

**Symplectic Fillings and Braid Factorizations**

By

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## Abstract

The purpose of this dissertation is to classify symplectic fillings for certain links of singularities. We study braided wiring diagrams, representations of complex curves, and a set of equivalence classes on these diagrams called Markov moves. We work in the braid group to classify braided wiring diagrams up to Markov moves and use these results to classify symplectic fillings via [PS23]. We first define braided wiring diagrams and an invariant called the braid monodromy, which assigns a braid to a curve. In addition, we provide some background from symplectic topology and algebraic geometry to explain why braided wiring diagrams can be used to classify symplectic fillings. More specifically, de Jong and van Straten introduced a relation between the deformation theory of complex plane curves and the deformation theory of a class of complex surface singularities. The techniques used in this paper were applied in the symplectic topology setting in [PS23]. We then review Markov moves defined in [CS97] and prove some interesting relations concerning three-stranded wiring diagrams using these moves. We apply these preliminary results to study various configurations to classify braided wiring diagrams. Finally, we state our results in terms of symplectic topology to piece everything together.

## CHAPTER 1

### Introduction

In this dissertation, the main objects of study are braided wiring diagrams, configurations of wires illustrating the behavior of complex curves. We review and expand a set of equivalence moves on these diagrams called Markov moves. We work in the braid group to classify certain braided wiring diagrams up to this equivalence. Then, we use our classification results to obtain classifications of symplectic fillings for certain links of singularities via [PS23].

#### 1.1. Context

In his foundational paper, Zariski investigated the following question:

Does there exist an algebraic function of two variables with a preassigned branch curve  $f$ ?

It was deduced that studying the fundamental group of the curve  $f$  was a key aspect of this problem. The braid monodromy was introduced by Moishezon to help with the computation of  $\pi_1(\mathbb{C}^2 \setminus f)$  where  $f$  is an algebraic curve in  $\mathbb{C}^2$ . It assigns a braid to a curve by tracking the behavior of the curve near and between the singularities. Subsequently, Moishezon and Teicher provided an algorithm for how to compute the local braid monodromy generated by branch points, cusp points, and transverse multi-intersection points. Given a polynomial  $f$  representing an arrangement of  $n$  complex curves in  $\mathbb{C}^2$ , the braid monodromy of a curve can be computed using a braided wiring diagram. A braided wiring diagram encodes equivalent information as the braid monodromy (See Section 3.1 for how to translate back and forth between these two). The wires give us information about the position of the curves while the braids illustrate how the curves behave before they come to intersect. A braided wiring diagram associated to a curve is not unique and can change depending on the way we construct our braid monodromy for a given line arrangement. We will see in Section 3.2 that different braided wiring diagrams for the same curve are related by Markov moves. Markov moves are also useful for simplifying braided wiring diagrams.

A relation between the deformation theory of complex plane curves with the deformation theory of a class of complex surface singularities was first discovered by de Jong and van Straten [dJvS98]. Recently, an analogue of this correspondence in symplectic topology was established in [PS23], which allows us to use classification results about braided wiring diagrams to understand symplectic fillings of certain contact 3-manifolds.

Given a complex surface  $X \subset \mathbb{C}^n$  with an isolated singularity at the origin, we can obtain a contact manifold  $(Y, \xi)$ , called the link of  $(X, 0)$ , by intersecting  $X$  with a sufficiently small  $(2n - 1)$ -sphere. We work with a specific type of surface singularity called rational surface singularity with reduced fundamental cycle. The dual resolution graph for this particular singularity is a tree satisfying an inequality involving the valency and weight of the vertices. After adding additional  $(-1)$ -curves to the dual resolution graph, we perform a blow-down procedure to obtain a weighted singular curve  $\mathcal{C}$  in  $\mathbb{C}^2$  (Section 2.4). We then deform the components of the weighted curve, and this altered weighted curve gives our braided wiring diagram. These rational surface singularities with reduced fundamental cycle are of significance because the links are supported by planar open books. The connection between Stein fillings and Lefschetz fibrations (Section 2.3) allows us to be able to obtain data on symplectic fillings by studying braid monodromy factorizations of the braided wiring diagrams.

## 1.2. Main results

As mentioned earlier, Markov moves provide us with a notion of equivalence between braided wiring diagrams. In this thesis, we provide a more generalized version of the moves from [CS97] and use this result to connect Markov moves and braid monodromy homomorphisms.

**THEOREM 1.2.1.** *Braided wiring diagrams  $\mathcal{W}$  and  $\hat{\mathcal{W}}$  are related by a sequence of Markov moves if and only if the braid monodromy homomorphisms are braid-equivalent.*

In our case, the main results will be in  $B_3$  so our braided wiring diagrams will have three strands. Furthermore, the process of changing the weighted singular curve  $\mathcal{C}$  makes all of the intersection points transverse (Section 2.4), meaning our braided wiring diagrams will just have transverse multi-intersection points. We refer to them as double points or triple points depending on the number of strands involved in the intersection.

The setup for the main theorem is as follows. We start with a symplectic line arrangement in  $\mathbb{C}^2$ , which consists of three symplectic graphical disks  $\{\Gamma_1, \Gamma_2, \Gamma_3\}$  such that

1.  $\Gamma_i$  intersects  $\Gamma_j$  positively transversally once for all  $i \neq j \in \{1, 2, 3\}$ ,
2. for a circle  $C_r$  with sufficiently large radius  $r$ , the braid  $(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) \cap \pi^{-1}(C_r)$  where  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  is the projection, called the total monodromy, is isotopic to the full twist on three strands.

Given a wiring diagram  $\mathcal{W}_0$  representing the symplectic line arrangement and a wiring diagram  $\mathcal{W}$  having the same singularities but with braids, the following theorem shows how these two wiring diagrams are related.

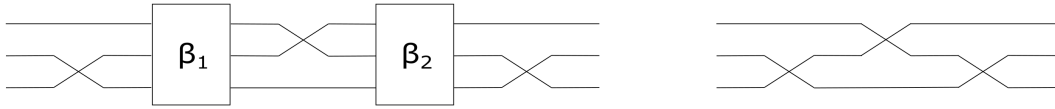


FIGURE 1.1.  $\mathcal{W}$  and  $\mathcal{W}_0$

**THEOREM 1.2.2.** *The total braid monodromy of a braided wiring diagram  $\mathcal{W}$  is equal to the full twist if and only if it is related by Markov moves to  $\mathcal{W}_0$ .*

We prove this theorem in two different ways. In the first method, we analyze  $\beta_i$  and use Garside theory to show  $\mathcal{W}$  and  $\mathcal{W}_0$  are Markov equivalent. The second method involves analyzing matrices using the isomorphism  $B_3/Z \cong PSL(2, \mathbb{Z})$ .

The goal then is to use this theorem and braid factorization techniques to study more arrangements that could not be done via other methods. We add  $m$  triple points to the beginning of  $\mathcal{W}$  and obtain a corollary.

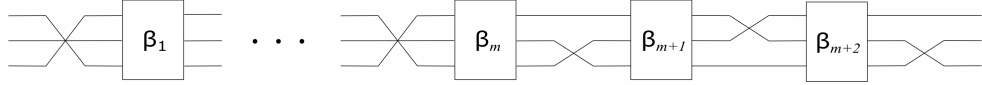


FIGURE 1.2.  $m$  triple points and three double points

**COROLLARY 1.2.1.** *The total braid monodromy of the braided wiring diagram above is  $\Delta^{2(m+1)}$  if and only if it is Markov equivalent to the standard wiring diagram.*

**REMARK 1.2.1.** *If we are given a braided wiring diagram, then we use the term standard wiring diagram to mean the same wiring diagram but with no braiding.*

In addition, we have a corollary for a wiring diagram with one more double point added to  $\mathcal{W}$ .

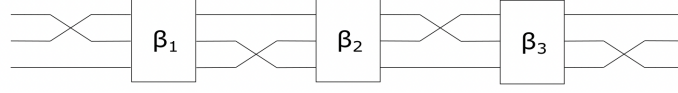


FIGURE 1.3. Four double points

COROLLARY 1.2.2. *The total braid monodromy of this braided wiring diagram is  $\Delta^2 \sigma_2^2$  if and only if it is related by Markov moves to the standard wiring diagram.*

Now consider a braided wiring diagram with  $n$  consecutive double points consisting of the top two strands and one triple point at the end. We are able to obtain the following result for  $1 \leq n \leq 4$  using the matrix method.

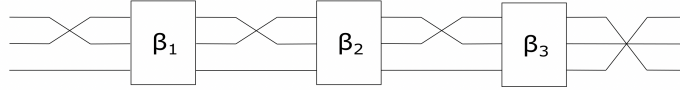


FIGURE 1.4.  $n = 3$

THEOREM 1.2.3. *The total braid monodromy of the braided wiring diagram when  $1 \leq n \leq 3$  is  $\Delta^2 \sigma_2^{2n}$  if and only if it is related by Markov moves to the standard wiring diagram.*

In Section 5, we rewrite the above results in terms of symplectic topology and deformation theory of complex curves using the following steps. This will then give us the complete picture.

- i) Find the resolution graphs for each braided wiring diagram (Figure 5.1).
- ii) Obtain incidence matrices using the diagrams and calculate the homology groups of the various fillings using a short exact sequence involving the incidence matrices (Figures 5.2–5.6).
- iii) Combine Theorem 1.5 from [PS23] and our results about braided wiring diagrams to classify symplectic fillings.

## CHAPTER 2

### Background

In this section, we provide the necessary tools to better understand the results of this dissertation.

#### 2.1. Braid group

Artin formally introduced the braid group  $B_n$  given by the presentation

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ 1 \leq i \leq n-2, \ \sigma_i \sigma_j = \sigma_j \sigma_i \ |i-j| \geq 2 \rangle.$$

Chow proved that the center of  $B_n$  is generated by the element  $\Delta^2$  where

$$\Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1.$$

REMARK 2.1.1. *We will use the notation  $\Delta^l \beta = \beta_{(l)} \Delta^l$  where  $\beta_{(l)} = \beta$  if  $l$  is even and  $\beta_{(l)}$  is obtained from  $\beta$  by replacing each  $\sigma_i$  with  $\sigma_{n-i}$  if  $l$  is odd.*

In his same paper, Artin presented the word problem for the braid group and suggested the idea of combing where he rewrote the braids in a certain unique form. The process to find this combed braid, however, was shown to be exponential in length and therefore not practical. Progress on the word problem has then been made using different methods by people including, but not limited to, Garside and Xu. A novel idea that helped with the problem was to consider working in the monoid  $B_n^+$  because we now only have to focus on the relations within the positive generators  $\sigma_i$ . In particular, Garside showed the existence of an embedding  $B_n^+ \rightarrow B_n$ .

THEOREM 2.1.1 ( [Gar69], Theorem 4). *If two positive words are equal, then there is a sequence of positive words from one to the other such that each word of the sequence is obtained by applying  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  for  $1 \leq i \leq n-2$  or  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for  $|i-j| \geq 2$ .*

Garside proved that every braid  $\beta$  can be written as  $\Delta^m \beta_+$  for  $m \in \mathbb{Z}$  and  $\beta_+ \in B_n^+$  where  $\beta_+$  is prime to  $\Delta$  [Gar69, Theorem 5]. The integer  $m$  is the Garside power and this expression is called

the Garside normal form for  $\beta$ . This representation is unique, namely two braids are equal if and only if their Garside normal forms are identical [Gar69, Theorem 6]. Given braids  $\Delta^{m_1}\beta_1$  and  $\Delta^{m_2}\beta_2$  in Garside normal form, it then suffices to look at the exponents  $m_i$  and the positive braids  $\beta_i$ . An easy way to then rule out the equivalence of two braids is if the exponents are different. Otherwise, we need to compare the positive braids, which is where Theorem 2.1.1 becomes helpful.

## 2.2. Braided wiring diagrams and braid monodromy

Braided wiring diagrams are used in [CS97] to study the fundamental group of plane algebraic curves in  $\mathbb{C}^2$ . Consider a projection  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  onto the first coordinate and let  $\{x_1, \dots, x_k\}$  be points in  $\mathbb{C}$  such that  $\pi^{-1}(x_i)$  contain singularities of a complex curve  $f(z_1, z_2)$ . Choose a point  $x_0$  with  $\Re\{x_0\} < \Re\{x_1\} < \dots < \Re\{x_k\}$  and a smooth path  $\phi : [0, 1] \rightarrow \mathbb{C}$  passing through the points  $x_i$  in order by increasing real part. Then, the braided wiring diagram corresponding to  $f$  is

$$\mathcal{W} = \{(z_1, z_2) \in \phi \times \mathbb{C} \mid f(z_1, z_2) = 0\}.$$

The braid monodromy associates a braid to a curve as we move around the singularities. Let  $s_i$  be a path in  $\mathbb{C}$  connecting  $x_0$  to  $q_i \in \partial D_i$  where  $D_i$  is a small counterclockwise-oriented disk centered at  $x_i$  and containing no other  $x_j$ . Set  $L_i = \pi^{-1}(q_i)$ . We have that  $L_i \cap f$  contains  $d = \deg(f)$  intersection points. Of these  $d$  points, there are  $m_i$  points, denoted  $\{x_{i,1}, \dots, x_{i,m_i}\}$ , that will approach the singular point as  $q_i$  becomes closer to  $x_i$ . In other words,  $m_i$  is the multiplicity of the singular point  $x_i$ . For example,  $m_i = 2$  if  $x_i$  is a double point. We illustrate these paths, disks, and points in the figure below.

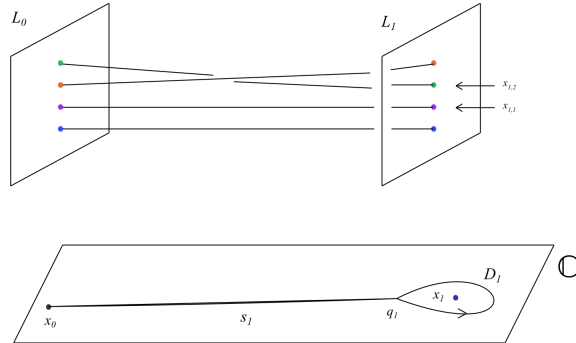


FIGURE 2.1. Braid monodromy setup

The system of segments in  $L_i$  connecting the points  $L_i \cap f$  represents the generators of the braid group on  $d$  strands,  $B_d[L_i, L_i \cap f]$ . More specifically, if  $L_i \cap f = \{p_1, \dots, p_d\}$ , then  $\sigma_i$  would be represented by the effect of a half twist between  $p_i$  and  $p_{i+1}$  on the segment connecting these two consecutive points. We write  $B_d[L_i, L_i \cap f]$  to be the group of homotopy equivalence classes of homeomorphisms  $\tau$  such that  $\tau$  preserves orientation,  $\tau(L_i \cap f) = L_i \cap f$ , and  $\tau = \text{Id}$  in the complement of an open disk containing  $L_i \cap f$ . Let  $\beta_i$  be the braid of  $x_i$ , meaning it tracks how the points in  $L_i \cap f$  behave as we move around  $\partial D_i \setminus \{q_i\}$ . We also obtain a braid  $\Phi_i$  associated to the path  $s_i$  connecting  $L_0 \cap f$  to  $L_i \cap f$ . We assume  $[L_0, L_0 \cap f]$  is identified with  $[L_i, L_i \cap f]$  under the projection to the second coordinate. If the identification is not exact, then we isotope the curve  $\{f = 0\}$ . We then have the diffeomorphism  $\Phi_i \beta_i \Phi_i^{-1}$ , which records the braiding of the system as we trace out an entire loop  $s_i * \partial D_i * s_i^{-1}$  associated to  $x_i$ , the  $i$ th loop. The homomorphism  $\pi_1(\mathbb{C} \setminus \{x_1, \dots, x_k\}) \rightarrow B_d$  sending the  $i$ th loop to  $\Phi_i \beta_i \Phi_i^{-1}$  is called the braid monodromy. For our purposes, we focus on the product of the individual loops  $\prod_{i=1}^k \Phi_i \beta_i \Phi_i^{-1}$ , which is homotopic to one large loop around all of the  $x_i$ 's.

REMARK 2.2.1. *Throughout this dissertation, we will use the term factorization of a braid  $\beta$  to mean a tuple of braid elements  $g = (g_1, g_2, \dots, g_k)$  where  $g_i = \Phi_i \beta_i \Phi_i^{-1}$  and  $g_1 g_2 \cdots g_k = \beta$ .*

There are various choices in our braid monodromy setup that are not fixed, so the braid monodromy of a curve is not unique. For example, we can define our paths or choose our basepoint  $x_0$  differently. We therefore need to define an equivalence notion for braid monodromies. Suppose  $f_1$  and  $f_2$  are equisingularly isotopic degree  $n$  complex curves. Let  $\{x_i\}$  and  $\{y_i\}$  be the set of points such that the preimages of these points under the projection  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  contain singular points of  $f_1$  and  $f_2$ , respectively. Libgober [Lib89] shows the existence of a homeomorphism  $j : \mathbb{C} \setminus \{x_i\} \rightarrow \mathbb{C} \setminus \{y_i\}$  such that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(\mathbb{C} \setminus \{x_i\}) & \xrightarrow{j_*} & \pi_1(\mathbb{C} \setminus \{y_i\}) \\ & \searrow & \downarrow \\ & & B_n \end{array}$$

Using this result, it was proved in Theorem 3.7 of [CS97] that the braid monodromy is well-defined up to braid-equivalence, which we now define.

DEFINITION 2.2.1. Two homomorphisms  $\alpha : F_s \rightarrow B_n$  and  $\alpha' : F_s \rightarrow B_n$  are *equivalent* if there exist  $\psi \in \text{Aut}(F_s)$  and  $\phi \in \text{Aut}(B_n)$  with  $\phi(B_n) \subset B_n$  satisfying  $\alpha' \circ \psi(g) = \phi^{-1} \cdot \alpha(g) \cdot \phi$ . In addition, the homomorphisms  $\alpha$  and  $\alpha'$  are *braid-equivalent* if  $\psi \in B_s$  and  $\phi \in B_n$ .

$$\begin{array}{ccc} F_s & \xrightarrow{\alpha} & B_n \\ \downarrow \psi & & \downarrow \text{conj}_\phi \\ F_s & \xrightarrow{\alpha'} & B_n \end{array}$$

For our case, we can set, for example, the top left  $F_s$  to be  $\pi_1(\mathbb{C} \setminus \{x_i\})$  and the other to be  $\pi_1(\mathbb{C} \setminus \{y_i\})$ . Then,  $\alpha : \pi_1(\mathbb{C} \setminus \{x_i\}) \rightarrow B_n$  and  $\alpha' : \pi_1(\mathbb{C} \setminus \{y_i\}) \rightarrow B_n$  would be the braid monodromy homomorphisms.

We also mention another type of equivalence that will be used later. Given a braid factorization, we can apply a Hurwitz move, which leaves the total monodromy fixed. There are two possible operations.

1.  $(g_1, g_2, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_k) \rightarrow (g_1, g_2, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_k)$
2.  $(g_1, g_2, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_k) \rightarrow (g_1, g_2, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_k)$

**2.2.1. Local braid monodromy of transverse multi-intersection points.** The algorithm for local braid monodromy computations has been generalized since its introduction by Moishezon and Teicher. For our purposes, we describe how to compute the local braid monodromy of a double point and triple point. Locally, a curve representing a double point in  $\mathbb{C}^2$  is given by the equation  $f(z_1, z_2) = (z_2 - z_1)(z_2 + z_1) = 0$ . Let  $D^2$  be the unit disk and  $F$  be a closed disk large enough so that  $f|_{D^2} \subset D^2 \times F$ . The unit circle  $\{z_1 = e^{2\pi i t} | t \in [0, 1]\}$  is a loop in  $\pi_1(D^2 \setminus \{0\})$ , and we lift this into  $f$ . In doing so, we obtain two paths that give us a braid as we track the roots of  $f$  for varying  $t$ . The cases plotted here include  $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ .

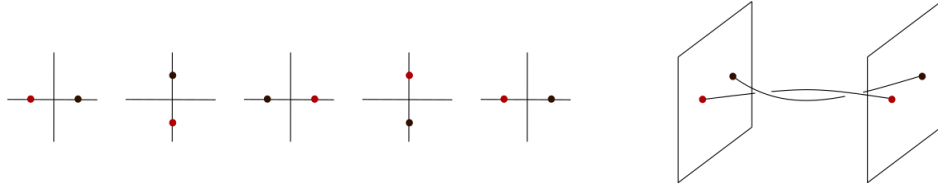


FIGURE 2.2. Braid monodromy computation for double point

The local braid monodromy around the origin is  $\sigma_1^2$ . We note the number in the index will vary based on which strand is involved in the singularity. Strands of a braided wiring diagram are counted from the bottom and a double point containing strands  $i$  and  $i + 1$  will have local braid monodromy  $\sigma_i^2$ .

A curve representing a triple point in  $\mathbb{C}^2$  is given by the equation  $f(z_1, z_2) = z_1(z_2 - z_1)(z_2 + z_1) = 0$ . We similarly plot the cases for  $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ .

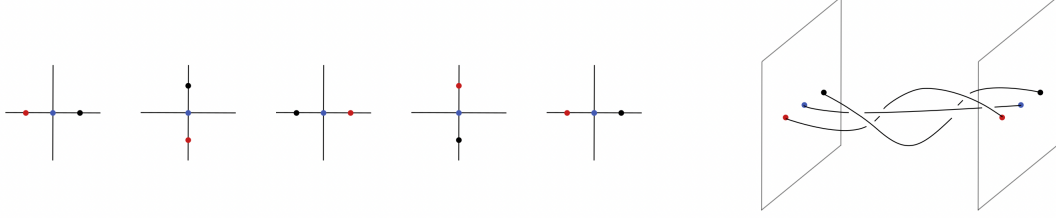


FIGURE 2.3. Braid monodromy computation for triple point

The local braid monodromy around the origin is  $\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 = \Delta^2$ . The same comment about the braid index applies here as well.

For an  $m$ -tuple point, the local braid monodromy is generated by the full twist on  $m$  strands; the general process is described in [MT91].

**2.2.2. Finding the total monodromy given a braided wiring diagram.** In this subsection, we explain how to read off the total monodromy of a braided wiring diagram. We use the following figure as an example.

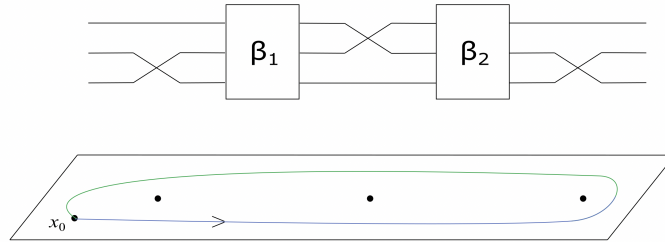


FIGURE 2.4. Computing the total monodromy

The braid associated to the first half of the loop in blue is  $\sigma_1\beta_1\sigma_2\beta_2\sigma_1$ . Recall the local braid monodromy of a double point is  $\sigma_i^2$  where  $i$  depends on the strands involved in the intersection. The first term  $\sigma_1$  then comes from this computation; it is not squared because we have only

traversed half of the loop around this projected singularity. The next braid in the wiring diagram is  $\beta_1$  and then we have  $\sigma_2$  from the double point involving local strands 2 and 3. Finally, we have  $\beta_2$  and  $\sigma_1$  where the latter is associated to the third double point. For the part of the loop in green, the braid is  $\sigma_1\beta_2^{-1}\sigma_2\beta_1^{-1}\sigma_1$ . The  $\sigma_i$ 's in this section come from traversing the remaining half of the loop around the projected singularities. However, we take the inverse of the  $\beta_i$ 's in the green path as we are reading the braids in the negative direction. The blue and green parts represent the push-off of the braid in the positive and negative imaginary directions in  $\mathbb{C}$ , respectively.

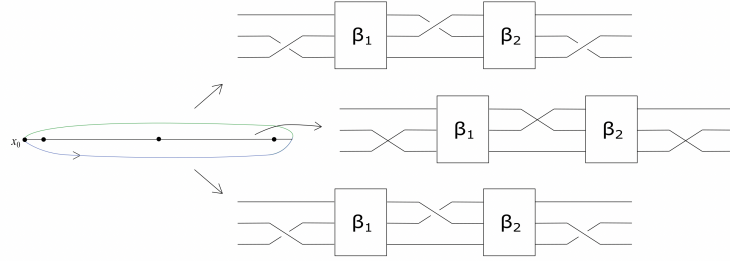


FIGURE 2.5. Total braid monodromy is the product of the bottom braid and inverse of the top braid

The total braid is then  $\sigma_1\beta_1\sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2\beta_1^{-1}\sigma_1$ . The figure below illustrates which part of the loop corresponds to which braid.

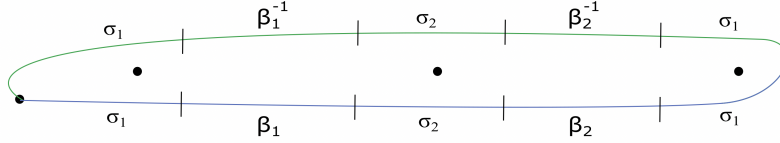


FIGURE 2.6. Total braid monodromy via the projected loop

### 2.3. Lefschetz fibrations and Stein fillings

The correspondence between Lefschetz fibrations and Stein fillings helps us understand why braid factorizations are connected to symplectic fillings. We describe this topic carefully in this subsection. We first review some definitions from symplectic and contact geometry.

**DEFINITION 2.3.1.** A 2-form  $\omega$  on a manifold  $X$  is *symplectic* if  $\omega$  is closed and non-degenerate. We call  $(X, \omega)$  a *symplectic manifold*.

If  $X$  is a symplectic manifold, then  $X$  must be even dimensional.

EXAMPLE 2.3.2. Let  $X = \mathbb{R}^{2n}$  with coordinates  $x_1, \dots, x_n, y_1, \dots, y_n$ . The form

$$\omega_{std} = \sum_{i=1}^n dx_i \wedge dy_i$$

is the standard symplectic form. If  $X = \mathbb{C}^n$ , then we have

$$\omega_{std} = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k.$$

Observe that these two forms are equivalent using the fact that  $z_k = x_k + iy_k$ .

For the rest of this section, we assume  $(X, \omega)$  is a symplectic manifold.

DEFINITION 2.3.3. A *contact structure* on a  $(2n+1)$ -dimensional manifold  $Y = \partial X$  is a hyperplane field  $\xi = \ker \alpha$  where  $\alpha$  is a 1-form satisfying  $\alpha \wedge (d\alpha)^n \neq 0$ . The 1-form  $\alpha$  is called a *contact form*.

In this case, we observe that contact manifolds must be odd dimensional.

EXAMPLE 2.3.4. Let  $Y = \mathbb{R}^{2n+1}$  with coordinates  $x_1, \dots, x_n, y_1, \dots, y_n, z$ . Then,

$$\alpha_{std} = dz + \sum_{i=1}^n x_i dy_i$$

is a contact form and  $\xi_{std} = \ker \alpha_{std}$  is the standard contact structure.

