

Two-Frequency Steady Streaming Around a Circular Cylinder

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To Dad, Mom, and Brother.

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Abstract

Steady streaming is the nonzero mean flow generated by an oscillating object. It is a classic problem that can be dated back to the experiments of Lord Rayleigh in 1883. Steady streaming has been studied both analytically and theoretically over the last 150 years, and there has been much recent interest in steady streaming due to its applications in microfluidics such as microchannel pumping and particle trapping. However, prior work has almost exclusively focused on single-frequency oscillations, which result in symmetric quadrupole-like flows. In our work, we demonstrate that dual-frequency oscillations induce asymmetric steady streaming with a nonzero net flux in a direction determined by the polarity of the oscillation – the oscillator serves as a pump. We use numerical simulations and asymptotic analysis at small amplitude to examine two-dimensional steady streaming around a cylinder, and our computational experiments show asymmetrical streaming and pumping, i.e., net flux downstream. It is well known from previous asymptotic analyses that steady streaming is second order in amplitude, and we show pumping is a higher order effect, but the solutions of the equations derived from the asymptotic expansion are analytically intractable beyond second order. Although the solutions are not available, we are able to derive the structure of the solutions from the asymptotic expansion at higher order, and identify the terms that are involved in producing a nonzero net force onto the surrounding fluid and leads to pumping. We give necessary conditions for pumping, and predict the order in amplitude at which pumping occurs. Moreover, we analyze the problem on an annular domain and reduce the equations from the asymptotic expansion to a series of boundary value problems of one spatial dimension, which can be numerically solved efficiently. For frequency ratio 2, we show that the net force on the cylinder is proportional to $\cos(\phi - \phi^*(\text{Re}))\beta/(1 + \beta^2)^{3/2}$, where β is the amplitude ratio, ϕ is the phase difference, Re is the Reynolds number, and $\phi^*(\text{Re})$ is the phase that gives the largest force. To test whether the force on an enclosed annulus serves as a useful proxy for pumping in an open system, we also performed numerical simulations of a cylinder oscillating with two-frequencies in a rectangular channel. Our computations demonstrate that the net flux through the channel is directly proportional to the net force on the annulus over all parameter ranges tested. Finally, we discuss the implications for using dual-mode vibrations to pump fluids in lab-on-a-chip and other applications.

Acknowledgments

To begin, I want to thank my close relatives for their unwavering support throughout undergrad and graduate school. Thanks to their encouragements, I was able to attend in high spirits.

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Spending more than a decade at UC Davis, I am quite certain that I have enjoyed my time at Davis, and I have even come to enjoy the weather as well. To describe the seasons in detail, Fall is the best time due to the combination of sunshine and the chilly breezes during the afternoon. Davis during the Winter always tested my will, and I will fondly remember biking through the rain, wind, or sometimes both. Spring is deceptive because of the allergies during March and April and the scorching rays of the sun during May and June. Summer is difficult due to the high temperatures, and I stayed at the library for the air conditioning. I will fondly remember working in the reading room at the library, and I hope that future students will come to enjoy it as well.

CHAPTER 1

Introduction

Steady streaming is the nonzero mean flow generated by an oscillating object [41], and it is a classical problem that can be traced back to the experiments of Faraday [15] and Lord Rayleigh [46]. The first experiments on streaming around a circular cylinder were performed by Carrière [10] and Andrade [4], who examined periodically oscillating air around a cylinder, and reported the now familiar quadrupole-like flow. Schlichting [43] carried out the first asymptotic analysis of steady streaming around a cylinder, and he compared the solution with his experiments of a vibrating cylinder in water. In the limit of small amplitude, he matched the inner boundary layer with the outer potential flow, and his solution became the basis for future analysis. Wang [55] adapted the layer analysis to include the curvature of the boundary, which is relevant at higher order. Holtsmark *et al.* [18] analyzed and solved the flow everywhere as a regular perturbation problem, and as noted in Chong *et al.* [11], Holtsmark's solution contains Schlichting's [43] and Wang's [55] solutions derived by matched asymptotic analysis. Riley [40] and Stuart [47] analyzed the problem at high Reynolds number where a second boundary layer exists, and their results agreed well with Schlichting's experiments.

Though the problem of steady streaming has been studied extensively, past works have primarily focused on single-frequency oscillations, with comparatively less attention to multi-frequency oscillations [13, 22, 24]. Davidson & Riley [13] analyzed steady streaming of multi-frequency oscillation around a cylinder, and found that the resultant flow is the superposition of steady streaming flows induced by each single-frequency oscillation. Similarly, Kotas *et al.* [22] experimentally examined the steady flow around a sphere made to oscillate with two frequencies, and they found that the observed flows were the sum of the streaming flows from the respective single-frequency oscillations. The experiments in both of these works were carried out for small-amplitude motion at which the streaming velocity is small, and the asymptotic analysis was performed only up to second order in amplitude. In a recent work of Lee *et al.* [24], we show, using asymptotic analysis and numerical computation, that a circular cylinder oscillating with two frequencies generates a net force on the

surrounding fluid, which produces an asymmetrical streaming in the oscillation direction and a net-directed flow. While steady streaming occurs at second order in amplitude, Lee *et al.* [24] showed that pumping occurs at third order or higher in amplitude, and thus, was not analyzed in the previous works. The different asymptotic scalings of the cylinder motion, steady streaming, and pumping are illustrated in Figure 1.1.

The work of Lee *et al.* [24] was inspired by experiments on a different physical system: an object sliding on a surface, undergoing two-frequency lateral oscillation [17]. Specifically, the surface displacement was $(\ell/2)[\sin(\omega t) + \sin(\alpha\omega t)]$, where ℓ is the amplitude, ω is the frequency, and α is a ratio of the two frequencies. For particular α values, the object exhibited a net translation. To help to understand the experimental observations, Hashemi *et al.* [17] analyzed a model in which the only force considered is the surface contact described by Coulomb’s friction law. Representing the frequency ratio α as p/q , they derived a necessary condition for net motion – one of p or q is odd, and the other is even – and validated this prediction with experiments. For example, $\alpha = 2, 3/2$ and $1/2$ showed net motion, while $\alpha = 1, 3$ and $5/3$ did not. We note that the same necessary conditions for directed motion were derived earlier by Reznik & Canny [39] using a simpler friction model [38]. Additional analysis and experiments for oscillations with different phase and amplitude were carried out in Zhang *et al.* [59], and Hui *et al.* [21] further corroborated the theory with experiments with granular media.

The main results of this dissertation are presented in Chapter 2 and Chapter 3. In Chapter 2, we simulate the flow around a cylinder oscillating with two frequencies, where the two oscillations are of equal amplitude and in phase. We first focus on frequency ratio 2 and measure the flux through a channel containing an oscillating cylinder, where we find that pumping is third order in amplitude. We then examine the asymptotic analysis of the problem at low amplitude, and even though we cannot solve the equations at third order, we show that a steady solution exists at third order and involves a net force responsible for pumping. We expand the analysis to general frequency ratios, and give necessary conditions for pumping, and we predict the order in amplitude at which pumping occurs. For select frequency ratios, the theoretical results are validated with computational simulations. In Chapter 3, we aim to understand how the streaming flow generated by two-frequency vibrations depends on amplitude ratio and phase difference of the two oscillations. We analyze the problem inside an annulus for frequency ratio $\alpha = 2$, and while there is no pumping

inside an annular domain, a net force is produced at third order in amplitude where the solutions are analytically intractable. We use the solution structure from the analysis to derive series of boundary value problems in one spatial dimension and efficiently solve the equations numerically. We carry out numerical simulations inside a channel that contains a cylinder oscillating with two frequencies and measure the flux through the channel. We show that the net force on the cylinder in the annular domain is proportional to the flux through the channel, and thus, the size of the net force serves as a proxy for the strength of pumping. Finally, we discuss microfluidic applications of two-frequency steady streaming and give potential research directions.

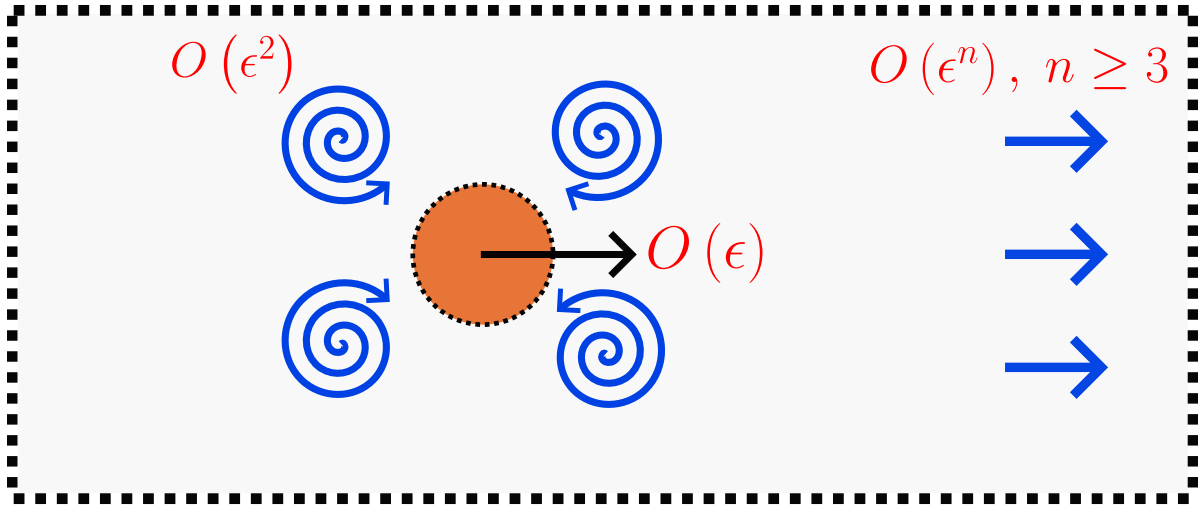


FIGURE 1.1. The different asymptotic scalings involved in a cylinder oscillating with two frequencies inside a channel: the horizontal motion of the cylinder is first order, $O(\epsilon)$; steady streaming occurs at second order, $O(\epsilon^2)$; and pumping is a higher order effect, $O(\epsilon^n)$, $n \geq 3$.

CHAPTER 2

Establishing Necessary Conditions for Pumping

2.1. Problem Statement

2.1.1. Cylinder Motion. We use two related formulations of the problem: one in which the cylinder moves, and the other in which the cylinder is stationary and the far-field flow oscillates. For the moving cylinder formulation, a cylinder of radius R and center $(X(t), 0)$ moves in a two-dimensional viscous fluid. The horizontal position of the cylinder's center is prescribed as

$$(2.1) \quad X(t) = \frac{A}{2}[\sin(\Omega t) + \sin(\alpha\Omega t)],$$

where A is the amplitude of oscillation, Ω is the angular frequency of the base oscillation, and $\alpha \geq 1$, the ratio of the two frequencies. The frequency ratio is assumed rational so that the flow is time periodic, and if we choose $\alpha = 1$, we obtain the single-frequency case.

For analysis, it is convenient to consider the cylinder fixed at the origin, and impose a horizontal flow at infinity of strength

$$(2.2) \quad U(t) = -\frac{A\Omega}{2}[\cos(\Omega t) + \alpha \cos(\alpha\Omega t)].$$

Furthermore, no-slip and no-penetration will be imposed, and the boundary conditions in two-dimensional cylindrical coordinates are

$$(2.3) \quad u_r(r = R) = 0, \quad u_\theta(r = R) = 0.$$

The two formulations are related by a change in reference frame. Switching our frame of reference from the inertial frame where the cylinder is oscillating to the non-inertial frame in which the cylinder is fixed results in a fictitious body force [7],

$$(2.4) \quad F = -\rho U'(t)e_x,$$

where ρ is the mass density and $\mathbf{e}_x$ is the direction vector in the horizontal direction. In the non-inertial frame, we incorporate the fictitious body force (2.4) in the modified pressure defined as,

$$(2.5) \quad \tilde{p} = p + \rho x U'(t).$$

For simplicity, we denote the modified pressure as p below.

2.1.2. Non-dimensionalization. The equations are non-dimensionalized using characteristic length scale R , time scale Ω^{-1} , and velocity scale $R\Omega$. The flow around the cylinder is governed by the incompressible Navier Stokes equations:

$$(2.6) \quad \text{Re} (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \Delta \mathbf{u},$$

$$(2.7) \quad \nabla \cdot \mathbf{u} = 0,$$

where the Reynolds number is

$$(2.8) \quad \text{Re} = \frac{R^2 \Omega}{\nu},$$

and ν is the kinematic viscosity. The boundary conditions (2.2) and (2.3) become

$$(2.9) \quad u_r(r=1) = 0, \quad u_\theta(r=1) = 0,$$

and the horizontal flow at infinity is

$$(2.10) \quad U(t) = -\frac{\epsilon}{2} [\cos(t) + \alpha \cos(\alpha t)],$$

where

$$(2.11) \quad \epsilon = \frac{A}{R}$$

is the dimensionless amplitude of oscillation.

The Reynolds number, $\text{Re} = R^2 \Omega / \nu$, is based on the length scale of the cylinder's radius and corresponds to the boundary layer in which viscous forces remain relevant. The streaming Reynolds number, Re_s , is based on the length scale of the oscillation amplitude, and it characterizes the

Reynolds number associated with the streaming flow. The two Reynolds numbers are related by

$$(2.12) \quad \text{Re}_s = \frac{A^2 \Omega}{\nu} = \epsilon^2 \text{Re}.$$

The flow structure depends on both Re and Re_s . We refer to Figure 1 from Chong *et al.* [11] (which is adapted from Figure 1 of Wang [55]) to illustrate the different flow regimes and associated analyses. Briefly, for high Reynolds numbers the flow exhibits a boundary layer [43, 55], and at high streaming Reynolds numbers, the streaming flow itself develops a second boundary layer [40, 47]. At low Reynolds number, the problem can be analyzed as a regular perturbation (Holtmark *et al.* [18]) – though as noted in Chong *et al.* [11], the regular perturbation solution from Holtmark *et al.* [18] contains the solution from the matched asymptotic analysis of Schlichting [43] and Wang [55], and is thus appropriate provided that the streaming Reynolds number is small. In our work, we focus on the low streaming Reynolds number regime ($\text{Re}_s \ll 1$) so that the problem can be analyzed as a regular perturbation, which allows us to compute the structure of solutions beyond second order needed to understand pumping from two-frequency oscillations.

2.2. Computational Studies of Frequency Ratio 2

We begin by examining the flow patterns of streaming for frequency ratio $\alpha = 2$ using numerical simulations. The numerical methods are described in Appendix C. We consider other frequency ratios in Section 2.4 where we show that $\alpha = 2$ yields the strongest pumping and the difference of the flow pattern from the single-frequency case is subtle for other frequency ratios.

2.2.1. Comparing Flow Patterns. We place the cylinder in a square 8 by 8 domain with periodic boundary conditions and solve for the flow at $\text{Re} = 10$. We solve for 250 periods which is sufficiently long to allow the initial transient to decay, and the resulting flows are visualized by the positions of passive marker particles from the previous 10 periods of time. As expected for the single-frequency case in Figure 2.1(a), the flow resembles the classical, four vortices associated with steady streaming. However, for two-frequency motion, shown in Figures 2.1(b,c), the vortices are no longer symmetric and there is a net horizontal flow. The bottom panels show the cylinder’s horizontal position for each case. In Figure 2.1(b), the net flow is left-to-right – away from the cylinder, the green particles (most current period) are to the right of the blue particles (past period),

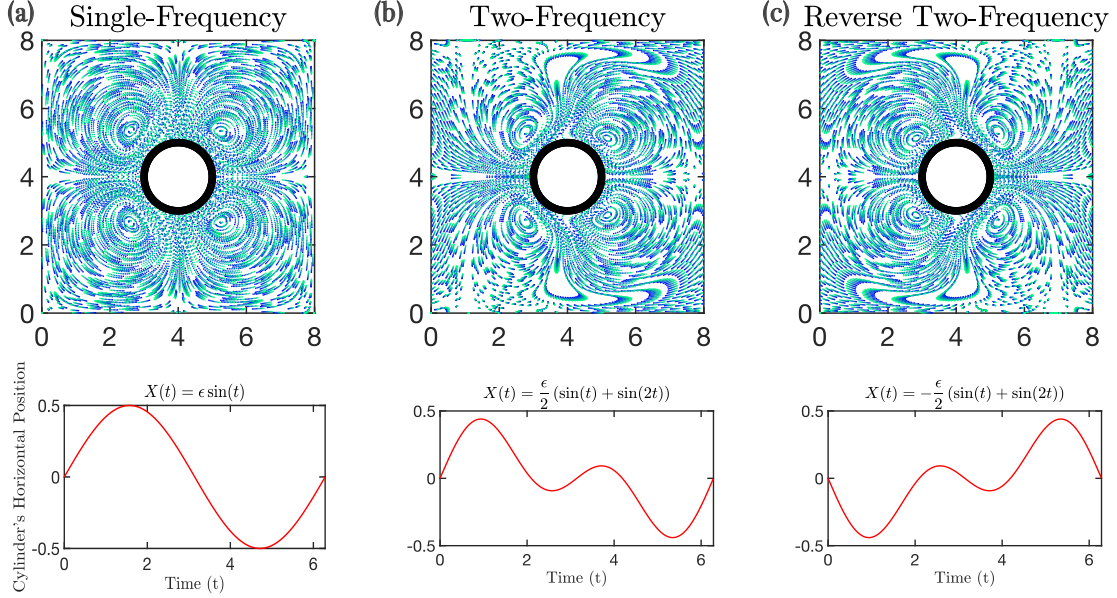


FIGURE 2.1. Streaming flows at time $T = 250$ for amplitude 0.5 and Reynolds number 10 for (a) single frequency motion, (b) two-frequency motion, and (c) two-frequency motion with time reversal. Shown are the positions of passive tracer particles over 10 periods where the current location is colored green and the location 10 periods prior is colored blue. The domain is a 8 by 8 with periodic boundary conditions. The bottom panels show the horizontal position of the cylinder's center over one period.

indicating a net rightward flow. We note that because of the periodic boundary conditions these simulations correspond to a lattice of oscillating cylinders.

Figure 2.1(c) shows the flow resulting from the time-reversed, two-frequency motion of the cylinder. In this case, the flow is moving right-to-left, and the flow pattern appears to be the reflection about the vertical axis of the flow pattern from Figure 2.1(b). The waveforms of the cylinders' positions for all three cases are depicted in the bottom row of Figure 2.1. Note that the time-reversed waveform of the two-frequency oscillation is not a phase shift of the original, as in the single-frequency case. For the time-reversed oscillation, the flow is moving right-to-left, and the flow pattern appears to be the reflection about the vertical axis of the flow pattern from Figure 2.1(b). See movie 1 in the supplementary materials to compare the flow patterns of single-frequency, two-frequency, and reverse two-frequency oscillations. The result from the time-reversed motion demonstrates that the direction of the net flow is determined by the time asymmetry of the cylinder's motion. In the discussion, we elaborate on how the time symmetry of the cylinder motion for different frequency ratios is related to pumping. We note that a similar result was observed in

the experiments of Hashemi *et al.* [17] and Hui *et al.* [21]. When they reversed the polarity of the motion, the object translated in the opposite direction.

2.2.2. Pumping in a Channel. The results in 2.1(b) suggest that cylinders oscillating with two frequencies can be used to pump fluids. To test that idea more directly, we repeated the computational experiments in a channel of height 8 and length 32 with no-slip boundary conditions on the top and bottom and periodic conditions in the horizontal direction. Figure 2.2 visualizes the resulting flow at $T = 250$ for single- and two-frequency motions of the cylinder. See movie 2 in the supplemental materials for the full time history of the flow. Again, for the single-frequency case, the four vortices align symmetrically around the cylinder, and particles away from the cylinder remain stationary. However, when the cylinder oscillates with two-frequencies, the vortices lose their symmetry, and fluid is pumped to the right. Applying the negative waveform again reversed the direction of motion (results not shown).

Figure 2.3(a) shows the flux through a channel of height H ,

$$(2.13) \quad Q(t) = \int_0^H u(x, y, t) dy,$$

over 5 periods for both the single- and two-frequency cases for amplitude $\epsilon = 0.9$. The flux for the two-frequency case shows a non-zero average flux on the scale of approximately 10% of the amplitude of the oscillation. We quantify pumping by the time average of the flux over the period T :

$$(2.14) \quad \langle Q(t) \rangle = \frac{1}{T} \int_{t-T}^t Q(\tau) d\tau.$$

Figure 2.3(b) shows the period-averaged flux as a function of time for different amplitudes. For each amplitude, the period-averaged flux approaches a steady value that depends on the amplitude. The time-reversed oscillation results in the negative of the flux for the non-reversed oscillation of the same amplitude.

Figure 2.3(c) shows that the steady flux grows like amplitude cubed. For single-frequency oscillations the magnitude of the steady flow is second order in amplitude and the oscillatory component is first order [11, 18, 55]. In Figure 2.3(d) we show that for the two-frequency case, these same scalings for the size of the steady and oscillatory flows result as measured by the 2-norm of the period-averaged flow and the time average of the 2-norm of the velocity, respectively.

Previously, analysis on steady streaming has only been carried out to second order in amplitude, and so to derive the terms responsible for pumping, we need to extend the analysis to third order.

2.2.3. Effect of Reynolds Number on Pumping. We repeat the computational studies from the previous section at amplitude $\epsilon = 0.5$ and time $T = 300$ to examine how the flux and flow structure change with Reynolds numbers. We note that at $\text{Re} = 0$, the time-averaged flow is zero because of the dynamic reversibility of Stokes flow, and thus there is no pumping at zero Reynolds number. Figure 2.4 shows the time-averaged flux for $10 \leq \text{Re} \leq 100$ at amplitude $\epsilon = 0.5$, frequency ratio $\alpha = 2$, and time $T = 300$. The flux grows with Reynolds number at a rate slightly larger than $\sqrt{\text{Re}}$.

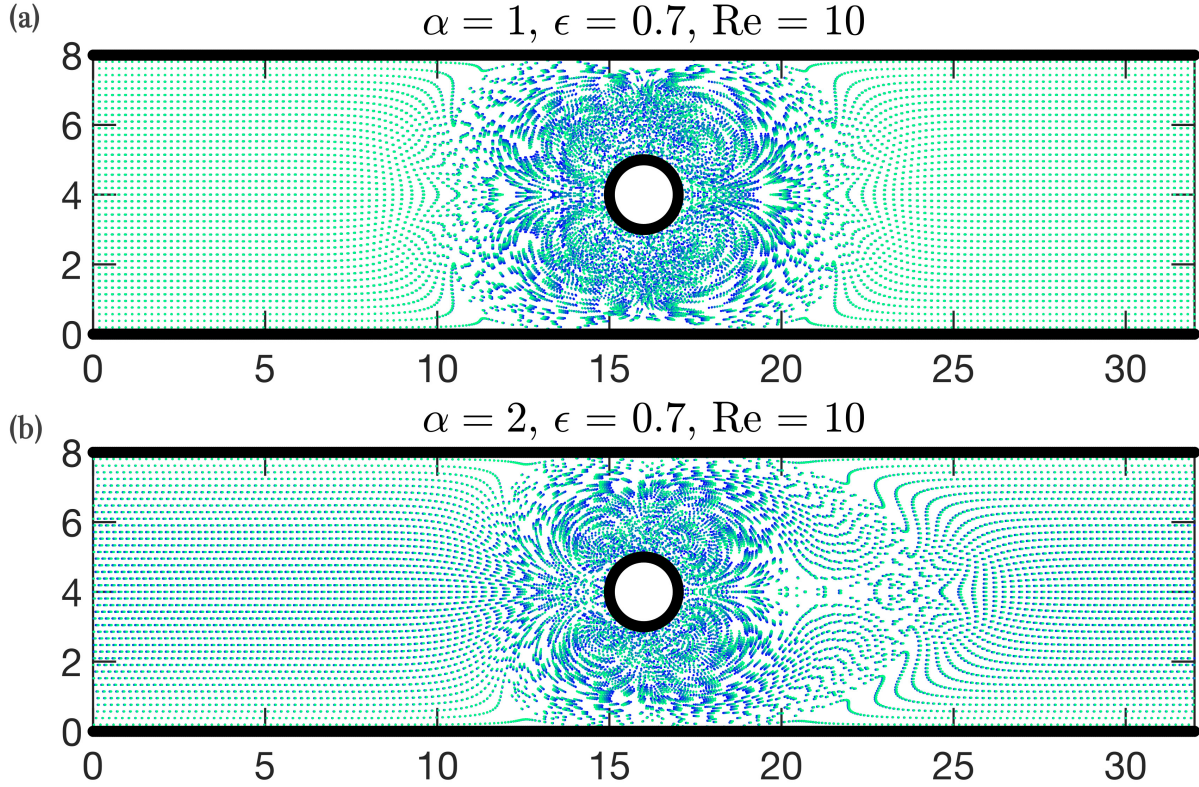


FIGURE 2.2. Streaming flows in a 32 by 8 channel for (a) single frequency ($\alpha = 1$) and (b) two-frequency ($\alpha = 2$) oscillations at time $T = 250$ visualized with amplitude $\epsilon = 0.7$ and Reynolds number 10. Passive tracer particles highlight the positions of the flow from the last 10 periods. Shown are the positions of passive tracer particles over 10 periods where the current location is colored green and the location 10 periods prior is colored blue.

