From Locality to Topology: Spectral and Transport Phenomena in Lattice Quantum Systems

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Abstract

This thesis studies the mathematical structure of quantum Hall transport in lattice systems, with an emphasis on relating mutiple definitions of Hall conductance in different mathematical branches: Geometrically as a bulk Chern number for a Bloch bundle; analytically as an edge spectral flow for a half-space Hamiltonian; and in operator algebra language as a many body charge transport index generated by magnetic flux insertion. The main results of the thesis reveal restrictions imposed on these indices by frustration-free structure, and improved locality estimates for interacting many body systems.

The first result concerns frustration free free-Fermion (F^4) Hamiltonians. For translation-invariant, strictly local, gapped F^4 Hamiltonians on $\mathbb{Z}^2$, we prove that the bulk index vanishes. More precisely, the two Bloch bundle associated with the image/kernel of the momentum space Hamiltonian is shown to be trivial by an algebraic argument using polynomial modules and a free resolution. Thus a frustration-free free-Fermion model cannot realize a nontrivial integer quantum Hall phase at the level of the bulk Chern index.

The second new result is the corresponding edge statement. For a translation-invariant F^4 Hamiltonian restricted to a half-plane, we prove that the edge index vanishes. We first identified that frustration free is equivalent to flat ground state band. Detailed analysis of spectral flow of the edge Hamiltonian shows that the edge mode is an analytic function and can be absorbed by the flat band. Consequently, the edge spectral flow across a regular energy in the bulk gap is constant and must equal its trivial value below the spectrum. This establishes, in this setting, that the frustration-free structure rules out nonzero edge transport as well as nonzero bulk transport.

The third new result is an exponential clustering theorem for finite-volume quantum spin systems with a possibly degenerate and split ground-state sector. Assuming only a uniform spectral gap above a low-energy spectral patch of sufficiently small width, we prove exponential decay of correlations in the form

$$\Big|\langle\Omega, AB\Omega\rangle - \langle\Omega, APB\Omega\rangle\Big| \leq C\|P\|_1\|A\|\|B\|\mathrm{e}^{-cd(X,Y)},$$

for local observables A and B with disjoint supports. This improves the locality control needed in many body Hall conductance arguments, where previous estimates under comparable hypotheses were superpolynomial. Together, these results clarify how locality and spectral gaps constrain both free-Fermion and interacting formulations of topological transport.

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CHAPTER 1

Introduction

The discovery of the integer quantum Hall effect revealed that certain experimentally observable transport coefficients are determined not merely by local material parameters, but by global topological structures encoded in quantum states of matter. In a two dimensional electron system, an in-plane electric field E_x may generate a transverse current J_y, and the corresponding Hall conductance is observed to be quantized with remarkable stability under perturbations and disorder. This phenomenon is the pioneering example of a topological phase of matter: a phase whose robust features are protected by a spectral gap and are measured by topological indices.

The goal of this thesis is to study several mathematical formulations of Hall conductance and to explain how they are connected through the common principle: on the microscopic models we have locality and abstractly the topology of certain verctor bundles. The models considered here range from translationally invariant free Fermion tight binding Hamiltonians on $\mathbb{Z}^2$, to half-space edge Hamiltonians, to interacting many body systems on finite tori. Although these settings use different mathematical languages, the same guiding question appears throughout: how does a local Hamiltonian with a spectral gap produce a quantized and stable transport coefficient.

There are three main perspectives on Hall conductance that will be used in this thesis. The first is the bulk, or linear response, perspective. In a translationally invariant free Fermion system, the Hamiltonian is diagonalized by the Fourier transform and becomes a family of finite dimensional matrices $H(\boldsymbol{k})$ over the Brillouin torus. If the Fermi energy lies in a gap, the occupied eigenspaces form a vector bundle over momentum space, called the Bloch bundle. The Hall conductance is then identified with the first Chern number of this bundle. This point of view originates in the work of Thouless, Kohmoto, Nightingale and den Nijs [37], building on the Kubo linear response formula [30]. It connects quantum Hall transport with curvature, characteristic classes, and vector bundle topology. The Haldane model [19] provides a particularly important lattice example, since it exhibits nonzero Chern number without Landau levels. In disordered systems, Bellissard, van

Elst and Schulz-Baldes [11] developed a noncommutative geometric formulation in which the Chern number is replaced by a cyclic cocycle expression for the Hall conductance.

The second perspective is the edge perspective. A bulk insulator may support conducting states near its boundary. For a half-plane system that remains translationally invariant along the edge, the edge Hamiltonian becomes a one-parameter family $\widehat{H}_e(k)$, and the edge spectrum may contain eigenvalue branches crossing the bulk spectral gap. The signed number of such crossings defines an edge index, equivalently a spectral flow. This gives a mathematical formulation of the physical idea that Hall current is carried by boundary modes. Early discussions of edge states and Hall conductance appear in Halperin's work [20], and the relation between Chern numbers and edge states was emphasized by Hatsugai [25]. Rigorous versions of the bulk–edge correspondence were developed using operator algebraic and K-theoretic methods by Kellendonk, Richter and Schulz-Baldes [29]. The approach most directly used in this thesis is the geometric method of Graf and Porta [18], where decaying solutions of a difference equation form a vector bundle whose transition functions encode the same topological information as the Bloch bundle. Related analytic formulations of the edge index are discussed in [16].

The third perspective is the many body charge transport perspective. Instead of relying on single-particle band theory, one considers a finite quantum system on a torus and threads magnetic flux through one cycle. The induced quasi-adiabatic evolution transports charge across a fiducial line, and the transported charge defines a many body index. This formulation is especially important for interacting systems, where there need not be a Bloch bundle or a single-particle Fermi projection. The mathematical control of this construction relies on locality estimates, beginning with the Lieb–Robinson bound [31], and on quasi-adiabatic continuation techniques [24]. Quantization of Hall conductance for interacting electrons on a torus was proved by Hastings and Michalakis [21]. A related many body index for charge transport was developed by Bachmann, Bols, De Roeck and Fraas [5], and rational quantization for ground state sectors was studied in [6]. These works provide the background for the many body results in Chapter 5.

The unifying theme of the thesis is that topological transport is controlled by the interaction between locality and spectral gaps. Locality ensures that perturbations, flux insertions, and boundary modifications have controlled spatial effects. A spectral gap stabilizes the relevant ground state or

Fermi projection. Together they allow one to define indices that are insensitive to local deformations. In the free Fermion setting, this stability appears through the topology of Bloch bundles and through exponentially localized Fermi projections. In the edge setting, it appears through the robustness of spectral flow under boundary perturbations. In the interacting setting, it appears through Lieb–Robinson bounds, exponential clustering, and quasi-local generators of adiabatic evolution.

Chapter 2.1 develops the free Fermion framework. We introduce tight binding Hamiltonians on $\ell^2(\mathbb{Z}^2, \mathbb{C}^N)$, finite range and translational invariance assumptions, and the Bloch decomposition. The occupied bands below the Fermi level define the Bloch bundle, whose first Chern number gives the bulk index. We also explain the Kubo formula and its reduction to the trace expression

$$\mathrm{i}\,\mathrm{Tr}\left(P\big[[\chi_1, P], [\chi_2, P]\big]\right),$$

where P is the Fermi projection and χ_1, χ_2 are switch functions. The Haldane model is used as a concrete example illustrating how changes in the Bloch Hamiltonian can change the Chern number through gap closing.

Chapter 3 studies the equality between the bulk and edge formulations. For a half-plane restriction of a tight binding Hamiltonian, the edge index is defined by the signed spectral flow of the edge eigenvalues across a regular energy in the bulk gap. Following the strategy of Graf and Porta [18], we introduce the decaying wave bundle and use it as an intermediate object relating the Bloch bundle to edge spectral flow. This chapter explains why a topological invariant computed from bulk eigenvectors is also visible in the boundary spectrum.

Chapter 4 applies these ideas to frustration free free Fermion Hamiltonians. Such models have a strong local positivity condition: the Hamiltonian is a sum of positive local terms whose kernels have a common nonzero intersection. We prove that, under the assumptions considered there, the edge index vanishes. While related conclusions may be inferred from flat band and K-theoretic considerations, the method used here is based on perturbative control of edge spectra in a particular lattice geometry. This result shows that strong locality and frustration free structure can obstruct nontrivial quantum Hall transport.

Chapter 5 turns to interacting quantum systems. The setting is a family of finite tori $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^2$ with uniformly gapped, short range, $U(1)$-invariant interactions. We review the quasi-local algebraic framework, the role of Lieb–Robinson bounds, and the construction of an almost inverse Liouvillian with exponential locality. These tools are then used to construct a flux-threading unitary and a charge transport operator. The main result proves that the many body Hall conductance, formulated as a charge transport index, is quantized up to an exponentially small error in the system size. This improves the superpolynomial error estimate of [6].

CHAPTER 2

Quantum Hall effect in free Fermionic system

In free Fermionic systems with translational invariance, the topological nature of quantum Hall effect is most readily apparent. Different phases are distinguished by different topology of certain vector bundles. We'll introduce vector bundles and their classifying index called Chern index. We refer to the textbook by [26] and lecture notes by [17] for the geometric definitions.

2.1. Free Fermions on a lattice

In this chapter we begin to construct a class of physics models that have a minimal setup assumptions, a clean mathematical structure, while being able to demonstrate most physical phenomena of quantum Hall effect. We call these models the tight binding models, following the terminology of solid state physics. They describe the energy eigenstates of non-interacting Fermions that occupy atomic orbitals in a crystal. This class of models are the setup in our work of proving zero Hall conductance in frustration free systems.

In our context, a lattice is a set of at most countable, discrete points embedded into $\mathbb{R}^d$ and equipped with the Euclidean metric from the embedding. In most cases the lattice will be denoted as Λ.

Definition 2.1

A tight binding model consists of

A lattice $\Lambda = \mathbb{Z}^d$,

A Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \cong \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$

A bounded self-adjoint operator called the Hamiltonian $H \in \mathcal{B}(H)$,

with additional assumptions:

- $\mathbb{Z}^d$ *is an abelian group by entry-wise integer addition, and a metric space with* $d(\boldsymbol{r}, \boldsymbol{s}) = \sum_{i=1}^{d} |r_i - s_i|$;
- *inner product of* $\mathcal{H}$ *is induced by those of* $\ell^2(\mathbb{Z}^d)$ *and* $\mathbb{C}^N$;
- $\mathbb{Z}^d$ *has an action on* $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ *as* $(T_{\boldsymbol{r}}\psi)(\boldsymbol{s}) = \psi(\boldsymbol{s} - \boldsymbol{r})\ \forall \boldsymbol{r}, \boldsymbol{s} \in \mathbb{Z}^d,\ \psi \in \mathcal{H}$;
- H *is translational invariant:* $T_{\boldsymbol{r}}\, H\, T_{\boldsymbol{r}}^{-1} = H$

Later we may also discuss tight binding models where the lattice is $\mathbb{N}\times\mathbb{Z}$, with translation group action in the $\mathbb{Z}$ component and monoid action on the $\mathbb{N}$ component. We call this a tight binding model with an edge.

A point in the lattice may not correspond to an atom in a physical crystal, but rather a translation group element of the crystal. And within a unit cell of the translation group, there may be more than one atom. Each of the atoms within a unit cell, together with each atomic orbital within an atom, contribute one of the N components in $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$.

Assumption 2.2

H is said to be gapped if $\sigma(H) \subseteq (-\infty, \gamma_1) \cup (\gamma_2, \infty)$ for some $\gamma_1 < \gamma_2$.

Proposition 2.1.1

If H is gapped and has short range, and the chemical potential is in the gap $\mu \in (\gamma_1, \gamma_2)$, then the Fermi projection $P_\mu := \chi_{(-\infty,\mu]}$ is exponentially localized

$$\exists C, \xi > 0,\ s.t.\ \|\left\langle \delta_{\boldsymbol{r}}\ ,\ P_\mu\, \delta_{\boldsymbol{s}} \right\rangle\| \,\leq\, C\exp(-d(\boldsymbol{r},\boldsymbol{s})/\xi),\ \forall \boldsymbol{r},\boldsymbol{s} \in \mathbb{Z}^2$$

This is established in [1].

Proof. A sketch of the proof is as follows: Let C be a CCW contour on the complex plane that encloses $\sigma(H) \cap (-\infty, \mu)$ and not intersecting $\sigma(H)$. Then

$$P(k) = \frac{1}{2\pi i}\oint_C (H(k) - z)^{-1} dz$$

$z \in C$ is not in $\sigma(H)$. And H has finite range so by Combes-Thomas estimate, the resolvent is also exponentially localized with some $L(z), l(z) > 0$ depending on z continuously

$$\left\langle \delta_{\boldsymbol{r}},\ (H-z)^{-1}\, \delta_{\boldsymbol{s}} \right\rangle < L(z)\ \exp(-l(z)\, d(\boldsymbol{r},\boldsymbol{s}))$$

And the compact contour allows the exponential localization of resolvent to pass to the Fermi projection $\square$

2.1.1. Finite interaction range and cell combination.

Definition 2.3

We say an operator $H \in \mathcal{B}(\ell^2(\Lambda, \mathbb{C}^N))$ has finite (interaction) range if there is an $R > 0$ such that

$$\left\langle \delta_{\boldsymbol{r}} \otimes \boldsymbol{u}\,, H\,\delta_{\boldsymbol{s}} \otimes \boldsymbol{v} \right\rangle = 0$$

for all $\boldsymbol{r}$, $\boldsymbol{s} \in \Lambda$ such that $d(\boldsymbol{r}, \boldsymbol{s}) > R$ and all $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{C}^N$. $\delta_{\boldsymbol{r}}$ is the Kronecker delta function supported on site $\boldsymbol{r} \in \Lambda$

By combining rectangular $a \times b$ cells into one cell, we have an isomorphism of Hilbert spaces

$$\phi_{ab} : \ell^2(\mathbb{Z}^2, \mathbb{C}^n) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^{a\,b\,n}) \tag{2.1}$$

$$\delta_{(pa+r, qb+s)} \otimes \boldsymbol{e}_j \mapsto \delta_{(p,q)} \otimes \boldsymbol{e}_{r\cdot s\cdot n+j}$$

where $(r < a,\ s < b,\ j \leq n) \in \mathbb{N}$ and $p, q \in \mathbb{Z}$. $\{\boldsymbol{e}_j\}$ are standard basis of $\mathbb{C}^n$. This leads naturally to the following:

Proposition 2.1.2 (Next nearest interaction)

Given $H \in \mathcal{B}(\ell^2(\Lambda, \mathbb{C}^N))$ having finite range and translational invariant, there are large enough $a, b \in \mathbb{N}$ such that $\tilde{H} := \phi_{ab}\, H\, \phi_{ab}^{-1}$ interacts among nearest and next-nearest neighbors:

$$\left\langle \delta_{\boldsymbol{r}}\,, \tilde{H}\,\delta_{\boldsymbol{s}} \right\rangle \neq 0 \ \textit{only if}\ \boldsymbol{r} - \boldsymbol{s} \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$$

As a result, every (finite range) tight binding model Hamiltonian can be decomposed into a sum of local terms

$$\tilde{H} = \sum_{\boldsymbol{x} \in \Lambda} T_{\boldsymbol{x}} \tilde{H}_0 T_{\boldsymbol{x}}^{-1}$$

such that $\tilde{H}_0 |\delta_{\boldsymbol{x}}\rangle$ only if $\boldsymbol{x} \in \{(1,0), (0,1), (1,1), (1,-1)\}$

Proof. For the first part, taking some $a, b > 2R$ greater than twice the interaction range will be sufficient.

For the local decomposition, let $\Pi_{\boldsymbol{x}}$ be the projection onto the internal degrees of freedom of block cell $\boldsymbol{x}$. Define

$$P := \{(1,0), (0,1), (1,1), (1,-1)\}.$$

One may choose the local term

$$\widetilde{H_0} = \Pi_0 \widetilde{H} \Pi_0 + \sum_{\boldsymbol{r} \in P} \left(\Pi_0 \widetilde{H} \Pi_{\boldsymbol{r}} + \Pi_{\boldsymbol{r}} \widetilde{H} \Pi_0 \right).$$

Then $\widetilde{H}_0$ is supported only on the block cells $\{(0,0),\ (1,0),\ (0,1),\ (1,1),\ (1,-1)\}$. And by translation invariance,

$$\widetilde{H} = \sum_{\boldsymbol{x} \in \mathbb{Z}^2} T_{\boldsymbol{x}} \widetilde{H}_0 T_{\boldsymbol{x}}^{-1},$$

where the sum is understood in the strong operator sense. Because the interaction range is finite, only finitely many translated local terms act nontrivially on any fixed basis vector, so this sum is well-defined. And it decomposes into translates of a local term supported on one cell together with the four positively oriented nearest/diagonal bonds. □

So without loss of generality, any tight binding model with finite range Hamiltonian can be seen as a model with next-nearest neighbors interaction. Specifically the Haldane model has next-nearest neighbors interaction. As a result of next-nearest neighbor interaction

2.1.2. Fourier transform. It is well-known that Fourier transform can be defined on any functional space whose domain has a finitely generated Abelian group action, as is our case of $\ell^2(\mathbb{Z}, \mathbb{C}^N)$. And the $\mathbb{Z}^d$ translational invariance of Hamiltonian in tight binding models brings more convenience. We denote $\mathcal{F}(\psi) = \widehat{\psi}$ and $\mathcal{F} H \mathcal{F}^{-1} = \widehat{H}$. The Fourier transform is the unitary map

$$\mathcal{F} : \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \to L^2(\mathbb{T}^d, \mathbb{C}^N), \quad \mathbb{T}^d = \big([0, 2\pi]/\langle 0 \sim 2\pi \rangle\big)^d$$

$$\psi \otimes \boldsymbol{e}_j \mapsto \widehat{\psi} \otimes \boldsymbol{e}_j, \quad \widehat{\psi}(\boldsymbol{k}) := \frac{1}{(2\pi)^{d/2}} \sum_{\boldsymbol{x} \in \mathbb{Z}^d} e^{-i\,\boldsymbol{k} \cdot \boldsymbol{x}} \psi(\boldsymbol{k})$$

Proposition 2.1.3

By translational invariance of H, we have a matrix valued function

$$\widehat{H} : \mathbb{T}^d \to M_N(\mathbb{C})$$

such that

$$\mathcal{F}(H\psi)(\boldsymbol{k}) = \widehat{H}(\boldsymbol{k}) \widehat{\psi}(\boldsymbol{k}),\ \forall \boldsymbol{k} \in \mathbb{T}^d$$

And if H has finite range then $\widehat{H}(\boldsymbol{k})$ has trig-polynomial entries.
In other words, $\widehat{H}$ acts as a fiberwise homomorphism on the trivial bundle $\mathbb{T}^d \times \mathbb{C}^N$.

Proof. $\ell^2(\mathbb{T}^d, \mathbb{C}^N)$ has an orthonormal basis $\{f_{x,j}(\cdot) := e^{i\,x(\cdot)} \otimes \boldsymbol{e}_j | x \in \mathbb{Z}^d,\, 1 \leq j\}$. It suffices to prove that $(\mathcal{F}H\mathcal{F}^{-1})f_{\boldsymbol{x},j}(\boldsymbol{k}) = \widehat{H}(\boldsymbol{k})f_{\boldsymbol{x},j}(\boldsymbol{k}),\ \forall \boldsymbol{k} \in \mathbb{T}^d$.

We know that translation act as a multiplication operator: $(\widehat{T}_{\boldsymbol{x}}\widehat{\psi})(\boldsymbol{k}) = e^{i\boldsymbol{x}\cdot\boldsymbol{k}}\,\widehat{\psi}(\boldsymbol{k})$. So the basis elements are generated by $f_{0,j}$ as $f_{\boldsymbol{x},j} = \widehat{T}_{\boldsymbol{x}}\, f_{0,j}$.

Let $\widehat{h}(\boldsymbol{k}) \in M_N(\mathbb{C})$ be a matrix with entries

$$\widehat{h}_{ij}(\boldsymbol{k}) := \left\langle \boldsymbol{e}_i\,, \widehat{H}\, f_{0,j} \right\rangle_{\mathbb{C}^N} (\boldsymbol{k})$$

By translational invariance:

$$\widehat{H} f_{x,j} = \widehat{H}\widehat{T}_x f_{0,j} = \widehat{T}_x \widehat{H} f_{0,j}$$

Evaluating at $\boldsymbol{k}$

$$\widehat{H} f_{x,j}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}(\widehat{H} f_{x,j})(\boldsymbol{k})$$

As an equation of $\mathbb{C}^N$ vector with $L^2(\mathbb{T}^d)$ entries, we can insert the $\boldsymbol{k}$-constant identity matrix $\sum_{i=1}^N \boldsymbol{e}_i\boldsymbol{e}_i^*$

$$\widehat{H} f_{x,j}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{i=1}^{N} \boldsymbol{e}_i\boldsymbol{e}_i^*(\widehat{H} f_{x,j})(\boldsymbol{k})$$

We can associate the $\boldsymbol{e}_i^*$ to the right

$$\widehat{H} f_{x,j}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{i=1}^{N} \boldsymbol{e}_i\,(\boldsymbol{e}_i^*\widehat{H} f_{x,j})(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{i=1}^{N} \boldsymbol{e}_i\widehat{h}_{ij}(\boldsymbol{k}) = \sum_i f_{x,j}(\boldsymbol{k})\widehat{h}_{ij}(\boldsymbol{k})$$

So we see $\widehat{H}\widehat{\psi}(\boldsymbol{k}) = \widehat{h}(\boldsymbol{k})\psi(\boldsymbol{k})$.

As functions of $\boldsymbol{k} \in \mathbb{T}^d$, the Fourier coefficients of $\widehat{H}(\boldsymbol{k})$ are exactly $\left\langle \delta_{\boldsymbol{0}}, H\,\delta_{\boldsymbol{x}} \right\rangle_{\ell^2(\mathbb{Z}^d)}$, $\boldsymbol{x} \in \mathbb{Z}^d$. If H has finite interaction range, then only finitely many Fourier coefficients are nonzero, so $\widehat{H}(\boldsymbol{k})$ has trig polynomial entries. □

2.2. Chern index of vector bundle

2.2.1. Vector bundle basics.

Let M be a real manifold that is connected and Hausdorff.

Definition 2.4

A fiber bundle $\xi = (E, \pi, M)$ is a triple consisting of a topological space E, the base manifold M, a surjective smooth map $\pi : E \to M$ such that all preimages $\pi^{-1}(x)\,\forall x \in M$ are diffeomorphic to the *same space F called the fiber.*

A $\mathbb{K}$-vector bundle is a fiber bundle whose fibers are diffeomorphic to a $\mathbb{K}$-vector space.

Throughout this thesis we always consider vector bundles over the complex field.

Given a vector bundle one can study its subsets that still have a bundle structure:

Definition 2.5

(E', π', M') is a sub-bundle of (E, π, M) if $E' \subset E$, $M' \subset M$ and $\pi' = \pi|_{E'}$.

In particular we are interested in the subbundle that has the same base space as parent bundle. In this case each fiber of the sub-bundle is a subspace of the fiber of parent bundle.

Maps between vector bundles that preserve the bundle structure are called bundle maps/homomorphisms:

Definition 2.6

A vector bundle homomorphism between (E, π, M) and (E', π', M') is a pair of linear maps $\phi : E \to E'$ and $f : M \to M'$ such that

$$\pi' \circ \phi = f \circ \pi$$

Since π is surjective, the map f between base manifolds is uniquely determined by the map ϕ through $f(x) = \pi' \circ \phi \circ \pi^{-1}(x)$. And the RHS is a well-defined map unaffected by different elements in $\pi^{-1}(x)$ because $\phi \circ \pi^{-1}(x) \subseteq \pi'^{-1} \circ f(x)$, i.e. the bundle map sends fiber $\pi^{-1}(x)$ into fiber $\pi'^{-1} \circ f(x)$.

We classify vector bundles up to isomorphism, that is

Definition 2.7

(ϕ, f) is called a bundle isomorphism if there is another pair of maps (ϕ', f') such that

$$\phi \circ \phi' = \mathbb{I}'_E,\ \phi' \circ \phi = \mathbb{I}_E;$$

$$f \circ f' = \mathbb{I}_N,\ f' \circ f = \mathbb{I}_M$$

Bundle map $(\phi, \mathbb{I}_M)$ between (E, π, M) and (E', π', M) over the same base M can be seen as a smooth family of linear maps $\phi_x : \mathrm{Hom}\big(\pi^{-1}(x),\ \pi'^{-1}(x)\big)$, $x \in M$. E and E' are isomorphic if and only if ϕ_x are linear isomorphisms for all x.

The idea of sections of a vector bundle generalizes $\mathbb{R}^N$ valued functions on the base manifold:

Definition 2.8

Given a bundle (E, π, M), a section $s : M \to E$ is a smooth map such that

$$\pi \circ s = \mathbb{I}_M$$

A straightforward criterion of trivial bundle is whether it has a global section of basis:

Proposition 2.2.1

A vector bundle (E, π, M) is trivial if and only if it has N sections $\{s_i\}_{i=1}^N$ such that at each base point $\{s_i(x)\}_{i=1}^N$ forms a basis of the fiber $\pi^{-1}(x)$.

PROOF. If $\{s_i(x)\}$ forms a basis at each x, then the decomposition coefficients

$$C_x : \pi^{-1}(x) \to \mathbb{C}^N$$

$$y = \sum_{i=1}^{N} c_i s_i \mapsto (c_1, ..., c_i, ...c_N)$$

is a smooth family of linear isomorphisms. Hence we have a bundle isomorphism $(C, \mathbb{I}_M)$: $(E, \pi, M) \to M \times F$.