Contact structures arise naturally on the boundary of symplectic manifolds with uniform structure on a collared neighborhood of the boundary. A Liouville vector field on  $(X, \omega)$  is a vector field  $V$  such that the Lie derivative preserves the symplectic form  $\mathcal{L}_V \omega = \omega$ . If there exists a Liouville vector field defined on a neighborhood of  $\partial X$  that is transverse to  $\partial X$ , then we say  $(X, \omega)$  has contact type boundary. The Cartan formula  $\mathcal{L}_V = d\omega + \iota_V d$  together with closedness of  $\omega$  gives  $d\iota_V \omega = \omega$ . By setting  $\alpha = i^*(\iota_V \omega)$  where  $i : \partial X \rightarrow X$  is the inclusion, we obtain a contact form  $\alpha$  on  $\partial X$  such that  $d\alpha = \omega|_{\partial X}$ . If the orientation induced by  $\alpha \wedge d\alpha$  aligns with the orientation of  $\partial X$ , then  $(X, \omega)$  is said to have convex boundary. Else, the symplectic manifold has concave boundary.

DEFINITION 2.3.5. A symplectic manifold with convex contact type boundary is a *(strong) symplectic filling* of the induced contact boundary.

With these definitions in mind, we now discuss Stein fillings.

DEFINITION 2.3.6. A *Stein manifold* is a complex manifold admitting a proper holomorphic embedding into  $\mathbb{C}^n$ . If we intersect this manifold with a large ball in  $\mathbb{C}^n$ , then we obtain a compact manifold with boundary called a *Stein domain*.

A Stein domain inherits the symplectic structure  $\omega_{std}$  and has convex contact type boundary. A Stein domain is a Stein filling of the induced contact boundary.

DEFINITION 2.3.7. A *Lefschetz fibration* on a 4-manifold  $M$  is a map  $\pi : M \rightarrow D^2$  having finitely many critical values  $t_1, \dots, t_n$  in the interior of  $D^2$  such that there is a unique  $s_i \in \pi^{-1}(t_i)$  and each critical point has a local coordinate chart on which  $\pi(z_1, z_2) = z_1^2 + z_2^2$ .

As the name suggests, the map  $\pi$  is a fibration away from the critical values. If  $M$  is a 4-manifold with boundary, then the fiber  $F$  is a surface with boundary. If the Lefschetz fibration has no critical points, then  $M$  is homeomorphic to  $F \times D^2$ . We now explain the effects of the addition of critical points. Let  $U$  be a neighborhood of a critical point at  $(0, 0)$  such that  $\pi(z_1, z_2) = z_1^2 + z_2^2$  in  $U$ . We substitute  $z_i = x_i + iy_i$  to obtain

$$\pi(x_1 + iy_1, x_2 + iy_2) = x_1^2 + x_2^2 - y_1^2 - y_2^2 + i(2x_1y_1 + 2x_2y_2).$$

We can see what happens to the fiber as we approach the critical value. Let  $c \in \mathbb{R}_+$ . Then,

$$\pi^{-1}(c) \cap U = \{(x_1 + iy_1, x_2 + iy_2) \mid x_1^2 + x_2^2 - y_1^2 - y_2^2 = c, 2x_1y_1 + 2x_2y_2 = 0\}.$$

The fiber is an annulus with core circle  $C_c = \{x_1^2 + x_2^2 = c, y_1 = y_2 = 0\}$ . The circle  $C_c$  is a vanishing cycle, and the collection of circles  $C_c$  with the origin is a disk called the thimble.

The boundary of a Lefschetz fibration  $\pi$  consists of two components. The vertical boundary is the union of the fibers  $F$  over points in the boundary of  $D^2$ . Since there are no critical values in  $\partial D^2$ , this is a fibration over  $S^1$  with fiber  $F$ . The fibers of the vertical boundary are called pages. The horizontal boundary is the union of the boundaries of all of the fibers. The boundary of each fiber is diffeomorphic to a disjoint union of circles and there are  $D^2$  copies. In other words, the horizontal boundary is diffeomorphic to  $\bigsqcup_n S^1 \times D^2$ . The circles  $S^1 \times \{0\}$  from the horizontal boundary are called the binding. The pages and binding form an open book decomposition; we can use this information with the monodromy  $\phi$  of the fibration over  $S^1$  and the diffeomorphism

type of  $F$  to construct a manifold diffeomorphic to  $\partial M$ . The following theorem, proven by Giroux, plays a crucial role in understanding filling properties of contact structures.

**THEOREM 2.3.8** ([Gir03]). *Let  $Y$  be a closed, oriented 3-manifold. There is a one-to-one correspondence between oriented contact structures on  $Y$  up to isotopy and open book decompositions of  $Y$  up to positive stabilization.*

Given a Lefschetz fibration on a 4-manifold, the theory of handle decompositions is applied to determine if the manifold admits a Stein structure [Eli90]. Conversely, Loi-Piergallini [LP01] proved we can find a Lefschetz fibration given a Stein filling. Hence, Stein fillings are in one-to-one correspondence with Lefschetz fibrations. We study Stein fillings of a contact manifold by looking at factorizations of the monodromy. We choose a specific singularity where the contact structure of the link is compatible with a planar open book. Wendl showed all Stein fillings of a planar contact manifold are given by Lefschetz fibrations whose fiber is the page of that specific open book [Wen10]. We need not consider all stabilizations and destabilizations of the given open book; Wendl's result allows us to work with a fixed page and monodromy. Monodromy factorizations can therefore be used to obtain data on symplectic fillings.

## 2.4. Dual resolution graphs and singular curves

As mentioned in Section 1.1, we choose  $(X, 0)$  to be a rational surface singularity with reduced fundamental cycle. Before defining this term, we go over some concepts from algebraic geometry.

**DEFINITION 2.4.1.** A *surface singularity*  $(X, 0)$  is defined as follows:

$$(X, 0) = (\{f_1 = \cdots = f_m = 0\}) \subset (\mathbb{C}^n, 0)$$

where  $f_i : \mathbb{C}^n \rightarrow \mathbb{C}$  are analytic functions such that

$$r(p) = \text{rank} \left[ \frac{\partial f_i}{\partial z_j}(p) \right]_{i=1, \dots, m; j=1, \dots, n} = n - 2$$

for any smooth  $p \in X$ . We say  $X$  has an *isolated singularity* at the origin if  $r(0) < n - 2$  and  $r(p) = n - 2$  for all  $p \in X \setminus \{0\}$ .

DEFINITION 2.4.2. A *resolution* of  $(X, 0)$  is a smooth complex 2-manifold  $\tilde{X}$  with a holomorphic map  $\pi : \tilde{X} \rightarrow X$  such that  $\pi : \pi^{-1}(X \setminus \{0\}) \rightarrow X \setminus \{0\}$  is an isomorphism. A resolution is *good* if each irreducible component  $E_i$  of the exceptional divisor  $E = \pi^{-1}(0) = E_1 \cup \dots \cup E_t$  is smooth.

The combinatorics of a resolution can be stored in a dual resolution graph. Assuming the resolution is good, we can construct the graph as follows. The vertices  $v \in V$  correspond to an irreducible component  $E_v$  of the exceptional divisor and two vertices are connected if the corresponding irreducible components intersect. Each vertex has two weights: the genus of  $E_v$  and the self-intersection of  $E_v$ .

Let  $X \subset \mathbb{C}^n$  be a complex surface with an isolated singularity at the origin. Fix a ball  $B \subset \mathbb{C}^n$ , called a Milnor ball, around 0 and consider  $X \cap B$ . This is a compact 4-manifold with boundary; the boundary of this intersection is a smooth 3-manifold called the link  $Y$  of the singularity and this 3-manifold inherits a contact structure  $\xi$  given by  $TY \cap i(TY)$ .

DEFINITION 2.4.3. A surface singularity  $(X, 0)$  is a *rational singularity with reduced fundamental cycle* if

1. the surface admits a normal crossing resolution with genus 0 exceptional curves,
2. the dual resolution graph  $G$  is a tree such that for each vertex  $v \in G$ , the inequality  $a(v) \leq -v \cdot v$  holds where  $a(v)$  denotes the valency of  $v$ .

EXAMPLE 2.4.4. We provide an example of a dual resolution graph for a rational singularity with reduced fundamental cycle.

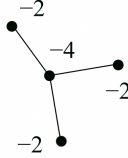


FIGURE 2.7. Dual resolution graph

For the vertices with valency one, the self intersection number is  $-2$ :  $1 \leq -(-2)$ . For the vertex with valency three, we have  $3 \leq -(-4)$ . Therefore, the inequality condition is satisfied for every vertex.

We choose  $(X, 0)$  to be this type of singularity as the links of these singularities are supported by planar open books. By Wendl, we know there is a one-to-one correspondence between Stein fillings of  $(Y, \xi)$  and Lefschetz fibrations whose boundary has a fixed planar open book supporting  $(Y, \xi)$ . Rational surface singularities with reduced fundamental cycle are a subset of sandwiched singularities. These singularities can be characterized by their resolutions by the conditions that one can add additional  $(-1)$  exceptional divisors onto the resolution to obtain a configuration that can be fully blown down to a smooth point. De Jong and van Straten studied the deformation theory of sandwiched singularities and showed it is determined by certain deformations of an associated weighted plane curve.

Given a dual resolution graph of  $(X, 0)$ , after adding additional  $(-1)$  exceptional divisors, a blow-down procedure can be applied to obtain de Jong-van Straten's weighted singular curve  $\mathcal{C} = \bigcup C_i$  in  $\mathbb{C}^2$ . The  $C_i$ 's are complex disks originally transverse to the  $(-1)$ -curves called curvetas, and the weight for each component represents the number of blow-down steps. We illustrate the blow-down procedure in the following examples.

EXAMPLE 2.4.5. We perform a blow-down procedure using the resolution graph in Figure 2.7. The additional  $(-1)$ -curves are red and the curvetas are green. As we can see, the weights for the curvetas (in green) increase each step.

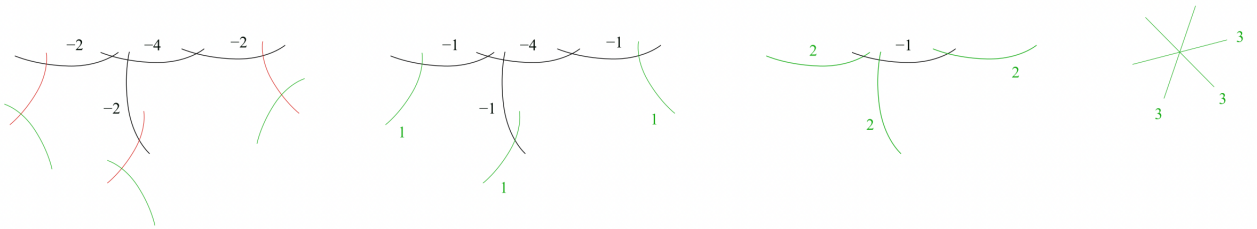


FIGURE 2.8. Blow-down procedure

EXAMPLE 2.4.6. We work out a more complicated example. The dual resolution graph is also drawn.

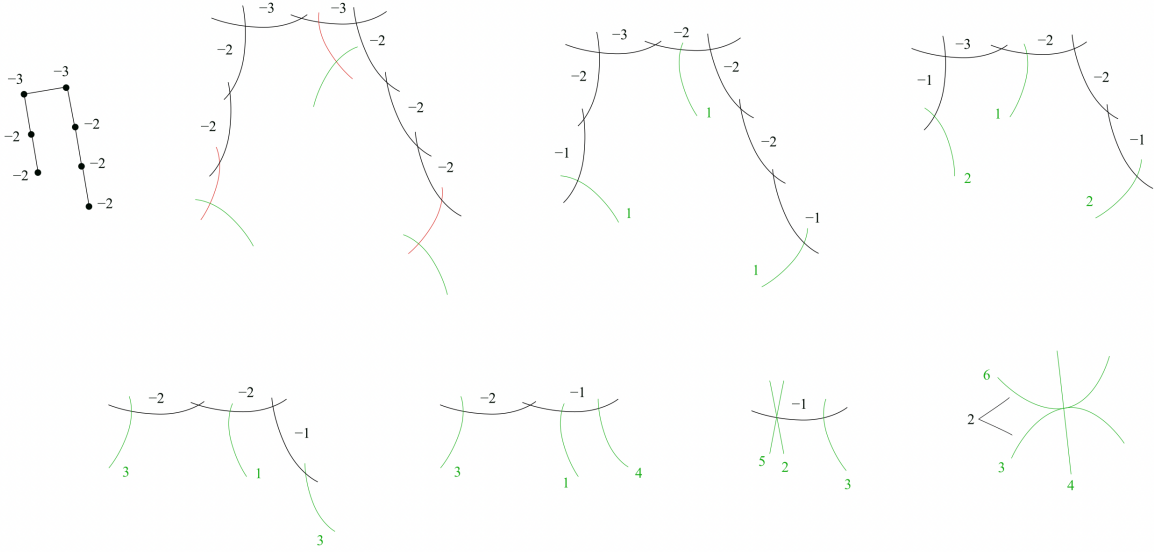


FIGURE 2.9. Blow-down procedure

Take a Milnor ball  $B$  so each branch  $C_i$  of our weighted curve intersects  $\partial B$  transversally. Up to change of coordinates, for sufficiently small  $B$ , each  $C_i$  can be expressed as  $z_2 = f_i(z_1)$ . Let  $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$  be a smooth graphical arrangement such that each  $\Gamma_i$  intersects  $\partial B$  transversally. Namely, each  $\Gamma_i$  is a smoothly embedded disk given by  $z_2 = f_i(z_1)$  for a smooth function  $f_i$  (not necessarily holomorphic).

**DEFINITION 2.4.7.** For a weighted curve  $\mathcal{C}$  with weights  $w_i = w(C_i)$ , a *smooth graphical homotopy* is a smooth homotopy of the branches  $C_i^t$  such that  $\mathcal{C} = \bigcup_i C_i^0$  with marked points and the following conditions hold in  $B$ .

1. We have  $C_i^t = \{z_2 = f_i^t(z_1)\}$  for smooth  $f_i(z_1, t)$  and  $C_i^t$  intersects  $\partial B$  transversally.
2. Intersections remain in the interior of  $B$ .
3. At  $t = 1$ , all intersections are transverse and positive, and every intersection is a marked point. Furthermore, the total number of marked points on  $C_i^1$  is  $w_i$ . (We may need to add additional free marked points. Note there are limitations on the number of intersections on a given component if the weight is small.)

**EXAMPLE 2.4.8.** We take the weighted curve from Figure 2.8 and apply a smooth graphical homotopy.

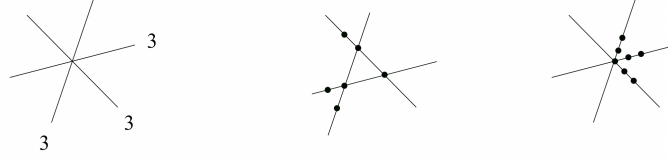


FIGURE 2.10. Smooth graphical homotopy

EXAMPLE 2.4.9. Now take the weighted curve from Figure 2.9.

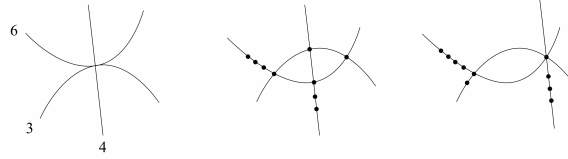


FIGURE 2.11. Smooth graphical homotopy

Roughly speaking, a smooth graphical homotopy takes a weighted singular curve  $\mathcal{C}$  and makes all of the intersections transverse while respecting the weight conditions. (Although in Figure 2.10, the original weighted curve was already transverse.) Lemma 3.2 in [PS23] produces a planar Lefschetz fibration from an arrangement  $\Gamma$  after applying a smooth graphical homotopy. We compose the projection map  $\pi$  with a blow-up (where we blow up all marked points of  $\Gamma$ ) and obtain the corresponding smooth 4-manifold by deleting the proper transforms of  $\Gamma_i$  from the blown-up surface. This planar Lefschetz fibration supports a Stein filling of  $(Y, \xi)$ , and the planar open book on  $Y$  arising from this constructed Lefschetz fibration is independent of the smooth graphical homotopy [PS23, Theorem 1.4].

It was shown that any Stein filling of  $Y$  is supported by a Lefschetz fibration coming from a configuration of symplectic disks  $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$  [PS23, Theorem 1.5]. Moreover,  $\Gamma$  and  $\mathcal{C}$ , the corresponding weighted curve, are related by a smooth graphical homotopy. The monodromy of the braid  $\bigcup_i \partial\Gamma_i \subset \partial B$  is the monodromy of the open book. Since  $\Gamma$  and  $\mathcal{C}$  are related by a smooth graphical homotopy, the boundary braids of  $\Gamma$  and  $\mathcal{C}$  are isotopic.

We now combine the ideas involving wiring diagrams and symplectic fillings. A wiring diagram is a representation of the graphical arrangement  $\Gamma$  obtained after applying (1) the blow-down procedure to a resolution graph of  $(X, 0)$  and (2) a smooth graphical homotopy to the resulting complex curve

$\mathcal{C}$ . For three-stranded wiring diagrams with transverse intersections, we have the following weighted curve for  $\mathcal{C}$ . Let the green and blue curves be tangent to each other with multiplicity  $m$ . Without loss of generality, let the red curve be tangent to the green and blue curves with multiplicity  $n \leq m$ .

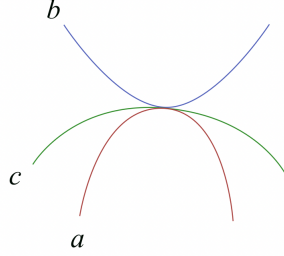


FIGURE 2.12. Weighted singular curve

By working backwards, we can find the resolution graphs for Figure 2.12. The number of branches of  $\Gamma$  as well as the blow-down procedure itself put constraints on the self-intersection numbers and valencies of nodes in the dual resolution graph.

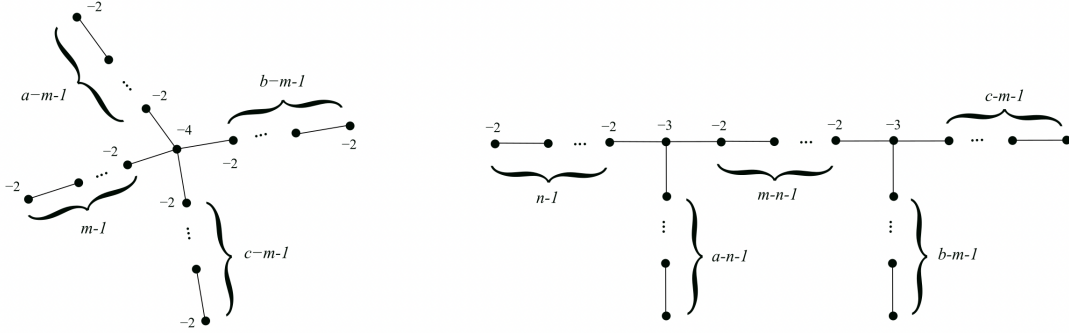


FIGURE 2.13. Corresponding dual resolution graphs

For the case where  $n = m$ , the resolution graph on the left hand side will produce the appropriate weighted singular curve. The branch with  $m - 1$  vertices increases the tangency between the three curves. We explain where to add the  $(-1)$  exceptional divisors to obtain the weighted curve.

1. If  $a - m - 1 \neq 0$ , then add one  $(-1)$  curve to the last vertex of this branch.
2. If  $a - m - 1 = 0$ , then add one  $(-1)$  curve to the vertex with self-intersection  $-4$ .

Repeat this for the branches with  $b$  and  $c$ .

If  $n \neq m$ , then we use the graph on the right hand side. The branch with  $m - n - 1$  vertices increases the tangency between the green and blue curve. We also explain where to add the  $(-1)$  curves for this case.

1. If  $a - n - 1 \neq 0$ , then add one  $(-1)$  curve to the last vertex of this branch.
2. If  $a - n - 1 = 0$ , then add one  $(-1)$  curve to the left vertex with self intersection  $-3$ .

Repeat this for the branches with  $b$  and  $c$ . Note that if  $b - m - 1 = 0$  (or  $c - m - 1 = 0$ ), then we add one  $(-1)$  curve to the other vertex with self intersection  $-3$ .

## CHAPTER 3

# Braided wiring diagrams, factorizations, and Markov moves

### 3.1. Braided wiring diagrams and factorizations

In this section, we explain how to construct a braided wiring diagram given a factorization and vice versa. First, suppose we are given a braided wiring diagram and want to obtain the corresponding factorization. Consider the following braided wiring diagram where  $p_k$  is the  $k$ th singularity, a transverse intersection point with local indices  $\{i_k, i_k + 1, \dots, i_k + n_k\}$ . Note this braided wiring diagram can have more than three strands.

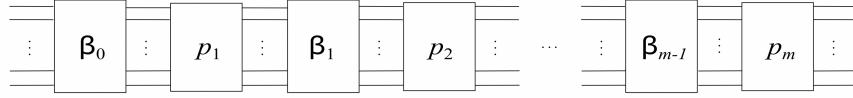


FIGURE 3.1. Braided wiring diagram

By our conventions, the basepoint is leftmost and loops in the projection are drawn counterclockwise. The product of the individual loops must equal the counterclockwise loop encircling the projected singularities, so the rightmost loop will be  $g_1$  where  $g = (g_1, g_2, \dots, g_m)$  is our factorization.

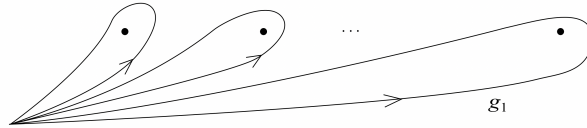


FIGURE 3.2. Loops in  $\mathbb{C}$

Given the braided wiring diagram in Figure 3.1, we have

$$g = (\tilde{\beta}_1 \mu_{i_m}^2 \tilde{\beta}_1^{-1}, \tilde{\beta}_2 \mu_{i_{m-1}}^2 \tilde{\beta}_2^{-1}, \dots, \tilde{\beta}_m \mu_{i_1}^2 \tilde{\beta}_m^{-1})$$

where

$$\mu_{i_k} = (\sigma_{i_k} \cdots \sigma_{i_k+n_k-1})(\sigma_{i_k} \cdots \sigma_{i_k+n_k-2}) \cdots (\sigma_{i_k} \sigma_{i_k+1}) \sigma_{i_k}.$$

REMARK 3.1.1. *Note that there are different ways to draw the loops in  $\mathbb{C}$  that will affect the factorization  $g$ .*

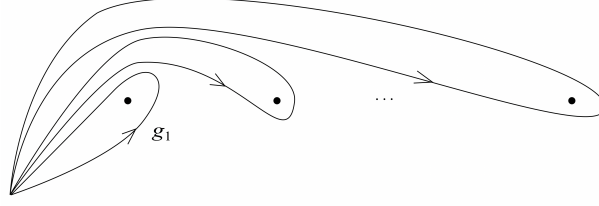


FIGURE 3.3. Different way to draw loops in  $\mathbb{C}$

*With this convention,  $g_1$  would be the leftmost loop, for example.*

We need to determine  $\tilde{\beta}_i$  in terms of  $\beta_i$  and the local braid monodromy generators of the singularities. Let us look at  $\tilde{\beta}_1$ . We trace the path of the first loop and look at the preimage of the projection. Using notation from Section 2.2, the braid associated to  $s_1$  is

$$\beta_0 \mu_{i_1} \beta_1 \mu_{i_2} \cdots \beta_{m-2} \mu_{i_{m-1}} \beta_{m-1}.$$

We do not square the  $\mu_{i_k}$ 's as we are not fully going around the singularities. Therefore,

$$\tilde{\beta}_k = \beta_0 \mu_{i_1} \cdots \beta_{m-k-1} \mu_{i_{m-k}} \beta_{m-k}.$$

We can check to see the product of the factors in  $g$  is equal to the monodromy of the big loop. We first compute

$$\tilde{\beta}_j^{-1} \tilde{\beta}_{j+1} = \beta_{m-j}^{-1} \mu_{i_{m-j}}^{-1} \beta_{m-j-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1} \beta_0 \mu_{i_1} \cdots \beta_{m-j-2} \mu_{i_{m-j-1}} \beta_{m-j-1} = \beta_{m-j}^{-1} \mu_{i_{m-j}}^{-1}.$$

Then,

$$\begin{aligned} g_1 \cdots g_m &= \tilde{\beta}_1 \mu_{i_m}^2 \tilde{\beta}_1^{-1} \tilde{\beta}_2 \mu_{i_{m-1}}^2 \tilde{\beta}_2^{-1} \cdots \tilde{\beta}_m \mu_{i_1}^2 \tilde{\beta}_m^{-1} \\ &= \tilde{\beta}_1 \mu_{i_m}^2 \beta_{m-1}^{-1} \mu_{i_{m-1}}^{-1} \mu_{i_{m-1}}^2 \beta_{m-2}^{-1} \mu_{i_{m-2}}^{-1} \cdots \mu_{i_2}^2 \beta_1^{-1} \mu_{i_1}^{-1} \mu_{i_1}^2 \tilde{\beta}_m^{-1} \\ &= \beta_0 \mu_{i_1} \beta_1 \mu_{i_2} \cdots \beta_{m-2} \mu_{i_{m-1}} \beta_{m-1} \mu_{i_m}^2 \beta_{m-1}^{-1} \mu_{i_{m-1}} \beta_{m-2}^{-1} \cdots \mu_{i_2} \beta_1^{-1} \mu_{i_1} \beta_0^{-1}, \end{aligned}$$

which is the braid associated to the one loop around all of the singularities.

Now we explain how to reverse engineer a braided wiring diagram given a factorization. Let

$$\mathbf{g} = (\beta_1 \mu_{i_1}^2 \beta_1^{-1}, \beta_2 \mu_{i_2}^2 \beta_2^{-1}, \dots, \beta_{m-1} \mu_{i_{m-1}}^2 \beta_{m-1}^{-1}, \beta_m \mu_{i_m}^2 \beta_m^{-1}) = (\mathbf{g}_1, \dots, \mathbf{g}_m).$$

Since  $\mathbf{g}_1$  is the rightmost loop in the wiring diagram, we look at the last factor  $\mathbf{g}_m$  to construct our wiring diagram from left to right. The braid  $\beta_m$  will be drawn before the first singularity labeled  $\mu_{i_m}$ , an  $(n_m + 1)$ -tuple point involving strands  $i_m$  to  $i_m + n_m$ .

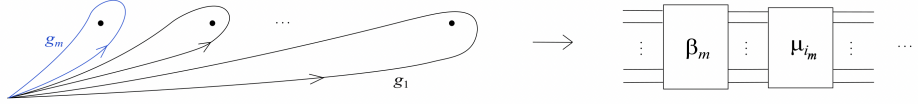


FIGURE 3.4. First part of the braided wiring diagram constructed using  $\mathbf{g}_m$

The braid drawn after vertex  $\mu_{i_m}$  can be deduced using  $\beta_{m-1}$ , the conjugating braid in the second to last factor of  $\mathbf{g}$ .