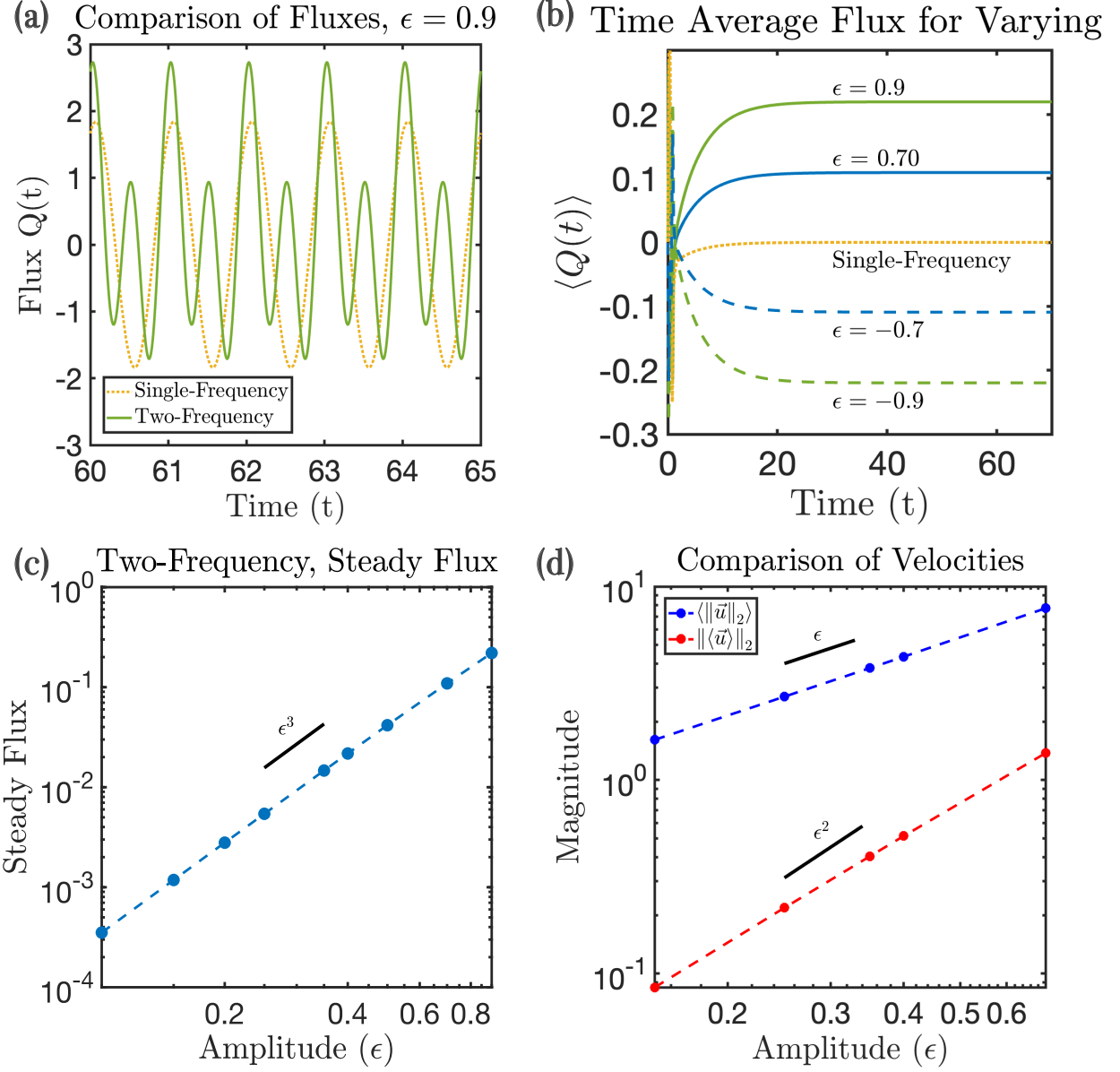


FIGURE 2.3. (a) Flux vs. time is presented for both the single- and two-frequency case for amplitude 0.9 and Reynolds number 10. (b) Time-averaged fluxes are shown for varying amplitudes (ϵ). (c) The steady time-averaged flux is third order in amplitude for two-frequency oscillations. (d) The size of the oscillatory component, $\langle \|u\|_2 \rangle$, is first order in amplitude, and the size of the steady streaming flow, $\|\langle u \rangle\|_2$, is second order in amplitude for the two-frequency oscillation.

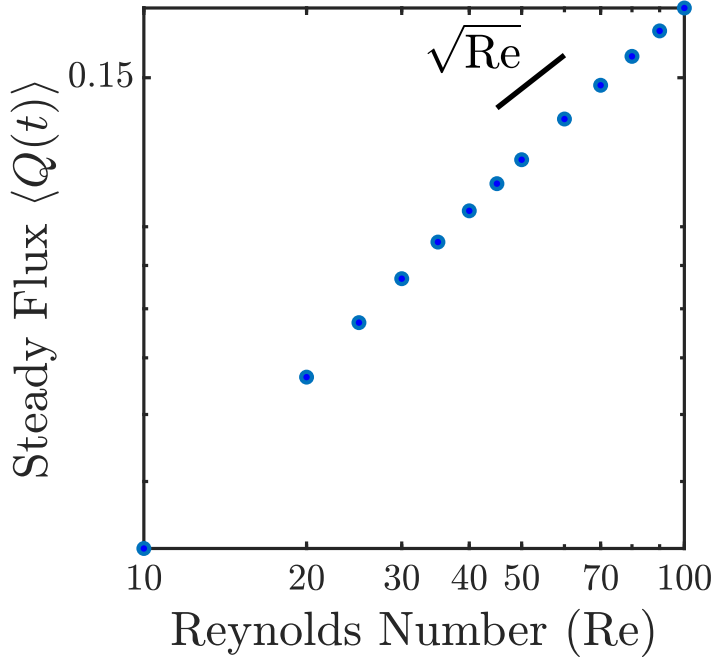


FIGURE 2.4. At amplitude $\epsilon = 0.5$, frequency ratio $\alpha = 2$, and time $T = 300$, the time-averaged fluxes for a 32 by 8 channel are measured for various Reynolds numbers. In general, the time-averaged fluxes appear to scale at a rate slightly larger than $\sqrt{\text{Re}}$.

In Figure 2.5, we show the streamlines and normalized velocity vectors of the time-averaged flow for both single and two-frequency cases at $\text{Re} = 10$ and $\text{Re} = 50$, which are above and below the critical Reynolds number at which a boundary layer forms [11]. At $\text{Re}=10$ for the single-frequency case, we see the classical quadrupole-like structure, where the red streamlines indicate counterclockwise flow, and the blue indicates clockwise flow (Figure 2.5(a)). For frequency ratio $\alpha = 2$, there is a clear flow to the right, and vortices are no longer left-right symmetric. At $\text{Re} = 50$, for both the single and two-frequency cases, there is a region near the cylinder consisting of four rotating vortices with counter-rotating vortices outside this area (Figure 2.5(c,d)). For two-frequency oscillations, the net flow passes between the inner and outer vortices downstream from the cylinder. It should be noted that the near-cylinder region containing the inner vortices in Figure 2.5(b,d) is sometimes referred to as the DC boundary layer to distinguish it from the thinner boundary layer associated with the oscillatory flow [8, 11, 28, 37].

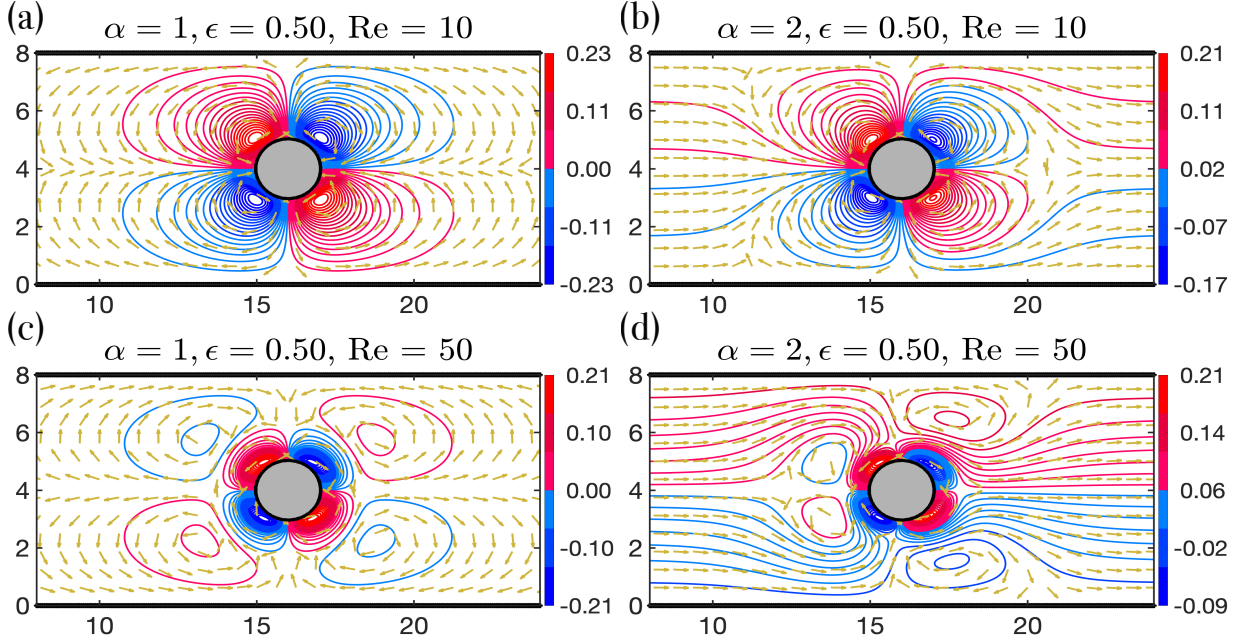


FIGURE 2.5. For a 32 by 8 channel at amplitude $\epsilon = 0.5$ and time $T = 300$, the streamlines and the flow directions (normalized velocity field) of the time-averaged flow are shown at frequency ratios $\alpha = 1$ (a, c) and $\alpha = 2$ (b, d), for $Re = 10$ (a, b) and $Re = 50$ (c, d).

2.3. Low-Amplitude Analysis

We now analyze the problem in the limit of small amplitude ($\epsilon \ll 1$), and we restrict our analysis to small streaming Reynolds numbers ($Re_s \ll 1$) to apply a regular perturbation. The single-frequency solution has been previously computed through second order in amplitude [11, 18], but extending the analysis to third order is not analytically tractable. However, we derive the structure of the higher order solutions, and, from the structure alone, we can understand why pumping is a third order effect for the two-frequency case.

2.3.1. Governing Equations. We analyze the problem in terms of the stream function, ψ , which is related to the velocity by

$$(2.15) \quad u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r},$$

and

$$(2.16) \quad \nabla \times \mathbf{u} = -\Delta \psi.$$

We take the curl of the momentum equation (2.6) to obtain

$$(2.17) \quad \left(\Delta - Re \frac{\partial}{\partial t} \right) \Delta \psi = Re \mathbf{u} \cdot \nabla (\Delta \psi).$$

We solve the problem in the reference frame in which the cylinder is stationary with an oscillatory flow at infinity that is defined by (2.2). The boundary conditions in terms of the stream function are

$$(2.18) \quad \psi(r=1) = 0, \quad \frac{\partial \psi}{\partial r}(r=1) = 0,$$

$$(2.19) \quad \psi \sim -\frac{\epsilon r}{2} (\cos(t) + 2 \cos(2t)) \sin(\theta), \quad r \rightarrow \infty.$$

We consider the low amplitude limit and expand the solution in terms of $\epsilon = A/R$:

$$(2.20) \quad \psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + O(\epsilon^4).$$

Substituting the expansion into (2.17), (2.18), and (2.19), we obtain the successive equations at each order:

First Order

$$(2.21) \quad \left(\Delta - Re \frac{\partial}{\partial t} \right) \Delta \psi_1 = 0,$$

$$(2.22) \quad \psi_1(r=1) = 0, \quad \frac{\partial \psi_1}{\partial r}(r=1) = 0,$$

$$(2.23) \quad \psi_1 \sim -\frac{r}{2} (\cos(t) + 2 \cos(2t)) \sin(\theta), \quad r \rightarrow \infty$$

Second Order

$$(2.24) \quad \left(\Delta - Re \frac{\partial}{\partial t} \right) \Delta \psi_2 = Re \mathbf{u}_1 \cdot \nabla (\Delta \psi_1),$$

$$(2.25) \quad \psi_2(r=1) = 0, \quad \frac{\partial \psi_2}{\partial r}(r=1) = 0,$$

$$(2.26) \quad \psi_2 \sim 0, \quad r \rightarrow \infty,$$

Third Order

$$(2.27) \quad \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \Delta \psi_3 = \text{Re} \mathbf{u}_1 \cdot \nabla (\Delta \psi_2) + \text{Re} \mathbf{u}_2 \cdot \nabla (\Delta \psi_1),$$

$$(2.28) \quad \psi_3(r=1) = 0, \quad \frac{\partial \psi_3}{\partial r}(r=1) = 0,$$

$$(2.29) \quad \psi_3 \sim 0, \quad r \rightarrow \infty.$$

2.3.2. Solution Structure.

2.3.2.1. *Single-Frequency Oscillation.* Before we consider the two-frequency case, we first examine the solution structure for single-frequency oscillation. The only difference in the equations for the single-frequency and two-frequency cases is the boundary condition at infinity at first order. For the single-frequency case, equation (2.23) is replaced by

$$(2.30) \quad \psi_1 \sim -r \cos(t) \sin(\theta).$$

This boundary condition determines the form of the solution at first order as

$$(2.31) \quad \psi_1 = \Re(a_1(r)e^{-it}) \sin(\theta),$$

where $\Re(z)$ denotes the real part of z . Analytically, $a_1(r)$ is a solution composed of Hankel functions [57]. We substitute (2.31) into the quadratic nonlinearity on the right side of Equation (2.24) to determine the form of the solution at second order. This computation involves all products of e^{-it} and e^{it} , which results in a steady term and terms proportional to $e^{\pm i2t}$. Therefore, at second order, we obtain

$$(2.32) \quad \psi_2 = \Re(b_2(r)e^{-2it}) \sin(2\theta) + b_0(r) \sin(2\theta).$$

Equation (2.27) at third order involves products of (2.31) and (2.32) on its right-hand side. This computation involves products of $e^{\pm it}$ with $\{1, e^{\pm 2it}\}$, which results in terms proportional to $e^{\pm i3t}$ and $e^{\pm it}$. The third order solution is of the form

$$(2.33) \quad \psi_3 = \Re(c_3(r)e^{-i3t} + c_1(r)e^{-it}) \sin(3\theta) + \Re(d_3(r)e^{-i3t} + d_1(r)e^{-it}) \sin(\theta).$$

Therefore, for the single-frequency case, there is no steady term at third order.

2.3.2.2. *Two-Frequency Oscillation.* For the two-frequency case, we obtain

$$(2.34) \quad \psi_1 = \Re \left(a_1(r) e^{-it} + a_2(r) e^{-2it} \right) \sin(\theta),$$

as the solution to (2.21) - (2.23). As before, we substitute the solution at first order into the right side of equation (2.24) to deduce the structure of the second order solution as

$$(2.35) \quad \begin{aligned} \psi_2 &= \Re \left(\sum_{k=1}^4 b_k(r) e^{-ikt} \right) \sin(2\theta) + b_0^{(1)}(r) \sin(2\theta) + b_0^{(2)}(r) \sin(2\theta) \\ &= \Re \left(\left(b_2(r) e^{-2it} + b_0^{(1)}(r) \right) + \left(b_4(r) e^{-4it} + b_0^{(2)}(r) \right) \right. \\ &\quad \left. + \left(b_1(r) e^{-it} + b_3(r) e^{-3it} \right) \right) \sin(2\theta). \end{aligned}$$

We express the solution in this second form with three pairs of terms grouped together to highlight the origins of the different terms in the sum. The first two pairs involving a steady term and unsteady term correspond to the single-frequency solutions from each of the two driving frequencies, and the last pair of unsteady terms arises from the interaction between the driving frequencies. Thus, as has been reported previously, the steady flow at second order is the superposition of two streaming fields caused by the individual, single-frequency oscillations [13, 22].

The nonlinear interaction between the two frequencies only affects the unsteady terms at second order, but these interactions produce a steady flow at third order. The solution at third order has the form

$$(2.36) \quad \begin{aligned} \psi_3 &= \Re \left(\sum_{k=1}^6 c_k(r) e^{-ikt} \right) \sin(3\theta) + c_0(r) \sin(3\theta) \\ &\quad + \Re \left(\sum_{k=1}^6 d_k(r) e^{-ikt} \right) \sin(\theta) + d_0(r) \sin(\theta). \end{aligned}$$

Unlike the single-frequency solution (2.33), there are steady terms at third order in amplitude. In the following sections, we explain how the third order, steady terms produce the observed left-right asymmetry and are responsible for producing a net force that pumps the fluid.

2.3.3. Symmetry of Steady Flow. To illustrate how the steady flows at third order create asymmetry, consider the steady solution,

$$(2.37) \quad \psi^s = \epsilon^2 b(r) \sin(2\theta) + \epsilon^3 c(r) \sin(3\theta) + \epsilon^3 d(r) \sin(\theta) + O(\epsilon^4).$$

We show that the familiar second order flow has a different spatial symmetry than the third order flow that arises from the interactions of the flows generated by the two different driving frequencies. Specifically, the spatial symmetry arises from the θ -dependence of the stream functions. To illustrate these symmetries, we examine the streamlines of the three stream functions,

$$(2.38) \quad \psi_2 = f(r) \sin(2\theta), \quad \psi_3^{(1)} = f(r) \sin(\theta), \quad \psi_3^{(3)} = f(r) \sin(3\theta),$$

where

$$(2.39) \quad f(r) = \frac{(r-1)^2}{1+r^4}.$$

This choice of $f(r)$ satisfies the boundary conditions on the cylinder and decays to zero at infinity, but it is not related to the solution of the equations.

The streamlines of $\psi_2, \psi_3^{(1)}$, and $\psi_3^{(3)}$ are shown in Figures 2.6(a)-(c), while the streamlines of the weighted sum is presented in Figure 2.6(d). All the terms share a common up-down symmetry where the values at the points reflected across the horizontal axis are opposite in sign. Specifically, the stream functions are odd in y :

$$(2.40) \quad \psi_2(x, -y) = -\psi_2(x, y), \quad \psi_3^{(1)}(x, -y) = -\psi_3^{(1)}(x, y), \quad \psi_3^{(3)}(x, -y) = -\psi_3^{(3)}(x, y),$$

and the sum of the stream functions maintain this up-down symmetry. However, ψ_2 has a different left-right symmetry than the symmetry shared by the ψ_3 's. Specifically, ψ_2 is odd in x while the ψ_3 's are even in x :

$$(2.41) \quad \psi_2(-x, y) = -\psi_2(x, y), \quad \psi_3^{(1)}(-x, y) = \psi_3^{(1)}(x, y), \quad \psi_3^{(3)}(-x, y) = \psi_3^{(3)}(x, y).$$

Therefore, the sum of the stream functions, illustrated in Figure 2.6(d), retains the up-down symmetry, but lacks the left-right symmetry.

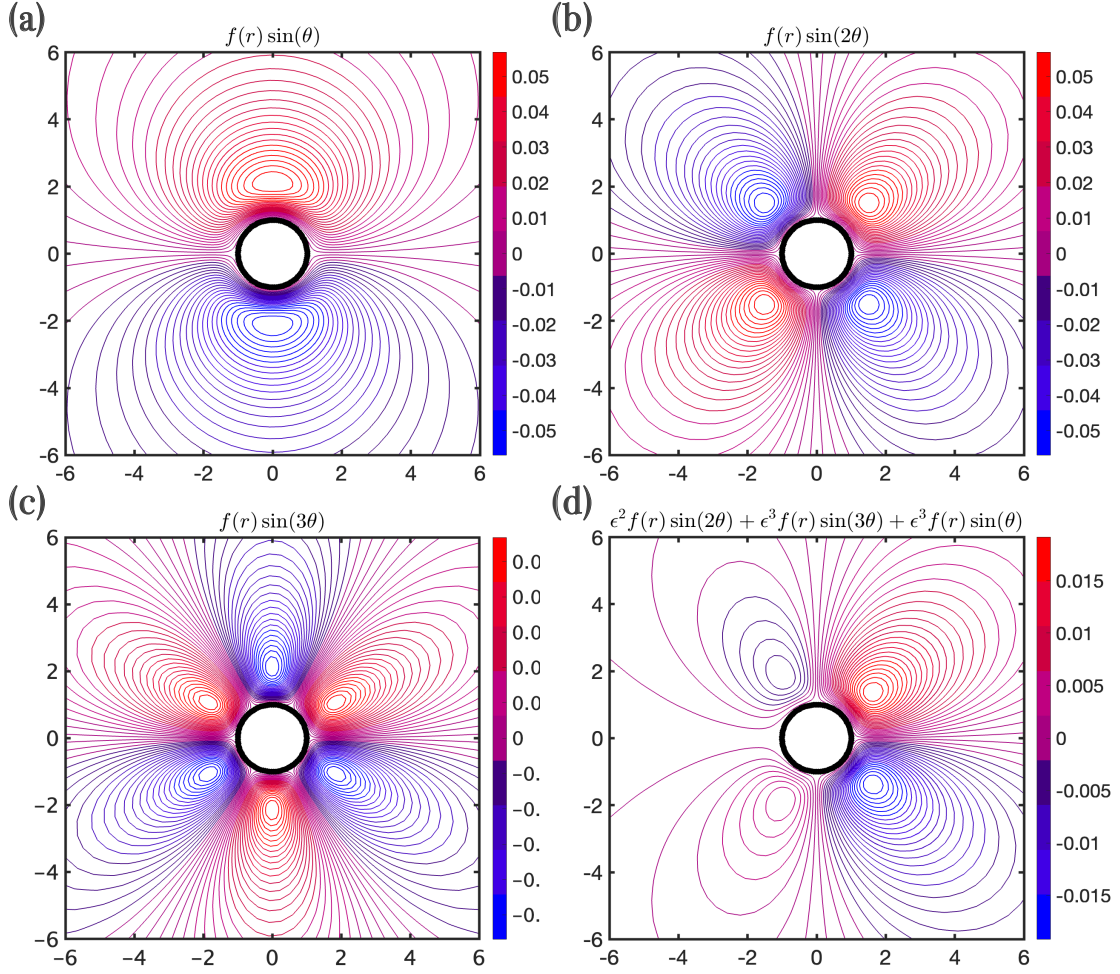


FIGURE 2.6. The contours of the stream functions are shown for (a) $\psi_2 = f(r) \sin(2\theta)$, (b) $\psi_3^{(1)} = f(r) \sin(\theta)$, and (c) $\psi_3^{(3)} = f(r) \sin(3\theta)$, where $f(r)$ is defined by (2.39). (d) Streamlines of the sum, $\psi = \epsilon^2 f(r) \sin(2\theta) + \epsilon^3 f(r) \sin(3\theta) + \epsilon^3 f(r) \sin(\theta)$, for $\epsilon = 0.45$ exhibit a left-right asymmetry.

2.3.4. Net Force. We compute the net force on the cylinder and show that the steady term proportional to $\sin(\theta)$ that appears at third order for two-frequency oscillations indicates that there is a net force on the cylinder. Assume a steady solution of the form,

$$(2.42) \quad \psi^s = \sum_{k=1}^{\infty} f_k(r) \sin(k\theta).$$

The pressure can be computed using the momentum equation (2.6) to obtain

$$(2.43) \quad p^s = p_{\infty} + \sum_{k=1}^{\infty} g_k(r) \cos(k\theta).$$

The traction force in the horizontal direction is

$$(2.44) \quad (\sigma \cdot \hat{e}_r) \cdot \hat{e}_x = \sigma_{rr} \cos(\theta) - \sigma_{r\theta} \sin(\theta),$$

where $\hat{e}_r$ and $\hat{e}_x$ are the direction vectors for the r - and x -directions, respectively, and σ is the stress tensor. Using the boundary conditions on the cylinder from equation (2.9), the components of the stress on the cylinder are

$$(2.45) \quad \sigma_{rr} = -p_\infty - \sum_{k=1}^{\infty} g_k(1) \cos(k\theta), \quad \sigma_{r\theta} = - \sum_{k=1}^{\infty} f_k''(1) \sin(k\theta).$$

Integrating the surface stress (2.44) over the cylinder, all the terms corresponding to $k \geq 2$ vanish due to symmetry, and the net horizontal force is

$$(2.46) \quad \text{Net Force} = \int_0^{2\pi} \sigma_{rr} \cos(\theta) - \sigma_{r\theta} \sin(\theta) d\theta = \pi (-g_1(1) + f_1''(1)).$$

Therefore the term proportional to $\sin(\theta)$ that appears at third order in equation (2.37) contributes a net force which is responsible for the observed pumping.

This force calculation gives insight as to why pumping does not occur for the single-frequency case. From Section 2.3.2.1, the only steady term through third order in amplitude is proportional to $\sin(2\theta)$, and the only nonzero term in the pressure expansion, equation (2.43), is proportional to $\cos(2\theta)$. Therefore there is no net force for single-frequency oscillations through third order in amplitude, though we have not yet considered the possibility of a net force resulting at higher order for the single-frequency case. In the next section, we analyze the form of the solution at higher orders, and we extend the analysis to general frequency ratio to derive necessary conditions on the frequency ratio for pumping.

2.4. Analysis of General Frequency Ratios

In the previous section, we showed that the steady stream function for frequency ratio $\alpha = 2$ contained a term at third order proportional to $\sin(\theta)$. In this section, we examine the solution structure of ψ_n , the solution to the n^{th} order perturbative equation (2.51), to determine whether there is potentially a net force on the fluid due to the presence of a $\sin(\theta)$ term in the expansion and if so, what order.