If $(C, \mathbb{I}_M) : (E, \pi, M) \to (M \times \mathbb{C}^N, \pi_1, M)$ is a bundle isomorphism, then C_x is a family of linear isomorphisms. $\pi_1 \circ C = \mathbb{I}_M \circ \pi$. Let e_i be the standard basis of $\mathbb{C}^N$ then $(x, e_i) \subset \pi_1^{-1}(x)$, $C_x^{-1}(x, e_i)$, are a basis of $\pi^{-1}(x)$. $\square$

The sections of a vector bundle carry the structure of a $C^\infty(M)$- module, as each section s can be pointwise scalar multiplied by $r \in C^\infty(M)$: $r \cdot s(x) = r(x) \cdot s(x)$; and two sections can perform pointwise vector addition $(s + t)(x) = s(x) + t(x)$, both operations produce sections.

Studying sections of a vector bundle reveals great amount of structures of the vector bundle. In fact taking sections is a functor, from the category of $\mathbb{K}$-vector bundle to the category of modules over $C^\infty(M)$. And Swan's theorem [36] shows that there is a one-to-one correspondence between vector bundles over compact base manifolds and projective modules over $C^\infty(M)$. Later this idea is developed by N.Read and later inspired us to prove the triviality of frustration free tight binding models.

Let $\mathbb{T}^2$ be the 2 torus: $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi] / \big((0, k_2) \sim (2\pi, k_2),\ (k_1, 0) \sim (k_1, 2\pi)\big)$.

2.2.2. Bloch bundles. The trivial bundle $\mathbb{T}^2 \times \mathbb{C}^N$ is of our interest since the sections of this bundle form the Hilbert space $L^2(\mathbb{T}^2, \mathbb{C}^N)$ that is isomorphic to the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C}^N)$ through Fourier transform.

Given a smooth family of $N \times N$ self-adjoint matrices parametrized by the compact manifold $\mathbb{T}^2$: $H : \mathbb{T}^2 \to \mathrm{Hom}(\mathbb{C}^N)$, there are finitely many eigenvalues, and all are uniformly bounded real numbers due to compactness of $\mathbb{T}^2$.

If there is a gap in the union of spectra:

$$\bigcup_{k \in \mathbb{T}^2} \sigma(H(k)) \subset (-\infty, \gamma_1) \cup (\gamma_2, \infty)$$

then the projections $P(k)_{1,2}$ on each part of the joint spectra has constant rank and varies smoothly with $k \in \mathbb{T}^2$ because it has a resolvent expression

$$P_{1,2}(k) = \frac{-1}{2\pi i} \oint_{C_{1,2}} (H(k) - z)^{-1} dz$$

where $C_{1,2}$ is a CCW contour on the complex plane that encloses the 1st or 2nd part of the joint spectra but intersects with none of them. Continuous functional calculus ensures that $(H(k) - z)^{-1}$ is smooth in k for all z, and compact integration contour C preserves the smoothness.

Definition 2.9 (Bloch bundles)

Given a smooth family of $N \times N$ self-adjoint matrices parametrized by the compact manifold $\mathbb{T}^2$: $H : \mathbb{T}^2 \to \mathrm{Hom}(\mathbb{C}^N)$, and the joint spectrum is separated by a gap into two parts, then the union of invariant subspaces is called Bloch bundles. Specifically

$$G := \bigcup_{k \in \mathbb{T}^2} \mathrm{Im}(P_1(k))$$

is the ground state bundle and

$$E := \bigcup_{k \in \mathbb{T}^2} \mathrm{Im}(P_2(k))$$

is the excited state bundle.

The base is $\mathbb{T}^2$ and the bundle projection is $\pi : \mathrm{Im}(P(k)) \mapsto k$.

As subsets of $\mathbb{T}^2 \times \mathbb{C}^N$, they carry the subset topology.

2.2.3. Curvature and Chern Index. In most classical texts [37] [38] of quantum Hall effect, the Chern index is defined through calculating the Berry connection, the curvature and then integrate its trace over the Brillouin zone. We'll take a hybrid of intrinsic and frame dependent approach to show its topological nature and to connect with literature in condensed matter physics. As a computational definition of Berry connection, one must first choose an orthonormal section of the ground state bundle G

$$v_j : U \subseteq \mathbb{T}^2 \to \mathrm{Im}(P),\ s.t.\ P(\boldsymbol{k}) = \sum_{j=1}^{N} |v_j(\boldsymbol{k})\rangle\langle v_j(\boldsymbol{k})|,\ \forall \boldsymbol{k} \in U \tag{2.2}$$

Upon choosing such a section as a chart, the Berry connection on the ground state bundle can be read out as a $M_N(\mathbb{C})$ valued 1-form, also known as the Christoffel symbol of a covariant derivative.

$$A(\boldsymbol{k})_{j,k} = \sum_{i=1}^{2} A^i(\boldsymbol{k}) dk_i = \sum_{i=1}^{2} \left\langle v_j(\boldsymbol{k}), \frac{\partial}{\partial k_i} v_k(\boldsymbol{k}) \right\rangle dk_i \tag{2.3}$$

$$\nabla_i \phi(\boldsymbol{k}) = \frac{\partial}{\partial k_i}\phi(\boldsymbol{k}) + A^i(\boldsymbol{k})\phi(\boldsymbol{k}) \tag{2.4}$$

Intrinsically this covariant derivative can be written as $\nabla_i \phi\,(\boldsymbol{k}) = P(\boldsymbol{k})\,\partial_i \phi(\boldsymbol{k})$ without reference to a choice of section of the ground state bundle. It has a clearer interpretation as

Remark 2.2.2

A section $\phi(\boldsymbol{k})$ is parallel to the connection, if its change in the ambient space $\mathbb{T}^2 \times \mathbb{C}^N$ is in the kernel of projection $P(\boldsymbol{k})$.

The curvature tensor is defined intrinsically as

$$F_{i,j} = [\nabla_i, \nabla_j] - \nabla_{[k_i,k_j]} \tag{2.5}$$

where the second term is zero if we take the $[0, 2\pi) \times [0, 2\pi)$ coordinate system on torus. Thanks to the frame independent expression of ∇ we can quickly derive the expression of curvature after a few Leibniz rule:

$$F_{i,j} = [P\frac{\partial}{\partial k_i}, P\frac{\partial}{\partial k_j}] = P[\frac{\partial}{\partial k_i}P, \frac{\partial}{\partial k_j}P] \tag{2.6}$$

The $\mathrm{Tr}(F)$ is an *invariant polynomial* of the Bloch bundle, namely:

$$T : M_N(\mathbb{C}) \to \mathbb{C},\ \text{a polynomial of } N \times N \text{ variables}$$

$$\text{such that} \quad T(U^{-1}FU) = T(F), \quad \forall U \in GL_N(\mathbb{C}),\ F \in M_N(\mathbb{C})$$

$\mathrm{Tr}(\cdot)$ is such an invariant polynomial. Taking any local chart on $\mathbb{T}^2$, the curvature tensor is a $M_N(\mathbb{C})$ valued 2-form, and the polynomial is

$$T(F_{i,j}) = \sum_i (F_{i,i}) = \mathrm{Tr}(F) \ \in \Omega^2(\mathbb{T}^2)$$

The $GL(N)$-invariance of trace implies that $\mathrm{Tr}(F)$ is a well-defined scalar 2-form unaffected by the transition between different frames 2.2 of the Bloch bundle.

Proposition 2.2.3

Let $F_{i,j} = [\nabla_i, \nabla_j] - \nabla_{[k_i,k_j]}$ be the curvature of a vector bundle, then $\mathrm{Tr}(F)$ is a closed form.

And thus the cohomology class $[\mathrm{Tr}(F)] \in H^2(\mathbb{T}^2)$ called the Chern class. Integrating the Chern class on the base manifold $\mathbb{T}^2$ gives the Chern index

$$\mathrm{Ch}_1\Big(\bigsqcup_{\boldsymbol{k}\in\mathbb{T}^2} \mathrm{Im}(P(\boldsymbol{k}))\Big) = i\int_{\boldsymbol{k}\in\mathbb{T}^2} \mathrm{Tr}_{\mathbb{C}^N}\Big(P(\boldsymbol{k})\left[\frac{\partial}{\partial k_1}P(\boldsymbol{k}),\ \frac{\partial}{\partial k_2}P(\boldsymbol{k})\right]\Big)\frac{dk}{(2\pi)^2}$$

The fact that such index is integer-valued is due to Gauss Bonnet theorem.

2.2.4. Gauge freedom in local observable versus in Bloch bundles. In classical texts that computes Hall conductance through Berry connection 2.3, a choice of gauge, namely a local frame of Bloch bundle, is always involved, and sometimes implicitly. Later difference in the gauge choice would disappear when calculating the Berry scalar curvature $\mathrm{Tr}(F)$.

Observation 2.2.4

An interesting trivia to the author is that decomposition of the Hamiltonian into local terms is a gauge freedom, and corresponds to the gauge freedom in Bloch bundle sections after a Fourier transform. In fact information of topological invariants is hidden in possibility of certain local decomposition, and more obviously the possibility of global frames of Bloch bundles. When generalizing to many body quantum system without translational symmetry, the gauge freedom of local decomposition becomes the only source of topological information.

Such gauge freedom of local decomposition, and its invariant classes, are extensively developed by A. Kapustin *et. al.* in a sequel of articles [27] [2] into a homology theory, where the chain complex is made of n-currents in the quasilocal algebra.

Assume that $\widehat{H}(\boldsymbol{k})$ has no degenerate or zero eigenvalues so that there are N Bloch bundles of rank 1. By finite interaction range, $\widehat{H}$ is a matrix with entries being polynomials of $e^{\pm ik_1}, e^{\pm ik_2}$ which we shall call it trig-polynomial. H has a local decomposition as

$$H_0 = \frac{1}{2}\Big(\sum_{d(\boldsymbol{x},\boldsymbol{0})\leq R} \Big\langle \delta_{\boldsymbol{x}},\, H\,\delta_0 \Big\rangle\, |\delta_{\boldsymbol{x}}\rangle\langle\delta_{\boldsymbol{0}}| + h.c.\Big), \quad H = \sum_{\boldsymbol{x}\in\mathbb{Z}^2} T_{\boldsymbol{x}} H_{\boldsymbol{0}} T_{\boldsymbol{x}}^{-1}$$

One may diagonalize the local term $H_{\boldsymbol{0}}$ within its supported subspace and yield $\{|v_j\rangle\} \in \ell^2(\mathrm{supp}(H_0), \mathbb{C}^N)$ that are mutually ℓ^2 orthogonal

$$H_{\boldsymbol{0}} = \sum_{j}^{\text{finite}} E_j |v_j\rangle\langle v_j|$$

Their Fourier transforms $\{\widehat{v}_j\}$ are of trig-polynomial entries as well, and they are sections of $\bigsqcup_{\boldsymbol{k}\in\mathbb{T}^2} \mathrm{Im}(\widehat{H}(\boldsymbol{k}))$. A gauge transformation such as:

$$S : \widehat{v}_j \mapsto \widehat{u}_j = e^{i\,\boldsymbol{s}_j\cdot\boldsymbol{k}}\,\widehat{v}_j$$

alters the section within the Bloch bundle, and leaves the full Hamiltonian invariant, as the phase $e^{i\,\boldsymbol{s}_j\cdot\boldsymbol{k}}\,\boldsymbol{v}_{i,j}$ cancels that of $e^{-i\,\boldsymbol{s}_j\cdot\boldsymbol{k}}\,\boldsymbol{v}_{i,j}^{\dagger}$. However this induces a different local decomposition of Hamiltonian,

$$S(H_{\boldsymbol{0}}) = \sum_j E_j |\mathcal{F}^{-1}(\widehat{u}_j)\rangle\langle\mathcal{F}^{-1}(\widehat{u}_j)| = T_{s_j}|v_j\rangle\langle v_j|T_{s_j}^{-1}$$

now $S(H_{\boldsymbol{0}})$ is supported near sites $\cup_j \boldsymbol{s}_j$.

It is tempting to build a frame of Bloch bundle by Fourier transforming $\{|v_j\rangle\}_j$. This may not work because $\{|v_j\rangle\}_j$ being an ℓ^2 basis of $\mathrm{Im}(H_0)$ does not imply $\{\widehat{v}(\boldsymbol{k})_j\}_j$ being an $\mathbb{C}^N$ basis of $\mathrm{Im}(\widehat{H}(k))$. On the other hands, projectors onto Bloch bundles may not be a trig-prolynomial matrix. And we arrive at the following observation

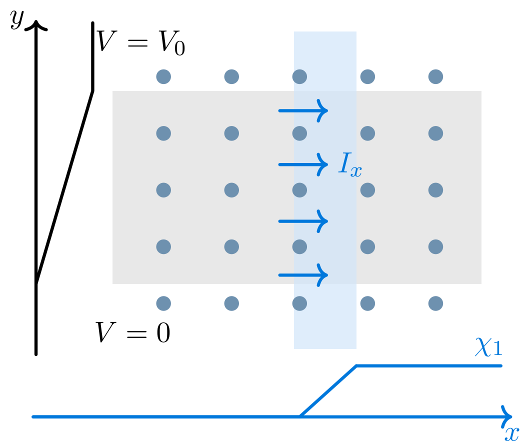
Observation 2.2.5

Let $\widehat{P}(\boldsymbol{k})$ be a spectral projection of $\widehat{H}(\boldsymbol{k})$ gapped away from the rest of $\sigma(\widehat{H}(\boldsymbol{k}))$. The Bloch bundle $\bigsqcup_{\boldsymbol{k}} \Big(\mathrm{Im}\,\widehat{P}(\boldsymbol{k})\Big)$ is trivial if P has a local decomposition.

This observation leads to our interest in flat band and frustration free Hamiltonians. We developed it into theorem 4.8 in Chapter 4.

2.3. Kubo formula: Hall conductance as a linear response coefficient

Among all the formulations of Hall conductance, the Kubo formula provides the most detailed depiction of the experimental setup.



We model a 2D crystalline material with the tight binding model $\big(\mathbb{Z}^2,\ \ell^2(\mathbb{Z}^2, \mathbb{C}^N),\ H\big)$. Assume a spectral gap $\sigma(H) \subseteq (-\infty, 0] \cup (\gamma, \infty)$. A time-dependent electric potential $V(t)$ is applied across the region $\mathbb{Z}_x \times \big(\mathbb{Z}_y \cap [-R, R]\big)$. The electric field was 0 at $t \to -\infty$ and gradually ($\epsilon << 1$) turned on to be $V_0\, e^{\epsilon t}$ until $t = 0$. As a multiplicative operator on $\ell^2(\mathbb{Z}^2, \mathbb{C}^N)$,

$$(V(t)\psi)\,(x, y) = V_0\, e^{\epsilon t}\ \tilde{\chi}_2(x, y)\, \psi(x, y)$$

where $\tilde{\chi}$ is a soft switch function such that

$$\tilde{\chi}(y < -R) = 0,\ \tilde{\chi}(y > R) = 1; \quad \tilde{\chi}_2(x, y) := \tilde{\chi}(y)$$

Let $P(t)$ be the instantaneous Fermi/ground state projection of the perturbed Hamiltonian, and P the unperturbed Hamiltonian

$$H(t) = H + V_0\, e^{\epsilon t}\, \tilde{\chi}_2\,; \quad P(t) = \chi_{(-\infty,\mu]} H(t); \quad P = \chi_{(-\infty,\mu]} H; \quad \mu \in (0, \gamma)$$

At $t = 0$ the current flowing into the region $\mathbb{N}_x \times \mathbb{Z}_y$ is measured. Let $\chi_1 = \chi_{\mathbb{N}\times\mathbb{Z}}$ be the indicator function supported on this region. The current expectation can be expressed as the in-region particle number increment rate:

$$\big\langle I_x \big\rangle = \frac{d}{dt}\Big|_{t=0} \mathrm{Tr}\big(P(t)\, \chi_1\big) = \mathrm{Tr}\big(P(0)\, i[H(0), \chi_1]\big)$$

Note that $P(0), H(0)$ both have V_0 and ϵ dependency.

Definition 2.10

The linear response coefficient κ_ϵ is the 1st order coefficient of perturbed expectation $\mathrm{Tr}\big(P(0)\, i[H(0), \chi_1]\big)$ in terms of perturbation V_0:

$$\mathrm{Tr}\big(P(0)\, i[H(0), \chi_1]\big) = \mathrm{Tr}\big(P\, i[H, \chi_1]\big) + \kappa_\epsilon\, V_0 + \mathcal{O}(V_0^2)$$

the Hall conductance is the limit $\kappa_H = \lim_{\epsilon \to 0} \kappa_\epsilon$

Theorem 2.11 (Kubo [30])

In the experimental setup as above, the Hall conductance has the expression

$$\kappa_H = i \int_0^\infty \mathrm{Tr}\Big(e^{itH}\, [H, \chi_1]\, e^{-itH}\, [\tilde{\chi}_2, P]\Big)\, dt \tag{2.7}$$

Proposition 2.3.1

The Hall conductance has another expression

$$\kappa_H = i\, \mathrm{Tr}\left(P\big[[\chi_1, P], [\tilde{\chi}_2, P]\big]\right) \tag{2.8}$$

And κ_H is independent of change in $\tilde{\chi} \mapsto \chi = \tilde{\chi} + g$, for any $g : \mathbb{Z} \to \mathbb{Z}$ compactly supported and $\sum_{x \in \mathbb{Z}} g(x) = 0$.

Derivation of this expression from the Kubo formula can be found in [17].

2.3.1. Trace class operators. Generalizing the concept of trace from finite dimensional matrices to operators on a (infinite dimensional) Hilbert space can lead to some surprising phenomena. In fact the expression $\kappa_H = i\, \mathrm{Tr}\left(P\big[[\chi_1, P], [\chi_2, P]\big]\right)$ deserves some sanity checking: we need to guarantee the sum of infinitely many diagonal entries converges and is independent of the choice of basis.

Note 2.3.2

When discussing the trace class properties on $\ell(\mathbb{Z}^2, \mathbb{C}^N)$ we shall hide the $\mathbb{C}^N$ tensor factor.

Definition 2.12

Given a separable Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_i\}_{i=1}^\infty$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called trace class if $\Big\langle e_i,\, \sqrt{A^*A}\, e_i \Big\rangle < \infty$. Denoted as $A \in \mathcal{J}_1$. Such A has a trace that is independent *of the basis.*

$$\mathrm{Tr}(A) := \sum_{i=1}^{\infty} \Big\langle e_i,\, Ae_i \Big\rangle$$

For another O.N.B. $\{f_i\}_{i=1}^{\infty}$

$$= \sum_{i=1}^{\infty} \Big\langle f_i,\, Af_i \Big\rangle$$

Proposition 2.3.3

*$\mathcal{J}_1$ is a two-sided *-ideal in $\mathcal{B}(\mathcal{H})$, that is:*

$\mathcal{J}_1$ is a vector space;

$A \in \mathcal{J}_1 \implies AB,\, BA \in \mathcal{J}_1,\ \forall B \in \mathcal{B}(\mathcal{H})$*;*

$A \in \mathcal{J}_1 \implies A^* \in \mathcal{J}_1$.

Proposition 2.3.4

If $A \in \mathcal{J}_1$ *then* $\forall B \in \mathcal{B}(\mathcal{H}),\ \mathrm{Tr}(AB) = Tr(BA)$

The proof of the previous two propositions can be found in standard textbook such as [34].

Now with the definition and properties of trace class operator we can prove that $\mathrm{Tr}\,\big(P\big[[\chi_1, P], [\chi_2, P]\big]\big)$ is well-defined, if we assume H to be gapped and has finite range:

Proposition 2.3.5

Let P be exponentially localized

$$\exists C, \xi > 0,\ s.t.\ \|\Big\langle \delta_{\boldsymbol{r}}\,,\, P\,\delta_{\boldsymbol{s}} \Big\rangle\| \,\leq\, C\exp(-d(\boldsymbol{r}, \boldsymbol{s})/\xi),\ \forall \boldsymbol{r}, \boldsymbol{s} \in \mathbb{Z}^2$$

then

$[\chi_1, P]\,[\chi_2, P] \in \mathcal{J}_1$*, as is* $P\big[[\chi_1, P], [\chi_2, P]\big] \in \mathcal{J}_1$

PROOF. For either $i = 1$ or 2 the component of $\mathbb{Z}^2$:

χ_i has delta function $\chi_i|\delta_{\boldsymbol{r}}\rangle = \chi_i(\boldsymbol{r})|\delta_{\boldsymbol{r}}\rangle$, $\forall \boldsymbol{r} \in \mathbb{Z}^2$ as eigenvectors,

$$\begin{aligned}
\|\Big\langle \delta_{\boldsymbol{r}},\, (\chi_i P - P\chi_i)\, \delta_{\boldsymbol{s}} \Big\rangle\| &= \|\Big\langle \delta_{\boldsymbol{r}}\chi_i,\, P\, \delta_{\boldsymbol{s}} \Big\rangle - \Big\langle \delta_{\boldsymbol{r}},\, P\chi_i\, \delta_{\boldsymbol{s}} \Big\rangle\| \\
&= \|\Big\langle \delta_{\boldsymbol{r}}\chi_i(\boldsymbol{r}),\, P\, \delta_{\boldsymbol{s}} \Big\rangle - \Big\langle \delta_{\boldsymbol{r}},\, P\chi_i(\boldsymbol{s})\, \delta_{\boldsymbol{s}} \Big\rangle\| \\
\text{eigenvalues } \chi_i(\boldsymbol{r}/\boldsymbol{s}) \text{ commute to the front} \quad &\le |\chi_i(\boldsymbol{r}) - \chi_i(\boldsymbol{s})|\, \|\Big\langle \delta_{\boldsymbol{r}}, P\delta_{\boldsymbol{s}} \Big\rangle\| \\
&\le \begin{cases} C\exp(-d(\boldsymbol{r},\boldsymbol{s})/\xi) & \text{if } sgn(r_i) \neq sgn(s_i) \\ 0 & \text{if } sgn(r_i) = sgn(s_i) \end{cases}
\end{aligned}$$

So in the nonzero case, different sign of $\mathbb{Z}^2$ components implies $d(\boldsymbol{r} - \boldsymbol{s}) = |r_1| + |s_1| + |r_2| + |s_2|$ and we are using the 1-norm on $\mathbb{Z}^2$. Sum over $\boldsymbol{s}$ gives

$$\Big\| \sum_{\boldsymbol{s}\in\mathbb{Z}^2} \Big\langle \delta_{\boldsymbol{r}},\, (\chi_i P - P\chi_i)\, \delta_{\boldsymbol{s}} \Big\rangle \Big\| \le C\,\xi^2 \exp\big(-\|\boldsymbol{r}\|/\xi\big)$$

Now we verify the trace class condition of $[\chi_1, P]\,[\chi_2, P]$:

Let $P_s := |\delta_s\rangle\langle\delta_s|$, then

$$\begin{aligned}
\Big|\mathrm{Tr}\big([\chi_1, P]\,[\chi_2, P]\big)\Big| &= \Big\|\Big\langle \delta_{\boldsymbol{r}}[\chi_2, P]^*\,[\chi_1, P]^*\,,\, [\chi_1, P]\,[\chi_2, P]\,\delta_{\boldsymbol{r}} \Big\rangle\Big\| \\
&\le \sum_{\boldsymbol{r}\in\mathbb{Z}^2} \sum_{\boldsymbol{s}_1, \boldsymbol{s}_2 \boldsymbol{s}_3} \Big\|\Big\langle \delta_{\boldsymbol{r}}[\chi_2, P]^*\, P_{s_1}\,[\chi_1, P]^*\, P_{s_2},\, [\chi_1, P]\, P_{s_3}\,[\chi_2, P]\,\delta_{\boldsymbol{r}} \Big\rangle\Big\| \\
&\le C\,\xi^8\, \exp(-1/\xi) < \infty
\end{aligned}$$

So $[\chi_1, P]\,[\chi_2, P] \in \mathcal{J}_1$, similarly $[\chi_2, P]\,[\chi_1, P] \in \mathcal{J}_1$ □

Another surprise is that the usual cyclicity within trace seems to imply trace of a commutator is always zero. And if one performs some algebra to the real lattice expression of Chern index:

$$
\begin{aligned}
& P\big[[\chi_1, P], [\chi_2, P]\big] \\
& = P\left((\chi_1 P - P\chi_1)(\chi_2 P - P\chi_2) - (\chi_2 P - P\chi_2)(\chi_1 P - P\chi_1)\right) \\
& \text{already we can cancel a pair in each line} \\
& = P(\chi_1 P\chi_2 P - \cancel{P\chi_1 P\chi_2} - P\chi_1\chi_2 P + \cancel{P\chi_1 P\chi_2}) \\
& - P(\chi_2 P\chi_1 P - \cancel{P\chi_2 P\chi_1} - P\chi_2\chi_1 P + \cancel{P\chi_2 P\chi_1}) \\
& \text{the multiplicative operators } \chi_1\, \chi_2 \text{ commute, so} \\
& = P(\chi_1 P^2 \chi_2 P) - P(\chi_2 P^2 \chi_1 P) = \Big[P\chi_1 P,\ P\chi_2 P\Big]
\end{aligned}
$$

and naively using cyclicity of trace this would imply

$$\kappa_H \sim \mathrm{Tr}\left((P\chi_1 P)\,(P\chi_2 P)\right) - \mathrm{Tr}\left((P\chi_2 P)\,(P\chi_1 P)\right) \sim 0$$

However there are obvious example of models with nonzero κ_H. Both can be explained by the concept of trace class operator.

The answer to our paradox is that $P\chi_1 P$ and $P\chi_2 P$ are not of trace class, so neither splitting κ_H into $\mathrm{Tr}\left((P\chi_1 P)\,(P\chi_2 P)\right) - \mathrm{Tr}\left((P\chi_2 P)\,(P\chi_1 P)\right)$ nor the cyclicity $\mathrm{Tr}\left((P\chi_1 P)\,(P\chi_2 P)\right) = \mathrm{Tr}\left((P\chi_2 P)\,(P\chi_1 P)\right)$ are well-defined.

