

A Virtual Element Method for Brittle Fracture of Thin Solids

By

BENJAMIN MAKOTO GODKIN
DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Applied Mathematics

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Joseph Teran, Chair

Elbridge Gerry Puckett

N. Sukumar

Committee in Charge

2026

To my grandmothers タマコ and Jill

Contents

Abstract	vi
Acknowledgments	viii
Chapter 1. Introduction	1
Chapter 2. Mathematical Preliminaries	8
2.1. Notions from Analysis	8
2.2. Partial Differential Equations	9
2.3. Some Functional Analysis Results	11
2.4. Weak Formulation of Partial Differential Equations	16
Chapter 3. Continuum Mechanics	22
3.1. The Continuum Body	22
3.2. Stress Tensor	26
3.3. Equilibrium	28
3.4. States of Stress	30
3.5. Kinematics	33
3.6. Balance Laws	41
Chapter 4. Rigid Body Dynamics	53
4.1. Rigid Body Dynamics	53
4.2. Rigid Body Representation	55
4.3. Quaternions	57
4.4. Rigid Body Simulation	60
4.5. Impulse Collection	65
Chapter 5. Elasticity	66
5.1. Solid Mechanics	66

5.2. Hyperelasticity	67
5.3. Variation of the Elastic Potential Energy	68
5.4. Infinitesimal Strain	70
5.5. Linear Elastic Solids	70
5.6. Elasticity Equations	73
Chapter 6. Finite Element Method	75
6.1. Continuous Problem	75
6.2. Discrete Problem	75
6.3. Basis Functions	77
6.4. Poisson	83
6.5. Elastodynamics	84
6.6. Nonconvex Quadrilaterals	86
Chapter 7. Virtual Element Method	88
7.1. Preliminaries	88
7.2. Virtual Element Space	89
7.3. Polynomial Projection Operators	91
7.4. Matrix Representation of Projection Operators	97
7.5. Poisson	98
7.6. Elastodynamics	103
Chapter 8. Fracture Mechanics	108
8.1. Griffith Energy Balance	108
8.2. Asymptotic Crack-tip Fields	110
8.3. Energy Release Rate and Stress Intensity Factor	120
8.4. Computing the Energy Release Rate	121
8.5. Dynamic J-Integral	123
8.6. Interaction Integral	126
8.7. Domain Integral	128
8.8. Crack Model	131
Chapter 9. Numerical Details	134

9.1. Newmark Method	134
9.2. Mesh Construction	135
9.3. Interaction Integral	138
9.4. Discrete Crack Propagation	138
9.5. Scaled Boundary Quadrature and Homogeneous Numerical Integration	139
Chapter 10. Validation Examples	142
10.1. Static Mode I Study	142
10.2. Qualitative Quasistatic Crack Propagation Examples	144
10.3. Dynamic Studies	147
10.4. Qualitative Dynamic Examples	152
Chapter 11. Rendered Examples	156
11.1. Dropped Bead	156
11.2. Dropped Decanter	156
11.3. Dropped SIGGRAPH Letters	157
11.4. Projectile Through Window	158
11.5. Dropped Cup	159
Chapter 12. Conclusions	160
12.1. Conclusions	160
Bibliography	161

Abstract

Blockbuster action movies and video games require eye-capturing destruction effects, and one costly component of a destruction engine is the fracture simulation. In computer graphics, the state of the art for incorporating runtime fracture effects into a scene involves prescoring an object and treating each fractured piece as a rigid body. If sufficient force is applied, the material breaks apart, but the cracks in the object are predetermined. Rather than using prescored bodies, the simulation of fracturing rigid bodies has been improved by implanting a fracture mechanics solver inside of a rigid body simulator which resolves contacts and collisions between rigid bodies. To efficiently propagate cracks in the object, we will employ an emerging element method, the virtual element method (VEM).

VEM is an element-based method in which the basis functions are not computed explicitly, and information about a given basis function is only known through its degrees of freedom. Since the VEM basis functions remain virtual, the polynomial projections of the element basis functions are first constructed, and these projections are then used to construct element matrices. VEM was first applied to the Laplace equation and has been extensively used in elasticity and fracture problems. A few advantages of VEM are that it may be readily used on polygonal meshes, meshes with a mix of convex and nonconvex elements, and elements with hanging nodes.

Crack evolution typically causes a change in topology of the underlying mesh which requires remeshing. Further, FEM encounters problems with nonconvex elements and hanging nodes, so further remeshing beyond splitting elements may be required. The mesh flexibilities of VEM allow cracks to evolve without the need to remesh.

We present a virtual element method (VEM) for the simulation of thin brittle materials (e.g. glass and ceramics). Objects are represented with triangle meshes and simulated as rigid bodies in-between collision events. Once a collision event occurs, we simulate our material using a linear elasticity formulation for thin solids and crack growth is conducted based on stress intensity factors. We use slightly thickened polyhedral meshes procedurally created from the original triangle meshes. Each triangle mesh is split into quadrilaterals during brittle fracture and volumetric polyhedra are updated accordingly. The quadrilateral geometry created by cracks (and subsequent thickened polyhedra) are not guaranteed to have good mesh quality; however, VEM is a recent stabilized Galerkin finite element technique that is surprisingly robust to very poor mesh element quality.

The basis functions in the VEM are not required to be computable (hence the name ‘virtual’), but allows the construction of a coercive bilinear form for linear elasticity. We show that the flexibility and robustness of VEM permits us to simulate a wide range of visually compelling (and physically accurate) brittle fractures.

Acknowledgments

I would like to thank Professor Joseph Teran for advising me throughout my PhD. This project has been an exciting combination of numerical PDEs, computational mechanics, and computer graphics. It has been a challenging endeavor, and I am excited by the many avenues for future exploration that this work has opened.

I would also like to thank Professor N. Sukumar for his guidance and advice while I worked through the virtual element method and fracture mechanics, and throughout this project. Thank you to Professor Elbridge Gerry Puckett for speaking with me on prospective student day, introducing me to the interdisciplinary nature of GGAM, and for your excellent lectures on finite difference methods and the multigrid method. Thank you to Professor John Hunter and Professor Julian Panetta for serving on my qualifying exam committee.

I am grateful for the generous support I received from the math department throughout my time in the PhD program by providing me with TA and AI appointments. I am sure there is much more happening behind the scenes than I realize, but I would especially like to thank Tina Denena, Victoria Whistler, Vanessa Bravo, Diana Bond, Minh Hoang, and Marji LeGrand for making the administrative side of graduate school run so smoothly.

Thank you to Alvin Chen for the many conversations we shared about the virtual element method, various PDE formulations, and many other topics. Thank you to Alex Lee and Girish Kumar for the many enjoyable conversations and good times. Thank you to Zhenhan Zhao for collaborating with me throughout the duration of this project. Finally, thank you to all of my office mates in MSB 2125 and to my lab mates over the years.

Most importantly, thank you to my partner, Katharina, and to my friends and family for their unwavering support and encouragement throughout my PhD.

CHAPTER 1

Introduction

Visual simulation of brittle fracture is essential for the creation of compelling computer graphics environments. Whether it be shattering window glass during an explosion or a simple cup falling to the ground and shattering, brittle fracture of thin ceramic or glass materials is very commonly needed when creating visually interesting destructible worlds. Simulation of these materials is complicated by the fast time scales of fracture dynamics (at near the speed of sound in stiff materials) as well as the efficient handling of topological changes and collisions between many small shards of material. Given that most brittle materials have very little visible change in shape, a common approach is to use a hybrid rigid body assumption where material stress and fracture events are only instantiated after collision events. We adopt this assumption and develop a new approach to simulating the collision driven brittle fracture dynamics. We develop a novel thin structure failure technique based on the virtual element method (VEM). VEM is a recent stabilized Galerkin finite element method (FEM) designed to support convex and nonconvex polyhedral elements with even poor element-quality. We show that these features are particularly advantageous for brittle fracture simulation of thin solids (see Fig. [1.1](#)).



FIGURE 1.1. A number of visually compelling fracture scenarios.

Many different approaches have been used for simulating mesh topological changes in computer graphics simulation of brittle fracture. Following O’Brien and Hodgins [64, 65], most techniques use some sort of adaptive mesh refinement [11, 15, 40, 67] to concentrate resolution at the evolving crack front and coarsen elsewhere, requiring only local mesh modifications. Phase-field methods [12, 23, 83, 88] and particle-based formulations such as the material point method [22, 33, 47, 84] and peridynamics [14, 43] avoid explicit topology changes by representing crack discontinuities implicitly. The extended finite element method (XFEM) [39, 76] enriches the FE basis with discontinuous functions, allowing cracks to propagate through element interiors without remeshing. Fragment-based methods [54, 55, 59, 66, 74] sidestep runtime topology changes by pre-fracturing objects into fragments using Voronoi decompositions or voxel representations. Polyhedral element methods [53, 81] simplify handling of the irregular geometries since mesh elements can be split into a wide variety of polyhedral shapes; however, this requires the computation of harmonic coordinates to define element shape functions (which is expensive).

The virtual element method (VEM) [4, 78] is a generalization of hourglass FEM on quadrilateral and hexahedral meshes to polytopal meshes. On polyhedral elements, the VEM avoids the explicit

construction of shape functions. In the VEM, only the polynomial projections of the shape functions are required and not the shape functions themselves. In addition, VEM is surprisingly robust to low-quality elements compared to classical FEM. It retains accuracy and efficiency even with elements that have very small volumes or high dynamic ranges of edge angles. VEM has been applied to solid continua: elasticity [3, 26], finite deformations [16], and fracture [9, 35, 63, 80] to name a few. We show that VEM is a natural extension of the promising polyhedral work in computer graphics [53, 81] and develop a novel technique for the simulation of nearly rigid, thin brittle solids in visually compelling fracture scenarios.

Fracture of thin brittle surfaces is complicated with conventional shell formulations. These techniques [48, 57, 70] define deformation models kinematically through a slightly thickened midsurface. Mindlin–Reissner shells [57, 70] is defined by extruding a midsurface along lines initially normal to it. These lines, referred to as fibers, retain the same length under deformation. Kinematics are defined by motion of the midsurface together with the rotation of the fibers. Kirchhoff–Love shells [48] impose the additional constraint that fibers neither shear nor compress relative to the midsurface. The constitutive behavior of the shell is then just that of a hyperelastic solid. However, these formulations typically require C^1 regularity which requires subdivision surfaces or NURBS. These models do not easily accommodate surface remeshing.

We show that a virtual element approach can prevent the need for these reduced, midsurface driven kinematics. This prevents the need for C^1 interpolation regularity and its associated complications for mesh geometry. We instead represent thin brittle materials with simple triangle meshes and thicken them into volumetric polyhedral meshes to create our solid model. This procedural construction is a convenient way to generate a volumetric discretization directly from a C^0 triangle mesh. We prevent the need for any assumptions about the kinematics of the thickened shell relative to the midsurface. We use the virtual node algorithm of [58] to split each surface triangle into a set of quadrilaterals, which are then extruded into a thin layer of hexahedral or dodecahedral elements that serve as a solid model. Crack propagation is governed by linear elastic fracture mechanics (LEFM): after each elastodynamic time step we extract mode I, II, and III stress intensity factors (SIFs) at each crack tip via the dynamic J-integral and interaction integral [71, 86], apply the mixed-mode propagation criterion of Richard [72] to determine whether a crack advances and at what angle, and bound the crack tip speed by the Rayleigh wave speed. Our thickened surface mesh

approach does not contain well-conditioned elements since we use only a few highly anisotropic elements in the normal directions. Element quality is particularly poor near crack tips where sliver quads can be created from triangle splits. However, VEM’s robustness to badly-shaped polyhedra allows us to retain accuracy and stability.

We summarize our contributions as:

- Novel virtual element discretization of thin-solid elasticity for brittle fracture that is robust to the degenerate and nonconvex elements arising from crack propagation.
- Procedural continuum 3D mesh construction using the virtual node algorithm [58] to split triangle meshes into extruded hexahedral/dodecahedral volumes from curved surfaces that retains simple planar polyhedral structure.
- Dynamic LEFM crack growth algorithm using the J-integral and interaction integral to extract mixed-mode SIFs, with crack direction and speed governed by Richard’s criterion [72].
- Verification against standard fracture benchmarks and demonstration on a variety of compelling fracture scenarios.

1.0.1. Finite Element and Extended Finite Element Methods. O’Brien and Hodgins [64,65] pioneered the use of the finite element method for physically-based fracture simulation of brittle and ductile materials in computer graphics. Bao et al. [6] combine rigid-body simulation with FEM to efficiently model brittle objects. Martin et al. [53] and Wicke et al. [81] introduced polyhedral finite element methods based on harmonic basis functions and mean value coordinates, respectively, to accommodate more general fracture geometry. The extended finite element method enriches the finite element basis with discontinuous functions, allowing cracks to propagate through element interiors without remeshing. Dick et al. [21] use adaptive refinement for topological changes in hexahedral FEM. Kaufmann et al. [39] use discontinuous Galerkin with XFEM basis functions to fracture in shell elements. Chitalu et al. [20] also use XFEM for brittle fracture. Koschier et al. [40] develop a reversible mesh refinement to adaptively refine near large tensile stresses. Chen et al. [15] enrich low-resolution fracture simulations with a material strength field. Mandal et al. [51,52] use graph-based FEM for anisotropic dynamic fracture without remeshing.

1.0.2. Phase-Field and Material Point Methods, and Particle-Based Methods. Phase-field fracture methods represent the crack as a diffuse damage field, naturally handling initiation, branching, and merging without explicit crack tracking. Wolper et al. [83] couple a phase-field formulation with the material point method (MPM) to simulate dynamic fracture with both brittle and plastic effects. Cao et al. [12] extend this to anisotropic materials using a frame-field concept. Zhao et al. [88] present a plastic phase-field method for ductile fracture with MPM. Fan et al. [23] apply a time-dependent phase-field method to Kirchhoff-Love thin shells in a hybrid Lagrangian-Eulerian framework, using moving least-squares to capture discontinuities near the crack tip. The material point method (MPM) is well-suited for fracture since topological changes are handled implicitly by the particle-grid transfer. Hegemann et al. [33] use MPM with a level set to simulate dynamic ductile fracture without remeshing. Wretborn et al. [84] extend MPM with a multi-body solver. Wang et al. [79] simulate elastoplastic ductile fracture with MPM. Fan et al. [22] combine elastodynamic MPM with rigid body methods, using continuum damage mechanics to track a damage field that drives crack propagation. Han et al. [32] use a Lagrangian particle method for fracture in thin membranes. Liu et al. [47] use MPM with continuum damage mechanics for dynamic elastoplastic fracture. Li et al. [44] simulate dynamic brittle fracture in anisotropic materials using adaptive smoothed particle hydrodynamics (SPH).

1.0.3. Peridynamics and Discrete Element Methods. Non-local formulations such as peridynamics replace the classical continuum with integral operators over a neighborhood which naturally allows for fracture. Levine et al. [43] apply peridynamics to fracture of elastoplastic materials. Chen et al. [14] develop a peridynamics-based method for elastoplastic deformation and fracture. The bonded discrete element method (DEM) models solids as collections of bonded particles, simulating fracture by breaking bonds. Lu et al. [49, 50] examine DEM’s ability to handle multiple fractures. Hahn and Thomaszewski [30, 31] simulate brittle fracture (quasistatic and dynamic loading) with the boundary element method, embedding the boundary element solver within a rigid-body simulation pipeline.

1.0.4. Rigid Body, Fragment-Based, and Real-Time Methods. A widely used strategy for animation is to pre-fracture objects into fragments and simulate the resulting pieces as rigid or deformable bodies. Müller et al. [59] present a real-time method using user-defined fracture

patterns driven by contact events. Oh et al. [66] organize fracture hierarchies for brittle objects in a tree-based structure. Zhu et al. [89] solve quasi-static linear elasticity via boundary integrals, initiating fracture at tensile stress thresholds and orienting cracks along principal stress eigenvectors. Meseure et al. [56] compute nodal stresses on rigid bodies to drive brittle fracture. Li et al. [46] reintroduce the energy stored by broken constraints as kinetic energy into the rigid body system. McGraw [54, 55] achieve real-time fracture with a volume-preserving voxel approach. Sellan et al. [74] precompute dominant fracture modes via eigenvalue analysis, enabling impact-driven runtime fracture effects. Najim et al. [60] implement GPU-based fracture calculations and use tessellation to increase fracture surface resolution.

1.0.5. Thin Structures. Fracture of thin shells, plates, and membranes present unique challenges due to their reduced dimensionality. Busaryev et al. [11] model fracture of thin plates using an adaptive Delaunay triangulation. Pfaff et al. [67] use dynamic adaptive meshing to capture crack detail in thin sheets. Lejemble et al. [42] present a real-time paper tearing method. Schreck et al. [73] simulate paper tearing with a geometric tear-initiation criterion, a physics-based propagation model, and a texturing method for fine surface detail. Ishitobi et al. [37] model weathering of thin-film-coated materials. Zhang et al. [87] simulate surface crack patterns in ceramic glazes using a quasi-static cracking node method.

Chapter 2 introduces the mathematical preliminaries that underpin the remainder of this work. We briefly review the mathematical framework underlying the numerical methods employed throughout the thesis, introducing partial differential equations and the concept of weak formulations.

Chapter 3 develops the language of continuum mechanics. Beginning from a small set of fundamental axioms, we describe the kinematics and dynamics of continuous bodies and derive the conservation laws governing the evolution of the materials considered in this work.

Chapter 4 presents the rigid body simulation framework adopted in this thesis. We describe the rigid body representation, the collision detection and contact resolution algorithms, and the impulse-based simulation method. Finally, we establish the connection between rigid body contact impulses and the boundary conditions imposed on the elastodynamic problem.

Chapter 5 introduces the formulation of solid mechanics used throughout this work. We present the governing equations of elastodynamics and formulate the coupled initial-boundary-value problem that incorporates fracture mechanics to describe crack evolution.

Chapter 6 develops the finite element method. We derive the linear and bilinear (trilinear) interpolation functions over triangular and quadrilateral (hexahedral) elements, respectively, and formulate the associated finite-dimensional problem. The chapter concludes by examining the principal limitation of the finite element method encountered during crack propagation: the emergence of non-convex polygonal elements.

Chapter 7 introduces the virtual element method as a natural extension capable of overcoming this limitation. We develop its underlying principles using the Poisson equation as a model problem before extending the formulation to the elastodynamic setting considered in this work.

Chapter 8 develops the fracture mechanics framework. We begin by introducing Griffith’s energy criterion and the stress intensity factors that govern crack growth. We then derive the J-integral and the interaction integral, which enable the extraction of stress intensity factors for bodies subjected to general loading conditions.

Chapter 9 presents the remaining numerical components of the proposed method, including implementation details and auxiliary algorithms required for robust simulation.

Chapter 10 validates the proposed approach through a series of quantitative benchmark problems drawn from the fracture mechanics literature. We also present qualitative examples demonstrating that the predicted crack paths and propagation behavior are physically consistent.

Chapter 11 showcases a collection of visually compelling fracture simulations that illustrate the capabilities of the proposed framework.

Finally, we conclude by summarizing the main contributions of this work and discussing several promising directions for future research.

CHAPTER 2

Mathematical Preliminaries

Analysis and partial differential equations (PDE) lay the foundation for the work presented in this dissertation. We present the basic background to familiarize the weak formulation of PDEs which is the framework for how we compute our numerical solutions. The Lax-Milgram theorem permits a concise proof of the existence of weak solutions for the Poisson equation which we present.

2.1. Notions from Analysis

We primarily work in the setting of a Hilbert space H which is endowed with inner product $(\cdot, \cdot) : H \times H \rightarrow \mathbb{R}$ and induced norm $\|\cdot\|$. We search for solutions in suitably defined function spaces based on the problem we have at hand. By $C^m(\Omega)$, we denote the collection of functions that are continuous, m -times differentiable, and whose m derivatives are also continuous. The functions in $L^p(\Omega)$ satisfy

$$(2.1) \quad \int_{\Omega} |f(x)|^p d\mathbf{x} < \infty,$$

that is their p -norm is finite. For the case $p = 2$, $L^2(\Omega)$ is a Hilbert space with inner product,

$$(2.2) \quad (f, v) := \int_{\Omega} f v d\mathbf{x}.$$

While the spaces $C^m(\Omega)$ have the familiar notion of continuity from calculus, they are not complete with respect to the L^2 -norm, so they do not provide the functional-analytic framework needed for many arguments in PDE theory. Instead we work with the Sobolev spaces $W^{k,p}(\Omega)$ which require the following definition which suitably weakens the notion of derivative.

DEFINITION 2.1.1. *We say that a function is weakly differentiable if*

$$(2.3) \quad \int_{\Omega} f \frac{\partial}{\partial x_{\alpha}} \phi(\mathbf{x}) d\mathbf{X} = \int_{\Omega} \frac{\partial f}{\partial x_{\alpha}} \phi d\mathbf{X},$$

and we call $\frac{\partial f}{\partial x_{\alpha}}$ the weak derivative of f .

The space $W^{k,p}(\Omega)$ is defined as the space of functions that have up to k -order weak derivatives defined and which live in $L^p(\Omega)$. Sobolev spaces are suitable for analysis of PDE since they form complete spaces, and in particular, $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space.

2.2. Partial Differential Equations

Physical phenomena are modeled using partial differential equations, and our interests lie in elliptic and hyperbolic PDEs. We consider the following PDEs over some domain Ω with boundary given by $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$ where $\partial\Omega_N \cap \partial\Omega_D = \emptyset$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a multi-index with $|\alpha| \leq k$. A k^{th} -order linear PDE is an equation of the form

$$(2.4) \quad \sum_{|\alpha| \leq k} a_\alpha(\mathbf{x}) \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} = f(\mathbf{x}),$$

where f is a given function. The goal of solving a PDE is to determine the unknown function $u : \Omega \rightarrow \mathbb{R}$.

2.2.1. Elliptic Partial Differential Equations. A second-order PDE may be expressed as

$$(2.5) \quad -\nabla \cdot (\mathbf{A}(\mathbf{x}) \cdot \nabla u) + \mathbf{b}(\mathbf{x}) \cdot \nabla u + c(\mathbf{x})u = f(\mathbf{x}),$$

where $\mathbf{A}$, $\mathbf{b}$ and c are given coefficient functions. We say that the PDE is elliptic if there is a constant $\beta > 0$ such that

$$(2.6) \quad \mathbf{v}^T \mathbf{A}(\mathbf{x}) \mathbf{v} \geq \beta |\mathbf{v}|^2,$$

for all $\mathbf{v} \in \mathbb{R}^n$. This condition states that the matrix $\mathbf{A}$ is positive definite with the smallest eigenvalue greater than or equal to β .

2.2.1.1. Poisson Equation. By taking $\mathbf{A} = \mathbf{I}$, $\mathbf{b} = \mathbf{0}$, and $c = 0$, we get the Poisson equation which is a classic example of an elliptic partial differential equation and is given by

$$(2.7) \quad -\Delta u = f.$$

Additionally, boundary data for the unknown function u is given in the form of Neumann boundary conditions

$$(2.8) \quad \nabla u \cdot \mathbf{n} = u_N$$

on $\partial\Omega_N$, and Dirichlet boundary conditions

$$(2.9) \quad u = u_D,$$

on $\partial\Omega_D$. The Neumann boundary conditions specify the normal derivative of u on $\partial\Omega_N$, and the Dirichlet boundary conditions specify the value of u on $\partial\Omega_D$.

2.2.1.2. *Linear Elastostatics.* A vectorial example of an elliptic PDE is the balance of linear momentum arising from elastostatics,

$$(2.10) \quad -\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f},$$

where

$$(2.11) \quad \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{u}), \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

Here, $C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$, and $\lambda > 0$ and $\mu > 0$ are constants. The function to be determined $\mathbf{u}$ now has d components, where d is the number of space dimensions. Boundary information for the unknown function $\mathbf{u}$ is given as Neumann boundary conditions

$$(2.12) \quad \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}_N$$

on $\partial\Omega_N$, and Dirichlet boundary conditions

$$(2.13) \quad \mathbf{u} = \mathbf{u}_D,$$

on $\partial\Omega_D$.

2.2.2. Second-Order Hyperbolic Partial Differential Equations. If we now consider one of our independent variables to be time, we now consider the domain $[0, T) \times \Omega$ where $T > 0$. We now consider second-order linear PDE of the form

$$(2.14) \quad \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\mathbf{A}(t, \mathbf{x}) \cdot \nabla u) + \mathbf{b}(t, \mathbf{x}) \cdot \nabla u + c(t, \mathbf{x})u = f(t, \mathbf{x}),$$

where $\mathbf{A}$, $\mathbf{b}$ and c are given coefficient functions that now depend on time. We say that the PDE is hyperbolic if there is a constant $\beta > 0$ such that

$$(2.15) \quad \mathbf{v}^T \mathbf{A}(t, \mathbf{x}) \mathbf{v} \geq \beta |\mathbf{v}|^2,$$

for all $\mathbf{v} \in \mathbb{R}^n$.

2.2.2.1. *Wave Equation.* Taking $\mathbf{A}(t, \mathbf{x}) = \mathbf{I}$, $\mathbf{b} = \mathbf{0}$, and $c = 0$, we have the wave equation,

$$(2.16) \quad \frac{\partial^2 u}{\partial t^2} = \Delta u + f.$$

Again, boundary data in the form of Dirichlet and Neumann boundary conditions are given, and the boundary conditions may depend on time. As a result of the time dimension entering the picture, we require the initial state be defined, and since the PDE is second-order in time, we require initial conditions on u and the first derivative of u with respect to t ,

$$(2.17) \quad u(0, \mathbf{X}) = u_0(\mathbf{X}), \quad \frac{\partial u}{\partial t}(0, \mathbf{X}) = \dot{u}_0(\mathbf{X}).$$

2.2.2.2. *Linear Elastodynamics.* The balance of linear momentum arising from elastodynamics is a vectorial hyperbolic equation given by,

$$(2.18) \quad \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f},$$

over Ω . Here $\boldsymbol{\sigma}(\mathbf{u})$ is given by (2.11), and $\rho \in \mathbb{R}$ is a fixed constant. Again, boundary and initial conditions for $\mathbf{u}$ are given.