For this analysis, it is convenient to express the motion of the cylinder as

$$(2.47) \quad X(t) = \frac{\epsilon}{2} (\sin(at) + \sin(bt)),$$

where $\gcd(a, b) = 1$. This expression is a slight modification from equation (2.1), and it amounts to a different definition of the Reynolds number than the one used in the previous sections. We again change reference frames by fixing the cylinder, and the governing equations have the form,

$$(2.48) \quad \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \Delta \psi = \text{Re} \mathbf{u} \cdot \nabla (\Delta \psi),$$

$$(2.49) \quad \psi(r=1) = 0, \quad \frac{\partial \psi}{\partial r}(r=1) = 0,$$

$$(2.50) \quad \psi \sim -\frac{\epsilon r}{2} (a \cos(at) + b \cos(bt)) \sin(\theta), \quad r \rightarrow \infty.$$

Expanding

$$\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + O(\epsilon^4)$$

in the low-amplitude limit, and substituting this expansion into Equation (2.48), we obtain successive equations of the form

$$(2.51) \quad \Delta \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \psi_n = \sum_{i+j=n} \text{Re} \mathbf{u}_i \cdot \nabla (\Delta \psi_j).$$

The boundary conditions on the cylinder at each order are

$$(2.52) \quad \psi_n(r=1) = 0, \quad \frac{\partial \psi_n}{\partial r}(r=1) = 0,$$

and the asymptotic condition at $r = \infty$ is

$$(2.53) \quad \psi_n \sim \begin{cases} \frac{-r}{2} (a \cos(at) + b \cos(bt)) \sin(\theta) & n = 1, \quad r \rightarrow \infty \\ 0 & n > 1, \quad r \rightarrow \infty. \end{cases}$$

The solution to (2.51)-(2.53) is of the form

$$(2.54) \quad \psi_n = \sum_{m=1}^{\infty} \sum_{k, m=-\infty}^{\infty} f_{k,m}^n(r) \sin(m\theta) e^{ikt},$$

but the number of nonzero terms is finite at each order. The boundary condition determines the nonzero terms at leading order, and at higher orders, the structure of the solution depends on the products of the lower order terms. In the following sections, we first determine which terms proportional to $\sin(m\theta)$ are nonzero at each order, and then we derive conditions for the presence of a steady term ($k = 0$).

2.4.1. The θ -dependent Terms. From the form of the far-field boundary condition (2.53), the leading order solution is of the form

$$(2.55) \quad \psi_1 = \Re \left(a_2(r) e^{-ibt} + a_1(r) e^{-iat} \right) \sin(\theta) = C_1(r, t) \sin(\theta).$$

The form of the right-hand side of (2.51) dictates the solution structure at higher orders. Substituting ψ_1 into the quadratic nonlinearity, the right-hand side of (2.51) is proportional to $\sin(\theta) \cos(\theta)$, and thus the solution is proportional to $\sin(2\theta)$. Therefore, ψ_2 is of the form

$$(2.56) \quad \begin{aligned} \psi_2 &= \Re \left(b_4(r) e^{-i2bt} + b_3(r) e^{-i(a+b)t} + b_2(r) e^{-i2at} + b_1(r) e^{-i(a-b)t} + b_0(r) \right) \sin(2\theta) \\ &= C_2(r, t) \sin(2\theta). \end{aligned}$$

The right-hand side of (2.51) for ψ_3 involves the products of derivatives of ψ_1 and ψ_2 . We obtain terms involving $\sin(2\theta) \cos(\theta)$ and $\sin(\theta) \cos(2\theta)$. Using the trigonometric identities,

$$(2.57) \quad 2 \sin(2\theta) \cos(\theta) = \sin(3\theta) + \sin(\theta), \quad 2 \sin(\theta) \cos(2\theta) = \sin(3\theta) - \sin(\theta),$$

ψ_3 can be expressed as

$$(2.58) \quad \psi_3 = C_3(r, t) \sin(3\theta) + C_4(r, t) \sin(\theta).$$

Similarly, the solution of ψ_4 has the form

$$(2.59) \quad \psi_4 = C_5(r, t) \sin(4\theta) + C_6(r, t) \sin(2\theta).$$

The first four orders show that the solution for odd values of n involve terms proportional to $\sin(m\theta)$ for odd value of m through $n = m$, and similarly for even values of n . In Appendix A.1 we prove a lemma that shows that this pattern holds for all n for a general multi-modal oscillation. Additionally, we prove an analogous result for the pressure. From Section 2.3.4, a net horizontal

force requires a term in the steady stream function proportional to $\sin(\theta)$, or a term in the steady pressure proportional to $\cos(\theta)$. Putting these two results together leads to the following necessary condition for pumping:

RESULT 2.4.1. *A necessary condition for pumping is the existence of an odd-valued n such that ψ_n has the steady component.*

In the next section, we examine the frequencies that occur at each order, and derive conditions for pumping based on the frequency ratio.

2.4.2. Necessary Conditions for Pumping.

2.4.2.1. No Pumping for Single Frequency. We demonstrate that single-frequency oscillations do not pump by showing that ψ_n cannot have a steady component when n is odd. We examine the frequencies that arise at each order, and deduce the orders at which the steady terms occur. For example, Figure 5 of [58] illustrates how the frequencies at each order arise from the lower order frequencies for the single-frequency case.

The structure of the leading order solution for the single-frequency case ($a = b = 1$) is given in Equation (2.31). Alternately, we can express the solution as

$$(2.60) \quad \psi_1 = G_1^1(r, \theta)e^{-it} + \overline{G_1^1(r, \theta)}e^{it},$$

where

$$(2.61) \quad G_1^1(r, \theta) = \frac{a_1(r) \sin(\theta)}{2}.$$

Similarly, the second order solution (2.32) is rewritten as

$$(2.62) \quad \psi_2 = G_2^2(r, \theta)e^{-i2t} + \overline{G_2^2(r, \theta)}e^{i2t} + G_0^2(r, \theta),$$

where

$$(2.63) \quad G_2^2(r, \theta) = \frac{b_2(r) \sin(2\theta)}{2}, \quad G_0^2(r, \theta) = b_0(r) \sin(2\theta).$$

The time dependence of the individual terms in ψ_2 , i.e., e^{-2it} , e^{2it} , and $1 = e^{i0t}$, result from the products of e^{-it} and e^{it} through the quadratic nonlinearity of the right-hand side of (2.51).

Extending the pattern to higher order n , if ψ_n has the term proportional to $e^{i\omega t}$, then

$$(2.64) \quad e^{i\omega t} = (e^{-it})^{n_1} (e^{it})^{n_2},$$

where $n_1 + n_2 = n$ and $0 \leq n_1, n_2 \leq n$.

Now, we show that the steady component cannot appear when n is odd. Using the fact that $n_2 = n - n_1$,

$$(2.65) \quad \omega = -n_1 + n_2 = -2n_1 + n.$$

For the steady term to appear ($\omega = 0$), n must be even. Combining this result with Result 2.4.1 shows that pumping does not occur for single-frequency oscillations.

2.4.2.2. Two Frequencies. The two-frequency case involves products of $e^{\pm iat}$ and $e^{\pm ibt}$ leading to a more complicated analysis. However, using a similar parity argument, we obtain the following results.

RESULT 2.4.2. *Suppose that a and b are both odd and ψ_n is the solution of (2.51)-(2.53). Steady terms only appear for n even, and thus pumping will not occur.*

RESULT 2.4.3. *Suppose that only one of a or b is even (and the other is odd) and ψ_n is the solution of (2.51)-(2.53). ψ_n can have the steady component for n odd if $n \geq (a + b)$. Therefore the minimum order at which pumping occurs is order $a + b$.*

See Appendix A.2 for the proofs of Results 2.4.2 and 2.4.3. In Result 2.4.3, we give the smallest order of amplitude at which pumping can occur; however, we did not guarantee pumping. For simplicity, we defined the radial components as coefficients in the analysis, but these coefficients could equate to zero.

2.4.3. Numerical Results for Other Frequency Ratios. According to Result 2.4.3, for small amplitude ϵ , the strength of pumping scales like ϵ^n , where $n = a + b$ is the sum of the two frequencies and must be odd. Thus, the strongest pumping results when $n = 3$, corresponding to frequency ratio $\alpha = 2$. The next smallest order at which pumping can occur is fifth order. From Result 2.4.3, a fifth order scaling can only occur when the frequency ratio is 3/2 or 4, respectively, corresponding to $a = 2, b = 3$ or $a = 1, b = 4$. Figure 2.7(a) shows the fluxes from numerical

simulations for different frequency ratios and amplitudes, and these results verify that pumping is fifth order in amplitude for frequency ratios $3/2$ and 4 . Moreover, the fluxes for frequency ratios $3/2$ and 4 are significantly smaller than the fluxes for frequency ratio 2 . For example, at amplitude 0.7 , the flux of frequency ratio 2 is approximately 41 times larger than the flux at frequency ratio $3/2$. In Figure 2.7(b) we show the passive tracer particles over 10 periods, and the net flow is not obvious without zooming in closely or seeing movie 3 from the supplementary materials. Though the fluxes for frequency ratios $3/2$ and 4 are small, they show clear scaling with amplitude. By contrast, the small fluxes for frequency ratio 3 are not correlated with the amplitude. According to Result 2.4.3, there is no pumping for frequency ratio 3 , and the small fluxes reported in Figure 2.7(a) arise from numerical error. Note that in Section 2.2.2 we examined how the flux scaled with amplitude for $\alpha = 2$ at $\text{Re} = 10$. Here, we use $\text{Re} = 40$ because at lower Reynolds numbers, the fluxes are small, and it is difficult to measure pumping at higher order. We note that the analysis was performed in the limit of small amplitude, and is expected to be valid for small streaming Reynolds numbers. The fluxes reported in Figure 2.7(a) were computed at streaming Reynolds numbers in the range 1.6-25.6, yet we obtained excellent agreement with the analysis.

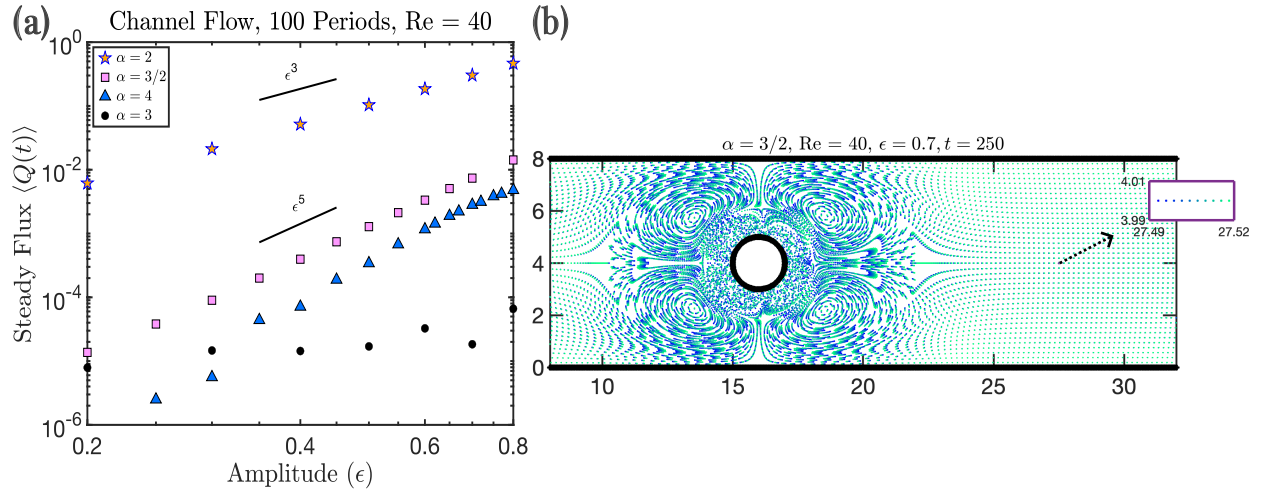


FIGURE 2.7. (a) Time-averaged flux vs. amplitude for frequency ratios $\alpha = 2, 3/2, 4$, and 3 . (b) Steady streaming in a channel for frequency ratio $\alpha = 3/2$ visualized at amplitude 0.7 and Reynolds number 40 . Shown are the positions of passive tracer particles over 10 periods where the current location is colored green, and the location 10 periods prior is colored blue. Though the fluid is being pumped, net motion is not obvious on this scale. The inset shows the region around a single tracer particle, which confirms that the fluid is moving slowly downstream.

2.4.4. Near-Miss Frequencies. An interesting question is what happens for frequency ratios that are close to, but not equal to, $\alpha = 2$. Figure 2.8 shows the fluxes averaged over one time unit for frequency ratios $\alpha = 2$, $\alpha = 1.999$, and $\alpha = 1.99$ at amplitude $\epsilon = 0.7$, and $\text{Re} = 10$. The fluxes for $\alpha = 1.99$ and $\alpha = 1.999$ oscillate on long time scales, with periods 100 and 1000, respectively, with amplitude similar to the steady flux for $\alpha = 2$. According to Result 2.4.3, for $\alpha = 1.99$ and $\alpha = 1.999$, the lowest order in amplitude pumping could occur is 299 and 2999, respectively. Over long periods we expect negligible net flux for these “near-miss” frequency ratios, but on a short time scale they exhibit fluxes similar to $\alpha = 2$.

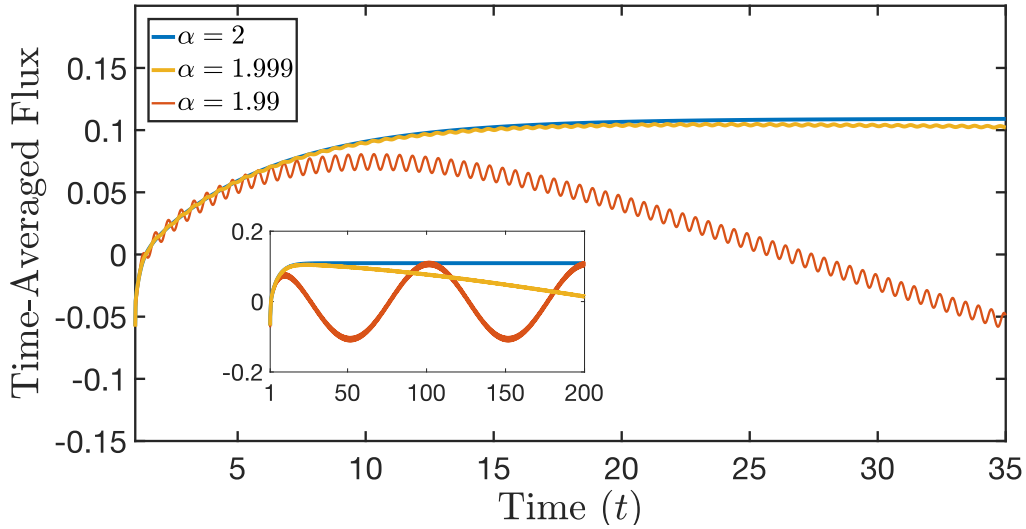


FIGURE 2.8. Time average of fluxes over one time unit for frequency ratios $\alpha = 2$, $\alpha = 1.999$, and $\alpha = 1.99$ at amplitude $\epsilon = 0.7$, and $\text{Re} = 10$. The inset shows the time-averaged fluxes to time 200. On short time scales the fluxes are similar, but on long time scales, the fluxes for $\alpha = 1.99$ and $\alpha = 1.999$ oscillate with amplitude similar to the steady flux for $\alpha = 2$.

CHAPTER 3

Pumping as a Function of Amplitude, Phase, and Reynolds Number

In Chapter 2, we showed that while steady streaming occurs at second order in amplitude, pumping is a higher order effect. The solutions of the equations from the asymptotic analysis at third order and above are analytically intractable, but based on the structure of the expansion alone, we derived necessary conditions on the frequency ratio for pumping. Moreover, we showed that the smallest order in amplitude at which pumping occurs is third order for frequency ratio $\alpha = 2$. Our work did not predict how the pumping rate varies with different parameters, which is important for microfluidic applications.

Specifically, there are a variety of microchannel pumps that combine the single-frequency steady streaming and the shape asymmetry of the device to pump fluid [19, 50, 51]. The asymmetry illustrated in Figure 2.2(b) is temporal rather than spatial, and pumping generated using two-frequency vibrations could lead to new designs of microfluidic pumps or potentially enhance the performance of existing designs. Developing devices that utilize two-frequency oscillations requires understanding how the pumping rate depends on the waveform, which was not previously addressed because the solutions to the asymptotic equations are analytically intractable. While we can quantify how the pumping rate changes in direction and strength with numerical simulations, it would be computationally expensive. Here, we use asymptotic analysis and computational methods to quantify how the streaming flow generated by two-frequency vibrations depends on amplitude ratio and phase difference of the two oscillations.

3.1. Problem Statement

We consider the 2D viscous flow around a cylinder that is oscillating in the horizontal direction centered at $(X(t), 0)$ where

$$(3.1) \quad X(t) = A_1 \sin(\Omega t) + A_2 \sin(\alpha \Omega t + \phi).$$

In Equation (3.1), A_1 and A_2 are the amplitudes of the two oscillations, Ω is the angular frequency of the base oscillation, ϕ is the phase difference, and $\alpha \geq 1$, the ratio of the two frequencies. We also consider the related problem of a stationary cylinder, immersed inside an oscillating horizontal far-field flow of strength,

$$(3.2) \quad U(t) = -A_1\Omega \cos(\Omega t) - \alpha A_2\Omega \cos(\alpha\Omega t + \phi).$$

To facilitate the analysis, we solve the problem on an annulus (Figure 3.1) where we fix the positions of both the inner and outer cylinders of radii R_{in} and R_{out} , correspondingly, and we impose the far-field flow of (3.2) around the outer cylinder. Specifically, the boundary conditions are

$$(3.3) \quad u_r(R_{\text{in}}) = 0, \quad u_\theta(R_{\text{in}}) = 0,$$

$$(3.4) \quad u_r(R_{\text{out}}) = U(t) \cos(\theta), \quad u_\theta(R_{\text{out}}) = -U(t) \sin(\theta).$$

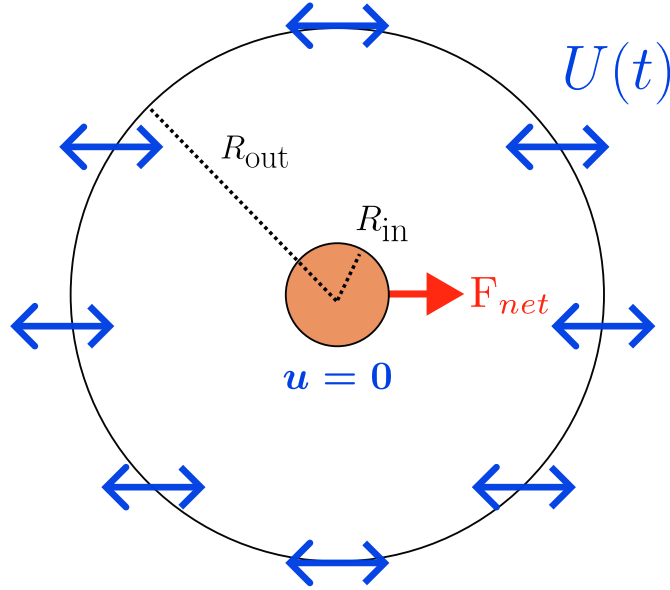


FIGURE 3.1. We solve for the flow inside an annulus where the radii of the inner and outer cylinder are R_1 and R_2 , respectively. We prescribe no-slip conditions at radius R_1 and a horizontal velocity of strength $U(t)$ at radius R_2 . There is no pumping inside an annulus, so we instead compute the time-averaged net force F_{net} on the inner cylinder.

Steady streaming in an annulus has been analyzed previously in the case one of the inner and the outer cylinder is fixed and the other is vibrating with a single frequency [8, 14, 44, 45]. The goal of our analysis is to quantify pumping from two-frequency oscillations, but there is no pumping inside an annulus, and so instead, we compute the net force on the inner cylinder. In the later sections, we use numerical simulations to measure the flux generated by an oscillating cylinder through a channel and show that the net force computed from the asymptotic analysis is proportional to the flux through the channel.

The equations are non-dimensionalized using the characteristic length scale R_{in} , time scale Ω^{-1} , and velocity scale $R_{\text{in}}\Omega$. The flow around the cylinder is governed by the incompressible Navier Stokes equations:

$$(3.5) \quad \text{Re} (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \Delta \mathbf{u},$$

$$(3.6) \quad \nabla \cdot \mathbf{u} = 0,$$

where the Reynolds number is

$$(3.7) \quad \text{Re} = \frac{R_{\text{in}}^2 \Omega}{\nu},$$

and ν is the kinematic viscosity. The dimensionless amplitude ratio, β and amplitude, ϵ are

$$(3.8) \quad \beta = \frac{A_2}{A_1}, \quad \epsilon = \frac{\sqrt{A_1^2 + A_2^2}}{R_{\text{in}}}.$$

Non-dimensionalizing the velocity $U(t)$ from Equation (3.2), we obtain

$$(3.9) \quad U(t) = -\frac{\epsilon}{\sqrt{1 + \beta^2}} (\cos(t) + \alpha\beta \cos(\alpha t + \phi)),$$

and its corresponding displacement,

$$(3.10) \quad X(t) = -\frac{\epsilon}{\sqrt{1 + \beta^2}} (\sin(t) + \beta \sin(\alpha t + \phi)).$$

In the limits of the amplitude ratio $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, the displacement is

$$(3.11) \quad \lim_{\beta \rightarrow 0} X(t) = -\epsilon \sin(t), \quad \lim_{\beta \rightarrow \infty} X(t) = -\epsilon \sin(\alpha t + \phi),$$

and the velocity is

$$(3.12) \quad \lim_{\beta \rightarrow 0} U(t) = -\epsilon \cos(t), \quad \lim_{\beta \rightarrow \infty} U(t) = -\epsilon \alpha \cos(\alpha t + \phi).$$

Thus single frequency motion results in both limits, but with frequencies one and α , correspondingly.

Finally, we define R as the ratio of the two radii of the annulus,

$$(3.13) \quad R = \frac{R_{\text{out}}}{R_{\text{in}}}.$$

Thus, the boundary conditions (3.3) and (3.4) become

$$(3.14) \quad u_r(r = 1) = 0, \quad u_\theta(r = 1) = 0,$$

$$(3.15) \quad u_r(r = R) = U(t) \cos(\theta), \quad u_\theta(r = R) = -U(t) \sin(\theta).$$

We define the streaming Reynolds number as,

$$(3.16) \quad \text{Re}_s = \frac{(A_1^2 + A_2^2)\Omega}{\nu} = \epsilon^2 \text{Re},$$

and we again focus on the low streaming Reynolds number regime ($\text{Re}_s \ll 1$) so that the problem can be analyzed as a regular perturbation.

3.2. Low Amplitude Analysis

We examine the problem in the limit of small amplitude ($\epsilon \ll 1$) and restrict the analysis to small streaming Reynolds number ($\text{Re}_s \ll 1$) to solve the problem as a regular perturbation. In Chapter 2, we used the structure of the regular perturbation analysis and derived necessary conditions on the frequency ratio α to produce a net force. Moreover, we showed that the lowest order in amplitude at which pumping occurs is third order for frequency ratio $\alpha = 2$, and thus, the largest pumping is generated for frequency ratio $\alpha = 2$. We aim to analyze the problem for general amplitude ratio β and phase ϕ at $\alpha = 2$ for varying Reynolds numbers. Thus, we carry out the asymptotic analysis to third order and compute the terms that generate a net force.

3.2.1. Governing Equations. We analyze the problem in terms of the stream function, ψ , which is related to the velocity by

$$(3.17) \quad u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r},$$

and

$$(3.18) \quad \nabla \times \mathbf{u} = -\Delta \psi.$$

We take the curl of the momentum equation (3.5) to obtain

$$(3.19) \quad \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \Delta \psi = \text{Re} \mathbf{u} \cdot \nabla (\Delta \psi).$$

The boundary conditions of (3.14)-(3.15) in terms of the stream function ψ are

$$(3.20) \quad \psi(r=1) = 0,$$

$$(3.21) \quad \frac{\partial \psi}{\partial r}(r=1) = 0,$$

$$(3.22) \quad \psi|_{r=R} = -\frac{\epsilon R}{\sqrt{1+\beta^2}} (\cos(t) + 2\beta \cos(2t + \phi)) \sin(\theta),$$

$$(3.23) \quad \frac{\partial \psi}{\partial r} \Big|_{r=R} = -\frac{\epsilon}{\sqrt{1+\beta^2}} (\cos(t) + 2\beta \cos(2t + \phi)) \sin(\theta).$$

We consider the low amplitude limit and expand the solution in terms of $\epsilon = \sqrt{A_1^2 + A_2^2}/R_{\text{in}}$:

$$(3.24) \quad \psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + O(\epsilon^4).$$

Substituting the expansion (3.24) into (3.19)-(3.23), we obtain successive equations at each order:

First Order

$$(3.25) \quad \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \Delta \psi_1 = 0,$$

$$(3.26) \quad \psi_1(r=1) = 0, \quad \frac{\partial \psi_1}{\partial r}(r=1) = 0,$$

$$(3.27) \quad \psi_1(r=R) = -\frac{R}{\sqrt{1+\beta^2}} (\cos(t) + 2\beta \cos(2t + \phi)) \sin(\theta),$$

$$(3.28) \quad \frac{\partial \psi_1}{\partial r}(r=R) = -\frac{1}{\sqrt{1+\beta^2}} (\cos(t) + 2\beta \cos(2t + \phi)) \sin(\theta),$$

Second Order

$$(3.29) \quad \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \Delta \psi_2 = \text{Re} \mathbf{u}_1 \cdot \nabla (\Delta \psi_1),$$

$$(3.30) \quad \psi_2(r=1) = 0, \quad \frac{\partial \psi_2}{\partial r}(r=1) = 0,$$

$$(3.31) \quad \psi_2(r=R) = 0, \quad \frac{\partial \psi_2}{\partial r}(r=R) = 0,$$

Third Order

$$(3.32) \quad \left(\Delta - \text{Re} \frac{\partial}{\partial t} \right) \Delta \psi_3 = \text{Re} \mathbf{u}_2 \cdot \nabla (\Delta \psi_1) + \text{Re} \mathbf{u}_1 \cdot \nabla (\Delta \psi_2),$$

$$(3.33) \quad \psi_3(r=1) = 0, \quad \frac{\partial \psi_3}{\partial r}(r=1) = 0,$$

$$(3.34) \quad \psi_3(r=R) = 0, \quad \frac{\partial \psi_3}{\partial r}(r=R) = 0.$$

3.2.2. Solution Structure. At first order in amplitude, the boundary conditions (3.27) and (3.28) define the solution structure as

$$(3.35) \quad \psi_1 = \Re (a_1(r)e^{-it} + a_2(r)e^{-i2t}) \sin(\theta),$$

where $\Re(z)$ denotes the real part of z . Substituting the first order solution (3.35) into the right-hand side of Equation (3.29), which is a quadratic nonlinearity, the products of $e^{\pm it}$ and $e^{\pm i2t}$ results in a steady term and terms proportional to $\{e^{\pm it}, e^{\pm i2t}, e^{\pm i3t}, e^{\pm i4t}\}$. Therefore, at second order, we

obtain

$$(3.36) \quad \psi_2 = \Re \left(\sum_{k=1}^4 b_k(r) e^{-ikt} \right) \sin(2\theta) + b_0(r) \sin(2\theta).$$

Similarly, we substitute (3.35) and (3.36) into the quadratic nonlinearity of (3.32), and the products of terms proportional to $\{e^{\pm it}, e^{\pm i2t}\}$ and $\{1, e^{\pm it}, e^{\pm i2t}, e^{\pm i3t}, e^{\pm i4t}\}$ result in a solution structure of (3.32)-(3.34) as

$$(3.37) \quad \psi_3 = \Re \left(\sum_{k=1}^6 c_k(r) e^{-ikt} \right) \sin(3\theta) + c_0(r) \sin(3\theta) + \Re \left(\sum_{k=1}^6 d_k(r) e^{-ikt} \right) \sin(\theta) + d_0(r) \sin(\theta).$$

Note that in (3.35)-(3.37), the dependence on time and angular coordinate, θ , is known, and only the radial dependence is unknown. We derive a series of one dimensional boundary value problems to determine how the solution depends on the radial variable, r .