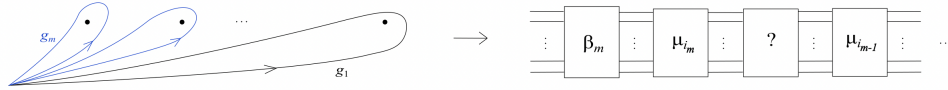


FIGURE 3.5. Determining the braid after  $\mu_{i_m}$

In other words, we want  $\beta_{m-1} = \beta_m \cdot \mu_{i_m} \cdot \beta$  where  $\beta$  is the unknown braid in Figure 3.5. Solving for  $\beta$ , we obtain  $\mu_{i_m}^{-1} \beta_m^{-1} \beta_{m-1}$  to be the braid we add to the wiring diagram.

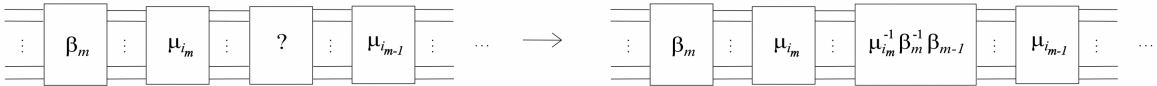


FIGURE 3.6. Braid after  $\mu_{i_m}$  is  $\mu_{i_m}^{-1} \beta_m^{-1} \beta_{m-1}$

Then,  $\mathbf{g}_{m-1} = \underbrace{\beta_m \cdot \mu_{i_m} \cdot \mu_{i_m}^{-1} \beta_m^{-1} \beta_{m-1}}_{\beta_{m-1}} \cdot \underbrace{\mu_{i_{m-1}}^2 \cdot \beta_{m-1}^{-1} \beta_m \mu_{i_m} \mu_{i_m}^{-1} \beta_m^{-1}}_{\beta_{m-1}^{-1}}$ , which matches what we have.

We continue this process to obtain the full wiring diagram. The braid  $\beta$  between  $\mu_{i_{m-1}}$  and  $\mu_{i_{m-2}}$  is given by

$$\beta_{m-2} = \beta_m \cdot \mu_{i_m} \cdot \mu_{i_m}^{-1} \beta_m^{-1} \beta_{m-1} \cdot \mu_{i_{m-1}} \cdot \beta \quad \Rightarrow \quad \beta = \mu_{i_{m-1}}^{-1} \beta_{m-1}^{-1} \beta_{m-2}.$$

From this, we deduce the braid between  $\mu_{i_k}$  and  $\mu_{i_{k-1}}$  is  $\mu_{i_k}^{-1}\beta_k^{-1}\beta_{k-1}$ . The entire braided wiring diagram constructed via  $\mathfrak{g}$  is drawn below.

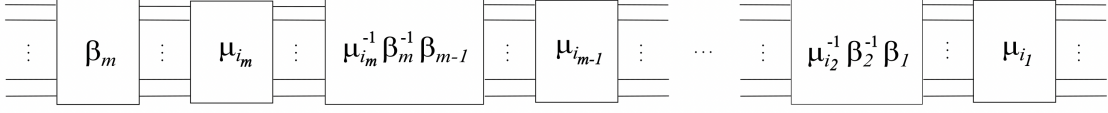


FIGURE 3.7. Braided wiring diagram constructed from  $\mathfrak{g}$

EXAMPLE 3.1.1. We perform an example of the above procedures for a three stranded wiring diagram. Suppose we want to write the factorization for the following wiring diagram.

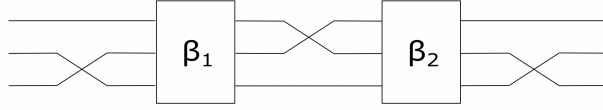


FIGURE 3.8. Braided wiring diagram

Given Figure 3.8, we have

$$g = (\tilde{\beta}_1\sigma_1^2\tilde{\beta}_1^{-1}, \tilde{\beta}_2\sigma_2^2\tilde{\beta}_2^{-1}, \tilde{\beta}_3\sigma_1^2\tilde{\beta}_3^{-1})$$

where

$$\tilde{\beta}_1 = \sigma_1\beta_1\sigma_2\beta_2, \quad \tilde{\beta}_2 = \sigma_1\beta_1, \quad \text{and} \quad \tilde{\beta}_3 = 1.$$

The product of the braids is

$$\sigma_1\beta_1\sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2^{-1}\beta_1^{-1}\sigma_1^{-1} \cdot \sigma_1\beta_1\sigma_2^2\beta_1^{-1}\sigma_1^{-1} \cdot \sigma_1^2 = \sigma_1\beta_1\sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2\beta_1^{-1}\sigma_1,$$

showing us the factorization is correct.

Next, we look at the translation in the opposite direction for three stranded braids. Set

$$\mathfrak{g} = (\beta_1\sigma_{i_1}^2\beta_1^{-1}, \beta_2\sigma_{i_2}^2\beta_2^{-1}, \beta_3\sigma_{i_3}^2\beta_3^{-1})$$

and let us draw the wiring diagram for this factorization. The braid  $\beta_3$  will be drawn before the first double point. This singularity will involve the bottom two strands if  $i_3 = 1$  (upper two strands if  $i_3 = 2$ ). The braid between the first and second double points is  $\sigma_{i_3}^{-1}\beta_3^{-1}\beta_2$ . Similarly, we obtain  $\sigma_{i_2}^{-1}\beta_2^{-1}\beta_1$  between the second and third double points. The strands for the second (third) double

point will depend on the value of  $i_2$  ( $i_3$ ) like the case with  $i_1$ . For the figure below, we set  $i_3 = 1 = i_1$  and  $i_2 = 2$ .

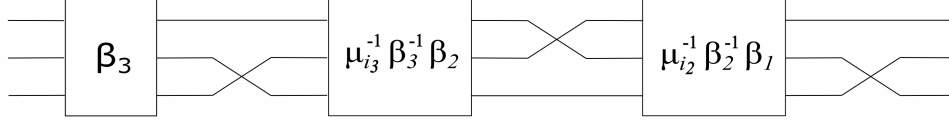


FIGURE 3.9. Braided wiring diagram constructed from  $g$

### 3.2. Markov moves

The effect of changing the choices made in the braid monodromy setup can be depicted pictorially using braided wiring diagrams. The following modifications to a braided wiring diagram are Cohen-Suciu's Markov moves [CS97][Section 5.6]. Braided wiring diagrams related by these moves represent the same complex curve/symplectic surface or an equisingularly isotopic one.

REMARK 3.2.1. *We write*

$$\{\beta_0, \text{vertex } \{a, \dots, c\}, \beta_1\}$$

*to indicate the part of the braided wiring diagram where the braid  $\beta_0$  precedes an  $n$ -tuple point with local indices  $\{a, \dots, c\}$  and the braid  $\beta_1$  comes after the same  $n$ -tuple point.*

1. Insert an arbitrary braid at the beginning of the wiring diagram.
2. Insert an arbitrary braid at the end of the wiring diagram.
3. Replace vertices  $\{i, \dots, j\}$  and  $\{k, \dots, l\}$  with
  - a. vertices  $\{k, \dots, l\}$  and  $\{i, \dots, j\}$ , respectively, if  $j < k$  or  $i > l$ .

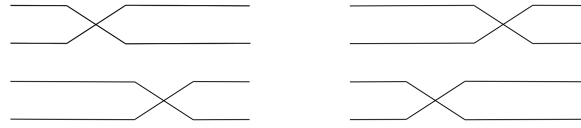


FIGURE 3.10. Move 3a

- $\tilde{b}$ .  $\{(\sigma_j \cdots \sigma_{l-1})(\sigma_{j-1} \cdots \sigma_{l-2}) \cdots (\sigma_{i+1} \cdots \sigma_k), \text{vertex } \{i, \dots, i+l-k\}\}$  and  $\{\text{vertex } \{i+l-k, \dots, l\}, (\sigma_{j-1}^{-1} \cdots \sigma_{l-2}^{-1})(\sigma_{j-2}^{-1} \cdots \sigma_{l-3}^{-1}) \cdots (\sigma_i^{-1} \cdots \sigma_{k-1}^{-1})\}$ , respectively, if  $j = k$ .

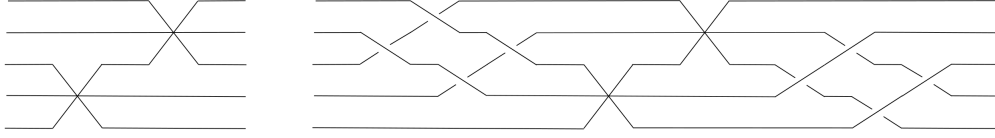


FIGURE 3.11. Move  $3\tilde{b}$

$\tilde{c}$ .  $\{(\sigma_{l-1} \cdots \sigma_{j-2})(\sigma_{l-2}^{-1} \cdots \sigma_{j-3}) \cdots (\sigma_k \cdots \sigma_{i-1}), \text{vertex } \{j+k-l, \dots, j\}\}$  and  $\{\text{vertex } \{k, \dots, j+k-l\}, (\sigma_l^{-1} \cdots \sigma_{j-1}^{-1})(\sigma_{l-1}^{-1} \cdots \sigma_{j-2}^{-1}) \cdots (\sigma_{k+1}^{-1} \cdots \sigma_i^{-1})\}$ , respectively, if  $i = l$ .

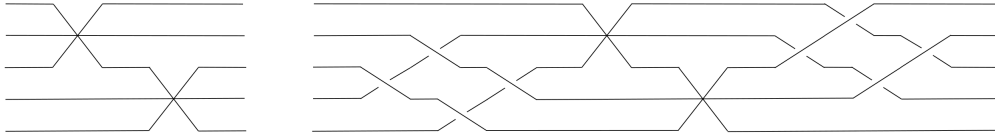


FIGURE 3.12. Move  $3\tilde{c}$

4. Simplify an intermediate braid.

REMARK 3.2.2. *We do not mention when this move is being used as it is clear when simplifications are happening.*

5. Replace  $\{\sigma_i, \text{vertex } \{j, \dots, k\}\}$  with

a.  $\{\text{vertex } \{j, \dots, k\}, \sigma_i\}$  if  $i < j - 1$  or  $i > k$ .

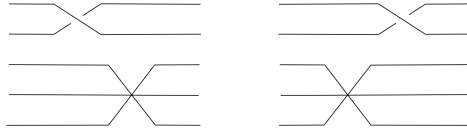


FIGURE 3.13. Move  $5a$

b.  $\{\sigma_{k-1}^{-1} \cdots \sigma_j^{-1}, \text{vertex } \{j+1, \dots, k+1\}, \sigma_j \cdots \sigma_k\}$  if  $i = k$ .

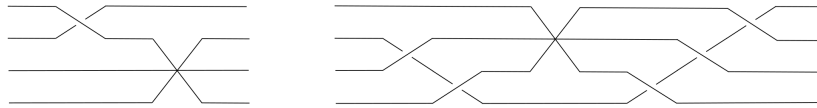


FIGURE 3.14. Move  $5b$

c.  $\{\sigma_j^{-1} \cdots \sigma_{k-1}^{-1}, \text{vertex } \{j-1, \dots, k-1\}, \sigma_{k-1} \cdots \sigma_{j-1}\}$  if  $i = j - 1$ .



FIGURE 3.15. Move 5c

d.  $\{\text{vertex } \{j, \dots, k\}, \sigma_{j+k-i-1}\}$  if  $j \leq i \leq k-1$ .



FIGURE 3.16. Move 5d

REMARK 3.2.3. *The braids can be inverted in moves 5. For instance, we can replace*

$$\{\sigma_i^{-1}, \text{vertex } \{j, \dots, k\}\} \quad \text{with} \quad \{\text{vertex } \{j, \dots, k\}, \sigma_i^{-1}\}$$

*for 5a. Also, we can switch the order of the braid and  $n$ -tuple point in moves 5b and 5c. For example, replace*

$$\{\text{vertex } \{j, \dots, k\}, \sigma_i\} \quad \text{with} \quad \{\sigma_k \cdots \sigma_j, \text{vertex } \{j+1, \dots, k+1\}, \sigma_j^{-1} \cdots \sigma_{k-1}^{-1}\}$$

*for 5b.*

**3.2.1. Generalizing Markov moves  $3\tilde{b}$  and  $3\tilde{c}$ .** The proof of Theorem 5.7 in [CS97] tells us how each Markov move affects the braid monodromy generators. Move 1 results in global conjugation by some braid  $\beta$  while moves 2, 4, and 5 do not affect the generators. For move 3, we see that the relations between the braid monodromy generators presented in the proof are exactly Hurwitz moves. However, there are conditions on the strands and the braid between the singularities, so we create a general version of these Markov moves. Given a braided wiring diagram and its factorization, we apply a Hurwitz move to the factorization and reconstruct the wiring diagram.

Let

$$\mathcal{W}_1 = \{\beta_0, \text{vertex } \{i_1, \dots, i_1 + n_1\}, \beta_1, \text{vertex } \{i_2, \dots, i_2 + n_2\}, \dots, \beta_{m-1}\}$$

and

$$\mathcal{W}_2 = \{\beta_{m+1}, \text{vertex } \{i_{m+2}, \dots, i_{m+2} + n_{m+2}\}, \dots, \beta_{r-1}, \text{vertex } \{i_r, \dots, i_r + n_r\}\}.$$

In words, the  $p$ th vertex is an  $(n_p + 1)$ -tuple point.

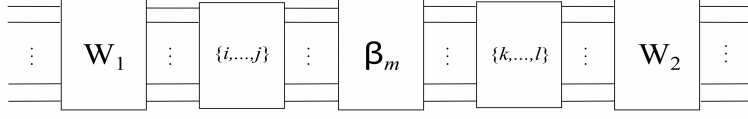


FIGURE 3.17. Braided wiring diagram

Set  $i_m = i$ ,  $i_m + n_m = j$  and  $i_{m+1} = k$ ,  $i_{m+1} + n_{m+1} = \ell$  (Figure 3.17). Note there are no conditions on the strands of the vertices for this new definition. We first focus on the Hurwitz move corresponding to Markov move  $3\tilde{b}$ .

The factorization for the braided wiring diagram is

$$\begin{aligned} \mathcal{F} = & (\beta_0 \mu_{i_1} \cdots \beta_{r-1} \mu_{i_r}^2 \beta_{r-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \dots, \beta_0 \mu_{i_1} \cdots \beta_{m+1} \mu_{i_{m+2}}^2 \beta_{m+1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \\ & \beta_0 \mu_{i_1} \cdots \beta_m \mu_{i_{m+1}}^2 \beta_m^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \beta_0 \mu_{i_1} \cdots \beta_{m-1} \mu_{i_m}^2 \beta_{m-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \\ & \beta_0 \mu_{i_1} \cdots \beta_{m-2} \mu_{i_{m-1}}^2 \beta_{m-2}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \dots, \beta_0 \mu_{i_1}^2 \beta_0^{-1}) \end{aligned}$$

where

$$\mu_{i_j} = (\sigma_{i_j} \cdots \sigma_{i_j+n_j-1})(\sigma_{i_j} \cdots \sigma_{i_j+n_j-2}) \cdots (\sigma_{i_j} \sigma_{i_j+1}) \sigma_{i_j}.$$

We apply Hurwitz move 1 to  $\mathcal{F}$  to switch the order at the  $m$ th and  $(m+1)$ st singularities to obtain

$$\begin{aligned} \mathcal{F}' = & (\beta_0 \mu_{i_1} \cdots \beta_{r-1} \mu_{i_r}^2 \beta_{r-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \dots, \beta_0 \mu_{i_1} \cdots \beta_{m+1} \mu_{i_{m+2}}^2 \beta_{m+1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \\ & \beta_0 \mu_{i_1} \cdots \beta_{m-1} \mu_{i_m}^2 \beta_{m-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \beta_0 \mu_{i_1} \cdots \beta_m \mu_{i_{m+1}}^2 \beta_m^{-1} \mu_{i_m} \beta_{m-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \\ & \beta_0 \mu_{i_1} \cdots \beta_{m-2} \mu_{i_{m-1}}^2 \beta_{m-2}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \dots, \beta_0 \mu_{i_1}^2 \beta_0^{-1}). \end{aligned}$$

We use the process explained in Section 3.1 to construct the corresponding wiring diagram.

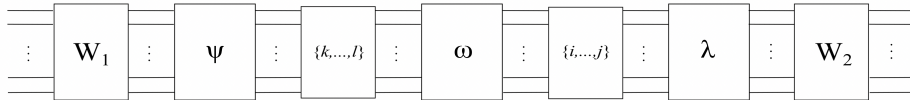


FIGURE 3.18. Applying Hurwitz move 1

Since  $\mathcal{W}_1$  and  $\mathcal{W}_2$  remain unaffected, we do not need to make changes to these portions of the braided wiring diagram. However, the red and orange factors in  $\mathcal{F}'$  will affect the part of the wiring

diagram between  $\mathcal{W}_1$  and  $\mathcal{W}_2$ . Since we draw the wiring diagram from left to right, we first consider the orange factor. We want the conjugating braid to be

$$\underbrace{\beta_0 \mu_{i_1} \cdots \beta_{m-1}}_{\text{from } \mathcal{W}_1} \mu_{i_m}^{-1} \beta_m,$$

meaning we need to add  $\mu_{i_m}^{-1} \beta_m = \psi$  after  $\mathcal{W}_1$ . In addition, the conjugated element of the orange factor is  $\mu_{i_{m+1}}^2$  so the local indices of the  $m$ th singularity change from  $\{i, \dots, j\}$  to  $\{k, \dots, \ell\}$ . To find the braid following vertex  $\{k, \dots, \ell\}$ , we now consider the red factor. The conjugating braid for this factor is

$$\beta_0 \mu_{i_1} \cdots \beta_{m-1}$$

whereas the conjugating braid read from the wiring diagram gives

$$\beta_0 \mu_{i_1} \cdots \beta_{m-1} \underbrace{\mu_{i_m}^{-1} \beta_m}_{\psi} \mu_{i_{m+1}} \omega.$$

Therefore, setting these equal to each other and solving for  $\omega$  gives

$$\omega = \mu_{i_{m+1}}^{-1} \beta_m^{-1} \mu_{i_m}.$$

The local indices of the  $(m+1)$ st singularity change from  $\{k, \dots, \ell\}$  to  $\{i, \dots, j\}$  because the conjugated element in the red factor is  $\mu_{i_m}^2$ . Observe we can omit  $\mu_{i_m}$  from  $\omega$  as that will commute with  $\mu_{i_m}^2$  and cancel out with its inverse. Finally, to find  $\lambda$ , we solve

$$\beta_0 \mu_{i_1} \cdots \beta_{m-1} \mu_{i_m} \beta_m \mu_{i_{m+1}} \beta_{m+1} = \beta_0 \mu_{i_1} \cdots \beta_{m-1} \underbrace{\mu_{i_m}^{-1} \beta_m}_{\psi} \mu_{i_{m+1}} \underbrace{\mu_{i_{m+1}}^{-1} \beta_m^{-1}}_{\omega} \mu_{i_m} \lambda \beta_{m+1}.$$

This gives  $\lambda = \mu_{i_m} \beta_m \mu_{i_{m+1}}$ . We fill in the unknown braids in the figure below.

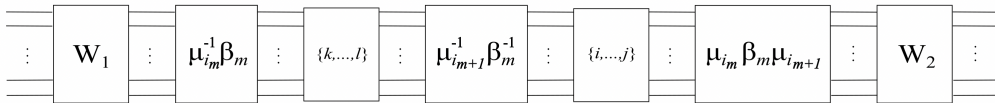


FIGURE 3.19. Applying Hurwitz move 1

DEFINITION 3.2.1. Markov move 3b changes Figure 3.17 to Figure 3.19.

In the three-stranded case, if these singularities are double points such that  $i \neq k$ , then observe that the local indices are now switched. For consistency, we can revert them to their original positions using Markov move 5 as we will see later in an example. This will also apply to Markov move 3c defined below.

We next consider Markov move  $3\tilde{c}$ .

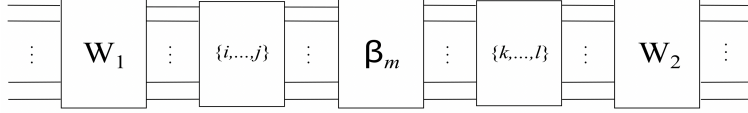


FIGURE 3.20. Braided wiring diagram

The factorization  $\mathcal{F}$  is the same as before since we wrote everything in terms of variables, but we apply Hurwitz move 2.

$$\begin{aligned} \mathcal{F}' = & (\beta_0 \mu_{i_1} \cdots \beta_{r-1} \mu_{i_r}^2 \beta_{r-1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \dots, \beta_0 \mu_{i_1} \cdots \beta_{m+1} \mu_{i_{m+2}}^2 \beta_{m+1}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \\ & \beta_0 \mu_{i_1} \cdots \beta_m \mu_{i_{m+1}}^2 \beta_m^{-1} \mu_{i_m}^2 \beta_m \mu_{i_{m+1}}^{-2} \beta_m^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \beta_0 \mu_{i_1} \cdots \beta_m \mu_{i_{m+1}}^2 \beta_m^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \\ & \beta_0 \mu_{i_1} \cdots \beta_{m-2} \mu_{i_{m-1}}^2 \beta_{m-2}^{-1} \cdots \mu_{i_1}^{-1} \beta_0^{-1}, \dots, \mu_{i_1}^2) \end{aligned}$$

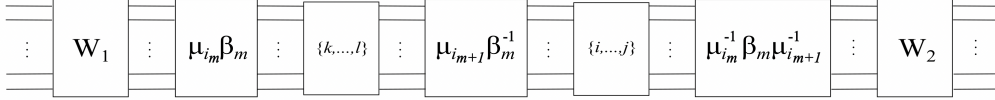


FIGURE 3.21. Applying Hurwitz move 2

DEFINITION 3.2.2. Markov move 3c changes Figure 3.20 to Figure 3.21.

EXAMPLE 3.2.3. Let us consider three-stranded wiring diagrams with only double points so  $n_j = 1$  for every  $j$ . Suppose we want to apply Markov move 3b to the following diagram.

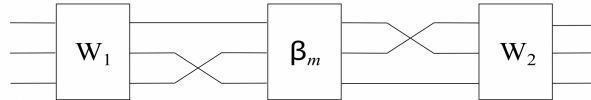


FIGURE 3.22. Braided wiring diagram

We see  $\mu_{i_m} = \sigma_1$  and  $\mu_{i_{m+1}} = \sigma_2$ . We use Figure 3.19 to draw the resulting wiring diagram.

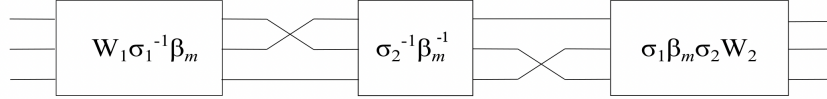


FIGURE 3.23. Applying Hurwitz move 1

By Markov move 5, we can change the local indices of the singularities so they match the original diagram.

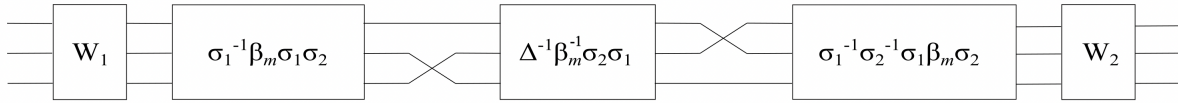


FIGURE 3.24. Shifting the double points

We check to make sure Markov moves  $3\tilde{b}$  and  $3b$  when  $\beta_m = 1$  yield the same diagram.

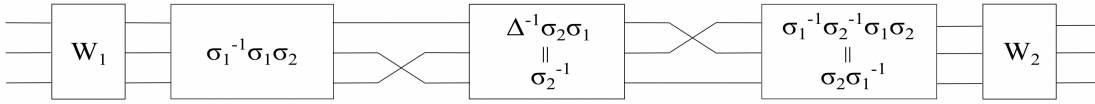


FIGURE 3.25.  $\beta_m = 1$

We move  $\sigma_2^{-1}$  to the right of vertex  $\{2, 3\}$  using Markov move  $5d$  and we arrive at the desired result.

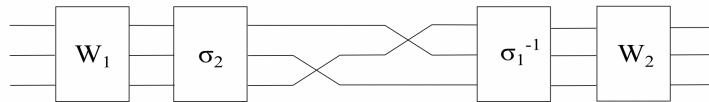


FIGURE 3.26. Markov move  $3\tilde{b}$

We now have all of the tools to understand braided wiring diagrams, so let us take a look at the above example where  $\mathcal{W}_1$  and  $\mathcal{W}_2$  are both empty and examine the braid monodromy generators.

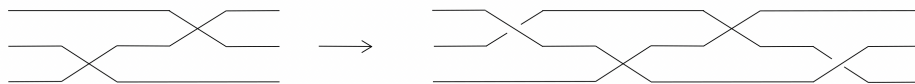


FIGURE 3.27. Example of move  $3b$

The factorizations are  $g = (\sigma_1 \sigma_2^2 \sigma_1^{-1}, \sigma_1^2)$  and  $\hat{g} = (\sigma_1^2, \sigma_2 \sigma_1^2 \sigma_2^{-1})$  for the original and new wiring diagrams, respectively. The total monodromy is the same and the relations between the factors are

$$\begin{aligned}\hat{g}_1 &= g_2 = \sigma_1^2, \\ \hat{g}_2 &= g_2^{-1} g_1 g_2 = \sigma_1^{-2} \cdot \sigma_1 \sigma_2^2 \sigma_1^{-1} \cdot \sigma_1^2 = \sigma_1^{-1} \sigma_2^2 \sigma_1.\end{aligned}$$

These new moves allow us to fill in some gaps from Theorem 5.7 and Corollary 5.8 in [CS97].

**THEOREM 3.2.4** (Theorem 1.2.1). *Braided wiring diagrams  $\mathcal{W}$  and  $\hat{\mathcal{W}}$  are related by Markov moves if and only if the braid monodromy homomorphisms are braid-equivalent.*

Before proving this theorem, we show a result connecting commuting relations in the braid group to Markov moves in braided wiring diagrams. We introduce some notation and results from [Gur84].

For  $1 \leq i \leq j$ , set

$$A_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_j, \quad B_{j,i} = \sigma_j \sigma_{j-1} \cdots \sigma_i, \quad \Delta_{i,j} = A_{i,j} A_{i,j-1} \cdots A_{i,i}.$$

Let  $Z_{\Delta_{i,j}}$  be the subgroup of  $B_n$  generated by the following elements.