2.3. Some Functional Analysis Results

The existence and unique of weak solutions to elliptic PDE have a particularly streamlined proof, and in Section 2.4.1 we prove the existence and uniqueness of weak solutions for the Poisson problem. The key tool used to prove existence and uniqueness for the Poisson problem is the Lax-Milgram theorem, but before we prove the theorem, we a number of results we require for the proof of the theorem.

The Riesz representation theorem states that any element of the dual space may be uniquely written as the inner product with an element of the Hilbert space.

THEOREM 2.3.1 (Riesz Representation Theorem). *If φ is a bounded linear functional on a Hilbert space H , then there exists a unique element $y \in H$ such that*

$$(2.19) \quad \varphi(x) = \langle y, x \rangle$$

for all $x \in H$.

For the proof of Lemma 2.3.2.1, we require the open mapping theorem.

THEOREM 2.3.2 (Open Mapping Theorem). *Let $T : X \rightarrow Y$ be a one-to-one, onto bounded linear map between Banach spaces X and Y . Then $T^{-1} : Y \rightarrow X$ is bounded.*

LEMMA 2.3.2.1. *Let $T : X \rightarrow Y$ be a bounded linear operator between Banach spaces. The following are equivalent:*

(a) *There is a constant $c > 0$ such that*

$$(2.20) \quad c\|x\| \leq \|Tx\|$$

for all $x \in X$.

(b) *T has closed range, and $Tx = 0$ only has the solution $x = 0$.*

PROOF. First, we show that (a) $\Rightarrow$ (b). If $Tx = 0$, then

$$(2.21) \quad 0 = \|Tx\| \geq c\|x\|$$

since T is bounded, so $\|x\| = 0$. Hence $x = 0$.

Now, suppose y_n is a convergent sequence in $\text{ran}(T)$ so that $y_n \rightarrow y \in \text{ran}(T)$. Then, there exists $x_n \in X$ with $Tx_n = y_n$. Since y_n is Cauchy and

$$(2.22) \quad \|x_n - x_m\| \leq \frac{1}{c} \|Tx_n - Tx_m\| = \frac{1}{c} \|y_n - y_m\|,$$

we see that x_n is also Cauchy. Therefore, $x_n \rightarrow x \in X$ since X is complete. Now,

$$(2.23) \quad Tx = \lim_{n \rightarrow \infty} Tx_n$$

$$(2.24) \quad = \lim_{n \rightarrow \infty} y_n$$

$$(2.25) \quad = y,$$

so $y \in \text{ran}(T)$. Hence, $\text{ran}(T)$ is closed.

Now, we show that (b) $\Rightarrow$ (a). Since $\text{ran}(T)$ is closed, $\text{ran}(T)$ is itself a Banach space. Since $\ker(T) = \{0\}$, $T : X \rightarrow Y$ is one-to-one, so $T : X \rightarrow \text{ran}(T)$ is one-to-one and onto. Then, by the Open Mapping Theorem, $T^{-1} : \text{ran}(T) \rightarrow X$ is bounded, so there exists $c > 0$ such that

$$(2.26) \quad \|T^{-1}y\| \leq c\|y\|,$$

for all $y \in \text{ran}(T)$. Upon letting $y = Tx$, we see that

$$(2.27) \quad \|x\| \leq \frac{1}{c}\|Tx\|,$$

for all $x \in X$. □

We now prove the Lax-Milgram theorem which does all the heavy lifting for proving the existence and uniqueness of weak solutions for the Poisson equation.

THEOREM 2.3.3 (Lax-Milgram Theorem). *Suppose $B : H \times H \rightarrow \mathbb{R}$ is a bilinear mapping for which there exists constants $\alpha, \beta > 0$ such that*

$$(2.28) \quad \|B(u, v)\| \leq \alpha\|u\|\|v\|$$

for all $u, v \in H$, and

$$(2.29) \quad \beta\|u\|^2 \leq B(u, u)$$

for all $u \in H$. Let $f : H \rightarrow \mathbb{R}$ be a bounded linear functional on H . Then there exists a unique $u \in H$ such that

$$(2.30) \quad B(u, v) = \langle f, v \rangle$$

for all $v \in H$.

PROOF. For fixed $u \in H$, $v \rightarrow B(u, v)$ is a bounded linear function on H , so the Riesz representation theorem says that there exists a unique $w \in H$ such that

$$(2.31) \quad B(u, v) = (w, v)$$

for all $v \in H$. Let $Au = w$. Then

$$(2.32) \quad B(u, v) = (Au, v)$$

for $u, v \in H$. We now show that $A : H \rightarrow H$ is a bounded linear operator. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$. Then,

$$(2.33) \quad (A(\lambda_1 u_1 + \lambda_2 u_2), v) = B(\lambda_1 u_1 + \lambda_2 u_2, v)$$

$$(2.34) \quad = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v)$$

$$(2.35) \quad = \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v)$$

$$(2.36) \quad = (\lambda_1 Au_1 + \lambda_2 Au_2, v),$$

so we see that A is indeed a linear operator. Next, we show that A is continuous. We have

$$(2.37) \quad \|Au\|^2 = (Au, Au)$$

$$(2.38) \quad = B(u, Au)$$

$$(2.39) \quad \leq \alpha \|u\| \|Au\|$$

by assumption. Thus,

$$(2.40) \quad \|Au\| \leq \alpha \|u\|$$

for all $u \in H$, so A is continuous. Now, we see that

$$(2.41) \quad \beta \|u\|^2 \leq B(u, u)$$

$$(2.42) \quad = (Au, u)$$

$$(2.43) \quad \leq \|Au\| \|u\|,$$

so

$$(2.44) \quad \beta \|u\| \leq \|Au\|.$$

Since A is bounded and we have the inequality (2.44), Lemma 2.3.2.1 implies that the range of A is closed. We now show that $\text{ran}(A) = H$. Suppose not. Since $\text{ran}(A)$ is closed, there exists nonzero

$w \in H$ with $w \in \text{ran}(A)^\perp$, but

$$(2.45) \quad \beta \|w\|^2 \leq B(w, w)$$

$$(2.46) \quad = (Aw, w)$$

$$(2.47) \quad = 0,$$

since $w \in \text{ran}(A)^\perp$. A contradiction. Next, the Riesz representation theorem says that there is a unique $w \in H$ such that

$$(2.48) \quad \langle f, v \rangle = (w, v),$$

for all $v \in H$. Since $\text{ran}(A) = H$ and A is one-to-one, there is a unique $u \in H$ such that $Au = w$. Then,

$$(2.49) \quad B(u, v) = (Au, v)$$

$$(2.50) \quad = (w, v)$$

$$(2.51) \quad = \langle f, v \rangle.$$

Finally, we establish uniqueness of u solely on the coercivity of $B(\cdot, \cdot)$. Suppose there exists $u, \tilde{u} \in H$ such that

$$(2.52) \quad B(u, v) = \langle f, v \rangle, \quad B(\tilde{u}, v) = \langle f, v \rangle.$$

Then,

$$(2.53) \quad B(u - \tilde{u}, v) = 0,$$

so

$$(2.54) \quad \beta \|u - \tilde{u}\|^2 \leq B(u - \tilde{u}, u - \tilde{u}) = 0.$$

Ergo, $u = \tilde{u}$. □

2.4. Weak Formulation of Partial Differential Equations

In Section 2.2, a number of (initial) boundary value problems were stated. We refer to the problem statements in Section 2.2 as the strong form since they require the solution u have classical derivatives. The requirement of the existence of classical derivatives can be weakened by considering weak derivatives and the associated weak formulation of the PDE.

2.4.1. Poisson Equation. We call (2.7) the strong form of the Poisson equation since it requires the solution u have two classical derivatives. This requirement is relaxed by consider weak solutions of the associated weak formulation. The weak form of (2.7) is found by multiplying by a suitable test function v and integrating over Ω ,

$$(2.55) \quad \int_{\Omega} -\Delta u v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}.$$

Now, rewriting the LHS of (2.55) and applying the divergence theorem, we have

$$(2.56) \quad \int_{\Omega} -\Delta u v \, d\mathbf{x} = \int_{\Omega} -\nabla \cdot (\nabla u v) + \nabla u \cdot \nabla v \, d\mathbf{x}$$

$$(2.57) \quad = \int_{\partial\Omega} -v \nabla u \cdot \mathbf{n} \, dS + \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x}.$$

Then, upon applying the boundary conditions, we see that the LHS of (2.55) becomes

$$(2.58) \quad \int_{\partial\Omega} -v \Delta u \cdot \mathbf{n} \, dS + \int_{\Omega} \Delta u \cdot \Delta v \, d\mathbf{x} = \int_{\partial\Omega_N} -v u_N \, dS + \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x}.$$

We define the bilinear form

$$(2.59) \quad a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x},$$

so the weak form of the Poisson equation is

$$(2.60) \quad a(u, v) = \int_{\partial\Omega_N} v u_N \, dS + (f, v).$$

We see that u and v each require one weak derivative, so the solution to the PDE lives in the space

$$(2.61) \quad V := \{v \in H^1(\Omega) : v = u_D \text{ on } \partial\Omega_D\},$$

and we take v to live in the space

$$(2.62) \quad V_0 := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}.$$

The statement of the weak formulation of the boundary value problem (2.7) is: find $u \in V$ such that

$$(2.63) \quad a(u, v) = \int_{\partial\Omega_N} u_N v \, dS + (f, v),$$

for all $v \in V_0$.

THEOREM 2.4.1. *There exists a unique solution to (2.63) provided $f \in L^2(\Omega)$, $u_D \in H^1(\partial\Omega_D)$, and $u_N \in L^2(\partial\Omega_N)$.*

PROOF. Let $\tilde{u} = u - w$, where $w \in H^1(\Omega)$ with $w|_{\partial\Omega_D} = u_D$. The existence of such a function w follows by the Trace Theorem [1]. Then, $\tilde{u} \in H^1(\Omega)$ and $\tilde{u}|_{\partial\Omega_D} = 0$, so $\tilde{u} \in H_0^1(\Omega)$. Then, we consider the following weak formulation: find $\tilde{u} \in H_0^1(\Omega)$ such that

$$(2.64) \quad a(\tilde{u}, v) = (f, v) + \int_{\partial\Omega_N} u_N v \, dS - a(w, v),$$

for all $v \in V_0$. Clearly $a(\cdot, \cdot)$ is bounded, since

$$(2.65) \quad a(\tilde{u}, v) \leq \|\nabla \tilde{u}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

The Poincaré inequality states

$$(2.66) \quad \|\tilde{u}\|_{L^2(\Omega)} \leq C \|\nabla \tilde{u}\|_{L^2(\Omega)}$$

for some constant $C > 0$, so

$$(2.67) \quad \|\tilde{u}\|_{H^1(\Omega)}^2 = \|\tilde{u}\|_{L^2(\Omega)}^2 + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2$$

$$(2.68) \quad \leq (1 + C^2) \|\nabla \tilde{u}\|_{L^2(\Omega)}^2.$$

Then, coercivity of $a(\cdot, \cdot)$ follows since

$$(2.69) \quad a(\tilde{u}, \tilde{u}) = \|\nabla u\|_{L^2(\Omega)}^2$$

$$(2.70) \quad \geq \frac{1}{1+C^2} \|\tilde{u}\|_{H^1(\Omega)}^2.$$

Now, let

$$(2.71) \quad F(v) := (f, v) + \int_{\partial\Omega_N} u_N v \, dS - a(w, v),$$

and we show that F is bounded on V_0 . By the trace inequality, we have that

$$(2.72) \quad \int_{\partial\Omega_N} u_N v \, dS \leq \|u_N\|_{L^2(\partial\Omega_N)} \|v\|_{L^2(\partial\Omega_N)}$$

$$(2.73) \quad \leq C_{\text{Trace}} \|u_N\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

$$(2.74) \quad \leq C_{\text{Trace}} \|u_N\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)},$$

so

$$(2.75) \quad |F(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C_{\text{Trace}} \|u_N\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

$$(2.76) \quad \leq (\|f\|_{L^2(\Omega)} + C_{\text{Trace}} \|u_N\|_{L^2(\Omega)} + \|w\|_{H^1(\Omega)}) \|v\|_{H^1(\Omega)}$$

Thus, by the Lax-Milgram theorem, a weak solution $\tilde{u}$ to (2.64) exists. Ergo, there exists a unique solution u to (2.63). $\square$

2.4.2. Wave Equation. The weak formulation of the wave equation (2.16) follows by multiplying by a time-dependent test function, integrating over Ω , and following the same steps as for the Poisson problem. The resulting weak form is

$$(2.77) \quad \left(\frac{\partial^2 u}{\partial t^2}, v \right) + a(u, v) = (f, v) + \int_{\partial\Omega} u_N v \, dS.$$

With the addition of time, the solution space becomes

$$(2.78) \quad V = \{v \in L^\infty(0, T; H^1(\Omega)) : v = u_D \text{ on } \partial\Omega_D\},$$

and the test space is

$$(2.79) \quad V_0 = \{v \in L^\infty(0, T; H^1(\Omega)) : v = 0 \text{ on } \partial\Omega_D\}.$$

The weak formulation is as follows: find $u \in V$ such that

$$(2.80) \quad \left(\frac{\partial^2 u}{\partial t^2}, v \right) + a(u, v) = (f, v) + \int_{\partial\Omega} u_N v \, dS.$$

for all $v \in V_0$.

Existence and uniqueness of weak solutions to (2.80) is proven using the following line of reasoning. First, a sequence of weak solutions is constructed using the Galerkin method with eigenfunctions which form an orthonormal basis of $L^2(\Omega)$. This results in a linear system of ordinary differential equations that are solvable on $[0, T]$ for any $T > 0$. Next, an energy estimate is established that bounds the sequence of approximate solutions and the first and second time derivatives of the approximate solutions. Therefore, there exists a convergent subsequence which allows the limit to be passed through the weak form.

2.4.3. Linear Elastostatics. The weak formulation of 2.10 follows by first taking the dot product of 2.10 with a suitable function $\mathbf{v}$,

$$(2.81) \quad \int_{\Omega} -(\nabla \cdot \boldsymbol{\sigma}(\mathbf{u})) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

Now, the LHS of 2.81 becomes

$$(2.82) \quad \int_{\Omega} -(\nabla \cdot \boldsymbol{\sigma}(\mathbf{u})) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} -\nabla \cdot (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{v}) + \boldsymbol{\sigma}(\mathbf{u}) : \nabla \mathbf{v} \, d\mathbf{x}$$

$$(2.83) \quad = \int_{\partial\Omega} -(\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{v}) \cdot \mathbf{n} \, dS + \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \nabla \mathbf{v} \, d\mathbf{x}$$

$$(2.84) \quad = \int_{\partial\Omega_N} -\mathbf{t}_N \cdot \mathbf{v} \, dS + \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \nabla \mathbf{v} \, d\mathbf{x}$$

upon applying the divergence theorem and the Neumann boundary conditions. Now,

$$(2.85) \quad \boldsymbol{\sigma}(\mathbf{u}) : \nabla \mathbf{v} = \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}),$$

and we define the bilinear form

$$(2.86) \quad a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{x}.$$

Therefore, (2.81) becomes

$$(2.87) \quad a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\partial\Omega_N} \mathbf{t}_N \cdot \mathbf{v} \, dS + (\mathbf{f}, \mathbf{v}).$$

We see that a suitable choice for the solution space is

$$(2.88) \quad V := \{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{u} = \mathbf{u}_D \text{ on } \partial\Omega_D\},$$

and we take the test space to be $V_0 \subset V$ which coincides with V but with zero trace on $\partial\Omega_D$. The weak formulation is as follows: find $\mathbf{u} \in V$ such that

$$(2.89) \quad a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\partial\Omega_N} \mathbf{t}_N \cdot \mathbf{v} \, dS + (\mathbf{f}, \mathbf{v}),$$

for all $\mathbf{v} \in V_0$.

Existence and uniqueness of a weak solution follows in the same manner as for the Poisson equation. The only complication that arises is establishing the coercivity of $a^\varepsilon(\cdot, \cdot)$, but this can be done using Korn's inequality which states that there exists a constant $C \geq 0$ such that for $\mathbf{v} \in [H^1(\Omega)]^d$,

$$(2.90) \quad \|\mathbf{v}\|_{H^1(\Omega)} \leq C \left(\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) + |\mathbf{v}|^2 \right)^{\frac{1}{2}}$$

2.4.4. Linear Elastodynamics. The weak formulation follows in the same manner as the previous weak formulations by taking the dot product of (2.18) with a suitable function $\mathbf{v}$,

$$(2.91) \quad \int_{\Omega} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} (\nabla \cdot \boldsymbol{\sigma}(\mathbf{u})) \cdot \mathbf{v} + \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

Upon using the divergence theorem to the RHS of (2.91) and applying the Neumann boundary condition, we have

$$(2.92) \quad \rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2}, \mathbf{v} \right) + a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\partial\Omega_N} \mathbf{t}_N \cdot \mathbf{v} \, dS + (\mathbf{f}, \mathbf{v}).$$

With the addition of time, the solution space becomes

$$(2.93) \quad V = \{\mathbf{v} \in L^\infty(0, T; [H^1(\Omega)]^d) : \mathbf{v} = \mathbf{u}_D \text{ on } \partial\Omega_D\},$$

and the test space is

$$(2.94) \quad V_0 = \{\mathbf{v} \in L^\infty(0, T; [H^1(\Omega)]^d) : \mathbf{v} = 0 \text{ on } \partial\Omega_D\}.$$

Therefore, the weak formulation is: find $\mathbf{u} \in V$ such that

$$(2.95) \quad \rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2}, \mathbf{v} \right) + a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\partial\Omega_N} \mathbf{t}_N \cdot \mathbf{v} \, dS + (\mathbf{f}, \mathbf{v}).$$

for all $\mathbf{v} \in V_0$.

Existence and uniqueness follows the same line of reasoning as the wave equation but with establishing the coercivity of $a^\varepsilon(\cdot, \cdot)$ using Korn's inequality.

CHAPTER 3

Continuum Mechanics

Our goal is to simulate the brittle fracture of materials, and we require a solid foundation in continuum mechanics to develop the language required to describe the mechanisms governing our simulations. If we consider any material body in real life, we know that at the microscopic level, the material is composed of collections atoms which define how the material behaves. Rather than taking an atomistic approach, we describe our material bodies using a continuum and consider the macroscopic behavior of the body. We will define notions of mass, force, equilibrium, and motions of the material body which will allow us to define our two simulation regimes: rigid body motions and elastic deformations. We present the main results we require to define the continuum mechanics we require, and we refer the reader to [27] for a practical and comprehensive introduction to continuum mechanics and to [77] for a mathematically verbose rendition.

3.1. The Continuum Body

We have a material body whose properties we study by representing the body using a continuum. That is, the material body is represented by an open subset $B \subset \mathbb{E}^3$ of three-dimensional Euclidean space, and each $\mathbf{X} \in B$ is called a material particle. We call B a configuration of the material body, and we will examine the properties satisfied by the material body B .

3.1.1. Mass and Mass Density. For a material body B , we define a notion of mass and mass density which describes the amount of matter contained in B . Let $\rho : B \rightarrow \mathbb{R}$ be the mass density function of the body B which gives the amount of mass per volume at a point $\mathbf{X} \in B$. Let m be the total mass of the body so that m is given by

$$(3.1) \quad m = \int_B \rho \mathbf{X} \, dV_{\mathbf{X}}.$$

That is, the total mass of the body is given by integrating the mass density function of B over the entirety of B . Let V denote the volume of the body so that V is given by

$$(3.2) \quad V = \int_B dV_{\mathbf{X}}.$$

Let $B_r(\mathbf{X})$ denote the open ball of radius r centered at $\mathbf{X} \in B$. Given the notions of total mass and volume, the mass density of B at $\mathbf{X}$ is defined as the limiting value of

$$(3.3) \quad \rho(\mathbf{X}) = \frac{\int_{B_r(\mathbf{X})} \rho(\mathbf{X}) dV_{\mathbf{X}}}{\int_{B_r(\mathbf{X})} dV_{\mathbf{X}}},$$

as $r \rightarrow 0$. The center of mass $\mathbf{X}_{\text{com}}$ is the mass-weighted center of B and is given by

$$(3.4) \quad \mathbf{X}_{\text{com}} = \frac{1}{m} \int_B \mathbf{X} \rho(\mathbf{X}) dV_{\mathbf{X}},$$

and the center of mass will be essential for defining rotation-based concepts of force and motion.

The centroid of B $\mathbf{X}_{\text{centroid}}$ is the geometric center of B and is given by

$$(3.5) \quad \mathbf{X}_{\text{centroid}} = \frac{1}{V} \int_B \mathbf{X} dV_{\mathbf{X}}.$$

3.1.2. Forces. As the material body interacts with its surrounding environment, the body will be subjected to forces which may distort or influence the motion of the material body. We consider two primary types of force: body forces and surface forces.

3.1.2.1. *Body Force.* Let $\mathbf{b} : B \rightarrow B$ be the body force per unit mass that acts throughout the entire material body. Common examples of body forces are the force due to gravity and the electromagnetic force. The resultant force $\mathbf{r}_b$, due to the body force $\mathbf{b}$ applied to the material body B is

$$(3.6) \quad \mathbf{r}_b = \int_B \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) dV_{\mathbf{X}},$$

and the resultant torque $\boldsymbol{\tau}_b$ about a point $\mathbf{p}$ due to the body force is given by

$$(3.7) \quad \boldsymbol{\tau}_b = \int_B (\mathbf{X} - \mathbf{p}) \times \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) dV_{\mathbf{X}}.$$

3.1.2.2. *Surface Forces.* Next, we consider surface forces which we may categorize as internal or external. An internal surface force is a surface force applied on a fictional surface inside the body

while an external surface force is a surface force applied on the boundary of the body. Internal surfaces forces represent the interaction of particles in the body which hold the body together, and external forces represent forces arising from the surrounding environment through contact. Let Γ be an oriented surface in B and $\mathbf{n}(\mathbf{X})$ the unit normal at a point $\mathbf{X} \in B$. The force per unit area exerted on Γ is called the traction field and denoted $\mathbf{t}_{\mathbf{n}}(\mathbf{X})$. The surface Γ is assigned a positive and negative side, and the traction on the positive side represents the forces exerted by the material upon the negative side of the material. When Γ is the boundary of B , we choose $\mathbf{n}$ to be the outward normal. The resultant force due to the traction is

$$(3.8) \quad \mathbf{r}_{\mathbf{t}} = \int_{\Gamma} \mathbf{t}(\mathbf{X}) \, dA_{\mathbf{X}},$$

and the resulting torque about a point $\mathbf{p}$ due to the traction is

$$(3.9) \quad \boldsymbol{\tau}_{\mathbf{t}} = \int_{\Gamma} (\mathbf{X} - \mathbf{p}) \times \mathbf{t}(\mathbf{X}) \, dA_{\mathbf{X}}.$$

The following theorem was taken to be a postulate during the development of continuum mechanics by Cauchy, but this postulate was eventually proven and is not required to be taken as an assumption [77].

THEOREM 3.1.1. *Cauchy's Postulate (Noll's theorem) The traction field $\mathbf{t}_{\mathbf{n}}$ on the oriented surface Γ in B depends only pointwise on $\mathbf{n}$. That is, there exists a traction function $\mathbf{t} : \mathcal{N} \times B \rightarrow \mathcal{V}$, where $\mathcal{N} \subset \mathcal{V}$ is the set of all unit vectors, so that*

$$(3.10) \quad \mathbf{t}_{\mathbf{n}}(\mathbf{X}) = \mathbf{t}(\mathbf{n}(\mathbf{X}), \mathbf{X}).$$

We call $\mathbf{t}$ the traction function for B .

The takeaway is that the traction function only depends on $\mathbf{X}$ and $\mathbf{n}$, and not on $\nabla \mathbf{n}$ which describes the curvature of the surface. With this postulate in hand, we may now prove Cauchy's lemma which is also known as the law of action and reaction. Cauchy's lemma states at any point on the surface, the traction function on the positive side and negative side are equal in magnitude and point in opposite directions.