Remark 2.3.6

$P[[\chi_1, P], [\chi_2, P]] \in \mathcal{J}_1$ *as well. Yet the splitting*

$$\mathrm{Tr}\Big(P\big[[\chi_1, P], [\chi_2, P]\big]\Big) = Tr\big(P[\chi_1, P]\,[\chi_2, P]\big) - \mathrm{Tr}\big(P[\chi_2, P]\,[\chi_1, P]\big)$$

may not be 0, because the cyclicity within trace cannot permute one term to the other, and the first P *is the barrier.*

2.3.2. Kubo formula is Chern index. The expression 2.8 visually resembles a curvature, and indeed after the Fourier transform, it is the integral of the curvature of the ground state bundle G, i.e. the Chern index.

Proposition 2.3.7

$$i\,\mathrm{Tr}\Big(P\big[[\chi_1,P],[\chi_2,P]\big]\Big) = Ch_1(\bigsqcup_{\boldsymbol{k}\in\mathbb{T}^2}\mathrm{Im}(P(\boldsymbol{k})))$$

PROOF.

$$\mathrm{Ch}_1(\bigsqcup_{\boldsymbol{k}\in\mathbb{T}^2}\mathrm{Im}(P(\boldsymbol{k}))) = i\int_{\boldsymbol{k}\in\mathbb{T}^2}\mathrm{Tr}_{\mathbb{C}^N}\Big(P(\boldsymbol{k})\Big[\frac{\partial}{\partial k_1}P(\boldsymbol{k}),\ \frac{\partial}{\partial k_2}P(\boldsymbol{k})\Big]\Big)\frac{dk}{(2\pi)^2}$$

The key observations to the left hand side are $\mathcal{F}\,i\,[x_i,P]\,\mathcal{F}^{-1}(\boldsymbol{k}) = \frac{\partial}{\partial k_i}P(\boldsymbol{k})$:

$$\begin{aligned}\forall\psi\in Ł^2(\mathbb{T}^2,\mathbb{C}^N),\quad Ad_{\mathcal{F}}\Big(i\,[x_i,\ P]\Big)\psi(k) =& i\Big[Ad_{\mathcal{F}}\,x_i,\ Ad_{\mathcal{F}}P\Big]\psi\,(k)\\ =&\Big[\frac{\partial}{\partial k_i},\ P(k)\Big]\psi\,(k)\\ =&\frac{\partial}{\partial k_i}\big(P(k)\psi(k)\big) - P(k)\,\frac{\partial}{\partial k_i}\psi(k)\\ =&\Big(\frac{\partial}{\partial k_i}P(k)\Big)\psi(k)\end{aligned}$$

As κ_H is independent of the details of the switch function χ_i, we replace χ_i with $\chi_{x_i\in[0,L]}\,\dfrac{x_i}{L}$ as the latter is also a switch function from 0 to 1. Now

$$\kappa_H = i\,\mathrm{Tr}\Big(\frac{1}{L^2}\chi_{x_i\in[0,L]}\,P\big[[x_1,P],[x_2,P]\big]\Big)$$

Taking the limit of $L\to\infty$,

$$\lim_{L\to\infty}\frac{1}{L^2}\mathrm{Tr}\Big(\chi_{x_i\in[0,L]}\quad\cdot\Big) = \int_{\boldsymbol{k}\in\mathbb{T}^2}\mathrm{Tr}_{\mathbb{C}^N}\Big(\cdot\Big)\frac{dk^2}{(2\pi)^2}$$

And we have the expression of Hall conductance in $L^2(\mathbb{Z}^2,\mathbb{C}^N)$. □

2.4. An example: Haldane model

The most well-known example of such tight binding models is the Haldane model, proposed by F.D.M. Haldane in 1988 [19] as a model that demonstrates integer quantum Hall effect with no external magnetic field and the Hamiltonian has explicit lattice translational invariance. The lattice, though often drawn as a tiling of right hexagons, is actually $\mathbb{Z}^2$, with a parallelogram unit cell containing 2 atoms at the vertices of hexagons, denoted as atomic orbital A and B of the unit cell. Using the isomorphism $\ell^2(\mathbb{Z}^2,\mathbb{C}^N)\cong\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^N$ we present the Hamiltonian as elements of

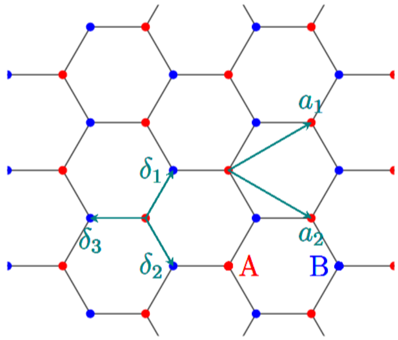


FIGURE 2.1. The lattice of Haldane's model is $\Lambda = \mathbb{Z}\boldsymbol{a}_1 \oplus \mathbb{Z}\boldsymbol{a}_2 \cong \mathbb{Z}^2$, there are two orbitals within each unit cell, labeled as A in red and B in blue.
To keep the hexagonal lattice appearance we choose the embedding of $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ as $(1,0) \mapsto \boldsymbol{a}_1 = \frac{1}{2}(3, \sqrt{3})$ and $(0,1) \mapsto \boldsymbol{a}_1 = \frac{1}{2}(3, -\sqrt{3})$

$$\left(\ell^2(\mathbb{Z}^2) \otimes \ell^2(\mathbb{Z}^2)^*\right) \otimes M_N(\mathbb{C})$$

$|\delta_r\rangle \in \ell(\mathbb{Z}^2)$ denotes a function supported in the unit cell at $\boldsymbol{r}$; $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are wavefunctions that indicate a fermion located on site A or B of the unit cell.

Now the Haldane Hamiltonian is:

$$\begin{aligned} H = \quad & t_1 \textstyle\sum_{\boldsymbol{r} \in \mathbb{Z}^2} \left(|\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}+a_1}| + |\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}+a_2}| + |\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}+a_1+a_2}| \right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + h.c. \\ + \quad & t_2 \textstyle\sum_{\boldsymbol{r} \in \mathbb{Z}^2} \left(|\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}+a_1}| + |\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}-a_2}| + |\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}-a_1+a_2}| \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + h.c. \\ + \quad & M \textstyle\sum_{\boldsymbol{r} \in \mathbb{Z}^2} |\delta_{\boldsymbol{r}}\rangle\langle\delta_{\boldsymbol{r}}| \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$t_1, t_2, M \in \mathbb{R}$ are tunable parameters to experimentalists. And the parameter space is separated into open regions, within which the Hall conductance is constant integers.

2.4.1. Haldane model phase diagram. The real-space Hamiltonian of the Haldane model was introduced above. Since the Hamiltonian is translationally invariant, after Fourier transform it decomposes into a family of two by two Hermitian matrices over the Brillouin torus $\mathbb{T}^2$:

$$\widehat{H}(\boldsymbol{k}) = d_0(\boldsymbol{k})\mathbb{I} + d_1(\boldsymbol{k})\sigma_1 + d_2(\boldsymbol{k})\sigma_2 + d_3(\boldsymbol{k})\sigma_3. \tag{2.9}$$

Here $d_0(\boldsymbol{k})$ only shifts both energy bands by the same amount, and hence does not affect the eigenvectors or the Chern index of the occupied band. The relevant topological data are therefore contained in

$$\boldsymbol{d}(\boldsymbol{k}) = (d_1(\boldsymbol{k}), d_2(\boldsymbol{k}), d_3(\boldsymbol{k})) \in \mathbb{R}^3.$$

$$d_0(k) = 2t_2 \cos\phi \sum_i \cos\left(k \cdot \overline{\delta}_i\right) \qquad d_1 = t_1 \sum_i \cos\left(k \cdot \delta_i\right)$$

$$d_2 = t_1 \sum_i \sin\left(k \cdot \delta_i\right) \quad d_3 = M - 2t_2 \sin\phi \left(\sum_i \sin\left(k \cdot \overline{\delta}_i\right)\right)$$

The two energy bands are

$$E_\pm(\boldsymbol{k}) = d_0(\boldsymbol{k}) \pm \left|\boldsymbol{d}(\boldsymbol{k})\right|.$$

Thus the model is gapped precisely when $\boldsymbol{d}(\boldsymbol{k}) \neq 0$ for all $\boldsymbol{k} \in \mathbb{T}^2$. In the gapped case, the occupied line bundle is the lower eigenbundle of $\widehat{H}(\boldsymbol{k})$, and its Chern index is constant under any deformation of parameters that does not close the spectral gap.

Let

$$\boldsymbol{n}(\boldsymbol{k}) = \frac{\boldsymbol{d}(\boldsymbol{k})}{\left|\boldsymbol{d}(\boldsymbol{k})\right|} : \mathbb{T}^2 \longrightarrow \mathbb{S}^2.$$

For a two-band Hamiltonian, the Berry curvature of the lower band can be written directly in terms of this map:

$$F_-(\boldsymbol{k}) = -\frac{1}{2}\boldsymbol{n}(\boldsymbol{k}) \cdot \left(\partial_{k_1}\boldsymbol{n}(\boldsymbol{k}) \times \partial_{k_2}\boldsymbol{n}(\boldsymbol{k})\right) dk_1 \wedge dk_2. \tag{2.10}$$

Equivalently, if $\boldsymbol{n}$ is expressed in spherical coordinates

$$\boldsymbol{n} = (\sin\theta\cos\varphi, , \sin\theta\sin\varphi, , \cos\theta),$$

then the lower-band Berry curvature is the pullback of the area form on the Bloch sphere:

$$F_- = -\frac{1}{2}\sin\theta \, d\theta \wedge d\varphi. \tag{2.11}$$

Therefore the Chern index is the degree of the map $\boldsymbol{n} : \mathbb{T}^2 \to \mathbb{S}^2$, up to the sign convention determined by the choice of lower band:

$$Ch_- = \frac{1}{2\pi}\int_{\mathbb{T}^2} F_- = -\frac{1}{4\pi}\int_{\mathbb{T}^2} \boldsymbol{n} \cdot \left(\partial_{k_1}\boldsymbol{n} \times \partial_{k_2}\boldsymbol{n}\right) dk_1, dk_2. \tag{2.12}$$

Thus the Haldane model is topologically nontrivial exactly when the image of the Brillouin torus under $\boldsymbol{n}$ wraps nontrivially around the Bloch sphere.

The phase boundaries can be found without evaluating the integral over the whole Brillouin zone. The gap can close only at the Dirac points K and K', where the first two components vanish:

$$d_1(K) = d_2(K) = 0, \qquad d_1(K') = d_2(K') = 0.$$

At these points, the remaining mass terms are

$$m_K = d_3(K) = M - 3\sqrt{3}, t_2 \sin\phi, \qquad m_{K'} = d_3(K') = M + 3\sqrt{3}, t_2 \sin\phi. \tag{2.13}$$

Hence the gap closes exactly when one of these two masses vanishes:

$$M = 3\sqrt{3}, t_2 \sin\phi \qquad \text{or} \qquad M = -3\sqrt{3}, t_2 \sin\phi. \tag{2.14}$$

These two curves divide the parameter space into regions with constant Chern index.

Near the two Dirac points, the Bloch Hamiltonian has the form of a massive Dirac Hamiltonian. Up to an irrelevant scalar term and higher order terms,

$$\widehat{H} * K(\boldsymbol{q}) \sim v(q_1\sigma_1 + q_2\sigma_2) + m_K\sigma_3, \qquad \widehat{H} * K'(\boldsymbol{q}) \sim v(-q_1\sigma_1 + q_2\sigma_2) + m_{K'}\sigma_3, \tag{2.15}$$

where $\boldsymbol{q}$ denotes the momentum measured from the corresponding Dirac point. The opposite signs in the linear parts express the fact that the two Dirac cones have opposite chirality. A change of sign of either mass changes the Chern index by one. Taking into account the opposite chirality of K and K', the lower-band Chern index is

$$Ch_- = \frac{1}{2}\left(\operatorname{sgn}(m_{K'}) - \operatorname{sgn}(m_K)\right) = \frac{1}{2}\Big(\operatorname{sgn}\big(M + 3\sqrt{3}, t_2 \sin\phi\big) - \operatorname{sgn}\big(M - 3\sqrt{3}, t_2 \sin\phi\big)\Big) \tag{2.16}$$

Consequently,

$$Ch_- = 0 \quad \text{if} \quad \Big|M\Big| > 3\sqrt{3}\Big|t_2 \sin\phi\Big|, \text{ while } Ch_- = \operatorname{sgn}(t_2\sin\phi) \quad \text{if} \quad \Big|M\Big| < 3\sqrt{3}\Big|t_2\sin\phi\Big|.$$

The topologically trivial region includes, for example, the atomic limit $\Big|M\Big| \gg \Big|t_2\Big|$, where the two sublattices are strongly separated in energy and the occupied states do not wrap around the Bloch sphere. The nontrivial region appears when the complex next-nearest-neighbor hopping produces opposite signs for the two Dirac masses. Thus the Haldane model realizes a Chern insulator without a net external magnetic field: the Hall conductance is determined by the Chern index of the occupied Bloch line bundle, and this integer changes only when the bulk gap closes along the phase-boundary curves (2.14).

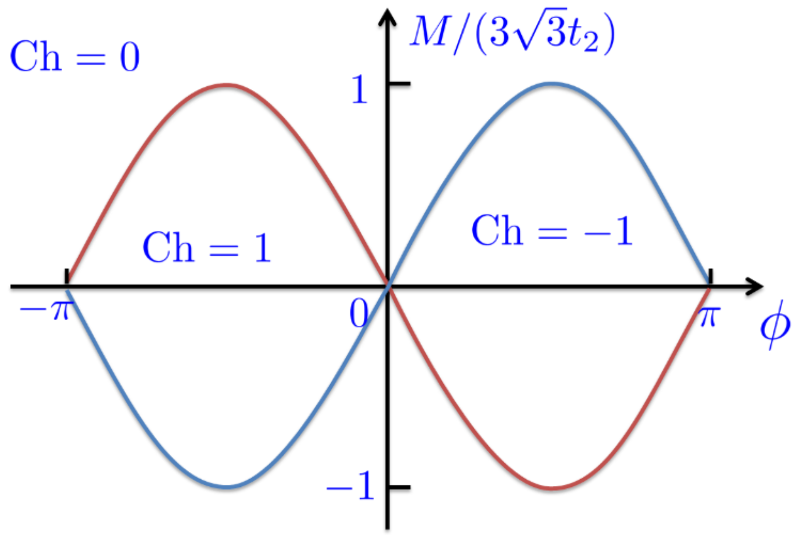


FIGURE 2.2. Chern number of Haldane model in the parameter space

2.5. Hall conductance as Edge index

In this section we'll describe the mathematical setup of a tight binding model with an edge, and define the edge index which is interpreted as the net current. By definition this edge index is robust under perturbation of the Hamiltonian which does not close the gap or lose translational invariance, though the latter can be dropped in the treatment of [20].

Though Kubo formula describes the experimental setup and the driving force of quantum Hall effect very well, it does not reveal the fact that Hall current is carried by states that are localized to the edge of material. Such edge states and their robustness are first discussed by [20]. In the next section we will prove the equallity of edge index and the bulk Chern index.

The edge system in our context is a restriction of an existing tight binding model defined as 2.1 onto a half plane lattice. We use the fact that tight binding Hamiltonians have a local decomposition.

Definition 2.13 (System with edge)

A tight binding model with edge consists of

A lattice $\Lambda = \mathbb{N} \times \mathbb{Z}$*,*

A Hilbert space $\mathcal{H}_e = \ell^2(\mathbb{N} \times \mathbb{Z}, \mathbb{C}^N) \cong \ell^2(\mathbb{N} \times \mathbb{Z}) \otimes \mathbb{C}^N$

The edge Hamiltonian is the bulk Hamiltonian omitting all local terms supported not on the half lattice.

$$H_e = \sum_{x \in \mathbb{N},\, y \in \mathbb{Z}} H_{(x,y)}$$

The additional assumptions become:

- *inner product of* $\mathcal{H}$ *is induced by those of* $\ell^2(\mathbb{N} \times \mathbb{Z})$ *and* $\mathbb{C}^N$*;*
- $\mathbb{Z}$ *has an action on* $\ell^2(\mathbb{N} \times \mathbb{Z}, \mathbb{C}^N)$ *as* $(T_y\psi)(\boldsymbol{s}) = \psi(s_1, s_2 - y)$ $\forall s_2,\, y \in \mathbb{Z}$ *and* $s_1 \in \mathbb{N}$*;*

The edge system retains $\mathbb{Z}$ direction translational invariance, hence there is a one-dimensional Fourier transformation

$$\mathcal{F}_2 : \ell^2(\mathbb{N} \times \mathbb{Z}) \to \ell^2(\mathbb{N}) \otimes L^2(\mathbb{S}^1)$$

$$\mathcal{F}_2(\psi)\,(k) := \frac{1}{(2\pi)^{1/2}} \sum_{n \in \mathbb{N}} e^{-i\,nk} \psi(k)$$

And the Hamiltonian H_e becomes a k-parametrized family of self-adjoint operators that depends analytically on k

$$\exists \widehat{H}_e(k) \in \mathcal{B}\big(\ell^2(\mathbb{N})\big), \quad \forall k \in \mathbb{S}^1,\ \mathcal{F}_2(H\psi)\,(k) = \widehat{H}_e(k)\ (\mathcal{F}_2\psi)(k) \tag{2.17}$$

2.5.1. Properties of edge Hamiltonian spectrum. We reinstate the gap assumption 2.2 on the bulk Hamiltonian.

Proposition 2.5.1

The essential spectrum of $\widehat{H}_e(k)$ *is equal to the essential spectrum of* $\widehat{H}(k)$*;*

The edge mode graph

$$\Gamma := \{(k, E) \in \mathbb{S}^1 \times (\gamma_1, \gamma_2) : E \in \sigma_p(\widehat{H}_e(k))\}$$

is an analytic set, i.e. a union of graphs of finitely many analytic functions.

This is discussed in [18]. And we'll give another proof in section 4.5.2. Each analytic function that forms Γ are referred to as (local) eigenvalue branches.

Globally, eigenvalue crossings may lead to a non-trivial permutation of the branches E_j between $k = 0$ and $k = 2\pi$, thus preventing a single global analytic labeling of eigenvalues on $\mathbb{S}^1$; however, as we will explain, this plays no role in what follows, and the reader may keep in mind the generic situation illustrated in Figure 2.3.

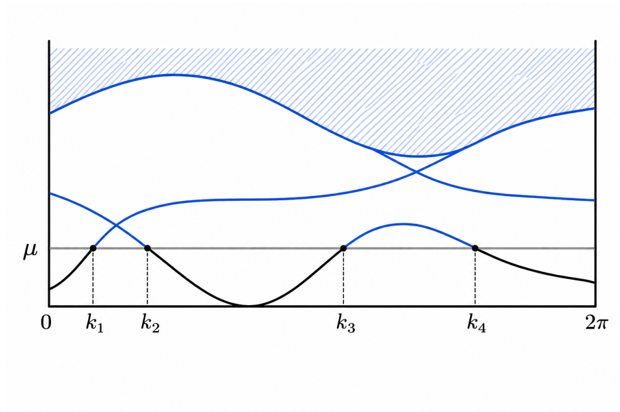


FIGURE 2.3. Eigenvalues of $H_e(k)$ below μ can be seen as functions defined on $[k_2, k_3]$ and $[k_4, k_1]$, while no single collection of functions defined on $\mathbb{S}^1$ could have graphs that include all eigenvalues of $H_e(k)$ because the blue/black curve is a self-braiding of one continuous curve.

Definition 2.14 (Regular energy)

A point $(k, E) \in \Gamma$ is called regular if locally Γ is the graph of a single analytic function $E(k)$ with $E'(k) \neq 0$. An energy $\mu \in (\gamma_1, \gamma_2)$ is called regular if the line $\mathbb{S}^1 \times \mu$ intersects Γ only at regular points.

All but finitely many energies in the gap are regular.

Definition 2.15 (Edge index)

The edge index $\mathcal{I}_e(H_e)$ of H_e is the total number of signed crossings of Γ across any regular energy μ.

This is well-defined. In particular, independent of the choice of regular energy μ follows from the observation that the edge index is the degree $\deg(f; \mu)$ of the map $E : \overline{\Gamma} \to \mathbb{R}$ defined by $(k, E) \to E$, and the fact that $\deg(E; \mu)$ is independent of the choice of regular value, see Theorem 13.1.2 in [15].

The edge index of H_e also has an analytic expression, see for example [16]. It is given by

$$\mathcal{I}_e(H_e) = \frac{1}{2\pi} \int_{\mathbb{S}^1} dk \, \mathrm{Tr} \left(g'(\widehat{H}_e(k)) \frac{d\widehat{H}_e(k)}{dk} \right), \tag{2.18}$$

for any $g : \mathbb{R} \to \mathbb{R}$, that is equal to 1 on $(-\infty, \gamma_1]$, equal to 0 on $[\gamma_2, \infty)$ and is differentiable on (γ_1, γ_2).

CHAPTER 3

Bulk edge correspondence

Having the bulk and edge index defined, and both supposed to describe the Hall conductance, it is natural to seek for proof that they are equal. And such studies are plentiful. Pioneering proofs were given by [25] and [29] using K-theory. We find the proof most relevant to our context is that of G.M. Graf and M. Porta [18], though it is focused on time reversal symmetric models with $\mathbb{Z}_2$ index. In this chapter we'll give a sketch of proof following its methods while only addressing the quantum Hall models.

This chapter is structured as follows: section 1 will recall assumptions that were introduced in the previous chapter, and introduce one more assumption about the hopping matrix invertibility. Then in section 2, the key definition of the decaying wave bundle will be introduced, which takes the place of the Bloch bundle and on which the bulk index can be defined. We declare Proposition 3.2.1 that such bulk index is equal to the Chern number of Bloch bundle. Next, section 3 will be the sketch of proof of bulk edge index equality. We leave the proof of equivalence of decaying wave bundle to Bloch bundle till the last section.

3.1. Assumptions

The edge (or bulk) Hamiltonian is defined as an operator $H_e(k)\,(H(k))$ on $\ell^2(\mathbb{N},\mathbb{C}^N)\otimes L^2(\mathbb{S}^1{}_y)$ (or $\ell^2(\mathbb{Z},\mathbb{C}^N)\otimes L^2(\mathbb{S}^1{}_y)$) after the partial Fourier transform on the lattice y direction and we omit the hat for simplicity in this section. We recall our assumptions including

- All properties in def 2.1 and 2.13.
- The gap in $\sigma(H)$ is assumed.
- The edge index $\mathcal{I}_e(H_e)$ is defined in the same way as def 2.15.

For the bulk index, we need a few preparation:

For each k the Hamiltonian acts as a finite difference operator on $\ell(\mathbb{N}, \mathbb{C}^N)$

$$(H\psi)(n) = A\psi(n-1) + A^*\psi(n+1) + V_n\psi(n); \quad A, A^*, V_n \in M_N(\mathbb{C}),\ \psi(n) \in \mathbb{C}^N,\ n \in \mathbb{Z}$$

And similarly for H_e.

$$(H_e\psi_e)(n) = A\psi_e(n-1) + A^*\psi_e(n+1) + V_n\psi(n), \quad n \in \mathbb{N}$$

We assume Dirichlet condition on the boundary: $\psi(-1) = 0$.

Assumption 3.1

We assume the hopping matrix A, A^* are invertible.

This assumption is necessary for the proof. By redefining lattice unit cell as in proposition 2.1.2 or edge direction in relation to lattice translation generators, one can have the same model with invertible or non-invertible hopping matrix A. But at least to the author's understanding, such reorientation does not guaranteed to make every model having an invertible hopping matrix.

3.2. Bulk index on Decaying wave bundle

Let $\mu \in (\gamma_1, \gamma_2)$ be a regular value in between the bulk spectral gap so that the edge spectrum cross μ as simple crossing with nonzero slope.

A closed contour $S \subset \rho(H(k_y)) \subset \mathbb{C}$ is parametrized by

$$z : (-\epsilon, 2\pi + \epsilon) \to S,$$
$$z(\alpha) = \|H\|e^{i\,\alpha} - \|H\| + \mu \tag{3.1}$$

so that $z(0) = \mu$ and S encloses the part of the spectrum of H below the gap. Recall that our Bloch bundle G is formed by eigenspaces of $H(\boldsymbol{k})$ with eigenvalue below the gap.

The 2 torus $\mathbb{T}^2$ is formed by Cartesian S with the Brillouin zone, and let $\mathbb{T}^2_c$ be a self-overlapping parametrization of the 2 torus:

$$\mathbb{T}^2 := S \times \mathbb{S}^1{}_y \subset \mathbb{C} \times \mathbb{S}^1{}_y; \quad \mathbb{T}^2_c := (-\epsilon, 2\pi + \epsilon) \times \mathbb{S}^1{}_y \tag{3.2}$$

Instead of the Bloch bundle, the authors use the following bundle to define the bulk index:

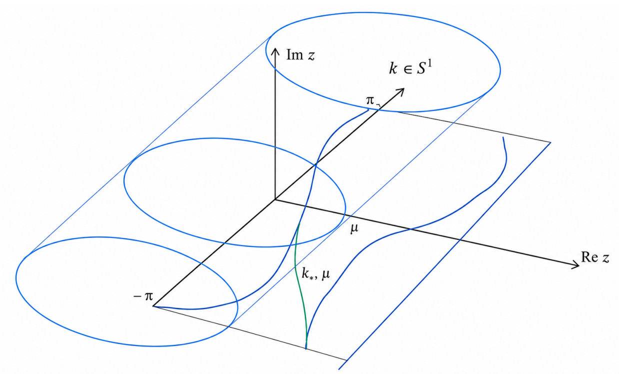


FIGURE 3.1. The torus $\mathbb{T}^2$ in blue encloses the lower part of bulk spectrum, and intersects with the edge spectrum (green) at (μ, k_*)

Definition 3.2 (Decaying wave bundles)

Let ψ be the solution to $(H(k)\psi)(n) = z\,\psi(n),\ n \in \mathbb{Z}$ that decays on the right. The decaying wave bundle of the bulk is

$$D := \left\{ \big((z,k),\psi\big) \,\Big|\, (z,k) \in \mathbb{T}^2,\ (H(k)\psi)(n) = z\,\psi(n),\ \lim_{n\to+\infty} \psi(n) = 0 \right\}$$

And similarly, the decaying wave bundle of the edge is

$$D_e := \left\{ \big((z,k),\psi\big) \,\Big|\, (z,k) \in \mathbb{T}^2,\ (H_e(k)\psi_e)(n) = z\,\psi_e(n),\ \lim_{n\to+\infty} \psi_e(n) = 0 \right\}$$

Both are rank N vector bundles over base $\mathbb{T}^2 \ni (z,k)$.