We begin by defining the differential operator,

$$(3.38) \quad D_\ell = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\ell^2}{r^2}.$$

Substituting the first order solution (3.35) into Equations (3.25)-(3.28), we obtain

First Order

$$(3.39) \quad D_1 (D_1 + ik \operatorname{Re}) a_k = 0,$$

$$(3.40) \quad a_k(1) = 0, \quad a'_k(1) = 0,$$

for $k = 1, 2$. The boundary conditions at $r = R$ are

$$(3.41) \quad a_1(R) = -\frac{R}{\sqrt{1+\beta^2}}, \quad a'_1(R) = -\frac{1}{\sqrt{1+\beta^2}},$$

and

$$(3.42) \quad a_2(R) = -\frac{R 2\beta e^{-i\phi}}{\sqrt{1+\beta^2}}, \quad a'_2(R) = -\frac{2\beta e^{-i\phi}}{\sqrt{1+\beta^2}}.$$

For the equations at second order, we substitute the second order solution (3.36) into Equations (3.29)-(3.31) and obtain

Second Order

$$(3.43) \quad D_2 (D_2 + ik \operatorname{Re}) b_k(r) = \Omega_k(r),$$

$$(3.44) \quad b_k(1) = 0, \quad b'_k(1) = 0,$$

$$(3.45) \quad b_k(R) = 0, \quad b'_k(R) = 0.$$

Similarly, we substitute the third order solution (3.37) into Equations (3.32)-(3.34) to derive the equations at third order as

Third Order

$$(3.46) \quad D_3 (D_3 + ik \operatorname{Re}) c_k(r) = \rho_k(r),$$

$$(3.47) \quad c_k(1) = 0, \quad c'_k(1) = 0,$$

$$(3.48) \quad c_k(R) = 0, \quad c'_k(R) = 0,$$

$$(3.49) \quad D_1 (D_1 + ik \operatorname{Re}) d_k(r) = \xi_k(r),$$

$$(3.50) \quad d_k(1) = 0, \quad d'_k(1) = 0,$$

$$(3.51) \quad d_k(R) = 0, \quad d'_k(R) = 0.$$

The terms $\Omega_k(r)$, $\rho_k(r)$, and $\xi_k(r)$ are defined in Appendix B.1. While the equations at first and second order can be solved analytically [11, 18], Equations (3.46)-(3.51) at third order are analytically intractable. Therefore, we solve the boundary value problems numerically, which we describe in detail in Appendix B.3.

In Figure 3.2, we show the contours of the stream function for single- and two-frequency ($\beta = 1$) oscillations for phase difference $\phi = 0$, $\epsilon = \sqrt{0.02} \approx 0.14$, $R = 8$, and $\operatorname{Re} = 50$. The streamlines for the single-frequency case ($\beta = 0$) show the classic quadrupole-like flow structure and exhibit symmetry across the lines $y = \pm x$. For the two-frequency case, like the flow through the channel shown in Figure 2.2(b), the streamlines in the annular domain are no longer left-right symmetric.

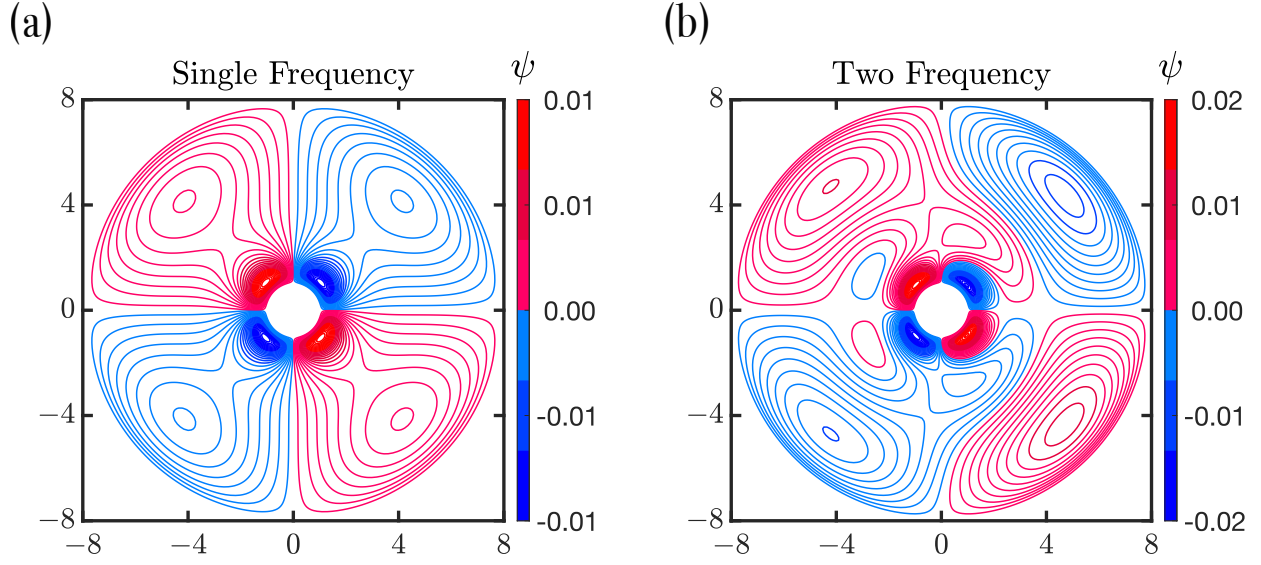


FIGURE 3.2. Contours of the steady stream function for (a) single-frequency and (b) two-frequency ($\beta = 1$) oscillations for phase difference $\phi = 0$, $\epsilon = \sqrt{0.02} \approx 0.14$, $R = 8$, and $\text{Re} = 50$.

Unlike the channel, there is no pumping through this closed domain, and we show in the following section that a net force is produced on the cylinder.

We solve Equations (3.49)-(3.51) with $k = 0$ and its corresponding pressure to compute the net force. As shown in Appendix B.1, the term ξ_0 of Equation (3.49) results from products involving $\{a_1(r), a_2(r), b_1(r), b_2(r)\}$ through the quadratic nonlinearity. Thus computing $d_0(r)$, requires the solutions of Equations (3.39)-(3.45) for $k = 1, 2$. In Appendix B.2, we show that the steady pressure at third order can be computed similarly by substituting our velocity components into the momentum equation (3.5).

3.3. Net Force as a Function of Amplitude Ratio, Phase, and Reynolds Number

Previously, we showed for frequency ratio $\alpha = 2$, that the steady term proportional to $\sin(\theta)$ at third order in amplitude produces a net force. Therefore, the net force scales like ϵ^3 , i.e., $F_{net} = \epsilon^3 \tilde{F}_{net}$, where $\tilde{F}_{net}$ is $O(1)$. Note that the cubic scaling in amplitude for the net force is independent of other parameters. When we display the net force for amplitude ratio, phase, and Reynolds number in the following sections, we neglect the ϵ^3 in the force, and thus, we plot

$$(3.52) \quad \tilde{F}_{net} = \epsilon^{-3} F_{net},$$

as the net force. In the numerical computation of the net force, we use 120 Chebyshev points on a domain size of $R = 8$.

3.3.1. Amplitude Ratio (β) and Phase Difference (ϕ). Figure 3.3(a) shows the net force at $\text{Re} = 10$ for amplitude ratio $0 \leq \beta \leq 3$ for different phases. As expected, the net force is zero at $\beta = 0$ and appears to approach zero for large β because there is no net force for single-frequency oscillations. Because of these limits, there must be a value of the amplitude ratio that corresponds to the peak force for each phase difference. What is notable is that the location of the extremum occurs at the same amplitude ratio, independent of the phase. In the next section we show that the extremum is achieved at $\beta = 1/\sqrt{2}$.

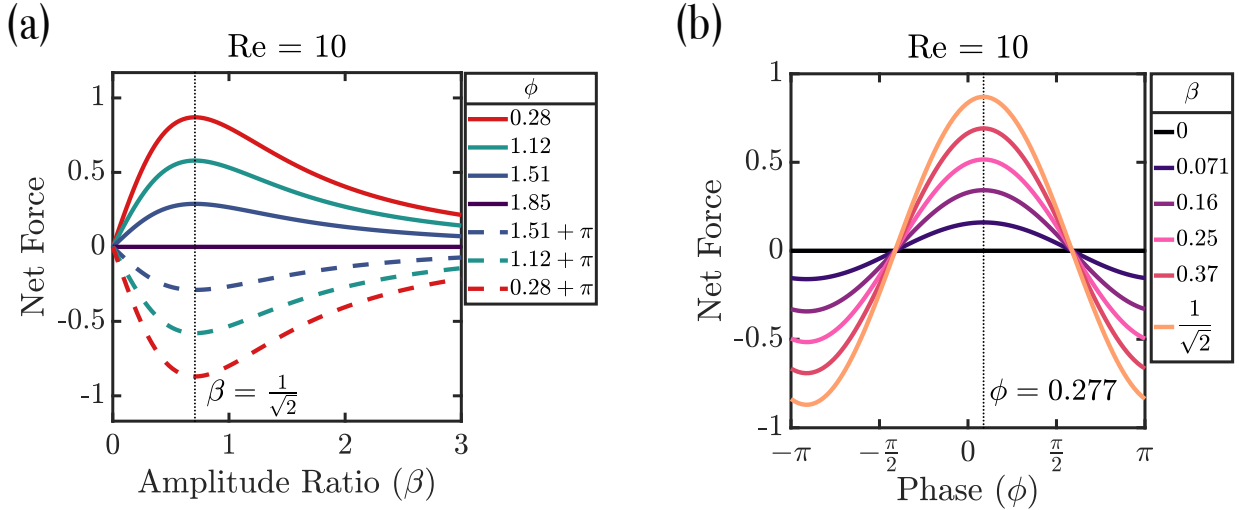


FIGURE 3.3. For $\text{Re} = 10$, net force as a function of (a) amplitude ratio $0 \leq \beta \leq 3$ for varying phase and (b) phase $-\pi \leq \phi \leq \pi$ for varying amplitude ratio. The vertical dashed lines show $\beta_{\max} = 1/\sqrt{2}$ and $\phi_{\max} = 0.277$ for (a) and (b) respectively.

The data also show that the force can be positive or negative, depending on the phase. We note that the value of the force at phases ϕ and $\phi + \pi$ are the negative of one another. We highlight this symmetry between the net force and its corresponding force at a phase shifted by $\Delta\phi = \pi$ with solid and dash dotted-lines in the figure. In the next section, we show how the symmetry results from the form of the asymptotic equation.

Figure 3.3(b) shows the net force as a function of phase $-\pi \leq \phi \leq \pi$ for different amplitude ratios. The curves appear to have a sinusoidal form with a maximal net force at approximately $\phi \approx 0.277$ independent of the amplitude ratio.

The different curves in Figure 3.3(a,b) appear to have the same basic shape. In Figure 3.4(a,b) we normalize the corresponding data by their maximum values, and show that the normalized data collapse onto a single curve. In the next subsection, we derive a form for the dependence on the amplitude ratio, β , by revisiting the asymptotic analysis. We confirm the sinusoidal dependence on phase by fitting the normalized data with a single Fourier mode (fitting error is below 10^{-8}). The resulting fit is

$$(3.53) \quad h(\phi) = \cos(\phi - \phi^*),$$

where $\phi^* \approx 0.277$ for $\text{Re} = 10$, and we later analyze the dependence of ϕ^* on Re .

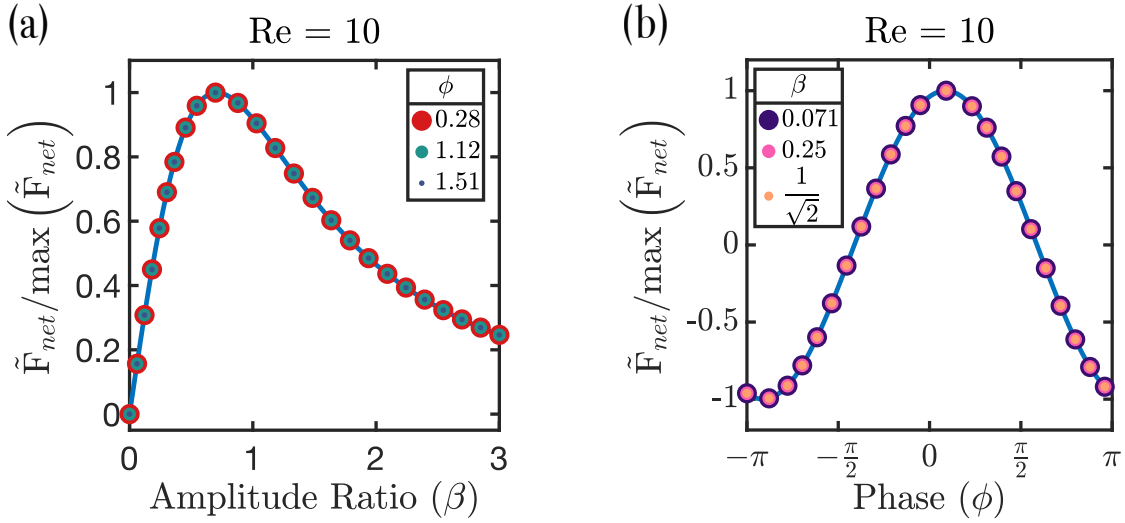


FIGURE 3.4. Scaled by the maximum value, the net force collapses onto a single curve as a function of (a) amplitude ratio $0 \leq \beta \leq 3$ and (b) phase $-\pi \leq \phi \leq \pi$. The solid blue lines represent the analytical functions (a) $f(\beta)$ from (3.58) and (b) $h(\phi)$ from (3.53).

3.3.1.1. Force Scaling with Amplitude Ratio. We aim to understand how the net force depends on the amplitude ratio β by revisiting the low amplitude analysis of Section 3.2. At third order in amplitude, the term involving $d_0(r)$ in the expression of ψ_3 of (3.37) is solely responsible for the net force. The scale of $d_0(r)$ with amplitude depends on the scale of the right-hand side of (3.49),

which involves a product of lower order terms. Thus, we begin by examining how the first and second order solutions, ψ_1 and ψ_2 , change with amplitude ratio β .

At first order, the scale of $a_1(r)$ and $a_2(r)$ in the expression of ψ_1 from (3.35) is determined by the boundary condition (3.41) and (3.42). To incorporate the scaling of these terms with amplitude ratio, we express the first order solution as

$$(3.54) \quad \psi_1 = \Re \left(C_1 \tilde{a}_1(r) e^{-it} + C_2 \tilde{a}_2(r) e^{-i2t} \right) \sin(\theta),$$

where

$$(3.55) \quad C_1 = \frac{1}{\sqrt{1 + \beta^2}}, \quad C_2 = \frac{\beta}{\sqrt{1 + \beta^2}},$$

account for the scaling.

At second order, we compute how each term $b_k(r)$ scales in (3.36) by noting that the right-hand side of (3.43), $\Omega_k(r)$, is obtained from the products of derivatives of ψ_1 . We rewrite the second order solution (3.36) as

$$(3.56) \quad \begin{aligned} \psi_2 &= \Re \left(\sum_{k=1}^4 b_k(r) e^{-ikt} + b_0^{(1)}(r) + b_0^{(2)}(r) \right) \sin(2\theta) \\ &= \Re \left(C_1^2 \left(\tilde{b}_0^{(1)}(r) + \tilde{b}_2(r) e^{-i2t} \right) + C_2^2 \left(\tilde{b}_0^{(2)}(r) + \tilde{b}_2(r) e^{-i4t} \right) \right. \\ &\quad \left. + C_1 C_2 \left(\tilde{b}_1(r) e^{-it} + \tilde{b}_3(r) e^{-i3t} \right) \right) \sin(2\theta). \end{aligned}$$

This expression for ψ_2 with three pairs of terms grouped together by amplitude ratio dependence highlights how the different scalings arise. The first two pairs involving a steady term and an unsteady term correspond to the single-frequency solution from each of the two driving frequencies, and thus, the pairs are proportional to C_1^2 and C_2^2 , correspondingly. The last two unsteady terms arise from the interaction between the driving frequencies, resulting in the scale factor $C_1 C_2$.

At third order the scaling of each term arises from products of the terms in ψ_1 and ψ_2 through the quadratic nonlinearity. Continuing the computation to third order we express ψ_3 as

$$(3.57) \quad \psi_3 = C_1^2 C_2 \left(\tilde{d}_0^{(1)}(r) e^{-it} e^{it} + \tilde{d}_0^{(2)}(r) e^{-i2t} e^{i2t} \right) \sin(\theta) + \dots,$$

where, for simplicity, we omit the terms that have no contribution to the net force. Therefore $d_0(r)$ in the expression of (3.37) is proportional to $C_1^2 C_2$, and thus, the net force from viscous stress

scales like $C_1^2 C_2$. As shown in Appendix B.2, the pressure contribution to the net force has the same scaling.

From this scaling, we define the normalized amplitude ratio dependence of the force as

$$(3.58) \quad f(\beta) = \frac{C_1^2 C_2}{\max_{\beta} (C_1^2 C_2)} = \frac{3\sqrt{3}\beta}{2(1 + \beta^2)^{3/2}},$$

which is plotted in in Figure 3.4(a) along with the normalized data from the numerical computation of the force. This function achieves its maximum value of 1 at $\beta = 1/\sqrt{2}$, and limits to zero as $\beta \rightarrow 0$ and $\beta \rightarrow \infty$.

3.3.1.2. Reversal of the Direction of the Force. We analytically show that the resulting forces generated by oscillation phase difference ϕ and $\phi + \pi$ are in the opposite directions and of the same strength, as suggested by the computational results in Figure 3.3(a). We denote the horizontal velocity on the outer cylinder for amplitude ratio β and phase difference ϕ by $U(t; \beta, \phi)$. We return to the velocity on the boundary from equation (3.9) to examine the effect of a change in phase difference by π :

$$(3.59) \quad \begin{aligned} U(t; \beta, \phi + \pi) &= -\frac{\epsilon}{\sqrt{1 + \beta^2}} (\cos(t) + \alpha\beta \cos(\alpha t + \phi + \pi)) \\ &= -\frac{\epsilon}{\sqrt{1 + \beta^2}} (\cos(t) - \alpha\beta \cos(\alpha t + \phi)) \\ &= U(t; -\beta, \phi). \end{aligned}$$

Thus a change of the phase difference by π is equivalent to replacing β with $-\beta$. From the previous section, the net force scale like $C_1^2 C_2$ where the scale factors are defined in equation (3.55). A change in sign in β results in a change in sign of C_2 . Therefore a change in phase difference of π results in a force of the same strength but of opposite sign.

3.3.2. Net Force for Varying Reynolds Numbers. Figure 3.5 shows the net force as a function of amplitude ratio β and phase ϕ for various Reynolds numbers. In Figure 3.5(a), we compute the force for amplitude ratio $0 \leq \beta \leq 3$ at a fixed phase $\phi = 0$, which shows that the net force increases with Reynolds number. As expected from the the analysis of the previous section, the net force achieves its maximum at $\beta = 1/\sqrt{2}$, and is proportional to $f(\beta)$ from (3.58).

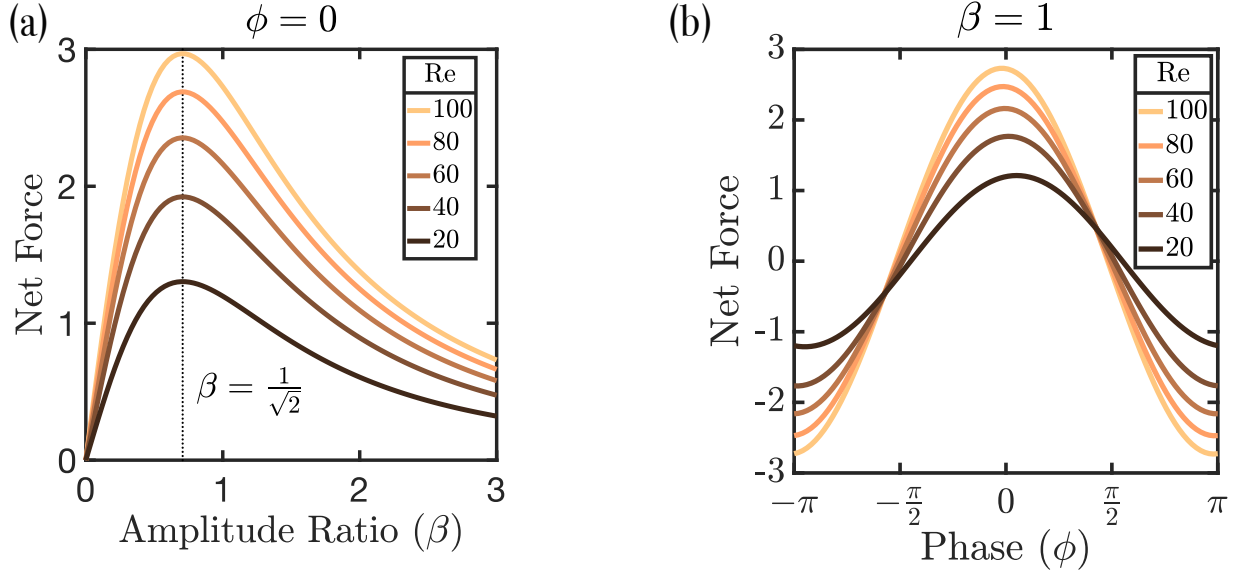


FIGURE 3.5. For varying Reynolds numbers, net force as a function of (a) amplitude ratio $0 \leq \beta \leq 3$ at $\phi = 0$ and (b) phase $-\pi \leq \phi \leq \pi$ at $\beta = 1$. The vertical dashed line in (a) shows $\beta_{max} = 1/\sqrt{2}$.

In Figure 3.5(b), we plot the net force as a function of phase $-\pi \leq \phi \leq \pi$, and we again observe that the force appears sinusoidal. The amplitude and phase at which the maximal force occurs both change with Reynolds number. We fit the function $\cos(\phi - \phi^*)$ to the normalized data for each Reynolds number to obtain how the optimal phase changes with Reynolds number. We plot the function $\phi^*(Re)$ in Figure 3.6(a), which shows that the optimal phase decreases from about 0.16 at Reynolds number 20 to about -0.06 at Reynolds number 100. For this range, the force generated with $\phi = 0$ is less than 2% lower than the force generated with $\phi = \phi^*$.

We let $g(Re)$ denote the maximum force as a function of the Reynolds number, which is found from computing the force at $\beta = 1/\sqrt{2}$ and $\phi = \phi^*(Re)$. Figure 3.6(b) shows that the size of the force scales at a rate little more than $\sqrt{Re}$. Finally, from this analysis we identified that the net force can be expressed as

$$(3.60) \quad F_{net} = \epsilon^3 f(\beta) g(Re) \cos(\phi - \phi^*(Re)),$$

where $f(\beta)$ is defined analytically in (3.58), and the optimal phase $\phi^*(Re)$ and amplitude $g(Re)$ are plotted in Figure 3.6.

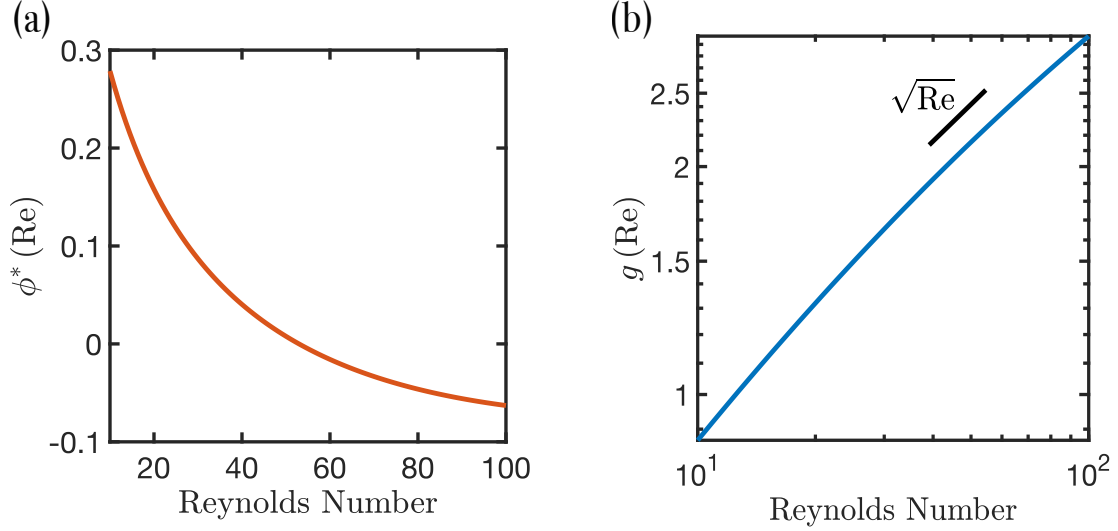


FIGURE 3.6. The optimal phase (a) $\phi^*(\text{Re})$ and maximum of the net force (scaled by ϵ^{-3}) on the cylinder in the annulus (b) $g(\text{Re})$ as a function of the Reynolds number.

