, then if (X, L) contains a homologically diverse surface that admits a primitive Bar-Natan filling, the knot K as a *local* link in $X \setminus \mathrm{int}(B^4)$ continues to bound exotic surfaces in the new 4-manifold. Furthermore, if the induced maps remain distinct after multiplication by H, the new exotic surfaces remain distinct after an internal stabilization.

For a 4-manifold X and a homologically diverse surface $S \subset X$ with primitive filling, it may be interesting to consider the effect of *external stabilization* on the double branched covers of pairs $(X, F_g \sqcup S)$ and $(X, F_g' \sqcup S)$. Recall that, for an exotic pair of 4-manifolds (X_0, X_1), an external stabilization is the operation corresponding to taking a connect-sum with $S^2 \times S^2$. The double branched cover of an internally stabilized surface corresponds to an external stabilization of its branched double cover, so such a question is natural to consider. We remark that the manifolds dealt with throughout this work generally all have non-empty boundary, and there exist many similar results on stabilization conjectures in the closed and relative case (see [**Lin23**, **LM21**, **KMT22**, **Kan24**, **Gut22**, **Auc23**]). One may also consider the effect of external stabilization on exotically knotted pairs of surfaces. Although the results presented in this paper are incapable of producing results for this conjecture with $S^2 \times S^2$, one may at least use Proposition 4.3.1 to show pairs of surfaces remain exotic after connect-summing with other 4-manifolds, given that the empty skein surface represents a non-zero element in the lasagna module.

COROLLARY 4.3.2. *Suppose that a primitive filling* $[S] \in \mathcal{S}_0^{\mathrm{BN}}(X;\emptyset)$ *is skein equivalent to the empty filling* $[\emptyset]$*, then the exotic knotted pair* (F_g, F_g') *remains exotic after a single internal stabilization after connect-summing with* X*.*

PROOF. This is an application of Proposition 4.3.1. $\square$

APPENDIX A

Appendix A: Categories

In this section, briefly discuss some constructions from the category theory of linear graded categories, including the Karoubi envelopes and Ind-completions (see [**HRW22**] for more details). This appendix contains the categorical constructions used throughout this thesis. For more on Karoubi envelopes and homotopy idempotents, a useful reference is Appendix A of [**GW23**]; see also the footnote under Definition 3.1 in [**BNM06**] for a brief history. For more on Ind-objects and Ind-completions, see [**KS06**], Definition 6.1. Finally, for an introduction to dg categories and twisted complexes, see [**Kel02**].

Definition A.0.1. *Let G be an abelian group, and let $\mathcal{C}$ be a G-graded $\mathbb{F}$-linear category. Then the* G-additive completion *of $\mathcal{C}$, denoted* $\mathrm{Mat}(\mathcal{C})$*, is the category whose objects are finite formal direct sums of objects in $\mathcal{C}$, and each morphism $f : \bigoplus_{i=1}^{n} A_i \to \bigoplus_{j=1}^{m} B_i$ is given by an $m \times n$ matrix of morphisms $f_{ij} : A_i \to B_j$ in $\mathcal{C}$.*

Unless specified otherwise, let $\mathcal{C}$ be a $\mathbb{Z} \oplus \mathbb{Z}$-graded $\mathbb{F}$-linear category. The additive completion of such a category formally adjoins grading shifts and finite sums, but we also need a completion that formally adjoins images of idempotent maps.

Definition A.0.2. *The* (graded) Karoubi envelope *of $\mathcal{C}$, denote* $\mathrm{Kar}(\mathcal{C})$*, is the category whose objects are pairs (A, e_A), where A is an object of $\mathcal{C}$ and $e_A \in \mathrm{End}^0_{\mathcal{C}}(A)$ is idempotent, i.e. $e_A^2 = e_A$. The morphisms are given by maps $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $f = e_B \circ f \circ e_A$.*

The colimits we study will be of the following form.

Definition A.0.3. *A* directed system *in $\mathcal{C}$ is a diagram in $\mathcal{C}$ indexed by a filtered small category. Furthermore, a* filtered colimit *of $\mathcal{C}$ is a colimit of a directed system in $\mathcal{C}$.*

The following completion formally adjoins filtered colimits.