LEMMA 3.1.1.1. (*Cauchy's lemma*) Suppose $\mathbf{t}(\mathbf{n}, \mathbf{x})$ is a continuous traction field defined over Γ . Consider a sequence of subsets $\Omega \subset B$ with volume going to zero, and suppose that

$$(3.11) \quad \frac{1}{|\Gamma|} \int_{\Gamma} \mathbf{t}(\mathbf{n}, \mathbf{X}) dA_{\mathbf{X}} \rightarrow 0,$$

as $|\Omega| \rightarrow 0$. Then,

$$(3.12) \quad \mathbf{t}(\mathbf{n}, \mathbf{x}) = \mathbf{t}(-\mathbf{n}, \mathbf{x}).$$

PROOF. Consider the disc $\mathcal{D}$ with unit normal $\mathbf{n}$ centered at $\mathbf{X}_0$. Now, let Ω_{δ} be the cylinder of height δ whose midplane is the disc $\mathcal{D}$. Let Γ^+ and Γ^- denote the top and bottom surfaces of Ω_{δ} and Γ_{δ} the circumferential surface. Then, the integral of the traction over the surface of the cylinder may be decomposed as

$$(3.13) \quad \int_{\partial\Omega_{\delta}} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} = \int_{\Gamma_{\delta}} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} + \int_{\Gamma^+} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} + \int_{\Gamma^-} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}}.$$

Now, $\tilde{\mathbf{n}} = \mathbf{n}$ on Γ^+ and $\tilde{\mathbf{n}} = -\mathbf{n}$ on Γ^- , so

$$(3.14) \quad \int_{\partial\Omega_{\delta}} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} = \int_{\Gamma_{\delta}} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{x}) dA_{\mathbf{x}} + \int_{\Gamma^+} \mathbf{t}(\mathbf{n}, \mathbf{X}) dA_{\mathbf{X}} + \int_{\Gamma^-} \mathbf{t}(-\mathbf{n}, \mathbf{X}) dA_{\mathbf{X}}.$$

As $\delta \rightarrow 0$, we have that

$$(3.15) \quad \int_{\Gamma_{\delta}} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} \rightarrow 0,$$

and

$$(3.16) \quad \int_{\partial\Omega_{\delta}} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} \rightarrow 0,$$

by assumption. Additionally, as $\delta \rightarrow 0$, $\Gamma^+ \rightarrow \mathcal{D}$ and $\Gamma^- \rightarrow \mathcal{D}$, so we are left with the relation

$$(3.17) \quad \int_{\mathcal{D}} \mathbf{t}(\mathbf{n}, \mathbf{X}) + \mathbf{t}(-\mathbf{n}, \mathbf{X}) dA_{\mathbf{X}} = 0.$$

Since the resulting integral must be zero for any arbitrary $\mathcal{D} \subset B$, we must have that

$$(3.18) \quad \mathbf{t}(\mathbf{n}, \mathbf{X}) = -\mathbf{t}(-\mathbf{n}, \mathbf{X})$$

for all $\mathbf{X} \in B$. □

3.2. Stress Tensor

Let $\mathbf{t}(\mathbf{n}, \mathbf{X})$ be the traction field for the material body B . Under the assumption that the traction over any surface vanishes as the volume contained by the surface shrinks to a point, Cauchy's theorem states that the traction function is linear in $\mathbf{n}$. Thus, there exists a second order tensor $\mathbf{S}$ that relates the traction and normal. The tensor $\mathbf{S}$ is called the Cauchy stress tensor, and is given by

$$(3.19) \quad \mathbf{S}(\mathbf{X}) = S_{ij}(\mathbf{X}) \mathbf{e}_i \otimes \mathbf{e}_j,$$

where $S_{ij}(\mathbf{X}) = t_i(\mathbf{e}_j, \mathbf{X})$ for $i, j = 1, 2, 3$. We see that the i, j -component of $\mathbf{S}$ is the i -th component of the traction function associated with the coordinate plane whose normal is $\mathbf{e}_j$.

THEOREM 3.2.1. (*Cauchy's theorem*) Let $\Omega \subset B$ be an arbitrary subset of the material body B . Let $\mathbf{t} : \mathcal{N} \times B \rightarrow \mathbf{V}$ be the traction function, and assume that the traction function is continuous. Additionally, assume that

$$(3.20) \quad \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{t}(\mathbf{n}, \mathbf{X}) dA_{\mathbf{X}} \rightarrow 0,$$

as $|\Omega| \rightarrow 0$. Then, $\mathbf{t}(\mathbf{n}, \mathbf{X})$ is linear in $\mathbf{n}$. Further, for $\mathbf{X} \in B$, there is a second order tensor $\mathbf{S}(\mathbf{X}) \in \mathcal{V}^2$ such that

$$(3.21) \quad \mathbf{t}(\mathbf{n}, \mathbf{X}) = \mathbf{S}(\mathbf{X})\mathbf{n}.$$

PROOF. Let $\{\mathbf{e}_i\}$ be unit basis vectors. Consider the tetrahedron Ω_δ which is formed by the region bounded by the coordinate planes and the triangular region centered around a point $\mathbf{p}$ with normal $\mathbf{n}$, and which has a maximum edge length of δ . We may decompose the boundary of Ω_δ as

$$(3.22) \quad \partial\Omega_\delta = \Gamma_\delta \cup \bigcup_{i=1}^3 \Gamma_i,$$

where Γ_i is the coordinate plane with normal $-\mathbf{e}_i$, and Γ_δ is the remaining surface which has normal $\mathbf{n}$. Now,

$$(3.23) \quad \int_{\partial\Omega_\delta} \mathbf{t}(\tilde{\mathbf{n}}, \mathbf{X}) dA_{\mathbf{X}} = \int_{\Gamma_\delta} \mathbf{t}(\mathbf{n}, \mathbf{X}) dA_{\mathbf{X}} + \sum_{i=1}^3 \int_{\Gamma_i} \mathbf{t}(-\mathbf{e}_i, \mathbf{X}) dA_{\mathbf{X}},$$

so

$$(3.24) \quad \lim_{\delta \rightarrow 0} \frac{1}{|\partial\Omega_\delta|} \left(\int_{\Gamma_\delta} \mathbf{t}(\mathbf{n}, \mathbf{X}) + \sum_{i=1}^3 \int_{\Gamma_i} \mathbf{t}(-\mathbf{e}_i, \mathbf{X}) dA_{\mathbf{X}} \right) = 0.$$

The area of Γ_δ and each Γ_i are related by $n_i|\Gamma_\delta| = |\Gamma_i|$, so

$$(3.25) \quad |\Omega_\delta| = \Gamma_\delta + \sum_{i=1}^3 |\Gamma_i| = \left(1 + \sum_{i=1}^3 n_i\right) |\Gamma_\delta|.$$

Each Γ_i has a linear mapping to Γ_δ with constant Jacobian $J = n_i$, so we have that

$$(3.26) \quad \lim_{\delta \rightarrow 0} \frac{1}{|\Gamma_\delta|} \left(\int_{\Gamma_\delta} \mathbf{t}(\mathbf{n}, \mathbf{X}) + \sum_{i=1}^3 \mathbf{t}(-\mathbf{e}_i, \mathbf{X}) n_i dA_{\mathbf{X}} \right) = 0.$$

Since $\mathbf{t}$ is continuous and Γ_δ shrinks down to $\mathbf{X}$, the mean value theorem for integrals gives

$$(3.27) \quad \mathbf{t}(\mathbf{n}, \mathbf{X}) + \sum_{i=1}^3 \mathbf{t}(-\mathbf{e}_i, \mathbf{X}) n_i = 0.$$

Now,

$$(3.28) \quad \mathbf{t}(\mathbf{n}, \mathbf{X}) = \sum_{i=1}^3 -\mathbf{t}(-\mathbf{e}_i, \mathbf{X}) n_i$$

$$(3.29) \quad = \sum_{i=1}^3 \mathbf{t}(\mathbf{e}_i, \mathbf{X}) n_i,$$

by Cauchy's lemma. Now,

$$(3.30) \quad \mathbf{t}(\mathbf{n}, \mathbf{X}) = \mathbf{t}(\mathbf{e}_i, \mathbf{X}) n_i$$

$$(3.31) \quad = (\mathbf{t}(\mathbf{e}_i, \mathbf{X}) \otimes \mathbf{e}_i) \mathbf{n}$$

$$(3.32) \quad = \mathbf{S}(\mathbf{X}) \mathbf{n},$$

so the second order tensor has entries

$$(3.33) \quad S_{ij}(\mathbf{X}) = t_i(\mathbf{e}_j, \mathbf{X}).$$

□

3.3. Equilibrium

Consider a material body in configuration B with mass density $\rho(\mathbf{X})$, body force per unit mass $\mathbf{b}(\mathbf{X})$, and applied surface traction per unit area $\mathbf{h}(\mathbf{X})$. Let Ω be an open subset of B and $\mathbf{t}$ the traction function acting on the boundary $\partial\Omega$. The resultant force on Ω due to the body force and applied traction is

$$(3.34) \quad \mathbf{r} = \mathbf{r}_b + \mathbf{r}_t$$

$$(3.35) \quad = \int_{\Omega} \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) \, dV_{\mathbf{X}} + \int_{\partial\Omega} \mathbf{t}(\mathbf{X}) \, dA_{\mathbf{X}},$$

and the resultant torque on Ω due to the body force and applied traction about a point $\mathbf{p}$ is

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{\tau}_b + \boldsymbol{\tau}_t \\ &= \int_{\Omega} (\mathbf{X} - \mathbf{p}) \times \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) \, dV_{\mathbf{X}} + \int_{\partial\Omega} (\mathbf{X} - \mathbf{p}) \times \mathbf{t}(\mathbf{X}) \, dA_{\mathbf{X}}. \end{aligned}$$

We say that the body B is in equilibrium if the resultant force and resultant torque are both zero for all open subsets of B . That is, if

$$(3.36) \quad \mathbf{r} = \mathbf{0}, \quad \boldsymbol{\tau} = \mathbf{0},$$

for all open $\Omega \subset B$. We now derive the pointwise consequences of the equilibrium conditions.

First, requiring the resultant force to vanish requires

$$(3.37) \quad \int_{\Omega} \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) \, dV_{\mathbf{X}} + \int_{\partial\Omega} \mathbf{t}(\mathbf{X}) \, dA_{\mathbf{X}} = \mathbf{0},$$

for all $\Omega \subset B$. Writing the traction function in terms of the Cauchy stress tensor gives

$$(3.38) \quad \int_{\Omega} \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) \, dV_{\mathbf{X}} + \int_{\partial\Omega} \mathbf{S}(\mathbf{X}) \mathbf{n} \, dA_{\mathbf{X}} = \mathbf{0},$$

and upon applying the divergence theorem, we find that we require

$$(3.39) \quad \int_{\Omega} \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) + \nabla \cdot \mathbf{S}(\mathbf{X}) \, dV_{\mathbf{X}} = \mathbf{0},$$

for all open $\Omega \subset B$. Therefore, we must have that

$$(3.40) \quad \nabla \cdot \mathbf{S}(\mathbf{X}) + \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) = \mathbf{0},$$

for all $\mathbf{X} \in B$.

Next, we consider the implications of the resultant torque vanishing. We require

$$(3.41) \quad \int_{\Omega} (\mathbf{X} - \mathbf{p}) \times \rho(\mathbf{X}) \mathbf{b}(\mathbf{X}) dV_{\mathbf{X}} + \int_{\partial\Omega} (\mathbf{X} - \mathbf{p}) \times \mathbf{t}(\mathbf{X}) dA_{\mathbf{X}} = \mathbf{0},$$

for all open $\Omega \subset B$. Writing $\mathbf{t}(\mathbf{X}) = \mathbf{S}(\mathbf{X})\mathbf{n}$ and using the fact that $\rho(\mathbf{X})\mathbf{b}(\mathbf{X}) = -\nabla \cdot \mathbf{S}(\mathbf{X})$ gives

$$(3.42) \quad \int_{\Omega} -(\mathbf{X} - \mathbf{p}) \times \nabla \cdot \mathbf{S}(\mathbf{X}) dV_{\mathbf{X}} + \int_{\partial\Omega} (\mathbf{X} - \mathbf{p}) \times \mathbf{S}(\mathbf{X})\mathbf{n} dA_{\mathbf{X}} = \mathbf{0}.$$

Define the second order tensor $\mathbf{R}$ so that $R_{il} = \epsilon_{ijk}(\mathbf{X}_j - \mathbf{p}_j)\mathbf{S}_{kl}(\mathbf{X})$ which allows us to write

$$(3.43) \quad \int_{\Omega} -(\mathbf{X} - \mathbf{p}) \times \nabla \cdot \mathbf{S}(\mathbf{X}) dV_{\mathbf{X}} + \int_{\partial\Omega} \mathbf{R}(\mathbf{X})\mathbf{n} dA_{\mathbf{X}} = \mathbf{0}.$$

Upon using the divergence theorem, we have

$$(3.44) \quad \int_{\Omega} -(\mathbf{X} - \mathbf{p}) \times \nabla \cdot \mathbf{S}(\mathbf{X}) + \nabla \cdot \mathbf{R}(\mathbf{X}) dV_{\mathbf{X}} = \mathbf{0},$$

for all open subsets $\Omega \subset B$, so we must have that

$$(3.45) \quad (\mathbf{X} - \mathbf{p}) \times \nabla \cdot \mathbf{S}(\mathbf{X}) = \nabla \cdot \mathbf{R}(\mathbf{X}),$$

for all $\mathbf{X} \in B$. Now, let us examine the divergence of $\mathbf{R}$. We see that

$$(3.46) \quad (\nabla \cdot \mathbf{R}(\mathbf{X}))_i = R_{ij,j}$$

$$(3.47) \quad = (\epsilon_{ikl}(X_k - p_k)S_{lj})_{,j}$$

$$(3.48) \quad = \epsilon_{ikl}(\delta_{kj})S_{lj} + \epsilon_{ikl}(X_k - p_k)S_{lj,j}$$

$$(3.49) \quad = \epsilon_{ijl}S_{lj} + \epsilon_{ikl}(X_k - p_k)S_{lj,j}$$

$$(3.50) \quad = \epsilon_{ijl}S_{lj} + ((\mathbf{X} - \mathbf{p}) \times \nabla \cdot \mathbf{S}(\mathbf{X}))_i,$$

so for the resultant torque to vanish, we must have that $\epsilon_{ijl}S_{lj} = 0$ for all $\mathbf{X} \in B$. Now, using the properties of the permutation symbol and index notation, we see that

$$(3.51) \quad 0 = \epsilon_{ijl}S_{lj}$$

$$(3.52) \quad = \frac{1}{2}(\epsilon_{ijl}S_{lj} + \epsilon_{ijl}S_{lj})$$

$$(3.53) \quad = \frac{1}{2}(\epsilon_{ijl}S_{lj} - \epsilon_{ilj}S_{lj})$$

$$(3.54) \quad = \frac{1}{2}(\epsilon_{ijl}S_{lj} - \epsilon_{ijl}S_{jl}),$$

so we must have that $S_{lj} = S_{jl}$ for all $\mathbf{X} \in B$. That is,

$$(3.55) \quad \mathbf{S}(\mathbf{X}) = \mathbf{S}^T(\mathbf{X}),$$

for all $\mathbf{X} \in B$. Thus, our two pointwise equilibrium conditions are

$$(3.56) \quad \nabla \cdot \mathbf{S}(\mathbf{x}) + \rho(\mathbf{X})\mathbf{b}(\mathbf{X}) = \mathbf{0}$$

$$(3.57) \quad \mathbf{S}^T(\mathbf{X}) = \mathbf{S}(\mathbf{X})$$

for all $\mathbf{X} \in B$.

3.4. States of Stress

We introduced the Cauchy stress tensor $\mathbf{S}$, and we now examine some simple states of stress to understand the physical actualization of the interaction between the traction applied to a body and the resulting stress.

3.4.1. Spherical Stress. We say that the material body is in a state of spherical stress if the Cauchy stress may be expressed as

$$(3.58) \quad \mathbf{S} = -\pi\mathbf{I}.$$

In this case, if $\mathbf{n}$ is the normal to the surface, the traction is

$$(3.59) \quad \mathbf{t} = \mathbf{S}\mathbf{n}$$

$$(3.60) \quad = -\pi\mathbf{n},$$

so the traction is always normal to the surface. If the surface of the body is a sphere, then the traction is always pointing toward the center of the sphere. The spherical state of stress gives rise to the notion of pressure,

$$(3.61) \quad p = -\frac{1}{3} \text{tr} \mathbf{S},$$

which characterizes how the body changes in volume without changing shape. By our sign convention, when p is positive, the material body is in a state of compression. Let

$$(3.62) \quad \mathbf{S}_S = -p\mathbf{I},$$

be the spherical stress tensor. The deviatoric stress tensor is

$$(3.63) \quad \mathbf{S}_D = \mathbf{S} - \mathbf{S}_S$$

and characterizes the change in shape of the material body without a change in volume, so the deviatoric stress captures the effects of shear stress.

3.4.2. Uniaxial Stress. We say that the stress is uniaxial if there is a unit vector $\boldsymbol{\alpha}$ so that the stress may be written as

$$(3.64) \quad \mathbf{S} = \sigma \boldsymbol{\alpha} \otimes \boldsymbol{\alpha},$$

where σ is some nonzero constant. When $\sigma > 0$, we say that the body is in pure tension, and when $\sigma < 0$, we say that the body is in pure compression. If $\mathbf{n}$ is the normal to some surface, we see that the traction is

$$(3.65) \quad \mathbf{t} = \mathbf{S}\mathbf{n} = (\boldsymbol{\alpha} \cdot \mathbf{n})\sigma\boldsymbol{\alpha},$$

so the traction is always parallel to $\boldsymbol{\alpha}$, and the traction vanishes when $\boldsymbol{\alpha}$ and $\mathbf{n}$ are orthogonal.

3.4.3. Pure Shear. We say that the body is in a state of pure shear if there are orthogonal unit vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ so that the stress tensor may be expressed as

$$(3.66) \quad \mathbf{S} = \tau(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha}),$$

where τ is some constant. In a state of pure shear, we see that the traction is

$$(3.67) \quad \mathbf{t} = \mathbf{S}\mathbf{n} = (\boldsymbol{\beta} \cdot \mathbf{n})\tau\boldsymbol{\alpha} + (\boldsymbol{\alpha} \cdot \mathbf{n})\tau\boldsymbol{\beta}.$$

When $\mathbf{n} = \boldsymbol{\beta}$, we see that

$$(3.68) \quad \mathbf{t} = \tau\boldsymbol{\alpha},$$

and when $\mathbf{n} = \boldsymbol{\alpha}$, we see that

$$(3.69) \quad \mathbf{t} = \tau\boldsymbol{\beta},$$

due to the orthogonality of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

The normal traction is the component of the traction in the normal direction,

$$(3.70) \quad \mathbf{t}_n = (\mathbf{t} \cdot \mathbf{n})\mathbf{n},$$

and the magnitude of the normal traction is called the normal stress

$$(3.71) \quad \sigma_n = |\mathbf{t}_n|.$$

The shear traction is the remaining part of the traction after the normal component has been removed,

$$(3.72) \quad \mathbf{t}_s = \mathbf{t} - \mathbf{t}_n,$$

and the magnitude of the shear traction is called the shear stress,

$$(3.73) \quad \sigma_s = |\mathbf{t}_s|.$$

3.4.4. Plane Stress. We say that a body is in a state of plane stress if there is an orthonormal coordinate system with $\mathbf{e}_1 = \boldsymbol{\alpha}$, $\mathbf{e}_2 = \boldsymbol{\beta}$, and $\mathbf{e}_3 = \boldsymbol{\alpha} \times \boldsymbol{\beta}$ so that the stress tensor takes the form

$$(3.74) \quad \mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where α and β are suitably chosen unit vectors. The state of plane stress reduces the three dimensional stress tensor to a two dimensional form, and this formulation is used to approximate thin materials.

3.4.5. Principal Stresses. The stress tensor can be understood in terms of its eigenvalue eigenvector pairs which are called the principal stresses and principal stress directions, respectively. Since the stress tensor is symmetric, there will be three real eigenvalues and three orthogonal eigenvectors. Let σ_1 be a principal stress with α_1 the principal stress direction. Then, the traction is

$$(3.75) \quad \mathbf{t} = \mathbf{S}\alpha = \sigma\alpha.$$

3.4.6. von Mises Stress. The von Mises stress, is stress measure that is commonly used as a material yield criterion. The von Mises stress is based on the second invariant of the deviatoric stress, and is given by

$$(3.76) \quad J_2 = \sqrt{\frac{3}{2} \mathbf{S}_D : \mathbf{S}_D} = \sqrt{\frac{(S_{11} - S_{22})^2 + (S_{22} - S_{33})^2 + (S_{33} - S_{11})^2 + 6(S_{23}^2 + S_{31}^2 + S_{12}^2)}{2}}$$

3.5. Kinematics

We have developed the language to describe the state of a material body at a fixed moment in time. Now, we define the tools we require to describe how the material body deforms and travels through space and time. That is, we want to describe the evolution of the configuration of the material body B at some later time t . We call B the reference configuration and let B^t be the time t configuration of B which we call the deformed configuration. We refer to points in the reference configuration as $\mathbf{X} = (X_1, X_2, X_3)^T$, and points in the deformed state are denoted as $\mathbf{x} = (x_1, x_2, x_3)^T$. The coordinates $\mathbf{X}$ are called material coordinates, and we refer to $\mathbf{x}$ as spatial coordinates.

3.5.1. Deformation Map. The material points in the reference configuration B are mapped to spatial coordinates in the deformed configuration B^t . We identify a mapping $\varphi : B \rightarrow B^t$ from the reference configuration to the time t deformed configuration which we call the deformation map. For now, we only consider the deformation map from the reference configuration to a fixed time t deformed configuration. Every material point $\mathbf{X}$ in the reference configuration is mapped to

a point $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$ in the deformed configuration, and the displacement of a material point is given by $\mathbf{u}(\mathbf{X}) = \boldsymbol{\varphi}(\mathbf{X}) - \mathbf{X}$. We require the following conditions to be satisfied by the deformation map:

- (1) $\boldsymbol{\varphi} : B \rightarrow B^t$ is one-to-one,
- (2) $\det(\nabla \boldsymbol{\varphi}(\mathbf{X})) > 0 \ \forall \mathbf{X} \in B$.

If the deformation map satisfies these two conditions, we call $\boldsymbol{\varphi}$ admissible. The first condition ensures that no two material points occupy the same region of space in the deformed state, and the second condition ensures that the body cannot be deformed in a manner which allows points to be passed through one another. That is, the reference configuration cannot be continuously deformed into its mirror image. Note that an admissible deformation map is onto and one-to-one, so an inverse map $\boldsymbol{\varphi}^{-1} : B^t \rightarrow B$ exists.

3.5.2. Strain. We would like to characterize how the material body changes locally when a material point is mapped from the reference configuration to the deformed configuration. Let $B_r(\mathbf{X}_0) \subset B$ be the open ball of radius r centered at $\mathbf{X}_0$ so that $\mathbf{x}_0 = \boldsymbol{\varphi}(\mathbf{X}_0)$. If we consider deformed ball

$$(3.77) \quad \tilde{B}_r(\mathbf{X}_0) = \{\mathbf{x} \in B^t \mid \mathbf{x} \in \boldsymbol{\varphi}(\mathbf{X}), \ \mathbf{X} \in B_r(\mathbf{X}_0)\},$$

the strain describes the difference between the open balls in the reference and deformed configurations and provides a description of the amount of stretching that occurs due to the mapping $\boldsymbol{\varphi}$. The measure of the deformation that occurs under $\boldsymbol{\varphi}$ is called the strain, and we now consider three strain measures.

3.5.2.1. *Deformation Gradient.* The deformation gradient is the second order tensor

$$(3.78) \quad \mathbf{F}(\mathbf{X}) = \nabla \boldsymbol{\varphi}(\mathbf{X}).$$

The inspiration for this measure comes from the Taylor expansion of $\boldsymbol{\varphi}$ about a point $\mathbf{X}_0 \in B$. We see that the Taylor series of $\boldsymbol{\varphi}$ is

$$(3.79) \quad \boldsymbol{\varphi}(\mathbf{X}) = \boldsymbol{\varphi}(\mathbf{X}_0) + \mathbf{F}(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) + \mathcal{O}(|\mathbf{X} - \mathbf{X}_0|^2),$$

and upon dropping higher order terms, we see that

$$(3.80) \quad \varphi(\mathbf{X}) = \varphi(\mathbf{X}_0) + \mathbf{F}(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0)$$

$$(3.81) \quad = \mathbf{c} + \mathbf{F}(\mathbf{X}_0)\mathbf{X},$$

where

$$(3.82) \quad \mathbf{c} = \varphi(\mathbf{X}_0) - \mathbf{F}(\mathbf{X}_0)\mathbf{X}_0,$$

is a vector that only depends on $\mathbf{X}_0$. We see that $\mathbf{F}$ characterizes how a material point $\mathbf{X}$ near $\mathbf{X}_0$ changes under the deformation map.

To gain a better understanding of the deformation gradient, let us suppose $\mathbf{F}$ is constant. In this case, we call φ a homogeneous deformation, and we have that

$$(3.83) \quad \varphi(\mathbf{X}) = \mathbf{c} + \mathbf{F}\mathbf{X}.$$

The three basic homogeneous deformations are translations, rotations, and stretches, and we will demonstrate that any homogeneous deformation may be understood as a composition of these three deformations. We call the homogeneous deformation φ a translation if

$$(3.84) \quad \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{c},$$

for some vector $\mathbf{c}$. When discussing rotations, we consider a rotation about some fixed point, and we call the point $\mathbf{P}$ a fixed point of φ if

$$(3.85) \quad \varphi(\mathbf{X}) = \mathbf{P} + \mathbf{F}(\mathbf{X} - \mathbf{P}),$$

that is, $\varphi(\mathbf{P}) = \mathbf{P}$. We call φ a rotation about the fixed point $\mathbf{P}$ if

$$(3.86) \quad \varphi(\mathbf{X}) = \mathbf{P} + \mathbf{Q}(\mathbf{X} - \mathbf{P}),$$

where $\mathbf{Q}$ is a rotation tensor. Under rotation, there is no change in the shape of the material body, but the net orientation of the body may change. We call φ a stretch with respect to the point $\mathbf{P}$ if

$$(3.87) \quad \varphi(\mathbf{X}) = \mathbf{P} + \mathbf{S}(\mathbf{X} - \mathbf{P}),$$

for some symmetric, positive-definite tensor $\mathbf{S}$. Using the polar decomposition theorem, we may express the deformation gradient as

$$(3.88) \quad \mathbf{F} = \mathbf{Q}\mathbf{S}$$

where $\mathbf{Q}$ is a rotation, and $\mathbf{S}$ is symmetric, positive-definite with $\mathbf{S} = \sqrt{\mathbf{F}^T \mathbf{F}}$. Therefore, we see that any homogeneous deformation may be written as

$$(3.89) \quad \varphi(\mathbf{X}) = \mathbf{c} + \mathbf{Q}\mathbf{S}\mathbf{X},$$

which consists of a stretch, rotation, and translation.