1.  $\sigma_1, \sigma_2, \dots, \sigma_{i-2}, \sigma_{j+2}, \sigma_{j+3}, \dots, \sigma_{n-1}$
2.  $B_{j+1,i} A_{i-1,j+1} \quad (1 < i \leq j < n-1)$
3.  $A_{i-1,j} A_{i-2,j-1} \cdots A_{i-s,j-s+1} B_{j-s+1,i-s} \cdots B_{j-1,i-2} B_{j,i-1} \quad (1 \leq s < i)$
4.  $B_{j+1,i} B_{j+2,i+1} \cdots B_{j+r,i+r-1} A_{i+r-1,j+r} \cdots A_{i+1,j+2} A_{i,j+1} \quad (1 \leq r < n-j)$
5.  $\sigma_{i+r} \sigma_{j-r} \quad \left(0 \leq r < \frac{j-i}{2}\right)$
6.  $\begin{cases} \sigma_{k+i} & j-i = 2k \\ \sigma_{k+i} \sigma_{k+i+1} \sigma_{k+i} & j-i = 2k+1 \end{cases}$

**THEOREM 3.2.5** (Theorem 3 in [Gur84]). *The centralizer of the braid word  $\Delta_{i,j}$  is  $Z_{\Delta_{i,j}}$ .*

**PROPOSITION 3.2.1.** *The braided wiring diagrams in the following figure are related by Markov move 5 if  $\beta'_1 = \beta_1 \alpha^{-1}$  and  $\beta'_2 = \alpha \beta_2$  where  $\alpha \in Z_{\Delta_{i,j}}$ .*

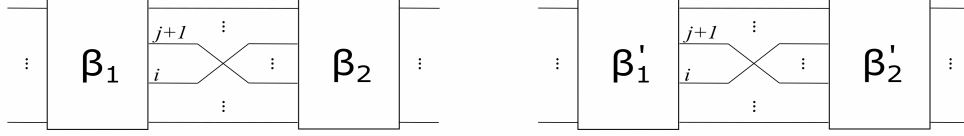


FIGURE 3.28. Braided wiring diagrams

PROOF. It suffices to show each braid from the list defining  $Z_{\Delta_{i,j}}$  can pass the singularity with vertex  $\{i, \dots, j+1\}$ . Markov move 5a proves the case for the braids in (1). The braids in (5) and (6) follow by Markov move 5d. The remaining elements in  $Z_{\Delta_{i,j}}$  will be explained more carefully. Figures 3.29 and 3.30 show two helpful Markov equivalences in braided wiring diagrams using Markov move 5.

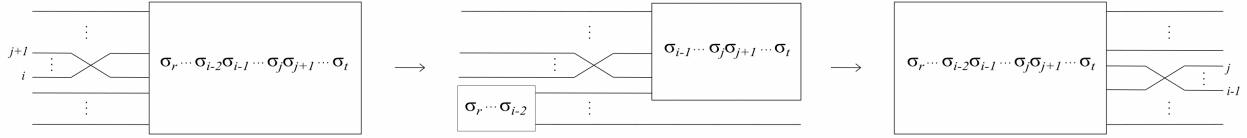


FIGURE 3.29. Let  $r < i \leq j \leq t$ . The first arrow uses Markov move 5a and the second arrow uses Markov moves 5a, 5c.

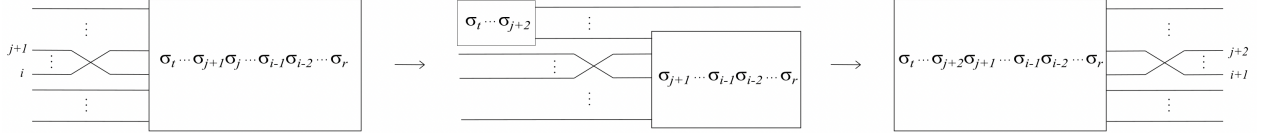


FIGURE 3.30. Let  $r \leq i \leq j < t$ . The first arrow uses Markov move 5a and the second arrow uses Markov moves 5a, 5b.

Let us look at (2)  $B_{j+1,i} A_{i-1,j+1}$ . We first focus on  $B_{j+1,i}$ . We use Figure 3.30 where  $t = j+1$  and  $r = i$ . The steps are written down below.

$$\{\{i, \dots, j+1\}, B_{j+1,i} A_{i-1,j+1}\} \xrightarrow{\text{Fig. 3.30}} \{B_{j+1,i}, \{i+1, \dots, j+2\}, A_{i-1,j+1}\}$$

For  $A_{i-1,j+1}$ , we look at Figure 3.29 where  $r = i-1$  and  $t = j+1$ .

$$\begin{aligned} \{B_{j+1,i}, \{i+1, \dots, j+2\}, A_{i-1,j+1}\} &\xrightarrow{\text{Move 5a}} \{B_{j+1,i} \sigma_{i-1}, \{i+1, \dots, j+2\}, A_{i,j+1}\} \\ &\xrightarrow{\text{Fig. 3.29}} \{B_{j+1,i} A_{i-1,j+1}, \{i, \dots, j+1\}\} \end{aligned}$$

For (3), we have  $A_{i-1,j}A_{i-2,j-1} \cdots A_{i-s,j-s+1}B_{j-s+1,i-s} \cdots B_{j-1,i-2}B_{j,i-1}$  with  $1 \leq s < i$ .

$$\begin{aligned} & \{\{i, \dots, j+1\}, A_{i-1,j}A_{i-2,j-1} \cdots A_{i-s,j-s+1}B_{j-s+1,i-s} \cdots B_{j-1,i-2}B_{j,i-1}\} \\ & \xrightarrow{\text{Fig. 3.29}} \{A_{i-1,j} \cdots A_{i-s,j-s+1}, \{i-s, \dots, j+1-s\}, B_{j-s+1,i-s} \cdots B_{j-1,i-2}B_{j,i-1}\} \\ & \xrightarrow{\text{Fig. 3.30}} \{A_{i-1,j}A_{i-2,j-1} \cdots A_{i-s,j-s+1}B_{j-s+1,i-s} \cdots B_{j-1,i-2}B_{j,i-1}, \{i, \dots, j+1\}\} \end{aligned}$$

This is simply a repeated application of our observation about Markov move 5 illustrated in the figure above. The braids in (4) can be shown to pass through the singularity in a similar manner so we omit the sequence of steps. The proof is now complete as we have covered every braid in the list.  $\square$

PROOF OF THEOREM 1.2.1. Suppose  $\mathcal{W}$  and  $\hat{\mathcal{W}}$  are Markov equivalent. As mentioned earlier, the proof of Theorem 5.7 in [CS97] states move 1 conjugates the braid monodromy whereas moves 2, 4, and 5 keep it fixed. Furthermore, the process we went through to define moves 3b and 3c tells us that a Hurwitz move is being applied in the case of move 3. Therefore, the braid monodromy homomorphisms are braid-equivalent. The other direction follows by definition of braid-equivalence. By [CS97, Theorem 3.7], we know braid monodromy is well-defined up to braid-equivalence. The proof of this theorem outlines how braid monodromy factorizations in the same braid-equivalence class are only affected by conjugation or Hurwitz moves. We claim braided wiring diagrams with identical factorizations are covered by the remaining moves. Suppose we have two equal factorizations  $\mathcal{F} = (g_1, \dots, g_k)$  and  $\hat{\mathcal{F}} = (\hat{g}_1, \dots, \hat{g}_k)$ . We first focus on the last factor from each factorization since the loops in the projection for these factors are leftmost:  $\beta_k \mu_{i_k}^2 \beta_k^{-1}$  and  $\hat{\beta}_k \hat{\mu}_{i_k}^2 \hat{\beta}_k^{-1}$ . If  $\mu_{i_k} = \hat{\mu}_{i_k}$ , then  $\hat{\beta}_k^{-1} \beta_k$  should commute with  $\mu_{i_k} = \Delta_{i_k, i_k + n_k - 1}$  for the factors to be equal. By Theorem 3.2.5, we know  $\hat{\beta}_k^{-1} \beta_k \in Z_{\Delta_{i_k, i_k + n_k - 1}}$ , and we can use Proposition 3.2.1 to conclude  $\hat{\beta}_k^{-1} \beta_k$  commutes with the singularity  $\mu_{i_k}$  via Markov moves. Suppose  $\mu_{i_k} \neq \hat{\mu}_{i_k}$ . The type of singularity must be the same because the sum of the exponents would otherwise change. Therefore, the inequality implies the singularity has shifted. Consequently, the conjugating braids in each factor are different. Markov move 5 would account for these various changes. By continuing to analyze each factor in this manner, we see that equal factors can be related by Markov moves. For the first factor, which is the rightmost loop in the projection, we also obtain Markov move 2 as

one of the consequences; this is the only case where adding a braid at the end of the wiring diagram would affect the factorization. Although ultimately, the added braid cancels and the factor remains unchanged. Thus, if the braid monodromy homomorphisms are braid-equivalent, then the associated braided wiring diagrams are Markov equivalent.  $\square$

### 3.3. Markov move results

We state some useful results involving Markov moves that will be used in the next section. Consider a double point with vertex  $\{1, 2\}$ , i.e.  $i = 1 = j$  given our notation in the previous section. We use Proposition 3.2.1 to obtain a corollary. It is known that the centralizer of  $\sigma_1$  is an abelian subgroup generated by  $\sigma_1$  and  $\sigma_2\sigma_1^2\sigma_2$ . If we use our list for  $\mathcal{Z}_{\Delta_{1,1}}$ , we indeed obtain  $B_{2,1}A_{1,2}$  and  $\sigma_1$  as our generators.

**COROLLARY 3.3.1.** *The braided wiring diagrams in the following figure are related by Markov move 5 if  $\beta'_1 = \beta_1\alpha^{-1}$  and  $\beta'_2 = \alpha\beta_2$  where  $\alpha \in C_{B_3}(\sigma_1)$ .*



FIGURE 3.31. Braided wiring diagrams

In other words, elements of the form  $\sigma_1^n(\sigma_2\sigma_1^2\sigma_2)^m$  pass through vertex  $\{1, 2\}$ . We also write a version involving vertex  $\{2, 3\}$ .

**COROLLARY 3.3.2.** *The braided wiring diagrams in the following figure are related by Markov move 5 if  $\beta'_1 = \beta_1\alpha^{-1}$  and  $\beta'_2 = \alpha\beta_2$  where  $\alpha \in C_{B_3}(\sigma_2)$ , an abelian subgroup generated by  $\sigma_2$  and  $\sigma_1\sigma_2^2\sigma_1$ .*

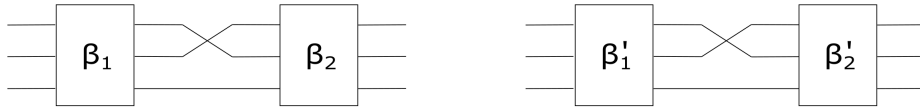


FIGURE 3.32. Braided wiring diagrams

Let  $\mathcal{W}_1$  and  $\mathcal{W}_2$  be defined as in Section 3.2.1. Recall the  $i$ th vertex is an  $(n_i + 1)$ -tuple point. Suppose  $n_m > n_{m+1}$ ,  $i_m \leq i_{m+1} \leq i_m + n_m - 1$ , and  $i_m + 1 \leq i_{m+1} + n_{m+1} \leq i_m + n_m$ , i.e. these two vertices are nested. An example is shown below.

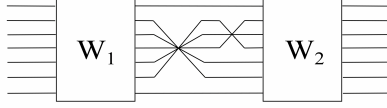


FIGURE 3.33. Braided wiring diagram

We prove a statement involving these types of wiring diagrams.

PROPOSITION 3.3.1. *The braided wiring diagrams in the following figure are related by Markov moves 3 and 5.*



FIGURE 3.34.  $k' = 1 + n - l$ ,  $l' = 1 + n - k$

Before proving this proposition, we use various Markov moves to show an interesting relationship between  $\Delta \in B_n$  and an  $m$ -tuple point where  $m < n$ .

PROPOSITION 3.3.2. *The braided wiring diagrams in the following figure are related by Markov move 5.*



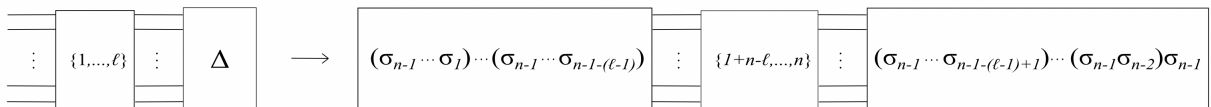
FIGURE 3.35. Braided wiring diagrams

PROOF. We use the observations from the proof of Proposition 3.2.1. There are two cases to consider:  $k = 1$  and  $k > 1$ .

Suppose  $k = 1$ . Recall  $\Delta$  can be written in multiple ways.

$$\Delta = (\sigma_{n-1} \cdots \sigma_1) \cdots (\sigma_{n-1} \sigma_{n-2}) \sigma_{n-1} = (\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1 \sigma_2) \sigma_1$$

We use Figure 3.30 until  $\{1, \dots, l\}$  becomes  $\{1 + n - l, \dots, n\}$ .

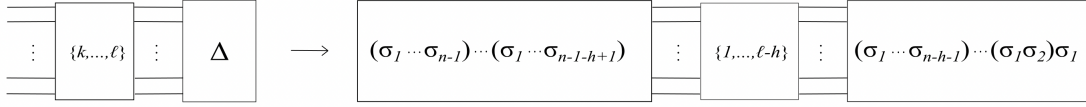


The remaining braids on the right hand side can be moved across the singularity by Markov move 5d. Moreover, we can show this braid remains unchanged using braid relations. Thus, the braid on the left hand side becomes

$$(\sigma_{n-1} \cdots \sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_{n-1-(l-1)}) \underbrace{(\sigma_{n-1} \cdots \sigma_{n-1-(l-1)+1}) \cdots (\sigma_{n-1} \sigma_{n-2}) \sigma_{n-1}}_{\text{Move 5d}} = \Delta$$

and we are done.

Else,  $k > 1$  so set  $k = 1 + h$ . We repeatedly apply Figure 3.29 so that  $\{k, \dots, l\}$  becomes  $\{1, \dots, l-h\}$ .



If  $l = n$ , then we can similarly use move 5d to shift the remaining braids to the left hand side and the statement is proved. If  $l < n$ , then we use Figure 3.30.

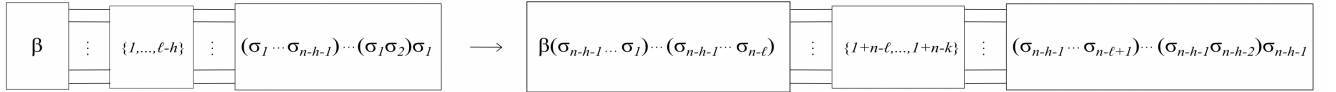


FIGURE 3.36.  $\beta = (\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1 \cdots \sigma_{n-1-h+1})$

By Markov move 5d, the braids following the vertex can be combined with the braids in the beginning, giving

$$\begin{aligned} & \beta \cdot \underbrace{[(\sigma_{n-h-1} \cdots \sigma_1) \cdots (\sigma_{n-h-1} \cdots \sigma_{n-l})] \cdot [(\sigma_{n-h-1} \cdots \sigma_{n-l+1}) \cdots (\sigma_{n-h-1} \sigma_{n-h-2}) \sigma_{n-h-1}]}_{(\sigma_1 \cdots \sigma_{n-h-1})(\sigma_1 \cdots \sigma_{n-h-2}) \cdots (\sigma_1 \sigma_2) \sigma_1} \\ &= \beta \cdot (\sigma_1 \cdots \sigma_{n-h-1})(\sigma_1 \cdots \sigma_{n-h-2}) \cdots (\sigma_1 \sigma_2) \sigma_1 \\ &= \Delta. \end{aligned}$$

□

We note this result still applies even if there are additional strands added to Figure 3.35. For example, suppose  $n = 4$  and we have vertex  $\{2, 3\}$ . In this case, the proposition tells us that  $\{\text{vertex } \{2, 3\}, \Delta\}$  is Markov equivalent to  $\{\Delta, \text{vertex } \{2, 3\}\}$ . Suppose we add extra strands at the

bottom and top so that  $n = 6$  and vertex  $\{2, 3\}$  turns into  $\{3, 4\}$ . We can think of our original  $\Delta$  as a subdelta in this case. Then, the proposition says  $\{\text{vertex } \{3, 4\}, \Delta_{2,4}\}$  is Markov equivalent to  $\{\Delta_{2,4}, \text{vertex } \{3, 4\}\}$ .

PROOF OF PROPOSITION 3.3.1. We use Markov move 3b or 3c to switch the order of the singularities. Both moves will give us the correct result so let us select Markov move 3c.

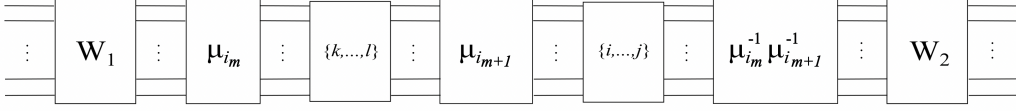


FIGURE 3.37. Markov move 3c

The braid  $\mu_{i_{m+1}}$  between the singularities can be shifted to the left of vertex  $\{k, \dots, l\}$  by Markov move 5d. The braid is unaffected and we draw the resulting wiring diagram.

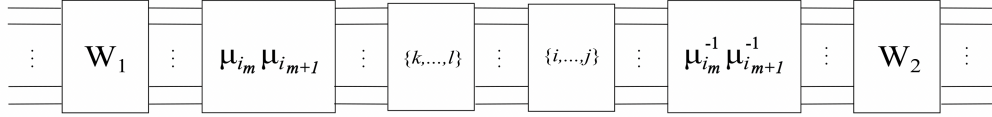


FIGURE 3.38. Markov move 5d

We move the braid  $\mu_{i_m}^{-1} \mu_{i_{m+1}}^{-1}$  across vertex  $\{i, \dots, j\}$  using move 5d. The first factor  $\mu_{i_m}^{-1}$  remains the same but

$$\mu_{i_{m+1}}^{-1} = \sigma_k^{-1}(\sigma_{k+1}^{-1} \sigma_k^{-1}) \cdots (\sigma_{l-2}^{-1} \cdots \sigma_k^{-1})(\sigma_{l-1}^{-1} \cdots \sigma_k^{-1})$$

becomes

$$\hat{\mu}_{i_{m+1}}^{-1} = \sigma_{i+j-k-1}^{-1} \left( \sigma_{i+j-(k+1)-1}^{-1} \sigma_{i+j-k-1}^{-1} \right) \cdots \left( \sigma_{i+j-(l-2)-1}^{-1} \cdots \sigma_{i+j-k-1}^{-1} \right) \left( \sigma_{i+j-(l-1)-1}^{-1} \cdots \sigma_{i+j-k-1}^{-1} \right).$$

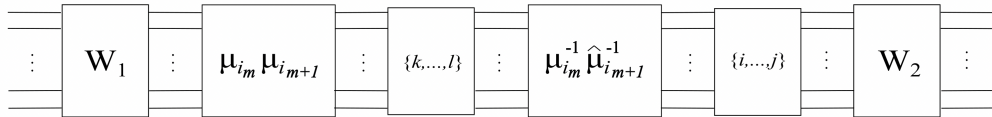


FIGURE 3.39. Markov move 5d

We wish to move the braid in the middle to the left of vertex  $\{k, \dots, l\}$  and cancel the remaining braids. By [Gur84, Lemma 10-c],

$$(3.1) \quad \Delta_{i,j} \sigma_r = \sigma_{i+j-r} \Delta_{i,j}$$

for  $i \leq r \leq j$ . We use this property to rewrite  $\mu_{i_m}^{-1} \hat{\mu}_{i_{m+1}}^{-1}$  as  $\mu_{i_{m+1}}^{-1} \mu_{i_m}^{-1}$ . By move 5d, we can then shift  $\mu_{i_{m+1}}^{-1}$  across vertex  $\{k, \dots, l\}$  to get

$$\{\mathcal{W}_1, \mu_{i_m}, \text{ vertex } \{k, \dots, l\}, \mu_{i_m}^{-1}, \text{ vertex } \{i, \dots, j\}, \mathcal{W}_2\}.$$

The final step is moving  $\mu_{i_m}^{-1}$  (or  $\mu_{i_m}$ ) across vertex  $\{k, \dots, l\}$ . By Proposition 3.3.2, we obtain

$$\{\mathcal{W}_1, \text{ vertex } \{i+j-l, \dots, i+j-k\}, \text{ vertex } \{i, \dots, j\}, \mathcal{W}_2\}.$$

□

## CHAPTER 4

### Proofs of our main results

Let  $\mathcal{W}_0$  be a wiring diagram representing a symplectic line arrangement of three lines intersecting generically in double points defined in Section 1.2 and  $\mathcal{W}$  be a wiring diagram having the same singularities but with braids in between. In this chapter, we will prove the following theorem using two methods.

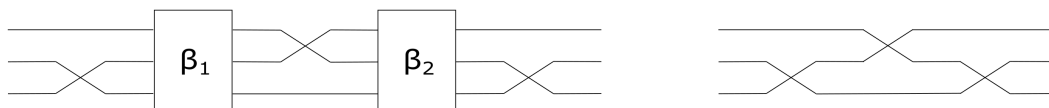


FIGURE 4.1.  $\mathcal{W}$  and  $\mathcal{W}_0$

**THEOREM 4.0.1** (Theorem 1.2.2). *The total braid monodromy of a braided wiring diagram  $\mathcal{W}$  is equal to the full twist if and only if it is related by Markov moves to  $\mathcal{W}_0$ .*

We will also take a look at a few more configurations and prove variations of Theorem 1.2.2.

#### 4.1. Markov moves and conditions for $\beta_i$

Before proving Theorem 1.2.2, we break down the statement and go over some preliminary results. If we want to show  $\mathcal{W}$  and  $\mathcal{W}_0$  are related by Markov moves, then it is necessary to understand how these moves affect braids and double points. We describe explicitly the possibilities for  $\beta_1$  and  $\beta_2$  such that the braided wiring diagram  $\mathcal{W}$  is related to  $\mathcal{W}_0$  by Markov moves.

**LEMMA 4.1.0.1.** *Let  $\beta_1 = \sigma_1^{m_1}(\sigma_2\sigma_1^2\sigma_2)^{k_1}(\sigma_1\sigma_2^2\sigma_1)^{l_1}\sigma_2^{n_1}$  and  $\beta_2 = \sigma_2^{n_2}(\sigma_1\sigma_2^2\sigma_1)^{l_2}(\sigma_2\sigma_1^2\sigma_2)^{k_2}\sigma_1^{m_2}$  in  $\mathcal{W}$  where  $(\sigma_1\sigma_2^2\sigma_1)^{l_1+l_2}\sigma_2^{n_1+n_2} \in Z(B_3)$  and all of the exponents are integers. Then,  $\mathcal{W}$  is related by Markov moves to  $\mathcal{W}_0$ .*

**PROOF.** Suppose  $\beta_1$  and  $\beta_2$  are given as in the statement. By Corollaries 3.3.1 and 3.3.2, we can move  $\sigma_1^{m_i}(\sigma_2\sigma_1^2\sigma_2)^{k_i}$  outward and  $\sigma_2^{n_i}(\sigma_1\sigma_2^2\sigma_1)^{l_i}$  through the middle singularity. The braids that were moved outward can be ignored by Markov moves 1 and 2. Without loss of generality,

we shift  $\sigma_2^{n_2}(\sigma_1\sigma_2^2\sigma_1)^{l_2}$  to the left so it becomes a part of  $\beta_1$ . The last braid  $\beta_2$  is now trivial so it suffices to focus on  $\beta_1 = (\sigma_1\sigma_2^2\sigma_1)^{l_1+l_2}\sigma_2^{n_1+n_2}$ . By assumption, this element is in the center of  $B_3$  so we use Proposition 3.3.2 and Markov move 1 to reduce  $\beta_1$  to the trivial braid. We outline the various steps.

$$\begin{aligned}
\mathcal{W} &= \{\text{vertex } \{1, 2\}, \sigma_1^{m_1}(\sigma_2\sigma_1^2\sigma_2)^{k_1}(\sigma_1\sigma_2^2\sigma_1)^{l_1}\sigma_2^{n_1}, \text{ vertex } \{2, 3\}, \sigma_2^{n_2}(\sigma_1\sigma_2^2\sigma_1)^{l_2}(\sigma_2\sigma_1^2\sigma_2)^{k_2}\sigma_1^{m_2}, \text{ vertex } \{1, 2\}\} \\
&\rightarrow \{\sigma_1^{m_1}(\sigma_2\sigma_1^2\sigma_2)^{k_1}, \text{ vertex } \{1, 2\}, (\sigma_1\sigma_2^2\sigma_1)^{l_1+l_2}\sigma_2^{n_1+n_2}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}, (\sigma_2\sigma_1^2\sigma_2)^{k_2}\sigma_1^{m_2}\} \\
&\rightarrow \{\text{vertex } \{1, 2\}, (\sigma_1\sigma_2^2\sigma_1)^{l_1+l_2}\sigma_2^{n_1+n_2}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}\} \\
&\rightarrow \{(\sigma_1\sigma_2^2\sigma_1)^{l_1+l_2}\sigma_2^{n_1+n_2}, \text{ vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}\} \\
&\rightarrow \{\text{vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}\} = \mathcal{W}_0
\end{aligned}$$

Hence,  $\mathcal{W}$  and  $\mathcal{W}_0$  are Markov equivalent. □

We can use the next result to piece everything together and prove Theorem 1.2.2.

**THEOREM 4.1.1.** *If the braids in  $\mathcal{W}$  are not of the form given in Lemma 4.1.0.1, then the total monodromy is not the full twist.*

**PROOF.** Let us assume the braid monodromy of  $\mathcal{W}$  is the full twist so

$$(4.1) \quad \sigma_1\beta_1\sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2\beta_1^{-1}\sigma_1 = \Delta^2.$$

We rewrite this equation in the following way:

$$(4.2) \quad \beta_1^{-1}\sigma_2\sigma_1^2\sigma_2\beta_1 = \sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2.$$

We write our unknowns in Garside normal form so we can work in the monoid  $B_3^+$ . Let  $\beta_i = \Delta^{k_i}\alpha_i$  and  $\beta_i^{-1} = \alpha_i^{-1}\Delta^{-k_i} = \Delta^{-k_i}\Delta^{-n_i}\tilde{\alpha}_i$  with  $\alpha_i, \tilde{\alpha}_i \in B_3^+$ . This gives the equation  $(\alpha_i)_{(k_i)}(\tilde{\alpha}_i)_{(n_i)} = \Delta^{n_i}$ . Recall  $\Delta^l\beta = \beta_{(l)}\Delta^l$  where  $\beta_{(l)} = \beta$  if  $l$  is even and  $\beta_{(l)}$  is obtained from  $\beta$  by replacing each  $\sigma_i$  with  $\sigma_{n-i}$  if  $l$  is odd. The relationship between  $\alpha_i$  and  $\tilde{\alpha}_i$  will be useful when solving for  $\tilde{\alpha}_i$  later. By substituting these Garside normal forms into Equation 4.2, we observe that the resulting equation depends on the parity of  $k_i$  and  $n_i$ :

$$\Delta^{-n_1}(\tilde{\alpha}_1\sigma_2\sigma_1^2\sigma_2)_{(k_1)}\alpha_1 = \Delta^{-n_2}(\sigma_2)_{(n_2)}(\alpha_2\sigma_1^2)_{(k_2+n_2)}\tilde{\alpha}_2\sigma_2.$$

We break this proof up into four cases and then examine subcases.