The general theory of difference equations shows that $(H(k)\psi)(n) = z\,\psi(n),\ n \in \mathbb{Z}$ has N linearly independent solutions that decay on the right, and N independent solutions grow on the right. This relies on the fact that A is invertible.

The two bundles D and D_e are isomorphic, if we identify the solutions that have matching tail

$$\psi \mapsto \psi_e \text{ if } \exists n_0 \text{ such that } \ \psi(n) = \psi_e(n)\, \forall n > n_0$$

Let Φ be a frame of D that is differentiable on $\mathbb{T}^2_c$, namely $\Phi(\alpha, k)$ is a N-tuple of sequences that form a basis of the solution space of $(H(k)\psi)(n) = z\,\psi(n)$. Evaluating the sequence at n produces a parametrized family of invertible matrix $\Phi(z,k,n)$.

Then it has a family of transition matrices to itself near the cut line

$$T(z,k) \in GL(N),\ \Phi(\alpha,k,n)T(e^{i\alpha},k) = \Phi(\alpha+2\pi,k,n),\ \alpha \in (-\epsilon,\epsilon)$$

Definition 3.3 (Bulk index from decaying wave bundle)
The bulk index is the total winding number of all eigenvalues of the transition matrix $T(\mu,k)$ as k winds around $\{\mu\} \times \mathbb{S}^1{}_y$ or any neighborhood band within the overlap of the charts.

$$\mathcal{I} := arg\big(det(T(\mu,k))\big)\Big|_{k=0}^{k=2\pi}$$

We claim a proposition that

Proposition 3.2.1
The winding number of the decaying wave bundle equals the Chern number of Bloch bundle.

3.3. Theorem and sketch of proof

Now the bulk-edge correspondence theorem can be stated as

Theorem 3.4 (G. M. Graf and M. Porta 2013)
For tight binding models H with spectral gap and invertible hopping matrix, and their corresponding models with an edge H_e,

$$\mathcal{I}(H) = \mathcal{I}_e(H_e)$$

3.3.1. A technical lemma. The proof can be made more structurally clear if we hide the technical part in the following lemma. It is Lemma (5.5) in the original paper.
It proves to be helpful to regard the frame Φ as a sequence of matrices that solve the difference equation

$$A(k)\Phi(n-1) + A(k)^*\Phi(n+1) + (V_n(k) - z)\Phi(n) = 0,\ n \in \mathbb{N} \tag{3.3}$$

Once we fix an initial condition at $\Phi(n=0)$ or $\Phi(n=1)$, and the decaying condition $\Phi(n \to \infty) \to 0$, the solution will always exist and be unique.

- This fact, together with points (1), (2) of the lemma help us define a pair of frames that covers $\mathbb{T}^2$ together;
- Point (3) of the lemma later will show that L are almost T;

- Point (4), (5) describes linear approximations of eigenvalues of L thus we can compute its winding numbers;
- (5) relates eigenvalues of L with edge spectrum crossing.

Lemma 3.3.1

We call it a cross point if $\mu \in \sigma(H_e(k_*))$. The following is true for a dense set of regular energy μ in the gap of $\sigma(H)$:

(1) Let Φ be a local frame of decaying wave bundle defined on B_ϵ an open ball around corssing point (μ, k_*). And consider its edge partner Φ_e evaluated at $n = 1$ then

$$det(\Phi_e(\mu, k_*, 1)) \neq 0$$

(2) Let Φ be a local frame of decaying wave bundle and consider its edge partner Φ_e evaluated at $n = 0$: $det(\Phi_e(\mu, k_*, 0)) = 0$ iff $\mu \in \sigma(H_e(k_*))$.

(3) The matrix

$$L(z, k) := \Phi_e^{-1}(\overline{z}, k, 1)\, A(k)\, \Phi_e(z, k, 0)$$

has the reflection property $L(z, k) = L(\overline{z}, k)^*$. Hence its eigenvalues are real for real z.

(4) Generically there is one eigenvalue branch $l(z, k)$ of $L(z, k)$ that vanishes at (μ, k_*) with nonzero derivatives.

(5) At (μ, k_*), the k-derivative of edge model energy ε has the same sign as that of l

$$sgn\Big(\frac{\partial l}{\partial k}\Big) = sgn\Big(\frac{\partial \varepsilon}{\partial k}\Big),\ and\ \frac{\partial l}{z} < 0$$

3.3.2. Defining transition map. Now let's define two local frames: Φ_e on $\mathbb{T}^2_c \setminus \{(\mu, k_{*,i})\}$, by imposing initial conditions

$$\Phi_e(z, k, n = 0) = \mathbb{I} \tag{3.4}$$

$\overline{\Phi}_{e,i}$ on $B_\epsilon(\mu, k_{*,i})$ by imposing initial conditions

$$\overline{\Phi}_{e,i}(z, k, n = 1) = \mathbb{I} \tag{3.5}$$

These actually fully determines Φ_e, Φ, $\overline{\Phi}_e$ and $\overline{\Phi}$ as the difference equation 3.3 governs, and the edge bundle is isomorphic to the bulk bundle. We shall suppress the label i and consider only one

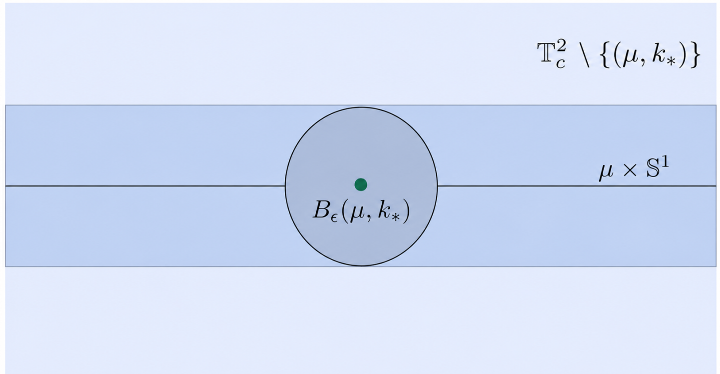


FIGURE 3.2. Φ on $\mathbb{T}_c^2 \setminus \{(\mu, k_*)\}$ in light blue, and $\overline{\Phi}$ on $B_\epsilon(\mu, k_*)$ in gray

crossing point for now. And we may abuse the notation $z = z(\alpha)$ for just z if no chart overlap is involved, or $z = z(\alpha)$ for α if we need to distinguish multiple covering of a point by charts, see 3.1 and 3.2

The transition map outside any $B_\epsilon(\mu, k_{*,i})$ are just the identity since initial condition 3.4 is the same for $z(\alpha)$ and $z(\alpha + 2\pi)$. So the winding number of transition map only has contributions from $T(z,k),\ (z,k) \in B_\epsilon(\mu, k_{*,i})$.

$$T(z,k) = \mathbb{I},\ \forall (z,k) \in z((-\epsilon, \epsilon)) \times \mathbb{S}^1 \setminus B_\epsilon(\mu, k_{*,i})$$

The transition map at $(z(\alpha), k) \in B_\epsilon(\mu, k_*)$ can be computed through two steps,

$$\overline{\Phi}(z,k) = \Phi(\alpha, k)\, T_-(z,k), \quad \text{and} \quad \Phi(\alpha + 2\pi, k) = \overline{\Phi}(z,k)\, T_+^{-1}(z,k) \tag{3.6}$$

$$\implies \quad T(z,k) = T_-^{(}z,k)\, T_+^{-1}(z,k) \tag{3.7}$$

Again by the difference equation and initial condition, if the above identity is true for $n = 0$ or $n = 1$, it holds true for all n and especially T is independent of n. So one can readily tell

$$\overline{\Phi}_e(z,k,0) = \Phi_e(z(\alpha), k, 0)\, T(z,k) = T(z,k), \quad (z,k) \neq (\mu, k_*) \in B_\epsilon(\mu, k_*)$$

Actually the transition map in the $B_\epsilon(\mu, k_*)$ is also fully expressed using the L matrix in the lemma, and hence fully expressed using Φ_e and A. Plugging equation 3.5 into the definition of L

gives $L(z,k) = -A(k)\overline{\Phi}_e(z,k,0)$ and further

$$T(z,k) = T_-(z,k)\,T_+(z,k) = \overline{\Phi}_e(z(\alpha),k,0)\;\overline{\Phi}_e(z(\alpha+2\pi),k,0)$$

$$= A(k)\,L(z(\alpha),k)\,L^{-1}(z(\alpha+2\pi),k)\,A^{-1}(k) \tag{3.8}$$

Eigenvalues of T are the same as those with A dropped. So we'll focus on eigenvalues of $L(z,k)$. Let's take a contour along the self-overlapping band of $\mathbb{T}^2_c \setminus \{(\mu,k_*)\}$ as

$$\mathbb{S}^1 \ni k \mapsto (z(k),k) = \begin{cases} (\mu,k) & if\ k \notin (k_*-\epsilon,k_*+\epsilon) \\ \partial B_\epsilon \cap (\mathrm{Im}(z)<0,k) & if\ k \in (k_*-\epsilon,k_*+\epsilon) \end{cases}$$

On such contour we abbreviate eq 3.8 as $T(z(k),k) \simeq L_-(k)L_+^{-1}(k)$. Now summarizing what we have so far

Proposition 3.3.2

The bulk index can be expressed as

$$\mathcal{I}(H) = \sum_{i\in crossing} arg\Big(det\big(L_-(k)L_+^{-1}(k)\big)\Big)\Big|_{k_{*i}-\epsilon}^{k_{*i}+\epsilon}$$

3.3.3. Local behavior of the transition map. Finally we simplify $L(z,k)$ by looking at its 0 and first-order terms in the expansion near the crossing point (μ,k_*). Let $(z,k) = (\mu,k_*)+\Delta$. And we use subscript 0 for zeroth-order term and subscript 1 for first-order term in Δ.

Recall (4) of lemma 3.3.1 there is an eigenvalue branch $l((\mu,k_*)+\Delta)$ that vanishes with Δ in first-order. Let $\Pi(z,k)$ be the rank 1 projection onto the eigenspace of $l(z,k)$ and $\overline{L}(z,k)$ is the rank (N-1) complement of $l(z,k)\Pi(z,k)$.

$$L(z,k) = l(z,k)\Pi(z,k) \oplus \overline{L}(z,k)$$

Then the expansion

$$L(z,k) = l(z,k)\Pi_0 + \overline{L}_0 + \overline{L}_1\,\Delta + \mathcal{O}(\Delta^2)$$

$$L(z,k)^{-1} = l(z,k)^{-1}\,(\Pi_0 + \Pi_1\,\Delta) + \overline{L}_0^{-1} + \mathcal{O}(\Delta)$$

Notice that upon choosing the contour and the abbreviation, taking the $\pm$ subscript is equivalent with complex conjugate on z: $L(z(k),k) = L_-(k)$, $L(\overline{z(k)},k) = L_+(k)$.

$$L_-(k)L_+^{-1}(k) = L(\overline{z(k)},k)L(z(k),k)^{-1} = \frac{l(\overline{z},k)}{l(z,k)}\,\Pi_0 + R + (1-\Pi_0) + \mathcal{O}(\Delta)$$

R is an $\mathcal{O}(1)$ term and $R = (1-\Pi_0)R\Pi$ is an off-diagonal block so R does not affect the eigenvalues. $N-1$ eigenvalues from $1-\Pi_0$ remain unchanged through the interval $(k_* \pm \epsilon)$.

Eventually by (5) of lemma 3.3.1, one eigenvalue from $\frac{l(\overline{z},k)}{l(z,k)}\Pi_0$ winds around 0 as k traverses $(k_*-\epsilon, k_*+\epsilon)$, and meanwhile $\varepsilon(k)$ passes through 0 with the same sign. That concludes the proof.

3.4. Bloch bundle and decaying wave bundle

One last step to connect the bulk edge correspondence to the usual Chern index is to prove prop 3.2.1 the Chern index of decaying wave bundle D is equal to the Chern number of Bloch bundle G. This can be found in the original paper [18] section 8. Here we'll provide a conceptual sketch.

Recall that our Bloch bundle on which the Chern index is computed is a rank-m bundle, whose fibers are eigenspaces of m eigenvalues of $\sigma(H(\boldsymbol{k}))$, separated from the others by a gap. While the decaying wave bundle is rank N, N being the size of $H(\boldsymbol{k})$ i.e. sites per unit cell.

The following argument works the same for any fixed k_y the momentum along edge. So we suppress it. We call the complexified x-direction momentum as ξ:

$$\xi := e^{i\,k_x} \subset \mathbb{C}^\times$$

For fixed k_y consider the characteristic polynomial of $H(\xi)$ extended to a complex neighborhood. It is the zeros of a Laurent polynomial in ξ of degree $[-N,...,N]$ and polynomial in z, of degree N.

$$P(\xi,z) = \det\big(H(\xi) - z\big)$$

The zero set forms a Riemann surface / 1D complex algebraic curve. We shall call it Bloch variety B:

$$B := \big\{(\xi,z) \in \mathbb{C}^\times \times \mathbb{C} \big| \det\big(H(\xi)-z\big) = 0\big\} \tag{3.9}$$

And two subsets of it $B_0 := B \cap \{(\xi,z) \mid |\xi| = 1\} \simeq \mathbb{S}^1$, $B_- := B \cap \{(\xi,z) \mid |\xi| < 1\}$. The former is a constant-k_y cycle in the base of Bloch bundle.

Now let's define a bundle-like parent structure

$$\tilde{E} := \left\{ \big((k,z),\psi\big) \middle| \ (z,k) \in B, \ (H(k)\psi)(n) = z\,\psi(n), \ \lim_{n\to+\infty} \psi(n) = 0 \right\} \tag{3.10}$$

$\ker(H(\xi)-z)$ as a decaying sequence is almost always one-dimensional so it resembles a line bundle over a Riemann surface. To truly have bundle structure, we take samples of $\tilde{E}$ on two constant slices of B_-:

$$\begin{aligned} \pi : B_- \to \mathbb{C}; &\quad \sigma : B_- \to \{(\xi,z) \mid |\xi| < 1\} \\ (\xi,z) \mapsto z; &\quad\quad\quad (\xi,z) \mapsto \xi \end{aligned}$$

In 3.2 we've argued using difference equation that $\mathrm{card}(\pi^{-1}(\xi)) = N$ iff $z \in \rho(H(\xi))$. So on the one-dimensional loop S as in 3.1

$$\Pi := \bigsqcup_{z\in S} \pi^{-1}(z) \ \subseteq B_-$$

is generally always a N-fold covering space of S, up to a perturbation in S. And let $\varepsilon_G(\xi)$ be the m eigenvalue branches of $H(\xi)$ in the Bloch bundle, then

$$\Sigma_0 := \left\{(\xi, \varepsilon_G(\xi)) \middle| \, |\xi| = 1\right\} \subseteq B$$

is an m-fold covering space of S. It can be analytically extended into B_-, specifically for some sufficiently small $\rho \in \mathbb{R}^+$

$$\Sigma := \left\{(\xi, \varepsilon_G(\xi)) \middle| \, |\xi| = e^{-\rho}\right\} \subseteq B_-$$

This is again an m fold covering space of the circle $S_- = \{\xi \in \mathbb{C}\,|\xi| = e^{-\rho}\}$. The two inclusions $i_\Pi : \Pi \to B$ and $i_\Sigma : \Sigma \to B$ have pullbacks that become true bundles:

$$D_k = i_\Pi^*(\tilde{E}) = \left\{ \big(z,\psi\big) \middle| \ z \in S, \ (H(k)\psi)(n) = z\,\psi(n), \ \lim_{n\to+\infty} \psi(n) = 0 \right\} \tag{3.11}$$

$$G_k \cong i_\Sigma^*(\tilde{E}) = \left\{ \big(\xi,\psi\big) \middle| \ |\xi| < 1, \ (H(k)\psi)(n) = \varepsilon_G\,\psi(n), \ \lim_{n\to+\infty} \psi(n) = 0 \right\} \tag{3.12}$$

D_k is a rank-1 bundle over base Π, $\Sigma^*(\tilde{E})$ is a rank-m bundle isomorphic to G_k. And they are the k_y-fibers of the decaying wave bundle / of deformed Bloch bundle respectively.

Again we hide some technical construction by the following lemmas:

Lemma 3.4.1

Covering spaces like Π *and* Σ *are cycles. So it makes sense to say:*

- *There are two subcycles* Π', Π'' *that concatenate into* Π*; correspondingly two pullbacks* $D'_k = i^*_{\Pi'}(\tilde{E})$, $D''_k = i^*_{\Pi''}(\tilde{E})$ *that sum into* $D_k = D'_k \oplus D''_k$*;*
- Π' *is homotopic to* Σ *in* B_-*; the homotopy is independent of* k *hence induces a bundle homotopy* $D'_k \cong G_k$*;*
- Π'' *is contractible to a point in* B_-*; the contraction is independent of* k*, so it induces a trivializing homotopy of* $D'' = \bigsqcup_{k \in \mathbb{S}^1} D''_k$.

So we have a sequence of homotopic bundles that concludes the proof of prop 3.2.1

$$D \cong D' \cong \bigsqcup_k i^*_\Sigma(\tilde{E}) \cong G$$

CHAPTER 4

Zero Hall conductance in frustration free tight binding models

We present the first result from the course of my PhD study. This chapter contains essentially the same results obtained in collaboration with, and under the advisory of, Martin Fraas and Sven Bachmann, and the manuscripts will be submitted to arxiv soon. We restate the assumptions for the tight binding models.

4.1. Recap of assumptions

Assumption 4.1

Recall our assumptions for tight binding models

- *We consider the lattice $\mathbb{Z}^2$ equipped with the l^∞ metric, and denote $\boldsymbol{e}_1 = (1,0)$, $\boldsymbol{e}_2 = (0,1)$;*
- *The single-particle Hilbert space is $\mathcal{H} = l^2(\mathbb{Z}^2, \mathbb{C}^N)$ for some $N \in \mathbb{N}$. The components of $\mathbb{C}^N$ are called spin components. For $\boldsymbol{x} \in \mathbb{Z}^2$, let $\delta_{\boldsymbol{x}}$ denote the projection onto site $\boldsymbol{x}$;*
- *We consider a finite-range Hamiltonian H, meaning that there is an upper bound $R > 0$ such that $\langle \delta_{\boldsymbol{x}},\, H\, \delta_{\boldsymbol{y}} \rangle = 0,\ \forall |\boldsymbol{x} - \boldsymbol{y}| > R$;*
- *And the Hamiltonian is translation invariant $T_{\boldsymbol{r}}\, H\, T_{\boldsymbol{R}}^{-1} = H$.*

After second quantization, the Hilbert space is the fermionic Fock space $\mathcal{F} = \Gamma(\mathcal{H})$ and the second-quantized Hamiltonian[1] is

$$d\Gamma(H) = \sum_{\boldsymbol{x},\boldsymbol{y},j,k} H_{\boldsymbol{x},\boldsymbol{y},j,k} a^*_{\boldsymbol{x},j} a_{\boldsymbol{y},k},$$

where $a_{\boldsymbol{x},j}$ is an annihilation operator at site $\boldsymbol{x}$ with on-site spin components labeled by j. The Hamiltonian is called frustration-free if there exists a decomposition

$$d\Gamma(H) = \sum_{\boldsymbol{x}} \Phi_{\boldsymbol{x}}$$

such that (i) the interactions are non-negative $\Phi_{\boldsymbol{x}} \geq 0$, (ii) $\Phi_{\boldsymbol{x}}$ has uniformly finite range and (iii) the kernels of $\{\Phi_{\boldsymbol{x}} : \boldsymbol{x} \in \mathbb{Z}^2\}$ have nonzero intersection. In the context of free Fermion systems

[1]Since this part of the text only serves the purpose of motivating the single-particle setting, we ignore technical details on the precise meaning to give to these formal second quantized expressions.

we will also assume that $\Phi_{\boldsymbol{x}} = d\Gamma(H_{\boldsymbol{x}})$ for some single-particle Hamiltonian $H_{\boldsymbol{x}}$. With this, the frustration-free condition can be rephrased in terms of the single-particle Hamiltonian.

4.2. Frustration free models

Assumption 4.2 (F^4 Hamiltonian)

A self-adjoint operator H on $\mathcal{H}$ is a frustration-free free Fermion Hamiltonian if there exists a decomposition $H = \sum_{\boldsymbol{x}\in\mathbb{Z}^2} H_{\boldsymbol{x}}$ such that:

(1) For all $\boldsymbol{x} \in \mathbb{Z}^2$, $H_{\boldsymbol{x}} \geq 0$,

(2) There exists $R > 0$, such that $H_{\boldsymbol{x}}\delta_{\boldsymbol{y}} = 0$ if $|\boldsymbol{x} - \boldsymbol{y}| > R$,

(3) $\cap_{\boldsymbol{x}\in\mathbb{Z}^2}\mathrm{Ker}H_{\boldsymbol{x}} \setminus \{0\} \neq \emptyset$.

We note that given i., iii. is equivalent to $0 \in \sigma(H)$.

Condition (ii) is not necessarily true in a finite ranged tight binding model. A decomposition of $H = \sum_{\boldsymbol{x}} H_{\boldsymbol{x}}$ into finite ranged terms $H_{\boldsymbol{x}}$ may not satisfy condition (3). In fact the Haldane model Hamiltonian in the nontrivial phase has a finite range decomposition, yet each local term is not positive, because our result prevent frustration free tight binding models to have non trivial phase. Another signature of non-positivity of Haldane model is revealed by the non-flat band structure as discussed below **??**.

To connect to the many-body framework, let P be the orthogonal projection onto the subspace $\ker H = \cap_{\boldsymbol{x}\in\mathbb{Z}^2}\mathrm{Ker}H_{\boldsymbol{x}}$. Then P or any nonzero sub-projection of P corresponds to a quasi-free ground state of $d\Gamma(H)$. The only scenario that will be interesting for us is when P is infinite dimensional, in which case $d\Gamma(H)$ has infinitely many ground states. This is in particular the case in the translation-invariant setting, see below. Specifically, we will consider the 'fully occupied' ground state, namely the one corresponding to the full projection P. In conjunction with the following gap assumption this is equivalent to having the Fermi energy in the gap between zero and the rest of the spectrum.

For compatibility with frustration free condition, the gap assumption in this chapter will be

Assumption 4.3 (Spectral gap)

There exists $\gamma > 0$ such that $\sigma(H) \subset \{0\} \cup (\gamma, \infty)$.

The conditions i. and ii. of the F^4 assumption imply that there is $M \in \mathbb{N}$ such that for any $\boldsymbol{x} \in \mathbb{Z}^2$, there exists $M_{\boldsymbol{x}} \in \mathbb{N}$ with $M_{\boldsymbol{x}} \leq M$, and compactly supported vectors $v_{j,\boldsymbol{x}}$, $j = 1, \ldots, M_{\boldsymbol{x}}$ such that

$$H_{\boldsymbol{x}} = \sum_{j=1}^{M_{\boldsymbol{x}}} |v_{j,\boldsymbol{x}}\rangle \langle v_{j,\boldsymbol{x}}|.$$

The vectors $|v_{j,\boldsymbol{x}}\rangle$ are neither orthogonal to each other nor normalized but they are all non-zero. We can obtain another valid frustration-free decomposition of H by relabeling the $\boldsymbol{x}$ and from now on we will assume that $\boldsymbol{x}$ in $|v_{j,\boldsymbol{x}}\rangle$ is the lower-left point in the support of $|v_{j,\boldsymbol{x}}\rangle$. Additionally, by

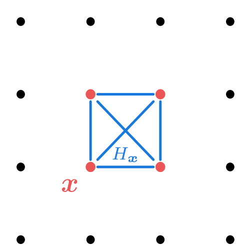


Figure 4.1. The term $H_{\boldsymbol{x}}$ describes the interaction between 4 sites, shown in blue lines. The subscript x is the bottom-left site of support of $H_{\boldsymbol{x}}$. The support of the $|v_{j,\boldsymbol{x}}\rangle$ is shown in red.

coarse-graining, i.e. grouping sites together into a site with more spin components, we can assume that the interaction range is one, see proposition **??**. In other words,

$$|v_{j,\boldsymbol{x}}\rangle = |\delta_{\boldsymbol{x}}\rangle a_{j,\boldsymbol{x}} + |\delta_{\boldsymbol{x}+e_1}\rangle b_{j,\boldsymbol{x}} + |\delta_{\boldsymbol{x}+e_2}\rangle c_{j,\boldsymbol{x}} + |\delta_{\boldsymbol{x}+e_1+e_2}\rangle d_{j,\boldsymbol{x}}, \tag{4.1}$$

where $|\delta_{\boldsymbol{x}}\rangle$ is an $\ell^2(\mathbb{Z})$ vector supported at $\boldsymbol{x}$. $a_{j,\boldsymbol{x}}, b_{j,\boldsymbol{x}}, c_{j,\boldsymbol{x}}, d_{j,\boldsymbol{x}}$ are vectors in $\mathbb{C}^N$ indexed by $j, \boldsymbol{x}$.

4.2.1. Edge Hamiltonian. The edge system is defined on $\mathcal{H}_e = l^2(\mathbb{N} \times \mathbb{Z}, \mathbb{C}^N)$. The edge Hamiltonian H_e is obtained from H by dropping all the terms $H_{\boldsymbol{x}}$ on the left-hand side of the edge,

$$H_e = \sum_{\boldsymbol{x} \in \mathbb{N} \times \mathbb{Z}} H_{\boldsymbol{x}}. \tag{4.2}$$

Notice that the edge Hamiltonian acts on $\mathcal{H}_e$ by our labeling convention. It satisfies the F^4 conditions but it is not necessarily gapped. In fact, we give in Section 4.6 an example of a gapped F^4 Hamiltonian whose edge Hamiltonian has no gap.