In the expression for F_{net} , the amplitude dependence enters through $\epsilon^3 f(\beta)$, which in terms of physical parameters is

$$(3.61) \quad \epsilon^3 f(\beta) = \frac{3\sqrt{3}}{2} \frac{A_1^2 A_2}{R_{\text{in}}}.$$

The optimal amplitude ratio $\beta = 1/\sqrt{2}$ arises from maximizing this expression with $\epsilon = (A_1^2 + A_2^2)^{1/2}/R_{\text{in}}$ fixed. If this quantity were optimized with a different constraint, a different optimal amplitude ratio would result.

3.4. Pumping Through a Channel

In the previous section, we used asymptotic analysis and numerical computation to derive the form of the net force on a cylinder from the streaming flow generated from two-frequency oscillations inside an annular domain. While there is no pumping inside an annulus, we show in this section that our analysis on the net force can be applied to understand the flow through a channel. Specifically, we set up numerical simulations of a cylinder oscillating with two frequencies inside a channel and measure the net-directed flow by computing the flux. We evaluate the flux for select parameters and show that the flux is proportional to the net force generated in the annular domain. Therefore, the results from analysis of the net force from Section 3.3 are directly applicable to pumping.

3.4.1. Measuring the Flux. We use numerical simulations to compute the flow inside a channel of height 8 and length 32 (length scaled by the cylinder radius) with no-slip boundary conditions on top and bottom and periodic conditions in the horizontal direction. The horizontal position of the cylinder's center is given by equation (3.10). We solve for 250 periods, which is sufficiently long to allow the initial transient to decay. Details of the numerical methodology are provided in Appendix C. The flux through a channel of height H is defined as

$$Q(t) = \int_0^H u(x, y, t) dy,$$

and we quantify pumping by the time-averaged flux over the period T :

$$(3.62) \quad \langle Q(t) \rangle = \frac{1}{T} \int_{t-T}^t Q(\tau) d\tau.$$

As shown in Figure 2.3(c), the flux exhibits third order scaling with amplitude.

Our results in Section 3.3 suggest that a cylinder oscillating with two frequencies will generate the largest pumping at amplitude ratio $\beta = 1/\sqrt{2}$. In Figure 3.7(a), we show the flux computed at

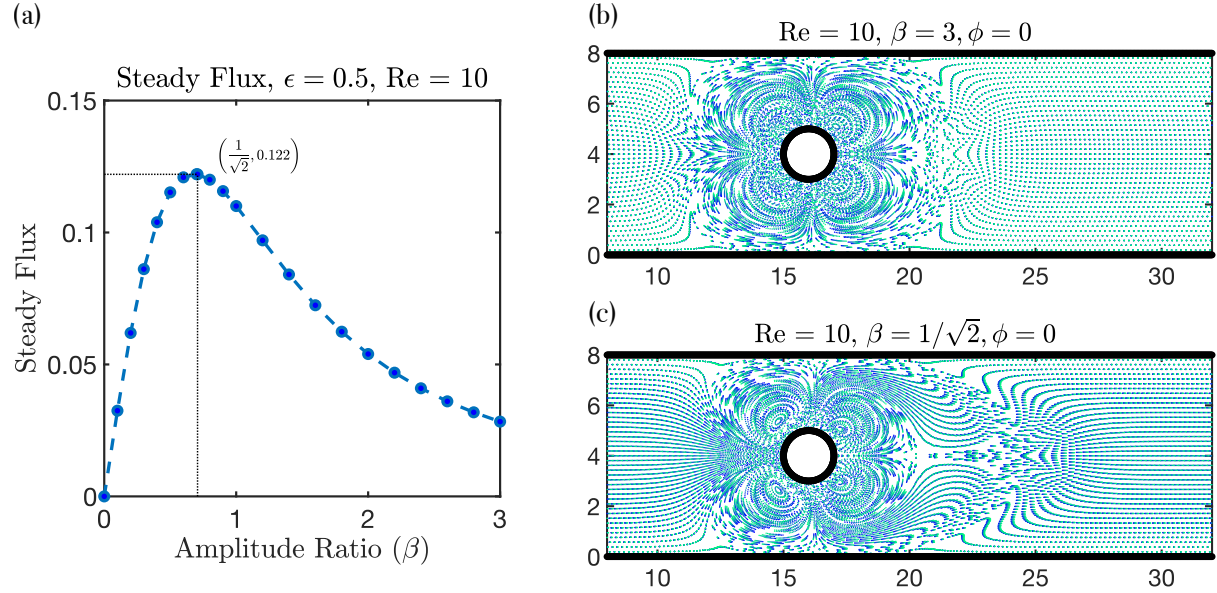


FIGURE 3.7. (a) The time-averaged flux for different amplitude ratios. The maximum flux is achieved at approximately $\beta \approx 1/\sqrt{2}$. Visualization of streaming flows in a 32 by 8 channel at time $T = 250$ with phase $\phi = 0$, amplitude $\epsilon = 0.5$, and $\text{Re} = 10$ for amplitude ratio (b) $\beta = 3$ and (c) $\beta = 1/\sqrt{2}$. Passive tracer particles highlight the positions of the flow from the last 10 periods, where the current location is colored green and the location 10 periods prior is colored blue.

$\phi = 0$, $\epsilon = 0.5$, and $\text{Re} = 10$ for different amplitude ratios, and the flux appears to be maximized at approximately $\beta \approx 1/\sqrt{2}$. The curve through these points resembles the graph of $f(\beta)$ from (3.58) which describes how the net force in the annulus changes with amplitude ratio. In the following section, we analyze the scaling between flux and force for different parameters.

We show the flow field at $\beta = 1/\sqrt{2}$ in Figure 3.7(b) at $\beta = 3$ in Figure 3.7(c) to examine how the flow pattern changes with amplitude ratio. In both cases the net flow is left-to-right – away from the cylinder, the green particles (most current period) are to the right of the blue particles (past period), indicating a net rightward flow. However, the flow for amplitude ratio $\beta = 1/\sqrt{2}$ results in a larger amount of tracer particles moving to the right, consistent with stronger pumping.

3.4.2. Comparing Flux and Net Force. We compare the net force on the cylinder in the annular domain from our analysis with the time-averaged flux through a channel from our simulations. Figure 3.8 shows the net force and time-averaged flux on the same plots as a function of amplitude ratio for fixed phase and as a function of phase for fixed amplitude ratio for two different Reynolds numbers. That flux data from the simulations (dots) show the same response to changes in parameters as the net force computed from the analysis (solid lines), and these plots suggest the flux and force are proportional to one another.

For each numerical simulation, we compute the net force from the analysis for the same amplitude, phase and Reynolds number, and we scatter plot the net force against the steady flux for each Reynolds number in Figure 3.8(c) and (f). These plots show that the flux is proportional to the net force with a proportionality constant that depends on the Reynolds number. Therefore, the results of the analysis on the net force in the annulus can be used to understand how the strength of pumping varies with parameters.

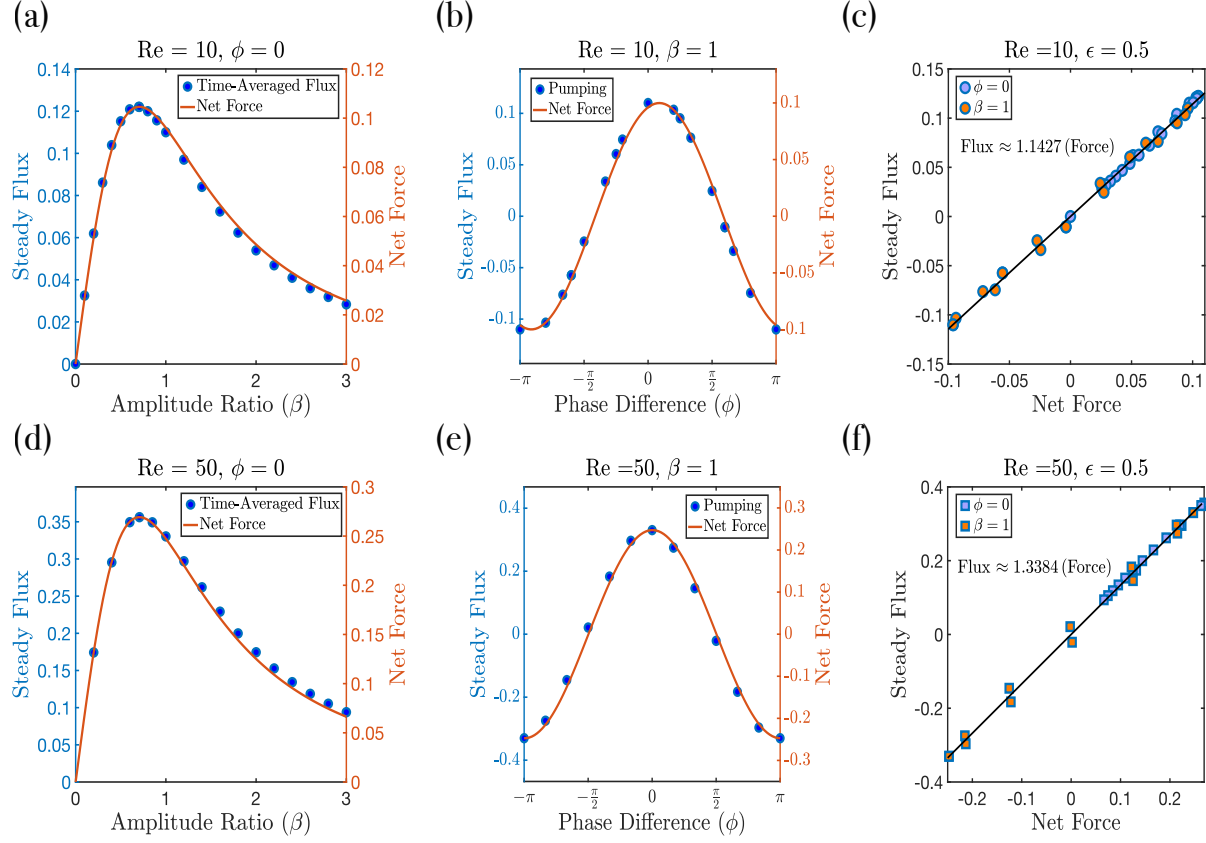


FIGURE 3.8. Net force on the cylinder computed on the annular domain (left axes) and steady time-averaged flux (right axes) through a channel for $\epsilon = 0.5$ for (a,b) $Re = 10$ and (d,e) $Re = 50$ as a function of (a,d) amplitude ratio and (b,e) phase. The net force versus the steady time-averaged flux at (c) $Re = 10$ and (f) $Re = 50$ are proportional to one another. The solid line is the best fit line through the origin.

CHAPTER 4

Discussion

Using asymptotic analysis and numerical simulations, we examined the flow around a cylinder moving with two-frequency oscillations, and we derived necessary conditions for pumping. Strikingly, the necessary conditions on the frequency ratio for non-zero net motion are identical to those observed in the physical system of objects sliding via frictional forces on a laterally vibrating surface [17, 39]. In Hashemi *et al.* [17], they identified the time asymmetry of the driving oscillation necessary for net motion. Specifically, the sliding object only exhibited net motion when the vibrations were *non-antiperiodic*. A 2π -periodic function $f(t)$ is *antiperiodic* if there exists $0 < \phi < 2\pi$ such that

$$(4.1) \quad f(t) = -f(t + \phi).$$

Conversely, $f(t)$ is non-antiperiodic if $f(t)$ does not satisfy Equation (4.1) for any ϕ . Hashemi *et al.* [17] showed that for frequency ratio $\alpha = b/a$, the two-frequency oscillation in (2.1) is antiperiodic if both a and b are odd. Otherwise, the motion is non-antiperiodic; i.e. one of a or b is odd and the other is even. For example, for $\alpha = 1/1$ and $\alpha = 3/1$, the motion is antiperiodic (Equation (4.1) is satisfied for $\phi = \pi$), and for $\alpha = 2/1$ the motion is non-antiperiodic.

To help illustrate the significance of antiperiodicity, in Figure 4.1 we plot the cylinder's horizontal position for $\alpha = 1/1$, $\alpha = 3/1$, and $\alpha = 2/1$ and highlight the paths between the maximum and minimum position. For an antiperiodic oscillation (e.g. $\alpha = 1$ and $\alpha = 3$), there is a symmetry between the path from the right-most to the left-most position (red dashed-line) and the path from the left-most to right-most position (blue dashed dotted-line). Based on this symmetry, one would not expect a net flow. For a non-antiperiodic oscillation (e.g. $\alpha = 2$), the cylinder's motion from the right-most to left-most position is distinct from its counterpart. In our simulations, we observed this motion produced net flow to the right. One may suspect that the direction of the flow results from the larger average velocity along the path from its left-most to right-most position compared to its opposite. This idea is consistent with the observed net flow to the left when $\alpha = -2$. The

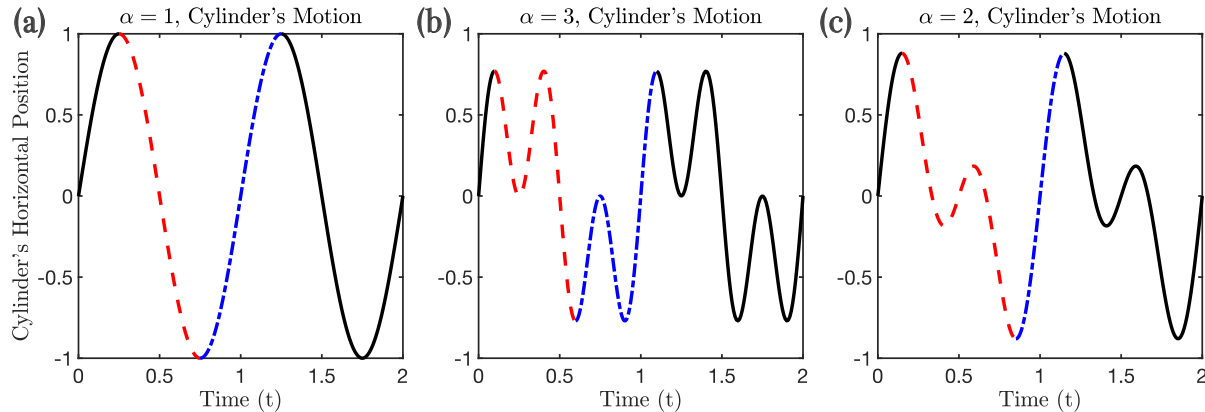


FIGURE 4.1. The displacement of the cylinder's center, defined by Equation (2.1), is plotted from $0 \leq t \leq 2$ for (a) $\alpha = 1$, (b) $\alpha = 3$, and (c) $\alpha = 2$ when $A = 1$. The blue dashed dotted-curve corresponds to the path from the left-most to the right-most position of the cylinder, and the red dashed-curve shows the path from the right-most to the left-most position.

relationship between the waveform and the direction of flow is likely not so simple, and further analysis is required.

In Chapter 2, we not only derived the necessary conditions for pumping but also obtained the smallest order in amplitude at which pumping occurs. Frequency ratio two results in pumping at third order in amplitude, and all other frequency ratios produce pumping at higher order. This result indicates that pumping is strongest for frequency ratio two at small amplitude. In numerical simulations, we observed (Figure 2.7(a)) that the flux for $\alpha = 2$ is at least 10 times larger than the flux from other frequency ratios. These results are consistent with those of Hashemi *et al.* [17] who reported both experimentally and numerically that $\alpha = 2$ generated the largest net motion. Likewise, Reznik & Canny [39] and Hui *et al.* [21] showed in their models that the largest net frictional force was produced when the frequency ratio was $\alpha = 2$.

Some insight into the connection between the flow around an oscillating cylinder and the frictional sliding problem is provided by Hashemi, Gilman & Khair [16] in which they analyze the movement of a particle in a tank of fluid undergoing two-frequency oscillations. The model involves linear and nonlinear drag forces induced by the relative motion of the particle and the surrounding fluid. They perform analysis in the small amplitude limit and show that only frequency ratio $\alpha = 2$ produces a nonzero time-average velocity at first-order in amplitude. Like the streaming problem we analyze, other frequency ratios produce net movement at higher order in amplitude. Hashemi

et al. [16] argues that both models of particle in an oscillating tank and of an object sliding on frictional surface are related to the same reduced model of a forced mechanical system with nonlinear dissipation.

In Chapter 3, we use asymptotic analysis and numerical simulations to analyze the pumping of a cylinder oscillating with two frequencies of different amplitudes and phases. We analyze the flow inside a fixed annular domain with a prescribed oscillatory horizontal velocity of (3.9) on the outer cylinder. While there is no pumping inside an annulus, we compute the net force and show that it has the form (3.60), which separates the effects of changes in amplitude, phase, and Reynolds number. For fixed amplitude, the net force achieves its maximal value at $\beta = 1/\sqrt{2}$, regardless of the phase difference and Reynolds number, and the optimal phase depends on the Reynolds number. The force can be reversed by changing the phase difference by π . By comparing with numerical simulations in a channel, we find that the net force inside an annular domain is proportional to the time-averaged flux through the channel. Thus our analysis of the force in the annulus can be used to understand how pumping varies with different parameters.

In our analysis of the flow between two concentric cylinders, we prescribe a two-frequency velocity at the outer wall and fix frequency ratio as $\alpha = 2$. Lee *et al.* [24] showed that the largest net force occurs at third-order in amplitude for frequency ratio $\alpha = 2$. Other frequency ratios result in a force at higher order, with the next largest being fifth order. Our analysis can be adapted to general frequency ratios; however, we would need to compute the solution structures to at least fifth order and numerically solve an additional number of boundary value problems.

In the frictional sliding problem, Zhang *et al.* [59] analytically and Hui *et al.* [21] experimentally studied the motion of an object sliding on a surface undergoing two-frequency lateral oscillation for different amplitude ratios and the phase differences. Zhang *et al.* [59] showed that the largest net motion occurred at an amplitude ratio $\beta = 1/4$ and at phase difference zero, independent of the friction coefficient. While a different physical system, we showed that the amplitude ratio $\beta = 1/\sqrt{2}$, corresponding to the largest pumping, is independent of Reynolds number, but the optimal phase difference changes with the Reynolds numbers.

In general, steady streaming has been exploited in many microfluidic applications such as drug delivery [48], particle trapping [1, 31, 34, 49, 52, 53, 60], bubble-driven flow [30, 36, 54], mixing [2, 20, 23, 27], and pumping [19, 29, 50, 51, 61]. Microchannel pumps driven by single-frequency

steady streaming rely on design asymmetry of the apparatus to produce a net-directed flow. For example, Tovar & Lee [50] designed a micropump in which cavities of trapped air bubbles were placed at an angle along the channel walls. The angling of the cavities produces an asymmetric streaming flow that results in a net force that drive the flow through the channel. In our work, we have shown that a cylinder vibrating with two-frequency oscillations can pump fluid in either direction. The asymmetry in this case is temporal rather than spatial, and this idea could lead to new designs of microfluidic pumps or potentially enhance the performance of existing designs.

The autonomous propulsion of microparticles is a means of transporting objects by steady streaming [3, 12, 25, 26, 32, 33, 42, 56]. The particle or the background flow vibrates with single-frequency oscillations, and variation in particle shape or density leads to an asymmetric streaming flow that induces a net propulsive force on the particle. The asymptotic analysis of this phenomenon involves computing how the particle asymmetry leads to a net force on itself [12, 32]. Our analysis illustrates how a net force can arise from temporal asymmetry through two-frequency oscillations of a symmetric object. Though our analysis was for a cylinder, it could be extended to a sphere. Based on our results, we predict that symmetric particles could be transported by employing two-frequency oscillations.

In the future, we aim to analyze the motion of fluid particles of different sizes, that are placed inside a channel vibrating with two frequencies. For frequency ratio 2 vibrations, we would expect objects of different sizes to move at different speeds, and this process would allow for the separation of particles. There have been previous works that used single-frequency steady streaming as a method for particle separation [6, 34, 49, 52]; however, we plan to show that separating particles through dual-mode oscillations can be achieved and does not require a spatial asymmetry.

APPENDIX A

Analysis of the Solution Structure

A.1. Spatial Structure of the Solution

We prove Result 2.4.1 in Section 2.4 by proving a more general case involving multifrequency oscillations. Suppose the cylinder is oscillating in the horizontal direction,

$$(A.1) \quad X(t) = \epsilon \sum_{k=1}^K a_k \sin(kt + \phi_k).$$

Then fixing the cylinder, the governing equations for multifrequency oscillation are

$$(A.2) \quad \begin{aligned} \left(\Delta - Re \frac{\partial}{\partial t} \right) \Delta \psi &= \epsilon Re \mathbf{u} \cdot \nabla (\Delta \psi), \\ \psi(r=1) &= 0, \quad \frac{\partial \psi}{\partial r}(r=1) = 0, \\ \psi &\sim \epsilon \sum_{k=-K}^K c_k r e^{ikt} \sin(\theta), \end{aligned}$$

where

$$(A.3) \quad c_k = \frac{-\text{sgn}(k) a_k k e^{i \text{sgn}(k) \phi_k}}{2}, \quad c_0 = 0.$$

Expanding in the small amplitude limit,

$$(A.4) \quad \psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + O(\epsilon^4),$$

the equations at the n^{th} order are

$$(A.5) \quad \Delta \left(\Delta - Re \frac{\partial}{\partial t} \right) \psi_n = \sum_{i+j=n} Re \mathbf{u}_i \cdot \nabla (\Delta \psi_j),$$

$$(A.6) \quad \psi_n(r=1) = 0, \quad \frac{\partial \psi_n}{\partial r}(r=1) = 0,$$

$$(A.7) \quad \psi_n \sim \begin{cases} \sum_{k=-K}^K (c_k r e^{ikt}) \sin(\theta) & n = 1 \\ 0 & n > 1. \end{cases}$$

From the stream function, the velocity is computed from (2.15), and the pressure can be found from substituting the velocity into the momentum equation (2.6).

To prove Result 2.4.1, we prove two lemmas in the following subsections that show that terms proportional to $\sin(\theta)$ and $\cos(\theta)$ only appear at odd order in the expansions for the velocity and pressure, respectively. In Section 2.3.4 we showed that a net force requires such terms, which together with the below lemmas prove Result 2.4.1.

A.1.1. Solution Structure of ψ_n . Equation (A.5) has the general solution

$$(A.8) \quad \psi_n = \sum_{k,m=-\infty}^{\infty} f_{k,m}^n(r) e^{ikt} e^{im\theta},$$

but the boundary conditions limit the solution to a finite number of terms. For example, the far-field boundary condition (A.7) dictates the structure of ψ_1 ,

$$(A.9) \quad \psi_1 = \left(\sum_{k=-K}^K f_{k,1}^1(r) e^{ikt} \right) \sin(\theta) = F_1^1(r, t) \sin(\theta).$$

Furthermore, the solution of ψ_2 is dependent on the quadratic nonlinearity of ψ_1 with itself, resulting in

$$(A.10) \quad \psi_2 = \left(\sum_{k=-2K}^{2K} f_{k,2}^2(r) e^{ikt} \right) \sin(2\theta) = F_2^2(r, t) \sin(2\theta).$$

Generally, the interaction of lower order terms determines the structure of ψ_n , and the following Lemma shows that the solution structure depends on the parity of n .

LEMMA A.1.0.1. For $n \in \mathbb{N}$, the solution of (A.5) with its boundary conditions (A.6) and (A.7) has the form,

$$(A.11) \quad \psi_n \sim \begin{cases} \sum_{l=1}^{n/2} F_{2l}^n(r, t) \sin(2l\theta) & \text{for } n \text{ even} \\ \sum_{l=0}^{(n-1)/2} F_{2l+1}^n(r, t) \sin((2l+1)\theta) & \text{for } n \text{ odd.} \end{cases}$$

Proof: We will proceed with induction, and choose $n = 1$ and $n = 2$ as the base cases. From the boundary conditions (A.6) and (A.7),

$$(A.12) \quad \psi_1 = F_1^1(r, t) \sin(\theta).$$

To solve for ψ_2 , we compute the right-hand side of (A.5),

$$(A.13) \quad Re \mathbf{u}_1 \cdot \nabla (\Delta \psi_1) = \frac{Re}{2} \left(\frac{1}{r} F_1^1(r, t) \cdot \frac{\partial}{\partial r} D_1 F_1^1(r, t) - \frac{1}{r} \frac{\partial}{\partial r} F_1^1(r, t) \cdot D_1 F_1^1(r, t) \right) \sin(2\theta),$$

where

$$(A.14) \quad D_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}.$$

Therefore,

$$(A.15) \quad \psi_2 = F_2^2(r, t) \sin(2\theta).$$

By induction, we assume (A.11) holds true up to ψ_{n-1} , and for simplicity, assume n is odd. Proving the even case involves the same procedure. Note, each term of the right-hand side of (A.5) consists of

$$(A.16) \quad \mathbf{u}_i \cdot \nabla (\Delta \psi_j) = \frac{1}{r} \frac{\partial \psi_i}{\partial \theta} \frac{\partial}{\partial r} (\Delta \psi_j) - \frac{1}{r} \frac{\partial \psi_i}{\partial r} \frac{\partial}{\partial \theta} (\Delta \psi_j),$$

where $i + j = n$. If we show (A.16) is a linear combination of $\{\sin(\theta), \sin(3\theta), \dots, \sin(n\theta)\}$, then we have completed the induction.