REMARK 4.1.1. *If we substitute the Garside normal forms into Equation 4.1, then we have*

$$(\sigma_1)_{(k_1)} \alpha_1 \sigma_2 (\alpha_2 \sigma_1^2 \alpha_2^{-1})_{(k_2)} \sigma_2 \alpha_1^{-1} (\sigma_1)_{(k_1)} = \Delta^2.$$

*We keep the local indices of the singularities the same as the standard wiring diagram. Hence, the parity of the  $k_i$ 's may affect the braids in the wiring diagram. For instance, if  $k_1$  and  $k_2$  are odd, then we have*

$$\sigma_1 (\sigma_2 \sigma_1 \alpha_1) \sigma_2 ((\alpha_2)_{(1)} \sigma_1 \sigma_2) \sigma_1^2 (\sigma_2^{-1} \sigma_1^{-1} (\alpha_2^{-1})_{(1)}) \sigma_2 (\alpha_1^{-1} \sigma_1^{-1} \sigma_2^{-1}) \sigma_1 = \Delta^2.$$

Case 1:  $k_1$  and  $k_2$  are even

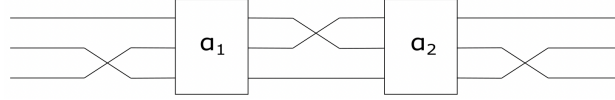


FIGURE 4.2. Braided wiring diagram for Case 1

To prove our theorem, we consider the possible beginnings and endings of the word  $\alpha_i$ , applying Markov moves to simplify until it is no longer possible to apply further Markov moves. The remaining  $\beta_i$ 's will not be in the desired forms by Lemma 4.1.0.1. It then suffices to show that Equation 4.2 is not true for those values of  $\alpha_i$ . Furthermore, we also consider when  $\alpha_1$  (or  $\alpha_2$ ) can end (or begin) with braids that move freely only between the second singularity. To account for this, we assume those braids are shifted to the beginning of the  $\alpha_2$ 's and we slightly modify the right side of Table 4.2 (See Table 4.4).

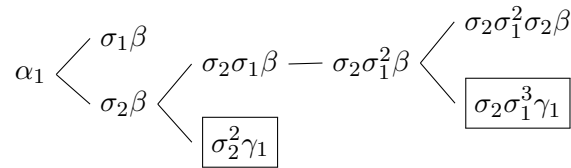


FIGURE 4.3. Options for  $\alpha_1$

We only describe the process for finding the beginning of  $\alpha_1$  as everything else is done in a similar manner. Since  $\sigma_1$  commutes with the leftmost singularity of Figure 4.2, if not further Markov moves can be applied,  $\alpha_1$  must start with  $\sigma_2$ . This is not enough, however, since there exist braids

in the centralizer of  $\sigma_1$  that start with  $\sigma_2$ . We then multiply  $\sigma_2$  by the two generators of  $B_3$  to obtain  $\sigma_2^2$  and  $\sigma_2\sigma_1$  as seen in the chart above. A positive braid prime to  $\Delta$  that does not commute with  $\sigma_1$  can begin with  $\sigma_2^2$  so this will be one of the choices for  $\alpha_1$ . The second choice is  $\sigma_2\sigma_1^3$  using the fact that  $\sigma_2\sigma_1^2\sigma_2 \in C_{B_3}(\sigma_1)$  and the assumption that  $\alpha_1$  is not divisible by  $\Delta$ . We use Lemma 3.10 from [Dat23] to determine  $\tilde{\alpha}_i$  in each of these cases. See Section 4.5.2 for a more detailed explanation on these concepts from [Dat23].

TABLE 4.1.  $\alpha_i$  and  $\tilde{\alpha}_i$

| $\alpha_1$                                     | $(\tilde{\alpha}_1)_{(n_1)}$                             | $\alpha_2$                                     | $(\tilde{\alpha}_2)_{(n_2)}$                             |
|------------------------------------------------|----------------------------------------------------------|------------------------------------------------|----------------------------------------------------------|
| $\sigma_2^2\gamma_1\sigma_1^2$                 | $\sigma_2\sigma_1\eta_1(\sigma_1\sigma_2)_{(n_1+1)}$     | $\sigma_1^2\gamma_2\sigma_2^2$                 | $\sigma_1\sigma_2\eta_2(\sigma_2\sigma_1)_{(n_2+1)}$     |
| $\sigma_2^2\gamma_1\sigma_2^3\sigma_1$         | $\sigma_2^2\sigma_1\eta_1(\sigma_1\sigma_2)_{(n_1+1)}$   | $\sigma_1^2\gamma_2\sigma_1^3\sigma_2$         | $\sigma_1^2\sigma_2\eta_2(\sigma_2\sigma_1)_{(n_2+1)}$   |
| $\sigma_2\sigma_1^3\gamma_1\sigma_1^2$         | $\sigma_2\sigma_1\eta_1(\sigma_1\sigma_2^2)_{(n_1+1)}$   | $\sigma_1\sigma_2^3\gamma_2\sigma_2^2$         | $\sigma_1\sigma_2\eta_2(\sigma_2\sigma_1^2)_{(n_2+1)}$   |
| $\sigma_2\sigma_1^3\gamma_1\sigma_2^3\sigma_1$ | $\sigma_2^2\sigma_1\eta_1(\sigma_1\sigma_2^2)_{(n_1+1)}$ | $\sigma_1\sigma_2^3\gamma_2\sigma_1^3\sigma_2$ | $\sigma_1^2\sigma_2\eta_2(\sigma_2\sigma_1^2)_{(n_2+1)}$ |

If  $\alpha_i$  does not fall under these cases, we can apply Corollaries 3.3.1 or 3.3.2 until it either falls under one of these cases or becomes the empty word. We now substitute these values into Equation 4.2 and rewrite them in Garside normal form. We claim each remaining positive braid in Table 4.2 is not divisible by  $\Delta$  using Theorem 2.1.1 and our assumption that the braids in parenthesis are not divisible by  $\Delta$ . An explanation is provided for one case. Consider the first braid in our table:  $(\sigma_2\sigma_1\eta_1\sigma_1)\sigma_1\sigma_2(\sigma_2^2\gamma_1\sigma_1^2)$ . By assumption,  $\sigma_2\sigma_1\eta_1\sigma_1$  is not divisible by  $\Delta$ . The product of  $\phi_1 = \sigma_2\sigma_1\eta_1\sigma_1$  and  $\phi_2 = \sigma_1\sigma_2$  cannot contain  $\Delta$  because the index of the last generator of  $\phi_1$  is equal to the index of the first generator of  $\phi_2$ . By the same logic, the product  $(\sigma_2\sigma_1\eta_1\sigma_1\sigma_1\sigma_2)(\sigma_2^2\gamma_1\sigma_1^2)$  is not divisible by  $\Delta$ . Now since the left and right hand sides of Equation 4.2 end with  $\sigma_1$  and  $\sigma_2$ , respectively, we conclude the equality does not hold.

TABLE 4.2. Writing both sides of Equation 4.2 in Garside normal form

|            | $\beta_1^{-1}\sigma_2\sigma_1^2\sigma_2\beta_1$                                                                   | $\sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2$                                                                                 |
|------------|-------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------|
| $n_i$ odd  | $\Delta^{-n_1+1}(\sigma_2\sigma_1\eta_1\sigma_1)\sigma_1\sigma_2(\sigma_2^2\gamma_1\sigma_1^2)$                   | $\Delta^{-n_2}\sigma_1(\sigma_2^2(\gamma_2)_{(1)}\sigma_1^2)\sigma_2^2(\sigma_2\sigma_1(\eta_2)_{(1)}\sigma_1\sigma_2)\sigma_2$ |
|            | $\Delta^{-n_1+1}(\sigma_2^2\sigma_1\eta_1\sigma_1)\sigma_1\sigma_2(\sigma_2^2\gamma_1\sigma_2^3\sigma_1)$         | $\Delta^{-n_2+1}\sigma_2(\sigma_1^2\gamma_2\sigma_1^2)\sigma_2(\sigma_2^2\sigma_1(\eta_2)_{(1)}\sigma_1\sigma_2)\sigma_2$       |
|            | $\Delta^{-n_1+2}(\sigma_1\sigma_2(\eta_1)_{(1)}\sigma_2)(\sigma_2\sigma_1^3\gamma_1\sigma_1^2)$                   | $\Delta^{-n_2+1}(\sigma_1^2(\gamma_2)_{(1)}\sigma_1^2)\sigma_2^2(\sigma_2\sigma_1(\eta_2)_{(1)}\sigma_1\sigma_2^2)\sigma_2$     |
|            | $\Delta^{-n_1+2}(\sigma_1^2\sigma_2(\eta_1)_{(1)}\sigma_2)(\sigma_2\sigma_1^3\gamma_1\sigma_2^3\sigma_1)$         | $\Delta^{-n_2+2}(\sigma_2^2\gamma_2\sigma_1^2)\sigma_2(\sigma_2^2\sigma_1(\eta_2)_{(1)}\sigma_1\sigma_2^2)\sigma_2$             |
| $n_i$ even | $\Delta^{-n_1+1}(\sigma_1\sigma_2(\eta_1)_{(1)}\sigma_1)\sigma_1\sigma_2(\sigma_2^2\gamma_1\sigma_1^2)$           | $\Delta^{-n_2}\sigma_2(\sigma_1^2\gamma_2\sigma_2^2)\sigma_1^2(\sigma_1\sigma_2\eta_2\sigma_1\sigma_2)\sigma_2$                 |
|            | $\Delta^{-n_1+1}(\sigma_1^2\sigma_2(\eta_1)_{(1)}\sigma_1)\sigma_1\sigma_2(\sigma_2^2\gamma_1\sigma_2^3\sigma_1)$ | $\Delta^{-n_2+1}\sigma_1(\sigma_2^2(\gamma_2)_{(1)}\sigma_2^2)\sigma_1(\sigma_1^2\sigma_2\eta_2\sigma_1\sigma_2)\sigma_2$       |
|            | $\Delta^{-n_1+2}(\sigma_2\sigma_1\eta_1\sigma_2)(\sigma_2\sigma_1^3\gamma_1\sigma_1^2)$                           | $\Delta^{-n_2+1}(\sigma_2^2\gamma_2\sigma_2^2)\sigma_1^2(\sigma_1\sigma_2\eta_2\sigma_1\sigma_2^2)\sigma_2$                     |
|            | $\Delta^{-n_1+2}(\sigma_2^2\sigma_1\eta_1\sigma_2)(\sigma_2\sigma_1^3\gamma_1\sigma_2^3\sigma_1)$                 | $\Delta^{-n_2+2}(\sigma_1^2(\gamma_2)_{(1)}\sigma_2^2)\sigma_1(\sigma_1^2\sigma_2\eta_2\sigma_1\sigma_2^2)\sigma_2$             |

We now consider the case when there are braids that move only between the second singularity. Since  $\alpha_2$  is prime to  $\Delta$ , it cannot start with  $\sigma_2(\sigma_1\sigma_2^2\sigma_1)$ , implying we can only add  $\sigma_2^s$  or  $(\sigma_1\sigma_2^2\sigma_1)^s$  for  $s > 0$ . We then use the indivisibility condition again to determine which of these two braids can be placed in the beginning of our original  $\alpha_2$ 's. The substitution  $\gamma_2' = \sigma_1^2\gamma_2$  and  $\gamma_2'' = \sigma_1\sigma_2^3\gamma_2$  is used to shorten some of the braids below.

 TABLE 4.3. Modified  $\alpha_2$  and  $\tilde{\alpha}_2$ 

|            | $\alpha_2$                                                   | $(\tilde{\alpha}_2)_{(n_2)}$                           |
|------------|--------------------------------------------------------------|--------------------------------------------------------|
| $s = 1$    | $\sigma_2^s\sigma_1^2\gamma_2\sigma_2^2$                     | $\sigma_1\sigma_2\eta_2(\sigma_2^2)_{(n_2+1)}$         |
|            | $\sigma_2^s\sigma_1^2\gamma_2\sigma_1^3\sigma_2$             | $\sigma_1^2\sigma_2\eta_2(\sigma_2^2)_{(n_2+1)}$       |
| $s \geq 2$ | $\sigma_2^s\sigma_1^2\gamma_2\sigma_2^2$                     | $\sigma_1\sigma_2\eta_2(\sigma_1\sigma_2)_{(n_2+1)}$   |
|            | $\sigma_2^s\sigma_1^2\gamma_2\sigma_1^3\sigma_2$             | $\sigma_1^2\sigma_2\eta_2(\sigma_1\sigma_2)_{(n_2+1)}$ |
| $s \geq 1$ | $(\sigma_1\sigma_2^2\sigma_1)^s\gamma_2'\sigma_2^2$          | $\sigma_1\sigma_2\eta_2(\sigma_1^2)_{(n_2+1)}$         |
|            | $(\sigma_1\sigma_2^2\sigma_1)^s\gamma_2'\sigma_1^3\sigma_2$  | $\sigma_1^2\sigma_2\eta_2(\sigma_1^2)_{(n_2+1)}$       |
|            | $(\sigma_1\sigma_2^2\sigma_1)^s\gamma_2''\sigma_2^2$         | $\sigma_1\sigma_2\eta_2(\sigma_1^2)_{(n_2+1)}$         |
|            | $(\sigma_1\sigma_2^2\sigma_1)^s\gamma_2''\sigma_1^3\sigma_2$ | $\sigma_1^2\sigma_2\eta_2(\sigma_1^2)_{(n_2+1)}$       |

We plug these modified values into the right hand side of Equation 4.2. The braids in the table below are in Garside normal form by Theorem 2.1.1 and our assumption but observe not all of them end with  $\sigma_2$ . We need to show the remaining ones, when paired with the computed braids from earlier, do not make Equation 4.2 true. Let us look at the beginning of these positive braids. We have a generator raised to (at least) the third power, which does not occur in any of the expressions corresponding to  $\beta_1$ . We can thus conclude Equation 4.2 does not hold for these braids as well.

TABLE 4.4. Rewriting the right hand side of Equation 4.2 in Garside normal form

|            |                                                                                                                                                                            |            |
|------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
|            | $\sigma_2 \beta_2 \sigma_1^2 \beta_2^{-1} \sigma_2$                                                                                                                        |            |
| $n_2$ odd  | $\Delta^{-n_2} \sigma_1 (\sigma_1 \sigma_2^2 (\gamma_2)_{(1)} \sigma_1^2) \sigma_2^2 (\sigma_2 \sigma_1 (\eta_2)_{(1)} \sigma_1^2) \sigma_2$                               | $s = 1$    |
|            | $\Delta^{-n_2+1} \sigma_2 (\sigma_2 \sigma_1^2 \gamma_2 \sigma_1^2) \sigma_2 (\sigma_2^2 \sigma_1 (\eta_2)_{(1)} \sigma_1^2) \sigma_2$                                     |            |
|            | $\Delta^{-n_2+1} \sigma_2 (\sigma_2^s \sigma_1^2 \gamma_2 \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2)$                                                               | $s \geq 2$ |
|            | $\Delta^{-n_2+2} \sigma_1 (\sigma_1^s \sigma_2^2 (\gamma_2)_{(1)} \sigma_2^2) \sigma_1 (\sigma_1^2 \sigma_2 \eta_2)$                                                       |            |
|            | $\Delta^{-n_2+1} (\sigma_1 \sigma_2 (\sigma_2 \sigma_1^2 \sigma_2)^{s-1} (\gamma_2')_{(1)} \sigma_1^2) \sigma_2^2 (\sigma_2 \sigma_1 (\eta_2)_{(1)} \sigma_2^2) \sigma_2$  | $s \geq 1$ |
|            | $\Delta^{-n_2+2} (\sigma_2 \sigma_1 (\sigma_1 \sigma_2^2 \sigma_1)^{s-1} \gamma_2' \sigma_1^2) \sigma_2 (\sigma_2^2 \sigma_1 (\eta_2)_{(1)} \sigma_2^2) \sigma_2$          |            |
|            | $\Delta^{-n_2+1} (\sigma_1 \sigma_2 (\sigma_2 \sigma_1^2 \sigma_2)^{s-1} (\gamma_2'')_{(1)} \sigma_1^2) \sigma_2^2 (\sigma_2 \sigma_1 (\eta_2)_{(1)} \sigma_2^2) \sigma_2$ |            |
|            | $\Delta^{-n_2+2} (\sigma_2 \sigma_1 (\sigma_1 \sigma_2^2 \sigma_1)^{s-1} \gamma_2'' \sigma_1^2) \sigma_2 (\sigma_2^2 \sigma_1 (\eta_2)_{(1)} \sigma_2^2) \sigma_2$         |            |
| $n_2$ even | $\Delta^{-n_2} \sigma_2 (\sigma_2 \sigma_1^2 \gamma_2 \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2 \sigma_1^2) \sigma_2$                                               | $s = 1$    |
|            | $\Delta^{-n_2+1} \sigma_1 (\sigma_1 \sigma_2^2 (\gamma_2)_{(1)} \sigma_2^2) \sigma_1 (\sigma_1^2 \sigma_2 \eta_2 \sigma_1^2) \sigma_2$                                     |            |
|            | $\Delta^{-n_2+1} \sigma_1 (\sigma_1^s \sigma_2^2 (\gamma_2)_{(1)} \sigma_1^2) \sigma_2^2 (\sigma_2 \sigma_1 (\eta_2)_{(1)})$                                               | $s \geq 2$ |
|            | $\Delta^{-n_2+2} \sigma_2 (\sigma_2^s \sigma_1^2 \gamma_2 \sigma_2^2) \sigma_2 (\sigma_2^2 \sigma_1 (\eta_2)_{(1)})$                                                       |            |
|            | $\Delta^{-n_2+1} (\sigma_2 \sigma_1 (\sigma_1 \sigma_2^2 \sigma_1)^{s-1} \gamma_2' \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2 \sigma_2^2) \sigma_2$                  | $s \geq 1$ |
|            | $\Delta^{-n_2+2} (\sigma_1 \sigma_2 (\sigma_2 \sigma_1^2 \sigma_2)^{s-1} (\gamma_2')_{(1)} \sigma_2^2) \sigma_1 (\sigma_1^2 \sigma_2 \eta_2 \sigma_2^2) \sigma_2$          |            |
|            | $\Delta^{-n_2+1} (\sigma_2 \sigma_1 (\sigma_1 \sigma_2^2 \sigma_1)^{s-1} \gamma_2'' \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2 \sigma_2^2) \sigma_2$                 |            |
|            | $\Delta^{-n_2+2} (\sigma_1 \sigma_2 (\sigma_2 \sigma_1^2 \sigma_2)^{s-1} (\gamma_2'')_{(1)} \sigma_2^2) \sigma_1 (\sigma_1^2 \sigma_2 \eta_2 \sigma_2^2) \sigma_2$         |            |

We look at one example in more detail. Let us look at the third braid when  $n_2$  is odd:

$$\Delta^{-n_2+1} \sigma_2 (\sigma_2^s \sigma_1^2 \gamma_2 \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2).$$

We have  $s \geq 2$  so

$$\sigma_2 (\sigma_2^s \sigma_1^2 \gamma_2 \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2) = (\underbrace{\sigma_2^t}_{t \geq 3} \sigma_1^2 \gamma_2 \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 \eta_2).$$

If we look at the entries on the left side of Table 4.2, then we observe none of the positive braids start with  $\sigma_2^t$  where  $t \geq 3$ .

Case 2:  $k_1$  and  $k_2$  are odd

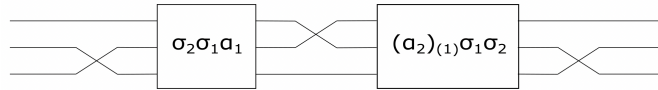


FIGURE 4.4. Braided wiring diagram for Case 2

We modify the values for  $\alpha_i$  and  $\tilde{\alpha}_i$  from Case 1. We substitute them into Equation 4.2 and write the expression in Garside normal form.

TABLE 4.5.  $\alpha_i$  and  $\tilde{\alpha}_i$ 

| $\alpha_1$                                         | $(\tilde{\alpha}_1)_{(n_1)}$                                 | $(\alpha_2)_{(1)}$                                 | $(\tilde{\alpha}_2)_{(n_2)}$                                 |
|----------------------------------------------------|--------------------------------------------------------------|----------------------------------------------------|--------------------------------------------------------------|
| $\sigma_1^2 \gamma_1 \sigma_1^2$                   | $\sigma_1 \sigma_2 \eta_1 (\sigma_1 \sigma_2)_{(n_1+1)}$     | $\sigma_1^2 \gamma_2 \sigma_1^2$                   | $\sigma_2 \sigma_1 \eta_2 (\sigma_2 \sigma_1)_{(n_2+1)}$     |
| $\sigma_1^2 \gamma_1 \sigma_2^3 \sigma_1$          | $\sigma_1^2 \sigma_2 \eta_1 (\sigma_1 \sigma_2)_{(n_1+1)}$   | $\sigma_1^2 \gamma_2 \sigma_2^3 \sigma_1$          | $\sigma_2^2 \sigma_1 \eta_2 (\sigma_2 \sigma_1)_{(n_2+1)}$   |
| $\sigma_1 \sigma_2^3 \gamma_1 \sigma_1^2$          | $\sigma_1 \sigma_2 \eta_1 (\sigma_1 \sigma_2^2)_{(n_1+1)}$   | $\sigma_1 \sigma_2^3 \gamma_2 \sigma_1^2$          | $\sigma_2 \sigma_1 \eta_2 (\sigma_2 \sigma_1^2)_{(n_2+1)}$   |
| $\sigma_1 \sigma_2^3 \gamma_1 \sigma_2^3 \sigma_1$ | $\sigma_1^2 \sigma_2 \eta_1 (\sigma_1 \sigma_2^2)_{(n_1+1)}$ | $\sigma_1 \sigma_2^3 \gamma_2 \sigma_2^3 \sigma_1$ | $\sigma_2^2 \sigma_1 \eta_2 (\sigma_2 \sigma_1^2)_{(n_2+1)}$ |

TABLE 4.6. Writing both sides of Equation 4.2 in Garside normal form

|            | $\beta_1^{-1} \sigma_2 \sigma_1^2 \sigma_2 \beta_1$                                                                         | $\sigma_2 \beta_2 \sigma_1^2 \beta_2^{-1} \sigma_2$                                                                                        |
|------------|-----------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|
| $n_i$ odd  | $\Delta^{-n_1+1} (\sigma_2 \sigma_1 (\eta_1)_{(1)} \sigma_2) \sigma_2 \sigma_1 (\sigma_1^2 \gamma_1 \sigma_1^2)$            | $\Delta^{-n_2} \sigma_1 (\sigma_2^2 (\gamma_2)_{(1)} \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2 (\eta_2)_{(1)} \sigma_1 \sigma_2) \sigma_2$ |
|            | $\Delta^{-n_1+1} (\sigma_2^2 \sigma_1 (\eta_1)_{(1)} \sigma_2) \sigma_2 \sigma_1 (\sigma_1^2 \gamma_1 \sigma_2^3 \sigma_1)$ | $\Delta^{-n_2+1} \sigma_2 (\sigma_1^2 \gamma_2 \sigma_2^2) \sigma_1 (\sigma_1^2 \sigma_2 (\eta_2)_{(1)} \sigma_1 \sigma_2) \sigma_2$       |
|            | $\Delta^{n_1+2} (\sigma_1 \sigma_2 \eta_1 \sigma_1) (\sigma_1 \sigma_2^3 \gamma_1 \sigma_1^2)$                              | $\Delta^{-n_2+1} (\sigma_1^2 (\gamma_2)_{(1)} \sigma_2^2) \sigma_1^2 (\sigma_1 \sigma_2) (\eta_2)_{(1)} \sigma_1 \sigma_2^2 \sigma_2$      |
|            | $\Delta^{-n_1+2} (\sigma_1^2 \sigma_2 \eta_1 \sigma_1) (\sigma_1 \sigma_2^3 \gamma_1 \sigma_2^3 \sigma_1)$                  | $\Delta^{-n_2+2} (\sigma_2^2 \gamma_2 \sigma_2^2) \sigma_1 (\sigma_1^2 \sigma_2 (\eta_2)_{(1)} \sigma_1 \sigma_2^2) \sigma_2$              |
| $n_i$ even | $\Delta^{-n_1+1} (\sigma_1 \sigma_2 \eta_1 \sigma_2) (\sigma_2 \sigma_1) (\sigma_1^2 \gamma_1 \sigma_1^2)$                  | $\Delta^{-n_2} \sigma_2 (\sigma_1^2 \gamma_2 \sigma_1^2) \sigma_2^2 (\sigma_2 \sigma_1 \eta_2 \sigma_1 \sigma_2) \sigma_2$                 |
|            | $\Delta^{-n_1+1} (\sigma_1^2 \sigma_2 \eta_1 \sigma_2) \sigma_2 \sigma_1 (\sigma_1^2 \gamma_1 \sigma_2^3 \sigma_1)$         | $\Delta^{-n_2+1} \sigma_1 (\sigma_2^2 (\gamma_2)_{(1)} \sigma_1^2) \sigma_2 (\sigma_2^2 \sigma_1 \eta_2 \sigma_1 \sigma_2) \sigma_2$       |
|            | $\Delta^{-n_1+2} (\sigma_2 \sigma_1 (\eta_1)_{(1)} \sigma_1) (\sigma_1 \sigma_2^3 \gamma_1 \sigma_1^2)$                     | $\Delta^{-n_2+1} (\sigma_2^2 \gamma_2 \sigma_1^2) \sigma_2^2 (\sigma_2 \sigma_1 \eta_2 \sigma_1 \sigma_2^2) \sigma_2$                      |
|            | $\Delta^{-n_1+2} (\sigma_2^2 \sigma_1 (\eta_1)_{(1)} \sigma_1) (\sigma_1 \sigma_2^3 \gamma_1 \sigma_2^3 \sigma_1)$          | $\Delta^{-n_2+2} (\sigma_1^2 (\gamma_2)_{(1)} \sigma_1^2) \sigma_1 (\sigma_2^2 \sigma_1 \eta_2 \sigma_1 \sigma_2^2) \sigma_2$              |

As in Case 1, these braids cannot be equivalent. We also include the braids that can pass through the second singularity to the beginning of  $\alpha_2$ . Let  $\gamma_2' = \sigma_1^2 \gamma_2$  and  $\gamma_2'' = \sigma_1 \sigma_2^3 \gamma_2$ .

TABLE 4.7. Modified  $\alpha_2$  and  $\tilde{\alpha}_2$ 

|            | $(\alpha_2)_{(1)}$                                                | $(\tilde{\alpha}_2)_{(n_2)}$                               |
|------------|-------------------------------------------------------------------|------------------------------------------------------------|
| $s = 1$    | $\sigma_2^s \sigma_1^2 \gamma_2 \sigma_1^2$                       | $\sigma_2 \sigma_1 \eta_2 (\sigma_2^2)_{(n_2+1)}$          |
|            | $\sigma_2^s \sigma_1^2 \gamma_2 \sigma_2^3 \sigma_1$              | $\sigma_2^2 \sigma_1 \eta_2 (\sigma_2^2)_{(n_2+1)}$        |
| $s \geq 2$ | $\sigma_2^s \sigma_1^2 \gamma_2 \sigma_1^2$                       | $\sigma_2 \sigma_1 \eta_2 (\sigma_1 \sigma_2)_{(n_2+1)}$   |
|            | $\sigma_2^s \sigma_1^2 \gamma_2 \sigma_2^3 \sigma_1$              | $\sigma_2^2 \sigma_1 \eta_2 (\sigma_1 \sigma_2)_{(n_2+1)}$ |
| $s \geq 1$ | $(\sigma_1 \sigma_2^2 \sigma_1)^s \gamma_2' \sigma_1^2$           | $\sigma_2 \sigma_1 \eta_2 (\sigma_1^2)_{(n_2+1)}$          |
|            | $(\sigma_1 \sigma_2^2 \sigma_1)^s \gamma_2' \sigma_2^3 \sigma_1$  | $\sigma_2^2 \sigma_1 \eta_2 (\sigma_1^2)_{(n_2+1)}$        |
|            | $(\sigma_1 \sigma_2^2 \sigma_1)^s \gamma_2'' \sigma_1^2$          | $\sigma_2 \sigma_1 \eta_2 (\sigma_1^2)_{(n_2+1)}$          |
|            | $(\sigma_1 \sigma_2^2 \sigma_1)^s \gamma_2'' \sigma_2^3 \sigma_1$ | $\sigma_2^2 \sigma_1 \eta_2 (\sigma_1^2)_{(n_2+1)}$        |

We get the following table for the right hand side of Equation 4.2. For the braids that do not end with  $\sigma_2$ , we use the argument about the exponent of the first term to show Equation 4.2 is not true.