4.2.2. Translation-invariant models. Let T_1, T_2 be the standard representation of translations of $\mathbb{Z}^2$ on $\mathcal{H}$. The F^4 model is translation-invariant if

$$H = \sum_{\boldsymbol{x}\in\mathbb{Z}^2} \sum_{j=1}^{M} T_{\boldsymbol{x}} |v_j\rangle\langle v_j| T_{\boldsymbol{x}}^{-1}, \tag{4.3}$$

namely the vectors $a_{j,\boldsymbol{x}}, b_{j,\boldsymbol{x}}, c_{j,\boldsymbol{x}}, d_{j,\boldsymbol{x}}$ in (4.1) are $\boldsymbol{x}$ independent, and $|v_j\rangle = |v_{(0,0),j}\rangle$ are some vectors supported near the lattice origin. We immediately remark that in archetypal magnetic situations such as the Harper model, translation invariance after coarse graining is possible only if the magnetic flux in a cell is rational.

The Fourier transform

$$\mathcal{F} : \mathcal{H} \to L^2(\mathbb{T}^2, \mathbb{C}^N) \quad , \quad f \mapsto \widehat{f}(k) := \frac{1}{2\pi} \sum_{\boldsymbol{x}\in\mathbb{Z}^2} e^{i\boldsymbol{k}\boldsymbol{x}} f(\boldsymbol{x}).$$

gives

$$\widehat{v}_{\boldsymbol{x},j}(k) = \frac{1}{2\pi} e^{i\boldsymbol{k}\boldsymbol{x}} \left(a_j + e^{ik_1} b_j + e^{ik_2} c_j + e^{i(k_1+k_2)} d_j \right).$$

The Hamiltonian $\widehat{H} = \mathcal{F} H \mathcal{F}^{-1}$ acts as an $N \times N$ matrix-valued function of k, $\widehat{H\psi}\ (k) = \widehat{H}(k)\widehat{\psi}(k)$, with

$$\widehat{H}(\boldsymbol{k}) = \sum_{j=1}^{M} \widehat{v}_{0,j}(\boldsymbol{k}) \widehat{v}_{0,j}(\boldsymbol{k})^* = \sum_{j=1}^{M} a_j a_j^* + b_j b_j^* + c_j c_j^* + d_j d_j^* + \tag{4.4}$$
$$+ e^{ik_1}(b_j a_j^* + d_j c_j^*) + e^{-ik_1}(a_j b_j^* + c_j d_j^*) + e^{ik_2}(c_j a_j^* + d_j b_j^*) + \dots .$$

We will not need the explicit expression for $\widehat{H}(\boldsymbol{k})$ but it will be crucial for us that, because of the finite range condition, $\widehat{H}(\boldsymbol{k})$ is a (matrix-valued) Laurent polynomial in e^{ik_1}, e^{ik_2}, which we shall call a trig-polynomial.

We now turn to the fact that a translation-invariant F^4 Hamiltonian has a flat ground state band.

Definition 4.4

We say that a translation-invariant Hamiltonian G has a flat ground state band if for all $\boldsymbol{k} \in \mathbb{T}^2$,

(1) $\widehat{G}(\boldsymbol{k}) \geq 0$,

(2) $\ker \widehat{G}(\boldsymbol{k}) \setminus \{0\} \neq \emptyset$.

In this case, any vector of the form $\int_{\mathbb{T}^2} \phi(\boldsymbol{k})\widehat{\Omega}(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}}d\boldsymbol{k}$ *with* $\phi \in L^2(\mathbb{T}^2)$ *and* $\widehat{\Omega}(\boldsymbol{k}) \in \ker \widehat{G}(\boldsymbol{k})$ *belongs to the kernel of* G*. This implies in particular that* $\ker G$ *is infinite dimensional. It corresponds to the flat ground state band at zero energy.*

The connection between frustration-freeness and flat bands is not new but we feel that it is worth stating explicitly.

Proposition 4.2.1

If H *is a translation-invariant gapped* F^4 *Hamiltonian then* H *has a flat ground state band. Conversely, if* G *is a translation-invariant finite range gapped Hamiltonian with a flat ground state band, then there exists a translation-invariant gapped* F^4 *Hamiltonian with the same kernel.*

Proof. If H is a translation-invariant F^4 Hamiltonian, then i. in Definition 4.4 follows immediately from (4.4). By same equation, $\boldsymbol{k} \mapsto \widehat{H}(\boldsymbol{k})$ is analytic. To show ii., we note that if $0 \neq \Omega \in \ker H$, then $\widehat{H}(\boldsymbol{k})\widehat{\Omega}(\boldsymbol{k}) = 0$ for all $\boldsymbol{k} \in \mathbb{T}^2$ since $\widehat{H}(\boldsymbol{k})$ is non-negative. This has a nonzero solution for any $\boldsymbol{k}$ such that $\widehat{H}(\boldsymbol{k})$ is not invertible. By analyticity of $\widehat{H}(\boldsymbol{k})$, this happens either for all $\boldsymbol{k} \in \mathbb{T}^2$ or at isolated points. In the latter case, $\widehat{\Omega} = 0$ almost everywhere, and hence $\Omega = 0$, which is a contradiction. Hence $\widehat{H}(\boldsymbol{k})$ is everywhere singular, which is ii. of Definition 4.4.
For the converse statement, let $\{\widehat{u_i}^*\}_{i=1}^N$ be the rows of $\widehat{G}$, with N trig-polynomial entries. Now let $\widehat{H}(k) = \sum_i \widehat{u}_i(k)\,\widehat{u}_i(k)^*$. Then $\ker(\widehat{G}(k)) \subseteq \ker(\widehat{H}(k))$. And $\mathrm{Rank}(\widehat{G}(k)) = \mathrm{Rank}(\widehat{H}(k))$. So their kernels are the same. And after Fourier transform, H has a frustration free decomposition

$$H = \sum_{\boldsymbol{x}\in\mathbb{Z}^2} H_x; \quad H_{\boldsymbol{x}} = T_{\boldsymbol{x}} \sum_i |u_i\rangle\langle u_i| T_{\boldsymbol{x}}^{-1}$$

with $H_{\boldsymbol{x}}$ positive by construction. $H_{\boldsymbol{x}}$ is finitely supported since $|u_i\rangle$ are inverse Fourier transforms of trig-polynomials. The joint kernel of $H_{\boldsymbol{x}}$ is the inverse Fourier transform of $\ker \widehat{G}$ and hence is nonzero and same as G. □

4.3. Result: Zero bulk Index

We define the bulk index of a gapped F^4 Hamiltonian as the Chern number of the ground state bundle $\bigcup_{\boldsymbol{k}\in\mathbb{T}^2} \left(\{\boldsymbol{k}\} \times \ker \widehat{H}(\boldsymbol{k})\right)$.

Theorem 4.5

The bulk index of a gapped F^4 *Hamiltonian vanishes.*

This theorem is not new. Using K-theory, [12] proved that flat bands from local Hamiltonians are trivial. In Section 4.7, we give an alternative proof of this statement using an approach similar to [14].

4.4. Result: Zero edge index

Recall the definition of edge index in section 2.5.1, while taking the gap of bulk Hamiltonian as $(0,\gamma)$.

The spectrum of $\widehat{H}_e(k)$ is manifestly non-negative and by the gap assumption, the spectrum of $\widehat{H}_e(k)$ in the interval $(0,\gamma]$ is discrete and finite, uniformly in k. Recall proposition 2.5.1 that Γ is an analytic set in $\mathbb{T}\times[0,\gamma)$ formed by eigenvalues of $\widehat{H}_e(k)$ that are below γ.

$$\Gamma=\{(k,E)\in S^1\times[0,\gamma]: E\in\sigma_p(\widehat{H}_e(k))\}$$

The closure $\overline{\Gamma}\subset\mathbb{S}^1\times[0,\gamma]$ is a one-dimensional analytic set with a boundary, hence any boundary point $(k,E)\in\partial\overline{\Gamma}$ satisfies $E\in\{0,\gamma\}$. It is worth noting that the energy 0 and the energy γ occur in the boundary of $\partial\overline{\Gamma}$ for different reasons. Energy γ is there as an arbitrary cutoff line and Γ can be extended analytically through γ. On the other hand $E=0$ is in the essential spectrum so perturbation theory no longer applies and a priori eigenvalue branches forming Γ might terminate there. The key technical result of this paper is that actually no boundary point occurs at $E=0$.

Lemma 4.4.1

Let H be a gapped, translation-invariant F^4 Hamiltonian. Then $E(\partial\overline{\Gamma})\subset\{\gamma\}$.

The proof of the lemma is in Section 4.5. By analyticity, see the proof of the lemma, branches that touch zero energy have a smooth continuation. Let $\gamma>0$ be such that all energies in $(0,\gamma)$ are regular and let $\mu\in(0,\gamma)$. Then the proposition is equivalent to saying that there exists a finite family of differentiable functions $E_l: D_l\to[0,\mu]$ defined on closed intervals $D_l\subset\mathbb{S}^1$ such that

$$\overline{\Gamma}=\cup_l\cup_{k\in D_l}\{(k,E_l(k))\}.$$

The edge index is independent of the choice of μ, and hence also the choice of γ.

It immediately follows that

Theorem 4.6 (S. Bachmann, Z. Du, M. Fraas 2026)
The edge index of a translation invariant F^4-Hamiltonian is 0.

Proof. The degree $\mu \mapsto \deg(E;\mu)$ is constant in any connected component of $\mathbb{R}\setminus E(\partial\overline{\Gamma})$. The claim of the theorem follows immediately from this and Lemma 4.4.1 since $\deg(E;\mu)=0$ for $\mu<0$ and $0\notin E(\partial\overline{\Gamma})$. The $\mathbb{Z}_2$ index of $\overline{\Gamma}$ is also constant when $\mu\in\mathbb{R}\setminus E(\partial\overline{\Gamma})$, so the same reasoning applies. □

It might be worth repeating the same reasoning analytically. The fundamental theorem of calculus applied to the expression (2.18) of the edge index yields that the index is equal to

$$\sum_l g\left(E_l(k)\right)\big|_{\partial D_l}.$$

By Lemma 4.4.1, an eigenvalue branch $E_l(k)$ can only terminate when $E_l(k)=\gamma$. Hence, $g\left(E_j(k)\right)|_{\partial D_j}=g(\gamma)=0$ by our choice of the function g, proving again that the edge index vanishes.

4.5. Proof of zero edge index

We now turn to the technical heart of this chapter, namely the proof of Lemma 4.4.1. We introduce an auxiliary Hamiltonian H_a, which is a bulk operator and can therefore be compared with H, but it corresponds to a system where two half-spaces have been disconnected, allowing for a relation with the edge Hamiltonian H_e.

4.5.1. Auxiliary Hamiltonian. Let H_a be the operator on $l^2(\mathbb{Z}^2,\mathbb{C}^N)$, defined by

$$H_a=\sum_{\boldsymbol{x}\in\mathbb{Z}^2, x_1\neq 0} H_x = H-\sum_{x_2\in\mathbb{Z}} H_{(0,x_2)}$$

The subspace $l^2(\mathbb{N}\times\mathbb{Z},\mathbb{C}^N)$ is an invariant subspace of H_a and the restriction of H_a to this subspace is the edge Hamiltonian H_e. H_a retains invariance in the vertical direction. The partial Fourier transform $\mathcal{F}_2$ yields the family of operators on $l^2(\mathbb{Z},\mathbb{C}^N)$

$$H_a(k)=\sum_{x_1\neq 0,\, j}\widehat{v}_{(x_1,0),j}(k)\,\widehat{v}^*_{(x_1,0),j}(k),$$

This is a finite rank perturbation of

$$H(k) = \sum_{x_1 \in \mathbb{Z},\, j} \widehat{v}_{(x_1,0),j}(k)\, \widehat{v}^*_{(x_1,0),j}(k).$$

$H(k)$ and $\widehat{v}_j(k)$ are the partial Fourier transforms of the Hamiltonian (4.3) and $|v_j\rangle$. It follows that the spectrum of $H_a(k)$ has at most finitely many eigenvalues in the gap $(0,\gamma)$. Again by analytic perturbation theory [28],

$$\Gamma_a = \{(k,E) \in \mathbb{S}^1 \times (0,\gamma] : E \in \sigma(H_a(k))\},$$

is a one-dimensional real analytic set, i.e. locally the set is a union of graphs of analytic functions. Defining again $E : \overline{\Gamma}_a \to \mathbb{R}$ by $(k,E) \to E$, we have the following analog of Lemma 4.4.1:

Lemma 4.5.1

Let H be a gapped, translation-invariant F^4 Hamiltonian and let Γ_a be as above. Then $\overline{\Gamma}_a$ is a one-dimensional real analytic set and $E(\partial\overline{\Gamma}_a) \subset \{\gamma\}$.

Proof. Consider the projection $P(k)$ onto the kernel of $H(k)$. By gap assumption and frustration free, 0 is an isolated point in the spectrum of $H(k)$ for all $k \in \mathbb{S}^1$, and so $k \mapsto P(k)$ is smooth. Since any complex vector bundle over $\mathbb{S}^1$ is trivial, there exists a smooth family of unitaries $U(k)$ such that $U(k)^* P(k) U(k) = P(0)$. The family of Hamiltonians $\tilde{H}(k) = U(k)^* H(k) U(k)$ acts trivially on $P(0)$, and by the gap assumption

$$\sigma\Big(\tilde{H}(k)(1 - P(0))\Big) \subset [\gamma, \infty).$$

Recalling that H_a is obtained from H by removing some interaction terms, the frustration-free condition implies that $\ker H_a \supset \ker H$ and so $H_a(k)P(k) = 0$. In particular, $H_a(k) = (1 - P(k))H_a(k)(1 - P(k))$. This implies that the range of $P(0)$ belongs to the kernel of

$$\tilde{H}_a(k) = U(k)^* H_a(k) U(k)$$

for all $k \in \mathbb{S}^1$. The eigenvalues $E(k) \in (0,\gamma)$ are in the spectrum of $\tilde{H}_a(k)|_{1-P(0)}$. This restriction is a finite rank perturbation of $\tilde{H}(k)|_{1-P(0)}$, which has no spectrum below γ. It follows from perturbation theory that for each $k \in S^1$, the spectrum of $\tilde{H}_a(k)|_{1-P(0)}$ has a finite number of eigenvalues in $(-\infty, \gamma]$. There is around each $k \in \mathbb{S}^1$ a closed interval I, and a finite number of analytic functions $\mathcal{E}_j(k)$ and analytic vectors $\psi_j(k)$ such that $\psi_j(k)$ is an eigenvector of $\tilde{H}_a(k)|_{1-P(0)}$ with energy $\mathcal{E}_j(k)$ and $\sum_j |\psi_j(k)\rangle\langle\psi_j(k)|$ is the spectral projection of $\tilde{H}_a(k)|_{1-P(0)}$ on the spectral interval $(-\infty,\gamma]$. In particular, the spectrum of $\tilde{H}_a(k)|_{1-P(0)}$ below γ is the union of $\mathcal{E}_j(k)$. It follows that the set

$$\tilde{\Gamma}_a = \{(k,E) \in \mathbb{S}^1 \times (-\infty,\gamma] : E \in \sigma(\tilde{H}_a(k)|_{1-P(0)})\}$$

is a real analytic set. Owing to the restriction on $(1-P(0))$, $\tilde{\Gamma}_a$ can have $E=\gamma$ as the only possible termination point for its branches.

Γ_a is a subset of $\tilde{\Gamma}_a$ that consists of all points (k,E) in $\tilde{\Gamma}_a$ for which $E>0$. By analyticity of $\tilde{\Gamma}_a$, $\overline{\Gamma}_a$ is also a subset of $\tilde{\Gamma}_a$. Points with $E=0$ that belong to $\tilde{\Gamma}_a$ might or might not belong to $\overline{\Gamma_a}$. Suppose that $(0,k_*) \in \tilde{\Gamma}_a$ so that locally $\tilde{\Gamma}_a$ is the union of graphs of $\mathcal{E}_j(k)$ for which $\mathcal{E}_j(k_*)=0$. Since $\tilde{H}_a(k)|_{1-P(0)}$ is a non-negative operator and $\mathcal{E}_j(k)$ are analytic, either $\mathcal{E}_j(k)=0$ for all k, so-called zero mode, or $\mathcal{E}_j(k)>0$ for all $k \neq k_*$ in some neighborhood of k_*, a mode that touches zero at k_*. So a point $(0,k_*)$ belongs to $\overline{\Gamma}_a$ if and only if there exists a mode that touches the line $E=0$ at k_*. It follows that $\overline{\Gamma}_a$ is real analytic and its branches cannot terminate at $E(\partial\overline{\Gamma}_a)=0$ but only at the boundary energy value $E(\partial\overline{\Gamma}_a)=\gamma$. $\square$

A consequence of the lemma is that if we can pick γ small enough so that all energies in $(0,\gamma]$ are regular, then there exists a finite family of analytic functions $\mathcal{E}_l : \mathcal{D}_l \to [0,\gamma]$ defined on closed intervals $\mathcal{D}_l \subset \mathbb{S}^1$, corresponding to the nonzero modes in the above proof, such that

$$\overline{\Gamma}_a = \cup_l \cup_{D_l} \{(k,\mathcal{E}_l(k))\} \tag{4.5}$$

the corresponding projections $\mathcal{P}_l(k)$ are analytic.

4.5.2. Proof of Lemma 4.4.1. We decompose $l^2(\mathbb{Z},\mathbb{C}^N) = l^2(\mathbb{N},\mathbb{C}^N) \oplus l^2(\mathbb{Z}\setminus\mathbb{N},\mathbb{C}^N)$. By construction,

$$H_a(k) = H_e(k) \oplus G(k).$$

Since eigenvalues and eigenvectors are locally analytic, we have that

$$\Gamma_a = \Gamma \cup \Gamma_G, \quad \text{and } \overline{\Gamma}_a = \overline{\Gamma} \cup \overline{\Gamma}_G,$$

and $\overline{\Gamma}$ and $\overline{\Gamma}_G$ are real analytic sets terminating only at $E = \gamma$. Here Γ_G is defined with respect to $G(k)$ analogously to Γ.

In the case where γ is small enough, we can explicitly pick the branches that belong to Γ. Let Q be the multiplication operator in $l^2(\mathbb{Z}, \mathbb{C}^N)$ by the characteristic function of the right half-space. The operator

$$\overline{H}_e(k) = H_a(k)Q = QH_a(k)Q$$

is the extension of $H_e(k)$ by 0 from $\ell^2(\mathbb{N}, \mathbb{C}^N)$ to $\ell^2(\mathbb{Z}, \mathbb{C}^N)$. Since $H_a(k)$ commutes with Q, so do its spectral projections and in particular, for all l, $\overline{P}_l(k) = Q\mathcal{P}_l(k)$ is a projection, possibly zero. If $\overline{P}_l(k)$ is nonzero, then it is a spectral projection of $\overline{H}_e(k)$, and any vector in its range is supported on the right half-line. It follows that $\overline{P}_l(k) = P_l(k)$, up to a relabeling of the branches which can be made consistently by continuity, and any spectral projector of $H_e(k)$ is obtained in this way. It follows that each branch of eigenvalue $E_l(k)$ of $H_e(k)$ is a branch of eigenvalue $\cancel{\mathcal{E}\text{of}}\,\mathcal{E}$ of $H_a(k)$, see (4.5). Since $k \mapsto \mathcal{E}_l(k)$ terminates only at energy γ by Lemma 4.5.1, so do $k \mapsto E_l(k)$.

4.6. Example with gapless edge states

In this section we construct an F^4 Hamiltonian, which has vanishing index by the previous sections, but nonetheless has a gapless edge spectrum.

Recalling the setup of Sections 2.5.1 and 4.5.1,

$$\widehat{H}(\boldsymbol{k}) = \sum_{j=1}^{M} \widehat{v}_j(\boldsymbol{k})\, \widehat{v}_j^*(\boldsymbol{k}), \quad H_e = \sum_{x \in \mathbb{N}} T_x H_0 T_x^{-1}, \quad H_a = \sum_{x \neq 0} T_x H_0 T_x^{-1},$$

we claim that

Proposition 4.6.1

$\widehat{H}_a(k_2)$ has an eigenvalue λ if and only if 1 is an eigenvalue of the two-parameter matrix $T(\lambda, k_2)$ with entries

$$T_{lj}(\lambda, k_2) = \left\langle \widehat{v}_l(k_2),\ \left(\widehat{H}(k_2) - \lambda\right)^{-1} \widehat{v}_j(k_2) \right\rangle_{l^2(\mathbb{Z}, \mathbb{C}^N)} \tag{4.6}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \widehat{v}_l(k_1, k_2)^* \left(\widehat{H}(k_1, k_2) - \lambda\right)^{-1} \widehat{v}_j(k_1, k_2) dk_1. \tag{4.7}$$

Note that we use the same notation for the partial and full Fourier transform. We postpone the proof of this technical result to Appendix A.1.3.

In other words, in order to have an edge state touching 0 at some k_2', we need that 1 is an eigenvalue of the matrix $T(0,k_2')$. For this, it suffices to pick $\{\widehat{v}_j(\boldsymbol{k}) : j = 1,\ldots,M\}$ to be an orthonormal set at k_2' for all k_1. Indeed, we then have for all l,j that

$$\widehat{\boldsymbol{v}}_l(k_1,k_2')^*\,(\widehat{H}(k_1,k_2'))^{-1}\widehat{\boldsymbol{v}}_j(k_1,k_2') = \widehat{\boldsymbol{v}}_l(k_1,k_2')^*(\sum_i \widehat{\boldsymbol{v}}_i(k_1,k_2')\widehat{\boldsymbol{v}}_i(k_1,k_2')^*)^{-1}\widehat{\boldsymbol{v}}_j(k_1,k_2') = \delta_{lj},$$

which implies that $T_{lj}(0,k_2') = \delta_{lj}$, and so 1 is its eigenvalues. An explicit example in $l^2(\mathbb{T}^2,\mathbb{C}^3)$ with $M = 2$ and $k_2' = \pi$ is given by

$$\widehat{\boldsymbol{v}}_1(\boldsymbol{k}) = \begin{pmatrix} 2(e^{-ik_2}-1)-e^{-ik_1} \\ 2i(e^{-ik_2}+1) \\ 1+ie^{-ik_1} \end{pmatrix}, \qquad \widehat{\boldsymbol{v}}_2(\boldsymbol{k}) = \begin{pmatrix} \frac{1}{2}(e^{ik_2}-1)+ie^{ik_1} \\ \frac{i}{2}(e^{ik_2}+1) \\ -4-e^{ik_1} \end{pmatrix}. \tag{4.8}$$

In $l^2(\mathbb{Z},\mathbb{C}^N)\otimes L^2(\mathbb{S}^1)$ local terms in the decomposition of $H_e(k_2)$ and $H_a(k_2)$ are:

$$\begin{aligned} H_0(k_2) = \quad & |\delta_{-1}\rangle\langle\delta_0| \otimes \begin{pmatrix} \left(-2+\frac{i}{2}\right)\left(-1+e^{-ik_2}\right) & -2i\left(1+e^{-ik_2}\right) & \frac{1}{2}\left(-1-e^{-ik_2}\right) \\ \frac{1}{2}\left(1+e^{-ik_2}\right) & 0 & \frac{1}{2}i\left(1+e^{-ik_2}\right) \\ -2i\left(1+e^{-ik_2}\right) & 2+2e^{-ik_2} & 4-i \end{pmatrix} \\ + \quad & |\delta_0\rangle\langle\delta_0| \otimes \begin{pmatrix} \frac{21}{2}-\frac{17}{2}\cos(k_2) & -\frac{15}{2}\sin(k_2) & 4i\sin(k_2) \\ -\frac{15}{2}\sin(k_2) & \frac{17}{2}(\cos(k_2)+1) & 4\sin(k_2) \\ -4i\sin(k_2) & 4\sin(k_2) & 19 \end{pmatrix} \\ + \quad & |\delta_1\rangle\langle\delta_0| \otimes \begin{pmatrix} \left(-2-\frac{i}{2}\right)\left(-1+e^{ik_2}\right) & \frac{1}{2}\left(1+e^{ik_2}\right) & 2i\left(1+e^{ik_2}\right) \\ 2i\left(1+e^{ik_2}\right) & 0 & 2\left(1+e^{ik_2}\right) \\ \frac{1}{2}\left(-1-e^{ik_2}\right) & -\frac{1}{2}i\left(1+e^{ik_2}\right) & 4+i \end{pmatrix}. \end{aligned} \tag{4.9}$$

We plot the full spectrum of H_e in Figure 4.2. The numerical computation was done using the PythTB package [13].

Finally, one may wonder if a similar example with $M = 2$ is possible. We show in Appendix A.2 that there is no such example with a gapless edge mode.

4.7. Proof of trivial Bloch bundle and zero bulk index

Recall our definition of Bloch bundle in 2.2.3 and proposition 4.2.1.

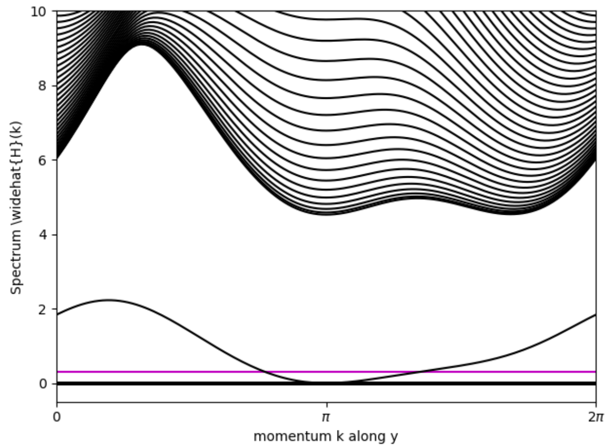


FIGURE 4.2. The gapless edge spectrum of an F^4 Hamiltonian. The edge index vanishes since the branch in the gap crosses the line of constant energy twice γ in opposite directions.