Because n is odd, one of i and j is even and the other is odd. Without a loss of generality, choose i even and j odd. Substituting ψ_i and ψ_j into (A.16), we obtain

$$(A.17) \quad \begin{aligned} \mathbf{u}_i \cdot \nabla (\Delta \psi_j) &= \sum_{s=1}^{i/2} \sum_{q=0}^{(j-1)/2} \frac{2s}{r} F_{2s}^i(r, t) \frac{\partial}{\partial r} D_{2q+1} F_{2q+1}^j(r, t) \sin((2q+1)\theta) \cos(2s\theta) \\ &\quad - \sum_{s=1}^{i/2} \sum_{q=0}^{(j-1)/2} \frac{(2q+1)}{r} D_{2q+1} F_{2q+1}^j(r, t) \frac{\partial}{\partial r} F_{2s}^i(r, t) \sin(2s\theta) \cos((2q+1)\theta), \end{aligned}$$

where

$$(A.18) \quad D_m = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2}.$$

We use the identity,

$$(A.19) \quad 2 \sin(a\theta) \cos(b\theta) = \sin((a+b)\theta) + \sin((a-b)\theta), \quad a, b \in \mathbb{N}$$

to simplify Equation (A.17) to

$$(A.20) \quad \mathbf{u}_i \cdot \nabla (\Delta \psi_j) = \sum_{s=1}^{i/2} \sum_{q=0}^{(j-1)/2} A_{s,q}^{i,j-}(r, t) \sin((2q+2s+1)\theta) + A_{s,q}^{i,j+}(r, t) \sin((2q-2s+1)\theta),$$

where

$$(A.21) \quad A_{s,q}^{i,j\pm}(r, t) = \frac{s}{r} F_{2s}^i(r, t) \frac{\partial}{\partial r} D_{2q+1} F_{2q+1}^j(r, t) \pm \frac{(2q+1)}{2r} D_{2q+1} F_{2q+1}^j(r, t) \frac{\partial}{\partial r} F_{2s}^i(r, t).$$

For $1 \leq s \leq i/2$ and $0 \leq q \leq (j-1)/2$,

$$(A.22) \quad 3 \leq |2q+2s+1| \leq i+j = n, \quad 1 \leq |2q-2s+1| \leq \max\{|i-3|, |j-2|\}.$$

In particular,

$$(A.23) \quad 1 \leq |2q+2s+1|, |2q-2s+1| \leq n.$$

Therefore, for some coefficients $F_{2l+1}^n(r, t)$,

$$(A.24) \quad \mathbf{u}_i \cdot \nabla (\Delta \psi_j) = \sum_{l=0}^{(n-1)/2} F_{2l+1}^n(r, t) \sin((2l+1)\theta),$$

which completes the induction.

A.1.2. Solution Structure of p_n . To compute the pressure, we convert the stream function $\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \dots$ into velocity components,

$$(A.25) \quad u_r = \epsilon u_{r,1} + \epsilon^2 u_{r,2} + \epsilon^3 u_{r,3} + O(\epsilon^4),$$

$$(A.26) \quad u_\theta = \epsilon u_{\theta,1} + \epsilon^2 u_{\theta,2} + \epsilon^3 u_{\theta,3} + O(\epsilon^4),$$

where

$$u_{r,i} = \frac{1}{r} \frac{\partial \psi_i}{\partial \theta}, \quad u_{\theta,i} = -\frac{\partial \psi_i}{\partial r}.$$

We expand the pressure in the small amplitude limit,

$$(A.27) \quad p = \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + O(\epsilon^4),$$

and substitute the velocity components (A.26) and the pressure (A.27) into the momentum equation (2.6) to obtain successive equations at each order n ,

$$(A.28) \quad \begin{aligned} \frac{\partial p_n}{\partial r} = & \Delta u_{r,n} - \frac{u_{r,n}}{r^2} - \frac{2}{r^2} \frac{\partial u_{\theta,n}}{\partial \theta} \\ & - Re \frac{\partial u_{r,n}}{\partial t} - Re \sum_{i+j=n} \left(u_{r,i} \frac{\partial u_{r,j}}{\partial r} + \frac{u_{\theta,i}}{r} \frac{\partial u_{r,j}}{\partial \theta} - \frac{u_{\theta,i} u_{\theta,j}}{r} \right), \end{aligned}$$

$$(A.29) \quad \begin{aligned} \frac{1}{r} \frac{\partial p_n}{\partial \theta} = & \Delta u_{\theta,n} - \frac{u_{\theta,n}}{r^2} + \frac{2}{r^2} \frac{\partial u_{r,n}}{\partial \theta} - Re \frac{\partial u_{\theta,n}}{\partial t} \\ & - Re \sum_{i+j=n} \left(u_{r,i} \frac{\partial u_{\theta,j}}{\partial r} + \frac{u_{\theta,i}}{r} \frac{\partial u_{\theta,j}}{\partial \theta} + \frac{u_{r,i} u_{\theta,j}}{r} \right). \end{aligned}$$

For future computations, it is convenient to work with the linear and nonlinear terms separately so that

$$(A.30) \quad \frac{\partial p_n}{\partial r} = L_{r,n} + Re \sum_{i+j=n} N L_{r,i,j},$$

$$(A.31) \quad \frac{1}{r} \frac{\partial p_n}{\partial \theta} = L_{\theta,n} + Re \sum_{i+j=n} N L_{\theta,i,j},$$

where

$$(A.32a) \quad L_{r,n} = \Delta u_{r,n} - \frac{u_{r,n}}{r^2} - \frac{2}{r^2} \frac{\partial u_{\theta,n}}{\partial \theta} - Re \frac{\partial u_{r,n}}{\partial t},$$

$$(A.32b) \quad L_{\theta,n} = \Delta u_{\theta,n} - \frac{u_{\theta,n}}{r^2} + \frac{2}{r^2} \frac{\partial u_{r,n}}{\partial \theta} - Re \frac{\partial u_{\theta,n}}{\partial t}$$

$$(A.32c) \quad NL_{r,i,j} = u_{r,i} \frac{\partial u_{r,j}}{\partial r} + \frac{u_{\theta,i}}{r} \frac{\partial u_{r,j}}{\partial \theta} - \frac{u_{\theta,i} u_{\theta,j}}{r},$$

$$(A.32d) \quad NL_{\theta,i,j} = u_{r,i} \frac{\partial u_{\theta,j}}{\partial r} + \frac{u_{\theta,i}}{r} \frac{\partial u_{\theta,j}}{\partial \theta} + \frac{u_{r,i} u_{\theta,j}}{r}.$$

The structure of the expansion of the pressure is given by the below lemma.

LEMMA A.1.0.2. *For $n \in \mathbb{N}$, the pressure from (A.28) has the form,*

$$(A.33) \quad p_n = \begin{cases} \sum_{l=0}^{n/2} G_{2l}^n(r, t) \cos(2l\theta) & \text{for } n \text{ even} \\ \sum_{l=0}^{(n-1)/2} G_{2l+1}^n(r, t) \cos((2l+1)\theta) & \text{for } n \text{ odd.} \end{cases}$$

Proof: We will examine the case when n is odd, as showing the case when n is even entails the same procedure. To prove the result, we show that r -component of (A.28) can be expressed as a linear combination of $\{\cos(\theta), \cos(3\theta), \dots, \cos(n\theta)\}$ and the θ -component can be expressed as a linear combination of $\{\sin(\theta), \sin(3\theta), \dots, \sin(n\theta)\}$. We show this is true first for the linear terms, and then for the nonlinear terms.

We begin with the linear terms. From Lemma (A.1.0.1), the stream function ψ_n has the form,

$$\psi_n = \sum_{l=1}^{(n-1)/2} F_{2l+1}^n(r, t) \sin((2l+1)\theta).$$

Substituting ψ_n into $L_{r,n}$, we obtain

$$(A.34) \quad L_{r,n} = \sum_{l=1}^{(n-1)/2} M_{2l+1}^n(r, t) \cos((2l+1)\theta),$$

where

$$(A.35) \quad M_m^n(r, t) = -\frac{4m}{r^3} \left(-\frac{r^2}{4} \frac{\partial^2 F_m^n}{\partial r^2} - \frac{r}{4} \frac{\partial F_m^n}{\partial r} + \frac{m^2}{4} F_m^n + \frac{Re}{4} r^2 \frac{\partial F_m^n}{\partial t} \right).$$

Similarly,

$$(A.36) \quad L_{\theta,n} = \sum_{l=1}^{(n-1)/2} S_{2l+1}^n(r, t) \sin((2l+1)\theta),$$

where

$$(A.37) \quad S_s^n(r, t) = -\frac{\partial^3}{\partial r^3} F_s^n - \frac{1}{r} \frac{\partial^2 F_s^n}{\partial r^2} + \left(\frac{s}{r}\right)^2 \frac{\partial F_s^n}{\partial r} + \frac{1}{r^2} \frac{\partial F_s^n}{\partial r} - \frac{2s^2}{r^3} F_s^n + Re \frac{\partial^2 F_s^n}{\partial t \partial r}.$$

We next examine the nonlinear terms $NL_{r,i,j}$ and $NL_{\theta,i,j}$ for $i+j=n$. Because n is odd, one of i or j is even and the other is odd. Without a loss of generality, choose i even and j odd. Using Lemma (A.1.0.1) for the form of the stream function, we substitute the velocity into Equation (A.32c) for $NL_{r,i,j}$ to obtain

$$(A.38) \quad NL_{r,i,j} = \sum_{s=1}^{i/2} \sum_{q=0}^{(j-1)/2} A_{i,j}(r, t) \sin(2s\theta) \sin((2q+1)\theta) + B_{i,j}(r, t) \cos(2s\theta) \cos((2q+1)\theta),$$

where

$$(A.39) \quad A_{i,j}(r, t) = -\frac{1}{r} \frac{\partial}{\partial r} F_{2s}^i(r, t) \frac{\partial}{\partial r} F_{2q+1}^j(r, t) + \frac{(2q+1)^2}{r^2} \frac{\partial}{\partial r} F_{2s}^i(r, t) F_{2q+1}^j(r, t),$$

and

$$(A.40) \quad B_{i,j}(r, t) = \frac{2s}{r^3} (2q+1) F_{2s}^i(r, t) \left(r \frac{\partial}{\partial r} F_{2q+1}^j(r, t) - F_{2q+1}^j(r, t) \right).$$

For Equation (A.38), we invoke the following trigonometric identity. For $a, b \in \mathbb{N}$,

$$(A.41) \quad 2 \sin(a\theta) \sin(b\theta) = \cos((a-b)\theta) - \cos((a+b)\theta),$$

$$(A.42) \quad 2 \cos(a\theta) \cos(b\theta) = \cos((a-b)\theta) + \cos((a+b)\theta).$$

Because $1 \leq s \leq i/2$ and $1 \leq q \leq (j-1)/2$,

$$(A.43) \quad 3 \leq |2q+2s+1| \leq i+j=n, \quad 1 \leq |2q-2s+1| \leq \max\{|i-3|, |j-2|\},$$

and Equation (A.38) is equivalent to

$$(A.44) \quad NL_{r,i,j} = \sum_{l=0}^{(n-1)/2} J_{2l+1}^n(r, t) \cos((2l+1)\theta),$$

for some functions $J_{2l+1}^n(r, t)$.

Similarly, we substitute the velocity into $NL_{\theta,i,j}$ of Equation (A.32d) to obtain

$$(A.45) \quad NL_{\theta,i,j} = \sum_{s=1}^{i/2} \sum_{q=0}^{(j-1)/2} E_{i,j}(r, t) \sin((2q+1)\theta) \cos(2s\theta) + F_{i,j}(r, t) \sin(2s\theta) \cos((2q+1)\theta),$$

where

$$(A.46) \quad E_{i,j}(r, t) = -\frac{2s}{r} F_{2s}^i(r, t) \frac{\partial^2}{\partial r^2} F_{2q+1}^j(r, t) - \frac{2s}{r^2} F_{2s}^i(r, t) \frac{\partial}{\partial r} F_{2q+1}^j(r, t),$$

and

$$(A.47) \quad F_{i,j}(r, t) = \frac{2q+1}{r} \frac{\partial}{\partial r} F_{2s}^i(r, t) \frac{\partial}{\partial r} F_{2q+1}^j(r, t).$$

We simplify Equation (A.45) using the trigonometric identity from Equation (A.19) to obtain

$$(A.48) \quad NL_{\theta,i,j} = \sum_{l=0}^{(n-1)/2} K_{2l+1}^n(r, t) \sin((2l+1)\theta)$$

for some functions $K_{2l+1}^n(r, t)$.

Finally, from (A.34) and (A.44), $\partial p_n / \partial r$ is composed of cosine functions with odd arguments, and similarly, from (A.36) and (A.48), $\partial p_n / \partial \theta$ is composed of sine functions with odd arguments. Integrating with respect to both r and θ , the solution structure of p_n is

$$(A.49) \quad p_n = \sum_{l=0}^{(n-1)/2} G_{2l+1}^n(r, t) \cos((2l+1)\theta).$$

Therefore, Equation (A.33) has been proven.

A.2. Establishing Necessary Conditions for Pumping

In Section 2.3.4, we showed that for frequency ratio $\alpha = 2$, the third order terms proportional to $\sin(\theta)$ in the steady, stream function (and $\cos(\theta)$ in the steady pressure) produce a net force. To extend the argument from Section 2.3.4 to general frequency ratios, we derive necessary conditions for the presence of a steady term proportional to $\sin(\theta)$ in the stream function or proportional to $\cos(\theta)$ in the pressure. Lemmas A.1.0.1 and A.1.0.2 state that $\sin(\theta)$ and $\cos(\theta)$ terms can only appear when n is odd. Therefore, the necessary condition for pumping is equivalent to finding order n so that n is odd, and ψ_n has a steady term.

A.2.1. Frequencies at each order. We first show which frequencies occur at each order.

Suppose the cylinder oscillates in the horizontal direction with the motion

$$(A.50) \quad X(t) = \frac{\epsilon}{2} (\sin(at) + \sin(bt)),$$

where $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$. Examining the first order solution (2.55), ψ_1 involves the terms proportional to $e^{\pm iat}$ and $e^{\pm ibt}$. To solve for the second order solution (2.56), we substitute ψ_1 into the right-hand side of (2.51). From the products of the exponential functions, the frequencies arise at second order are all sums of $\pm a$ and $\pm b$. Generally, the solution structure of ψ_n will be determined by the products of exponentials from lower order solutions. If ψ_n contains a term proportional to e^{ift} , then e^{ift} is related to $e^{\pm iat}$ and $e^{\pm ibt}$ by

$$(A.51) \quad e^{ift} = (e^{iat})^{\xi_1} (e^{-iat})^{\xi_2} (e^{ibt})^{\xi_3} (e^{-ibt})^{\xi_4},$$

and

$$(A.52) \quad \xi_1 + \xi_2 + \xi_3 + \xi_4 = n, \quad \xi_i \geq 0, \quad i = 1, 2, 3, 4.$$

Therefore,

$$(A.53) \quad f = \xi^{(1)} a + \xi^{(2)} b,$$

where

$$(A.54) \quad \xi^{(1)} = (\xi_1 - \xi_2),$$

$$(A.55) \quad \xi^{(2)} = (\xi_3 - \xi_4).$$

The above analysis is based on the structure of equation (2.51) for the stream function. We next consider the pressure and show the pressure at n^{th} order involves the same frequencies as the stream function. To begin, we take the divergence of the Navier Stokes equation, and we use the divergence-free condition to obtain

$$(A.56) \quad \Delta p = -Re \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}).$$

After expanding the solution in power of ϵ , the pressure at n^{th} order satisfies

$$(A.57) \quad \Delta p_n = -Re \sum_{i+j=n} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_j).$$

The right side of this equation involves the same quadratic nonlinearity of the velocity that appears on the right side of equation (2.51) for the stream function. Thus, the stream function and the pressure involve the same frequencies at each order.

A.2.2. Existence of Pumping. We return to equations (A.52)-(A.55) which give the frequencies that occur at each order. A steady solution ($f = 0$) occurs when

$$(A.58) \quad 0 = \xi^{(1)}a + \xi^{(2)}b.$$

Solving this equation for $\xi^{(1)}$, we obtain

$$(A.59) \quad \xi^{(1)} = -\frac{b\xi^{(2)}}{a}.$$

Because $\gcd(a, b) = 1$, and $\xi^{(1)}$ and $\xi^{(2)}$ are integers, solutions to this equation can be expressed as

$$(A.60) \quad \xi^{(1)} = \xi_1 - \xi_2 = cb, \quad \xi^{(2)} = \xi_3 - \xi_4 = -ca, \quad c \in \mathbb{Z}.$$

Using these equations to eliminate ξ_1 and ξ_2 from (A.52) gives the order at which steady solutions occur as

$$(A.61) \quad n = 2(\xi_2 + \xi_3) + c(a + b).$$

We use this last equation to prove Results 2.4.2 and 2.4.3.

A.2.2.1. *Proof of Result 2.4.2.* Let a and b be both odd. From (A.61), because $a + b$ is even, steady terms can only occur at even orders. From Result 2.4.1, there is no pumping.

A.2.2.2. *Proof of Result 2.4.3.* Without a loss of generality, choose a odd and b even. From (A.61), steady terms occur at odd order n for c odd. The smallest odd order n is obtained by choosing $c = 1$ and $\xi_2 = \xi_3 = 0$, which is order $n = a + b$.

APPENDIX B

Computation of the Net Force

B.1. Evaluating the Right-Hand Sides of (3.43), (3.46), and (3.49)

To begin, we substitute ψ_1 from (3.35) into the right-hand side of Equation (3.29), which is a quadratic nonlinearity, and the structure of the right-hand side is defined as

$$(B.1) \quad \text{Re } \mathbf{u}_1 \cdot \nabla (\Delta \psi_1) = \Re \left(\sum_{k=1}^4 \Omega_k(r) e^{-ikt} \right) \sin(2\theta) + \Omega_0(r) \sin(2\theta).$$

We simplify our notation of $\Omega_k(r)$ by defining the following operators,

$$(B.2) \quad Q_1[f(r), g(r), \ell_1, \ell_2] = \frac{\ell_1}{r} \left(f \cdot \frac{d^3 g}{dr^3} \right) + \frac{\ell_1}{r^2} \left(f \cdot \frac{d^2 g}{dr^2} \right) - \frac{\ell_1}{r^3} \left(f \cdot \frac{dg}{dr} \right) - \frac{\ell_1 \ell_2^2}{r^3} \left(f \cdot \frac{dg}{dr} \right) + \frac{2\ell_1 \ell_2^2}{r^4} (f \cdot g),$$

and

$$(B.3) \quad Q_2[f(r), g(r), \ell_2] = -\frac{\ell_2}{r} \left(\frac{df}{dr} \cdot \frac{d^2 g}{dr^2} \right) - \frac{\ell_2}{r^2} \left(\frac{df}{dr} \cdot \frac{dg}{dr} \right) + \frac{\ell_2^3}{r^3} \left(\frac{df}{dr} \cdot g \right).$$

We use the operators (B.2) and (B.3) to define Ω_k as,

$$(B.4) \quad \begin{aligned} \Omega_0 = & \frac{\text{Re}}{8} (Q_1[a_1^*(r), a_1(r), 1, 1] + Q_1[a_1(r), a_1^*(r), 1, 1] \\ & + Q_1[a_2^*(r), a_2(r), 1, 1] + Q_1[a_2(r), a_2^*(r), 1, 1] \\ & + Q_2[a_1^*(r), a_1(r), 1] + Q_2[a_1(r), a_1^*(r), 1] \\ & + Q_2[a_2^*(r), a_2(r), 1] + Q_2[a_2(r), a_2^*(r), 1]), \end{aligned}$$

$$(B.5) \quad \begin{aligned} \Omega_1 = & \frac{\text{Re}}{4} (Q_1[a_1^*(r), a_2(r), 1, 1] + Q_1[a_2(r), a_1^*(r), 1, 1] \\ & + Q_2[a_1^*(r), a_2(r), 1] + Q_2[a_2(r), a_1^*(r), 1]), \end{aligned}$$

$$(B.6) \quad \Omega_2 = \frac{\text{Re}}{4} (Q_1[a_1(r), a_1(r), 1, 1] + Q_2[a_1(r), a_1(r), 1]),$$

$$\begin{aligned}
\Omega_3 &= \frac{\text{Re}}{4} (Q_1[a_1(r), a_2(r), 1, 1] + Q_1[a_2(r), a_1(r), 1, 1] \\
&\quad + Q_2[a_1(r), a_2(r), 1] + Q_2[a_2(r), a_1(r), 1]),
\end{aligned}
\tag{B.7}$$

$$\Omega_4 = \frac{\text{Re}}{4} (Q_1[a_2(r), a_2(r), 1, 1] + Q_2[a_2(r), a_2(r), 1]),
\tag{B.8}$$

where $a_1(r)$ and $a_2(r)$ are the solutions of (3.39)-(3.42). The $*$ denotes the complex conjugate.

Similarly, we substitute ψ_1 and ψ_2 from (3.35) and (3.36) into the right-hand side of (3.32) and obtain its structure as,

$$\begin{aligned}
\text{Re } \mathbf{u}_2 \cdot \nabla (\Delta \psi_1) + \text{Re } \mathbf{u}_1 \cdot \nabla (\Delta \psi_2) &= \Re \left(\sum_{k=1}^6 \rho_k(r) e^{-ikt} \right) \sin(3\theta) + \rho_0(r) \sin(3\theta) \\
&\quad + \Re \left(\sum_{k=1}^6 \xi_k(r) e^{-ikt} \right) \sin(\theta) + \xi_0(r) \sin(\theta).
\end{aligned}
\tag{B.9}$$

It should be noted that we will only solve for $\rho_k(r)$ and $\xi_k(r)$ from (B.9) at $k = 0$, as $\rho_0(r)$ and $\xi_0(r)$ are the terms in the right-hand side that lead to a steady solution at third order. In (3.46)-(3.51), we solve for $d_0(r)$ to compute the net force and solve for $c_0(r)$ as well to generate the contours of the steady flow. Therefore, we will omit the derivation of $\rho_k(r)$ and $\xi_k(r)$ for other k values. In general, we define

$$\xi_k(r) = \tau_k^{(1)} - \tau_k^{(2)}, \quad \rho_k(r) = \tau_k^{(1)} + \tau_k^{(2)},
\tag{B.10}$$

where

$$\begin{aligned}
\tau_0^{(1)} &= \frac{\text{Re}}{8} (Q_1[a_1^*(r), b_1(r), 1, 2] + Q_2[b_1(r), a_1^*(r), 1] \\
&\quad + Q_1[a_1(r), b_1^*(r), 1, 2] + Q_2[b_1^*(r), a_1(r), 1] \\
&\quad + Q_1[a_2^*(r), b_2(r), 1, 2] + Q_2[b_2(r), a_2^*(r), 1] \\
&\quad + Q_1[a_2(r), b_2^*(r), 1, 2] + Q_2[b_2^*(r), a_2(r), 1]),
\end{aligned}
\tag{B.11}$$

and

$$\begin{aligned}
\tau_0^{(2)} &= \frac{\text{Re}}{8} (Q_2[a_2^*(r), b_2(r), 2] + Q_1[b_2(r), a_2^*(r), 2, 1] \\
&+ Q_2[a_2(r), b_2^*(r), 2] + Q_1[b_2^*(r), a_2(r), 2, 1] \\
&+ Q_2[a_1(r), b_1^*(r), 2] + Q_1[b_1^*(r), a_1(r), 2, 1] \\
&+ Q_2[a_1^*(r), b_1(r), 2] + Q_1[b_1(r), a_1^*(r), 2, 1]).
\end{aligned}
\tag{B.12}$$

$a_k(r)$ and $b_k(r)$ are the solutions of (3.39)-(3.42) and (3.43)-(3.45), correspondingly.