Definition A.0.4 ([**HRW22**], Def. 2.22). *The* Ind-completion *of $\mathcal{C}$ (denoted* $\operatorname{Ind}(\mathcal{C})$*) is the category whose objects are directed systems $\alpha : \mathcal{I} \to \mathcal{C}$ (where $\mathcal{I}$ is a directed indexing set). Given objects $\alpha : \mathcal{I} \to \mathcal{C}$ and $\beta : \mathcal{J} \to \mathcal{C}$, the morphism set is given by*

$$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(\alpha, \beta) := \lim{}_{i \in \mathcal{I}} \operatorname{colim}{}_{j \in \mathcal{J}} \operatorname{Hom}_{\mathcal{C}}(\alpha(i), \beta(j)).$$

Definition A.0.5. *A $\mathbb{Z}$-graded $\mathbb{F}$-linear category $\mathcal{C}_{dg}$ is a* dg-category *if its morphism spaces are differential graded $\mathbb{Z}$-modules, and morphism compositions $\operatorname{Hom}_{\mathcal{C}_{dg}}(X, Z) \otimes \operatorname{Hom}(Y, X) \to \operatorname{Hom}(Y, Z)$ are differential graded $\mathbb{Z}$-module homomorphisms. That is, each morphism space decomposes as a direct sum of graded pieces $\operatorname{Hom}(X, Y) = \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}^k(X, Y)$, where $\operatorname{Hom}^k(X, Y)$ is the space of homogeneous degree-k morphisms from X to Y. The morphism spaces $\operatorname{Hom}(X, Y)$ form cochain complexes $(\operatorname{Hom}(X, Y), d)$, where $d : \operatorname{Hom}^k(X, Y) \to \operatorname{Hom}^{k+1}(X, Y)$ satisfies $d^2 = 0$ and morphism compositions are chain maps.*

For the category of chain complexes $\operatorname{Kom}(\mathcal{C})$ over a $\mathbb{Z} \oplus \mathbb{Z}$-graded $\mathbb{F}$-linear category $\mathcal{C}$, the differentials of morphism spaces are defined as commutators with internal differentials, i.e. for $f \in \operatorname{Hom}^k_{\operatorname{Kom}(\mathcal{C})}(M^\bullet, N^\bullet)$, the differential is $d(f) = d_N \circ f - (-1)^k f \circ d_M$.

Definition A.0.6. *A* one-sided twisted complex *in the dg-category* $\operatorname{Kom}(\mathcal{C})$ *is a collection of chain complexes and chain maps $\{B_i, g_{i,j} : B_i \to B_j\}$ such that if $i \geq j$, then $g_{i,j} = 0$, and the morphisms satisfy*

$$(-1)^j d(g_{i,j}) + \sum_k g_{k,j} \circ g_{i,k} = 0. \tag{A.1}$$

Throughout, all twisted complexes will be one-sided, so we refer to them simply as twisted complexes. Let $\operatorname{Tw}(\mathcal{C})$ denote the dg-category of twisted complexes over $\mathcal{C}$.

Definition A.0.7. *There is a functor* $\operatorname{Tot} : Tw(\mathcal{C}) \to \operatorname{Kom}(\mathcal{C})$ *sending a twisted complex $B = \{\{B_i\}, g_{i,j} : B_i \to B_j\}$ to its* total complex, *denoted* $\operatorname{Tot}(B)$, *given by* $\operatorname{Tot}(B) := \{\bigoplus_i B_i[i], d\}$. *The*

$$\begin{tikzcd}
 & A_0[0] & A_1[0] & A_2[0] & \\
\mathcal{D}_{\mathcal{A}} := & A_0[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_0"] & A_1[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_1"] & A_2[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_2"] & \cdots
\end{tikzcd}$$

Figure A.1. The 2-term double complex associated to a directed system $\mathcal{A}$.

brackets denote homological degree shifts and the differential d *is given by*

$$d := \begin{bmatrix} d_{B_0} & 0 & 0 & \cdots \\ g_{0,1} & -d_{B_1} & 0 & \cdots \\ g_{0,2} & g_{1,2} & d_{B_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We use the same notation, $\mathrm{Tot}(A)$, to denote the *total complex* of a double complex A.

The *homotopy category* of chain complexes $K(\mathcal{C})$ is the category with the same objects as $\mathrm{Kom}(\mathcal{C})$, but with morphisms taken up to chain homotopy. In fact, the morphisms are precisely given by $H^0(\mathrm{Hom}_{\mathrm{Kom}(\mathcal{C})}(X, Y))$.

Let $\mathcal{A}$ denote the following directed system (A_k, d_k), where each A_k is a chain complex with internal differential d_k:

$$\mathcal{A} := A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots . \tag{A.2}$$

Let $\mathcal{D}_{\mathcal{A}}$ denote the double complex in Figure A.1. The differentials from the bottom row to the top row (id and $-f_k$ maps) commute with the internal differentials d_k. The square brackets indicate homological shift, and also serve to differentiate the vertices of the diagram.

For such a double complex $\mathcal{D}_{\mathcal{A}}$, in the Ind-completion $\mathrm{Ind}(K(\mathcal{C}))$ we have the *totalization* of $\mathcal{D}_{\mathcal{A}}$, denoted $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$, with signs chosen as in Figure A.2.