Definition 4.7

We define the excited band bundle to be the subset of $\mathbb{T}^2 \times \mathbb{C}^N$

$$E := \bigsqcup_{k \in [0,2\pi)^2} Im_{\mathbb{C}}\big(H(\boldsymbol{k})\big) \tag{4.10}$$

with the subset topology. The base manifold is $\mathbb{T}^2$ *and fibers are vector spaces* $Im_{\mathbb{C}}\big(H(\boldsymbol{k})\big)$*. Similarly, the ground state bundle is*

$$G := \bigsqcup_{k \in [0,2\pi)^2} ker_{\mathbb{C}}\big(H(\boldsymbol{k})\big) \tag{4.11}$$

The bulk index refers to the first Chern number of the bundle G *and is known to be the Hall conductance at the ground state [37] [3].*

By the gap condition, the dimension of kernel and image of $\widehat{H}(\boldsymbol{k})$ are constant over $k \in \mathbb{T}^2$, because otherwise there will be some $\boldsymbol{k}_1$ and a 0-eigenvalue of $\widehat{H}(\boldsymbol{k}_1)$ departing from 0 continuously and closing the gap. So let

$$M = \mathrm{Rank}_{\mathbb{C}}(\widehat{H}(\boldsymbol{k})),\ \forall k \in \mathbb{T}^2 \tag{4.12}$$

E is a rank M vector bundle.

It is more convenient to study the bundle E whose Chern number is the negative of G. E is characterized by a module over the ring R of polynomials: Let

$$X := e^{ik_x},\ Y := e^{ik_y} \text{ and } R := \mathbb{C}[X, Y]$$

Owing to compact support of H_x in lattice space, the momentum space $\widehat{H}$ is an $n \times n$ matrix of trig polynomial entries, i.e. Laurent polynomial [2] of X, Y. Its image

$$\mathrm{Im}_R(\widehat{H}) = \{\widehat{H}\psi \mid \psi \in R^N\}$$

is a finitely generated $R-$module.

Theorem 4.8

Under the assumption that H is non-negative, gapped, frustration-free and strictly local; the excited state bundle $E = \bigsqcup_k Im_{\mathbb{C}}\big(\widehat{H}(\boldsymbol{k})\big)$ must be trivial. As a consequence, the 1st Chern number is zero.

4.7.1. Proof of theorem 4.8. We sketch the proof as following: First, by the Syzygy theorem there is an exact sequence (4.13) of R modules ending at the module $\mathrm{Im}_R(\widehat{H})$; Next, when evaluating the polynomials at any $\boldsymbol{k} \in \mathbb{T}^2$, the exact sequence of R-modules gives rise to a family of exact sequences of $\mathbb{C}$-vector spaces; Finally, we show Corollary (4.7.3) E is a successive quotient of a trivial bundle by a trivial subbundle.

By the Hilbert syzygy theorem [35], every finitely generated module over the regular ring $\mathbb{C}[X, Y]$ has a *free resolution*:

$$0 \to R^{N_2} \xrightarrow{h_2} R^{N_1} \xrightarrow{h_1} R^N \xrightarrow{h_0 = \widehat{H}} \mathrm{Im}_R(\widehat{H}) \to 0_R \tag{4.13}$$

meaning $\ker(h_j) = \mathrm{Im}(h_{j+1})$ as R-modules.

Lemma 4.7.1

For some j, let $Rank_{\mathbb{C}}(h_j(\boldsymbol{k})) = M_j\ \forall \boldsymbol{k} \in \mathbb{T}^2$ be constant upon evaluation, and consider the degree-j term in a sequence of R-module: $\xrightarrow{h_{j+1}} R^{N_j} \xrightarrow{h_j}$, then the rank of degree $j+1$ term is also constant and full:

$$Rank_{\mathbb{C}}(h_{j+1}(\boldsymbol{k})) = Null_{\mathbb{C}}(h_j(\boldsymbol{k})) = N_j - M_j$$

[2]To avoid Laurent polynomials, one can also multiply the negative power entries of $\widehat{H}$ so that $\tilde{\widehat{H}} := X^p Y^q \widehat{H}$ has proper polynomial entries and is a matrix of entries in R. But it does not make any difference algebraically or geometrically.

Since $\ker_R(h_j) = \mathrm{Im}_R(h_{j+1}) \implies \mathrm{Im}_{\mathbb{C}}(h_{j+1}(\boldsymbol{k})) \subset \ker_{\mathbb{C}}(h_j(\boldsymbol{k}))$, the upshot of this lemma is that the proper containment $\mathrm{Rank}_{\mathbb{C}}(h_{j+1}(\boldsymbol{k})) < \mathrm{Null}_{\mathbb{C}}(h_j(\boldsymbol{k}))$ never happens. We'll delay the proof of this lemma to the end of this chapter.

A frustration-free non-interacting fermion Hamiltonian $\widehat{H}(\boldsymbol{k})$ has constant rank M. Applying the lemma inductively, we see that $\mathrm{Rank}_{\mathbb{C}}(h_{j+1}(\boldsymbol{k})) = \mathrm{Null}_{\mathbb{C}}(h_j(\boldsymbol{k})) = N_j - M_j$ for all j, and hence we have the following exact sequence of vector spaces:

Lemma 4.7.2

The sequence of $\mathbb{C}$-vector spaces after evaluating (4.13) at $\forall \boldsymbol{k} \in \mathbb{T}^2$

$$0 \to \mathbb{C}^{N_2} \xrightarrow{h_2(\boldsymbol{k})} \mathbb{C}^{N_1} \xrightarrow{h_1(\boldsymbol{k})} \mathbb{C}^{N} \xrightarrow{h_0(\boldsymbol{k})} Im_{\mathbb{C}}(\widehat{H}(\boldsymbol{k})) \to 0_{\mathbb{C}} \tag{4.14}$$

is still exact.

Then the theorem follows:

Corollary 4.7.3

Exact sequence (4.14) implies the vector bundle E is trivial.

Proof. $\{0\} = \ker(h_2(\boldsymbol{k}))$, $\forall k$ and h_2 has polynomial entries so $h_2 : \mathbb{T}^2 \times \mathbb{C}^{N_2} \to \bigsqcup_k \mathrm{Im}(h_2(\boldsymbol{k}))$ is a smooth family of fiberwise isomorphisms, i.e. a bundle isomorphism. So $\bigsqcup_k \mathrm{Im}(h_2(\boldsymbol{k}))$ is a trivial bundle. Taking quotient from $\mathbb{T}^2 \times \mathbb{C}^{N_1}$ in each fiber yields a trivial bundle $\bigsqcup_k \left(\mathbb{C}^{N_1}/\mathrm{Im}\, h_2(\boldsymbol{k})\right)$. $\mathrm{Im}(h_2(\boldsymbol{k})) = \ker(h_1(\boldsymbol{k}))$ so h_1 induces a bundle isomorphism $\bigsqcup_k \mathbb{C}^{N_1}/\mathrm{Im}(h_2(\boldsymbol{k})) \to \bigsqcup_k \mathrm{Im}(h_1(\boldsymbol{k}))$. Repeating this argument once more leads to the last bundle $E = \bigsqcup_k \mathrm{Im}(\widehat{H}(\boldsymbol{k}))$ being trivial. $\square$

4.7.2. Proof of lemma 4.7.1. We start from the constant rank of deg 0 term in the sequence

Proposition 4.7.4

$Rank_{\mathbb{F}}(h_0(\boldsymbol{k})) = Rank_{\mathbb{C}}(h_0(\boldsymbol{k})) = Rank_{\mathbb{C}}(\widehat{H}(\boldsymbol{k})) = M$ where $\mathbb{F}$ is the field of rational functions

Proof. For any $\boldsymbol{k}_0$, there is a set of polynomial vectors $\{w_i\}_{i=1}^M$ that is a $\mathbb{C}$-basis of $\mathrm{Im}_{\mathbb{C}}(\widehat{H}(\boldsymbol{k}_0))$ when evaluating in a neighborhood of $\boldsymbol{k}_0$. These bases exist because rank of bundle E is M. $\widehat{H}$ has Laurent polynomial matrix entries so one can multiply the common divisor to the arguments of $\widehat{H}$ until all entries are polynomial. $\{w_i(k_0)\}$ are $\mathbb{C}$-linearly independent so $\bigwedge_i w_i(k) \neq 0$. If w_i

are $\mathbb{F}$-linearly dependent then $\bigwedge_i w_i = 0_{\mathbb{F}}$, which contradicts $\bigwedge_i w_i(k) \neq 0$. So $\{w_i\}$ are linearly independent as $\mathbb{F}$ vectors; $\mathrm{Rank}_{\mathbb{F}}(\widehat{H}) = M$.

The rank of upstream term h_1 can be deduced in the rational field:

$$\mathrm{Rank}_{\mathbb{F}}(h_1) = \mathrm{Null}_{\mathbb{F}}(\widehat{H}) = N - \mathrm{Rank}_{\mathbb{F}}(\widehat{H}) = N - M \tag{4.15}$$

because the localization functor $loc : Mod_R \to Mod_{\mathbb{F}}$ preserves exact sequences [35], $\ker_R(\widehat{H}) = \mathrm{Im}_R(h_1) \implies \ker_{\mathbb{F}}(\widehat{H}) = \mathrm{Im}_{\mathbb{F}}(h_1)$ and then it follows from proposition 4.7.4.

Our concern for lemma 4.7.1 is whether evaluation at some $\boldsymbol{k}$ may decrease the rank: $\mathrm{Rank}_{\mathbb{C}}(h_1(\boldsymbol{k})) \leq \mathrm{Null}_{\mathbb{F}}(\widehat{H})$. However, one can find explicitly $N - M$ independent vectors in $\mathrm{Im}(\widehat{H}(k))$:
For any $\boldsymbol{k} \in \mathbb{T}^2$, one performs two successive row reductions of $\widehat{H}$ with coefficients first in $\mathbb{C}$ and then in the rational field $\mathbb{F}$. This gives us an $m \times m$ identity matrix on the top-left corner, and a rational matrix G on top-right:

$$V\widehat{H}(\boldsymbol{k}) = \begin{pmatrix} \mathbb{I} & g(\boldsymbol{k}) \\ 0 & 0 \end{pmatrix}_{M_{N\times N}(\mathbb{C})} \quad \text{for some } V \in M_{N\times N}(\mathbb{C}) \tag{4.16}$$

$$V\widehat{H} = \begin{pmatrix} i & g \\ f & j \end{pmatrix} \quad \text{where } i, j, f, g \text{ are polynomial matrices} \tag{4.17}$$

$$UV\widehat{H} = \begin{pmatrix} \mathbb{I}_{M\times M} & G \\ 0 & 0 \end{pmatrix}_{M_{N\times N}(\mathbb{F})} \quad \text{for some } U \in M_{N\times M}(\mathbb{F}) \tag{4.18}$$

Let $e_i = (0, ..., 1, ..., 0)^T$, $M \leq i \leq N$ be the standard basis of $\mathbb{F}^{N-M}$. Let $\Delta \in R$ be the least common multiple of denominators of G, then

$$W_i := \Delta^2 \begin{pmatrix} Ge_i \\ -e_i \end{pmatrix} \in R^N$$

satisfies $UV\widehat{H}W_i = 0_R$; U, V being invertible so $W_i \in \ker_R(\widehat{H})$.
For this specific $\boldsymbol{k}$, $\Delta(\boldsymbol{k}) \neq 0$ because rows of G are $\mathbb{F}$-linear combinations of g and j, and all denominators in entries of G come from dividing by diagonal entries of polynomial matrix i. When

evaluating at $\boldsymbol{k}$, diagonal entries of i become 1 as in (4.18) and thus denominators of entries of G are nonzero.

$\{W_i(\boldsymbol{k})\}_{i=1}^{N-M}$ are $\mathbb{C}$-independent because they are nonzero and the lower sector $\{e_i\}$ are $\mathbb{C}$-independent. By exactness (4.13) $W_i \in \ker_R(\widehat{H}) = \mathrm{Im}_R(h_1)$. So their evaluations are also in the image $W_i(\boldsymbol{k}) \in \mathrm{Im}_{\mathbb{C}}(h_1(\boldsymbol{k}))$ and we have the constant $\mathrm{Rank}_{\mathbb{C}}(h_1(\boldsymbol{k})) = N - M \ \forall \boldsymbol{k}$. Lemma 4.7.1 holds at deg 1 term. h_1 is a polynomial matrix with constant rank $N - M$ just like h_0, so we can proceed by induction up through the degree and prove lemma 4.7.1.

CHAPTER 5

Quantum Hall effect in many body system

In this chapter we shift our scope to many-body quantum systems. The operator algebra that describes quanum spin system and interaction will be introduced in section 5.1. In section 5.4.2 a sequence of literatures will be studied to present various related expressions of Hall conductance and their established results. We'll briefly repeat the construction of many body index in [5] and recognize its interpretation as charge transport through a fiducial line while the system's ground state undergoes a unitary evolution. Then we'll further identify the unitary evolution as a magnetic flux insertion through a loop of the toric lattice. Finally we identify that such index is also a generalized Chern number in many-body systems.

Among those studies, the technique of spectral filtering is used extensively, specifically in the adiabatic evolution of ground-state projector during flux threading, and in the inverse Liouvillian used to prove approximate quantization of the many-body charge transport index. In section 5.4 we present a new result that improves the approximate quantization error due to finite system size. This chapter contains essentially the same content as a co-authored article [10] with Sven Bachmann, Martin Fraas and Tom Wessel.

5.1. Algebra of quantum many-body systems

Unlike in the setup of free Fermions, we'll study a family of finite-sized lattices Λ. Mostly we will work on the discrete 2 torus

$$\Lambda = \mathbb{Z}/(L\mathbb{Z}) \times \mathbb{Z}/(L\mathbb{Z})$$

with the 1 norm induced metric on torus $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{2} \min_n(|x_i - (y_i + nL)|)$. In preparation for the Lieb-Robinson bound, we verify that

Proposition 5.1.1

The family of discrete toric lattices is a $D-$regular graph: there is a constant $\mathcal{C}_{\mathrm{vol}} > 0$ such that *the volume of any ball*

$$B_x(r) := \{z \in \Lambda \mid d(z,x) \le r\}$$

scales as the square of the radius

$$\Big|B_x(r)\Big| \le \mathcal{C}_{\mathrm{vol}}\,(r+1)^2 \qquad \textit{for all } x \in \Lambda \textit{ and } r \ge 0.$$

In particular, we note that there exists a constant $\mathcal{C}_{\mathrm{vol},b,k} \ge 1$ such that for all sets $Z \subset \Lambda$

$$\Big|Z\Big|^k \mathrm{e}^{-b\,\mathrm{diam}(Z)} \le \mathcal{C}_{\mathrm{vol},b,k}.$$

A quantum spin system of n local components consists of the Hilbert space

$$\mathcal{H} = \mathcal{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathbb{C}^n_x$$

and the algebra of observables

$$\mathcal{A} = \mathcal{L}(H)$$

and specifically we call $\mathcal{A}_X$ the subalgebra that acts nontrivially on a subset X of the lattice:

$$\mathcal{A}_X = \left\{A_X \otimes \mathbb{I}_{\Lambda\setminus X} \middle| A \in \mathcal{L}(\mathcal{H}_X)\right\}$$

A Fermionic system of n local components consists of the Fock space

$$\mathcal{H} = \mathcal{H}_F = \bigoplus_{n=0}^{\infty} \bigwedge^{n} \ell^2(\Lambda, \mathbb{C}^n)$$

and the algebra of observables generated by the canonical anticommutation relation

$$c^\dagger_{x,i}\,1_{\mathbb{C}} = |\delta_x \otimes e_i\rangle \in \ell^2(\Lambda,\mathbb{C}^n); \quad \{c_{x,i}, c^\dagger_{y,j}\} = \delta_{x,y}\delta_{i,j}; \quad \{c,c\} = \{c^\dagger, c^\dagger\} = 0$$

The notion of local subalgebra only exists in the even degree, as operators made of even degree product of $c_{x,i}, c^\dagger_{x,i}$ for $x \in X$.

We shall not distinguish the local algebra of quantum spin system or Fermionic system because both satisfy the locality property: $[A,B] = 0$ for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_{\Lambda\setminus X}$; and for a dynamic generated by a finite range interaction, the Lieb-Robinson bound holds true, see lemma 5.1.2.

A finite range interaction on Λ is a map that sends subsets of Λ to self adjoint operators

$$\Phi : \mathcal{P}(\Lambda) \to \mathcal{A}_\Lambda, \quad Z \mapsto \Phi(Z) \in \mathcal{A}_Z \quad \text{with} \quad \Phi(Z) = \Phi(Z)^* \tag{5.1}$$

And there is an $R_\Phi > 0$, $s.t.\, \Phi(X) = 0 \;\forall\, \mathrm{diam}(X) > R_\Phi$. Such R_Φ is assumed to be universal for all system sizes L.

For $b \geq 0$, we define interaction b-norms

$$\|\Phi\|_b := \sup_{z\in\Lambda} \sum_{Z \in z} \|\Phi(Z)\|_{op}\, \mathrm{e}^{b\,\mathrm{diam}(Z)}. \tag{5.2}$$

An interaction Φ gives rise to the corresponding operator

$$H := \sum_{Z \subset \Lambda} \Phi(Z),$$

which generates the Heisenberg evolution $\tau_{s,t}$ defined as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\tau_{s,t}(A) = \tau_t\big(\mathrm{i}\;[H, A]\big), \qquad \tau_0(A) = A \quad \forall\, A \in \mathcal{A}_\Lambda \tag{5.3}$$

An important property of the Heisenberg evolution is its locality as captured by Lieb-Robinson bounds, which originated in [**31**] and were generalized in [**33**] and many other works. Here we state a version for the norm (5.2), whose time-independent version appeared in [**8**, Theorem A.1], and the time-dependent one is in [**32**, Theorem 7.3.3].

Lemma 5.1.2 (Lieb-Robinson bound)
Let Λ be a finite regular D-graph with $\mathcal{C}_{\mathrm{vol}} > 0$, and $b' > b > 0$. Then, for all intervals $I \subset \mathbb{R}$, time-dependent interactions Φ such that $\|\Phi\|_{b'} < \infty$, disjoint subsets X, $Y \subset \Lambda$, observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, and s, $t \in I$ it holds that

$$\big\|[\tau_{s,t}(A), B]\big\| \leq 2\,\mathcal{C}^{-1}_{\mathrm{vol},1,b'-b}\,\|A\|\,\|B\|\,\big(\mathrm{e}^{b\,v\,|t-s|} - 1\big)\,D(X,Y), \tag{5.4}$$

where $v = 2\,\mathcal{C}_{\mathrm{vol},1,b'-b}\,\|\Phi\|_{b'}/b$ is the Lieb-Robinson velocity and

$$\begin{aligned} D(X,Y) &:= \min\{\sum_{x\in X} \mathrm{e}^{-bd(x,Y)}, \sum_{y \in Y} \mathrm{e}^{-bd(y,X)}\} \\ &\leq \min\{|X|, |Y|\}\,\mathrm{e}^{-bd(X,Y)}. \end{aligned}$$

5.2. Almost inverse Liouvillian with exponential locality

The Heisenberg time evolution $A \mapsto \tau_t(A) = \mathrm{e}^{\mathrm{i}Ht} A \, \mathrm{e}^{-\mathrm{i}Ht}$ preserves locality of operator A. Propagation is strictly within the light cone when the underlying dynamics follow the wave equation; it is diffusive for the heat equation, and the Lieb-Robinson bound [31] provides an effective ballistic propagation bound for the Schrödinger equation on sufficiently regular lattices. Among others, these results are essential for understanding both thermal phases at finite temperature and ground state phases, see [22] for many examples. In this context, the smearing map[1]

$$\tau_f(A) = \int_{-\infty}^{\infty} \mathrm{d}t \, f(t) \, \tau_t(A) \tag{5.5}$$

has seen many uses, since the real space decay of f controls the locality of $A \mapsto \tau_f(A)$ while its Fourier transform $\widehat{f}$ controls the spectral properties of the map.

In this section, we consider the use of Gaussian filter functions as $f(t)$, providing exact bounds both on the spatial locality and on the spectral errors. In particular, we define a spectral flow which is exponentially local and almost exact. Our Ansatz is similar to the one used in [24]. Here, the smearing is applied term-by-term on an interaction and the width of the Gaussian must be chosen in a spatially inhomogeneous way. As expected, better locality yields worse spectral mapping properties. What is more, a properly chosen Gaussian filter convolved with a step function yields exponential decay of correlations for gapped spectral patches whose width does not need to vanish in the thermodynamic limit. These results are similar to those of [?], although the methods are completely different. As a corollary of all the above, we conclude that the Hall conductance is quantized in finite systems up to errors that are exponentially small in the system size. In this statement and the rest of the chapter, the interaction norm and the evolution implicitly depend on Λ and this dependence is understood from the context. Importantly, all the constants will be independent of the size L of the lattice $\mathbb{Z}/(L\mathbb{Z})^2$ and only depend on the lattice through $\mathcal{C}_{\mathrm{vol}}$. In fact, one can define an interaction Φ and the interaction norm $\|\cdot\|_b$ on an infinite lattice $\Gamma = \mathbb{Z}^D$ and the restrictions $\Phi|_\Lambda$ to finite lattices $\Lambda \subset \Gamma$ satisfy $\|\Phi|_\Lambda\|_b \leq \|\Phi\|_b$. Our results are then uniform in $\Lambda \subset \Gamma$.

5.2.1. The almost inverse Liouvillian.

[1] When the subscript of τ is a real number, it denotes the time evolution and when the subscript is a function, it denotes this integral.

Assumption 5.1 (Gap Assumption)

Let Λ be finite. Let Φ be an interaction satisfying $\|\Phi\|_{b'} < \infty$ for some $b' > 0$, and let H be the corresponding Hamiltonian. We assume that the spectrum of H is of the form

$$\sigma(H) = \sigma_0 \cup \sigma_1$$

with $\gamma = d(\sigma_1, \sigma_0) > 0$.

From here onwards, we shall drop the subscript Λ for notational clarity. We denote $P = \chi_{\sigma_0}(H)$ the spectral projection of H corresponding to the patch σ_0.
For $\beta > 0$ let

$$\phi_\beta(t) := \frac{\beta}{\sqrt{\pi}} \, \mathrm{e}^{-\beta^2 t^2}.$$

It is such that $\int \phi_\beta = 1$ and the Fourier transform is

$$\hat{\phi}_\beta(\omega) := \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t \, \phi_\beta(t) \, \mathrm{e}^{-\mathrm{i}t\omega} = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-\frac{\omega^2}{4\beta^2}} \tag{5.6}$$

for which $\int \hat{\phi}_\beta = \beta \sqrt{\frac{2}{\pi}}$. Then, for any Hamiltonian H and observable A

Definition 5.2 (Almost-inverse-Liouvillian)

$$\mathcal{I}_{H,\beta}(A) := \int_{-\infty}^{\infty} \mathrm{d}t \, \phi_\beta(t) \int_0^t \mathrm{d}s \, \tau_s^H(A), \tag{5.7}$$

where $\tau_s^H(A) = \mathrm{e}^{\mathrm{i}Hs} \, A \, \mathrm{e}^{-\mathrm{i}Hs}$.

To quantify the properties of the almost inverse Liouvillian, it will be helpful to define the exact inverse Liouvillian $\mathcal{I}_H$ along the same lines.

Definition 5.3 (Inverse Liouvillian)

$$\mathcal{I}_H(A) := \int_{-\infty}^{\infty} \mathrm{d}t \, w(t) \int_0^t \mathrm{d}s \, \tau_s^H(A) \tag{5.8}$$

With $\widehat{w}$ being a continuous function supported on $(-\delta, \delta)$, w an even function and $\int_{\mathbb{R}} w(t)dt = 1$

See [9] for a concrete definition of such w. Note that the compact support of $\widehat{w}$ allows w to decay faster than any inverse power but not exponentially. Whenever $\mathcal{I}_H$ involves a gapped Hamiltonian with gap γ, we implicitly choose $\delta < \gamma$.

Proposition 5.2.1

The inverse Liouvillian enjoys the defining property that

$$\mathcal{I}_H \circ \mathcal{L}_H(P\,A\,P^\perp) = P\,A\,P^\perp \tag{5.9}$$

Proof. The fundamental theorem of calculus on $\tau_t^H\big(\mathcal{L}_H(P\,A\,P^\perp)\big) = -\frac{d}{dt}\tau_t^H(P\,A\,P^\perp)$ gives

$$\tau_t^H(P\,A\,P^\perp) - P\,A\,P^\perp = \int_0^t \frac{d}{ds}\tau_s^H(P\,A\,P^\perp)\,ds = -\int_0^t \tau_s^H\big(\mathcal{L}_H(P\,A\,P^\perp)\big)\,ds$$

$$\implies \mathcal{I}_H \circ \mathcal{L}_H(P\,A\,P^\perp) = \int_{\mathbb{R}} dt\,(-w(t)) \int_0^t ds\,\tau_s^H\big(\mathcal{L}_H(P\,A\,P^\perp)\big)$$

Inserting resolution of identity $\mathbb{I} = \sum_{\mu\in\sigma(H)} P_\mu$ gives

$$\mathcal{I}_H \circ \mathcal{L}_H(P\,A\,P^\perp) = \sum_{\mu,\sigma_0}\sum_{\nu\in\sigma_1}(-\widehat{w}(\mu-\nu))P_\mu A\,P_\nu \;-\int_{\mathbb{R}} dt\,(-w(t))P\,A\,P^\perp = P\,A\,P^\perp$$

where we used $\int_R dt\,w(t) = 1$ and $\widehat{w}(\mu-\nu) = 0$ for $|\mu-\nu| > \gamma$ □

Remark 5.2.2

Since we are dealing with a fixed finite volume here, the rank of P is finite. For the applications we have in mind however, it is crucial to have that P_Λ remains uniformly bounded as $\Lambda \to \Gamma$.