3.5.2.2. *Cauchy-Green Strain Tensor.* The next measure of strain we consider is the Cauchy-Green strain tensor,

$$(3.90) \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}.$$

We immediately see that $\mathbf{C}$ is symmetric and positive-definite, and the polar decomposition of $\mathbf{C}$ is

$$(3.91) \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}$$

$$(3.92) \quad = (\mathbf{Q}\mathbf{S})^T (\mathbf{Q}\mathbf{S})$$

$$(3.93) \quad = \mathbf{S}^T \mathbf{Q}^T \mathbf{Q} \mathbf{S}$$

$$(3.94) \quad = \mathbf{S}^T \mathbf{S},$$

so $\mathbf{C}$ only contains information about stretches.

We say that φ is a rigid deformation if

$$(3.95) \quad \varphi(\mathbf{X}) = \mathbf{c} + \mathbf{Q}\mathbf{X},$$

where $\mathbf{c}$ is a translation and $\mathbf{Q}$ is a rotation. Under a rigid deformation, we see that the deformation gradient is a rotation, so the Cauchy-Green tensor is

$$(3.96) \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}$$

$$(3.97) \quad = \mathbf{Q}^T \mathbf{Q}$$

$$(3.98) \quad = \mathbf{I}.$$

Therefore, under a rigid deformation, there is no strain according to the Cauchy-Green strain tensor.

3.5.2.3. *Infinitesimal Strain Tensor.* The final measure of strain we consider is the infinitesimal strain tensor. Let $\mathbf{u}$ be the displacement field and $\nabla \mathbf{u}$ the displacement gradient. Then, the infinitesimal strain tensor is given by

$$(3.99) \quad \mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

We see that the infinitesimal strain tensor is symmetric, and $\mathbf{E}$ may be written in terms of the deformation gradient and Cauchy-Green strain tensor. First, we relate the infinitesimal strain tensor to the deformation gradient,

$$(3.100) \quad \nabla \mathbf{u} = \nabla(\varphi(\mathbf{X}) - \mathbf{X})$$

$$(3.101) \quad = \mathbf{F} - \mathbf{I},$$

so we see that

$$(3.102) \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T + \mathbf{F} - 2\mathbf{I}).$$

Next, we relate the infinitesimal strain tensor to the Cauchy-Green strain tensor. We have

$$(3.103) \quad \mathbf{F} = \nabla \mathbf{u} + \mathbf{I},$$

so

$$(3.104) \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}$$

$$(3.105) \quad = (\nabla \mathbf{u} + \mathbf{I})^T (\nabla \mathbf{u} + \mathbf{I})$$

$$(3.106) \quad = \nabla \mathbf{u} + \nabla \mathbf{u}^T + \nabla \mathbf{u}^T \nabla \mathbf{u} + \mathbf{I}.$$

Thus,

$$(3.107) \quad \mathbf{C} = 2\mathbf{E} + \nabla \mathbf{u}^T \nabla \mathbf{u} + \mathbf{I},$$

or

$$(3.108) \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) - \frac{1}{2}\nabla \mathbf{u}^T \nabla \mathbf{u}.$$

We call the deformation small if there exists $0 \leq \epsilon \ll 1$ such that $|\nabla \mathbf{u}| = \mathcal{O}(\epsilon)$. In this case, we see that

$$(3.109) \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) + \mathcal{O}(\epsilon^2),$$

so if we drop terms of order $\mathcal{O}(\epsilon^2)$, we see that the infinitesimal strain tensor and Cauchy-Green tensor are identical up to a scaling and offset.

3.5.3. Motions. We have established the language to describe how a material body changes under a single deformation map, and we now extend this to multiple deformations that are each associated with some time t . That is, the body will have some deformed state at time t that is described by $\varphi : B \times [0, \infty) \rightarrow \mathbb{E}^3$ called a motion. At time t , the deformed state is denoted B^t , and for $\mathbf{x} \in B^t$, we have $\mathbf{x} = \varphi(\mathbf{X}, t)$. The reference configuration is the time $t = 0$ configuration so that $\varphi(\mathbf{X}, 0) = \mathbf{X}$. Now, fields such as the velocity, acceleration, stress, and strain are defined over the material body which evolve as time progresses. We refer to fields defined over the time t configuration as the spatial field, and fields defined over the reference configuration as material fields. Since at each time t , φ is invertible, we may freely travel between the material and spatial descriptions. We denote spatial fields by lowercase variables and material fields by uppercase variables. If $\Theta(\mathbf{X}, t)$ is a material field, the spatial description of Θ is given by the pullback of the

flow map,

$$(3.110) \quad \Theta_s(\mathbf{x}, t) = \Theta(\varphi^{-1}(\mathbf{x}, t), t).$$

Similarly, if we have the spatial field $\boldsymbol{\theta}(\mathbf{x}, t)$, the material description is

$$(3.111) \quad \boldsymbol{\theta}_m(\mathbf{X}, t) = \boldsymbol{\theta}(\varphi(\mathbf{X}, t), t).$$

For coordinate derivatives, derivatives with respect to material and spatial coordinates are denoted $\nabla^{\mathbf{X}}$ and $\nabla^{\mathbf{x}}$, respectively. The deformation map now depends on time, and we are interested in how the fields evolve with the material body over time.

3.5.4. Velocity and Acceleration. The velocity of a material particle $\mathbf{X}$ at time t is

$$(3.112) \quad \mathbf{V}(\mathbf{X}, t) = \frac{\partial \varphi}{\partial t}(\mathbf{X}, t),$$

and the acceleration of a material particle at time t is

$$(3.113) \quad \mathbf{A}(\mathbf{X}, t) = \frac{\partial^2 \varphi}{\partial t^2}(\mathbf{X}, t).$$

The expressions for the velocity and acceleration are both given in the material space description, and the spatial counterparts are given by the pullback of the flow map,

$$(3.114) \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{V}_s(\mathbf{x}, t) = \mathbf{V}(\varphi^{-1}(\mathbf{x}, t), t),$$

and

$$(3.115) \quad \mathbf{a}(\mathbf{x}, t) = \mathbf{A}_s(\mathbf{x}, t) = \mathbf{A}(\varphi^{-1}(\mathbf{x}, t), t).$$

3.5.5. Total Time Derivative. The total time derivative of a spatial field quantity $\boldsymbol{\theta}(\mathbf{x}, t)$ is given by

$$(3.116) \quad \frac{D}{Dt} \boldsymbol{\theta}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\theta}}{\partial t}(\mathbf{x}, t) + \nabla^{\mathbf{x}} \boldsymbol{\theta}(\mathbf{x}, t) \mathbf{v},$$

where $\mathbf{v}$ is the spatial velocity of the body. We see that the total time derivative is composed of the change with respect to time of the field quantity at a given spatial coordinate and the change due to the field quantity being advected by the spatial velocity.

3.5.6. Rate of Strain and Rotation. The rate of strain is

$$(3.117) \quad \mathbf{L} = \frac{1}{2}(\nabla^x \mathbf{v} + \nabla^x \mathbf{v}^T),$$

and the rate of rotation is

$$(3.118) \quad \boldsymbol{\omega} = \frac{1}{2}(\nabla^x \mathbf{v} - \nabla^x \mathbf{v}^T).$$

3.5.7. Piola-Kirchhoff Stress. The nominal or first Piola-Kirchhoff stress field is

$$(3.119) \quad \mathbf{P}(\mathbf{X}, t) = J(\mathbf{X}, t) \mathbf{S}_m(\mathbf{X}, t) \mathbf{F}^{-T}(\mathbf{X}, t),$$

where the field $J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t)$ is the Jacobian field of $\boldsymbol{\varphi}$. The second Piola-Kirchhoff stress field is

$$(3.120) \quad \boldsymbol{\Sigma}(\mathbf{X}, t) = \mathbf{F}^{-1}(\mathbf{X}, t) \mathbf{P}(\mathbf{X}, t).$$

3.5.8. Rigid Motions. A motion is said to be rigid if

$$(3.121) \quad \boldsymbol{\varphi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{R}(t) \mathbf{X},$$

where $\mathbf{c}(t)$ represents a translation and $\mathbf{R}(t)$ represents a rotation. The velocity of a rigid motion is,

$$(3.122) \quad \mathbf{V}(\mathbf{X}, t) = \mathbf{c}'(t) + \mathbf{R}'(t) \mathbf{X}.$$

The next chapter will be dedicated to rigid body motions.

3.5.9. Reynold's Transport Theorem. Let Ω^t be an open subset of B^t and Ω the corresponding open subset of the reference configuration B . We can relate an integral over the deformed configuration Ω^t to an integral over the reference configuration Ω using the relation

$$(3.123) \quad \int_{\Omega^t} \boldsymbol{\theta}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega} \boldsymbol{\theta}(\mathbf{X}, t) J(\mathbf{X}, t) dV_{\mathbf{X}},$$

and the time derivative of the Jacobian is given by

$$(3.124) \quad \frac{\partial}{\partial t} J(\mathbf{X}, t) = J(\mathbf{X}, t) \nabla^x \cdot \mathbf{v}(\mathbf{x}, t).$$

Putting these two results together gives us the Reynold Transport theorem,

$$(3.125) \quad \frac{d}{dt} \int_{\Omega^t} \boldsymbol{\theta}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega^t} \frac{\partial}{\partial t} \boldsymbol{\theta}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \boldsymbol{\theta}(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n} dA_{\mathbf{x}}.$$

3.5.10. Transformation of Surface Integrals. We will require the transformation of surface integral between the deformed configuration and reference configuration. For a scalar field quantity, we have the identity

$$(3.126) \quad \int_{\partial\Omega^t} \theta(\mathbf{x}, t) \mathbf{n}(\mathbf{x}) dA_{\mathbf{x}} = \int_{\partial\Omega^0} \theta_m(\mathbf{X}, t) J(\mathbf{X}, t) \mathbf{F}^{-T}(\mathbf{X}, t) \mathbf{N}(\mathbf{X}) dA_{\mathbf{X}},$$

and the identity for vector and tensor fields is defined analogously.

3.6. Balance Laws

We now establish a collection of balance laws that will give rise to the mechanics of our continuum body. We are interested in whether the mass, linear momentum, angular momentum, and energy of the material body is increased, decreased, or conserved as the material body interacts with its environment. The mass of Ω^t is

$$(3.127) \quad m = \int_{\Omega^t} \rho(\mathbf{x}, t) dV_{\mathbf{x}},$$

the linear momentum is

$$(3.128) \quad \boldsymbol{\ell} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}},$$

and the angular momentum is

$$(3.129) \quad \mathbf{j}_{\mathbf{p}} = \int_{\Omega^t} (\mathbf{x} - \mathbf{p}) \times \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}}.$$

We take as an axiom that the mass of any open subset of the continuum body does not change as the body undergoes deformations, that is

$$(3.130) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) dV_{\mathbf{x}} = 0,$$

for all open subsets Ω_t of B_t . If the material body is subject to a body force per unit mass $\mathbf{b}(\mathbf{x}, t)$ and an external traction field $\mathbf{t}(\mathbf{x}, t)$, the resultant force on Ω_t is

$$(3.131) \quad \mathbf{r} = \int_{\Omega_t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega_t} \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}},$$

and the resultant torque on Ω_t with respect to $\mathbf{p}$ is

$$(3.132) \quad \boldsymbol{\tau} = \int_{\Omega_t} (\mathbf{x} - \mathbf{p}) \times \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega_t} (\mathbf{x} - \mathbf{p}) \times \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}}.$$

An additional axiom we adopt is that the change in linear momentum is equal to the resultant force,

$$(3.133) \quad \frac{d}{dt} \boldsymbol{\ell} = \mathbf{r},$$

and the change in angular momentum is equal to the resultant torque,

$$(3.134) \quad \frac{d}{dt} \mathbf{j} = \boldsymbol{\tau}.$$

Let $\psi(\mathbf{x}, t)$ be the internal energy density field per unit mass so that the internal energy over an arbitrary subset $\Omega^t \subset B^t$ is given by

$$(3.135) \quad U = \int_{\Omega^t} \rho(\mathbf{x}, t) \psi(\mathbf{x}, t) dV_{\mathbf{x}}.$$

The kinetic energy is

$$(3.136) \quad K = \int_{\Omega^t} \frac{1}{2} \rho(\mathbf{x}, t) |\mathbf{v}(\mathbf{x}, t)|^2 dV_{\mathbf{x}},$$

and the power of external forces is

$$(3.137) \quad \mathcal{P} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \mathbf{t}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dA_{\mathbf{x}}.$$

The final axiom we adopt states that the change in internal and kinetic energy is equal to the power of external forces,

$$(3.138) \quad \frac{d}{dt} (U + K) = \mathcal{P}.$$

We now derive the local forms of the balance laws in the Eulerian (spatial) and Lagrangian (material) forms.

3.6.1. Conservation of Mass.

THEOREM 3.6.1. (Lagrangian Form) *Let B be a material body with deformation map φ and mass density field $\rho(\mathbf{x}, t)$. The conservation of mass requires that*

$$(3.139) \quad \rho_m(\mathbf{X}, t)J(\mathbf{X}, t) - \rho_m(\mathbf{X}, 0) = 0$$

for all $\mathbf{X} \in B$.

PROOF. For any $\Omega^t \subset B_t$, conservation of mass requires

$$(3.140) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) dV_{\mathbf{x}} = 0,$$

at any time t . Equivalently, we have

$$(3.141) \quad \frac{d}{dt} \int_{\Omega^0} \rho_m(\mathbf{X}, t)J(\mathbf{X}, t) dV_{\mathbf{X}} = 0,$$

and integrating (3.141) from 0 to t , we have

$$(3.142) \quad \int_{\Omega^0} \rho_m(\mathbf{X}, t)J(\mathbf{X}, t) - \rho_m(\mathbf{X}, 0)J(\mathbf{X}, 0) dV_{\mathbf{X}} = 0,$$

and since $J(\mathbf{X}, 0) = 1$,

$$(3.143) \quad \int_{\Omega^0} \rho_m(\mathbf{X}, t)J(\mathbf{X}, t) - \rho_m(\mathbf{X}, 0) dV_{\mathbf{X}} = 0.$$

Now, 3.143 must hold for any $\Omega^0 \subset B$, so we must have that

$$(3.144) \quad \rho_m(\mathbf{X}, t)J(\mathbf{X}, t) - \rho_m(\mathbf{X}, 0) = 0$$

for all $\mathbf{X} \in B$. □

THEOREM 3.6.2. (Eulerian Form) *Let φ be a motion of a material body B with velocity field $\mathbf{v}(\mathbf{x}, t)$ and mass density field $\rho(\mathbf{x}, t)$. The conservation of mass requires*

$$(3.145) \quad \frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \nabla^{\mathbf{x}} \cdot (\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)) = 0,$$

for all $\mathbf{x} \in B^t$.

PROOF. Let Ω^t be an open subset of the material body B_t and Ω^0 the corresponding subset in B . Let m_0 denote the mass of Ω^0 and m_t denote the mass of Ω^t . By the conservation of mass axiom, we require $m_t = m_0$, so

$$(3.146) \quad \int_{\Omega^t} \rho(\mathbf{x}, t) \, d\mathbf{x} = \int_{\Omega^0} \rho(\mathbf{X}, 0) \, d\mathbf{X}.$$

Taking the time derivative of (3.146), we see that

$$(3.147) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \, d\mathbf{x} = 0$$

since the right hand side of (3.146) does not depend on time. Now, examining the left hand side of (3.147), we have

$$(3.148) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \, d\mathbf{x} = \frac{d}{dt} \int_{\Omega^0} \rho(\varphi(\mathbf{X}, t), t) J(\mathbf{X}, t) \, d\mathbf{X}$$

$$(3.149) \quad = \int_{\Omega^0} \frac{\partial}{\partial t} \rho(\varphi(\mathbf{X}, t), t) J(\mathbf{X}, t) + \rho(\varphi(\mathbf{X}, t), t) \frac{\partial}{\partial t} J(\mathbf{X}, t) \, d\mathbf{X}.$$

Now,

$$(3.150) \quad \frac{\partial}{\partial t} \rho(\varphi(\mathbf{X}, t), t) = \frac{\partial \rho}{\partial t}(\varphi(\mathbf{X}, t), t) + \frac{\partial \rho}{\partial \mathbf{x}} \cdot \mathbf{v}(\varphi(\mathbf{X}, t), t)$$

and

$$(3.151) \quad \frac{\partial J}{\partial t} = J(\mathbf{X}, t) \nabla^{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t),$$

so we have that

$$(3.152) \quad \int_{\Omega^0} \left(\frac{\partial \rho}{\partial t}(\varphi(\mathbf{X}, t), t) + \frac{\partial \rho}{\partial \mathbf{x}} \cdot \mathbf{v}(\varphi(\mathbf{X}, t), t) \right) J(\mathbf{X}, t) + \rho(\varphi(\mathbf{X}, t), t) J(\mathbf{X}, t) \nabla^{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) \, d\mathbf{X} = 0.$$

We see that each term in (3.152) has a factor of $J(\mathbf{X}, t)$, so we return to integrating over Ω^t ,

$$(3.153) \quad \int_{\Omega^t} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) + \rho(\mathbf{x}, t) \nabla^{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) \right) \, d\mathbf{x} = 0.$$

Since (3.153) must hold for any $\Omega^t \subset B^t$, we require

$$(3.154) \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) + \rho(\mathbf{x}, t) \nabla^{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t),$$

for all $\mathbf{x} \in B^t$. □

3.6.2. Balance of Linear Momentum.

THEOREM 3.6.3. (*Lagrangian Form*) The balance of linear momentum requires

$$(3.155) \quad \rho(\mathbf{X}, 0) \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}(\mathbf{X}, t) = \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X}, t) + \rho(\mathbf{X}, 0) \mathbf{b}(\mathbf{X}, t)$$

for all $\mathbf{X} \in B$ and $t \geq 0$.

PROOF. The balance of linear momentum requires that the time rate of change of the linear momentum is equal to the resultant force due to tractions and body forces,

$$(3.156) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}},$$

for any subset Ω^t of B^t . Now,

$$(3.157) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \frac{d}{dt} \int_{\Omega^0} \rho(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) J(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.158) \quad = \frac{d}{dt} \int_{\Omega^0} \rho(\mathbf{X}, 0) \frac{\partial \boldsymbol{\varphi}}{\partial t}(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.159) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}(\mathbf{X}, t) dV_{\mathbf{X}},$$

where we used the Lagrangian form of the conservation of mass. Now, turning our attention to the RHS of (3.156), we have

$$(3.160) \quad \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \mathbf{S}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}) dA_{\mathbf{x}}$$

$$(3.161) \quad = \int_{\Omega^0} \rho(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{b}(\boldsymbol{\varphi}(\mathbf{X}, t), t) J(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.162) \quad + \int_{\partial\Omega^0} \mathbf{S}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{F}(\mathbf{X}, t)^{-T} J(\mathbf{X}, t) \mathbf{N}(\mathbf{X}) dA_{\mathbf{X}}$$

$$(3.163) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\partial\Omega^0} \mathbf{P}(\mathbf{X}, t) \mathbf{N}(\mathbf{X}) dA_{\mathbf{X}}$$

$$(3.164) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\Omega^0} \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X}, t) dV_{\mathbf{X}},$$

so we see that

$$(3.165) \quad \int_{\Omega_0} \rho(\mathbf{X}, 0) \frac{\partial^2 \varphi}{\partial t^2}(\mathbf{X}, t) dV_{\mathbf{X}} = \int_{\Omega_0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) + \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X}, t) dV_{\mathbf{X}}.$$

This must hold for any $\Omega_0 \subset B$, so we must have that

$$(3.166) \quad \rho(\mathbf{X}, 0) \frac{\partial^2 \varphi}{\partial t^2}(\mathbf{X}, t) = \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) + \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X}, t),$$

for all $\mathbf{X} \in B$. □

THEOREM 3.6.4. (*Eulerian Form*) *The balance of linear momentum requires*

$$(3.167) \quad \rho(\mathbf{x}, t) \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla^{\mathbf{x}} \mathbf{v} \mathbf{v} \right) = \nabla^{\mathbf{x}} \cdot \mathbf{S} + \rho \mathbf{b},$$

for all $\mathbf{x} \in B_t$.

PROOF. The balance of linear momentum states that the change in linear momentum is equal to the resultant force

$$(3.168) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial \Omega^t} \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}},$$

for any open $\Omega^t \subset B^t$. Recalling that the Cauchy stress field $\mathbf{S}(\mathbf{x}, t)$ and outward unit normal $\mathbf{n}(\mathbf{x})$ are related to the applied traction by

$$(3.169) \quad \mathbf{t}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}),$$

we may write

$$(3.170) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial \Omega^t} \mathbf{S}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}) dA_{\mathbf{x}}.$$

Using the divergence theorem, we find that

$$(3.171) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) + \nabla^{\mathbf{x}} \cdot \mathbf{S}(\mathbf{x}, t) dV_{\mathbf{x}}.$$

Now,

$$(3.172) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \frac{d}{dt} \int_{\Omega^0} \rho(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{J}(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.173) \quad = \frac{d}{dt} \int_{\Omega} \rho(\mathbf{X}, 0) \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) dV_{\mathbf{X}}$$

$$(3.174) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) \frac{\partial}{\partial t} \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) dV_{\mathbf{X}}$$

$$(3.175) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) (\nabla^{\mathbf{x}} \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}) dV_{\mathbf{X}}$$

$$(3.176) \quad = \int_{\Omega^0} \rho(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{J}(\mathbf{X}, t) (\nabla^{\mathbf{x}} \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}) dV_{\mathbf{X}}$$

$$(3.177) \quad = \int_{\Omega^t} \rho(\mathbf{x}, t) (\mathbf{X}, t) (\nabla^{\mathbf{x}} \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}) dV_{\mathbf{x}},$$

so we must have that

$$(3.178) \quad \int_{\Omega^t} \rho(\mathbf{x}, t) (\nabla^{\mathbf{x}} \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}) dV_{\mathbf{x}} = \int_{\Omega_t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) + \nabla^{\mathbf{x}} \cdot \mathbf{S}(\mathbf{x}, t) dV_{\mathbf{x}},$$

for all open $\Omega^t \subset B^t$. Thus, we must have that

$$(3.179) \quad \rho \frac{D}{Dt} \mathbf{v} = \rho \mathbf{b} + \nabla^{\mathbf{x}} \cdot \mathbf{S},$$

for all $\mathbf{x} \in B^t$. □

3.6.3. Balance of Angular Momentum.

THEOREM 3.6.5. (Eulerian Form) *The conservation of angular momentum requires*

$$(3.180) \quad \mathbf{S} = \mathbf{S}^T,$$

for all $\mathbf{x} \in B^t$.

PROOF. The axiom of the conservation of angular momentum states that the time rate of change of the the angular momentum is equal to the resultant torque,

$$(3.181) \quad \frac{d}{dt} \int_{\Omega^t} \mathbf{x} \times \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\Omega^t} \mathbf{x} \times \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial \Omega^t} \mathbf{x} \times \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}},$$

where we have taken the origin to be the point about which the angular momentum and resultant torque are computed. The LHS of (3.181) becomes

$$(3.182) \quad \frac{d}{dt} \int_{\Omega^t} \mathbf{x} \times \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \frac{d}{dt} \int_{\Omega^0} \boldsymbol{\varphi}(\mathbf{X}, t) \times \rho_m(\mathbf{X}, t) \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) J(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.183) \quad = \frac{d}{dt} \int_{\Omega^0} \boldsymbol{\varphi}(\mathbf{X}, t) \times \rho_m(\mathbf{X}, 0) \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) dV_{\mathbf{X}}$$

$$(3.184) \quad = \int_{\Omega^0} \boldsymbol{\varphi}(\mathbf{X}, t) \times \rho_m(\mathbf{X}, 0) \frac{D}{Dt} \mathbf{v}(\boldsymbol{\varphi}(\mathbf{X}, t), t) dV_{\mathbf{X}}$$

$$(3.185) \quad = \int_{\Omega^t} \mathbf{x} \times \rho(\mathbf{x}, t) \frac{D}{Dt} \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}},$$

since $\mathbf{v} \times \mathbf{v} = 0$. Next, expressing the traction in terms of the Cauchy stress and using the balance of linear momentum, we find that the RHS of (3.181) may be written as

$$(3.186) \quad \int_{\Omega^t} \mathbf{x} \times \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \mathbf{x} \times \mathbf{t}(\mathbf{x}, t) dA_{\mathbf{x}} = \int_{\Omega^t} \mathbf{x} \times \left(\rho \frac{D}{Dt} \mathbf{v} - \nabla^x \cdot \mathbf{S} \right) dV_{\mathbf{x}} + \int_{\partial\Omega^t} \mathbf{x} \times \mathbf{S}(\mathbf{x}, t) \mathbf{n} dA_{\mathbf{x}}.$$

Therefore, (3.181) is equivalent to

$$(3.187) \quad \int_{\Omega^t} \mathbf{x} \times \nabla^x \cdot \mathbf{S} dV_{\mathbf{x}} - \int_{\partial\Omega^t} \mathbf{x} \times \mathbf{S}(\mathbf{x}, t) \mathbf{n} dA_{\mathbf{x}} = 0.$$

The expression in (3.187) is identical to the expression in (3.42), so if we proceed as we did for (3.42), we find that the pointwise requirement for the conservation of angular momentum is that for all $\mathbf{x} \in B^t$,

$$(3.188) \quad \mathbf{S} = \mathbf{S}^T.$$

□

THEOREM 3.6.6. (Lagrangian Form) *The conservation of angular momentum requires that*

$$(3.189) \quad \mathbf{P} \mathbf{F}^T = \mathbf{F} \mathbf{P}^T,$$

for all $\mathbf{X} \in B$. Equivalently, conservation of angular momentum is satisfied if

$$(3.190) \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T,$$

for all $\mathbf{X} \in B$.