Proposition A.0.1. *[**Hog18**, Proposition 2.28] Let* $\mathcal{A} = (A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots)$ *be a directed system of chain complexes such that* $f_{i+1} \circ f_i = 0$ *for all* i*. Assume also that each chain complex* A_i *is homotopy equivalent to a corresponding chain complex* B_i*, then* $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$ *is homotopy equivalent*

$$\begin{tikzcd}
A_0[0] \arrow[loop above, "d_0"] & A_1[0] \arrow[loop above, "d_1"] & A_2[0] \arrow[loop above, "d_2"] & {} \\
A_0[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_0"'] \arrow[loop below, "-d_0"] & A_1[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_1"'] \arrow[loop below, "-d_1"] & A_2[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_2"'] \arrow[loop below, "-d_2"] &
\end{tikzcd}$$

FIGURE A.2. The total complex $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$ of the double complex $\mathcal{D}_{\mathcal{A}}$. The components of the total differential ∂^{Tot} are depicted as the arrows.

to the total complex of a twisted complex of the form

$$\begin{tikzcd}
B_0 \arrow[r, "g_{0,1}"] \arrow[rr, bend right, "g_{0,2}"'] \arrow[rrr, bend right=60, "g_{0,3}"'] & B_1 \arrow[r, "g_{1,2}"] \arrow[rr, bend right, "g_{1,3}"'] & B_2 \arrow[r, "g_{2,3}"] & B_3 \arrow[r] & \cdots ,
\end{tikzcd}$$

where each $g_{i,j}$ *is a map of homological degree* $j-i-1$*.*

If the homotopy equivalences are given by the collection of maps $\phi_i : A_i \to B_i$, with homotopy inverses $\overline{\phi}_i$, then the maps $g_{i,i+1}$ may be chosen to be $\phi_{i+1} \circ f_i \circ \overline{\phi}_i$, so that the following diagram commutes (on the nose):

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r] \arrow[d, "\phi_3"] & \cdots \\
B_0 \arrow[r, "g_{0,1}"] & B_1 \arrow[r, "g_{1,2}"] & B_2 \arrow[r, "g_{2,3}"] & B_3 \arrow[r] & \cdots
\end{tikzcd} \tag{A.3}$$

Then $g_{i+1,i+2} \circ g_{i,i+1} = (\phi_{i+2} \circ f_{i+1} \circ \overline{\phi}_{i+1}) \circ (\phi_{i+1} \circ f_i \circ \overline{\phi}_i) \sim \phi_{i+2} \circ (f_{i+1} \circ f_i) \circ \overline{\phi}_i \sim 0$, which is why we include the higher homotopies $g_{i,j}$ where $j - i > 1$ in the statement of Proposition A.0.1.

DEFINITION A.0.8. *(Universal property for colimits) A* colimit C *of the directed system* (A.2) *is an object in* $\mathrm{Ind}(K(\mathcal{C}))$ *satisfying the following two properties:*

(C-1) There are chain maps (ϕ_k) *such that*

$$\begin{tikzcd}
A_k \arrow[rr, "f_k"] \arrow[dr, "\phi_k"'] & & A_{k+1} \arrow[dl, "\phi_{k+1}"] \\
& C &
\end{tikzcd}$$

commutes up to homotopy for all $k \in \mathbb{N}$. That is, there are homotopies $(h_k : A_k \to C)$ such that

$$\phi_k - \phi_{k+1} \circ f_k = h_k \circ d_{A_k} + d_C \circ h_k. \tag{A.4}$$

(C-2) Let C' be an object satisfying (C-1), with structure maps (ϕ'_k) and homotopies (h'_k). Then there exists a chain map ξ, unique up to homotopy, making the following diagram homotopy-commute for all $k \in \mathbb{N}$:

$$\begin{tikzcd}
A_k \arrow[rr, "f_k"] \arrow[dr, "\phi'_k"'] \arrow[ddr, bend right, "\phi_k"'] & & A_{k+1} \arrow[dl, "\phi'_{k+1}"] \arrow[ddl, bend left, "\phi_{k+1}"] \\
& C' & \\
& C \arrow[u, dotted, "\xi"] &
\end{tikzcd}$$

By standard category theory arguments, the reader may check that if an object C satisfies the conditions in Definition A.0.8 exists, then it is unique up to homotopy equivalence.

Definition A.0.9. *Let $\mathcal{A} = (A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots)$ be a directed system of chain complexes. Suppose that the homotopy category $K(\mathcal{C})$ contains infinite direct sums, then the* homotopy colimit *of $\mathcal{A}$, denoted* $\operatorname{hocolim} \mathcal{A}$, *can be identified with the total complex of the 2-term complex $\mathcal{D}_{\mathcal{A}}$ (see Figure A.1).*

We now show $\operatorname{hocolim}(\mathcal{A})$ and $\operatorname{colim}(\mathcal{A})$ are equivalent in this context.