TABLE 4.8. Rewriting the right hand side of Equation 4.2 in Garside normal form

|            | $\sigma_2\beta_2\sigma_1^2\beta_2^{-1}\sigma_2$                                                                                                               |            |
|------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| $n_2$ odd  | $\Delta^{-n_2}\sigma_1(\sigma_1\sigma_2^2(\gamma_2)_{(1)}\sigma_2^2)\sigma_1^2(\sigma_1\sigma_2(\eta_2)_{(1)}\sigma_1^2)\sigma_2$                             | $s = 1$    |
|            | $\Delta^{-n_2+1}\sigma_2(\sigma_2\sigma_1^2\gamma_2\sigma_2^2)\sigma_1(\sigma_1^2\sigma_2(\eta_2)_{(1)}\sigma_1^2)\sigma_2$                                   |            |
|            | $\Delta^{-n_2+1}\sigma_2(\sigma_2^s\sigma_1^2\gamma_2\sigma_1^2)\sigma_2^2(\sigma_2\sigma_1\eta_2)$                                                           | $s \geq 2$ |
|            | $\Delta^{-n_2+2}\sigma_1(\sigma_1^s\sigma_2^2(\gamma_2)_{(1)}\sigma_1^2)\sigma_2(\sigma_2^2\sigma_1\eta_2)$                                                   |            |
|            | $\Delta^{-n_2+1}(\sigma_1\sigma_2(\sigma_2\sigma_1^2\sigma_2)^{s-1}(\gamma_2')_{(1)}\sigma_2^2)\sigma_1^2(\sigma_1\sigma_2(\eta_2)_{(1)}\sigma_2^2)\sigma_2$  | $s \geq 1$ |
|            | $\Delta^{-n_2+2}(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^{s-1}\gamma_2'\sigma_2^2)\sigma_1(\sigma_1^2\sigma_2(\eta_2)_{(1)}\sigma_2^2)\sigma_2$          |            |
|            | $\Delta^{-n_2+1}(\sigma_1\sigma_2(\sigma_2\sigma_1^2\sigma_2)^{s-1}(\gamma_2'')_{(1)}\sigma_2^2)\sigma_1^2(\sigma_1\sigma_2(\eta_2)_{(1)}\sigma_2^2)\sigma_2$ |            |
|            | $\Delta^{-n_2+2}(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^{s-1}\gamma_2''\sigma_2^2)\sigma_1(\sigma_1^2\sigma_2(\eta_2)_{(1)}\sigma_2^2)\sigma_2$         |            |
| $n_2$ even | $\Delta^{-n_2}\sigma_2(\sigma_2\sigma_1^2\gamma_2\sigma_2^2)\sigma_2^2(\sigma_2\sigma_1\eta_2\sigma_1^2)\sigma_2$                                             | $s = 1$    |
|            | $\Delta^{-n_2}\sigma_2(\sigma_2\sigma_1^2\gamma_2\sigma_1^3\sigma_2)\sigma_2^2(\sigma_2^2\sigma_1\eta_2\sigma_1^2)\sigma_2$                                   |            |
|            | $\Delta^{-n_2+1}\sigma_1(\sigma_1^s\sigma_2^2(\gamma_2)_{(1)}\sigma_1^2)\sigma_1^2(\sigma_1\sigma_2(\eta_2)_{(1)})$                                           | $s \geq 2$ |
|            | $\Delta^{-n_2+1}\sigma_1(\sigma_1^s\sigma_2^2(\gamma_2)_{(1)}\sigma_2^3\sigma_1)\sigma_1^2(\sigma_1^2\sigma_2(\eta_2)_{(1)})$                                 |            |
|            | $\Delta^{-n_2+1}(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^{s-1}\gamma_2'\sigma_2^2)\sigma_2^2(\sigma_2\sigma_1\eta_2\sigma_2^2)\sigma_2$                  | $s \geq 1$ |
|            | $\Delta^{-n_2+1}(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^{s-1}\gamma_2'\sigma_1^3\sigma_2)\sigma_2^2(\sigma_2^2\sigma_1\eta_2\sigma_2^2)\sigma_2$        |            |
|            | $\Delta^{-n_2+1}(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^{s-1}\gamma_2''\sigma_2^2)\sigma_2^2(\sigma_2\sigma_1\eta_2\sigma_2^2)\sigma_2$                 |            |
|            | $\Delta^{-n_2+1}(\sigma_2\sigma_1(\sigma_1\sigma_2^2\sigma_1)^{s-1}\gamma_2''\sigma_1^3\sigma_2)\sigma_2^2(\sigma_2^2\sigma_1\eta_2\sigma_2^2)\sigma_2$       |            |

Case 3:  $k_1$  is even and  $k_2$  is odd

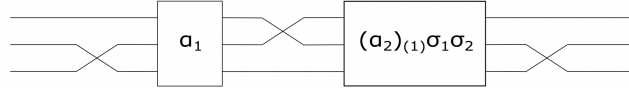


FIGURE 4.5. Braided wiring diagram for Case 3

The necessary calculations were completed in the earlier two cases so we refer to them for the rest of the proof. The values for  $\alpha_1$ ,  $\tilde{\alpha}_1$  and  $\alpha_2$ ,  $\tilde{\alpha}_2$  are from Case 1 and Case 2, respectively, and we see Equation 4.2 does not hold.

Case 4:  $k_1$  is odd and  $k_2$  is even

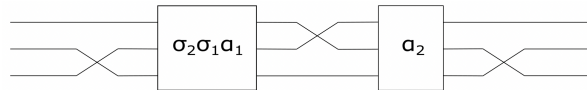


FIGURE 4.6. Braided wiring diagram for Case 4

The values for  $\alpha_1, \tilde{\alpha}_1$  and  $\alpha_2, \tilde{\alpha}_2$  are obtained from Case 2 and Case 1, respectively. We again have that none of the pairs make Equation 4.2 true.

□

PROOF OF THEOREM 1.2.2. Suppose the total monodromy of  $\mathcal{W}$  is equal to the full twist on three strands. Combining Theorem 4.1.1 and Lemma 4.1.0.1 (in this order) gives us that  $\mathcal{W}$  and  $\mathcal{W}_0$  are Markov equivalent. Now assume  $\mathcal{W}$  and  $\mathcal{W}_0$  are related by Markov moves. Then, Theorem 1.2.1 tells us the braid monodromy of  $\mathcal{W}$  is the full twist.

□

Although we are working in one of the simplest cases ( $B_3$  with total monodromy  $\Delta^2$ ), the proof became quite cumbersome. As we increase the number of singularities and braids to a wiring diagram, this proof technique will not be efficient. The hope is that this result can be used to rule out possibilities for  $\beta_i$  in arrangements that are slightly more complicated than  $\mathcal{W}$ .

#### 4.2. A different way to prove Theorem 1.2.2

Before looking at other arrangements, we provide an easier way to prove Theorem 1.2.2 using the isomorphism  $B_3/Z \cong PSL(2, \mathbb{Z})$ . This allows us to express elements in  $B_3$  as matrices and we use facts about conjugate matrices to solve the problem. The isomorphism is constructed using the surjection  $\phi : B_3 \rightarrow PSL(2, \mathbb{Z})$  defined by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

As we will see later, we need to know how the generators of the centralizers of  $\sigma_i$  behave under the map  $\phi$ . Recall the centralizer of  $\sigma_i$  is an abelian group generated by  $\sigma_i$  and  $\sigma_{i'}\sigma_i^2\sigma_{i'}$ . Note

$$\sigma_2\sigma_1^2\sigma_2 \mapsto \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1\sigma_2^2\sigma_1 \mapsto \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}.$$

LEMMA 4.2.0.1. *Let  $n \in \mathbb{Z}$ . Then,*

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^n &= \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}^n &= \begin{pmatrix} (-1)^n & (-1)^{n+1}2n \\ 0 & (-1)^n \end{pmatrix}, & \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}^n &= \begin{pmatrix} (-1)^n & 0 \\ (-1)^n 2n & (-1)^n \end{pmatrix}. \end{aligned}$$

PROOF. We show the proof for the first equality as the process is the same for the rest. Let  $n \geq 0$ . If  $n = 0$ , then we have the identity matrix and the equality holds. Suppose the equality holds for  $n = k$ . Consider  $n = k + 1$ .

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+k \\ 0 & 1 \end{pmatrix}$$

If  $n < 0$ , then we first compute

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We rewrite the expression so the exponent becomes positive and use induction like before. Let  $n = -m$ . We want to show

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^m = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

When  $m = 1$ , we observe the above is true so suppose it holds for  $m = k$ .

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1-k \\ 0 & 1 \end{pmatrix}$$

□

We now proceed with the proof of the theorem.

ALTERNATE PROOF OF THEOREM 1.2.2. We already know the backwards direction is true so assume the braid monodromy of  $\mathcal{W}$  is the full twist.

$$\sigma_1 \beta_1 \sigma_2 \beta_2 \sigma_1^2 \beta_2^{-1} \sigma_2 \beta_1^{-1} \sigma_1 = \Delta^2$$

We rewrite this equation to obtain

$$(4.3) \quad \beta_1^{-1} \sigma_2 \sigma_1^2 \sigma_2 \beta_1 = \sigma_2 \beta_2 \sigma_1^2 \beta_2^{-1} \sigma_2.$$

We claim the equation being true implies  $\beta_1 = ab$  and  $\beta_2 = cd$  where  $a, d \in C_{B_3}(\sigma_1)$ ,  $b, c \in C_{B_3}(\sigma_2)$ , and  $bc \in Z$ . We first determine the values for  $\beta_2$  that will make the right hand side of the equation conjugate to  $\sigma_2 \sigma_1^2 \sigma_2$ . Set

$$\phi(\beta_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $ad - bc = 1$ . Then,

$$\sigma_2 \sigma_1^2 \sigma_2 \mapsto \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = A$$

and

$$\sigma_2 \beta_2 \sigma_1^2 \beta_2^{-1} \sigma_2 \mapsto \begin{pmatrix} 1 - 2ac - 2a^2 & 2a^2 \\ 2a^2 - 2c^2 - 2 & -2a^2 + 2ac + 1 \end{pmatrix} = B.$$

Conjugate matrices have the same trace so we set the traces of  $A$  and  $B$  equal to each other. However, since matrices  $M$  in  $PSL(2, \mathbb{Z})$  have the identification  $M = -M$ , we obtain two equalities:  $-2 = 2 - 4a^2$  and  $2 = 2 - 4a^2$ . This gives us  $a = \pm 1$  and  $a = 0$ . We claim when  $a = 0$ , an appropriate matrix representation for  $\beta_2$  that satisfies Equation 4.3 does not exist. If  $a = 0$ , then

$$\phi(\beta_2) = \left\{ \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \right\}.$$

Since the determinant of this matrix is 1, this implies  $-bc = 1$ . The only integer solutions are  $b = \pm 1$  and  $c = \mp 1$ . Then,

$$B = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}.$$

Set

$$\beta_1 \mapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix} = C$$

to get

$$CAC^{-1} = \begin{pmatrix} -1 - 2xz & 2x^2 \\ -2z^2 & -1 + 2xz \end{pmatrix} = B.$$

The bottom left entry says  $-2z^2 = -4$  (up to multiplication by  $-1$ ). Since an integral solution for  $z$  does not exist, our claim is proved. We now consider the case when  $a = \pm 1$ . Set

$$\mathcal{H} = \left\{ \begin{pmatrix} \pm 1 & b \\ c & d \end{pmatrix} \mid \pm d - bc = 1 \right\}.$$

We claim

$$\mathcal{G} = \phi(C_{B_3}(\sigma_2)C_{B_3}(\sigma_1)) = \left\{ \begin{pmatrix} \pm 1 & b \\ c & d \end{pmatrix} \mid b, c, d \in \mathbb{Z} \right\}.$$

By Lemma 4.2.0.1, we know

$$\begin{aligned} \sigma_1^{n_1} &\mapsto \begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix}, & (\sigma_2 \sigma_1^2 \sigma_2)^{m_1} &\mapsto \begin{pmatrix} (-1)^{m_1} & (-1)^{m_1+1} 2m_1 \\ 0 & (-1)^{m_1} \end{pmatrix}, \\ \sigma_2^{n_2} &\mapsto \begin{pmatrix} 1 & 0 \\ -n_2 & 1 \end{pmatrix}, & (\sigma_1 \sigma_2^2 \sigma_1)^{m_2} &\mapsto \begin{pmatrix} (-1)^{m_2} & 0 \\ (-1)^{m_2} 2m_2 & (-1)^{m_2} \end{pmatrix}. \end{aligned}$$

Multiplying these matrices appropriately gives

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ -n_2 & 1 \end{pmatrix} \begin{pmatrix} (-1)^{m_2} & 0 \\ (-1)^{m_2} 2m_2 & (-1)^{m_2} \end{pmatrix} \begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (-1)^{m_1} & (-1)^{m_1+1} 2m_1 \\ 0 & (-1)^{m_1} \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{m_2} & 0 \\ (-1)^{m_2}(-n_2) + (-1)^{m_2} 2m_2 & (-1)^{m_2} \end{pmatrix} \begin{pmatrix} (-1)^{m_1} & (-1)^{m_1+1} 2m_1 + (-1)^{m_1} n_1 \\ 0 & (-1)^{m_1} \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{m_1+m_2} & (-1)^{m_1+m_2}(-2m_1 + n_1) \\ (-1)^{m_1+m_2}(-n_2 + 2m_2) & (-1)^{m_1+m_2}(2m_1 n_2 - n_1 n_2 - 4m_1 m_2 + 2m_2 n_1 + 1) \end{pmatrix}. \end{aligned}$$

We conclude from this calculation that  $\mathcal{G} \subset \mathcal{H}$ . To get the other inclusion, we find  $m_1, m_2, n_1, n_2$  such that the above matrix realizes a given  $\begin{pmatrix} \pm 1 & b \\ c & d \end{pmatrix}$ . If the first entry is 1 and we are given  $b$  and  $c$ , then set  $m_1 = 0 = m_2, n_1 = b$ , and  $n_2 = -c$ . On the other hand, if the first entry is  $-1$ , then set  $m_1 = 0, m_2 = 1, n_1 = -b$ , and  $n_2 = c + 2$ . This allows us to find the corresponding braid in  $C_{B_3}(\sigma_2)C_{B_3}(\sigma_1)$  given an arbitrary matrix of this form.

Although  $PSL(2, \mathbb{Z})$  is only isomorphic to a quotient group of  $B_3$ , these calculations are enough as information is not lost if we mod out by the center  $Z(B_3)$ . If  $\beta_2$  is a solution to Equation 4.3 for some  $\beta_1$  and  $z \in Z(B_3)$ , then  $\beta_2' = \beta_2 z$  is also a solution to the equation for the same  $\beta_1$ . Additionally,

if  $\beta_2 \in C_{B_3}(\sigma_2)C_{B_3}(\sigma_1)$ , then  $\beta_2 z \in C_{B_3}(\sigma_2)C_{B_3}(\sigma_1)$ . Hence,  $\phi(\beta_2) \in \phi(C_{B_3}(\sigma_2)C_{B_3}(\sigma_1))$  implies  $\beta_2 \in C_{B_3}(\sigma_2)C_{B_3}(\sigma_1)$ . We now have that if the right hand side of Equation 4.3 is conjugate to  $\sigma_2\sigma_1^2\sigma_2$ , then  $\beta_2 = cd$  where  $c \in C_{B_3}(\sigma_2)$  and  $d \in C_{B_3}(\sigma_1)$ . This allows us to directly solve for  $\beta_1$ .

$$\beta_1^{-1}\sigma_2\sigma_1^2\sigma_2\beta_1 = c\sigma_2\sigma_1^2\sigma_2c^{-1}$$

We deduce  $\beta_1 \cdot c \in C_{B_3}(\sigma_1) = C_{B_3}(\sigma_2\sigma_1^2\sigma_2)$  where the equality follows from Theorem 4 in [Gur84]. Written differently,  $\beta_1 = ac^{-1}$  where  $a \in C_{B_3}(\sigma_1)$ . Thus, if Equation 4.3 is true, then  $\beta_1 = ab$  and  $\beta_2 = cd$  where these variables are defined in the same manner as earlier. By Markov moves 2 and 5, this braided wiring diagram is equivalent to the standard diagram.  $\square$

### 4.3. Applications of Theorem 1.2.2

In this section, we introduce additional configurations to apply our theorem. Suppose we are given a braided wiring diagram consisting of one triple point and three double points with total monodromy  $\Delta^4$ . We determine  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  using our results. Consider the wiring diagram where the first singularity is a triple point.

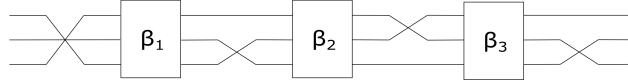


FIGURE 4.7. One triple point and three double points

The total monodromy equation corresponding to this diagram is

$$\sigma_1\sigma_2\sigma_1\beta_1\sigma_1\beta_2\sigma_2\beta_3\sigma_1^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_1\beta_1^{-1}\sigma_1\sigma_2\sigma_1 = \Delta^4.$$

By rearranging this equation, we are left with

$$\sigma_1\beta_2\sigma_2\beta_3\sigma_1^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_1 = \Delta^2$$

where  $\beta_1$  is arbitrary. We can then use Theorem 4.1.1 to say  $\beta_2 = ab$ , and  $\beta_3 = cd$  where  $a, d \in C_{B_3}(\sigma_1)$ ,  $b, c \in C_{B_3}(\sigma_2)$ , and  $bc \in Z(B_3)$ .

**COROLLARY 4.3.1.** *The total braid monodromy of the braided wiring diagram in Figure 4.7 is  $\Delta^4$  if and only if it is Markov equivalent to the standard wiring diagram.*

We can keep adding an arbitrary number of triple points to the above wiring diagram.

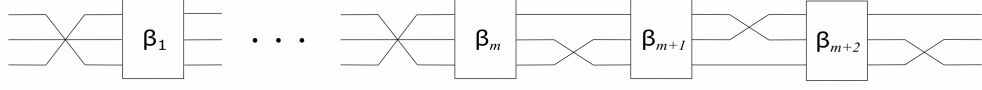


FIGURE 4.8.  $m$  triple points and three double points

The total monodromy equation is

$$\underbrace{\sigma_1 \sigma_2 \sigma_1 \beta_1 \cdots \sigma_1 \sigma_2 \sigma_1}_{m \text{ times}} \beta_m \sigma_1 \beta_{m+1} \sigma_2 \beta_{m+2} \sigma_1^2 \beta_{m+2}^{-1} \sigma_2 \beta_{m+1}^{-1} \sigma_1 \beta_m^{-1} \underbrace{\sigma_1 \sigma_2 \sigma_1 \cdots \beta_1^{-1} \sigma_1 \sigma_2 \sigma_1}_{m \text{ times}} = \Delta^{2(m+1)}.$$

If we solve this equation, then  $\beta_i \in B_3$  for  $1 \leq i \leq m$ ,  $\beta_{m+1} = ab$ , and  $\beta_{m+2} = cd$ . Therefore, Corollary 4.3.1 can be generalized as follows.

COROLLARY 4.3.2 (Corollary 1.2.1). *The total braid monodromy of the braided wiring diagram in Figure 4.8 is  $\Delta^{2(m+1)}$  if and only if it is Markov equivalent to the standard wiring diagram.*

Now let us examine a braided wiring diagram with four double points having total monodromy  $\Delta^2 \sigma_2^2$ .

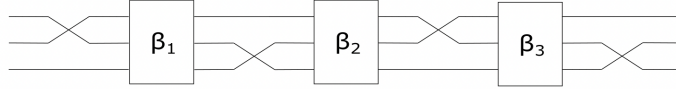


FIGURE 4.9. Four double points

We have

$$\sigma_2 \beta_1 \sigma_1 \beta_2 \sigma_2 \beta_3 \sigma_1^2 \beta_3^{-1} \sigma_2 \beta_2^{-1} \sigma_1 \beta_1^{-1} \sigma_2 = \Delta^2 \sigma_2^2$$

and after making some cancellations, we obtain an equation we can easily work with:

$$\sigma_1 \beta_2 \sigma_2 \beta_3 \sigma_1^2 \beta_3^{-1} \sigma_2 \beta_2^{-1} \sigma_1 = \Delta^2.$$

We arrive at the same constraints as the previous arrangement where  $\beta_1$  is arbitrary,  $\beta_2 = ab$ , and  $\beta_3 = cd$ .

COROLLARY 4.3.3 (Corollary 1.2.2). *The total braid monodromy of the braided wiring diagram in Figure 4.9 is  $\Delta^2 \sigma_2^2$  if and only if it is related by Markov moves to the standard wiring diagram.*

#### 4.4. More braided wiring diagrams using $B_3/Z \cong PSL(2, \mathbb{Z})$

We continue with examples of more arrangements and obtain classification results using the proof technique involving matrices. Suppose we have a braided wiring diagram with  $n$  double points and one triple point as in Figure 1.4.

**THEOREM 4.4.1** (Theorem 1.2.3). *The total braid monodromy of the braided wiring diagram when  $1 \leq n \leq 3$  is  $\Delta^2 \sigma_2^{2n}$  if and only if it is related by Markov moves to the standard wiring diagram.*

For these results, we note the backward direction follows by Theorem 1.2.1.

**PROOF OF THEOREM 1.2.3 FOR  $n = 1$ .** The total braid monodromy is  $\Delta^2 \sigma_2^2$ .

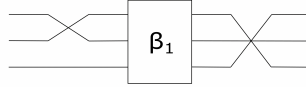


FIGURE 4.10. One double point and one triple point

This case is simple as we can directly solve for  $\beta_1$ .

$$\sigma_2 \beta_1 \Delta^2 \beta_1^{-1} \sigma_2 = \Delta^2 \sigma_2^2 \quad \Leftrightarrow \quad \beta_1 \in B_3$$

The wiring diagram is Markov equivalent to the standard diagram by Markov moves 5d and 2.  $\square$

**PROOF OF THEOREM 1.2.3 FOR  $n = 2$ .** The total monodromy is  $\Delta^2 \sigma_2^4$ , and we can also directly solve for the braids.

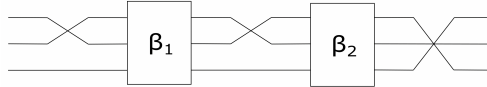


FIGURE 4.11. Two double points and one triple point

$$\sigma_2 \beta_1 \sigma_2 \beta_2 \Delta^2 \beta_2^{-1} \sigma_2 \beta_1^{-1} \sigma_2 = \Delta^2 \sigma_2^4 \quad \Leftrightarrow \quad \beta_1 \in C_{B_3}(\sigma_2), \beta_2 \in B_3$$

We explain how this equation was reduced. Since  $\Delta^2$  commutes with any braid,  $\beta_2, \beta_2^{-1}$  cancel and we obtain  $\beta_1 \sigma_2^2 \beta_1^{-1} = \sigma_2^2$ . Markov moves 5 and 2 can be used to show Markov equivalence.  $\square$

PROOF OF THEOREM 1.2.3 FOR  $n = 3$ . If  $n = 3$  (Figure 1.4), then the proof becomes more interesting as we cannot solve for all of the braids.

$$(4.4) \quad \sigma_2 \beta_1 \sigma_2 \beta_2 \sigma_2 \beta_3 \Delta^2 \beta_3^{-1} \sigma_2 \beta_2^{-1} \sigma_2 \beta_1^{-1} \sigma_2 = \Delta^2 \sigma_2^6 \Rightarrow \sigma_2 \beta_2 \sigma_2^2 \beta_2^{-1} \sigma_2 = \beta_1^{-1} \sigma_2^4 \beta_1$$

In particular,  $\beta_3$  can be any braid so we focus on  $\beta_1$  and  $\beta_2$ . We claim this equation being true implies  $\beta_1, \beta_2 \in C_{B_3}(\sigma_2)$ . We first determine the values for  $\beta_2$  that will make the left hand side of the equation conjugate to  $\sigma_2^4$ . Recall the surjection  $\phi : B_3 \rightarrow PSL(2, \mathbb{Z})$  from Section 4.2. Set

$$\phi(\beta_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $ad - bc = 1$ . In addition, we have

$$\sigma_2^4 \mapsto \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} = A$$

and

$$\sigma_2 \beta_2 \sigma_2^2 \beta_2^{-1} \sigma_2 \mapsto \begin{pmatrix} 1 - 2bd - 2b^2 & 2b^2 \\ 2b^2 - 2d^2 - 2 & 1 + 2bd - 2b^2 \end{pmatrix} = B.$$

We use the same argument about traces to prove this result. By setting the traces equal, we get  $b = 0$  and  $b = \pm 1$ . We claim when  $b = \pm 1$ , an appropriate matrix representation for  $\beta_2$  satisfying Equation 4.4 does not exist. When we plug in  $b = \pm 1$ , we have

$$B = \begin{pmatrix} -1 \mp 2d & 2 \\ -2d^2 & -1 \pm 2d \end{pmatrix}.$$

If

$$\beta_1 \mapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix} = C,$$

then

$$CAC^{-1} = \begin{pmatrix} 1 - 4yw & 4y^2 \\ -4w^2 & 1 + 4yw \end{pmatrix} = B.$$

We conclude  $4y^2 = 2$  (up to multiplication by  $-1$ ) has no integral solution. Now we consider when  $b = 0$ . Set

$$\phi(\beta_2) = \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix}.$$

We show

$$\phi(C_{B_3}(\sigma_2)) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ c & \pm 1 \end{pmatrix} \mid c \in \mathbb{Z} \right\}.$$

We already know how the generators of  $C_{B_3}(\sigma_2)$  behave under  $\phi$ . We take those matrices and multiply them.

$$\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} \begin{pmatrix} (-1)^m & 0 \\ (-1)^m 2m & (-1)^m \end{pmatrix} = \begin{pmatrix} (-1)^m & 0 \\ (-1)^{m+1}n + (-1)^m 2m & (-1)^m \end{pmatrix}$$

This matrix matches the form of  $\phi(\beta_2)$ . Let  $c \in \mathbb{Z}$  and suppose the diagonal entries are 1. We can then set  $m = 0$  and  $n = -c$  to obtain

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

If the diagonal entries are  $-1$ , then set  $m = 1$  and  $n = c + 2$ . In other words, we can find a corresponding braid in  $C_{B_3}(\sigma_2)$  whose image under  $\phi$  is the given matrix.