PROOF. We found that the Lagrangian form of conservation of angular momentum required that

$$(3.191) \quad \mathbf{S}(\mathbf{x}, t) = \mathbf{S}^T(\mathbf{x}, t),$$

for all $\mathbf{x} \in B^t$. We see that

$$(3.192) \quad \mathbf{P}\mathbf{F}^T = J(\mathbf{X}, t)\mathbf{S}_m(\mathbf{X}, t)\mathbf{F}^{-T}(\mathbf{X}, t)\mathbf{F}^T(\mathbf{X}, t)$$

$$(3.193) \quad = J(\mathbf{X}, t)\mathbf{S}_m^T(\mathbf{X}, t)$$

$$(3.194) \quad = \mathbf{F}(\mathbf{X}, t)\mathbf{F}^{-1}(\mathbf{X}, t)J(\mathbf{X}, t)\mathbf{S}_m^T(\mathbf{X}, t)$$

$$(3.195) \quad = \mathbf{F}(\mathbf{X}, t)(J(\mathbf{X}, t)\mathbf{S}_m(\mathbf{X}, t)\mathbf{F}^{-T}(\mathbf{X}, t))^T$$

$$(3.196) \quad = \mathbf{F}(\mathbf{X}, t)\mathbf{P}^T(\mathbf{X}, t)$$

which is our desired result. The equivalent statement for the second Piola-Kirchhoff stress follows in the exact same fashion. $\square$

3.6.4. Conservation of Energy.

THEOREM 3.6.7. (*Lagrangian Form*) *The conservation of energy requires that*

$$(3.197) \quad \rho(\mathbf{X}, 0)\frac{\partial \Psi}{\partial t} = \mathbf{P} : \frac{\partial \mathbf{F}}{\partial t},$$

for all $\mathbf{X} \in B$.

PROOF. The conservation of energy states

$$(3.198) \quad \frac{d}{dt}(U + K) = \mathcal{P}.$$

First, we compute the derivative of the internal and kinetic energies,

$$(3.199) \quad \frac{d}{dt}(U + K) = \frac{d}{dt} \int_{\Omega^0} \rho(\mathbf{X}, 0) \Psi(\mathbf{X}, t) + \frac{1}{2} \rho(\mathbf{X}, 0) |\mathbf{V}(\mathbf{X}, t)|^2 dV_{\mathbf{X}}$$

$$(3.200) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) \frac{\partial \Psi}{\partial t}(\mathbf{X}, t) + \rho(\mathbf{X}, 0) \mathbf{V}(\mathbf{X}, t) \cdot \mathbf{A}(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.201) \quad = \int_{\Omega^0} \rho(\mathbf{X}, 0) \frac{\partial \Psi}{\partial t}(\mathbf{X}, t) + \mathbf{V}(\mathbf{X}, t) \cdot (\nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X}, t) + \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t)) dV_{\mathbf{X}}$$

$$(3.202)$$

Next, the RHS may be expressed as

$$(3.203)$$

$$\mathcal{P} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial \Omega^t} \mathbf{t}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dA_{\mathbf{x}}$$

$$(3.204)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\partial \Omega^t} \mathbf{S}^T(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{n} dA_{\mathbf{x}}$$

$$(3.205)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\partial \Omega^0} \mathbf{S}_m^T(\mathbf{X}, t) \mathbf{V}(\mathbf{X}, t) \cdot \mathbf{J}(\mathbf{X}, t) \mathbf{F}^{-T}(\mathbf{X}, t) \mathbf{N} dA_{\mathbf{X}}$$

$$(3.206)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\partial \Omega^0} \mathbf{V}(\mathbf{X}, t) \cdot \mathbf{J}(\mathbf{X}, t) \mathbf{S}_m(\mathbf{X}, t) \mathbf{F}^{-T}(\mathbf{X}, t) \mathbf{N} dA_{\mathbf{X}}$$

$$(3.207)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\partial \Omega^0} \mathbf{V}(\mathbf{X}, t) \cdot \mathbf{P}(\mathbf{X}, t) \mathbf{N} dA_{\mathbf{X}}$$

$$(3.208)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) dV_{\mathbf{X}} + \int_{\partial \Omega^0} \mathbf{P}^T(\mathbf{X}, t) \mathbf{V}(\mathbf{X}, t) \cdot \mathbf{N} dA_{\mathbf{X}}$$

$$(3.209)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) + \nabla^{\mathbf{X}} \cdot (\mathbf{P}^T(\mathbf{X}, t) \mathbf{V}(\mathbf{X}, t)) dV_{\mathbf{X}}$$

$$(3.210)$$

$$= \int_{\Omega^0} \rho(\mathbf{X}, 0) \mathbf{b}_m(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) + (\nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X}, t)) \cdot \mathbf{V}(\mathbf{X}, t) + \mathbf{P} : \nabla^{\mathbf{X}} \mathbf{V} dV_{\mathbf{X}}.$$

Upon comparing the resulting LHS and RHS, we see that we are left with

$$(3.211) \quad \int_{\Omega^0} \rho(\mathbf{X}, 0) \frac{\partial \Psi}{\partial t}(\mathbf{X}, t) dV_{\mathbf{X}} = \int_{\Omega^0} \mathbf{P} : \nabla^{\mathbf{X}} \mathbf{V} dV_{\mathbf{X}},$$

for all $\Omega^0 \subset B$. Since

$$(3.212) \quad \nabla^{\mathbf{X}} \mathbf{V} = \nabla^{\mathbf{X}} \frac{\partial \boldsymbol{\varphi}}{\partial t} = \frac{\partial}{\partial t} \nabla^{\mathbf{X}} \boldsymbol{\varphi} = \frac{\partial \mathbf{F}}{\partial t},$$

we see that must have

$$(3.213) \quad \rho(\mathbf{X}, 0) \frac{\partial \Psi}{\partial t}(\mathbf{X}, t) = \mathbf{P} : \frac{\partial \mathbf{F}}{\partial t},$$

for all $\mathbf{X} \in B$. □

THEOREM 3.6.8. (Eulerian Form) *The conservation of energy requires*

$$(3.214) \quad \rho(\mathbf{x}, t) \left(\frac{\partial \psi}{\partial t} + \nabla^{\mathbf{x}} \psi \cdot \mathbf{v} \right) = \mathbf{S} : \mathbf{L},$$

for all $\mathbf{x} \in B^T$.

PROOF. The conservation of energy axiom states

$$(3.215) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \psi(\mathbf{x}, t) + \frac{1}{2} \rho(\mathbf{x}, t) |\mathbf{v}(\mathbf{x}, t)|^2 dV_{\mathbf{x}} = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} + \int_{\partial \Omega^t} \mathbf{t}(\mathbf{x}, t) \cdot \mathbf{v} dA_{\mathbf{x}}.$$

Examining the LHS of (3.215), we have

$$(3.216) \quad \frac{d}{dt} \int_{\Omega^t} \rho(\mathbf{x}, t) \psi(\mathbf{x}, t) + \frac{1}{2} \rho(\mathbf{x}, t) |\mathbf{v}(\mathbf{x}, t)|^2 dV_{\mathbf{x}} = \frac{d}{dt} \int_{\Omega^0} \rho_m(\mathbf{X}, 0) \psi_m(\mathbf{X}, t) + \frac{1}{2} \rho(\mathbf{X}, 0) |\mathbf{v}_m(\mathbf{X}, t)|^2 dV_{\mathbf{X}}$$

$$(3.217) \quad = \int_{\Omega^0} \rho_m(\mathbf{X}, 0) \frac{D}{Dt} \psi_m(\mathbf{X}, t) + \rho(\mathbf{X}, 0) \mathbf{v}_m(\mathbf{X}, t) \cdot \frac{D}{Dt} \mathbf{v}_m(\mathbf{X}, t) dV_{\mathbf{X}}$$

$$(3.218) \quad = \int_{\Omega^t} \rho(\mathbf{x}, t) \frac{D}{Dt} \psi(\mathbf{x}, t) + \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot \frac{D}{Dt} \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}}.$$

Now, we take the dot product of the Eulerian form of the conservation of linear momentum with $\mathbf{v}$ and integrate over Ω^t , and we see that

$$(3.219) \quad \int_{\Omega^t} \rho(\mathbf{x}, t) \left(\frac{D}{Dt} \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} \right) = \int_{\Omega^t} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) + \nabla^{\mathbf{x}} \cdot \mathbf{S}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}}.$$

Using (3.219) in (3.215), we see that

$$(3.220) \quad \int_{\Omega^t} \rho(\mathbf{x}, t) \frac{D}{Dt} \psi(\mathbf{x}, t) + \nabla^{\mathbf{x}} \cdot \mathbf{S}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}} = \int_{\partial\Omega^T} \mathbf{t}(\mathbf{x}, t) \cdot \mathbf{v} dA_{\mathbf{x}}.$$

Now, using the identity

$$(3.221) \quad \nabla^{\mathbf{x}} \cdot (\mathbf{S}^T \mathbf{v}) = (\nabla^{\mathbf{x}} \cdot \mathbf{S}) \cdot \mathbf{v} + \mathbf{S} : \nabla^{\mathbf{x}} \mathbf{v}$$

in (3.220) and using the divergence theorem, we find that

$$(3.222) \quad \int_{\Omega^t} \rho(\mathbf{x}, t) \frac{D}{Dt} \psi(\mathbf{x}, t) - \mathbf{S} : \nabla^{\mathbf{x}} \mathbf{v} dV_{\mathbf{x}} = 0.$$

Finally, using the conservation of angular momentum and some tensor manipulation, we find that

$$(3.223) \quad \mathbf{S} : \nabla^{\mathbf{x}} \mathbf{v} = \frac{1}{2} (\mathbf{S} : \nabla^{\mathbf{x}} \mathbf{v} + \mathbf{S} : \nabla^{\mathbf{x}} \mathbf{v})$$

$$(3.224) \quad = \frac{1}{2} (\mathbf{S} : \nabla^{\mathbf{x}} + \mathbf{S}^T : (\nabla^{\mathbf{x}} \mathbf{v})^T)$$

$$(3.225) \quad = \frac{1}{2} (\mathbf{S} : \nabla^{\mathbf{x}} + \mathbf{S} : (\nabla^{\mathbf{x}} \mathbf{v})^T)$$

$$(3.226) \quad = \mathbf{S} : \frac{1}{2} (\nabla^{\mathbf{x}} + (\nabla^{\mathbf{x}} \mathbf{v})^T)$$

$$(3.227) \quad = \mathbf{S} : \mathbf{L}.$$

Thus, conservation of energy requires

$$(3.228) \quad \rho \left(\frac{\partial \psi}{\partial t} + \nabla^{\mathbf{x}} \psi \cdot \mathbf{v} \right) = \mathbf{S} : \mathbf{L},$$

for all $\mathbf{x} \in B^T$. □

CHAPTER 4

Rigid Body Dynamics

A key assumption we make is that the objects we simulate have negligible elastic deformations outside of collision events. This assumption allows us to assume the material bodies live in the regime of rigid body dynamics which greatly reduces the computational load required to perform our simulations. Rather than expecting predictive behavior from our rigid body algorithm, we aim to achieve simulations which produce plausible motions. We employ the approach of [29] for our rigid body simulation which consists of three main steps: time integration, collision detection and modeling, and contact resolution.

4.1. Rigid Body Dynamics

Consider a material body with initial configuration B composed of particles $\mathbf{X}$ that evolves under rigid deformations. The deformation mapping $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$ describes the evolution the body and is of the form

$$(4.1) \quad \boldsymbol{\varphi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{R}(t)\mathbf{X},$$

where $\mathbf{c}(t)$ and $\mathbf{R}(t)$ describe the translation and orientation, respectively, of the rigid body. We represent the orientation $\mathbf{R}(t)$ using a unit quaternion $\mathbf{q}(t)$ which we discuss in 4.3. Let $\mathbf{X}_{\text{com}}$ be the center of mass of the the rigid body, and we describe the evolution of B under rigid deformations with respect to the evolution of the center of mass,

$$(4.2) \quad \boldsymbol{\varphi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{R}(t)\mathbf{X} + \mathbf{x}_{\text{com}}(t) - \mathbf{x}_{\text{com}}(t)$$

$$(4.3) \quad = \mathbf{c}(t) + \mathbf{R}(t)\mathbf{X}_{\text{com}} + \mathbf{R}(t)(\mathbf{X} - \mathbf{X}_{\text{com}})$$

$$(4.4) \quad = \mathbf{x}_{\text{com}}(t) + \mathbf{R}(t)(\mathbf{X} - \mathbf{X}_{\text{com}}),$$

where $\mathbf{x}_{\text{com}}(t) = \boldsymbol{\varphi}(\mathbf{X}_{\text{com}}, t)$. We see that the material description of the velocity of the rigid body is

$$(4.5) \quad \mathbf{V}(\mathbf{X}, t) = \mathbf{x}'_{\text{com}}(t) + \mathbf{R}'(t)(\mathbf{X} - \mathbf{X}_{\text{com}}),$$

and the Eulerian description is

$$(4.6) \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{v}_{\text{com}}(t) + \mathbf{R}'(t)\mathbf{R}^T(t)(\mathbf{x} - \mathbf{x}_{\text{com}}),$$

where $\mathbf{v}_{\text{com}} = \mathbf{x}'_{\text{com}}(t)$. Since $\mathbf{R}'(t)\mathbf{R}^T(t)$ is skew-symmetric, there is a unique vector $\boldsymbol{\omega}$ such that

$$(4.7) \quad \mathbf{R}'(t)\mathbf{R}^T(t)(\mathbf{x} - \mathbf{x}_{\text{com}}) = \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_{\text{com}}).$$

Therefore, the Eulerian description of the velocity is

$$(4.8) \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{v}_{\text{com}}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_{\text{com}}),$$

and we call $\boldsymbol{\omega}(t)$ the spatial angular velocity. We see that the velocity is composed of the linear velocity $\mathbf{v}_{\text{com}}(t)$ and angular velocity $\boldsymbol{\omega}(t)$.

The linear momentum of B is

$$(4.9) \quad \mathbf{p}(\mathbf{x}, t) = \int_{B^t} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dV_{\mathbf{x}}$$

$$(4.10) \quad = \int_{B^t} \rho(\mathbf{x}, t) (\mathbf{v}_{\text{com}}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_{\text{com}})) dV_{\mathbf{x}}$$

$$(4.11) \quad = \int_{B^t} \rho(\mathbf{x}, t) dV_{\mathbf{x}} \mathbf{v}_{\text{com}}(t) + \boldsymbol{\omega}(t) \times \left(\int_{B^t} \rho(\mathbf{x}, t) \mathbf{x} dV_{\mathbf{x}} + \int_{B^t} \rho(\mathbf{x}, t) dV_{\mathbf{x}} \mathbf{x}_{\text{com}} \right)$$

$$(4.12) \quad = m \mathbf{v}_{\text{com}}(t),$$

and the angular momentum with respect to the center of mass is

(4.13)

$$\mathbf{L}(t) = \int_{B^t} \rho(\mathbf{x}, t) (\mathbf{x} - \mathbf{x}_{\text{com}}) \times \mathbf{v}(t) dV_{\mathbf{x}}$$

$$(4.14) \quad = \int_{B^t} \rho(\mathbf{x}, t) (\mathbf{x} - \mathbf{x}_{\text{com}}) \times (\mathbf{v}_{\text{com}}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_{\text{com}})) dV_{\mathbf{x}}$$

$$(4.15) \quad = \int_{B^t} \rho(\mathbf{x}, t) (\mathbf{x} - \mathbf{x}_{\text{com}}) \times \mathbf{v}_{\text{com}}(t) dV_{\mathbf{x}} + \int_{\Omega^t} \rho(\mathbf{x}, t) (\mathbf{x} - \mathbf{x}_{\text{com}}) \times \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_{\text{com}}) dV_{\mathbf{x}}$$

$$(4.16) \quad = \int_{B^t} \rho(\mathbf{x}, t) ((\mathbf{x} - \mathbf{x}_{\text{com}}) \cdot (\mathbf{x} - \mathbf{x}_{\text{com}}) \mathbf{1} - (\mathbf{x} - \mathbf{x}_{\text{com}}) \otimes (\mathbf{x} - \mathbf{x}_{\text{com}})) dV_{\mathbf{x}} \boldsymbol{\omega}(t),$$

where $\mathbf{1}$ is the identity tensor. We see that the angular momentum may be expressed as

$$(4.17) \quad \mathbf{L}(t) = \mathbf{I} \boldsymbol{\omega}(t),$$

where $\mathbf{I}$ is the inertia tensor given by

$$(4.18) \quad \mathbf{I} = \int_B \rho(\mathbf{X}, 0) (|\mathbf{X} - \mathbf{X}_{\text{com}}|^2 \mathbf{1} - (\mathbf{X} - \mathbf{X}_{\text{com}}) \otimes (\mathbf{X} - \mathbf{X}_{\text{com}})) dV_{\mathbf{X}}.$$

Recall that the change in linear and angular momentum of B is equal to the resultant force and torque, respectively. Therefore,

$$(4.19) \quad \frac{\partial \mathbf{v}_{\text{com}}}{\partial t} = \frac{1}{m} \mathbf{F}, \quad \frac{\partial \mathbf{L}}{\partial t} = \boldsymbol{\tau},$$

where $\mathbf{F}$ and $\boldsymbol{\tau}$ are the resultant force and torque, respectively. The only contribution to the resultant force we consider is the body force due to gravity, for which $\mathbf{F} = m\mathbf{g}$.

Therefore, the evolution of the rigid body is governed by the system of equations,

$$(4.20) \quad \frac{\partial \mathbf{x}_{\text{com}}}{\partial t} = \mathbf{v}_{\text{com}}, \quad \frac{\partial \mathbf{q}}{\partial t} = \frac{1}{2} \boldsymbol{\omega} \mathbf{q},$$

$$(4.21) \quad \frac{\partial \mathbf{v}_{\text{com}}}{\partial t} = \mathbf{g}, \quad \frac{\partial \mathbf{L}}{\partial t} = \boldsymbol{\tau}.$$

4.2. Rigid Body Representation

We have established the system of equations that dictate the evolution of a material body under rigid deformations, and the state of a rigid body is determined by the following five quantities:

- center of mass $\mathbf{x}_{\text{com}}$,

- center of mass velocity $\mathbf{v}_{\text{com}}$,
- orientation $\mathbf{q}$,
- angular velocity $\boldsymbol{\omega}$,
- inertia tensor $\mathbf{I}$.

The rigid body B is taken to be composed of N_p particles whose reference positions are given by the vertices of a hexahedral mesh. Associated with the rigid body is a mass density function $\rho(\mathbf{X}, 0)$ which is used to determine the particle masses m_i , and the total mass of the rigid body is the sum of the particle masses. The particle masses are determined using the lumped mass matrix of the hexahedral mesh. After computing the center of mass,

$$(4.22) \quad \mathbf{X}_{\text{com}} = \frac{1}{m} \sum_{i=1}^{N_p} m_i \mathbf{X}_i,$$

construction of the inertia tensor proceeds using the discrete analogue of (4.18),

$$(4.23) \quad \mathbf{I} = \sum_{i=1}^{N_p} m_i (|\mathbf{X} - \mathbf{X}_{\text{com}}|^2 \mathbf{1} - (\mathbf{X} - \mathbf{X}_{\text{com}}) \otimes (\mathbf{X} - \mathbf{X}_{\text{com}})) dV_{\mathbf{X}}.$$

The quantities that determine the state of the rigid body are stored in material space which is defined by taking $\mathbf{X}_{\text{com}}$ to be at the origin and the coordinate axes aligned in a manner which makes the inertia tensor diagonal. Since the inertia tensor is real symmetric, the diagonalization of the inertia tensor is

$$(4.24) \quad \mathbf{I} = \mathbf{R} \mathbf{D} \mathbf{R}^T,$$

where $\mathbf{R}$ is orthogonal and $\mathbf{D}$ is diagonal. Associated with $\mathbf{R}$ is a unit quaternion $\mathbf{q}$ which we take to be the initial orientation of the rigid body.

During the rigid body simulation, the stored quantities for each rigid body are:

- the center of mass $\mathbf{x}_{\text{com}}$,
- the linear velocity $\mathbf{v}_{\text{com}}$,
- the angular velocity $\boldsymbol{\omega}$,
- inverse of the inertia tensor $\mathbf{I}^{-1}$,
- total mass m
- orientation $\mathbf{q}$.

The inverse of the inertia tensor will be used to determine the impulses that arise during the collision and contact resolution steps. Aside from the quantities that determine the evolution of the rigid body, we store a level set representing the surface of the rigid body that is used to determine interference between rigid bodies. The level set is stored in material space coordinates, and the zero isocontour is constructed from the triangulation of the quadrilateral boundary mesh of the hexahedral mesh used to construct the mass matrices.

4.3. Quaternions

Quaternions provide a more efficient and numerically stable representation of rotations than rotation matrices. We briefly state the operations and properties we require for representing the orientation of a rigid body.

A quaternion is of the form

$$(4.25) \quad \mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where $a, b, c, d \in \mathbb{R}$, and $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are symbols that behave in the same manner as the standard unit basis vectors in $\mathbb{R}^3$. The conjugate quaternion is

$$(4.26) \quad \mathbf{q}^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k},$$

and the norm of a quaternion is

$$(4.27) \quad \|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^*} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

A unit quaternion has length one, so a unit quaternion with the same direction as $\mathbf{q}$ is given by $\frac{\mathbf{q}}{\|\mathbf{q}\|}$. The quaternions form a four dimensional vector space with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and are endowed with component-wise addition and scalar multiplication. Quaternion multiplication is defined using the

following properties of the basis vectors,

$$(4.28) \quad 1\mathbf{i} = \mathbf{i}, \quad 1\mathbf{j} = \mathbf{j}, \quad 1\mathbf{k} = \mathbf{k}$$

$$(4.29) \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$(4.30) \quad \mathbf{i}\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\mathbf{i} = \mathbf{j}$$

$$(4.31) \quad \mathbf{j}\mathbf{i} = -\mathbf{k}, \quad \mathbf{k}\mathbf{j} = -\mathbf{i}, \quad \mathbf{i}\mathbf{k} = -\mathbf{j},$$

and two quaternions $\mathbf{q}_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $\mathbf{q}_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ have product

$$(4.32) \quad \mathbf{q}_1\mathbf{q}_2 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})$$

$$(4.33) \quad = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i}$$

$$(4.34) \quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}.$$

4.3.1. Rotations. Unit quaternions are used to represent rotations in $\mathbb{R}$. Consider a rotation of θ radians about the axis defined by the unit vector $\mathbf{u} = (u_1, u_2, u_3)$. The quaternion representation of this action is

$$(4.35) \quad \mathbf{q} = \cos \frac{\theta}{2} + (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \sin \frac{\theta}{2}.$$

A pure quaternion is a quaternion of the form $(0, a, b, c)$, so given a vector $\mathbf{p} = (p_1, p_2, p_3)$, we can associate with it a pure quaternion $\bar{\mathbf{p}} = (0, p_1, p_2, p_3)$. Then, the rotation represented by $\mathbf{q}$ is applied to the vector $\mathbf{p}$ by computing

$$(4.36) \quad L(\mathbf{p}) = \mathbf{q}\bar{\mathbf{p}}\mathbf{q}^*.$$

For pure quaternions, the cross product is given by

$$(4.37) \quad \mathbf{q}_1 \times \mathbf{q}_2 = (c_1d_2 - c_2d_1)\mathbf{i} + (d_1b_2 - b_1d_2)\mathbf{j} + (b_1c_2 - c_2b_1)\mathbf{k},$$

with the identity

$$(4.38) \quad \mathbf{q}_1 \times \mathbf{q}_2 = \frac{1}{2}(\mathbf{q}_1\mathbf{q}_2 - \mathbf{q}_2\mathbf{q}_1).$$

A rotation matrix representing a rotation by θ about $\mathbf{u}$ is given by

$$(4.39) \quad \mathbf{R} = \cos \theta \mathbf{I} + \sin \theta \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} + (1 - \cos \theta)(\mathbf{u} \otimes \mathbf{u}),$$

so given the quaternion representation of a rotation, we may express it as a rotation matrix. In the other direction, if we are given a rotation matrix with entries R_{ij}

$$(4.40) \quad a = \frac{1}{2} \sqrt{1 + \text{tr}(\mathbf{R})}, \quad b = \frac{(R_{32} - R_{23})}{2\sqrt{1 + \text{tr}(\mathbf{R})}}, \quad c = \frac{R_{13} - R_{31}}{2\sqrt{1 + \text{tr}(\mathbf{R})}}, \quad d = \frac{Q_{21} - Q_{12}}{2\sqrt{1 + \text{tr}(\mathbf{R})}}.$$

If the trace of $\mathbf{R}$ is negative, we need to use a different conversion since numerical issue may crop up due to $1 + \text{tr}(\mathbf{R}) \approx 0$. When $\text{tr}(\mathbf{R}) < 0$ and R_{11} is the largest diagonal entry

$$(4.41) \quad a = \frac{R_{32} - R_{23}}{2\sqrt{1 + Q_{00} - Q_{22} - Q_{33}}}, \quad b = \frac{1}{2} \sqrt{1 + R_{11} - R_{22} - R_{33}},$$

$$c = \frac{Q_{12} - Q_{21}}{2\sqrt{1 + Q_{11} - Q_{22} - Q_{33}}}, \quad d = \frac{Q_{31} + Q_{13}}{2\sqrt{1 + Q_{11} - Q_{22} - Q_{33}}}.$$

When R_{22} or R_{33} is the largest diagonal entry, a cyclic permutation of (4.41) is used.