Proposition A.0.2. *We have that* $\operatorname{hocolim}(\mathcal{A}) = \operatorname{Tot}(\mathcal{D}_{\mathcal{A}})$ *satisfies the conditions of Definition A.0.8.*

Note that this is true, as the indexing category is freely generated by the graph $(\bullet \to \bullet \to \bullet \to \cdots)$, hence the homotopy colimit of $\mathcal{A}$ is the representing object of the homotopy-commutative version of the cocone of our directed system [**Shu06**, Section 10]. However, we prove it now explicitly.

Proof. To check condition (C-1) in Definition A.0.8, let $\mathcal{A} = (A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots)$ be a directed system in $\mathcal{C}$, and let $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$ denote the 2-term chain complex given in Figure A.2. Note that we may define the following collection of maps $\phi^{\mathrm{Tot}} := \{\phi^{\mathrm{Tot}}_k : A_i \to \mathrm{Tot}(\mathcal{D}_{\mathcal{A}})\}$ by:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{k-1} \arrow[loop above, "d_{A_{k-1}}"] \arrow[r, "f_{k-1}"] \arrow[dd, red, "\phi^{\mathrm{Tot}}_{k-1}"] & A_k \arrow[loop above, "d_{A_k}"] \arrow[r, "f_k"] \arrow[dd, red, "\phi^{\mathrm{Tot}}_{k}"] & A_{k+1} \arrow[loop above, "d_{A_{k+1}}"] \arrow[r] \arrow[dd, red, "\phi^{\mathrm{Tot}}_{k+1}"] & \cdots \\
 & & & & \\
\cdots & A_{k-1}[0] \arrow[loop above] & A_k[0] \arrow[loop above] & A_{k+1}[0] \arrow[loop above] & \cdots \\
\cdots \arrow[ur] & A_{k-1}[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_{k-1}"] \arrow[loop below, "-d_{A_{k-1}}"] & A_k[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_k"] \arrow[loop below, "-d_{A_k}"] & A_{k+1}[-1] \arrow[u, "\mathrm{id}"] \arrow[ur] \arrow[loop below, "-d_{A_{k+1}}"] & \cdots
\end{tikzcd} \tag{A.5}$$

where $\phi^{\mathrm{Tot}}_k := -\mathrm{id}_{A_k}$. We first verify that ϕ^{Tot}_k is a chain map. Letting d_{A_k} denote the internal differential of A_k, note that $\partial^{\mathrm{Tot}} \circ \phi^{\mathrm{Tot}}_k = d_{A_k} \circ -\mathrm{id}_{A_k} = -\mathrm{id}_{A_k} \circ d_{A_k} = \phi^{\mathrm{Tot}}_k \circ d_{A_k}$. We then verify that the ϕ^{Tot}_k maps commute with f_k maps up to homotopy; $\phi^{\mathrm{Tot}}_k \sim \phi^{\mathrm{Tot}}_{k+1} \circ f_k$. We define a new collection of maps $h^{\mathrm{Tot}} := \{h^{\mathrm{Tot}}_k\}$, where $h^{\mathrm{Tot}}_k := -\mathrm{id}_{A_k}$ from A_k to the copy of A_k in degree -1.

(A.6)

$$\begin{tikzcd}
\cdots \arrow[r] & A_{k-1} \arrow[loop above, "d_{A_{k-1}}"] \arrow[r, "f_{k-1}"] \arrow[d, red, "\phi^{\mathrm{Tot}}_{k-1}"] \arrow[dd, blue, bend right, "h^{\mathrm{Tot}}_{k-1}"'] & A_k \arrow[loop above, "d_{A_k}"] \arrow[r, "f_k"] \arrow[d, red, "\phi^{\mathrm{Tot}}_{k}"] \arrow[dd, blue, bend right, "h^{\mathrm{Tot}}_{k}"'] & A_{k+1} \arrow[loop above, "d_{A_{k+1}}"] \arrow[r] \arrow[d, red, "\phi^{\mathrm{Tot}}_{k+1}"] \arrow[dd, blue, bend right, "h^{\mathrm{Tot}}_{k+1}"'] & \cdots \\
\cdots & A_{k-1}[0] \arrow[loop above] & A_k[0] \arrow[loop above] & A_{k+1}[0] \arrow[loop above] & \cdots \\
\cdots \arrow[ur] & A_{k-1}[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_{k-1}"] \arrow[loop below, "-d_{A_{k-1}}"] & A_k[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_k"] \arrow[loop below, "-d_{A_k}"] & A_{k+1}[-1] \arrow[u, "\mathrm{id}"] \arrow[ur] \arrow[loop below, "-d_{A_{k+1}}"] & \cdots
\end{tikzcd}$$