Although  $\phi$  has kernel so the braid whose image is a given matrix is not unique, information is not lost if we mod out by the center  $Z(B_3)$ . If  $\beta_2$  is a solution to Equation 4.4 for some  $\beta_1$  and  $z \in Z(B_3)$ , then  $\beta'_2 = \beta_2 z$  is also a solution to the equation for the same  $\beta_1$ . Furthermore, if  $\beta_2 \in C_{B_3}(\sigma_2)$ , then  $\beta_2 z \in C_{B_3}(\sigma_2)$ . Therefore,  $\phi(\beta_2) \in \phi(C_{B_3}(\sigma_2))$  implies  $\beta_2 \in C_{B_3}(\sigma_2)$ . We can now solve for  $\beta_1$ . The equation becomes

$$\sigma_2^4 = \beta_1^{-1} \sigma_2^4 \beta_1 \quad \Rightarrow \quad \beta_1 \in C_{B_3}(\sigma_2)$$

where  $C_{B_3}(\sigma_2^4) = C_{B_3}(\sigma_2)$  follows from [Gur84][Theorem 4]. Hence, we have concluded that Equation 4.4 is true if and only if  $\beta_1, \beta_2 \in C_{B_3}(\sigma_2)$  and  $\beta_3 \in B_3$ . Markov moves 2 and 5 can then be used to show this braided wiring diagram is Markov equivalent to the standard diagram.  $\square$

We devote the rest of this section to the case when  $n = 4$ . We do not have a complete result and we explain later which parts make the proof incomplete. We first state a result from number theory that will be useful.

PROPOSITION 4.4.1. *The only integral solution to the equation  $a^2 + b^2 + c^2 = a^2b^2$  is  $a = b = c = 0$ .*

PROOF. For any  $x \in \mathbb{Z}$ , we know  $x^2 \equiv 0 \pmod{4}$  or  $x^2 \equiv 1 \pmod{4}$ . Therefore,  $a^2b^2 = (ab)^2$  is equivalent to 0 or 1 modulo 4. We claim it must be the first option. Suppose the latter is true. Then, both  $a$  and  $b$  must be odd so  $a^2 \equiv 1 \pmod{4} \equiv b^2$ . The equation becomes  $1 + 1 + c^2 \equiv 1$ , which is not possible. Therefore,  $a^2b^2 \equiv 0$  so  $a$  or  $b$  is even. Suppose, without loss of generality, that  $a$  is even. Then,  $0 + b^2 + c^2 \equiv 0 \pmod{4}$ , which forces both  $b$  and  $c$  to be even. Set  $a = 2a_1$ ,  $b = 2b_1$ , and  $c = 2c_1$ . If we plug this into the original equation, then

$$a_1^2 + b_1^2 + c_1^2 = 4a_1^2b_1^2.$$

We apply the above procedure again. We have  $a_1^2b_1^2$  is equivalent to 0 or 1 modulo 4. We again claim it must be zero. If it was congruent to 1 modulo 4, then  $a_1, b_1$  are odd so  $a_1^2 \equiv 1 \pmod{4} \equiv b_1^2$ . The equation becomes  $1 + 1 + c^2 \equiv 4$ , which is not possible. Thus,  $a_1^2b_1^2 \equiv 0$ . If we similarly assume, without loss of generality, that  $a$  is even, then  $0 + b_1^2 + c_1^2 \equiv 0$ . This forces  $b_1$  and  $c_1$  to be even like before. Set  $a_2 = 2a_1$ ,  $b_2 = 2b_1$ , and  $c_2 = 2c_1$ . We have

$$a_2^2 + b_2^2 + c_2^2 = 16a_2^2b_2^2$$

We can repeat this indefinitely but nothing is infinitely divisible by 2, so we deduce the only solution is when  $a = b = c = 0$ .  $\square$

We now proceed with the incomplete proof for the case when  $n = 4$ . The total monodromy equation is

$$\sigma_2\beta_1\sigma_2\beta_2\sigma_2\beta_3\sigma_2\beta_4\Delta^2\beta_4^{-1}\sigma_2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_2\beta_1^{-1}\sigma_2 = \Delta^2\sigma_2^8.$$

We simplify this equation to get the following equality.

$$(4.5) \quad \sigma_2\beta_2\sigma_2\beta_3\sigma_2^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_2 = \beta_1^{-1}\sigma_2^6\beta_1$$

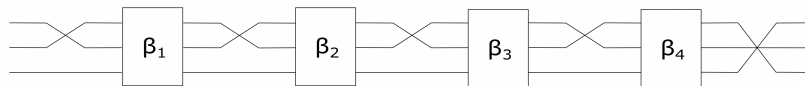


FIGURE 4.12. Four double points and one triple point

Let  $\phi$  send

$$\beta_3 \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \beta_2 \mapsto \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

We compute the trace of  $\phi(\sigma_2\beta_2\sigma_2\beta_3\sigma_2^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_2)$  using Sage. We write down the code below.

```
var('x','y','z','w','a','b','c','d')
A=matrix([[1,0],[-1,1]])
B=matrix([[x,y],[z,w]])
Binverse=matrix([[w,-y],[-z,x]])
C=matrix([[1-2*b*d-2*b^2,2*b^2],[2*b^2-2*d^2-2,1+2*b*d-2*b^2]])
D=A*B*C*Binverse*A
E=D.trace()
E.expand()
```

The matrix labeled  $C$  is the matrix  $\phi(\sigma_2\beta_2\sigma_2^2\beta_2^{-1}\sigma_2)$  computed in the previous case. Given these calculations, the trace of  $\phi(\sigma_2\beta_2\sigma_2\beta_3\sigma_2^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_2)$  is then

$$-4b^2 - 4b^2x^2 - 8bdxy + 4b^2y^2 - 4d^2y^2 + 2 - 4y^2.$$

This should be equal to  $\text{Tr}(\phi(\sigma_2^6)) = 2$  up to multiplication by  $\pm 1$ . We first describe the case when the trace of  $\phi(\sigma_2\beta_2\sigma_2\beta_3\sigma_2^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_2)$  is equal to  $-2$ . In this case, we can rewrite the equation as

$$(bx + dy)^2 + (b + y)^2 = (by + 1)^2.$$

Our hope is to show for the variables that satisfy the equation above, the matrix representations for  $\beta_2$  and  $\beta_3$  do not satisfy Equation 4.5. However, the author is unable to solve this.

The other case is when the trace of  $\phi(\sigma_2\beta_2\sigma_2\beta_3\sigma_2^2\beta_3^{-1}\sigma_2\beta_2^{-1}\sigma_2)$  is equal to 2, which we rewrite as follows.

$$b^2y^2 = b^2 + y^2 + \underbrace{(bx + dy)^2}_{m^2}$$

By Proposition 4.4.1, we conclude  $b = y = m = 0$ . In other words, the lower left entries of  $\phi(\beta_2)$  and  $\phi(\beta_3)$  are zero. The rest of the proof is now similar to the  $n = 3$  case. We have  $\beta_2, \beta_3 \in C_{B_3}(\sigma_2)$ ,

which allows us to conclude  $\beta_1 \in C_{B_3}(\sigma_2)$ . The braided wiring diagrams are then equivalent by Markov moves 2 and 5.

#### 4.5. Generalization of Theorem 1.2.2

In this section we discuss some attempts to generalize Theorem 1.2.2 to analyze the three-stranded braided wiring diagram with six double points having total monodromy  $\Delta^4$ . We would like to prove a version of Theorem 1.2.2 for this case but it proves to be more difficult. As in the original result, the forward direction is the difficult statement to show.

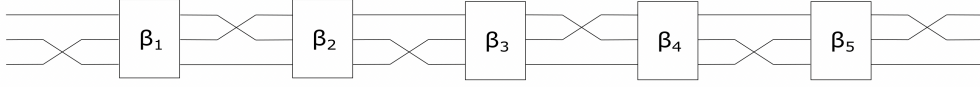


FIGURE 4.13. Six double points

We are unable to complete the proof, but we document here two different strategies with partial progress which may help in future work on this problem. When proving Theorem 1.2.2 using Markov moves, we were able to rewrite the braids in Garside normal form as seen in Table 4.2, for example. When applying the same technique for this case, we run into a few problems. Mainly, the number of unknown braids increases so the casework becomes complicated and as a result, it is not possible to confirm whether or not the resulting braids are in Garside normal form. Hence, the method does not work.

**4.5.1. Method one: Markov moves.** We look at the total monodromy equation and try to see if we can conclude anything about the braids.

$$\sigma_1 \beta_1 \sigma_2 \beta_2 \sigma_1 \beta_3 \sigma_2 \beta_4 \sigma_1 \beta_5 \sigma_2^2 \beta_5^{-1} \sigma_1 \beta_4^{-1} \sigma_2 \beta_3^{-1} \sigma_1 \beta_2^{-1} \sigma_2 \beta_1^{-1} \sigma_1 = \Delta^4$$

Unfortunately, unlike the braided wiring diagrams in Figures 4.7 and 4.9, the equation cannot be simplified nicely. Therefore, we make some simplifying assumptions to see if we can build from there. We look at specific cases which we force certain  $\beta_i$ 's to be empty and then solve the total monodromy equation to see if this will give us hints on the conditions for the braids.

Suppose only one braid is nonempty. Then, we can easily solve for each  $\beta_i$  using the equation. For example,

$$\sigma_1 \beta_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2^2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \beta_1^{-1} \sigma_1 = \Delta^4 \quad \Rightarrow \quad \beta_1 \sigma_2 \sigma_1^2 \sigma_2 \beta_1^{-1} = \sigma_2 \sigma_1^2 \sigma_2 \quad \Rightarrow \quad \beta_1 \in C_{B_3}(\sigma_1).$$

We compute each braid in a similar manner assuming the other braids are trivial:

$$\beta_1 \in C_{B_3}(\sigma_1), \quad \beta_2 \in C_{B_3}(\sigma_1), \quad \beta_3 \in B_3, \quad \beta_4 \in C_{B_3}(\sigma_2), \quad \text{or} \quad \beta_5 \in C_{B_3}(\sigma_2).$$

We want to show all of these braided wiring diagrams are Markov equivalent to the standard wiring diagram. We first show this is true when  $\beta_3$  is an arbitrary braid. Then, this result can be applied to the  $\beta_2$  and  $\beta_4$  cases. The following will help us achieve this.

LEMMA 4.5.0.1. *The braided wiring diagrams in the following figure are related by Markov moves.*



FIGURE 4.14. Braided wiring diagrams

PROOF. Suppose  $\beta$  ends with  $\sigma_2$  so  $\beta = \beta' \cdot \sigma_2$ . Applying Markov move 3b and 5d gives us

$$\begin{aligned} & \{\beta' \sigma_2 \cdot \sigma_2^{-1}, \text{ vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \sigma_1, \text{ vertex } \{1, 2\}\} \\ & \rightarrow \{\beta', \text{ vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}, \sigma_1\}. \end{aligned}$$

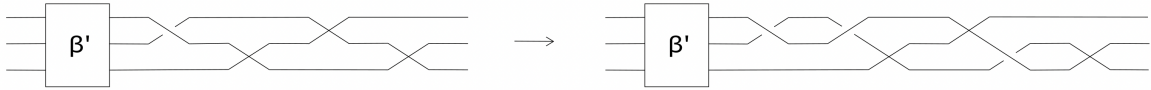


FIGURE 4.15. Markov move 3b

If  $\beta$  ends with  $\sigma_2^{-1}$ , then we analogously obtain

$$\{\beta', \text{ vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}, \sigma_1^{-1}\}$$

where we skipped some steps. If  $\beta = \beta' \cdot \sigma_1^{\pm 1}$ , then the wiring diagram is

$$\{\beta', \text{ vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}, \sigma_2^{\pm 1}\}$$

after Markov move 3c and 5d. For all of these cases, the braids we obtain from Markov move 3 either cancel with the braid at the end of  $\beta$  or is moved to the end of the wiring diagram by Markov move 5d. Moreover, the braid at the end of the wiring diagram has the opposite index. We continue this process until the braid on the left hand side is trivial.  $\square$

We keep in mind this statement also holds if the local indices of the singularities are switched, i.e.

$$\{\beta, \text{vertex } \{2, 3\}, \text{vertex } \{1, 2\}, \text{vertex } \{2, 3\}\} \sim \{\text{vertex } \{2, 3\}, \text{vertex } \{1, 2\}, \text{vertex } \{2, 3\}, (\beta)_{(1)}\}.$$

Observe that this is another way to express the relation  $\Delta\beta = (\beta)_{(1)}\Delta$  in terms of braided wiring diagrams. In Proposition 3.3.2, the half twist was in the braid whereas in this result, the half twist is represented by the singularities.

**COROLLARY 4.5.1.** *The wiring diagram given in Figure 4.16 is Markov equivalent to the standard wiring diagram.*

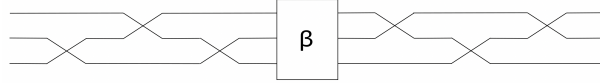


FIGURE 4.16.  $\beta$  is arbitrary

**PROOF.** By Lemma 4.5.0.1, we can move  $\beta$  to the end (or beginning) of the wiring diagram. We then have our result by Markov move 2 (or 1).  $\square$

Let  $\beta_2 \in C_{B_3}(\sigma_1)$  (resp.  $\beta_4 \in C_{B_3}(\sigma_2)$ ). Then, Corollary 3.3.1 (Corollary 3.3.2) tells us we can move this braid to the center of the wiring diagram so  $\beta_3$  becomes the only nonempty braid. From the result we just proved, we can conclude this braided wiring diagram is Markov equivalent to the standard wiring diagram. For the remaining two cases, we appropriately apply Corollary 3.3.1 or 3.3.2 to move the braid to the beginning or end of the wiring diagram. Then, the resulting braided wiring diagram is Markov equivalent to the standard wiring diagram by Markov move 1 or 2.

Observe that we can obtain a solution for the braids if we do not have move 3. The form will be similar to the one given in Lemma 4.1.0.1. More specifically, if the wiring diagram of Figure 4.13 is related to the standard diagram by a sequence of Markov moves 1, 2, 4, and 5, then

$$\beta_1 = a_1 b_1, \quad \beta_2 = b_2 a_2, \quad \beta_3 = a_3 b_3, \quad \beta_4 = b_4 a_4, \quad \text{and} \quad \beta_5 = a_5 b_5$$

where

$$a_i \in C_{B_3}(\sigma_1) \quad \text{and} \quad b_i \in C_{B_3}(\sigma_2) \quad \text{such that} \quad a_\ell a_{\ell+1}, b_j b_{j+1} \in Z(B_3) \quad \text{for} \quad \ell \in \{2, 4\}, j \in \{1, 3\}.$$

Recall Markov moves 1 and 2 deal with braids at the beginning and end of a wiring diagram, respectively. Moreover, move 4 concerns braid simplifications. The conditions for  $a_i$  and  $b_i$  then come from Corollaries 3.3.1 and 3.3.2. The addition of move 3 makes the problem more complicated, and this is even evident in the simplest case where only one braid is nonempty when we showed Figure 4.16 is Markov equivalent to the standard wiring diagram. We provide one more example that showcases how Markov move 3 can affect the forms of the braids.

EXAMPLE 4.5.1. Suppose  $\beta_2, \beta_3, \beta_5$  are empty and the other nonempty braids satisfy  $(\beta_1)_{(1)}\beta_4 \in C_{B_3}(\sigma_2)$ . We claim this is Markov equivalent to the standard diagram. By Lemma 4.5.0.1, move  $\beta_1$  so that it becomes multiplied with  $\beta_4$ . The only nonempty braid is now between the fourth and fifth singularity. This nonempty braid  $(\beta_1)_{(1)}\beta_4$  is hypothesized to commute with  $\sigma_2$ . We then use Corollary 3.3.2 to move this braid to the left. Finally, we apply Corollary 4.5.1 to arrive at the standard diagram.

**4.5.2. Method two: factorizations.** For this method, we study the factorization of the wiring diagram  $g = (g_1, \dots, g_6)$ . We introduce some theory and notation following [Dat23]. Note that [Dat23] also proved Theorem 1.2.2 from this braid monodromy factorization point of view. Here we give an expository introduction to Datta's techniques and see how far these methods can go towards the more general problem.

DEFINITION 4.5.2. Given  $\rho, \tau \in B_3^+$  indivisible by  $\Delta$ , we say  $\rho$  is *dual* to  $\tau$  if  $\rho\tau = \Delta^{\omega(\tau)}$ .

EXAMPLE 4.5.3. Let  $\rho = \sigma_1^2\sigma_2$  and  $\tau = \sigma_1^2\sigma_2$ . Both braids are indivisible by  $\Delta$  and  $\rho\tau = \Delta^2$ .

We note this theory of duality was used in our proof of Theorem 1.2.2 when we calculated  $\tilde{a}_i$ . We referred to Lemma 3.10 from [Dat23], which we explain more in detail here. This result gives an algorithm for computing  $\tau$  given  $\rho = \prod_{j=1}^k \rho_j$  where  $k = \omega(\tau)$  and  $\rho_j$  is an Artin generator or a product of two distinct generators. The lemma states  $\tau = \prod_{j=1}^k (\tau_{k+1-j})_{(j+1)}$  where  $\rho_j$  is dual to  $\tau_j$ .

EXAMPLE 4.5.4. Let  $\rho = \sigma_1 \sigma_2^3 \sigma_1^2$ . We use Lemma 3.10 to find  $\tau$ . The first step is to decompose  $\rho$  as follows:

$$\rho = \underbrace{\sigma_1 \sigma_2}_{\rho_1} \cdot \underbrace{\sigma_2}_{\rho_2} \cdot \underbrace{\sigma_2 \sigma_1}_{\rho_3} \cdot \underbrace{\sigma_1}_{\rho_4}.$$

From this, we observe  $\omega(\tau) = 4$  and we apply the formula for  $\tau$ .

$$\tau = \prod_{j=1}^4 (\tau_{k+1-j})_{(j+1)} = \tau_4(\tau_3)_{(1)} \tau_2(\tau_1)_{(1)} = \sigma_2 \sigma_1 \cdot (\sigma_2)_{(1)} \cdot \sigma_1 \sigma_2 \cdot (\sigma_1)_{(1)} = \sigma_2 \sigma_1^3 \sigma_2^2$$

We check to confirm  $\rho\tau = \Delta^4$ .

$$\sigma_1 \sigma_2^3 \sigma_1^2 \cdot \sigma_2 \sigma_1^3 \sigma_2^2 = \sigma_1 \sigma_2^2 \underbrace{\sigma_2 \sigma_1^2 \sigma_2 \sigma_1^2}_{\Delta^2} \sigma_1 \sigma_2^2 = \Delta^2 \underbrace{\sigma_1 \sigma_2^2 \sigma_1 \sigma_2^2}_{\Delta^2} = \Delta^4$$

Theorem 3.24 in [Dat23] shows we can write  $\beta \sigma_i^e \beta^{-1}$  in a specific form involving these dual braids. This is exactly the form of our factors where  $e = 2$ , and we use this result in our setup. Let  $\mathcal{F} = (g_1, \dots, g_n)$  be a factorization of  $g_1 \cdots g_n$ . The theorem says we can write each factor as

$$g_j = \Delta^{-\omega(\tau_j)} \underbrace{(\rho_j \sigma_{i_j}^2 \tau_j)_{(\omega_j)}}_{\xi_j}$$

where

$$\omega_j = \sum_{\ell=1}^j \omega(\tau_\ell).$$

DEFINITION 4.5.5. The *complexity* of a factorization  $\mathcal{F}$  is the sum of the absolute values of the Garside powers of each factor. Moreover, a factorization has *minimal complexity* if its complexity is minimal in its Hurwitz equivalence class.

EXAMPLE 4.5.6. Let

$$\mathcal{F} = (\sigma_1^2, \sigma_1^{-1} \sigma_2^2 \sigma_1, \sigma_2^2)$$

be a factorization of  $\Delta^2$ . We write each  $g_j$  in the specific form using the fact that  $\omega(\tau_1) = 0$ ,  $\omega(\tau_2) = 1$ , and  $\omega(\tau_3) = 0$ .

$$g_1 = \sigma_1^2, \quad g_2 = \Delta^{-1} \left( \underbrace{\sigma_2 \sigma_1}_{\rho_2} \cdot \underbrace{\sigma_1^2 \cdot \sigma_2}_{\tau_2} \right)_{(1)}, \quad g_3 = (\sigma_1^2)_{(1)}.$$

As we can see, the complexity of this factorization is one.

We first state the generalized theorem as a conjecture and go over smaller results that can help verify this result.

CONJECTURE 4.5.1. *Let  $\mathcal{F}$  be a factorization of  $\Delta^4$  where each factor is of the form  $g_j = \beta_j \sigma_j^2 \beta_j^{-1}$ . Assume for some  $1 \leq j \leq 6$ , the factor  $g_j$  is not conjugated and  $\xi_{j-1} \xi_j$  or  $\xi_j \xi_{j+1}$  is divisible by  $\Delta$ . Then,  $\mathcal{F}$  is Markov equivalent to the standard braided wiring diagram (Figure 4.13 with  $\beta_i = \emptyset$ ).*

This next result will give us some information about the positive braid from the Garside normal form of each factor.

LEMMA 4.5.6.1. *Let  $\mathcal{F}$  be a factorization of  $\Delta^{2k}$  with double point singularities. Then,  $\xi_j \xi_{j+1}$  is divisible by  $\Delta$  for some  $j$ .*

PROOF. We assume our singularities are double points so we will have  $3k$  factors. By construction,

$$\Delta^{2k} = g_1 \cdots g_{3k} = \Delta^{-\omega(\tau_1)} (\xi_1)_{(\omega_1)} \cdots \Delta^{-\omega(\tau_{3k})} (\xi_{3k})_{(\omega_{3k})} = \xi_1 \cdots \xi_{3k} \Delta^{-\omega_{3k}}$$

or

$$\Delta^{2k+\omega_{3k}} = \xi_1 \cdots \xi_{3k}.$$

Suppose  $\xi_j \xi_{j+1}$  is indivisible by  $\Delta$  for all  $j$ . Since the product of the  $\xi_i$ 's is a power of  $\Delta$ , Lemma 3.20 in [Dat23] says there exists a  $j$  such that  $\xi_{j-1} \xi_j \xi_{j+1}$  is divisible by  $\Delta$  and  $\xi_j$  is an Artin generator. However,  $\xi_j$  cannot be an Artin generator by definition of  $\xi_j$  in this setting so it follows that at least one  $\xi_j \xi_{j+1}$  must be divisible by  $\Delta$ .  $\square$

The proof of the following result is incomplete and uses Lemma 4.5.6.1. We write down the conditions for when this result holds and explain the missing components.

CONJECTURE 4.5.2. *Let  $\mathcal{F}$  be a minimal complexity factorization for  $\Delta^{2k}$ . Then, one of the corresponding factors  $g_j$  or  $g_{j+1}$  (same  $j$  from Lemma 4.5.6.1) is not conjugated, i.e. equal to  $\sigma_i^2$ .*

By the previous lemma, we know  $\xi_j \xi_{j+1}$  is divisible by  $\Delta$  for some  $j$ . Assume the result is false so both  $g_j$  and  $g_{j+1}$  are conjugated. Let  $g_j = h^{-1} \sigma_i^2 h$  where  $h = \Delta^{-\ell} \tilde{\tau}_j$  is the Garside normal form for  $h$ . We write the Garside normal form for  $g_j$ .

$$g_j = (\Delta^{-\ell} \tilde{\tau}_j)^{-1} \cdot \sigma_i^2 \cdot \Delta^{-\ell} \tilde{\tau}_j = \tilde{\tau}_j^{-1} \Delta^\ell \cdot \sigma_i^2 \cdot \Delta^{-\ell} \tilde{\tau}_j = \Delta^{-\omega(\tilde{\tau}_j)} \tilde{\rho}_j(\sigma_i^2)_{(\ell)} \tilde{\tau}_j = \Delta^{-\omega(\tau_j)} (\rho_j \sigma_{i_j}^2 \tau_j)_{(\omega_j)}$$

From this series of equalities, we observe

$$\omega(\tilde{\tau}_j) = \omega(\tau_j), \quad \tilde{\rho}_j = (\rho_j)_{(\omega_j)}, \quad (\sigma_i)_{(\ell)} = (\sigma_{i_j})_{(\omega_j)}, \quad \text{and} \quad \tilde{\tau}_j = (\tau_j)_{(\omega_j)}.$$

Now suppose we apply a global conjugation move by  $h$  to  $\mathcal{F}$ . Since  $\mathcal{F}$  is a factorization for  $\Delta^{2k}$ , this move will not affect the total braid. However, it may affect the complexity of  $\mathcal{F}$  so assume the complexity stays constant. We denote the new factorization as  $\mathcal{F}' = (g'_1, \dots, g'_{3k})$ . Then,  $g'_j = \sigma_i^2$  and

$$\begin{aligned} g'_{j+1} &= h g_{j+1} h^{-1} = \Delta^{-\ell} \tilde{\tau}_j \cdot \Delta^{-\omega(\tau_{j+1})} (\xi_{j+1})_{(\omega_{j+1})} \cdot \Delta^{-\omega(\tau_j)} \tilde{\rho}_j \Delta^\ell \\ &= \Delta^{-\omega(\tau_{j+1}) - \omega(\tau_j)} (\tilde{\tau}_j)_{(\omega(\tau_{j+1}) + \omega(\tau_j) + \ell)} (\xi_{j+1})_{(\omega_{j+1} + \omega(\tau_j) + \ell)} (\tilde{\rho}_j)_{(\ell)} \\ &= \Delta^{-\omega(\tau_{j+1}) - \omega(\tau_j)} (\tau_j \xi_{j+1})_{(\omega_{j+1} + \omega(\tau_j) + \ell)} (\rho_j)_{(\ell + \omega_j)}. \end{aligned}$$

Consider the product of  $g'_j$  and  $g'_{j+1}$ .

$$\begin{aligned} g'_j g'_{j+1} &= \sigma_i^2 \cdot \Delta^{-\omega(\tau_{j+1}) - \omega(\tau_j)} (\tau_j \xi_{j+1})_{(\omega_{j+1} + \omega(\tau_j) + \ell)} (\rho_j)_{(\ell + \omega_j)} \\ &= \left[ (\sigma_{i_j}^2 \tau_j \xi_{j+1})_{(\ell + \omega_j)} (\rho_j)_{(\ell + \omega_{j+1} + \omega(\tau_j))} \right] \Delta^{-\omega(\tau_{j+1}) - \omega(\tau_j)} \end{aligned}$$

Recall  $\Delta$  divides  $\xi_j \xi_{j+1} = \rho_j \sigma_{i_j}^2 \tau_j \cdot \xi_{j+1}$ , so we can conclude by the blue braid that  $\xi'_j \xi'_{j+1}$  is also divisible by  $\Delta$ . However, we are assuming the final expression for  $g'_{j+1}$  is in Garside normal form.

In other words,

$$(\tau_j \xi_{j+1})_{(\omega_{j+1} + \omega(\tau_j) + \ell)} (\rho_j)_{(\ell + \omega_j)}$$

is indivisible by  $\Delta$ . If it is divisible by  $\Delta$ , then we cannot use the above argument as the expression in brackets is not equal to  $\xi'_j \xi'_{j+1}$ . In this case, we want to show  $\mathcal{F}$  does not have minimal complexity.