4.3.2. Rigid Body Orientation. Let $\mathbf{r}(t) = \mathbf{x}(t) - \mathbf{x}_{\text{com}}(t)$ so that

$$(4.42) \quad \mathbf{r}(t) = \mathbf{R}(t)(\mathbf{X} - \mathbf{X}_{\text{com}}).$$

Let $\mathbf{q}(t)$ be a unit quaternion that corresponds to $\mathbf{R}(t)$. Note that since $\mathbf{q}(t)$ is a unit quaternion, we have $\mathbf{q}(t)\mathbf{q}^*(t) = 1$, so

$$(4.43) \quad \dot{\mathbf{q}}(t)\mathbf{q}^*(t) + \mathbf{q}(t)\dot{\mathbf{q}}^*(t) = 0.$$

Now, expressing $\mathbf{r}$, $\mathbf{X}$, and $\mathbf{X}_{\text{com}}$ as a pure quaternions, we have

$$(4.44) \quad \bar{\mathbf{r}}(t) = \mathbf{q}(t)(\bar{\mathbf{X}} - \bar{\mathbf{X}}_{\text{com}})\mathbf{q}^*(t),$$

and upon taking the time derivative, we find that

$$(4.45) \quad \dot{\bar{\mathbf{r}}}(t) = \dot{\mathbf{q}}(t)(\bar{\mathbf{X}} - \bar{\mathbf{X}}_{\text{com}})\mathbf{q}^*(t) + \mathbf{q}(t)(\bar{\mathbf{X}} - \bar{\mathbf{X}}_{\text{com}})\dot{\mathbf{q}}^*(t)$$

$$(4.46) \quad = \dot{\mathbf{q}}(t)\mathbf{q}^*(t)\mathbf{q}(t)(\bar{\mathbf{X}} - \bar{\mathbf{X}}_{\text{com}})\mathbf{q}^*(t) + \mathbf{q}(t)(\bar{\mathbf{X}} - \bar{\mathbf{X}}_{\text{com}})\mathbf{q}^*(t)\mathbf{q}(t)\dot{\mathbf{q}}^*(t)$$

$$(4.47) \quad = \dot{\mathbf{q}}(t)\mathbf{q}^*(t)\bar{\mathbf{r}}(t) - \bar{\mathbf{r}}(t)\dot{\mathbf{q}}(t)\mathbf{q}^*(t).$$

Next, the time derivative of $\mathbf{r}$ is related to the angular velocity, so

$$(4.48) \quad \dot{\bar{\mathbf{r}}} = \bar{\boldsymbol{\omega}}(t) \times \bar{\mathbf{r}}(t)$$

$$(4.49) \quad = \frac{1}{2}(\bar{\boldsymbol{\omega}}(t)\bar{\mathbf{r}}(t) - \bar{\mathbf{r}}(t)\bar{\boldsymbol{\omega}}(t)).$$

Upon comparing (4.47) and (4.49), we see that

$$(4.50) \quad \dot{\mathbf{q}}(t) = \frac{1}{2}\bar{\boldsymbol{\omega}}(t)\mathbf{q}(t).$$

4.4. Rigid Body Simulation

The equations governing the state of the rigid body are solved using the forward Euler method,

$$(4.51) \quad \mathbf{x}_{\text{com}}^{n+1} = dt\mathbf{v}_{\text{com}}^n + \mathbf{x}_{\text{com}}^n, \quad \mathbf{q}^{n+1} = \frac{dt}{2}\boldsymbol{\omega}^n\mathbf{q}^n + \mathbf{q}^n,$$

$$(4.52) \quad \mathbf{v}_{\text{com}}^{n+1} = dt\mathbf{g} + \mathbf{v}_{\text{com}}^n, \quad \mathbf{L}^{n+1} = dt\boldsymbol{\tau} + \mathbf{L}^n,$$

where dt is the time step size of the rigid body simulation. While more accurate and stable time stepping schemes exist, forward Euler is computationally cheap and priority is placed on resolving contacts and collisions. When simulating the rigid bodies, our goal is to achieve visual realism rather than the exact behavior, so if any unstable behavior arises or fast moving objects pass through one another, we simply decrease the time step size.

For each time step of the rigid body simulation, the following processes are performed:

- (1) collision detection and modeling,
- (2) update velocities,
- (3) contact resolution,
- (4) update positions.

The order of the simulation steps is delicately chosen to ensure that objects do not chatter on the ground, bounce uphill, or exhibit other nonphysical behavior. The key step is that the contact resolution occurs directly after the velocity update which prevents objects from bouncing or falling through the ground plane. Contact resolution after the velocity update is ideal since it resolves forces, and the velocity update is where forces are incorporated into the dynamics.

4.4.1. Collision Resolution. Collisions are simultaneously resolved rather than chronologically, but our goal is plausible rather than predictive dynamics, so the efficiency gains of simultaneous resolution are welcome. Collisions are detected by temporarily moving bodies to their predicted (candidate) locations and checking for interference. The new velocities are used to determine the candidate positions so that the collision and contact resolutions steps detect the same interference points. The important distinction is that the old velocities are used to process collisions and new velocities are used to process contacts.

Interference between pairs of rigid bodies is checked, and the deepest point of interpenetration is chosen as the collision point. The collision resolution modifies the current velocity of the rigid body, so new collisions may occur that were not originally present. The collision resolution is repeated a specified number of times, and the bodies are moved to the newly predicted locations. After several iterations, there may be more interpenetrating pairs, but these will be resolved during contact processing.

Interference is checked using the candidate positions,

$$(4.53) \quad \mathbf{x}^{\text{new}} = \mathbf{x} + dt(\mathbf{v} + dt\mathbf{g}).$$

If interference is detected, a collision impulse is computed using the current velocity and the current velocity is subsequently updated. Contact processing uses the predicted position

$$(4.54) \quad \mathbf{x}^{\text{new}} = \mathbf{x} + dt\mathbf{v}^{\text{new}},$$

and applies impulses to the new velocity $\mathbf{v}^{\text{new}}$,

$$(4.55) \quad \mathbf{v}^{\text{new}} = \mathbf{v} + dt\mathbf{g},$$

which is set in the velocity update step, so the candidate positions match and the interference is consistent.

We now discuss the construction of the collision impulses. Let $\mathbf{N}$ be the unit collision normal and $\mathbf{u}_{\text{rel}}$ be the relative velocity at the collision point. The normal component of the relative velocity $\mathbf{u}_{\text{rel},n}$ is

$$(4.56) \quad \mathbf{u}_{\text{rel},n} = \mathbf{u}_{\text{rel}} \cdot \mathbf{N},$$

so we may write the tangential relative velocity as

$$(4.57) \quad \mathbf{u}_{\text{rel},t} = \mathbf{u}_{\text{rel}} - \mathbf{u}_{\text{rel},n} \mathbf{N}.$$

The linear and angular velocity of each rigid body are updated using an impulse $\mathbf{j}$,

$$(4.58) \quad \mathbf{v}^{\text{new}} = \mathbf{v} \pm \frac{1}{m} \mathbf{j},$$

$$(4.59) \quad \boldsymbol{\omega}^{\text{new}} = \boldsymbol{\omega} \pm \mathbf{I}^{-1}(\mathbf{r} \times \mathbf{j})$$

where $\mathbf{r}$ points from the center of mass of each body to the collision location. The new velocity at the point of collision is

$$(4.60) \quad \mathbf{u}^{\text{new}} = \mathbf{u} \pm \mathbf{K} \mathbf{j},$$

where

$$(4.61) \quad \mathbf{K} = \frac{1}{m} \mathbf{1} + (\mathbf{r}^*)^T \mathbf{I}^{-1} \mathbf{r}^*,$$

and

$$(4.62) \quad \mathbf{r} = \begin{bmatrix} 0 & -r_2 & r_1 \\ r_2 & 0 & -r_0 \\ -r_1 & r_0 & 0 \end{bmatrix},$$

is the cross product matrix. Taking the dot product with the unit collision normal, we find that

$$(4.63) \quad \mathbf{u}_{\text{rel},n}^{\text{new}} = \mathbf{u}_{\text{rel},n} \pm \mathbf{N}^T \mathbf{K} \mathbf{j},$$

where $\mathbf{K}_T = \mathbf{K}_1 + \mathbf{K}_2$ is the sum of the two $\mathbf{K}$ matrices. The coefficient of restitution ε characterizes how elastic the collision is, and the normal component of the updated velocity at the collision point is given by

$$(4.64) \quad \mathbf{u}_{\text{rel},n}^{\text{new}} = -\varepsilon \mathbf{u}_{\text{rel},n}.$$

Each body has its own coefficient of restitution, and the minimum of the two bodies is used to process the collision. For immovable static objects such as the ground plane, we set $\mathbf{K} = \mathbf{0}$ and do not update the velocity.

4.4.1.1. *Frictionless Collision Model.* In the absence of friction, the impulse only acts in the normal direction, so the impulse may be written as

$$(4.65) \quad \mathbf{j} = j_n \mathbf{N}.$$

Then, we have that

$$(4.66) \quad -\varepsilon \mathbf{u}_{\text{rel},n} = \mathbf{u}_{\text{rel},n} \pm \mathbf{N}^T \mathbf{K}_T j_n \mathbf{N},$$

so

$$(4.67) \quad j_n = \mp \frac{(1 + \varepsilon) \mathbf{u}_{\text{rel},n}}{\mathbf{N}^T \mathbf{K}_T \mathbf{N}}.$$

4.4.1.2. *Static and Kinetic Friction Collision Model.* The effects of friction may be incorporated into the collision model. Each rigid body has a coefficient of friction, and the maximum of the two bodies is used to process collisions. First, the rigid bodies are assumed to stick due to static friction, and we determine the resulting impulse. The rigid bodies sticking implies

$$(4.68) \quad u_{\text{rel},t}^{\text{new}} = 0,$$

so the new relative velocity is

$$(4.69) \quad \mathbf{u}_{\text{rel}}^{\text{new}} = -\varepsilon u_{\text{rel},n} \mathbf{N}.$$

The new velocity at the collision point is given by

$$(4.70) \quad \mathbf{u}_{\text{rel}}^{\text{new}} = \mathbf{u}_{\text{rel}} + \mathbf{K}_T \mathbf{j},$$

so

$$(4.71) \quad \mathbf{j} = -\mathbf{K}_T^{-1} (\varepsilon u_{\text{rel},n} \mathbf{N} + \mathbf{u}_{\text{rel}}).$$

Now, we check the friction cone condition to determine whether the two bodies stick or whether we apply sliding friction. If

$$(4.72) \quad |\mathbf{j} - (\mathbf{j} \cdot \mathbf{N})| \leq \mu \mathbf{j} \cdot \mathbf{N},$$

then the collision points are sticking due to static friction, and we use the impulse $\mathbf{j}$. Otherwise, we apply kinetic friction results in an impulse given by

$$(4.73) \quad \mathbf{j} = j_n \mathbf{N} - \mu j_n \mathbf{T},$$

where

$$(4.74) \quad \mathbf{T} = \frac{1}{\|\mathbf{u}_{\text{rel},t}\|} \mathbf{u}_{\text{rel},t},$$

is the unit tangent to the collision point. Now, the relative velocity is updated by

$$(4.75) \quad \mathbf{u}_{\text{rel}}^{\text{new}} = \mathbf{u}_{\text{rel}} + \mathbf{K}_T \mathbf{j},$$

and upon computing the dot product of each side with the collision, normal, we have

$$(4.76) \quad u_{\text{rel},n}^{\text{new}} = u_{\text{rel},n} + \mathbf{N}^T \mathbf{K}_T \mathbf{j}.$$

Since $u_{\text{rel},n}^{\text{new}} = -\varepsilon u_{\text{rel},n}$, we have that

$$(4.77) \quad -\varepsilon u_{\text{rel},n} = u_{\text{rel},n} + \mathbf{N}^T \mathbf{K}_T \mathbf{j},$$

so the kinetic friction impulse is given by

$$(4.78) \quad j_n = -\frac{(\varepsilon + 1)u_{\text{rel},n}}{\mathbf{N}^T \mathbf{K}_T (\mathbf{N} - \mu \mathbf{T})}.$$

4.4.2. Contact Resolution. During contact resolution, the goal is to resolve forces between objects. The contact modeling is the same as the collision modeling except a zero coefficient of restitution is used, so any collisions are interpreted as inelastic collisions. As in the collision

resolution step, we move the bodies to the predicted locations, and check for interference. Multiple iterations may be required to process all contacts, and once the contact resolution is complete, the positions of the rigid bodies are updated.

4.5. Impulse Collection

The impulses that arise from the collision and contact resolution steps are collected and used as traction boundary conditions for the elastodynamic phase. Impulse and force are related by $\mathbf{j} = \Delta t \mathbf{F}$, so we scale the collected impulses by the time over which the elastodynamic problem is solved. Let $\mathbf{j}_i$ be the impulse applied at $\mathbf{x}_i$ and T the total time of the elastodynamic problem. Then, the traction boundary condition is

$$(4.79) \quad \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \begin{cases} \frac{1}{\pi r_s^2} \frac{1}{T} \mathbf{j} & |\mathbf{x} - \mathbf{x}_c| < r_s \\ \mathbf{0} & \text{otherwise,} \end{cases}, \quad \mathbf{x} \in \partial\Omega_N^t.$$

Here, r_s is the radius of a small conceptual circle on which we assume the impulse is applied.

CHAPTER 5

Elasticity

Once a collision is detected during the rigid body simulation, an elastodynamic simulation kicks off with traction boundary conditions coming from the collision impulses and initial state arising from the collision state of the rigid body. We assume the rigid bodies behave as linear elastic solids on the timescale of the crack speed. Starting from general elastic solids, we narrow our scope in this chapter to modeling homogeneous isotropic linear elastic bodies under the small strain assumption.

5.1. Solid Mechanics

When modeling different material behaviors, the stress response of the material may obey different laws, so the Cauchy stress tensor is defined using a given stress response function $\hat{\mathbf{S}}$ with the stress function being a function of the deformation gradient and reference position. The stress response function is called homogeneous if it does not vary throughout the material body. We call the continuum body an elastic solid if the Cauchy stress tensor satisfies the following two properties:

- $\mathbf{S}_m(\mathbf{X}, t) = \hat{\mathbf{S}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X})$,
- $\hat{\mathbf{S}}^T(\mathbf{F}, \mathbf{X}) = \hat{\mathbf{S}}(\mathbf{F}, \mathbf{X})$.

The first condition is called the generalized Hooke's law and establishes a stress-strain relation. Additionally, the generalized Hook's law states that the stress at a point only depends on the strain at the present moment, and does not depend on the past deformations or the strain rate. The second condition states that the Cauchy stress tensor must be symmetric so that the balance of angular momentum is maintained. The stress response functions $\hat{\mathbf{S}}(\mathbf{F})$, $\hat{\mathbf{P}}(\mathbf{F})$, and $\hat{\mathbf{\Sigma}}(\mathbf{F})$ give rise to the stress fields $\mathbf{S}_m(\mathbf{X}, t)$, $\mathbf{P}(\mathbf{X}, t)$, and $\mathbf{\Sigma}(\mathbf{X}, t)$, respectively, and the stress response function is an intrinsic property of the material body. A common example of a stress response function is that of the St. Venant-Kirchhoff model,

$$(5.1) \quad \hat{\mathbf{\Sigma}}(\mathbf{F}) = \lambda \text{tr}\left(\frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})\right) \mathbf{I} + 2\mu \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}),$$

which models an isotropic material, and notice that $\hat{\mathbf{\Sigma}}$ is a linear function of $\frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})$.

5.2. Hyperelasticity

We call a solid hyperelastic if the stress response function can be constructed from a strain energy density function. That is, the stress response function is of the form

$$(5.2) \quad \hat{\mathbf{P}}(\mathbf{F}) = DW(\mathbf{F}),$$

where W is the strain energy density of the material and $DW(\mathbf{F})$ is the derivative of W at $\mathbf{F}$. Satisfying the balance of angular momentum requires that

$$(5.3) \quad DW(\mathbf{F})\mathbf{F}^T = \mathbf{F}DW(\mathbf{F})^T,$$

$\forall \mathbf{F} \in \mathcal{V}^2$. We list some common strain energy density models that are functions of the Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$.

5.2.1. St. Venant-Kirchhoff Model. St. Venant-Kirchhoff model

$$\begin{aligned} \bar{W}(\mathbf{C}) &= \frac{\lambda}{2} (\text{tr} \mathbf{G})^2 + \mu \text{tr}(\mathbf{G}^2) \\ \mathbf{G} &= \frac{1}{2} (\mathbf{C} - \mathbf{I}) \end{aligned}$$

for $\lambda, \mu > 0$.

5.2.2. Neo-Hookean Model.

$$\begin{aligned} \bar{W}(\mathbf{C}) &= a \text{tr} \mathbf{C} + \Gamma \left(\sqrt{\det \mathbf{C}} \right) \\ \Gamma(s) &= cs^2 - d \ln s \end{aligned}$$

for $a, c, d > 0$.

5.2.3. Mooney-Rivlin Model.

$$\begin{aligned} \bar{W}(\mathbf{C}) &= a \text{tr} \mathbf{C} + b \text{tr} \mathbf{C}_* + \Gamma(\sqrt{\det \mathbf{C}}) \\ \mathbf{C}_* &= (\det \mathbf{C}) \mathbf{C}^{-1} \end{aligned}$$

$a, b > 0$.

5.2.4. Ogden Model.

$$\overline{W}(\mathbf{C}) = \sum_{i=1}^M a_i \text{tr}(\mathbf{C}^{\frac{\gamma_i}{2}}) + \sum_{j=1}^N b_j \text{tr}(\mathbf{C}_*^{\frac{\delta_j}{2}}) + \Gamma(\sqrt{\det(\mathbf{C})})$$

$M, N \geq 1$, $a_i, b_j > 0$, $\gamma_i, \delta_j \geq 1$.

5.3. Variation of the Elastic Potential Energy

By assuming a material is hyperelastic, a variational framework is inherited, and we show that $\nabla \cdot DW(\mathbf{F})$ extremizes the elastic potential energy.

Let $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$ be the deformation map of the material body B . Let $\psi : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be an energy density function so that the potential energy is given by

$$(5.4) \quad \mathcal{U} = \int_B \psi\left(\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}}\right) d\mathbf{x}.$$

Now, consider a variation of $\boldsymbol{\varphi}$, $\boldsymbol{\varphi} + \varepsilon \delta \boldsymbol{\varphi}$ so that,

$$(5.5) \quad g(\varepsilon) = \int_B \psi\left(\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} + \varepsilon \frac{\partial \delta \boldsymbol{\varphi}}{\partial \mathbf{X}}\right) d\mathbf{X}.$$

Now, we want to extremize g , so we find the stationary points of g ,

$$(5.6) \quad \frac{d}{d\varepsilon} g(\varepsilon) = \frac{d}{d\varepsilon} \int_B \psi\left(\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} + \varepsilon \frac{\partial \delta \boldsymbol{\varphi}}{\partial \mathbf{X}}\right) d\mathbf{X}$$

$$(5.7) \quad = \int_B \frac{\partial \psi}{\partial F_{\alpha\beta}} \left(\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} + \varepsilon \frac{\partial \delta \boldsymbol{\varphi}}{\partial \mathbf{X}} \right) \frac{\partial \delta \varphi_\alpha}{\partial X_\beta} d\mathbf{X},$$

where $F_{\alpha\beta} = \frac{\partial \varphi_\alpha}{\partial X_\beta}$. Now,

$$(5.8) \quad \frac{dg}{d\varepsilon}(0) = \int_B \frac{\partial \psi}{\partial F_{\alpha\beta}} \frac{\partial \delta \varphi_\alpha}{\partial X_\beta} d\mathbf{X}$$

$$(5.9) \quad = \int_{\partial B} \frac{\partial \psi}{\partial F_{\alpha\beta}} \delta \varphi_\alpha N_\beta dS - \int_B \frac{\partial}{\partial X_\beta} \frac{\partial \psi}{\partial F_{\alpha\beta}} \delta \varphi_\alpha d\mathbf{X}.$$

Therefore, if $\boldsymbol{\varphi}$ extremizes the potential energy, then we must have that

$$(5.10) \quad \frac{\partial \psi}{\partial F_{\alpha\beta}} N_\beta = 0$$

on the boundary of B and

$$(5.11) \quad \frac{\partial}{\partial X_\beta} \frac{\partial \psi}{\partial F_{\alpha\beta}} = 0$$

in B .

We adopt the St. Venant-Kirchhoff model as our strain energy density function, and now, we explicitly compute $\frac{\partial \psi}{\partial \mathbf{F}}$. The strain energy density function for the St. Venant-Kirchhoff model is

$$(5.12) \quad \psi(\mathbf{F}) = \frac{\lambda}{2} (\text{tr}(\frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})))^2 + \mu \text{tr}(\frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}))^2,$$

so we compute

$$(5.13) \quad \frac{\partial \psi}{\partial F_{\alpha\beta}} = \frac{\partial}{\partial F_{\alpha\beta}} \left(\frac{\lambda}{2} (\text{tr}(\frac{1}{2}(F_{\gamma\delta}^T F_{\delta\epsilon} - \delta_{\gamma\epsilon})))^2 + \mu \text{tr}(\frac{1}{2}(F_{\gamma\delta}^T F_{\delta\epsilon} - \delta_{\gamma\epsilon}))^2 \right).$$

First,

$$(5.14) \quad \frac{\lambda}{2} (\text{tr}(\frac{1}{2}(F_{\gamma\delta}^T F_{\delta\epsilon} - \delta_{\gamma\epsilon})))^2 = \frac{\lambda}{2} (\frac{1}{2}(F_{\gamma\delta}^T F_{\delta\gamma} - \delta_{\gamma\gamma}))^2$$

$$(5.15) \quad = \frac{\lambda}{2} (\frac{1}{2}(F_{\delta\gamma}^2 - \delta_{\gamma\gamma}))^2,$$

and

$$(5.16) \quad \frac{\partial}{\partial F_{\alpha\beta}} \frac{\lambda}{2} (\frac{1}{2}(F_{\delta\gamma}^2 - \delta_{\gamma\gamma}))^2 = \lambda (\frac{1}{2}(F_{\delta\gamma}^2 - \delta_{\gamma\gamma})) F_{\alpha\beta}.$$

Next,

$$(5.17) \quad \mu \text{tr}(\frac{1}{2}(F_{\gamma\delta}^T F_{\delta\epsilon} - \delta_{\gamma\gamma}))^2 = \frac{\mu}{4} (F_{\epsilon\delta} F_{\epsilon\phi} - \delta_{\delta\phi})(F_{\iota\phi} F_{\iota\delta} - \delta_{\phi\delta}),$$

so

$$(5.18) \quad \frac{\partial}{\partial F_{\alpha\beta}} \frac{\mu}{4} (F_{\epsilon\delta} F_{\epsilon\phi} - \delta_{\delta\phi})(F_{\iota\phi} F_{\iota\delta} - \delta_{\phi\delta}) = \mu (F_{\alpha\delta} F_{\epsilon\delta} F_{\epsilon\beta} - F_{\alpha\beta}).$$

Thus, we see that

$$(5.19) \quad \hat{\mathbf{P}}(\mathbf{F}) = \frac{\partial \psi}{\partial \mathbf{F}} = \lambda \text{tr}(\frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})) \mathbf{F} + 2\mu \mathbf{F} \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}).$$

5.4. Infinitesimal Strain

We restrict ourselves to small strain kinematics for which $|\nabla \mathbf{u}| = O(\epsilon)$ where $0 < \epsilon \ll 1$. The displacement is $\mathbf{u} = \boldsymbol{\varphi} - \mathbf{X}$ and

$$(5.20) \quad \nabla \mathbf{u} = \mathbf{F} - \mathbf{I},$$

so up to leading order, we have $\mathbf{F} = \mathbf{I}$. Next, the Cauchy-Green strain tensor becomes

$$(5.21) \quad \mathbf{F}^T \mathbf{F} = \mathbf{I} + (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + O(\epsilon^2),$$

so $\frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) = \boldsymbol{\varepsilon}(\mathbf{u})$ upon dropping the higher order terms. Recall $\boldsymbol{\varepsilon}$ is called the infinitesimal strain tensor given by

$$(5.22) \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

Now, using the infinitesimal strain tensor in the stress response function for the St. Venant Kirchhoff model, we find that

$$(5.23) \quad \hat{\mathbf{P}} = \lambda \text{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}).$$

5.5. Linear Elastic Solids

The St. Venant-Kirchhoff model results in a stress response function that is linear in the infinitesimal strain tensor. Therefore, there is a fourth-order tensor $\mathbf{C}$ that relates the stress $\boldsymbol{\sigma}(\mathbf{u})$ and strain $\boldsymbol{\varepsilon}(\mathbf{u})$,

$$(5.24) \quad \boldsymbol{\sigma}(\mathbf{u}) := \mathbf{C} : \boldsymbol{\varepsilon}.$$

We call the tensor $\mathbf{C}$ is the elasticity tensor, and for the St. Venant-Kirchhoff model,

$$(5.25) \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}).$$

Here, μ and λ are the two parameters that determine the stiffness tensor called Lamé's first and second parameters, respectively. We can express the strain in terms of the stress by inverting the

stress-strain relation. We have

$$(5.26) \quad \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij},$$

and computing the trace of the expression, we find that

$$(5.27) \quad \sigma_{kk} = 3\lambda \varepsilon_{kk} + 2\mu \varepsilon_{kk}.$$

Therefore, the trace of the strain tensor is

$$(5.28) \quad \varepsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu}.$$

Now,

$$(5.29) \quad \sigma_{ij} = \frac{\lambda \sigma_{kk}}{3\lambda + 2\mu} \delta_{ij} + 2\mu \varepsilon_{ij},$$

so

$$(5.30) \quad \varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda \sigma_{kk}}{2\mu(3\lambda + 2\mu)} \delta_{ij}.$$

5.5.1. Young's Modulus and Poisson's Ratio. An equivalent pair of physically motivated constants that determine the elasticity tensor are the Young's modulus E and Poisson's ratio ν .