Note that $\phi^{\mathrm{Tot}}_k - \phi^{\mathrm{Tot}}_{k+1} \circ f_k = -\mathrm{id}_{A_k} + \mathrm{id}_{A_{k+1}} \circ f_k = -\mathrm{id}_{A_k} + f_k$. However, $\partial^{\mathrm{Tot}} \circ h^{\mathrm{Tot}}_k + h^{\mathrm{Tot}}_k \circ d_{A_k} = -(-d_{A_k} + \mathrm{id}_{A_k} - f_k) - d_{A_k} = \phi^{\mathrm{Tot}}_k - \phi^{\mathrm{Tot}}_{k+1} \circ f_k$, thus, the diagram in Equation (A.5) commutes up-to-homotopy. Next, for A.0.8 (C-2), let B be an arbitrary chain complex in Kom with internal differential d_B, and suppose that we have structure chain maps $\phi^B := \{\phi^B_i : A_i \to B\}$, and homotopies $h^B := \{h^B_i\}$, such that $\phi^B_k - \phi^B_{k+1} \circ f_k = d_B \circ h^B_k + h^B_k \circ d_{A_k}$. Let $\overline{\phi}^B_k := -\phi^B_k$ and $\overline{h}^B_k := -h^B_k$. We may define a the chain map $\xi : \mathrm{Tot}(\mathcal{D}_{\mathcal{A}}) \to B$ by assembling both collections $\{\overline{h}^B_k\}$ and $\{\overline{\phi}^B_k\}$:

(A.7)

$$\begin{tikzcd}
\cdots \arrow[r] & B \arrow[loop above, "d_B"] \arrow[r, "id"] & B \arrow[loop above, "d_B"] \arrow[r, "id"] & B \arrow[loop above, "d_B"] \arrow[r] & \cdots \\
\cdots & A_{k-1}[0] \arrow[loop above] \arrow[u, purple, "\overline{\phi}^B_{k-1}"] & A_k[0] \arrow[loop above] \arrow[u, purple, "\overline{\phi}^B_{k}"] & A_{k+1}[0] \arrow[loop above] \arrow[u, purple, "\overline{\phi}^B_{k+1}"] & \cdots \\
\cdots \arrow[ur] & A_{k-1}[1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_{k-1}"] \arrow[loop below, "-d_{A_{k-1}}"] \arrow[uu, teal, bend left, "\overline{h}^B_{k-1}"] & A_k[1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_k"] \arrow[loop below, "-d_{A_k}"] \arrow[uu, teal, bend left, "\overline{h}^B_{k}"] & A_{k+1}[1] \arrow[u, "\mathrm{id}"] \arrow[ur] \arrow[loop below, "-d_{A_{k+1}}"] \arrow[uu, teal, bend left, "\overline{h}^B_{k+1}"] & \cdots
\end{tikzcd}$$

Note first that ξ is a chain map:

$$\begin{aligned} d_B \circ \xi - \xi \circ \partial^{\mathrm{Tot}} &= d_B \circ \overline{h}_k^B - \xi \circ (-d_{A_k} + \mathrm{id}_{A_k} - f_k) \\ &= -d_B \circ h_k^B - (-\overline{h}_k^B \circ d_{A_k} + \overline{\phi}_k^B - \overline{\phi}_{k+1}^B \circ f_k) \\ &= -d_B \circ h_k^B + \overline{h}_k^B \circ d_{A_k} - \overline{\phi}_k^B + \overline{\phi}_{k+1}^B \circ f_k \\ &= -(d_B \circ h_k^B + h_k^B \circ d_{A_k}) - (-(d_B \circ h_k^B + h_k^B \circ d_{A_k})) \\ &= 0. \end{aligned}$$

Also, ξ is clearly a chain homotopy equivalence map, as $\xi \circ \phi_k^{\mathrm{Tot}} = -\xi \circ \mathrm{id}_{A_k} = -\overline{\phi}_k^B = \phi_k^B$. Thus, $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$ satisfies the universal property of the colimit of $\mathcal{A}$ up to homotopy, and Proposition A.0.2 follows. □

The specific directed systems we study will satisfy the property that $f_{i+1} \circ f_i$ is zero. The corresponding homotopy colimits then admit useful properties, which we now discuss.

Proposition A.0.3. *Let $\mathcal{D}_{\mathcal{A}}$ be the double complex associated to a directed system $\mathcal{A} := (A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots)$ and let $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$ denote the corresponding homotopy colimit (as in (A.2)). If $f_{i+1} \circ f_i \sim 0$ for all $i \in \mathbb{Z}_{\geq 0}$, then $\mathrm{Tot}(\mathcal{D}_{\mathcal{A}})$ is contractible.*

Proof. The proof of this proposition follows directly from the following lemma.

Lemma A.0.9.1. *There exists an endomorphism $F : \mathcal{D}_{\mathcal{A}} \to \mathcal{D}_{\mathcal{A}}$, given by chain maps $F_i : A_i \to A_{i+1}$, where each F_i is a copy of the chain map f_i. The endomorphism F is denoted by the dotted arrows in the following figure.*

$$\begin{tikzcd} A_0[0] \arrow[r, dashed, "F_0"] & A_1[0] \arrow[r, dashed, "F_1"] & A_2[0] \arrow[r, dashed] & \cdots \\ A_0[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_0"] \arrow[r, dashed, "F_0"'] & A_1[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_1"] \arrow[r, dashed, "F_1"'] & A_2[-1] \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_2"] \arrow[r, dashed] & \cdots \end{tikzcd}$$

The endomorphism $F : \mathcal{D}_{\mathcal{A}} \to \mathcal{D}_{\mathcal{A}}$ is homotopic to $\mathrm{id}_{\mathcal{D}_{\mathcal{A}}}$.