One way to try to show this would be to apply a Hurwitz move to  $g_j$  and  $g_{j+1}$  and observe what

happens to the complexity.

$$\begin{aligned}
(g_j, g_{j+1}) &\mapsto (g_j g_{j+1} g_j^{-1}, g_j) \\
&= \left( h^{-1} \sigma_i^2 h \cdot \Delta^{-\omega(\tau_{j+1})} (\rho_{j+1} \sigma_{i_{j+1}} \tau_{j+1})_{(\omega_{j+1})} \cdot h^{-1} \sigma_i^{-2} h, g_j \right) \\
&= \left( \Delta^{-\omega(\tau_j)} \tilde{\rho}_j \Delta^\ell \sigma_i^2 \Delta^{-\ell} \tilde{\tau}_j \cdot \Delta^{-\omega(\tau_{j+1})} (\rho_{j+1} \sigma_{i_{j+1}} \tau_{j+1})_{(\omega_{j+1})} \cdot \Delta^{-\omega(\tau_j)} \tilde{\rho}_j \Delta^\ell \Delta^{-2} \sigma_{i'} \sigma_i^2 \sigma_{i'} \Delta^{-\ell} \tilde{\tau}_j, g_j \right) \\
&= \left( \Delta^{-2\omega(\tau_j) - 2 - \omega(\tau_{j+1})} (\rho_j \sigma_{i_j}^2 \tau_j \xi_{j+1})_{(\omega_{j+1} + \omega(\tau_j))} (\rho_j \sigma_{i'_j} \sigma_{i_j}^2 \sigma_{i'_j} \tau_j)_{(\omega_j)}, g_j \right) \\
&= \left( \Delta^{-\omega(\tau_j)} (\xi_j)_{(\omega_j)} \Delta^{-\omega(\tau_{j+1})} (\xi_{j+1})_{(\omega_{j+1})} \Delta^{-\omega(\tau_j)} \rho'_j \sigma_{i'_j}^2 \tau'_j, g_j \right)
\end{aligned}$$

It is difficult to deduce anything precise about the complexity so perhaps another technique would be better.

In this proof, we assumed i) the complexity of the factorization does not change if we apply a global conjugation move by  $h$  and ii) our expression for  $g'_{j+1}$  is written in Garside normal form. Although not mentioned above, we also know at least one  $\xi_k \xi_{k+1}$  is indivisible by  $\Delta$  due to Lemma 4.29 in [Dat23]. However, the author is unclear as to how to use this information with everything else to arrive at a contradiction.

If Conjecture 4.5.2 is true, then we can prove Conjecture 4.5.1, a version of [Dat23, Theorem 4.38]. Without loss of generality, set  $g_j = \sigma_1^2$  after applying a global conjugation move by  $\Delta$  to  $\mathcal{F}$ . Moreover, since the total monodromy is in the center of the braid group, we cyclically permute the factors to let  $g_6 = \sigma_1^2$ . Then,  $g_5 = \sigma_1^{-1} \sigma_2^2 \sigma_1$  by [Dat23, Proposition 4.33] so

$$\mathcal{F} = (g_1, g_2, g_3, g_4, \sigma_1 \sigma_2^2 \sigma_1^{-1}, \sigma_1^2).$$

This tells us

$$g_1 g_2 g_3 g_4 = \Delta^2 \sigma_2^2.$$

We consider the subfactorization  $(g_1, g_2, g_3, g_4)$  and the corresponding wiring diagram given by Figure 4.9. Corollary 1.2.2 tells us Figure 4.9 is Markov equivalent to the standard wiring diagram. Then, we can concatenate the two subfactorizations  $(g_1, g_2, g_3, g_4)$  and  $(g_5, g_6)$ .

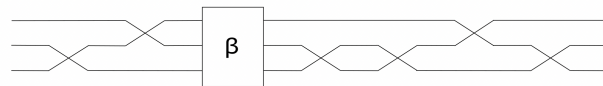


FIGURE 4.17. Combining the subfactorizations

We insert an unknown braid  $\beta$  between the subfactorizations and solve for  $\beta$ .

$$\sigma_1\sigma_2\beta\sigma_1^2\sigma_2\sigma_1^2\sigma_2\sigma_1^2\beta^{-1}\sigma_2\sigma_1 = \Delta^4 \quad \Rightarrow \quad \beta\sigma_1^2\beta^{-1} = \sigma_1^2 \quad \Rightarrow \quad \beta \in C_{B_3}(\sigma_1)$$

Since  $\beta \in C_{B_3}(\sigma_1)$ , Corollary 3.3.1 gives us

$$\{\text{vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}, \beta, \text{ vertex } \{1, 2\}, \text{ vertex } \{2, 3\}, \text{ vertex } \{1, 2\}\}.$$

Moreover, we extract  $\Delta$  from the end of the wiring diagram, which can be done by Markov move 2. Then, we use Proposition 3.3.2 to move  $\Delta$  so it becomes multiplied with  $\beta$ . Our braided wiring diagram now looks like Figure 4.16, and this wiring diagram is Markov equivalent to the standard wiring diagram by Corollary 4.5.1.

## CHAPTER 5

### Braided wiring diagrams to symplectic fillings

Our results in this thesis so far have been in terms of braided wiring diagrams. We now provide more context using language from symplectic topology and complex deformation theory of sandwiched singularities. We first provide the dual resolution graphs for the sandwiched singularities which have corresponding curvetta germs with picture deformations giving the configurations from our main results.

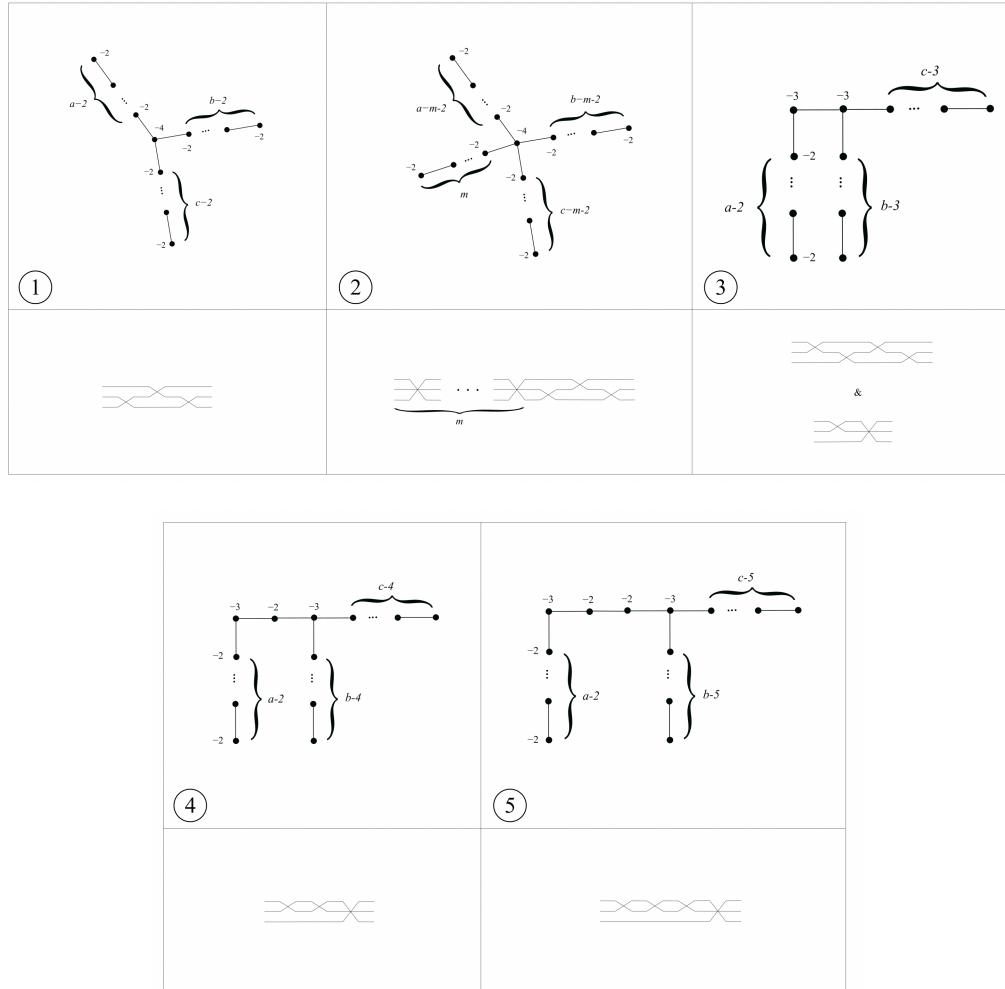


FIGURE 5.1. Resolution graphs

Recall  $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$  is the arrangement of symplectic curvetas with marked points  $\{p_j\}_{j=1}^n$  that is related to the weighted singular curve  $\mathcal{C} = \{C_1, \dots, C_m\}$  by a smooth graphical homotopy. We find the incidence matrix  $\mathcal{I}(\Gamma, \{p_j\})$  for each configuration in our table. This  $m \times n$  matrix is defined by

$$a_{ij} = \begin{cases} 1 & p_j \in \Gamma_i \\ 0 & \text{else} \end{cases}.$$

Since the way we label our marked points will affect the columns of the matrix, incidence matrices are only defined up to permutation of the columns. Two arrangements are said to be combinatorially equivalent if their incidence matrices coincide.

Let  $\mathcal{L}$  be the Lefschetz fibration associated to  $(\Gamma, \{p_j\})$ . The fiber is a disk with  $m$  holes as we have  $m$  curvetas, and the vanishing cycles of  $\mathcal{L}$  correspond to the marked points  $\{p_j\}$ . If  $\Gamma_{i_1}, \dots, \Gamma_{i_k}$  are curvetas intersecting at  $p_j$ , then the vanishing cycle encloses the holes  $h_{i_1}, \dots, h_{i_k}$ . We cite a result that describes the relationship between the homology classes of the vanishing cycles of  $\mathcal{L}$  and the incidence matrix.

**PROPOSITION 5.0.1** (Proposition 6.1 in [PS23]). *If the  $j$ th column of  $\mathcal{I}(\Gamma, \{p_j\})$  has 1's in rows  $i_1, \dots, i_k$ , then the corresponding vanishing cycle of  $\mathcal{L}$  encloses the holes  $h_{i_1}, \dots, h_{i_k}$  in the fiber.*

The concept of combinatorial equivalence is used to obtain an additional result.

**COROLLARY 5.0.1** (Corollary 6.2 in [PS23]). *Let  $\mathcal{L}$  and  $\mathcal{L}'$  be Lefschetz fibrations for the combinatorially equivalent arrangements  $(\Gamma, \{p_j\})$  and  $(\Gamma', \{p'_j\})$ , respectively. Then, the vanishing cycles for  $\mathcal{L}$  and  $\mathcal{L}'$  are in one-to-one correspondence and the vanishing cycles corresponding to each other are given by homologous curves in the fiber.*

The incidence matrix determines  $H_1(W)$  and  $H_2(W)$  where  $W$  denotes the filling. Set  $\mathbb{Z}\langle\{p_j\}\rangle$  and  $\mathbb{Z}\langle\{\Gamma_j\}\rangle$  to be the free abelian group generated by  $\{p_j\}$  and  $\{\Gamma_j\}$ , respectively.

**PROPOSITION 5.0.2** (Theorem 5.2 in [dJvS98], Proposition 6.4 in [PS23]). *There is a short exact sequence*

$$0 \rightarrow H_2(W) \rightarrow \mathbb{Z}\langle\{p_j\}\rangle \xrightarrow{\mathcal{I}} \mathbb{Z}\langle\{\Gamma_i\}\rangle \rightarrow H_1(W) \rightarrow 0.$$

In other words,  $H_1(W) = \mathbb{Z}\langle\{\Gamma_i\}\rangle/\text{im}(\mathcal{I})$  and  $H_2(W) = \ker(\mathcal{I})$ . We find the incidence matrices and homology groups for the wiring diagrams in our table. Since our results only involve three-stranded wiring diagrams, we note every matrix will have three rows. However, since the weights are arbitrary, the number of columns varies. More specifically, the matrix will have varying numbers of columns with

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For each wiring diagram, we observe there are different combinations of transverse singularities that will give the same total monodromy. For example, a braided wiring diagram with one triple point has the same total monodromy as the braided wiring diagram with three double points (first wiring diagram from our chart). Therefore, for the wiring diagrams above, we list all of the possible factorization combinations and their corresponding data. We use the following definition and result to find the various factorizations.

DEFINITION 5.0.1. Let  $\varepsilon : B_3 \rightarrow \mathbb{Z}$  be the abelianization homomorphism sending each Artin generator to 1.

PROPOSITION 5.0.3. Let  $\mathcal{F} = (g_1, \dots, g_k)$  be a factorization of  $g = g_1 \cdots g_k \in B_3$  with  $g_j = \beta_j \mu_{i_j}^2 \beta_j^{-1}$ . Let  $\nu_n$  be the number of factors in  $\mathcal{F}$  such that  $\varepsilon(\mu_{i_j}) = n$ . In particular, we just have  $\nu_1$  and  $\nu_3$  as three-stranded wiring diagrams can only have double points or triple points as transverse singularities. Then,  $\varepsilon(g) = 2\nu_1 + 6\nu_3$ .

PROOF. We have  $\varepsilon(g_j) = \varepsilon(\beta_j \mu_{i_j}^2 \beta_j^{-1}) = 2\varepsilon(\mu_{i_j})$ . Since  $\varepsilon$  is a homomorphism,

$$\varepsilon(g_1 \cdots g_k) = \varepsilon(g_1) + \cdots + \varepsilon(g_k) = 2\varepsilon(\mu_{i_1}) + \cdots + 2\varepsilon(\mu_{i_k}).$$

Given our definition of  $\nu_n$ , the result follows. □

We now go over each wiring diagram from our main results. For the wiring diagram with total monodromy  $\Delta^2$ , Proposition 5.0.3 says  $6 = 2\nu_1 + 6\nu_3$ . The solutions are (1)  $\nu_1 = 3$  and  $\nu_3 = 0$  or (2)  $\nu_1 = 0$  and  $\nu_3 = 1$ . This is the case we described earlier with the three double points.

|                                                                                   |                                                                                                                                                                                                                                                                                                         |                                                                                                                                                                |
|-----------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | $\begin{pmatrix} 1 & 1 & & 1 & 0 & & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$<br>$\underbrace{\hspace{1.5cm}}_{a-1} \quad \underbrace{\hspace{1.5cm}}_{b-1} \quad \underbrace{\hspace{1.5cm}}_{c-1}$                         | $H_1(W) = 0$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-5}$                                                                   |
|  | $\begin{pmatrix} 1 & 1 & 0 & 1 & & 1 & 0 & & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$<br>$\underbrace{\hspace{1.5cm}}_{a-2} \quad \underbrace{\hspace{1.5cm}}_{b-2} \quad \underbrace{\hspace{1.5cm}}_{c-2}$ | $H_1(W) = \begin{cases} \mathbb{Z}_2 & a=b=c=2 \\ 0 & \text{else} \end{cases}$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-6}$ |

FIGURE 5.2. Wiring diagrams with total monodromy  $\Delta^2$

The second braided wiring diagram has total monodromy  $\Delta^{2(m+1)}$  so  $6(m+1) = 2\nu_1 + 6\nu_3$ . Suppose we first start with only triple points so  $\nu_3 = m+1$  and  $\nu_1 = 0$ . By increasing the number of double points and decreasing the number of triple points, the next pairing we obtain is  $\nu_1 = 3$  and  $\nu_3 = m$ . In other words, each triple point is replaced with three double points. By continuing in this manner, we eventually get  $\nu_1 = 3(m+1)$  and  $\nu_3 = 0$ .

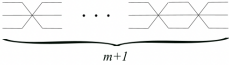
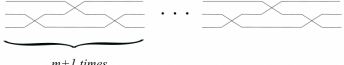
|                                                                                     |                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                                                                          |
|-------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | $\begin{pmatrix} 1 & 1 & 1 & & 1 & 0 & & 0 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$<br>$\underbrace{\hspace{1.5cm}}_{m+1} \quad \underbrace{\hspace{1.5cm}}_{a-m-2} \quad \underbrace{\hspace{1.5cm}}_{b-m-2} \quad \underbrace{\hspace{1.5cm}}_{c-m-2}$                                                                         | $H_1(W) = 0$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-2m-5}$                                                                          |
|  | $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & & 1 & 0 & & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$<br>$\underbrace{\hspace{1.5cm}}_m \quad \underbrace{\hspace{1.5cm}}_{a-m-2} \quad \underbrace{\hspace{1.5cm}}_{b-m-2} \quad \underbrace{\hspace{1.5cm}}_{c-m-2}$                                                 | $H_1(W) = \begin{cases} \mathbb{Z}_2 & a=b=c=m+2, m=0 \\ 0 & \text{else} \end{cases}$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-2m-6}$ |
|  | $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & & 1 & 0 & & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$<br>$\underbrace{\hspace{1.5cm}}_{m-1} \quad \underbrace{\hspace{1.5cm}}_{a-m-2} \quad \underbrace{\hspace{1.5cm}}_{b-m-2} \quad \underbrace{\hspace{1.5cm}}_{c-m-2}$ | $H_1(W) = \begin{cases} \mathbb{Z}_2 & a=b=c=m+2, m=1 \\ 0 & \text{else} \end{cases}$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-2m-4}$ |
| $\vdots$                                                                            | $\vdots$                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                          |
|  | $\begin{pmatrix} 1 & 1 & 0 & & 1 & 1 & 0 & 1 & & 1 & 0 & & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & 0 & 1 & 1 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$<br>$\underbrace{\hspace{1.5cm}}_{m+1 \text{ times}} \quad \underbrace{\hspace{1.5cm}}_{a-m-2} \quad \underbrace{\hspace{1.5cm}}_{b-m-2} \quad \underbrace{\hspace{1.5cm}}_{c-m-2}$           | $H_1(W) = \begin{cases} \mathbb{Z}_2 & a=b=c=m+2 \\ 0 & \text{else} \end{cases}$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-6}$         |

FIGURE 5.3. Wiring diagrams with total monodromy  $\Delta^{2m+2}$

The third wiring diagram has total monodromy  $\Delta^2\sigma_2^2$  so  $8 = 2\nu_1 + 6\nu_3$ . We obtain (1)  $\nu_1 = 4$  and  $\nu_3 = 0$  or (2)  $\nu_1 = 1 = \nu_3$ .

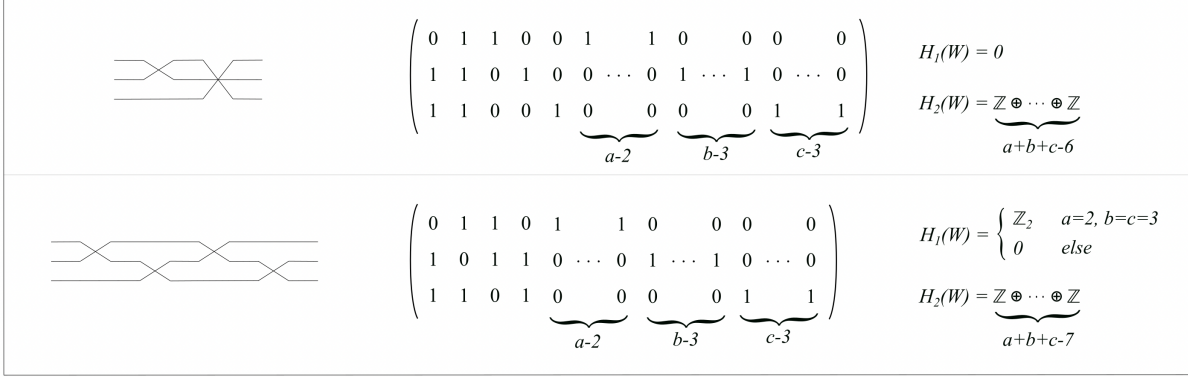


FIGURE 5.4. Wiring diagrams with total monodromy  $\Delta^2\sigma_2^2$

The fourth wiring diagram has total monodromy  $\Delta^2\sigma_2^4$  so  $10 = 2\nu_1 + 6\nu_3$ . We have (1)  $\nu_1 = 2$  and  $\nu_3 = 1$  or (2)  $\nu_1 = 5$  and  $\nu_3 = 0$ .

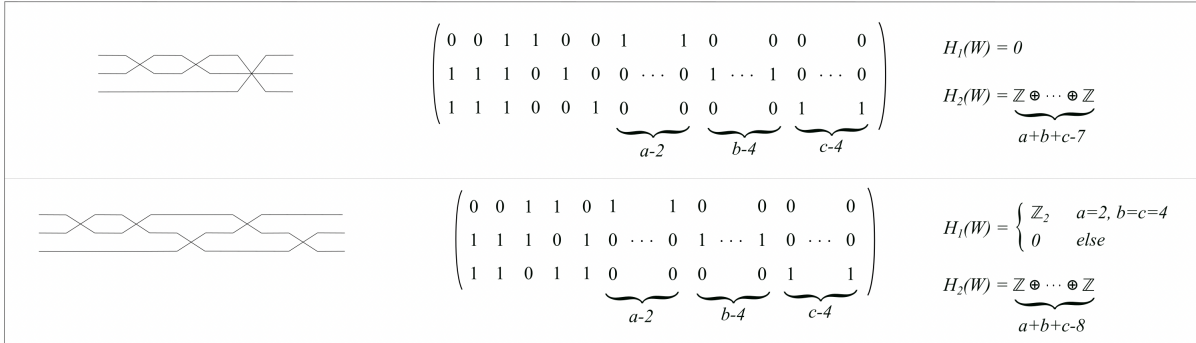


FIGURE 5.5. Wiring diagrams with total monodromy  $\Delta^2\sigma_2^4$

The fifth wiring diagram has total monodromy  $\Delta^2\sigma_2^6$  so  $12 = 2\nu_1 + 6\nu_3$ . This gives (1)  $\nu_1 = 6$  and  $\nu_3 = 0$ , (2)  $\nu_1 = 3$  and  $\nu_3 = 1$ , or (3)  $\nu_1 = 0$  and  $\nu_3 = 2$ . However, the last option is not possible because we would have

$$\beta_1 \Delta^2 \beta_1^{-1} \cdot \beta_2 \Delta^2 \beta_2^{-1} = \Delta^4$$

and we cannot write  $\Delta^2\sigma_2^6$  as a product involving only  $\Delta$ 's. Therefore, we only consider the first two options.

|                                                                                   |                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | $\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & & 1 & 0 & & 0 & 0 & & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$ <p style="text-align: center;"> <math>\underbrace{\hspace{10em}}_{a-2} \quad \underbrace{\hspace{10em}}_{b-5} \quad \underbrace{\hspace{10em}}_{c-5}</math> </p> | $H_1(W) = 0$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-8}$                                                                      |
|  | $\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & & 1 & 0 & & 0 & 0 & & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & & 0 & 0 & & 0 & 1 & & 1 \end{pmatrix}$ <p style="text-align: center;"> <math>\underbrace{\hspace{10em}}_{a-2} \quad \underbrace{\hspace{10em}}_{b-5} \quad \underbrace{\hspace{10em}}_{c-5}</math> </p>             | $H_1(W) = \begin{cases} \mathbb{Z}_2 & a=2, b=c=5 \\ 0 & \text{else} \end{cases}$<br>$H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-9}$ |

FIGURE 5.6. Wiring diagrams with total monodromy  $\Delta^2\sigma_2^6$

Our main results can now be reformulated as follows. Let  $(X, 0)$  be a rational surface singularity with reduced fundamental cycle with contact link  $(Y, \xi)$  and decorated germ  $(\mathcal{C}, w)$ . As mentioned before,  $\Gamma$  is an arrangement of symplectic curvetas with marked points  $\{p_j\}$  related to  $\mathcal{C}$  by a smooth graphical homotopy, and  $\Gamma$  gives rise to a Stein filling  $W$  of  $Y$ . We calculated  $H_1(W)$  and  $H_2(W)$  associated to the braided wiring diagrams, and our results allow us to conclude the  $\Gamma$ 's corresponding to the diagrams from our main results have a unique filling. We write down the specific statement for each case.

**THEOREM 5.0.2.** *Consider the singularity corresponding to the first resolution graph in Figure 5.1. Then, the link of the singularity has exactly two symplectic fillings, distinguished by their homology groups:*

$$(1) \ H_1(W) = 0 \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-5},$$

$$(2) \ H_1(W) = \begin{cases} \mathbb{Z}_2 & a = b = c = 2 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-6}.$$

**PROOF.** Let  $(Y, \xi)$  be the link of the singularity. By [PS23][Theorem 1.5], every symplectic filling corresponds to a configuration  $\Gamma$  obtained from a germ  $\mathcal{C}$  by a smooth graphical homotopy. By our above analysis about  $\varepsilon$  (Proposition 5.0.3), anything obtained from  $\mathcal{C}$  by a smooth graphical homotopy has singularities with either one triple point or three double points. By Theorem 1.2.2 and the fact that a smooth graphical homotopy preserves the total monodromy, we can conclude

there is a unique filling with the homology groups given above by (2). A similar explanation applies for the case with one triple point.  $\square$

**THEOREM 5.0.3.** *Consider the singularity corresponding to the second resolution graph in Figure 5.1. Then, the link of the singularity has a unique symplectic filling with homology*

$$H_1(W) = 0 \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-2m-5},$$

*a unique symplectic filling with homology*

$$H_1(W) = \begin{cases} \mathbb{Z}_2 & a = b = c = m + 2, m = 0 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-2m-6},$$

*and at least one (but potentially more than one) symplectic filling with the other homology groups from Figure 5.3.*

The proof of this theorem is similar but we are unable to verify uniqueness for all of the fillings. When considering the last braided wiring diagram in Figure 5.3, for example, there may be a wiring diagram with braids between the double points such that the total monodromy is  $\Delta^{2m+2}$  but the wiring diagram is not Markov equivalent to the standard one. We only have definitive results concerning Markov moves for the first two wiring diagrams in Figure 5.3.

**THEOREM 5.0.4.** *The link of the singularity corresponding to the third resolution graph in Figure 5.1 has exactly two symplectic fillings. They are distinguished by their homology which is given as follows:*

$$(1) \quad H_1(W) = 0 \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-6},$$

$$(2) \quad H_1(W) = \begin{cases} \mathbb{Z}_2 & a = 2, b = c = 3 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-7}.$$

THEOREM 5.0.5. *The link of the singularity corresponding to the fourth resolution graph in Figure 5.1 has a unique symplectic filling with homology*

$$H_1(W) = 0 \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-7}$$

*and at least one (but potentially more than one) symplectic filling with homology*

$$H_1(W) = \begin{cases} \mathbb{Z}_2 & a = 2, b = c = 4 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-8}$$

THEOREM 5.0.6. *The singularity corresponding to the fifth resolution graph in Figure 5.1 has a unique symplectic filling with homology*

$$H_1(W) = 0 \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-8}$$

*and at least one (but potentially more than one) symplectic filling with homology*

$$H_1(W) = \begin{cases} \mathbb{Z}_2 & a = 2, b = c = 5 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad H_2(W) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a+b+c-9}$$

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