B.2. Equations for Pressure

In this section, we obtain the equations at third order that involve a steady pressure, which contributes a net force. To compute the pressure p , we convert our stream function,

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + O(\epsilon^4) \tag{B.13}$$

into velocity components,

$$u_r = \epsilon u_{r,1} + \epsilon^2 u_{r,2} + \epsilon^3 u_{r,3} + O(\epsilon^4), \tag{B.14}$$

$$u_\theta = \epsilon u_{\theta,1} + \epsilon^2 u_{\theta,2} + \epsilon^3 u_{\theta,3} + O(\epsilon^4), \tag{B.15}$$

where

$$u_{r,n} = \frac{1}{r} \frac{\partial \psi_n}{\partial \theta}, \quad u_{\theta,n} = -\frac{\partial \psi_n}{\partial r}. \tag{B.16}$$

We expand the pressure in the limit of small amplitude,

$$p = \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + O(\epsilon^4), \tag{B.17}$$

and substitute the u_r , u_θ , and p of Equations (B.14)-(B.17) into the momentum equation,

$$\frac{\partial p}{\partial r} = \Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \text{Re} \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} \right), \tag{B.18}$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \text{Re} \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} \right). \tag{B.19}$$

Consequently, we obtain successive equations at each order:

First Order

$$(B.20) \quad \frac{\partial p_1}{\partial r} = \Delta u_{r,1} - \frac{u_{r,1}}{r^2} - \frac{2}{r^2} \frac{\partial u_{\theta,1}}{\partial \theta} - \text{Re} \frac{\partial u_{r,1}}{\partial t},$$

$$(B.21) \quad \frac{1}{r} \frac{\partial p_1}{\partial \theta} = \Delta u_{\theta,1} - \frac{u_{\theta,1}}{r^2} + \frac{2}{r^2} \frac{\partial u_{r,1}}{\partial \theta} - \text{Re} \frac{\partial u_{\theta,1}}{\partial t},$$

Second Order

$$(B.22) \quad \frac{\partial p_2}{\partial r} = \Delta u_{r,2} - \frac{u_{r,2}}{r^2} - \frac{2}{r^2} \frac{\partial u_{\theta,2}}{\partial \theta} - \text{Re} \frac{\partial u_{r,2}}{\partial t} - \text{Re} \left(u_{r,1} \frac{\partial u_{r,1}}{\partial r} + \frac{u_{\theta,1}}{r} \frac{\partial u_{r,1}}{\partial \theta} - \frac{u_{\theta,1}^2}{r} \right),$$

$$(B.23) \quad \frac{1}{r} \frac{\partial p_2}{\partial \theta} = \Delta u_{\theta,2} - \frac{u_{\theta,2}}{r^2} + \frac{2}{r^2} \frac{\partial u_{r,2}}{\partial \theta} - \text{Re} \frac{\partial u_{\theta,2}}{\partial t} - \text{Re} \left(u_{r,1} \frac{\partial u_{\theta,1}}{\partial r} + \frac{u_{\theta,1}}{r} \frac{\partial u_{\theta,1}}{\partial \theta} + \frac{u_{r,1} u_{\theta,1}}{r} \right),$$

Third Order

$$(B.24) \quad \begin{aligned} \frac{\partial p_3}{\partial r} = & \Delta u_{r,3} - \frac{u_{r,3}}{r^2} - \frac{2}{r^2} \frac{\partial u_{\theta,3}}{\partial \theta} - \text{Re} \frac{\partial u_{r,3}}{\partial t} - \text{Re} \left(u_{r,2} \frac{\partial u_{r,1}}{\partial r} + \frac{u_{\theta,2}}{r} \frac{\partial u_{r,1}}{\partial \theta} - \frac{u_{\theta,2} u_{\theta,1}}{r} \right) \\ & - \text{Re} \left(u_{r,1} \frac{\partial u_{r,2}}{\partial r} + \frac{u_{\theta,1}}{r} \frac{\partial u_{r,2}}{\partial \theta} - \frac{u_{\theta,1} u_{\theta,2}}{r} \right), \end{aligned}$$

$$(B.25) \quad \begin{aligned} \frac{1}{r} \frac{\partial p_3}{\partial \theta} = & \Delta u_{\theta,3} - \frac{u_{\theta,3}}{r^2} + \frac{2}{r^2} \frac{\partial u_{r,3}}{\partial \theta} - \text{Re} \frac{\partial u_{\theta,3}}{\partial t} - \text{Re} \left(u_{r,2} \frac{\partial u_{\theta,1}}{\partial r} + \frac{u_{\theta,2}}{r} \frac{\partial u_{\theta,1}}{\partial \theta} + \frac{u_{r,2} u_{\theta,1}}{r} \right) \\ & - \text{Re} \left(u_{r,1} \frac{\partial u_{\theta,2}}{\partial r} + \frac{u_{\theta,1}}{r} \frac{\partial u_{\theta,2}}{\partial \theta} + \frac{u_{r,1} u_{\theta,2}}{r} \right). \end{aligned}$$

We convert ψ_1 from (3.35) as velocity components, $u_{r,1}$ and $u_{\theta,1}$, and substitute the values into the right-hand side of (B.20) and (B.21) to obtain the solution structure of p_1 as

$$(B.26) \quad p_1 = \Re (A_1(r) e^{-it} + A_2(r) e^{-i2t}) \cos(\theta).$$

Similarly, we use both ψ_1 and ψ_2 from (3.35) and (3.36) to obtain the right-hand side of (B.22) and (B.23). We define the form of the solution at second order as,

$$(B.27) \quad p_2 = \left(\sum_{k=1}^4 B_k(r) e^{-ikt} \right) \cos(2\theta) + B_0(r) \cos(2\theta) + \left(\sum_{k=1}^4 E_k(r) e^{-ikt} \right) + E_0(r).$$

Finally, we use ψ_1 , ψ_2 , and ψ_3 from (3.35)-(3.37) to obtain the right-hand side of (B.24) and (B.25) and derive the solution structure of p_3 as,

$$(B.28) \quad p_3 = \left(\sum_{k=1}^6 C_k(r) e^{-ikt} \right) \cos(3\theta) + C_0(r) \cos(3\theta) + \left(\sum_{k=1}^6 D_k(r) e^{-ikt} \right) \cos(\theta) + D_0(r) \cos(\theta).$$

From the net force computation of (2.46), $D_0(r)$ is the only term in p_3 that contributes a net force. From the right-hand side of (B.24) and (B.25), the $D_0(r)$ is involved in a PDE that linearly depends on ψ_3 and contains a quadratic nonlinearity of ψ_1 and ψ_2 . Specifically, we define a linear operators of the form,

$$(B.29) \quad L_1[f] = r \frac{d^3 f}{dr^3} + \frac{d^2 f}{dr^2} - \frac{2}{r} \frac{df}{dr} + \frac{2}{r^2} f,$$

and

$$(B.30) \quad L_2[f] = \frac{1}{r} \frac{d^2 f}{dr^2} + \frac{1}{r^2} \frac{df}{dr} - \frac{1}{r^3} f.$$

We also define the nonlinear operators as

$$(B.31) \quad NL_1^{m,n}[f, g] = \text{sgn}(n - m) \left(\frac{m}{2} f \frac{d^2 g}{dr^2} + \frac{n}{2} \frac{df}{dr} \frac{dg}{dr} + \frac{m}{2r} f \frac{dg}{dr} \right),$$

and

$$(B.32) \quad NL_2^{m,n}[f, g] = \frac{mn}{2r^2} f \frac{dg}{dr} - \frac{mn}{2r^3} f g + \frac{n^2}{2r^2} \frac{df}{dr} g - \frac{1}{2r} \frac{df}{dr} \frac{dg}{dr}.$$

Substituting (B.28) and (3.35)-(3.37) into (B.25), we obtain

$$(B.33) \quad D_0(r) = L_1[d_0] - \text{Re } NL^{(1)},$$

where

$$(B.34) \quad \begin{aligned} NL^{(1)} = & \frac{1}{4} \left(NL_1^{2,1} [b_2, a_2^*] + NL_1^{2,1} [b_1, a_1^*] + NL_1^{2,1} [b_2^*, a_2] + NL_1^{2,1} [b_1^*, a_1] + NL_1^{1,2} [a_2, b_2^*] \right. \\ & \left. + NL_1^{1,2} [a_1, b_1^*] + NL_1^{1,2} [a_2^*, b_2] + NL_1^{1,2} [a_1^*, b_1] \right). \end{aligned}$$

B.3. Numerical Solutions of (3.39)-(3.51)

We compute the solutions of Equations (3.39)-(3.51) using pseudo-spectral methods. The differentiation matrices are constructed by using Chebyshev polynomials, which are defined on $s \in [-1, 1]$. We solve the problem on $r \in [1, R]$, and thus, we apply a linear transformation,

$$(B.35) \quad r = 1 + \frac{(R-1)(s+1)}{2},$$

which maps $s = -1$ and $s = 1$ to $r = 1$ and $r = R$, correspondingly. Moreover, the derivative with respect r is related to those of s by

$$(B.36) \quad \frac{d}{dr} = \frac{2}{R-1} \frac{d}{ds}.$$

B.3.1. First Order Computation. In the following, we carry out the steps to compute solution $a_k(r)$ of Equations (3.39)-(3.42) for $k = 1$ and $k = 2$. Expanding the differential operator (3.38) in Equation (3.39), we obtain

$$(B.37) \quad a_k^{(4)} + \frac{2}{r} a_k^{(3)} + \left(ik \operatorname{Re} - \frac{3}{r^2} \right) a_k'' + \left(\frac{ik \operatorname{Re}}{r} + \frac{3}{r^3} \right) a_k' + \left(-\frac{ik \operatorname{Re}}{r^2} - \frac{3}{r^4} \right) a_k = 0.$$

Substituting the derivatives in Equation (B.37) with derivatives (B.36) that are dependent on $s \in [-1, 1]$, we obtain

$$(B.38) \quad \begin{aligned} & \frac{16}{(R-1)^4} a_k^{(4)} + \frac{16}{(R-1)^3 r} a_k^{(3)} + \left(\frac{ik \operatorname{Re}}{(R-1)^2} - \frac{12}{(R-1)^2 r^2} \right) a_k'' \\ & + \left(\frac{ik \operatorname{Re}}{(R-1)r} + \frac{6}{(R-1)r^3} \right) a_k' + \left(-\frac{3}{r^4} - \frac{ik \operatorname{Re}}{r^2} \right) a_k = 0. \end{aligned}$$

Because of the inhomogeneous boundary conditions at $r = R$ or $s = 1$ at (3.41) and (3.42), we define a polynomial interpolant, $p_k(s) = a_k(s) - v_k(s)$ where $v_k(s)$ satisfies homogeneous boundary conditions. To simplify the notation, we define the constants $c_1^{(k)}$ and $c_2^{(k)}$ where

$$(B.39) \quad c_1^{(1)} = -\frac{R}{\sqrt{1+\beta^2}}, \quad c_2^{(1)} = -\frac{1}{\sqrt{1+\beta^2}},$$

and

$$(B.40) \quad c_1^{(2)} = -\frac{R 2\beta e^{-i\phi}}{\sqrt{1+\beta^2}}, \quad c_2^{(2)} = -\frac{2\beta e^{-i\phi}}{\sqrt{1+\beta^2}},$$

which implies that $p_k(s)$ has the form,

$$(B.41) \quad p_k(s) = \left(\frac{c_2^{(k)}}{4} - \frac{c_1^{(k)}}{4} \right) s^3 + \frac{c_2^{(k)}}{4} s^2 + \left(\frac{3c_1^{(k)}}{4} - \frac{c_2^{(k)}}{4} \right) s + \left(\frac{c_1^{(k)}}{2} - \frac{c_2^{(k)}}{4} \right).$$

Equation (B.38) in terms of $v(s)$ results in

$$(B.42) \quad \begin{aligned} & \frac{16}{(R-1)^4} v_k^{(4)} + \frac{16}{(R-1)^3 r} v_k^{(3)} + \left(\frac{i4k \operatorname{Re}}{(R-1)^2} - \frac{12}{(R-1)^2 r^2} \right) v_k'' \\ & + \left(\frac{i2k \operatorname{Re}}{(R-1)r} + \frac{6}{(R-1)r^3} \right) v_k' + \left(-\frac{3}{r^4} - \frac{ik \operatorname{Re}}{r^2} \right) v_k = \gamma_k, \end{aligned}$$

where $\gamma_k = \delta_1 c_1^{(k)} + \delta_2 c_2^{(k)}$ and

$$(B.43) \quad \begin{aligned} \delta_1 = & \left(-\frac{3}{4r^4} - \frac{ik \operatorname{Re}}{4r^2} \right) s^3 + \left(\frac{9}{2r^3(R-1)} + \frac{i3k \operatorname{Re}}{2r(R-1)} \right) s^2 \\ & + \left(\frac{9}{4r^4} - \frac{18}{r^2(R-1)^2} + \frac{i6k \operatorname{Re}}{(R-1)^2} + \frac{i3k \operatorname{Re}}{4r^2} \right) s \\ & - \frac{9}{2r^3(R-1)} + \frac{24}{r(R-1)^3} - \frac{i3k \operatorname{Re}}{2r(R-1)} + \frac{ik \operatorname{Re}}{2r^2} + \frac{3}{2r^4}, \end{aligned}$$

$$(B.44) \quad \begin{aligned} \delta_2 = & \left(\frac{3}{4r^4} + \frac{ik \operatorname{Re}}{4r^2} \right) s^3 + \left(\frac{3}{4r^4} - \frac{9}{2r^3(R-1)} + \frac{ik \operatorname{Re}}{4r^2} - \frac{i3k \operatorname{Re}}{2r(R-1)} \right) s^2 \\ & + \left(-\frac{3}{4r^4} - \frac{3}{r^3(R-1)} + \frac{18}{r^2(R-1)^2} - \frac{ik \operatorname{Re}}{4r^2} - \frac{i6k \operatorname{Re}}{(R-1)^2} - \frac{ik \operatorname{Re}}{r(R-1)} \right) s \\ & + \frac{3}{2r^3(R-1)} + \frac{6}{r^2(R-1)^2} - \frac{24}{r(R-1)^3} - \frac{ik \operatorname{Re}}{4r^2} - \frac{3}{4r^4} - \frac{i2k \operatorname{Re}}{(R-1)^2} + \frac{ik \operatorname{Re}}{2r(R-1)}. \end{aligned}$$

Finally, we define

$$(B.45) \quad v_k(s) = (1-s^2)q_k(s),$$

which limits the number of boundary conditions to two:

$$(B.46) \quad q_k(-1) = 0, \quad q_k(1) = 0,$$

and Equation (B.42) becomes

$$(B.47) \quad \sigma_4 q_k^{(4)} + \sigma_3 q_k^{(3)} + \sigma_2 q_k'' + \sigma_1 q_k' + \sigma_0 q_k = \gamma_k,$$

where

$$(B.48) \quad \sigma_4 = \frac{16(1-s^2)}{(R-1)^4},$$

$$(B.49) \quad \sigma_3 = -\frac{128s}{(R-1)^4} + \frac{16(1-s^2)}{r(R-1)^3},$$

$$(B.50) \quad \begin{aligned} \sigma_2 = & \left(\frac{12}{r^2(R-1)^2} - \frac{i4k \operatorname{Re}}{(R-1)^2} \right) s^2 - \left(\frac{96s}{r(R-1)^3} \right) s \\ & - \frac{192}{(R-1)^4} - \frac{12}{r^2(R-1)^2} + \frac{i4k \operatorname{Re}}{(R-1)^2}, \end{aligned}$$

$$(B.51) \quad \begin{aligned} \sigma_1 = & \left(-\frac{6}{r^3(R-1)} - \frac{i2k \operatorname{Re}}{r(R-1)} \right) s^2 + \left(\frac{48}{r^2(R-1)^2} - \frac{i16k \operatorname{Re}}{(R-1)^2} \right) s \\ & - \frac{96}{r(R-1)^3} + \frac{6}{r^3(R-1)} + \frac{i2k \operatorname{Re}}{r(R-1)}, \end{aligned}$$

$$(B.52) \quad \begin{aligned} \sigma_0 = & \left(\frac{3}{r^4} + \frac{ik \operatorname{Re}}{r^2} \right) s^2 + \left(-\frac{12}{r^3(R-1)} - \frac{i4k \operatorname{Re}}{r(R-1)} \right) s \\ & + \frac{24}{r^2(R-1)^2} - \frac{i8k \operatorname{Re}}{(R-1)^2} - \frac{3}{r^4} - \frac{ik \operatorname{Re}}{r^2}. \end{aligned}$$

Constructing the differentiation matrices with two homogeneous boundary conditions, we solve for q_k by using MATLAB's backslash command and obtain a_k by substituting q_k into

$$(B.53) \quad a_k(s) = p_k(s) + (1-s^2)q_k(s).$$

B.3.2. Second and Third Order Solutions. The computations required to solve $b_k(r)$, $c_k(r)$, and $d_k(r)$ of Equations (3.43)-(3.45), (3.46)-(3.48), and (3.49)-(3.51) are equivalent to each other.

For simplicity, we solve for $f_{k,\ell}$, which is a solution of

$$(B.54) \quad D_\ell (D_\ell + ik \operatorname{Re}) f_{k,\ell}(r) = \text{RHS},$$

$$(B.55) \quad f_{k,\ell}(1) = 0, \quad f'_{k,\ell}(1) = 0,$$

$$(B.56) \quad f_{k,\ell}(R) = 0, \quad f'_{k,\ell}(R) = 0.$$

It should be noted that RHS of Equation (B.54) denotes the right-hand side of Equations (3.43), (3.46), and (3.49), i.e., $\Omega_k(r)$, $\rho_k(r)$, and $\xi_k(r)$. We apply a linear transformation (B.35) that results Equation (B.54) to have derivatives of $s \in [-1, 1]$:

$$(B.57) \quad a_4^{k,\ell} f_{k,\ell}^{(4)}(s) + a_3^{k,\ell} f_{k,\ell}^{(3)}(s) + a_2^{k,\ell} f_{k,\ell}^{(2)}(s) + a_1^{k,\ell} f_{k,\ell}^{(1)}(s) + a_0^{k,\ell} f_{k,\ell}^{(0)}(s) = \text{RHS},$$

where

$$(B.58) \quad a_4^{k,\ell} = \frac{16}{(R-1)^4},$$

$$(B.59) \quad a_3^{k,\ell} = \frac{16}{r(R-1)^3},$$

$$(B.60) \quad a_2^{k,\ell} = \frac{4}{(R-1)^2} \left(\frac{ikr^2 \operatorname{Re} - 2\ell^2 - 1}{r^2} \right),$$

$$(B.61) \quad a_1^{k,\ell} = \frac{2}{R-1} \left(\frac{ikr^2 \operatorname{Re} + 2\ell^2 + 1}{r^3} \right),$$

$$(B.62) \quad a_0^{k,\ell} = -\frac{\ell^2 (ikr^2 \operatorname{Re} - \ell^2 + 4)}{r^4}.$$

Due to the homogeneous boundary conditions of Equations (B.55) and (B.56), a polynomial interpolant is not needed. We limit the boundary conditions to two by defining

$$(B.63) \quad f_{k,\ell}(s) = (1 - s^2)q_{k,\ell}(s).$$

Then, we write Equation (B.57) as

$$(B.64) \quad b_4^{k,\ell} q_{k,\ell}^{(4)}(s) + b_3^{k,\ell} q_{k,\ell}^{(3)}(s) + b_2^{k,\ell} q_{k,\ell}^{(2)}(s) + b_1^{k,\ell} q_{k,\ell}^{(1)}(s) + b_0^{k,\ell} q_{k,\ell}^{(0)}(s) = \text{RHS},$$

where

$$(B.65) \quad b_4^{k,\ell} = a_4^{k,\ell}(-s^2 + 1),$$

$$(B.66) \quad b_3^{k,\ell} = -8s a_4^{k,\ell} + a_3^{k,\ell}(-s^2 + 1),$$

$$(B.67) \quad b_2^{k,\ell} = -12 a_4^{k,\ell} - 6s a_3^{k,\ell} + a_2^{k,\ell}(-s^2 + 1),$$

$$(B.68) \quad b_1^{k,\ell} = -6 a_3^{k,\ell} - 4s a_2^{k,\ell} + a_1^{k,\ell}(-s^2 + 1),$$

$$(B.69) \quad b_0^{k,\ell} = -2 a_2^{k,\ell} - 2s a_1^{k,\ell} + a_0^{k,\ell}(-s^2 + 1).$$

Equation (B.64) is computed by using MATLAB's backslash command, and we obtain our solution $f_{k,\ell}(s)$ by plugging in $q_{k,\ell}(s)$ in Equation (B.63).

B.4. Numerical Validation of the Net Force

From the asymptotic analysis of Section 3.2, we compute the steady solutions up to third order in amplitude and obtain the net force. We validate the results from our analysis by comparing with the numerical solution of the incompressible, time-dependent Navier Stokes equations on an annular domain with a horizontal velocity of strength (3.9) on the outer cylinder. The numerical solutions were generated by using the Dedalus software [9], which is an open-sourced library that uses a parallelized framework to solve PDE's with spectral methods. The velocity is computed on a Fourier-Chebyshev mesh with 256 Fourier modes and 512 Chebyshev points. We solve for the flow to time $T = 105$ to allow the initial transient to decay and compute the steady flow by taking the time-average of the last 5 periods.

Figure B.1 shows the stream functions computed from the asymptotic analysis and from the numerical solution of the Navier Stokes equations at amplitude ratio $\beta = 1$, phase difference $\phi = 0$, $\epsilon = \sqrt{0.02} \approx 0.14$, and $\text{Re} = 50$. The relative difference between two stream functions is approximately 6.11%, and the relative difference of the net force is about 12.45%. For smaller amplitude $\epsilon = \sqrt{0.0002}$, the relative error from the stream function and the net force is around 0.0618% and 0.12%, correspondingly.

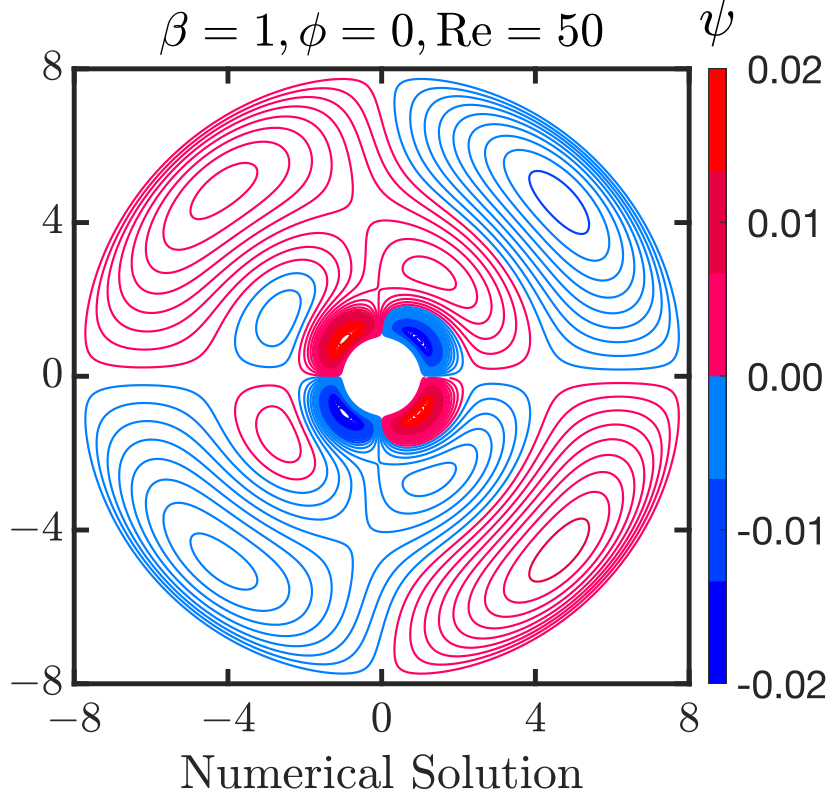


FIGURE B.1. Contour of the numerical solution is shown at amplitude ratio $\beta = 1$, phase difference $\phi = 0$, $\epsilon = \sqrt{0.02} \approx 0.14$, and $\text{Re} = 50$.

B.5. Analytical and Numerical Solutions at First and Second Order

In (3.49), the steady solution that is involved in the computation of the net force is $d_0(r)$, and the value of this term depends on the right-hand side $\xi_0(r)$. Note that $\xi_0(r)$ is the sum of products of interacting terms, $a_1(r)$ and $a_2(r)$ at first order and $b_1(r)$ and $b_2(r)$ at second order. The analytical solutions of the equations at first and second order are solved in [11, 18]. We perform a refinement study using these analytical solutions to validate our computational method.

B.5.1. Analytical Solution.

B.5.1.1. *First Order Equations* (3.39)-(3.42) for $k = 1$ and $k = 2$. The solutions of (3.39)-(3.42) involve Hankel functions, which are complex conjugate pairs of Bessel functions [57]. For simplicity, we define v_k for $k = 1$ and $k = 2$ as,

$$(B.70) \quad v_k = \frac{k\beta^{k-1}e^{-i\phi(k-1)}}{\sqrt{1+\beta^2}}.$$

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Given

$$(B.71) \quad \gamma_k = \sqrt{ik \operatorname{Re}},$$

we define

$$(B.72) \quad c^{(1)} = H_2^{(1)}(\gamma_k) - R^2 H_2^{(1)}(\gamma_k R), \quad c^{(2)} = H_2^{(2)}(\gamma_k) - R^2 H_2^{(2)}(\gamma_k R),$$

$$(B.73) \quad M = \left(H_0^{(1)}(\gamma_k) - H_0^{(1)}(\gamma_k R) \right) c^{(2)} - \left(H_0^{(2)}(\gamma_k) - H_0^{(2)}(\gamma_k R) \right) c^{(1)},$$

where R is the ratio of the two radii in the annular domain. Then, we can write $a_k(r)$, the solution of (3.39)-(3.42) as,

$$(B.74) \quad a_k(r) = AH_1^{(1)}(\gamma_k r) + BH_1^{(2)}(\gamma_k r) + Cr + \frac{D}{r},$$

where

$$(B.75) \quad A = \frac{2v_k}{\gamma} \frac{c_2}{M}, \quad B = -\frac{2v_k}{\gamma} \frac{c_1}{M},$$

$$(B.76) \quad C = -\frac{A\gamma_k H_0^{(1)}(\gamma_k)}{2} - \frac{B\gamma_k H_0^{(2)}(\gamma_k)}{2}, \quad D = -\frac{A\gamma_k H_2^{(1)}(\gamma_k)}{2} - \frac{B\gamma_k H_2^{(2)}(\gamma_k)}{2}.$$

B.5.1.2. *Steady Second Order Equations* (3.43)-(3.45). The steady analytic solution at second order in amplitude is composed of a homogeneous and a particular solution,

$$(B.77) \quad b_k(r) = b_k^{(h)}(r) + b_k^{(p)}(r).$$

For $k = 0$, we obtain the form of the solution as,

$$(B.78) \quad b_0^{(h)}(r) = A_0 r^4 + B_0 r^2 + C_0 + \frac{D_0}{r^2}$$

and

$$(B.79) \quad b_0^{(p)}(r) = r^4 \int_1^r \frac{W_{0,4}(x)}{W_0(x)} dx + r^2 \int_1^r \frac{W_{0,3}(x)}{W_0(x)} dx + \int_1^r \frac{W_{0,2}(x)}{W_0(x)} dx + \frac{1}{r^2} \int_1^r \frac{W_{0,1}(x)}{W_0(x)} dx,$$

where the unknown coefficients in (B.78) are chosen to satisfy the boundary conditions and the Wronskian determinants in (B.79) are

$$(B.80) \quad W_0(x) = \frac{768}{x^2},$$

$$(B.81) \quad W_{0,4}(x) = \frac{16\Omega_0(x)}{x^3},$$

$$(B.82) \quad W_{0,3}(x) = -\frac{48\Omega_0(x)}{x},$$

$$(B.83) \quad W_{0,2}(x) = 48x\Omega_0(x),$$

$$(B.84) \quad W_{0,1}(x) = -16x^3\Omega_0(x),$$

where $\Omega_0(x)$ is the right-hand side of (3.43). Plugging in (B.80)-(B.84) into (B.79), we obtain

$$(B.85) \quad b_0^{(p)}(r) = \frac{r^4}{48} \int_1^r \frac{\Omega_0(x)}{x} dx - \frac{r^2}{16} \int_1^r x\Omega_0(x) dx + \frac{1}{16} \int_1^r x^3\Omega_0(x) dx - \frac{1}{48r^2} \int_1^r x^5\Omega_0(x) dx.$$

Therefore, the analytical solution of $b_0(r)$ has the form,

$$(B.86) \quad \begin{aligned} b_0(r) = & r^4 \left(\frac{1}{48} \int_1^r \frac{\Omega_0(x)}{x} dx + A_0 \right) + r^2 \left(-\frac{1}{16} \int_1^r x\Omega_0(x) dx + B_0 \right) \\ & + \left(\frac{1}{16} \int_1^r x^3\Omega_0(x) dx + C_0 \right) + \frac{1}{r^2} \left(-\frac{1}{48} \int_1^r x^5\Omega_0(x) dx + D_0 \right). \end{aligned}$$

Finally, we solve A_0 , B_0 , C_0 , and D_0 subject to the boundary conditions (3.44) and (3.45) for $k = 0$.