Proof. The chain homotopy maps are given by identity maps $\mathrm{id}_{A_i}^{-1} : A_i[0] \to A_i[-1]$. Note that $[\partial^{\mathrm{Tot}}, \mathrm{id}_{A_i}^{-1}] = \partial^{\mathrm{Tot}} \circ \mathrm{id}_{A_i}^{-1} + \mathrm{id}_{A_i}^{-1} \circ d_{A_i} = (-d_{A_i} - f_i + \mathrm{id}_{A_i}) + d_{A_i} = -f_i + \mathrm{id}_{A_i}$; thus, the chain map F is homotopic to $\mathrm{id}_{\mathcal{D}_{\mathcal{A}}}$. □

To complete the proof of Proposition A.0.3, note that the condition $f_{i+1} \circ f_i \sim 0$ implies that $F^2 \sim 0$. Thus, by Lemma A.0.9.1, we have that $\mathrm{id}_{\mathcal{D}_{\mathcal{A}}} \sim F \sim F^2 \sim 0$, proving the claim. ☐

Recall that homotopy equivalences of objects in a directed system can be extended to chain homotopy equivalences of homotopy colimits of said directed system.

Lemma A.0.9.2. *Let* $\mathcal{A} := \{A_i, f_i\}_{i\in\mathbb{Z}_{\geq 0}}$ *and* $\mathcal{B} := \{B_i, g_i\}_{i\in\mathbb{Z}_{\geq 0}}$ *be directed systems in* $\mathcal{C}$. *Suppose we have a collection of chain maps* $\alpha_i : A_i \to B_i$ *such that* $\alpha_{i+1} \circ f_i \sim g_i \circ \alpha_i$, *and let* C_i *denote the cone* $\mathrm{Cone}(A_i \xrightarrow{\alpha_i} B_i)$.

(a) There exists a chain map $\alpha : \mathrm{hocolim}\,(\mathcal{A}) \to \mathrm{hocolim}\,(\mathcal{B})$ *corresponding to the collection* $\{\alpha_i\}$.

(b) We have that $\mathrm{Cone}(\alpha) = \mathrm{hocolim}\,(\mathrm{Cone}(A_i \xrightarrow{\alpha_i} B_i))$.

(c) If α_i *is a homotopy equivalence for all* $i \in \mathbb{Z}_{\geq 0}$, *then* $C_i \simeq 0$ *and* $\mathrm{hocolim}\,(C_i) \simeq 0$ *for all* i, *so* $\mathrm{Cone}(\alpha) \simeq 0$.

Proof. (a) The induced chain map α is given by each $\alpha_i : A_i[k] \to B_i[k]$ for all $k \in \{0, 1\}$. The relevant homotopy maps are given by $h_i : A_i[0] \to B_{i+1}[1]$.

(b) Define a map $\Phi_i : C_i \to C_{i+1}$ by the following diagram:

$$\begin{tikzcd}
A_i \arrow[r, "\alpha_i"] \arrow[d, "f_i"] \arrow[dr, "h_i"] & B_i \arrow[d, "g_i"] \\
A_{i+1} \arrow[r, "\alpha_{i+1}"] & B_{i+1}
\end{tikzcd}$$

Then $\mathrm{Cone}(\alpha) = \mathrm{hocolim}\,(C_0 \xrightarrow{\Phi_1} C_1 \xrightarrow{\Phi_2} C_2 \to \cdots)$.

(c) Suppose that each α_i is a homotopy equivalence. Then each cone C_i is contractible, and therefore the homotopy colimit is contractible:

$$\mathrm{hocolim}\,(C_0 \xrightarrow{\Phi_0} C_1 \xrightarrow{\Phi_1} C_2 \to \cdots) \simeq 0.$$

This implies $\mathrm{Cone}(\alpha) \simeq 0$ by part (b); therefore α is a homotopy equivalence.

□

Finally, we recall the following standard lemma from homological algebra.