The almost inverse Liouvillian enjoys almost the same properties of inverting $\mathcal{L}_H(P\,A\,P^\perp)$:

Proposition 5.2.3

Let H satisfy the Gap Assumption. Then for all $A \in \mathcal{A}$ such that $A = P\,A\,P^\perp$ or $A = P^\perp\,A\,P$, and for all $q \in [1,\infty]$,

$$\big\|\mathcal{I}_{H,\beta} \circ \mathcal{L}_H(A) - A\big\|_q \le \|P_\mu\|_1\,\|A\|\,\mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Moreover, for all $A \in \mathcal{A}$ and $q \in [1,\infty]$, it holds that

$$\big\|[\mathcal{I}_{H,\beta} \circ \mathcal{L}_H(A) - A, P]\big\|_q \le 2\,\|P_\mu\|_1\,\|A\|\,\mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Proof. For the first statement, since the calculation is exactly analogous, we only consider the case $A = P\,A\,P^{\perp}$. From the definition of the almost spectral flow, the fact that $\tau_s^H(\mathcal{L}_H(A)) = -\frac{\mathrm{d}}{\mathrm{d}s}\,\tau_s^H(A)$ and the spectral theorem for H, we obtain

$$
\begin{aligned}
\big(\mathcal{I}_{H,\beta} - \mathcal{I}_H\big) \circ \mathcal{L}_H(A) &= \int_{-\infty}^{\infty} \mathrm{d}t\, \big(w(t) - \phi_\beta(t)\big)\, \big(\tau_t^H(A) - A\big) \\
&= \sum_{\mu\in\sigma_0} \sum_{\nu\in\sigma_1} \int_{-\infty}^{\infty} \mathrm{d}t\, \big(w(t) - \phi_\beta(t)\big)\, \mathrm{e}^{\mathrm{i}t(\mu-\nu)}\, P_\mu\, A\, P_\nu \\
&= \sqrt{2\pi} \sum_{\mu\in\sigma_0} P_\mu\, A \sum_{\nu\in\sigma_1} \hat{\phi}_\beta(\nu-\mu)\, P_\nu,
\end{aligned}
$$

where we used $\int w = 1 = \int \phi_\beta$ and $\widehat{w}(\nu-\mu) = 0$ because $\left|\nu-\mu\right| \geq \gamma$ by the Gap Assumption. By the triangle and Hölder inequalities, in particular using that $\|P_\mu\|_q \leq \|P_\mu\|_1$ for all q and μ and $\sum_{\mu\in\sigma_0} \|P_\mu\|_1 = \|P\|_1$ we obtain,

$$
\begin{aligned}
\left\|\mathcal{I}_{H,\beta} \circ \mathcal{L}_H(A) - A\right\|_q &\leq \sqrt{2\pi} \sum_{\mu\in\sigma_0} \|P_\mu\|_q\, \|A\| \left\|\sum_{\nu\in\sigma_1} \hat{\phi}_\beta(\nu-\mu)\, P_\nu\right\| \\
&\leq \sqrt{2\pi}\, \|P\|_q\, \|A\|\, \hat{\phi}_\beta(\gamma).
\end{aligned}
$$

With (5.6), this yields the claim.

For the second statement, note that

$$
[\mathcal{I}_{H,\beta} \circ \mathcal{L}_H(A) - A, P] = \mathcal{I}_{H,\beta} \circ \mathcal{L}_H([A,P]) - [A,P]
$$

because H and P commute. The statement follows by applying the first part since $[A,P] = P^{\perp}\,A\,P - P\,A\,P^{\perp}$. $\square$

Proposition 5.2.4

Let H satisfy the Gap Assumption and assume that $\sigma_0 = \{E_0\}$ is a single eigenvalue. Then

$$
\mathcal{I}_{H,\beta}(P\,A\,P) = 0,
$$

for all $A \in \mathcal{A}_\Lambda$.

Proof.

$$
\mathcal{I}_H(P\,A\,P) = \int_{-\infty}^{\infty} \mathrm{d}t\, w(t) \int_0^t \mathrm{d}s\, \tau_s^H(P\,A\,P) = \int_{-\infty}^{\infty} \mathrm{d}t\, w(t)\, t\, P\,A\,P = 0,
$$

because the time-evolution is trivial and w is an even function. $\square$

Remark 5.2.5

The fact that $\mathcal{I}_{H,\beta}$ vanishes exactly on the range of P depends on the assumption $\sigma_0 = \{E_0\}$. For spectral patches with $\delta = \mathrm{diam}(\sigma_0) > 0$ sufficiently small with respect to γ, a similar result could be obtained, although with an error bound, by replacing $\hat{\phi}_\beta$ with two Gaussians centred at $\pm\frac{\delta+\gamma}{2}$.

The point of defining $\mathcal{I}_{H,\beta}$ is to trade the inversion accuracy for better locality properties.

Lemma 5.2.6 (Locality of the almost inverse Liouvillian)

Let $D \in \mathbb{N}$, $\mathcal{C}_{\mathrm{vol}} > 0$ and $\Lambda \in \mathcal{G}(D, \mathcal{C}_{\mathrm{vol}})$ be finite. Let $b' > b > 0$, Φ be an interaction such that $\|\Phi\|_{b'} < \infty$ and H the corresponding Hamiltonian. Then, for all disjoint X, $Y \subset \Lambda$, $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$,

$$\big\|[\mathcal{I}_{H,\beta}(A), B]\big\| \leq 2\,\min\{\big|X\big|, \big|Y\big|\}\,\|A\|\,\|B\|\,\inf_{T>0}\left(\frac{2\,\beta}{\sqrt{\pi}\,b^2\,v^2}\,\mathrm{e}^{b(vT-d(X,Y))} + \frac{1}{\sqrt{\pi}\,\beta}\,\mathrm{e}^{-\beta^2 T^2}\right), \tag{5.10}$$

where v is the Lieb-Robinson velocity from Lemma 5.1.2.

One may choose $T = \frac{d(X,Y)}{2v}$ and then use $d(X,Y)^2 \geq d(X,Y)$ to get

$$\big\|[\mathcal{I}_{H,\beta}(A), B]\big\| \leq 2\,\min\{\big|X\big|, \big|Y\big|\}\,\|A\|\,\|B\|\left(\frac{2\,\beta}{\sqrt{\pi}\,b^2\,v^2} + \frac{1}{\sqrt{\pi}\,\beta}\right)\mathrm{e}^{-b(\beta)d(X,Y)}, \tag{5.11}$$

where $b(\beta) = \min\{\frac{b}{2}, \frac{\beta^2}{4v^2}\}$.

Remark 5.2.7

The last expression (5.11), in particular $b(\beta)$, emphasizes the double origin of the quasi-locality of the map $\mathcal{I}_{H,\beta}$, namely the locality of the Hamiltonian through the Lieb-Robinson bound yielding the dependence on b and the locality expressed by the Gaussian filter yielding the dependence on β. In particular, the locality cannot be improved further than β of the order of $\sqrt{b}$.

Proof. We bound

$$\big\|[\mathcal{I}_{H,\beta}(A), B]\big\| \leq \int_{-\infty}^{\infty}\mathrm{d}t\,\phi_\beta(t)\int_0^t \mathrm{d}s\,\big\|\big[\tau_s^H(A), B\big]\big\| = \inf_{T>0}(I_{<T} + I_{\geq T}),$$

and split the integral into a part $I_{<T}$ where $\big|t\big| < T$ and the rest $I_{\geq T}$ where $\big|t\big| \geq T$. For the first part we use $\phi_\beta(t) \leq \beta/\sqrt{\pi}$ and the Lieb-Robinson bound for τ^H, Lemma 5.1.2, where we bound

$\mathcal{C}_{\mathrm{vol},1,b'-b}^{-1} \leq 1$, to obtain

$$I_{<T} \leq \frac{\beta}{\sqrt{\pi}}\, 4\, \min\{\left|X\right|, \left|Y\right|\}\, \|A\|\, \|B\|\, \mathrm{e}^{-bd(X,Y)} \int_0^T \mathrm{d}t\, \int_0^t \mathrm{d}s\, \left(\mathrm{e}^{bvs} - 1\right)$$
$$= \frac{\beta}{\sqrt{\pi}}\, 4\, \min\{\left|X\right|, \left|Y\right|\}\, \|A\|\, \|B\|\, \frac{1}{(bv)^2} \left(\mathrm{e}^{bvT} - 1 - bvT - \tfrac{1}{2}(bvT)^2\right) \mathrm{e}^{-bd(X,Y)}.$$

For the second term we use the decay of ϕ_β and the trivial bound $\left\|\left[\tau_s^H(A), B\right]\right\| \leq 2\, \|A\|\, \|B\|$ to get

$$I_{\geq T} \leq 4\, \|A\|\, \|B\|\, \frac{\beta}{\sqrt{\pi}} \int_T^\infty \mathrm{d}t\, \mathrm{e}^{-\beta^2 t^2}\, t \leq 2\, \|A\|\, \|B\|\, \frac{1}{\sqrt{\pi}\, \beta} \left(\mathrm{e}^{-\beta^2 T^2} - 1\right).$$

Combining both bounds concludes the proof. □

It will later also be helpful to compare the almost inverse Liouvillian to the exact inverse Liouvillian directly.

Lemma 5.2.8

Let Λ be finite and $H \in \mathcal{A}_\Lambda$ satisfy the Gap Assumption. For all $A \in \mathcal{A}$ such that $A = P\, A\, P^\perp$ or $A = P^\perp\, A\, P$, and for all $q \in [1, \infty]$,

$$\left\|\mathcal{I}_{H,\beta}(A) - \mathcal{I}_H(A)\right\|_q \leq \|P\|_1\, \|A\|\, \gamma^{-1}\, \mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Moreover, for all $A \in \mathcal{A}$ and $q \in [1, \infty]$,

$$\left\|[\mathcal{I}_{H,\beta}(A) - \mathcal{I}_H(A), P]\right\|_q \leq 2\, \|P\|_1\, \|A\|\, \gamma^{-1}\, \mathrm{e}^{-\frac{\gamma^2}{4\beta^2}}.$$

Proof. As in the proof of Proposition 5.2.3, we obtain for $A = P\, A\, P^\perp$

$$\mathcal{I}_{H,\beta}(A) - \mathcal{I}_H(A) = \sum_{\mu \in \sigma_0} \sum_{\nu \in \sigma_1} \int_{-\infty}^\infty \mathrm{d}t\, \left(\phi_\beta(t) - w(t)\right) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}(\mu - \nu)s}\, P_\mu\, A\, P_\nu$$
$$= \sum_{\mu \in \sigma_0} P_\mu\, A \sum_{\nu \in \sigma_1} \frac{\sqrt{2\pi}}{\mathrm{i}(\mu - \nu)}\, \hat{\phi}_\beta(\nu - \mu)\, P_\nu,$$

by the Gap Assumption. It follows that

$$\left\|\mathcal{I}_{H,\beta}(A) - \mathcal{I}_H(A)\right\|_q \leq \sqrt{2\pi} \sum_{\mu \in \sigma_0} \|P_\mu\|_q\, \|A\|\, \left\|\sum_{\nu \in \sigma_1} \frac{\hat{\phi}_\beta(\nu - \mu)}{\nu - \mu}\, P_\nu\right\| \leq \sqrt{2\pi}\, \|P_\mu\|_1\, \|A\|\, \frac{\hat{\phi}_\beta(\gamma)}{\gamma}.$$

The second statement follows as in Proposition 5.2.3. □

5.3. Decay of correlation in gapped ground state

Gaussian filters were already used in previous proofs of the exponential clustering theorem: Ground state correlations decay in the presence of a spectral gap. In the present context of quantum spin systems, exponential decay of correlations was proved in [33] in the infinite-volume limit (the gap refers in this case to the gap of the GNS Hamiltonian), in the limit of finite volumes in [23], where the splitting of the ground state energies in finite volume is assumed to vanish in the limit, while the latter assumption is removed in [6] but the decay is only superpolynomial. In this section, we prove that correlations decay indeed exponentially under the sole assumption of a spectral gap, even if there is eigenvalue splitting in the ground state.

Assumption 5.4

Let Λ be finite. Let Φ be an interaction satisfying $\|\Phi\|_{b'} < \infty$ for some $b' > 0$, and let H be the corresponding Hamiltonian. We assume that the spectrum of H is of the form

$$\sigma(H) = \sigma_0 \cup \sigma_1$$

with $\inf(\sigma_1) - \sup(\sigma_0) \geq \gamma > 0$ *and* $\operatorname{diam}(\sigma_0) \leq \Delta < \gamma/4$.

As above, we denote by P the spectral projection associated with the spectral patch σ_0. Note that the condition $\Delta < \gamma/4$ is not tight but will simplify the estimates. What is needed in the proof is that $-\Delta + \gamma/2 \geq c > 0$.

Theorem 5.5 (Exponential clustering ground state)

Let $D \in \mathbb{N}$, $\mathcal{C}_{\mathrm{vol}} > 0$, $b' > 0$, $C^{\mathrm{int}} > 0$, $\gamma > 0$ and $\Delta > 0$. Then there exist constants C, $c > 0$, such that the following holds. For all $\Lambda \in \mathcal{G}(D, \mathcal{C}_{\mathrm{vol}})$ finite and Hamiltonians H that satisfy Assumption 5.4 with gap γ and width Δ and are given by interactions Φ such that $\|\Phi\|_{b'} < C^{\mathrm{int}}$ the following holds:

For any normalized state $\Omega \in \operatorname{Ran}(P)$,

$$\Big| \langle \Omega, A\, B\, \Omega \rangle - \langle \Omega, A\, P\, B\, \Omega \rangle \Big| \leq C\, \|P\|_1 \|A\|\, \|B\|\, \mathrm{e}^{-c d(X,Y)}$$

for all disjoint X, $Y \subset \Lambda$ and $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$.

PROOF. For the proof, we define the filter function

$$g_\beta(t) = \frac{\mathrm{e}^{-\mathrm{i}t\frac{\gamma}{2}}}{\sqrt{2}\beta}\left(\sqrt{\frac{\pi}{2}}\,\delta_0(t) + \frac{\mathrm{i}}{\sqrt{2\pi}}\ \mathrm{p.\,v.}\Big(\frac{1}{t}\Big)\right)\phi_\beta(t),$$

where $\mathrm{p.\,v.}\big(\frac{1}{t}\big)$ denotes the principal value distribution, and let

$$\mathcal{J}_{H,\beta}(A) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}t\,g_\beta(t)\,\tau_t^H(A).$$

The Fourier transform $\widehat{g}_\beta$ is given by

$$\widehat{g}_\beta(\omega) = \frac{1}{\sqrt{2}\beta}\,(\Theta_{\gamma/2}\star\widehat{\phi}_\beta)(\omega) = \frac{1}{2\beta\sqrt{\pi}}\int_{-\infty}^{\omega-\gamma/2}\mathrm{d}\xi\,\mathrm{e}^{-\frac{\xi^2}{4\beta^2}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{\omega-\gamma/2}{2\beta}}\mathrm{d}x\,\mathrm{e}^{-x^2},$$

where Θ_a is the Heaviside step function with discontinuity at a.

Before we continue the proof, we note some facts about the Gaussian error function. We have

$$1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{z}\mathrm{d}x\,\mathrm{e}^{-x^2} = \frac{1}{\sqrt{\pi}}\int_{z}^{\infty}\mathrm{d}x\,\mathrm{e}^{-x^2} \le \frac{1}{2\sqrt{\pi}\,z}\int_{z}^{\infty}\mathrm{d}x\,2\,x\,\mathrm{e}^{-x^2} = \frac{1}{2\sqrt{\pi}}\,\frac{\mathrm{e}^{-z^2}}{z}$$

whenever $z > 0$, and similarly

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{z}\mathrm{d}x\,\mathrm{e}^{-x^2} \le \frac{1}{2\sqrt{\pi}}\,\frac{\mathrm{e}^{-z^2}}{\big|z\big|}$$

if $z < 0$. Hence, the Fourier transform of the filter function satisfies

$$\widehat{g}_\beta(\omega) \le \frac{\beta}{\sqrt{\pi}\,\big|\omega-\gamma/2\big|}\,\mathrm{e}^{-\frac{(\omega-\gamma/2)^2}{4\beta^2}} \qquad \text{for } \omega < \gamma/2$$

and

$$1-\widehat{g}_\beta(\omega) \le \frac{\beta}{\sqrt{\pi}\,(\omega-\gamma/2)}\,\mathrm{e}^{-\frac{(\omega-\gamma/2)^2}{4\beta^2}} \qquad \text{for } \omega > \gamma/2.$$

We now decompose the correlation into three terms

$$\langle\Omega, A\,B\Omega\rangle - \langle\Omega, A\,P\,B\,\Omega\rangle = \left\langle\Omega, A\,P^\perp\,B\,\Omega\right\rangle$$

$$= \big\langle\Omega, [\mathcal{J}_{H,\beta}(A), B]\ \Omega\big\rangle \tag{5.12a}$$

$$+\big\langle\Omega, B\,\mathcal{J}_{H,\beta}(A)\,\Omega\big\rangle \tag{5.12b}$$

$$+\big\langle\Omega, \big(P\,A\,P^\perp - P\,\mathcal{J}_{H,\beta}(A)\big)\,B\,P\,\Omega\big\rangle. \tag{5.12c}$$

In the following we will show that each term decays exponentially in $d(X,Y)$ with the choice $\beta = \frac{\gamma}{2\sqrt{d(X,Y)}}$.

We start by bounding (5.12b), for which we observe

$$\mathcal{J}_{H,\beta}(A)\, P = \sum_{\mu\in\sigma(H)} \sum_{\nu\in\sigma_0} \widehat{g}_\beta(\nu-\mu)\, P_\mu\, A\, P_\nu.$$

Since $\nu - \mu \le \Delta < \gamma/4$, we obtain the bound

$$\|\mathcal{J}_{H,\beta}(A)\, P\| = \|A\| \sum_{\nu\in\sigma_0} \Big\| \sum_{\mu\in\sigma(H)} \widehat{g}_\beta(\nu-\mu)\, P_\mu \Big\| \le \frac{4\,\big|\sigma_0\big|}{\sqrt{\pi}}\, \frac{\beta}{\gamma}\, \mathrm{e}^{-\frac{1}{64}\left(\frac{\gamma}{\beta}\right)^2}\, \|A\|. \tag{5.13}$$

For (5.12c) we bound

$$\|P\, A\, P^\perp - P\, \mathcal{J}_{H,\beta}(A)\, P^\perp\| = \Big\| \sum_{\mu\in\sigma_0} \sum_{\nu\in\sigma_1} \big(1 - \widehat{g}_\beta(\nu-\mu)\big)\, P_\mu\, A\, P_\nu \Big\| \le \frac{2\,\big|\sigma_0\big|}{\sqrt{\pi}}\, \frac{\beta}{\gamma}\, \mathrm{e}^{-\frac{1}{16}\left(\frac{\gamma}{\beta}\right)^2}\, \|A\|$$

because $\nu - \mu \ge \gamma$. And together with (5.13), we obtain

$$\|P\, A\, P^\perp - P\, \mathcal{J}_{H,\beta}(A)\| \le \frac{6\,\big|\sigma_0\big|}{\sqrt{\pi}}\, \frac{\beta}{\gamma}\, \mathrm{e}^{-\frac{1}{64}\left(\frac{\gamma}{\beta}\right)^2}\, \|A\|. \tag{5.14}$$

Finally, the commutator $[\mathcal{J}_{H,\beta}(A), B]$ in (5.12a) is bounded using an argument similar to that of Lemma 5.2.6. As we do there, we decompose the integral defining $\mathcal{J}_{H,\beta}(A)$ into $\big|t\big| \le T$ and $\big|t\big| > T$ and use the Lieb-Robinson bound, Lemma 5.1.2, to estimate the short time part while the long time contribution can be bounded by the Gaussian decay. Since $d(X,Y) > 0$, we have that $[A,B] = 0$ and so the δ_0-contribution vanishes. Therefore,

$$\begin{aligned}
&\Big\| \int_{|t|<T} \mathrm{d}t\, g_\beta(t)\; \big[\tau_t^H(A), B\big] \Big\| \\
&\qquad \le \frac{2\,\mathcal{C}_{\mathrm{vol},1,b'-b}^{-1}}{\beta\sqrt{\pi}}\, \|A\|\, \|B\|\, \min\big\{\big|X\big|, \big|Y\big|\big\}\, \sup_t \big|\phi_\beta(t)\big|\, \mathrm{e}^{-b d(X,Y)}\, \lim_{\epsilon\to 0} \int_\epsilon^T \mathrm{d}t\, \frac{1}{t}\, \big(\mathrm{e}^{bvt} - 1\big).
\end{aligned}$$

The mean value theorem implies that $t^{-1}\,(\mathrm{e}^{bvt} - 1) \le b\, v\, \mathrm{e}^{bvt}$ and so

$$\lim_{\epsilon\to 0} \int_\epsilon^T \mathrm{d}t\, \frac{1}{t}\, \big(\mathrm{e}^{bvt} - 1\big) \le \big(\mathrm{e}^{bvT} - 1\big).$$

Since $\sup\left|\phi_\beta(t)\right| = \frac{\beta}{\sqrt{\pi}}$, we conclude that

$$\left\|\int_{\left|t\right|<T} \mathrm{d}t\, g_\beta(t)\ \left[\tau_t^H(A), B\right]\right\| \leq \frac{2\,\mathcal{C}_{\mathrm{vol},1,b'-b}^{-1}\,\|A\|\,\|B\|}{\pi}\ \min\left\{\left|X\right|, \left|Y\right|\right\} \mathrm{e}^{-bd(X,Y)}\left(\mathrm{e}^{bvT}-1\right). \tag{5.15}$$

For $\left|t\right| \geq T$, we use the simple norm bound on the commutator and

$$\int_T^\infty \mathrm{d}t\, \frac{\phi_\beta(t)}{t} \leq \frac{1}{T}\int_T^\infty \mathrm{d}t\, \phi_\beta(t) \leq \frac{1}{2\,T^2\sqrt{\pi}}\,\mathrm{e}^{-\beta^2T^2}$$

to conclude that

$$\left\|\int_{\left|t\right|>T} \mathrm{d}t\, g_\beta(t)\ \left[\tau_t^H(A), B\right]\right\| \leq \frac{2\,\|A\|\,\|B\|}{T^2\sqrt{\pi}}\,\mathrm{e}^{-\beta^2T^2}. \tag{5.16}$$

Together, (5.15), (5.16) and the choice $T = \frac{d(X,Y)}{2v}$ yield the following bound on (5.12a):

(5.17)

$$\left\|\int_{-\infty}^\infty \mathrm{d}t\, g_\beta(t)\ \left[\tau_t^H(A), B\right]\right\| \leq \frac{2\,\|A\|\,\|B\|}{\pi}\left(\frac{\min\left\{\left|X\right|, \left|Y\right|\right\}}{\mathcal{C}_{\mathrm{vol},1,b'-b}}\,\mathrm{e}^{-\frac{bd(X,Y)}{2}} + \frac{4\,v^2\sqrt{\pi}}{d(X,Y)^2}\,\mathrm{e}^{-\frac{\beta^2 d(X,Y)^2}{4v^2}}\right).$$

Gathering (5.17), (5.13) and (5.14) to bound (5.12a), (5.12b) and (5.12c), the claim of the theorem follows by bounding $\left|\sigma_0\right| \leq \|P\|_1$ and setting

$$\beta = \frac{\gamma}{2\sqrt{d(X,Y)}},$$

which makes all exponents proportional to $d(X,Y)$, and using that $d(X,Y) \geq 1$ in the prefactors.

□

5.4. Implication on many body Hall conductance

We briefly recall the setting of [6] applied to the quantum Hall effect. The lattice is a sequence of discrete tori $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^2$. The Hamiltonian is given by a finite range interaction Φ that is invariant under a strictly local $U(1)$-action. This means that there is a family $q_x = q_x^* \in \mathcal{A}_x$ with integer spectrum and such that $[Q_{\Lambda_L}, \Phi(Z)] = 0$ for all $Z \subset \Lambda_L$. The q_x are the local charges. The "ground state space" is the range of a spectral projection P_L of H_{Λ_L} whose dimension is constant and equal to p for all L large enough. Moreover, H_{Λ_L} is assumed to satisfy the Gap Assumption, uniformly in L for L large enough.

The proof of quantization of the Hall conductance in [6] relies heavily on the inverse Liouvillian on the one hand, and on clustering on the other hand. The inverse Liouvillian is used to construct a unitary U_L describing a magnetic flux threading and its locality allows for the definition of a charge transport operator T_L across a fiducial line of the torus. Replacing the exact inverse Liouvillian by the almost inverse Liouvillian introduced in Section 5.2.1 yields a unitary U_L^β and in turn an exponentially localized charge transport operator T_L^β. This improved localization and the exponential clustering of Section 5.3 yield the following.

Theorem 5.6
Let $\beta = L^{-1/2}$. There is an integer $n_L \in \mathbb{Z}$ and constants C, $c > 0$ such that

$$\left| n_L - \mathrm{Tr}\big(P_L\, T_L^\beta\big) \right| \leq C\, \mathrm{e}^{-cL}$$

for all L. If, moreover, the sequence of states $p^{-1}\mathrm{Tr}(P_L(\cdot))$ is convergent, then $n_L = n$ for L large enough and $2\,\pi\,\kappa = \frac{n}{p}$, where κ is the Hall conductance.

We now explain the arguments with a focus on the changes from using the almost inverse Liouvillian, referring to [6, Section IV] for more details.

5.4.1. Geometric setup. We name the following sets in the lattice, see Figure 5.1:
The right half torus is $R = \{(x_1, x_2)\,|0 < x_1 < L/2\}$. Its right boundary: $\{(x_1, x_2)\big|\,|x_1 - L/2| < R_\Phi\}$ and left boundary $\{(x_1, x_2)\big|\,|x_1| < R_\Phi\}$
The upper half torus is $U = \{(x_1, x_2)\,|0 < x_2 < L/2\}$, its upper boundary $\{(x_1, x_2)\big|\,|x_2 - L/2| < R_\Phi\}$ and lower boundary $\{(x_1, x_2)\big|\,|x_2| < R_\Phi\}$. We again drop the L dependence to simplify notation.