First, we establish the reasoning behind Young's modulus. Suppose we load a body so that the only nonzero component of $\boldsymbol{\sigma}$ is $\sigma_{11} = \sigma$. Then, the Young's modulus E is defined to be the ratio between the axial stress and axial strain,

$$(5.31) \quad E = \frac{\sigma_{11}}{\varepsilon_{11}}.$$

From the inverted stress strain relation, we see that

$$(5.32) \quad \varepsilon_{11} = \frac{\sigma_{11}}{2\mu} - \frac{\lambda \sigma_{11}}{2\mu(3\lambda + 2\mu)},$$

so in terms of the Lamé parameters, we see that

$$(5.33) \quad E = \frac{1}{\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

Next, Poisson's ratio is defined to be the ratio of the lateral to axial strain,

$$(5.34) \quad \nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}.$$

From the inverted stress strain relation, we see that

$$(5.35) \quad \varepsilon_{22} = -\frac{\lambda\sigma_{11}}{2\mu(3\lambda + 2\mu)},$$

so

$$(5.36) \quad \nu = \frac{\frac{\lambda}{2\mu(3\lambda+2\mu)}}{\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda+2\mu)}} = \frac{\lambda}{2(\lambda + \mu)}.$$

5.5.2. Plane Strain. Suppose we are analyzing a specimen such as a bar which is much thicker in the x_3 -direction, and we consider a cross-section of the bar in the $x_1 x_2$ plane. If we assume that there are only deformations in the x_1 and x_2 directions, we say that the cross-section being analyzed is in a state of plane strain. We assume that $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$, so

$$(5.37) \quad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda(\varepsilon_{11} + \varepsilon_{22})\delta_{ij}.$$

We see that when $i \neq j$, and $j = 3$, $\sigma_{i3} = 0$, but when $i = j = 3$,

$$(5.38) \quad \sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22}).$$

5.5.3. Plane Stress. Alternatively, suppose we consider thin sheet, and we assume that all of the components of stress in the x_3 direction vanish. Under this assumption, we say that the body is in a state of plane stress. We have that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$, so

$$(5.39) \quad \sigma_{i3} = 2\mu\varepsilon_{i3} + \lambda\varepsilon_{kk}\delta_{i3}.$$

For $i = 1, 2$ the stress strain relation gives $\varepsilon_{13} = \varepsilon_{23} = 0$, but for $i = 3$, we see that

$$(5.40) \quad 0 = 2\mu\varepsilon_{33} + \lambda\varepsilon_{kk}.$$

Therefore,

$$(5.41) \quad \varepsilon_{33} = \frac{-\lambda(\varepsilon_{11} + \varepsilon_{22})}{2\mu + \lambda}.$$

5.5.4. Voigt Notation. Voigt notation is a convenient way to represent the stress strain relation. Letting

$$(5.42) \quad \bar{\sigma} := \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}, \quad \bar{\varepsilon} := \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

allows us to represent the elasticity tensor as

$$(5.43) \quad \bar{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

5.6. Elasticity Equations

We now state the system of PDEs for the resulting displacement $\mathbf{u}$ of a body under a given external force $\mathbf{f}$ and boundary conditions. Let $\mathbf{u}_D$ be the prescribed displacement boundary condition so that $\mathbf{u} = \mathbf{u}_D$ on Γ_D , and $\mathbf{t}$ the traction boundary condition defined on Γ_N .

5.6.1. Elastostatics. When inertial effects are negligible, the balance of linear momentum gives the time independent problem,

$$(5.44) \quad -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f}.$$

Quasistatic deformation of the body may also be considered where the evolution of the body is seen as a sequence of static solutions by letting the external forcing or boundary conditions vary with time.

5.6.2. Elastodynamics. When inertial effects are considered, the balance of linear momentum requires

$$(5.45) \quad \rho \ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f}.$$

Due to the presence of second order time derivatives, we require the initial conditions

$$(5.46) \quad \mathbf{u}(0) = \mathbf{u}_0$$

$$(5.47) \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0.$$

Note that the external forcing and boundary conditions may also be time dependent.

CHAPTER 6

Finite Element Method

The Finite Element Method (FEM) is the workhorse for solving the equations of solid mechanics. Rather than working with the strong form of a PDE, the FEM constructs weak solutions of the PDE by decomposing the domain of interest into a collection of elements. The FEM is well developed technology with extensions such as the discontinuous Galerkin FEM, extended FEM, and mixed FEM. We outline the essential ideas of using the FEM to solve the Poisson and Elastodynamic problems, and conclude with a limitation of the FEM: nonconvex elements. For further details, we refer the reader to [34] for getting up and running with the FEM and to [10] for a more mathematical approach to the FEM.

6.1. Continuous Problem

Starting from the strong form of a given PDE, we saw in section 2.4 that we can state an equivalent weak formulation that admits a larger class of solutions called weak solutions. Let $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ be a bilinear form where V is a suitably chosen function space and $f \in V^*$. The weak formulation is as follows: find $u \in V$ such that

$$(6.1) \quad a(u, v) = (f, v),$$

for all $v \in V$. If a is coercive and bounded, we know that by the Lax-Milgram theorem, a solution to (6.1) exists.

6.2. Discrete Problem

We solve the weak formulation over Ω which has been discretized into a collection of elements $\mathcal{T}_h = \{E_i\}$ so that $\Omega = \cup_{i=1}^{N_E} E_i$. Here, h is some parameter characterizing the size of the elements E_i and N_E is the total number of elements. Now, associated with the continuous problem is a finite dimensional problem, and the solution lives in $V_h \subset V$ which is a finite dimensional subspace

of dimension N_{DoF} . The degrees of freedom (DoF) establish the basis functions that allow any $u_h \in V_h$ to be decomposed as a linear combination of basis functions $\{N_i\}_{i=1}^{N_{\text{DoF}}}$.

Now, the statement of the discrete problem is: find $u_h \in V_h$ such that

$$(6.2) \quad a(u_h, v_h) = (f, v_h),$$

for all $v_h \in V_h$. Note that $a(\cdot, \cdot)$ and $(\cdot, \cdot)$ are now evaluated over V_h . Any $u_h, v_h \in V_h$ may be expressed as a linear combination of the basis elements, so

$$(6.3) \quad u_h = \sum_{j=1}^{N_{\text{DoF}}} c_j N_j, \quad v_h = \sum_{i=1}^{N_{\text{DoF}}} d_i N_i.$$

Using (6.3) in (6.2), we see that

$$(6.4) \quad a(u_h, v_h) = a\left(\sum_{j=1}^{N_{\text{DoF}}} c_j N_j, \sum_{i=1}^{N_{\text{DoF}}} d_i N_i\right)$$

$$(6.5) \quad = \sum_{j=1}^{N_{\text{DoF}}} \sum_{i=1}^{N_{\text{DoF}}} c_j d_i a(N_j, N_i),$$

and

$$(6.6) \quad (f, v_h) = (f, \sum_{i=1}^{N_{\text{DoF}}} d_i N_i)$$

$$(6.7) \quad = d_i \sum_{i=1}^{N_{\text{DoF}}} (f, N_i).$$

Thus, the discrete problem boils down to solving the system of equations

$$(6.8) \quad \sum_{j=1}^{N_{\text{DoF}}} a(N_j, N_i) c_j = (f, N_i),$$

or

$$(6.9) \quad \mathbf{K} \mathbf{c} = \mathbf{b},$$

where

$$(6.10) \quad K_{ij} = a(N_j, N_i), \quad b_j = (f, N_j).$$

Often, the right hand side terms physically represent body forces or applied external forces, so we call $\mathbf{b}$ the load vector. Here, $\mathbf{K}$ and $\mathbf{b}$ contain global information about the discrete problem. The construction of the load vector and $\mathbf{K}$ may be performed entry-wise, but this is usually not the most efficient method for constructing $\mathbf{b}$ and $\mathbf{K}$. Typically, the support of a given basis function N_i is local to its associated DoF, so this results in $\mathbf{K}$ having a sparse structure. Therefore, we assume that $a(\cdot, \cdot)$ may be decomposed as

$$(6.11) \quad a(u_h, v_h) = \sum_{E \in \mathcal{T}_h} a^E(u_h, v_h),$$

so that we may construct $\mathbf{K}$ using the element matrices $\mathbf{K}^E$,

$$(6.12) \quad K_{ij}^E = a^E(N_j, N_i),$$

where $i, j = 1, 2, \dots, N_{\text{DoF}}^E$. Here, N_{DoF}^E is the number of DoFs associated with element E . Once the element matrices are computed, we construct the global matrix $\mathbf{K}$, so the bulk of the computational work is spent on element-level computations. This aspect of element-style methods lends itself to be readily parallelized.

6.3. Basis Functions

The basis functions are the key tools used in building the components necessary to solve the discrete problem, and we will consider two standard choices of FE basis functions. Over triangles and tetrahedra, we will construct the linear basis functions, and for quadrilaterals and hexahedra, we build the bilinear and trilinear functions, respectively. To construct element matrices and the load vector, we must evaluate the basis functions over the element, and since quadrature schemes are performed on some parameterized domain, we construct the basis functions in a parent domain. Therefore each element $E \in \mathcal{T}_h$ is mapped to a parent element.

Three properties we ask of the basis functions are to be $C^m(E)$ inside the element, C^{m-1} at element interfaces, and polynomial complete. The $C^m(E)$ condition depends on the problem being solved since we typically integrate derivatives over the element, and continuity at element interfaces ensures that the derivatives of the basis functions only have finite jump discontinuities at element interfaces. Elements which satisfy these two conditions are called compatible or conforming. Let

$u_h \in V_h$ with

$$(6.13) \quad u_h = \sum_{i=1}^{N_{\text{DoF}}^E} d_i N_i(\mathbf{x}).$$

The polynomial completeness condition is satisfied if for $i = 1, 2, \dots, N_{\text{DoF}}^E$,

$$(6.14) \quad d_i = c_0 + c_1 x_i + c_2 y_i + c_3 z_i,$$

implies

$$(6.15) \quad u_h(\mathbf{x}) = c_0 + c_1 x + c_2 y + c_3 z.$$

That is, the basis functions be able to reproduce any arbitrary polynomial. Equivalently, we see that

$$(6.16) \quad u_h = \sum_{i=1}^{N_{\text{DoF}}^E} d_i N_i(\mathbf{x})$$

$$(6.17) \quad = c_0 \sum_{i=1}^{N_{\text{DoF}}^E} N_i(\mathbf{x}) + c_1 \sum_{i=1}^{N_{\text{DoF}}^E} x_i N_i(\mathbf{x}) + c_2 \sum_{i=1}^{N_{\text{DoF}}^E} y_i N_i(\mathbf{x}) + c_3 \sum_{i=1}^{N_{\text{DoF}}^E} z_i N_i(\mathbf{x}),$$

so satisfaction of the linear reproducibility conditions,

$$(6.18) \quad \sum_{i=1}^{N_{\text{DoF}}^E} N_i(\mathbf{x}) = 1$$

$$(6.19) \quad \sum_{i=1}^{N_{\text{DoF}}^E} x_i N_i(\mathbf{x}) = x$$

$$(6.20) \quad \sum_{i=1}^{N_{\text{DoF}}^E} y_i N_i(\mathbf{x}) = y$$

$$(6.21) \quad \sum_{i=1}^{N_{\text{DoF}}^E} z_i N_i(\mathbf{x}) = z,$$

results in (linear) polynomial completeness being satisfied.

We denote coordinates in the original domain by $\mathbf{x} = (x, y)$ and $\mathbf{x} = (x, y, z)$ in $2D$ and $3D$, respectively, and coordinates in the parent domain by $\boldsymbol{\xi}$ with components $\boldsymbol{\xi} = (\xi, \eta)$ and $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ in $2D$ and $3D$, respectively. Each basis function N_i is associated with a DoF, and

we have the interpolation property $\text{dof}_j(N_i) = \delta_{ij}$ where $\text{dof}_j(\cdot)$ is the degree of freedom operator which evaluates N_i at the j -th DoF. We now proceed with deriving the basis functions for triangles and quadrilaterals in $2D$ and tetrahedra and hexahedra in $3D$.

6.3.1. Triangles. We construct the linear basis functions over triangles, and as the parent domain, we use the triangle whose vertices are $\boldsymbol{\xi}_1 = (0, 0)$, $\boldsymbol{\xi}_2 = (1, 0)$, and $\boldsymbol{\xi}_3 = (0, 1)$. We denote the parent triangle by $\triangle$, and an affine function in $2D$ is of the form

$$(6.22) \quad N(\boldsymbol{\xi}) = \alpha\xi + \beta\eta + \delta,$$

so three parameters define N . We require three conditions to establish the values of α , β , and δ , and a natural choice is to choose the value of N at the vertices of the triangle as the DoFs and demand the interpolation property be satisfied. Then, there are three basis functions defined over $\triangle$, and each is associated with a vertex of the triangle. The three basis functions are

$$(6.23) \quad N_1(\boldsymbol{\xi}) = 1 - \xi - \eta$$

$$(6.24) \quad N_2(\boldsymbol{\xi}) = \xi$$

$$(6.25) \quad N_3(\boldsymbol{\xi}) = \eta.$$

When performing quadrature over the element E , we need to determine a suitable change of variables which maps the element E to the parent domain $\triangle$. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be the vertices of E which will be mapped to the vertices of $\triangle$. We hope to establish an affine transformation so that

$$(6.26) \quad x = \alpha_1\xi + \alpha_2\eta + \alpha_3$$

$$(6.27) \quad y = \beta_1\xi + \beta_2\eta + \beta_3$$

for some unknown constants α_i , β_i , δ_i , $i = 1, 2$. The change of variables results in the two linear systems

$$(6.28) \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Now, when we compute integrals over E , we have

$$(6.29) \quad \int_E f(\mathbf{x}) \, d\mathbf{x} = \int_{\Delta} f(\mathbf{x}(\boldsymbol{\xi})) \, |\mathbf{J}| d\boldsymbol{\xi},$$

where $\mathbf{J}$ is the Jacobian of the transformation,

$$(6.30) \quad \mathbf{J} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \beta_1 & \beta_2 \end{bmatrix}.$$

6.3.2. Quadrilaterals. Next, we consider quadrilateral elements. Let the parent domain be the square centered around the origin with vertices $\boldsymbol{\xi}_1 = (1, 1)$, $\boldsymbol{\xi}_2 = (-1, 1)$, $\boldsymbol{\xi}_3 = (-1, -1)$, and $\boldsymbol{\xi}_4 = (1, -1)$ and let $\square$ denote the parent domain. Again, we want the basis functions to satisfy the property that $N_i(\boldsymbol{\xi}_j) = \delta_{ij}$, so the suitable basis functions over $\square$ are

$$(6.31) \quad N_1(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$(6.32) \quad N_2(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

$$(6.33) \quad N_3(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$(6.34) \quad N_4(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta).$$

Now, the change of variables from $\square$ to E is of the form

$$(6.35) \quad x = \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \xi \eta + \alpha_4$$

$$(6.36) \quad y = \beta_1 \xi + \beta_2 \eta + \beta_3 \xi \eta + \beta_4$$

which yields the linear systems

$$(6.37) \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

Now, when we compute integrals over E , we have

$$(6.38) \quad \int_E f(\mathbf{x}) \, d\mathbf{x} = \int_{\square} f(\boldsymbol{\xi}) \, |\mathbf{J}| d\boldsymbol{\xi},$$

where the Jacobian is now

$$(6.39) \quad \mathbf{J}(\boldsymbol{\xi}) = \begin{bmatrix} \alpha_1 + \alpha_3\eta & \alpha_2 + \alpha_3\xi \\ \beta_1 + \beta_3\eta & \beta_2 + \beta_3\xi \end{bmatrix}.$$

6.3.3. Tetrahedra. Let the parent domain be the tetrahedron in the first quadrant with vertices $\boldsymbol{\xi}_0 = (0, 0, 0)$, $\boldsymbol{\xi}_1 = (1, 0, 0)$, $\boldsymbol{\xi}_2 = (0, 1, 0)$, and $\boldsymbol{\xi}_3 = (0, 0, 1)$, and we demand that $N_j(\boldsymbol{\xi}_j) = \delta_{ij}$. The resulting basis functions are

$$(6.40) \quad N_1(\boldsymbol{\xi}) = 1 - \xi - \eta - \zeta$$

$$(6.41) \quad N_2(\boldsymbol{\xi}) = \xi$$

$$(6.42) \quad N_3(\boldsymbol{\xi}) = \eta$$

$$(6.43) \quad N_4(\boldsymbol{\xi}) = \zeta.$$

The mapping from E to the parent domain is of the form

$$(6.44) \quad x = \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta + \alpha_4$$

$$(6.45) \quad y = \beta_1\xi + \beta_2\eta + \beta_3\zeta + \beta_4$$

$$(6.46) \quad z = \delta_1\xi + \delta_2\eta + \delta_3\zeta + \delta_4,$$

which results in the three linear systems

$$(6.47) \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}.$$

Now, integrals over E are

$$(6.48) \quad \int_E f(\mathbf{x}) \, d\mathbf{x} = \int_{\Delta} f(\mathbf{x}(\boldsymbol{\xi})) \, |\mathbf{J}| \, d\boldsymbol{\xi},$$

and the Jacobian of the transformation is now,

$$(6.49) \quad \mathbf{J} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix}.$$

6.3.4. Hexahedra. Let the parent domain be the cube centered around the origin with vertices $\boldsymbol{\xi}_1 = (-1, -1, -1)$, $\boldsymbol{\xi}_2 = (1, -1, -1)$, $\boldsymbol{\xi}_3 = (1, 1, -1)$, $\boldsymbol{\xi}_4 = (-1, 1, -1)$, $\boldsymbol{\xi}_5 = (-1, -1, 1)$, $\boldsymbol{\xi}_6 = (1, -1, 1)$, $\boldsymbol{\xi}_7 = (1, 1, 1)$, $\boldsymbol{\xi}_8 = (-1, 1, 1)$. Demanding that $N_j(\boldsymbol{\xi}_i) = \delta_{ij}$ results in the trilinear basis functions

$$(6.50) \quad N_1(\boldsymbol{\xi}) = \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \zeta)$$

$$(6.51) \quad N_2(\boldsymbol{\xi}) = \frac{1}{8}(1 + \xi)(1 - \eta)(1 - \zeta)$$

$$(6.52) \quad N_3(\boldsymbol{\xi}) = \frac{1}{8}(1 + \xi)(1 + \eta)(1 - \zeta)$$

$$(6.53) \quad N_4(\boldsymbol{\xi}) = \frac{1}{8}(1 - \xi)(1 + \eta)(1 - \zeta)$$

$$(6.54) \quad N_5(\boldsymbol{\xi}) = \frac{1}{8}(1 - \xi)(1 - \eta)(1 + \zeta)$$

$$(6.55) \quad N_6(\boldsymbol{\xi}) = \frac{1}{8}(1 + \xi)(1 - \eta)(1 + \zeta)$$

$$(6.56) \quad N_7(\boldsymbol{\xi}) = \frac{1}{8}(1 + \xi)(1 + \eta)(1 + \zeta)$$

$$(6.57) \quad N_8(\boldsymbol{\xi}) = \frac{1}{8}(1 - \xi)(1 + \eta)(1 + \zeta).$$

Now, the mapping from E to the parent domain is of the form

$$(6.58) \quad x = \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta + \alpha_4\xi\eta + \alpha_5\xi\zeta + \alpha_6\eta\zeta$$

$$(6.59) \quad y = \beta_1\xi + \beta_2\eta + \beta_3\zeta + \beta_4\xi\eta + \beta_5\xi\zeta + \beta_6\eta\zeta$$

$$(6.60) \quad z = \delta_1\xi + \delta_2\eta + \delta_3\zeta + \delta_4\xi\eta + \delta_5\xi\zeta + \delta_6\eta\zeta,$$

and the Jacobian is

$$(6.61) \quad \mathbf{J}(\boldsymbol{\xi}) = \begin{bmatrix} \alpha_1 + \alpha_4\eta + \alpha_5\zeta & \alpha_2 + \alpha_4\xi + \alpha_6\zeta & \alpha_3 + \alpha_5\xi + \alpha_6\eta \\ \beta_1 + \beta_4\eta + \beta_5\zeta & \beta_2 + \beta_4\xi + \beta_6\zeta & \beta_3 + \beta_5\xi + \beta_6\eta \\ \delta_1 + \delta_4\eta + \delta_5\zeta & \delta_2 + \delta_4\xi + \delta_6\zeta & \delta_3 + \delta_5\xi + \delta_6\eta \end{bmatrix}.$$

6.4. Poisson

Now, as our primer for using the FEM, we consider the Poisson equation with Dirichlet boundary conditions $u = u_D$ on $\partial\Omega_D$ and Neumann boundary conditions $\nabla u \cdot n = u_N$ on $\partial\Omega_N$. Recall the weak formulation of the Poisson problem states: find $u \in V = \{u \in H^1(\Omega) : u = u_D \text{ on } \partial\Omega_D\}$ such that

$$(6.62) \quad a(u, v) = (f, v) + \int_{\partial\Omega_N} u_N v \, dS,$$

for all $v \in H_0^1(\Omega)$. Here,

$$(6.63) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{x},$$

and

$$(6.64) \quad (f, v) = \int_{\Omega} f v \, d\mathbf{x}.$$

Now, choosing a finite dimensional subspace $V_h \subset V$, the discrete problem is to find $u_h \in V_h$ such that

$$(6.65) \quad a(u_h, v_h) = (f, v_h) + \int_{\partial\Omega_N} u_N v_h \, dS,$$

for all $v_h \in V_0$ where $V_0 \subset V_h$ consists of functions in V_h with zero trace. Finally, we decompose $a(\cdot, \cdot)$ and $(\cdot, \cdot)$ into its element-level bilinear forms,

$$(6.66) \quad a(u_h, v_h) = \sum_{E \in \mathcal{T}_h} a^E(u_h, v_h), \quad (f, v_h) = \sum_{E \in \mathcal{T}_h} (f, v_h)_E,$$

and express u_h and v_h in terms of the basis functions,

$$(6.67) \quad u = \sum_{i=1}^{N_{\text{DoF}}} d_i N_i, \quad v = \sum_{i=1}^{N_{\text{DoF}}} c_i N_i.$$

Then, we have that

$$(6.68) \quad a^E(u_h, v_h) = \int_E \nabla u_h \cdot \nabla v_h \, d\mathbf{x}$$

$$(6.69) \quad = \sum_{i=1}^{N_{\text{DoF}}} \sum_{j=1}^{N_{\text{DoF}}} c_i d_j \int_E \nabla N_i \cdot \nabla N_j \, d\mathbf{x},$$

so the entries of the element matrix are

$$(6.70) \quad K_{ij}^E = \int_{\tilde{E}} \nabla N_i \cdot \nabla N_j |\mathbf{J}| d\boldsymbol{\xi},$$

where $\tilde{E}$ is the parent domain of E . Next, for the load vector, we have

$$(6.71) \quad (f, v_h)_E = \sum_{i=1}^{N_{\text{DoF}}} \int_E f N_i d\mathbf{x},$$

so

$$(6.72) \quad b_{ij}^E = \int_{\tilde{E}} f N_i d\boldsymbol{\xi}.$$

Thus, we arrive at the global system

$$(6.73) \quad \mathbf{K} \mathbf{d} = \mathbf{b},$$

which is a linear system of N_{DoF} equations in N_{DoF} unknowns.

6.5. Elastodynamics

Next, we consider elastodynamics which is a time dependent vectorial problem. Extension to time dependent problems follows by assuming that the coefficients in the basis function expansion of $u_h \in V_h$ are time dependent,

$$(6.74) \quad \mathbf{u}_h(\mathbf{x}, t) = \sum_{i=1}^{N_{\text{DoF}}} d_i(t) \mathbf{N}_i(\mathbf{x}),$$

and using the scalar basis functions $\{N_i\}_{i=1}^{N_{\text{DoF}}}$, we extend to the vectorial regime by considering the vectorial basis defined by

$$(6.75) \quad (\mathbf{N}_{di+j})_k = \delta_{jk} N_i.$$

Let $\mathbf{u}_D$ be the prescribed traction boundary conditions on $\partial\Omega_D$, $\mathbf{t}$ the applied traction on $\partial\Omega_N$, and $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0$ and $\dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0$ the initial conditions. Recall that the weak form of the elastodynamic problem is as follows: find $\mathbf{u} \in V = \{\mathbf{u} \in [H^1(\Omega)]^3 : \mathbf{u}_D \text{ on } \partial\Omega_D\}$ such that

$$(6.76) \quad \rho(\ddot{\mathbf{u}}, \mathbf{v}) + a^\varepsilon(\mathbf{u}, \mathbf{v}) = (\mathbf{b}, \mathbf{v}) + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} d\mathbf{X},$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^3$. Here,

$$(6.77) \quad a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{X},$$

and

$$(6.78) \quad (\mathbf{b}, \mathbf{v}) = \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \, d\mathbf{X}.$$

Now, the finite dimensional problem is as follows: find $u_h \in V_h$ such that

$$(6.79) \quad \rho(\ddot{\mathbf{u}}, \mathbf{v}) + a^\varepsilon(\mathbf{u}, \mathbf{v}) = (\mathbf{b}, \mathbf{v}) + \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} \, d\mathbf{X},$$

for all $\mathbf{v} \in V_0 \subset V$. Now, expressing $\mathbf{u}$ and $\mathbf{v}$ in terms of the basis functions,

$$(6.80) \quad \mathbf{u} = \sum_{i=1}^{N_{\text{DoF}}} d_i(t) \mathbf{N}_i(\mathbf{x}), \quad \mathbf{v} = \sum_{j=1}^{N_{\text{DoF}}} c_j(t) \mathbf{N}_j(\mathbf{x}),$$

we have the linear system

$$(6.81) \quad \mathbf{M} \ddot{\mathbf{d}} + \mathbf{K} \mathbf{d} = \mathbf{f},$$

where

$$(6.82) \quad M_{ij} = \rho(\mathbf{N}_i, \mathbf{N}_j), \quad K_{ij} = a^\varepsilon(\mathbf{N}_i, \mathbf{N}_j), \quad f_i = (\mathbf{b}, \mathbf{N}_i) + \int_{\partial\Omega_N} \mathbf{t} \cdot \mathbf{N}_i \, dS.$$

We call $\mathbf{M}$ and $\mathbf{K}$ the global mass and stiffness matrices, respectively. Construction of the global mass and stiffness matrices are done at the element-level by decomposing $a^\varepsilon(\cdot, \cdot)$ and the L^2 inner product,

$$(6.83) \quad a^\varepsilon(u, v) = \sum_{i=1}^{N_E} a^{\varepsilon, E}(u, v), \quad (u, v) = \sum_{i=1}^{N_E} (u, v)_E,$$

so that the element mass and stiffness matrices are

$$(6.84) \quad M_{ij}^E = \rho(N_i, N_j)_E, \quad K_{ij}^E = a^{\varepsilon, E}(N_i, N_j).$$

6.6. Nonconvex Quadrilaterals

Consider a quadrilateral element with fixed vertices $\mathbf{x}_2 = (-1, 1)$, $\mathbf{x}_3 = (-1, -1)$, and $\mathbf{x}_4 = (1, -1)$ consider the node that varies as $\mathbf{x}_1(t) = (1, 1) + 2t(-1, -1)$ for $t \in [0, 1]$. For $t = 0$, we have a square, for $t = \frac{1}{2}$ the quadrilateral degenerates into a triangle, and for $\frac{1}{2} < t < 1$ the quadrilateral becomes nonconvex. See Figure 6.1 for the evolution of the square from square to triangle to nonconvex quadrilateral.