Lemma A.0.9.3. *Let X, Y be complexes of vector spaces over $\mathbb{F}$, and let $f : X \to Y$ be a chain map. Then*

$$H^*(\mathrm{Cone}(f)) \cong H^*\left(\mathrm{Cone}\left(H^*(X) \xrightarrow{f^*} H^*(Y)\right)\right).$$

Proof. The short exact sequence of chain complexes

$$0 \to Y \xrightarrow{\iota} \mathrm{Cone}(f) \xrightarrow{\pi} X[1] \to 0$$

induces a long exact sequence on homology

$$\cdots \xrightarrow{f^*} H^i(Y) \xrightarrow{\iota^*} H^i(\mathrm{Cone}(f)) \xrightarrow{\pi^*} H^{i+1}(X) \xrightarrow{f^*} \cdots,$$

i.e. there is an exact triangle

$$\begin{tikzcd}
H^*(Y) \arrow[rr, "\iota^*"] & & H^*(\mathrm{Cone}(f)) \arrow[dl, "\pi^*"] \\
& H^*(X[1]) \arrow[ul, "f^*"] &
\end{tikzcd}$$

□

In Section 3.1, we use Lemma A.0.9.3 to compute homotopy colimits of complexes by passing to graded vector spaces:

Corollary A.0.1. *Let $\mathcal{C} = \{C_i, f_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be a directed system of complexes (C_i, d_i) of vector spaces over $\mathbb{F}$. Then the associated homotopy colimit can be computed by first computing the homology of each C_i:*

$$\begin{tikzcd}
 & H^*(C_0) & H^*(C_1) & H^*(C_2) & \cdots \\
\mathrm{hocolim}(\mathcal{C}) \quad \simeq & H^*(C_0) \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_0^*"] & H^*(C_1) \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_1^*"] & H^*(C_2) \arrow[u, "\mathrm{id}"] \arrow[ur, "-f_2^*"] &
\end{tikzcd}$$

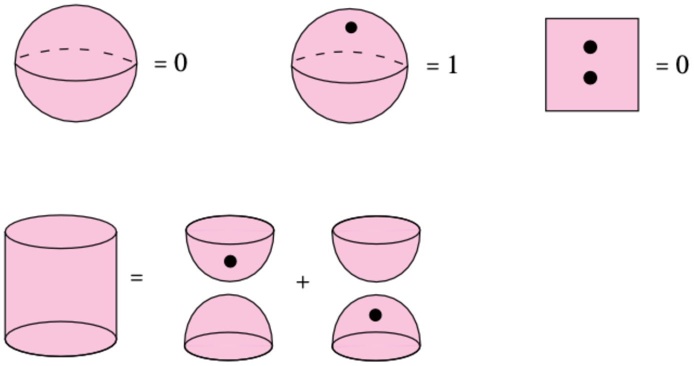


FIGURE A.3. Local relations in the Bar-Natan cobordism category. The bottom relation is called *neck cutting*.

A.1. Bar-Natan Categories

Here we recall some preliminary definitions about the cobordism categories associated to the categorification of the Temperley-Lieb algebra ([**BN02**], [**BN05**], and subsequent works) with the grading conventions used in the skein lasagna literature.

For $n \geq 0$, let D^2_n denote the disk with a fixed set of $2n$ marked points $X_n \subset D^2_n$ on the boundary. A *planar tangle* $T \subset D^2_n$ is a properly embedded 1-manifold in D^2_n with boundary $\partial T = X_n$.

On the other hand, a *tangle* in general may have crossings, and is to be regarded as properly embedded in $D^2_n \times (-\varepsilon, \varepsilon)$ with $X_n \subset \partial D^2_n \times \{0\}$. These will be represented using chain complexes built from the planar tangles above, which we discuss in the following sections. We use the same notation for the homological and quantum shift operators on tangles (e.g. $h^k q^\ell [\![T]\!]_{\mathrm{BN}} = [\![T]\!]_{\mathrm{BN}}[k]\{\ell\}$).

A *(dotted) cobordism* $F : q^i T_0 \to q^j T_1$ between (quantum-shifted) planar tangles $T_0, T_1 \subset D^2_n$ is a properly embedded surface $F \subset D^2_n \times [0,1]$ with boundary $\partial F = (T_0 \times \{0\}) \cup (T_1 \times \{1\}) \cup (X_n \times [0,1])$, possibly decorated with a finite number of dots.

The *quantum degree* of the cobordism F is

$$\deg_q(F) = n + j - i - \chi(F) + 2(\#\text{ of dots}) \tag{A.8}$$

where $\chi(F)$ is the Euler characteristic of the surface.

Furthermore, $\deg_q(F) = -\chi(F)$ for a closed surface F without dots viewed as a cobordism from the unshifted $\emptyset$ to itself. The degree of a dot is $\deg_q(\bullet) = +2$. The category $\mathbf{Cob}_n$ is then defined as follows.

Definition A.1.1. *The objects* $\mathrm{Ob}(\mathbf{Cob}_n)$ *are formal shifts of planar tangles* $T \subset D^2_n$. *A morphism* $f : q^i T_0 \to q^j T_1$ *in* $\mathrm{Mor}(\mathbf{Cob}_n)$ *is a formal* $\mathbb{Z}$*-linear combination of dotted cobordisms, modulo isotopy rel boundary, movement of dots in the same connected component, and Bar-Natan's local relations, shown in Figure A.3. The morphisms of* $\mathbf{Cob}_n$ *are composed by vertical stacking. Occasionally, we require cobordisms categories of planar tangles with different numbers of specified endpoints. For planar tangles with* n *bottom endpoints and* k *top endpoints, the corresponding cobordism category is denoted* $\mathbf{Cob}_{n,k}$.