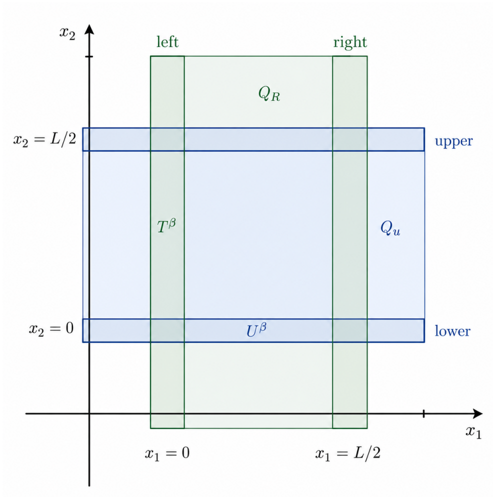


FIGURE 5.1

5.4.2. Constructing a flux threading unitary. Similarly to [7, Section 2] and [4, Section 3.4] we first construct a unitary U^β that models the threading of one unit of flux through the torus. In our case U^β will be exponentially localized near a line along the torus.
Let Q_{U} be the charge on the region R and define

$$\overline{Q}_{\mathrm{U}}^{\beta} := Q_{\mathrm{U}} - \mathcal{I}_{H,\beta} \circ \mathcal{L}_H(Q_{\mathrm{U}}).$$

By the $U(1)$-invariance and the interaction range R_Φ of the Hamiltonian, the operator $\mathcal{L}_H(Q_{\mathrm{U}})$ is strictly localized in two strips of finite width around the boundaries of the half-torus,

$$\mathcal{L}_H(Q_{\mathrm{U}}) = \mathcal{L}_H(Q_{\mathrm{U}})_{\mathrm{lower}} + \mathcal{L}_H(Q_{\mathrm{U}})_{\mathrm{upper}}$$

. The locality result, Lemma 5.2.6, implies that $\mathcal{I}_{H,\beta}(\mathcal{L}_H(Q_{\mathrm{U}})_{\mathrm{lower}})$ is exponentially localized around the lower boundary. Specifically, the choice $\beta = L^{-1/2}$ in 5.11 yields a localization estimate of the form $C\,\mathrm{e}^{-cL}$ for $d(X,Y)$ of order L, namely

$$\mathcal{I}_{H,\beta} \circ \left(\mathcal{L}_H(Q_{\mathrm{U}})_{\mathrm{lower}}\right)$$

can be approximated by an operator that is strictly localized in a strip of width $\frac{L}{8}$ at the lower boundary (or any other smaller fraction of L), up to errors that are exponentially small in L. This and the fact that $\sigma Q_{\mathrm{U}} \in \mathbb{Z}$ imply that the unitary $\mathrm{e}^{2\pi\mathrm{i}\overline{Q}_{\mathrm{U}}^{\beta}}$ factorizes up to exponentially small errors into parts at the lower and upper boundary. We let

$$U^\beta := \big(\mathrm{e}^{2\pi\mathrm{i}\overline{Q}_{\mathrm{U}}^\beta}\big)_{\mathrm{lower}} \tag{5.18}$$

be the factor localized on the lower boundary of the half-torus. This is the unitary that implements the flux threading through the constant x_1 cycle of the torus.

Next, we consider the charge transport observable along the torus. Let Q_{R} be the charge on the right half of the torus. By charge conservation and again the locality lemma, the operator $(U^\beta)^*\, Q_{\mathrm{R}}\, U^\beta - Q_{\mathrm{R}}$ decomposes into two contributions at each boundary of the right half-torus. We define the operator of charge transport as the left one

$$T^\beta := \Big((U^\beta)^*\, Q_{\mathrm{R}}\, U^\beta - Q_{\mathrm{R}}\Big)_{\mathrm{left}}. \tag{5.19}$$

More precisely, $\mathrm{Tr}\big(P_L\, T_L^\beta\big)$ measures the amount of charge transported in the ground state across one fiducial line across the torus after threading one unit of flux in the torus – up to exponentially small errors in L, because the splittings we used are only unique up to exponentially small terms.

We briefly pause the argument to compare explicitly with [6]. There, $\overline{Q}_{\mathrm{U}}$ is defined using the exact inverse Liouvillian $\mathcal{I}_H$ rather than $\mathcal{I}_{H,\beta}$. As a result, $\big[\overline{Q}_{\mathrm{U}}, P\big] = 0$ while here by Proposition 5.2.3:

$$\|\Big[\overline{Q}_{\mathrm{U}}^\beta, P\Big]\| \leq C\, L^2\, \mathrm{e}^{-c\beta^{-2}} = C\, \mathrm{e}^{-cL}$$

The same holds for the exponentials, and by exponential clustering, Theorem 5.5, for their restrictions to the lower boundary of the half-torus. What is more, Lemma 5.2.8 implies that $\overline{Q}_{\mathrm{U}}^\beta$ and $\overline{Q}_{\mathrm{U}}$ are exponentially close and therefore so are U^β and U. As a consequence, U^β almost preserves the ground state space and almost implements 2π-flux threading.

5.4.3. Proof of exponentially small quantization error. With these definitions, the proof of quantization of this quantity follows exactly the original argument of [6, Section IV.B]. We only recall the main steps and illustrate where the results of the previous sections yield improved bounds. The unitary

$$Z^\beta(\phi) = (U^\beta)^*\, \mathrm{e}^{\mathrm{i}\phi\overline{Q}_{\mathrm{R}}^\beta}\, U^\beta\, \mathrm{e}^{-\mathrm{i}\phi\overline{Q}_{\mathrm{R}}^\beta}$$

factorizes as $Z^\beta(\phi) = Z^\beta(\phi)_{\mathrm{left}}\, Z^\beta(\phi)_{\mathrm{right}}$ as in the discussion above. Note that the "$-$" and "$+$" in [6] are the analogues of "left" and "right" here, respectively. Since $\mathcal{I}_{H,\beta}$ is an almost inverse

Liouvillian, Proposition 5.2.3, $Z^\beta(\phi)$ commutes with P (in operator norm) up to exponentially small errors in $\beta^{-2} = L$. By exponential clustering, Theorem 5.5, it follows that $\left[Z^\beta(\phi)_{\mathrm{left}}, P\right] \overset{\exp}{=} 0$, where we use the notation $\overset{\exp}{=}$ to indicate equality up to exponentially small errors in L. From there on, the argument runs without change, but the errors are always exponential rather than superpolynomial, yielding that

$$P\, Z^\beta(\phi)_{\mathrm{left}}\, P \overset{\exp}{=} \mathrm{e}^{\mathrm{i}\phi\left(P\left(T^\beta-((U^\beta)^* K^\beta_{\mathrm{left}} U^\beta - K^\beta_{\mathrm{left}})\right)P + \overline{Q}^\beta_{\mathrm{R}}\right)}\, \mathrm{e}^{\mathrm{i}\phi \overline{Q}^\beta_{\mathrm{R}}}\, P, \tag{5.20}$$

where $K^\beta_{\mathrm{left}} = \mathcal{I}_{H,\beta} \circ \big(\mathcal{L}_H(Q_{\mathrm{R}})_{\mathrm{left}}\big)$. At $\phi = 2\pi$, an independent argument yields that $\det_P Z^\beta(2\pi)_{\mathrm{left}} \overset{\exp}{=} 1$, where the determinant is on the range of P. With this, the claim that $\mathrm{Tr}\big(P\, T^\beta\big)$ is exponentially close to an integer follows from (5.20) by computing the trace of the exponent and using the unitary invariance of the trace.

Finally, if the sequence of states is convergent, then $p^{-1}\,\mathrm{Tr}\,(P_L\, T_L)$ is convergent because T_L is a sufficiently local operator, see [5, Corollary 2.3]. Note that T_L is obtained with the exact inverse Liouvillian here. The Laughlin argument then implies that the limit of $p^{-1}\,\mathrm{Tr}\,(P_L\, T_L)$ is equal to $2\,\pi\,\kappa$ where κ is the Hall conductance, see [5, Theorem 3.2]. By Lemma 5.2.8, we further have that $\|T_L^\beta - T_L\| \to 0$ as $L \to \infty$, again with $\beta^{-2} = L$. We conclude that $p^{-1}\,\mathrm{Tr}\big(P_L(T_L^\beta - T_L)\big)$ converges to 0, and hence that $p^{-1}\,\mathrm{Tr}\left(P_L\, T_L^\beta\right)$ is convergent. In particular, the sequence $\left(\frac{n_L}{p}\right)_L$ is eventually constant and equal to $2\pi\kappa$.

APPENDIX A

Some scattered results

A.1. Frame transfer matrix as detector of gapless edge mode

A.1.1. F^4 Models in 1D. One dimensional version of F^4 models appears in the analysis of 2D edge Hamiltonian. In order not to break the main text, we provide the relevant analysis in this appendix. It is convenient to not burden ourselves with the notation used in the main part so the notation in the appendix might sometimes differ from the notation used in the main text.

We recall the notion of a frame. Let $\mathcal{H}$ be a Hilbert space and $S \subset \mathcal{H}$ a countable set. We denote by P_S the orthogonal projection on the (closed) linear span L_S of S. The set S is a frame of L_S if there exists numbers A, B such that for any $\psi \in L_S$

$$A\|\psi\|^2 \leq \sum_{v \in S} |\langle v|\psi\rangle|^2 \leq B\|\psi\|^2.$$

If S is a frame then the associated Hamiltonian

$$H = \sum_{v \in S} |v\rangle\langle v|$$

is bounded from below on its range.

In what follows we consider $\mathcal{H} = l^2(\mathbb{Z}, \mathbb{C}^N)$. The support of a vector ψ, $\operatorname{supp}(\psi)$, is the set of all $x \in \mathbb{Z}$ for which $\psi(x) \neq 0$.

Definition A.1 (Wannier Frame)

A frame S of L_S is called a Wannier frame if

$$\sup_{v \in S} \operatorname{diam} \operatorname{supp}(v) < \infty.$$

We will assume that each vector of a Wannier frame is supported on a single site or at two consecutive sites, which can be always achieved by coarse graining. In terms of Wannier frames, H is a gapped, F^4-Hamiltonian if and only if S is a Wannier frame and L_S is a proper subspace of $\mathcal{H}$.

For $x \in \mathbb{Z}$, we will denote S_x the vectors in S for which x is the left-most point of their support. This give s a decomposition of S into disjoint subsets, $S = \cup_x S_x$. We put $H_x = \sum_{v \in S_x} |v\rangle\langle v|$ and obtain a frustration-free decomposition,

$$H = \sum_{x \in \mathbb{Z}} H_x.$$

Finally, we can always replace vectors in S_x by non-zero eigenvectors of H_x, scaled by the corresponding eigenvalue. This leaves L_S invariant and the new set S is a Wannier frame with same constants A, B. So from now on we will assume that S_x is an orthogonal set.
Thanks to the assumption that the support of each vector is maximally two sites, the Hamiltonian $H_d = H - H_0$ describes two disconnected half-line Hamiltonians.

A.1.2. Translation Invariant Hamiltonians. In the translation invariant case

$$S = \cup_{n \in \mathbb{Z}} T^n S_0,$$

where T is a translation by one site and $S_0 = \{v_1, v_2, \ldots, v_M\}$ is a finite set of orthogonal vectors supported on sites $\{0, 1\}$. The Fourier transform of S is

$$\widehat{S} = \cup_n e^{ikn} \widehat{S}_0.$$

Hence, in the Fourier space, we obtain a frame $\widehat{S}_0(k)$ at each fiber of the bundle $\cup_k L_{\widehat{S}_0(k)}$. The Hamiltonian in the Fourier space is fibered in k, and

$$\widehat{H}(k) = \sum_{v \in \widehat{S}_0(k)} |v\rangle\langle v|.$$

The gap condition implies that the dimension of $\mathrm{Ran}\widehat{H}(k)$ is constant.

A.1.3. Edge Spectrum, proof of prop 4.6.1. We will call λ an edge eigenvalue if λ is in the gap of H. The eigenvalue equation for H_e is

$$(H - H_0)\psi = \lambda\psi. \tag{A.1}$$

Writing $H_0\psi = \sum_j c_j v_j$, we get

$$(H - \lambda)\psi = \sum_j c_j v_j.$$

Let λ be in the gap of H, then

$$\psi = \sum_j c_j (H-\lambda)^{-1} v_j. \tag{A.2}$$

Plugging this back into Eq. (A.1), we get

$$\sum_l c_l v_l = H_0 \sum_j c_j (H-\lambda)^{-1} v_j = \sum_l \sum_j v_l c_j \Big\langle v_l (H-\lambda)^{-1} v_j . \Big\rangle$$

By the assumption that v_l are linearly independent we get an equation,

$$c_l = \sum_j T_{lj} c_j, \quad T_{lj}(\lambda) := \langle v_l | \frac{1}{H-\lambda} | v_j \rangle. \tag{A.3}$$

To summarize, λ is an edge eigenvalue if $T_{lj}(\lambda)$ has eigenvalue 1, and (A.2) gives the corresponding eigenvector.

In the k-space,

$$(\widehat{H}\psi)(k) = \widehat{H}(k)\psi(k), \quad \widehat{H}(k) = 2\pi \sum_j |v_j(k)\rangle\langle v_j(k)|.$$

and

$$T_{lj}(\lambda) = \int_0^{2\pi} v_l(k)^* \frac{1}{\widehat{H}(k)-\lambda} v_j(k) dk.$$

An important observation for our analysis

Lemma A.1.1

The matrix $T(\lambda)$ is continuous for $\lambda \in (-\infty, \gamma)$.

Proof. Continuity for $\lambda \neq 0$ follows from continuity of the resolvent. There is no singularity at $\lambda = 0$ because the vectors v_j belong to the range of H. □

A.2. Impossibility of gapless edge state on $\ell^2(\mathbb{N} \times \mathbb{Z}, \mathbb{C}^2)$

A.2.1. 1D lattice Edge mode. We place ourselves in the setting of single body solid state physics, with lattice $\mathbb{Z}$ and 2 spin components per site. The Hamiltonian is a sum of compactly supported rank one operator $|v\rangle\langle v|$ and all translations of it.

One can creat an edge in the lattice by removing a few terms from the Hamiltonian:

$$H_e = \sum_{x\in\mathbb{Z}} T_x|v\rangle\langle v|T_x^{-1} - |v\rangle\langle v|$$

This Hamiltonian is supported on $\mathbb{Z} - \{0\}$ thus can be considered as two copies of half line, each having an edge.

The spectrum of $H = \sum_x T_x|v\rangle\langle v|T_x^{-1}$ is $\sigma(H) = \{0\} \cup \{\|\widehat{v}(k)\|^2 \mid k \in \mathbb{T}\} \subseteq \mathbb{R}_+$. H_e is still non negative so as its spectrum. Any 0 eigenvector of H is still an eigenvector of H_e by the frustration-free nature of H. So $\ker(H) \subset \ker(H_e)$. We may regard the edge mode of H_e as a quotient:

Definition A.2

We define the edge mode as representative of $\ker(H_e)$ *mod* $\ker(H)$.

Proposition A.2.1

$\psi_e(k) = \frac{\widehat{v}(k)}{\|\widehat{v}(k)\|^2}$ *is a 0 eigenvector of* H_e *iff* H *is gapped.*

Clearly in order for the 0 eigenvector to be L^2, $\|\widehat{v}(k)\|^2 \neq 0$, $\forall k$ i.e. the original Hamiltonian H must be gapped.

Let $\widehat{\psi_e}$ be the new 0 eigenvector in momentum space, W.O.L.G. for each fiber k the $\widehat{\psi}_e(k)$ decompose into $\widehat{v}(k)$ component and its normal component:

$$\widehat{\psi}_e = c(k)\widehat{v}(k) \ mod \ \ker(H(k))$$

$$\text{(A.4)} \qquad \Longrightarrow \qquad \widehat{H}(k)\widehat{\psi}_e(k) - \left(\int_{\mathbb{T}} \tfrac{dk'}{2\pi}\, \widehat{v}^{\dagger}(k')\widehat{\psi}_e(k')\right)\widehat{v}(k) = 0$$

$$\Longleftrightarrow \qquad c(k)\,\|\widehat{v}(k)\|^2\,\widehat{v}(k) - \left(\int_{\mathbb{T}} \tfrac{dk'}{2\pi}\, \|\widehat{v}(k')\|^2\, c(k')\right)\widehat{v}(k) = 0$$

$$\text{(A.5)} \qquad \Longleftrightarrow \qquad c(k) \propto \tfrac{1}{\|\widehat{v}(k)\|^2} \ \text{ and } \ \psi_e(k) = \tfrac{1}{\|\widehat{v}(k)\|^2}\widehat{v}(k) \ mod \ \ker(H(k))$$

A.2.2. We suspect that there is no new positive eigenvalues introduced by the edge $\sigma(H_e) - \sigma(H)\cap^+ = \emptyset$. Let's illustrate in details:

If $H_e\psi = \lambda\psi$ for some $\lambda > 0$ then:

$$(H-\lambda)|\psi\rangle = |v\rangle\Big\langle v|\psi\Big\rangle \tag{A.6}$$

$$\text{LHS } H \text{ being } k \text{ diagonal thus } \left(|\widehat{v}(k)\rangle\langle\widehat{v}(k)| - \lambda\right)|\widehat{\psi}(k)\rangle$$

$$= \int \frac{dk'}{2\pi}\left(\widehat{v}^\dagger(k')\widehat{\psi}(k')\right)\widehat{v}(k) \tag{A.7}$$

$$\iff \left(\widehat{v}^\dagger(k)\widehat{\psi}(k)\right)\widehat{v}(k) - \lambda\widehat{\psi}(k) = \int \frac{dk'}{2\pi}\left(\widehat{v}^\dagger(k')\widehat{\psi}(k')\right)\widehat{v}(k) \tag{A.8}$$

$$\text{for all } k \tag{A.9}$$

(A.8) is seen as a k-family of 2d inhomogeneous linear equations. A solution $\widehat{\psi(k)}$ is an eigenvector of H_e if it is $L^2(\mathbb{T})$.

A.2.2.1. *solution with $\widehat{v}$ component.* If solution $\widehat{\psi}$ has parallel component of $\widehat{v}$ in $L^2(BZ, \mathbb{C}^2)$, then RHS of A.8 is a non 0 scalar multiple of $\widehat{v}$. We can normalize $\widehat{\psi}$ so that $\int \frac{dk'}{2\pi}\left(\widehat{v}^\dagger(k')\widehat{\psi}(k')\right) = 1$:

$$\left(\widehat{v}^\dagger(k)\widehat{\psi}(k)\right)\widehat{v}(k) - \lambda\widehat{\psi}(k) = 1\widehat{v}(k)$$

$$\text{then the ansatz } \widehat{\psi} \text{ solves this 2d linear equation for every } k$$

$$\begin{aligned}\widehat{\psi}(k) &= \left(\widehat{v}(k)\widehat{v}^\dagger(k) - \lambda\right)^{-1}_{2\times 2}\widehat{v}(k)\\ &= \left(\|\widehat{v}(k)\|^2 - \lambda\right)^{-1}\widehat{v}(k)\end{aligned}$$

But when we plug the ansatz back into the normalization we find:

$$\begin{aligned}1 &= \int \frac{dk'}{2\pi}\left(\widehat{v}^\dagger(k')\widehat{\psi}(k')\right)\\ &= \int \frac{dk'}{2\pi}\underbrace{\frac{\|\widehat{v}(k')\|^2}{\|\widehat{v}(k')\|^2 - \lambda}}_{>1}\\ &> 1\end{aligned}$$

The integrand is greater than 1 if not diverging to $+\infty$, thus in any case it can not be normalized to 1.

Proposition A.2.2

So for $\lambda > 0$, the eigenvector as a solution of A.8 has no $\widehat{v}$ component. This is regardless of the gap of H.

A.2.2.2. $\widehat{v}^{\perp}$ *component.* In $L^2(BZ, \mathbb{C}^2)$, the $\widehat{v}^{\perp}$ component of $\widehat{\psi}$ means RHS of A.8 is 0. This is in general possible even $\forall k$, $\widehat{\psi_e}(k) = c(k)\widehat{v}(k)$ for example $\int c(k)dk = 0$ and $\|\widehat{v}(k)\| = const$. Since we are looking for eigenvectors that are not ground states of H, there must be some k' s.t. $\left\langle \widehat{v}(k') | \widehat{\psi}(k') \right\rangle \neq 0$:

$$
\begin{aligned}
|\widehat{v}(k')\rangle \left\langle \widehat{v}(k') | \widehat{\psi}(k') \right\rangle &= \lambda \widehat{\psi}(k') \\
\implies \widehat{\psi}(k') &= \frac{1}{\lambda} \left\langle \widehat{v}(k') | \widehat{\psi}(k') \right\rangle \widehat{v}(k') \\
\text{and plug back } \frac{1}{\lambda} \cancel{\left\langle \widehat{v}(k') | \widehat{\psi}(k') \right\rangle} \|\widehat{v}(k')\|^2 \, \widehat{v}(k') &= \cancel{\left\langle \widehat{v}(k') | \widehat{\psi}(k') \right\rangle} \widehat{v}(k') \\
\implies \|\widehat{v}(k')\|^2 &= \lambda
\end{aligned}
$$

$\widehat{v}$ being a trig polynomial of k, and $\|\widehat{v}(k')\|^2 = \lambda$ being constant of k', together implies $v \in l^2(\mathbb{Z}, \mathbb{C}^2)$ is supported on a single site, because:

$$
\text{trig polynomial } \widehat{v}(k) = \sum_{p=-d}^{d} \begin{pmatrix} w_p X^p \\ u_p X^p \end{pmatrix}, \; X := e^{ik}
$$

thus

$$
\|\widehat{v}(k)\|^2 = \sum_p w_p^* X^{-p} \sum_p w_q X^q \; + \sum_p u_p^* X^{-p} \sum_p u_q X^q
$$

$\|\widehat{v}(k)\|^2 = \lambda$ being a constant implies the only non zero term is X^0, so

$$
\sum_{p+q=k} w_p^* w_q + u_p^* u_q = \lambda \delta_k
$$

Realizing that w_p, u_p are the Fourier coeff of $\widehat{v}(k)$ i.e. the real space matrix element of the real space projector, this means $|v\rangle\langle v|$ is supporten on a single site:

$$
[|v\rangle\langle v|]_{p,q} = \delta_{p-q}
$$

If ψ_e wuth positive eigenvalue has non zero $\widehat{v}^{\perp}$ component of ψ_e, it must be local excited states of H and H must has no interaction between sites. Thus indeed the edge only introduce a 0-eigenstate given by A.5.

A.2.3. 2D Edge mode. Now our lattice is $\mathbb{Z}^2$ and Hamiltonian on (both copies of) half lattice is

$$H_e = \sum_{(x,y)} |T_{(x,y)}v\rangle\langle T_{(x,y)}v| - \sum_{y} |T_{(0,y)}v\rangle\langle T_{(0,y)}v| \tag{A.10}$$

$$= \sum_{y} T_{(0,y)} \underbrace{\left(\sum_{x} |T_x v\rangle\langle T_x v| - |v\rangle\langle v|\right)}_{H_{e,1d}} T_{(0,y)}^{-1} \tag{A.11}$$

The middle $H_{e,1d}$ is recognized as the 1D Hamiltonian with an edge. After a partial Fourier transform on y direction, we would get a k_y-family of Hamiltonian that are subject to our conclusion A.2.1.

We define the partial Fourier transform of as the unitary map

$$\mathcal{F}_2 : l^2(\mathbb{Z}^2) \to L^2(\mathbb{T}) \otimes l^2(\mathbb{Z})$$
$$\psi \mapsto \int \frac{dy}{\sqrt{2\pi}} \psi(x,y)\, e^{-i\,k_y\,y} = \psi_{\widehat{2}}(x,k_y)$$

We use the subscript $\widehat{2}$ to indicate adjoint action of y-partial Fourier transform on operators, or image of a vector under y-partial Fourier transform.

Under adjoint by such unitary map, the Hamiltonian on $l^2(\mathbb{Z}) \otimes L^2(\mathbb{T})$ is called $H_{e,\widehat{2}}$ and preserves each k_y sector/fiber:

$$\left(H_{e,\widehat{2}}\psi_{\widehat{2}}\right)(x,k_y) = H_{e,\widehat{2}}(k_y)\psi_{\widehat{2}}(x,k_y)$$

On each k_y fiber

$$H_{e,\widehat{2}}(k_y) = \sum_{x\in\mathbb{Z}\setminus 0} T_x\, |v_{\widehat{2}}(\cdot,k_y)\rangle\langle v_{\widehat{2}}(\cdot,k_y)|\, T_x^* \tag{A.12}$$

$$= H_{\widehat{2}}(k_y) - |v_{\widehat{2}}(0,k_y)\rangle\langle v_{\widehat{2}}(0,k_y)| \tag{A.13}$$

A.13 is a Hamiltonian on each k_y fiber that looks like $l^2(\mathbb{Z})$ and fit into the setup of 1D analysis. In particular:

Proposition A.2.3

For any k_y:

- $\sigma\left(H_{\widehat{2}}(k_y)\right)$ *has a gap since* $\sigma(H) = \cup_{k_y}\, \sigma\left(H_{\widehat{2}}(k_y)\right)$ *and* $\sigma(H)$ *has a gap.*

- $v_{\widehat{2}}(x, k_y) \neq 0 \; \forall x \in \mathbb{Z}$
- *Given $H_{\widehat{2}}(k_y)$ is gapped, the presence of edge only introduce a 0 eigenstate to $H_{e,\widehat{2}}(k_y)$:*

$$\widehat{\psi}_e(k_x, k_y) = \frac{\widehat{v}(k_x, k_y)}{\|\widehat{v}(k_x, k_y)\|^2_{\mathbb{C}^2}}$$

- *If $H_{\widehat{2}}(k_y)$ is gapless then no new eigenvalue can be found.*

If $\widehat{\psi}_e(k_x, k_y)$ is well defined for all k_y and $L^2([0, 2\pi]^2)$, then we call it an edge state.

Proposition A.2.4

We see edge state exist iff H is gapped for all k_x, k_y

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