B.5.1.3. *Oscillatory Second Order Equations* (3.43)-(3.45). For $k = 1, \dots, 4$, we use Hankel functions, $H_2^{(1)}(r)$ and $H_2^{(2)}(r)$, to obtain the form of the solution as,

$$(B.87) \quad b_k^{(h)}(r) = A_k r^2 + \frac{B_k}{r^2} + C_k H_2^{(1)}(r\gamma_k) + D_k H_2^{(2)}(r\gamma_k)$$

and

$$(B.88) \quad b_k^{(p)}(r) = r^2 \int_1^r \frac{W_{k,4}(x)}{W_k(x)} dx + \frac{1}{r^2} \int_1^r \frac{W_{k,3}(x)}{W_k(x)} dx + H_2^{(1)}(\gamma_k r) \int_1^r \frac{W_{k,2}(x)}{W_k(x)} dx + H_2^{(2)}(\gamma_k r) \int_1^r \frac{W_{k,1}(x)}{W_k(x)} dx,$$

where γ_k is defined by (B.71).

To compute the Wronskian determinants of (B.88), we use the following identities from Holtsmark *et al.* [18], which was adapted from Watson [57]:

$$(B.89) \quad H_2^{(1)}(\gamma_k r) \frac{d}{dr} \left(H_2^{(2)}(\gamma_k r) \right) - \frac{d}{dr} \left(H_2^{(1)}(\gamma_k r) \right) H_2^{(2)}(\gamma_k r) = -\frac{4i}{\pi r},$$

$$(B.90) \quad H_2^{(1)}(\gamma_k r) \frac{d^2}{dr^2} \left(H_2^{(2)}(\gamma_k r) \right) - \frac{d^2}{dr^2} \left(H_2^{(1)}(\gamma_k r) \right) H_2^{(2)}(\gamma_k r) = \frac{4i}{\pi r^2},$$

$$(B.91) \quad \frac{d}{dr} \left(H_2^{(1)}(\gamma_k r) \right) \frac{d^2}{dr^2} \left(H_2^{(2)}(\gamma_k r) \right) - \frac{d^2}{dr^2} \left(H_2^{(1)}(\gamma_k r) \right) \frac{d}{dr} \left(H_2^{(2)}(\gamma_k r) \right) = \frac{4i}{\pi r} \left(\frac{4}{r^2} - \gamma_k^2 \right),$$

$$(B.92) \quad \frac{d}{dr} \left(H_2^{(1)}(\gamma_k r) \right) \frac{d^3}{dr^3} \left(H_2^{(2)}(\gamma_k r) \right) - \frac{d^3}{dr^3} \left(H_2^{(1)}(\gamma_k r) \right) \frac{d}{dr} \left(H_2^{(2)}(\gamma_k r) \right) = -\frac{4i}{\pi r^2} \left(\frac{12}{r^2} - \gamma_k^2 \right),$$

$$(B.93) \quad H_2^{(1)}(\gamma_k r) \frac{d^3}{dr^3} \left(H_2^{(2)}(\gamma_k r) \right) - \frac{d^3}{dr^3} \left(H_2^{(1)}(\gamma_k r) \right) H_2^{(2)}(\gamma_k r) = -\frac{4i}{\pi r} \left(\frac{6}{r^2} - \gamma_k^2 \right),$$

$$(B.94) \quad \frac{d^2}{dr^2} \left(H_2^{(1)}(\gamma_k r) \right) \frac{d^3}{dr^3} \left(H_2^{(2)}(\gamma_k r) \right) - \frac{d^3}{dr^3} \left(H_2^{(1)}(\gamma_k r) \right) \frac{d^2}{dr^2} \left(H_2^{(2)}(\gamma_k r) \right) = -\frac{48i}{\pi r^5} + \frac{36i}{\pi r^3} \gamma_k^2 - \frac{4i}{\pi r} \gamma_k^4.$$

We use (B.89)-(B.94) to obtain the Wronskian determinants as,

$$(B.95) \quad W_k(x) = \frac{i16\gamma_k^4}{\pi x^2},$$

$$(B.96) \quad W_{k,4}(x) = \frac{i4\gamma_k^2\Omega_k(x)}{\pi x^3},$$

$$(B.97) \quad W_{k,3}(x) = -\frac{i4\gamma_k^2\Omega_k(x)x}{\pi},$$

$$(B.98) \quad W_{k,2}(x) = -\frac{4\gamma_k^2\Omega_k(x)H_2^{(2)}(\gamma_k x)}{x},$$

$$(B.99) \quad W_{k,1}(x) = \frac{4\gamma_k^2\Omega_k(x)H_2^{(1)}(\gamma_k x)}{x},$$

where $\Omega_k(x)$ is the right-hand side of (3.43). Plugging in (B.95)-(B.99) into (B.88), we obtain

$$(B.100) \quad \begin{aligned} b_k^{(p)}(r) &= \frac{r^2}{4\gamma_k^2} \int_1^r \frac{\Omega_k(x)}{x} dx - \frac{1}{4\gamma_k^2 r^2} \int_1^r x^3 \Omega_k(x) dx \\ &+ \frac{i\pi H_2^{(1)}(\gamma_k r)}{4\gamma_k^2} \int_1^r x H_2^{(2)}(\gamma_k x) \Omega_k(x) dx - \frac{i\pi H_2^{(2)}(\gamma_k r)}{4\gamma_k^2} \int_1^r x H_2^{(1)}(\gamma_k x) \Omega_k(x) dx. \end{aligned}$$

Therefore, the analytical solution of $b_k(r)$ has the form,

$$\begin{aligned}
b_k(r) = & r^2 \left(\frac{1}{4\gamma_k^2} \int_1^r \frac{\Omega_k(x)}{x} dx + A_k \right) + \frac{1}{r^2} \left(-\frac{1}{4\gamma_k^2} \int_1^r x^3 \Omega_k(x) dx + B_k \right) \\
& + H_2^{(1)}(\gamma_k r) \left(\frac{i\pi}{4\gamma_k^2} \int_1^r x H_2^{(2)}(\gamma_k x) \Omega_k(x) dx + C_k \right) \\
& + H_2^{(2)}(\gamma_k r) \left(-\frac{i\pi}{4\gamma_k^2} \int_1^r x H_2^{(1)}(\gamma_k x) \Omega_k(x) dx + D_k \right).
\end{aligned}
\tag{B.101}$$

Finally, we solve A_k , B_k , C_k , and D_k subject to the boundary conditions (3.44) and (3.45) for $k = 1, \dots, 4$.

B.5.2. Convergence of the Numerical Solutions at First and Second Order. We compare our numerical solutions at first and second order with the analytical solutions from above.

B.5.2.1. Convergence for First Order Solutions. We compute the max norm of the error between $a_1(r)$ and $a_2(r)$ and their analytical solutions at amplitude ratio $\beta = 1$ and phase difference $\phi = 0$ for different Reynolds numbers and radii. In Figure B.2, our first order solutions converge spectrally, and specifically, we show in Figure B.2(a,b) that for both $\text{Re} = 10$ and $\text{Re} = 50$, an error of size 10^{-13} is achieved at 40 Chebyshev points. Similarly, Figure B.2(c,d) shows that our numerical solutions converge spectrally at $R = 32$ and $R = 64$. To obtain error of size 10^{-13} , we require 70 Chebyshev points for $R = 32$ and 100 Chebyshev points for $R = 64$.

B.5.2.2. Convergence for Second Order Solutions. For amplitude ratio $\beta = 1$ and phase difference $\phi = 0$, we numerically compute the solutions $b_k(r)$ for $k = 0, \dots, 4$ and compare the values with the analytical solutions. At $\text{Re} = 10$ and $R = 8$, we show in Figure B.3 that our numerical solutions converge spectrally. For example, the max norm of the error for $b_0(r)$ achieves an error of size 10^{-12} with 40 Chebyshev points. Note that the error for $b_k(r)$ becomes larger for increasing k and grows by at least a factor of 10^2 . The analytical solution in (B.101) contains terms proportional to $H_2^{(1)}(\gamma_k r)$ which decay exponentially and terms proportional to $H_2^{(2)}(\gamma_k r)$ which grow exponentially. As the coefficient γ_k in (B.71) increases with k , the modulus of these Hankel functions are changing exponentially, and the cancellation of terms in (B.101) leads to errors. $b_0(r)$ in (B.86) does not involve second order Hankel functions, and we achieve an error that is comparable to those from our first order solutions. Thus, because of numerical error in the evaluation of the analytic solution, the actual errors at second order may be smaller than reported. Due to

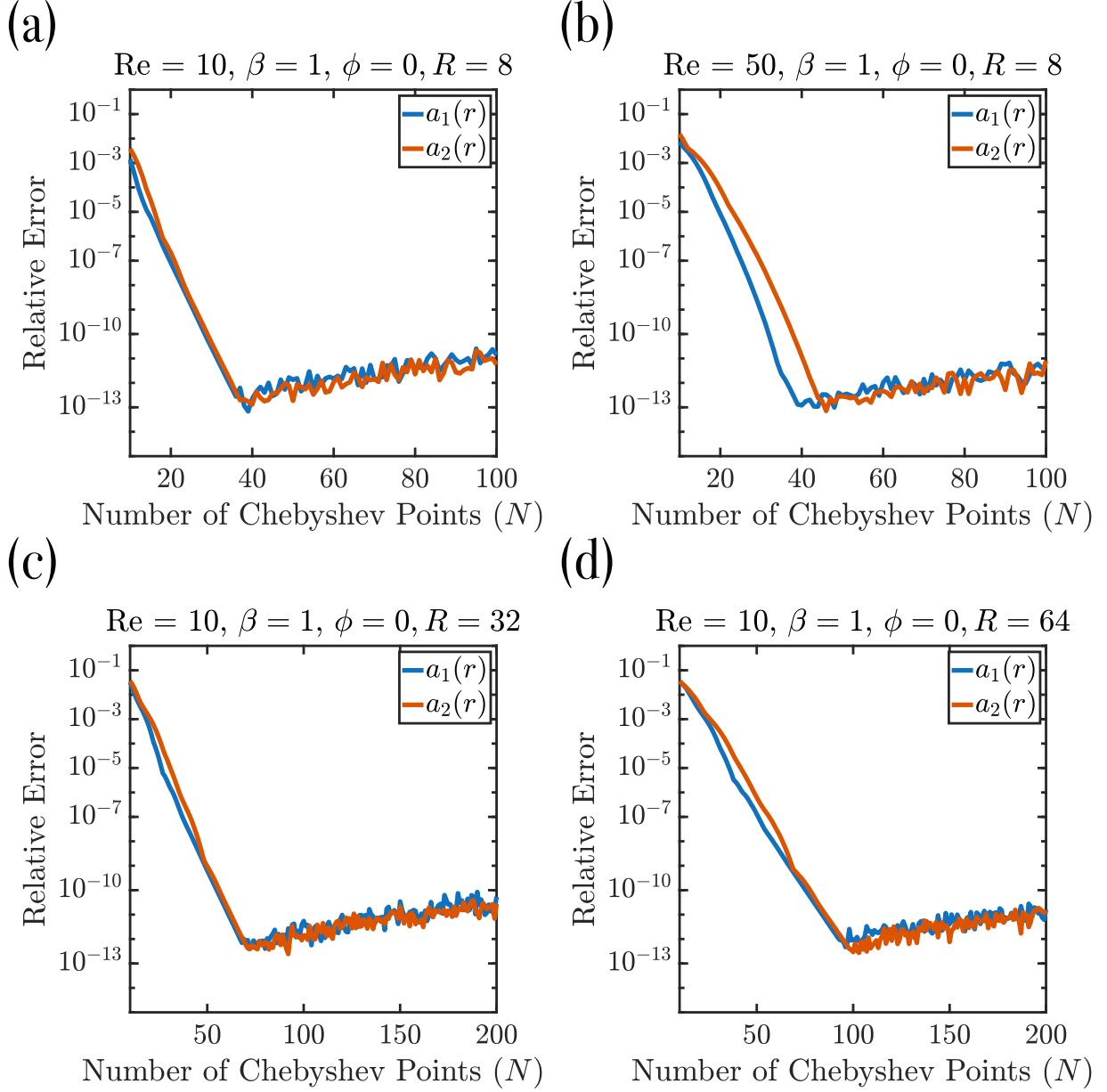


FIGURE B.2. Shown are relative errors between the analytical and the numerical solutions at amplitude ratio $\beta = 1$ and phase difference $\phi = 0$ for Reynolds numbers (a) $\text{Re} = 10$ and (b) $\text{Re} = 50$ at $R = 8$. Similarly, given are the errors for (c) $R = 32$ and (d) $R = 64$ at $\text{Re} = 10$. The relative errors are computed using the max norm.

cancellation errors, we do not present the errors for the second order solutions at larger Reynolds numbers and radii.

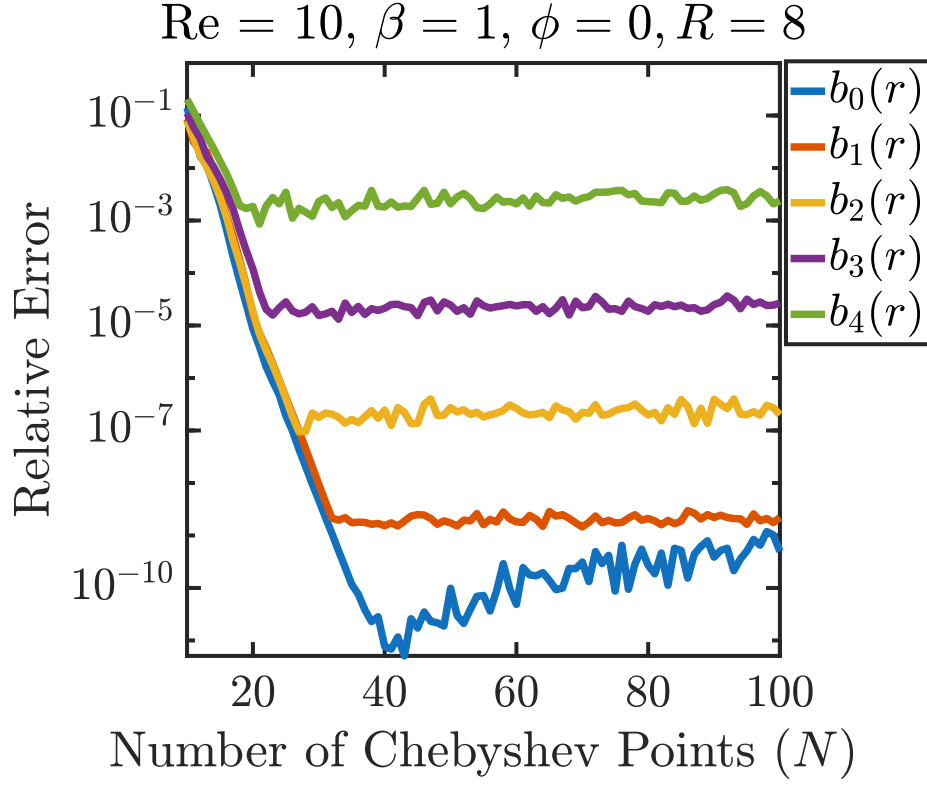


FIGURE B.3. Shown are relative errors between the analytical and the numerical solutions at $\text{Re} = 10$, amplitude ratio $\beta = 1$, phase difference $\phi = 0$, and $R = 8$. The relative errors are computed using the max norm.

APPENDIX C

Numerical Methods

C.1. Immersed Boundary Method

We use the Immersed Boundary (IB) method to solve the Navier-Stokes equations. The IB method uses an Eulerian coordinate system for the fluid and a Lagrangian coordinate system for the immersed structures (i.e. cylinder or channel walls) [35]. Let s be the parametric coordinate of a structure and $\mathbf{X}(s, t)$ be its position. We use capital letters $\mathbf{X}(s, t)$, $\mathbf{U}(s, t)$, and $\mathbf{F}(s, t)$ to define position, velocity, and force density in Lagrangian coordinates, and similarly we use lower case $p(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, and $\mathbf{f}(\mathbf{x}, t)$ for pressure, velocity, and force density in Eulerian coordinates.

The forces on the structures are applied to the surrounding fluid, and the fluid and structure move with the same velocity on the structure. The structural force density $\mathbf{F}(s, t)$ in Eulerian coordinates is given by

$$(C.1) \quad \mathbf{f}(\mathbf{x}, t) = S\mathbf{F} = \int_{\text{structure}} \mathbf{F}(s, t) \delta(\mathbf{x} - \mathbf{X}(s, t)) ds,$$

where $\delta(\mathbf{x})$ is the Dirac delta function. The operator S “spreads” the the force density from the immersed structure to the surrounding fluid. Similarly the fluid velocity is interpolated to the immersed structure by the adjoint of the spreading operator:

$$(C.2) \quad \mathbf{U}(s, t) = S^* \mathbf{u} = \int_{\text{fluid}} \mathbf{u}(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{X}(s, t)) d\mathbf{x}.$$

In our simulations, the motion of the structure is prescribed, and the force density on the structure, $\mathbf{F}(s, t)$, is determined implicitly by requiring the fluid velocity match the prescribed velocity of the boundary:

$$S^* \mathbf{u} = \mathbf{U}_b.$$

The full system describing the fluid and immersed boundaries is

$$(C.3) \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u} + S\mathbf{F},$$

$$(C.4) \quad \nabla \cdot \mathbf{u} = 0,$$

$$(C.5) \quad S^* \mathbf{u} = \mathbf{U}_b,$$

where ρ is the fluid density and μ is the fluid viscosity.

C.2. Discretization

We solve on a doubly periodic domain discretized into points equally spaced by Δx , and structures are discretized by points equally spaced by $\Delta s \approx \Delta x$. We approximate the differential operators using Fourier pseudo-spectral methods. The discrete delta function is

$$\delta = \delta_{\Delta x}(x) \delta_{\Delta x}(y),$$

where

$$(C.6) \quad \delta_{\Delta x}(x) = \begin{cases} \frac{1}{4\Delta x} (1 + \cos(\frac{\pi x}{2\Delta x})) & |x| < 2\Delta x, \\ 0 & \text{else.} \end{cases}$$

The discretized spread (C.1) and interpolating (C.2) operators are

$$f_{i,j} = S\mathbf{F} = \Delta s \sum_k F_k \delta_{\Delta x}(x_i - X_k) \delta_{\Delta x}(y_j - Y_k),$$

$$U_k = S^* \mathbf{u} = \Delta x^2 \sum_{i,j} u_{i,j} \delta_{\Delta x}(x_i - X_k) \delta_{\Delta x}(y_j - Y_k).$$

For the temporal discretization, we use a second-order IMEX scheme, named SBDF in [5], in which the nonlinear terms are treated explicitly in time, and the terms for the viscous force and

structure force are treated implicitly in time with BDF2. The discretized system is

$$(C.7) \quad \rho \left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} + 2\mathbf{u}^n \cdot G\mathbf{u}^n - \mathbf{u}^{n-1} \cdot G\mathbf{u}^{n-1} \right) = -Gp^{n+1} + \mu L\mathbf{u}^{n+1} + S\mathbf{F}^{n+1},$$

$$(C.8) \quad D\mathbf{u}^{n+1} = 0,$$

$$(C.9) \quad S^*\mathbf{u}^{n+1} = \mathbf{U}_b^{n+1},$$

where G , D , and L represent the discrete gradient, divergence, and Laplacian, respectively. Because the nonlinear terms are treated explicitly in time, the resulting system to solve at each time step is linear. In block form, the system is

$$(C.10) \quad \begin{bmatrix} A & G & -S \\ D & 0 & 0 \\ S^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \\ \mathbf{F} \end{bmatrix}^{n+1} = \begin{bmatrix} \mathbf{g} \\ 0 \\ \mathbf{U}_b^{n+1} \end{bmatrix},$$

where

$$A = \frac{3\rho}{2\Delta t}I - \mu L,$$

and the known terms are

$$\mathbf{g} = \rho \left(-2\mathbf{u}^n \cdot G\mathbf{u}^n + \mathbf{u}^{n-1} \cdot G\mathbf{u}^{n-1} - \frac{-4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} \right).$$

We solve (C.10) at each time step by first solving for the force density which satisfies

$$(C.11) \quad S^*\mathcal{L}^{-1}S\mathbf{F} = \mathbf{U}_b^{n+1} - S^*\mathcal{L}^{-1}\mathbf{g},$$

where we denote the operator which maps the fluid force density to the fluid velocity by $\mathcal{L}^{-1}$. Specifically, suppose $\mathbf{u}$ and p satisfy the system

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} + \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} = 0,$$

and $\mathcal{L}^{-1}$ is then

$$\mathbf{u} = \mathcal{L}^{-1}\mathbf{f}.$$

The operator $\mathcal{L}^{-1}$ can be applied efficiently using the FFT. We solve equation (C.11) with a Krylov method, which requires a preconditioner for efficiency. For preconditioning, we explicitly form the

Δx	Δt	Flux	Difference	Ratio	$\log_2(\text{Ratio})$
8/256	1.00×10^{-2}	9.6044×10^{-2}	—	—	—
8/512	5.00×10^{-3}	1.0130×10^{-1}	5.2573×10^{-3}	—	—
8/1024	2.50×10^{-3}	1.0279×10^{-1}	1.4937×10^{-3}	3.5197	1.8155
8/2048	1.25×10^{-3}	1.0324×10^{-1}	4.4149×10^{-4}	3.3833	1.7584

TABLE C.1. Refinement study for the flux with simultaneous refinement of space and time for the flow around an oscillating cylinder in a square 8 by 8 domain with periodic boundary conditions at $\text{Re} = 10$, amplitude $\epsilon = 0.5$, and time $T = 100$.

Δt	Flux	Difference	Ratio	$\log_2(\text{Ratio})$
1.000×10^{-2}	9.6044×10^{-2}	—	—	—
5.000×10^{-3}	1.0086×10^{-1}	4.8185×10^{-3}	—	—
2.500×10^{-3}	1.0175×10^{-1}	8.8700×10^{-4}	5.4324	2.4416
1.250×10^{-3}	1.0191×10^{-1}	1.6291×10^{-4}	5.4447	2.4448
6.250×10^{-4}	1.0194×10^{-1}	3.2425×10^{-5}	5.0243	2.3289
3.125×10^{-4}	1.0195×10^{-1}	7.1196×10^{-6}	4.5543	2.1872

TABLE C.2. Refinement study in time only for the flux with the space step fixed at $\Delta x = 8/256$ for the flow around an oscillating cylinder in a square 8 by 8 domain with periodic boundary conditions at $\text{Re} = 10$, amplitude $\epsilon = 0.5$, and time $T = 100$.

dense matrix representing $S^* \mathcal{L}^{-1} S$ at time 0 and compute its Cholesky factorization. This work is performed before running the simulation.

The grid resolution for channel simulations was $\Delta x = 8/256$, so that there were 256 grid points in the vertical direction, or equivalently 64 points along the cylinder diameter. The time step was $\Delta t = 5.00 \times 10^{-4}$. The tolerance for the conjugate gradient method used to solve (C.11) was 1×10^{-3} .

For the simulations presented in Figure 2.2, the grid resolution was $\Delta x = 8/512$ and the time step was $\Delta t = 5.00 \times 10^{-3}$. Table C.1 shows the results of a mesh refinement study on this problem, which shows that the time-averaged flux converges at approximately second-order. Using the finest mesh ($\Delta x = 8/2048, \Delta t = 1.25 \times 10^{-3}$) as the solution, we estimate the error in the flux for $\Delta x = 8/512$ and $\Delta t = 5.00 \times 10^{-3}$ is approximately 1.9%.

In the channel simulations, we used a coarser spatial resolution, $\Delta x = 8/256$, but a smaller time step $\Delta t = 5.00 \times 10^{-4}$. Fixing the grid resolution as $\Delta x = 8/256$ and refining the time step, we find that the flux converges approximately second-order in Δt (Table C.2). For the smallest time step, the relative changes in flux are on the scale of 10^{-4} , and thus the spatial error is dominant.

Comparing with the fluxes from the space-time refinement study in Table C.1, we estimate the error in our numerical simulations to be on the scale of $1 - 2\%$.

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