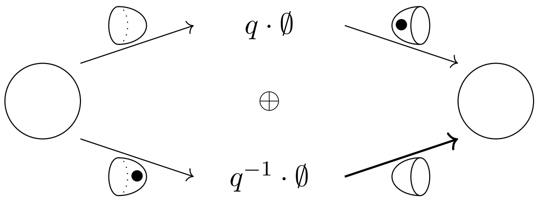
Remark A.1.1. *Let* T_1 *and* T_2 *be tangle diagrams. We use the notation* $[\![T_1]\!]_{\mathrm{BN}} \sqcup [\![T_2]\!]_{\mathrm{BN}}$ *(resp.* $[\![T_1]\!]_{\mathrm{KhR}} \sqcup [\![T_2]\!]_{\mathrm{KhR}}$*) to denote the chain complex associated to the horizontal composition of tangles* $T_1 \sqcup T_2$.

Let $\mathcal{BN}_n$ denote $\mathrm{Mat}(\mathbf{Cob}_n)$, and let $\mathrm{Kom}(\mathcal{BN}_n)$ denote the category of chain complexes over $\mathrm{Mat}(\mathbf{Cob}_n)$ where the morphisms are quantum degree 0 chain maps, and where differentials have homological degree $+1$. Following [**Hog19**], we generally drop the brackets, with the understanding that all instances of tangles should be interpreted as chain complexes in $\mathrm{Kom}(\mathcal{TL}_n)$, defined below. In the following sections we will often want to consider (n, n) *planar tangles*, or planar tangles in $D^2_n \cong [0, 1] \times [0, 1]$ where the boundary points X_n are split into two sets, with n each (equally spaced, say) along $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. In this case, we write $\mathcal{TL}_n$ in place of $\mathrm{Mat}(\mathbf{Cob}_n)$, and write $\mathrm{Kom}(\mathcal{TL}_n)$ for the category of chain complexes and degree-preserving chain maps. If we instead work with tangles with different numbers of top and bottom boundary points, we write $\mathcal{TL}^k_n$ instead.

Given two (n, n) planar tangles T, T', stacking T' on top of T gives a composition operation, forming the new planar tangle $T' \otimes T$. This composition induces a composition operation in $\mathcal{TL}_n$ and $\mathrm{Kom}(\mathcal{TL}_n)$.

An (n, n) *tangle*, which may contain crossings, is regarded as properly embedded in $D^2_n \times (-\varepsilon, \varepsilon) \cong [0, 1]^2 \times (-\varepsilon, \varepsilon)$ with the marked points along $\{0\} \times [0, 1] \times \{0\}$ and $\{1\} \times [0, 1] \times \{0\}$.

We now compile some of the techniques fundamental to the computation of Khovanov homology using Bar-Natan's local techniques. The *delooping* operation, depicted in the following figure, describes an isomorphism in $\mathcal{TL}_n$ between an object with a closed loop and the same object with the closed loop removed. This operation is used to remove disjoint circles from diagrams.



This operation is used in conjunction with Gaussian elimination:

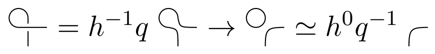
Lemma A.1.1.1 ([**BN07**], Lemma 4.2). *(Gaussian Elimination) Let* $C \in \mathrm{Kom}(\mathcal{A})$ *be a chain complex over an additive category* $\mathcal{A}$*, and suppose that* C *contains the subcomplex*

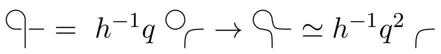
$$A \xrightarrow{\begin{bmatrix}0\\ f\end{bmatrix}} \begin{matrix}B\\ \oplus\\ C\end{matrix} \xrightarrow{\begin{bmatrix}\Phi & \alpha\\ \beta & \gamma\end{bmatrix}} \begin{matrix}D\\ \oplus\\ E\end{matrix} \xrightarrow{\begin{bmatrix}0 & g\end{bmatrix}} F$$

where $\Phi : B \to D$ *is an isomorphism. Then complex* C *is chain homotopy equivalent to the complex* $C' \in \mathrm{Kom}(\mathcal{A})$ *with the above portion of the complex replaced by*

$$A \xrightarrow{f} C \xrightarrow{\gamma-\beta\Phi^{-1}\alpha} E \xrightarrow{g} F.$$

By delooping and the unoriented skein relation (2.2), we have the following Reidemeister I chain homotopy equivalences for KhR_2:





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For the chain maps associated to the other Reidemeister moves, we use the conventions set in [**MWW22**, Section 3.3].

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