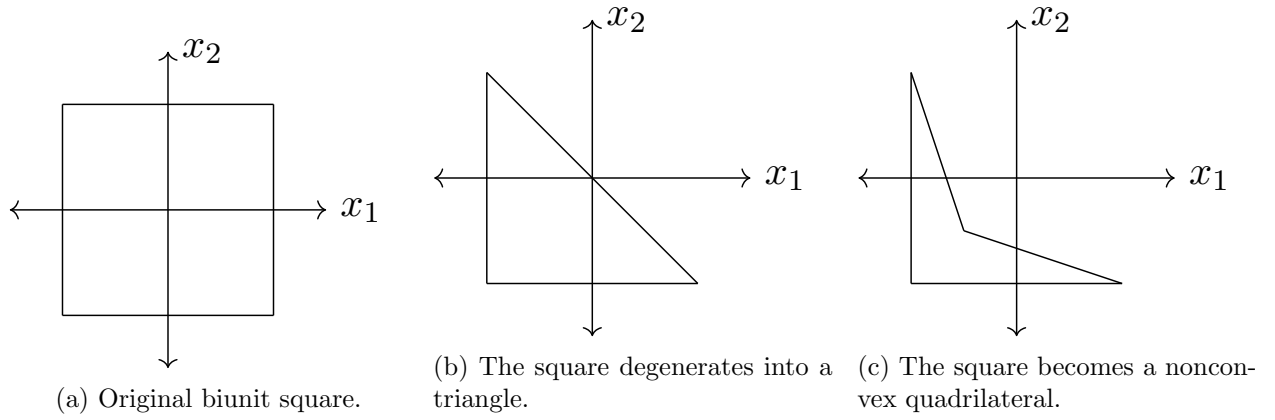


FIGURE 6.1. Evolution of the square into the nonconvex quadrilateral.

The mapping from the parent element (biunit square) to the quadrilateral element is given by

$$(6.85) \quad x = (1 - \frac{t}{2})\xi - \frac{t}{2}\eta - \frac{t}{2}\xi\eta - \frac{t}{2}$$

$$(6.86) \quad y = -\frac{t}{2}\xi + (1 - \frac{t}{2})\eta - \frac{t}{2}\xi\eta - \frac{t}{2},$$

so the Jacobian is

$$(6.87) \quad \mathbf{J}(\xi, \eta; t) = \begin{bmatrix} 1 - \frac{t}{2}(1 + \eta) & -\frac{t}{2}(1 + \xi) \\ -\frac{t}{2}(1 + \eta) & 1 - \frac{t}{2}(1 + \xi) \end{bmatrix}.$$

Now,

$$(6.88) \quad |\mathbf{J}| = 1 - \frac{t}{2}(2 + \xi + \eta) \geq 1 - 2t,$$

and for $0 < t < \frac{1}{2}$, we see that $|\mathbf{J}| > 0$. For $\frac{1}{2} \leq t < 1$, $|\mathbf{J}|$ is not longer strictly positive since along the line

$$(6.89) \quad \eta = -\xi - 2\left(t - \frac{1}{t}\right),$$

$|\mathbf{J}|$ is zero. When $t = \frac{1}{2}$, $|\mathbf{J}|$ is zero at $\boldsymbol{\xi} = (1, 1)$, and for $\frac{1}{2} < t < 1$, $|\mathbf{J}| < 0$ in the region

$$(6.90) \quad \eta > -\xi - 2\left(t - \frac{1}{t}\right).$$

In the regime where $\frac{1}{2} \leq t < 1$, the basis functions are no longer valid, so element matrix construction cannot proceed.

This drawback of the FEM is a fundamental issue when the geometry of the mesh is dynamically changing. Traditionally, to avoid the nonconvex elements that appear, remeshing would be required at each step that the geometry of the mesh changes.

CHAPTER 7

Virtual Element Method

The virtual element method (VEM) is an element method arising from the mimetic finite difference method where the goal is to perform computations solely using the DoFs. The VE space consists of functions which are k -degree polynomials on the boundary of an element and which satisfy a Poisson equation inside the element. The basis functions are not computed explicitly and hence, remain virtual. Since basis functions are never explicitly constructed, the projection of the VE basis onto the space of k -degree polynomials is used to construct element matrices. We begin by establishing notation, then introduce the classical VE space. Next, we construct the polynomial projections and conclude with the VE formulations for the Poisson and Elastodynamic problems.

7.1. Preliminaries

We use $\mathcal{T}_h$ to denote a polygonal discretization in two space dimensions and polyhedral discretization in three space dimensions of the domain Ω where h is some characteristic length of the mesh. $E \in \mathcal{T}_h$ denotes a polyhedral element that has N_V^E vertices. In two space dimensions, e_i represents the i -th edge of E . In three space dimensions, we use f_i^E to denote the i -th face of element E , and e_i^f represents the i -th edge of f_i^E . The diameter of element E is h_E , and $\mathbf{x}_E$ is its centroid.

The diameter of an element is the maximum distance between any two vertices of the element and is given by

$$(7.1) \quad h_E = \max_{i \in \{1, \dots, N_{v_E}\}} \|\mathbf{X}_i - \mathbf{X}_j\|,$$

and the centroid, or geometric center, of the element is given by

$$(7.2) \quad \mathbf{X}_E = \frac{1}{N_{v_E}} \sum_{i=1}^{N_{v_E}} \mathbf{X}_i.$$

7.1.1. Scaled Monomial Basis. The projection of functions in the virtual element space onto a polynomial space is a fundamental tool when using the VEM. Let $\mathbb{P}_k(E)$ be the collection of degree k monomials. We use the scaled element monomials $\mathcal{M}_k(E)$ as a basis for $\mathbb{P}_k(E)$. We restrict ourselves to the case $k = 1$, and in two dimensions, $\mathcal{M}_1(E)$ consists of

$$(7.3) \quad \xi = \frac{x - x_E}{h_E}, \quad \eta = \frac{y - y_E}{h_E},$$

and in three dimensions, we have the additional scaled monomial

$$(7.4) \quad \zeta = \frac{z - z_E}{h_E}.$$

The basis we employ for $\mathbb{P}_1(E)$ in two dimensions is

$$(7.5) \quad \{1, \xi, \eta\},$$

and

$$(7.6) \quad \{1, \xi, \eta, \zeta\},$$

in three dimensions. For two dimensional vectorial problems, we use

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\eta \\ \xi \end{bmatrix}, \begin{bmatrix} \eta \\ \xi \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \eta \end{bmatrix} \right\}$$

as our basis for $[\mathbb{P}_1(E)]^2$, and the basis for $[\mathbb{P}_1(E)]^3$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\zeta \\ \eta \end{bmatrix}, \begin{bmatrix} \zeta \\ 0 \\ -\xi \end{bmatrix}, \begin{bmatrix} -\eta \\ \xi \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \zeta \\ \eta \end{bmatrix}, \begin{bmatrix} \zeta \\ 0 \\ \xi \end{bmatrix}, \begin{bmatrix} \eta \\ \xi \\ 0 \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \eta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \zeta \end{bmatrix} \right\}.$$

This choice and order of the bases is convenient and will be evidenced when we construct the polynomial projections.

7.2. Virtual Element Space

The classical local virtual element space $V_k(E)$ consists of functions that are k degree polynomials on the boundary of an element and whose Laplacian are a $k - 2$ degree polynomial over an

element. This setting is sufficient for solving the Poisson equation in two dimensions, but difficulties arise when extending the approach to three dimensions or computing L^2 -type inner products. The enhanced virtual element space [2] overcomes these issues by enlarging $V_k(E)$ and subsequently restricting the enlarged space to a suitable subspace. In this section, we construct the classical virtual element space in two space dimensions.

First, let

$$(7.7) \quad B_k(\partial E) := \{v \in C^0(\partial E) : v|_e \in \mathbb{P}_k(e) \ \forall e \subset \partial E\},$$

be the function space which consists of functions that are continuous over the boundary of an element and a degree k polynomial when restricted to an edge. The local virtual element space is then defined to be

$$(7.8) \quad V_k(E) = \{v \in H^1(E) : v|_{\partial E} \in B_k(\partial E), \Delta v|_E \in \mathbb{P}_{k-2}(E)\}.$$

A function $v \in V_k(E)$ is continuous over the boundary of E , is a k -degree polynomial on each edge of E , and has Laplacian that is a $k - 2$ -degree polynomial over E , and the dimension of $V_k(E)$ is

$$(7.9) \quad \dim V_k(E) = N_{v_E} + (k - 1)N_{v_E} + \frac{k(k - 1)}{2}.$$

This follows from recognizing the Laplace problem with the $N_{v_E} + (k - 1)N_{v_E}$ unique Dirichlet boundary conditions and the $\frac{k(k-1)}{2}$ Poisson problems with homogeneous boundary conditions that arise from the definition of $V_k(E)$. These problems are never solved when using the VEM.

The DoFs for $V_k(E)$ are taken to be the following:

- the value of v at the N_{v_E} vertices of E ,
- the value of v at $k - 1$ interior points on each edge e of E ,
- the $k - 2$ moments

$$(7.10) \quad \frac{1}{|E|} \int_E m_\alpha v \, d\mathbf{X},$$

for $\alpha = 0, 1, \dots, k - 2$.

The above DoFs are sufficient to uniquely identify a function $v \in V_k(E)$ [78], and a function $v \in V_k(E)$ is evaluated at a given DoF using the DoF operator $\text{dof}_i(\cdot)$. For the vertex and edge

DoFs, $\text{dof}_i(v)$ evaluates v at the vertices and interior edge points, and for the moment DoFs,

$$(7.11) \quad \text{dof}_{kN_{v_E} + \frac{(k-1)k}{2} + \alpha}(v) = \frac{1}{|E|} \int_E m_\alpha v \, d\mathbf{X},$$

for $\alpha = 0, 1, \dots, k-2$.

Now, the global VE space V^h is defined by demanding that the restriction of $v \in V^h$ to an element E lies in the local VE space $V_k(E)$,

$$(7.12) \quad V^h = \{v \in H_0^1(\Omega) : v|_E \in V_k(E) \, \forall E \in \mathcal{T}_h\}.$$

The global DoFs are:

- the value of v at internal vertices,
- the value of v at $k-1$ points of each internal edge,
- the $k-2$ moments in each element.

Let $\{\varphi_i\}$ be the canonical basis of V^h which satisfies the Kronecker-delta property $\text{dof}_i(\varphi_j) = \delta_{ij}$. The key stipulation of the VEM is that the basis functions remain virtual, so they are never explicitly constructed. Instead, the projection of the VE basis onto the space of polynomials is the key tool used to construct element matrices.

7.3. Polynomial Projection Operators

In element-type methods, the global matrices that are used to solve the finite dimensional problem are assembled using element matrices. Typically, construction of element matrices require evaluating bilinear forms using the basis functions, but the VE basis is never explicitly constructed. Since the VE basis remains virtual, evaluation of bilinear forms is performed using the projection of the VE basis onto polynomials. Construction of the projection matrix requires solving a linear system for the coefficients that express the projection of a function in the VE space in terms of the polynomial basis functions. That is, the projector $\Pi_k^{\cdot, E} : V_k(E) \rightarrow \mathbb{P}_k(E)$ allows $v \in V_k(E)$ to be approximated by a linear combination of k degree polynomials,

$$(7.13) \quad \Pi_k^{\cdot, E} v = \sum_{\alpha=1}^{n_k} c_\alpha m_\alpha.$$

The projector is constructed by enforcing orthogonality of the projection error $v - \Pi_k^E v$ to any monomial $m \in \mathbb{P}_k(E)$ with respect to a chosen bilinear form. We now construct the elliptic, L^2 , and energy projections.

7.3.1. Elliptic Projection Operator. The elliptic projection of a function $v \in H^1(E)$ is the unique element $\Pi_1^{\nabla, E} v \in \mathbb{P}_1(E)$ that satisfies the orthogonality condition

$$(7.14) \quad a^E(m_\alpha, \Pi_k^{\nabla, E} v^h - v^h) = 0,$$

for all $m_\alpha \in \mathbb{P}_k(E)$, where,

$$(7.15) \quad a^E(u, v) := \int_E \nabla u \cdot \nabla v \, d\mathbf{X}.$$

Note that for m_1 , (7.14) does not provide a meaningful equation since it leads to the trivial identity $0 = 0$, so we require the additional condition that

$$(7.16) \quad \int_{\partial E} (u - \Pi_1^{\nabla, E} u) \cdot \mathbf{m}_1 \, dS = 0.$$

Writing out (7.14), we have

$$(7.17) \quad \int_E \nabla \Pi_k^{\nabla, E} v \cdot \nabla m \, d\mathbf{X} = \int_E \nabla v \cdot \nabla m \, d\mathbf{X},$$

and upon using (7.13) in the LHS of (7.17), we see that

$$(7.18) \quad \int_E \nabla \Pi_k^{\nabla, E} v \cdot \nabla m \, d\mathbf{X} = \sum_{\alpha=0}^k c_\alpha \int_E \nabla m_\alpha \cdot \nabla m \, d\mathbf{X},$$

which is a polynomial integral, so it is computable using a suitable cubature scheme [17]. Next, the integral on the RHS of (7.17) requires information about ∇v over E , but we only have explicit information about v on the boundary ∂E . This impasse is overcome by using the divergence theorem to write the RHS of (7.17) as

$$(7.19) \quad \begin{aligned} \int_E \nabla v \cdot \nabla m \, d\mathbf{X} &= \int_E \nabla \cdot (v \nabla m) \, d\mathbf{X} - \int_E v \Delta m \, d\mathbf{X} \\ &= \int_{\partial E} v \nabla m \cdot \mathbf{n} \, dS - \int_E v \Delta m \, d\mathbf{X}. \end{aligned}$$

The first integral on the RHS of (7.19) is computable since we know that on each edge of E , v is a k -degree polynomial, and we know the value of v at each vertex of E and the value of v at $k - 1$ interior points of each edge. Ergo, we may readily construct v on the boundary of E . The second integral on the RHS of (7.19) is also computable since $\Delta m \in \mathbb{P}_{k-2}$, and we know the $k - 2$ moments of v . Thus, we have established the computability of the elliptic projection of $v \in V_k(E)$ using the DoFs.

7.3.2. L^2 Projection Operator. In time-dependent problems, we require construction of the mass matrix which is constructed by computing an L^2 -type inner product of the basis functions. That is, we must compute integrals of the form

$$(7.20) \quad (u, v) = \int_E uv \, d\mathbf{X}.$$

To construct the mass matrix, we use the $L^2(E)$ projection $\Pi_k^{0,E} : W_k(E) \rightarrow \mathbb{P}_k(E)$, where $W_k(E)$ is the enhanced virtual element space. Before we define the enhanced space $W_k(E)$, we illustrate the roadblock we encounter if we do not make use of $W_k(E)$.

Suppose we proceed as we did for the elliptic projection. The orthogonality condition requires

$$(7.21) \quad (\Pi_k^{0,E} v - v, m_\alpha) = 0,$$

for all $m_\alpha \in \mathbb{P}_k(E)$. Writing out (7.21), we have

$$(7.22) \quad \int_E \Pi_k^{0,E} v m_\alpha \, d\mathbf{X} = \int_E v m_\alpha \, d\mathbf{X},$$

but we see that we may no longer apply the divergence theorem to integral on the RHS, and we only know the $k - 2$ moments of v . Therefore, the RHS is not computable from the DoFs.

The technique that will allow us to define a computable L^2 projection will be to first enlarge $V_k(E)$ and then restrict the enlarged space to a desirable subspace [2]. We enlarge $V_k(E)$ by softening the requirement that Δv be an at most $k - 2$ degree polynomial and instead require that $\Delta v \in \mathbb{P}_k(E)$ so that the enlarged space is

$$(7.23) \quad \tilde{V}_k(E) := \{v \in H^1(\Omega) : v|_{\partial E} \in B_k(\partial E), \Delta v \in \mathbb{P}_k(E)\}.$$

Next, to restrict the enlarged space $\tilde{V}_k(E)$ to a suitable subspace, recall that the issue at hand is the computability of the integral

$$(7.24) \quad \int_E v m_\alpha d\mathbf{X},$$

for all $m_\alpha \in \mathbb{P}_k(E)$. We know the $k-2$ moments of v , so it remains to compute (7.24) for $m_\alpha \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E)$. To resolve this issue, we restrict $\tilde{V}_k(E)$ to a subspace in which

$$(7.25) \quad \int_E v m_\alpha d\mathbf{X} = \int_E \Pi_k^{\nabla, E} v m_\alpha d\mathbf{X},$$

for $m_\alpha \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E)$. Essentially, we have solved the computability problem of (7.24) by moving to a space in which the elliptic projection of v is a suitable approximation of v . Thus, the restricted space over which we define the L^2 projection is

$$(7.26) \quad W_k(E) := \{w \in \tilde{V}_k(E) : (w - \Pi_k^{\nabla, E} w, m_\alpha) = 0 \forall m_\alpha \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E)\},$$

and the DoFs that identify $w \in W_k(E)$ are the same as those that identify $v \in V_k(E)$.

The cases $k=1$ and $k=2$ are special in that $\Pi_k^{0, E} \equiv \Pi_k^{\nabla, E}$. First, for $k=1$, the restriction condition (7.25) states

$$(7.27) \quad \int_E v m_\alpha d\mathbf{X} = \int_E \Pi_1^{\nabla, E} v m_\alpha d\mathbf{X},$$

for all $m_\alpha \in \mathbb{P}_1(E)$, and upon comparing (7.27) with (7.22), we see that

$$(7.28) \quad \int_E \Pi_1^{0, E} v m_\alpha d\mathbf{X} = \int_E \Pi_1^{\nabla, E} v m_\alpha d\mathbf{X},$$

for all $m_\alpha \in \mathbb{P}_1$. For $k=2$, the restriction condition (7.25) applies to $m_\alpha \in \mathbb{P}_2(E) \setminus \mathbb{P}_0(E)$, so we have

$$(7.29) \quad \int_E \Pi_2^{0, E} v m_\alpha d\mathbf{X} = \int_E \Pi_2^{\nabla, E} v m_\alpha d\mathbf{X},$$

for $m_\alpha \in \mathbb{P}_2(E) \setminus \mathbb{P}_0(E)$. The 0th moment DoF states

$$(7.30) \quad \int_E \Pi_2^{0, E} v d\mathbf{X} = \int_E \Pi_2^{\nabla, E} v d\mathbf{X},$$

so we see that $\Pi_2^{\nabla, E} \equiv \Pi_2^{0, E}$.

7.3.3. Energy Projection Operator. To construct the element stiffness matrix for linear elasticity in d space dimensions, we use the energy projection $\Pi_1^{\varepsilon,E} : V_k(E) \rightarrow [\mathbb{P}_1(E)]^d$ which is defined by the orthogonality of the projection error to the degree 1 polynomials with respect to the bilinear form,

$$(7.31) \quad a^{\varepsilon,E}(\mathbf{m}_\alpha, \mathbf{v} - \Pi_1^{\varepsilon,E} \mathbf{v}) = 0,$$

where

$$(7.32) \quad a^{\varepsilon,E}(\mathbf{u}, \mathbf{v}) := \int_E \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{X}.$$

Note that for $\mathbf{m}_\alpha \in [\mathbb{P}_1(E)]^3 (\alpha = 1, 2, \dots, 6)$, (7.31) does not provide a meaningful equation since it leads to the trivial identity $0 = 0$, so we require the additional condition that

$$(7.33) \quad \int_{\partial E} (\mathbf{v} - \Pi_1^{\varepsilon,E} \mathbf{v}) \cdot \mathbf{m}_\alpha \, dS = 0$$

for $\mathbf{m}_\alpha \in [\mathbb{P}_1(E)]^d (\alpha = 1, 2, \dots, 6)$.

Proceeding from the orthogonality condition, we have

$$(7.34) \quad a^{\varepsilon,E}(\mathbf{m}_\alpha, \mathbf{v}) = a^{\varepsilon,E}(\mathbf{m}_\alpha, \Pi_1^{\varepsilon,E} \mathbf{v})$$

for $\mathbf{m}_\alpha \in [\mathbb{P}_1(E)]^d$. The RHS of (7.34) is computable since the integrand is a polynomial, so it remains to establish the computability of the LHS of (7.34). Upon using the divergence theorem, we see that

$$(7.35) \quad \begin{aligned} a^{\varepsilon,E}(\mathbf{m}_\alpha, \mathbf{v}) &= \int_E \boldsymbol{\sigma}(\mathbf{m}_\alpha) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{X} \\ &= \int_{\partial E} \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{m}_\alpha) \cdot \mathbf{v} \, dS - \int_E (\nabla \cdot \boldsymbol{\sigma}(\mathbf{m}_\alpha)) \cdot \mathbf{v} \, d\mathbf{X} \\ &= \int_{\partial E} \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{m}_\alpha) \cdot \mathbf{v} \, dS, \end{aligned}$$

where the last line follows from $\nabla \cdot \boldsymbol{\sigma}(\mathbf{m}_\alpha) = \mathbf{0}$ since $\mathbf{m}_\alpha \in [\mathbb{P}_1(E)]^d$. For $d = 2$, this suffices for establishing a computable energy projection since we may explicitly construct $\mathbf{v}$ on the boundary ∂E .

7.3.4. Face Projection Operator. When considering the VE projection operators in three space dimensions, the boundary of E is composed of polygonal faces. When establishing the computability of the projection operators, the divergence theorem results in a boundary integral that requires information about v over ∂E . In two dimensions, v may be explicitly constructed from the DoFs on each edge of E , but in three dimensions, we do not have enough information to construct v over a polygonal face. We illustrate this difficulty using the elliptic projection.

The natural extension of the two-dimensional approach is to define the space

$$(7.36) \quad \tilde{\mathbb{B}}_k(\partial E) := \{v \in C^0(\partial E) : v^h|_f \in V_k(f) \forall f \subset \partial E\},$$

where $V_K(f)$ is the two dimensional VE space (7.8). Then, the next step would be to define

$$(7.37) \quad \tilde{\mathbb{V}}_k(E) = \{v \in H^1(E) : v^h|_{\partial E} \in \tilde{\mathbb{B}}_k(\partial E), \Delta v^h|_E \in \mathbb{P}_{k-2}(E)\},$$

but when we attempt to construct the elliptic projection of $v \in \tilde{\mathbb{V}}_k(E)$, the RHS of (7.19) is not computable from the DoFs. The uncomputable term is

$$(7.38) \quad \int_{\partial E} v \nabla m_\alpha \cdot \mathbf{n} d\mathbf{X} = \sum_{i=1}^{N_{fE}} \int_{f_i} v \nabla m_\alpha \cdot \mathbf{n} d\mathbf{X},$$

for $m_\alpha \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-1}(E)$. On a given face, we only know the $k-2$ moments of v , so we would require an additional k DoFs to be specified on each face of E . Rather than using this natural extension of the two-dimensional approach, we use the enhanced VE space (7.26) as this allows the number of DoFs to remain unchanged. Over each face, let $\Pi_k^{\nabla, f} : W_k(f) \rightarrow \mathbb{P}_k(f)$ denote the elliptic projection defined over the enhanced VE space $W_k(f)$. Now, (7.38) is computable since

$$(7.39) \quad \int_{\partial E} v \nabla m \cdot \mathbf{n} dS = \sum_{i=1}^{N_{fE}} \int_{f_i} \Pi_k^{\nabla, f} v \nabla m \cdot \mathbf{n} d\mathbf{X},$$

for $m \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E)$, and the computability of $\Pi_k^{\nabla, f} v$ has already been established.

Now, let

$$(7.40) \quad \mathbb{B}_k(\partial E) := \left\{ v \in C^0(\partial E) : v|_f \in W_k(f) \forall f \in \partial E \right\},$$

so that the VE space for three space dimensions is

$$(7.41) \quad \mathbb{V}_k(E) := \left\{ v \in H^1(E) : v|_f \in \mathbb{B}_k(\partial E) \ \forall f \in \partial E, \right. \\ \left. \Delta v|_E \in \mathbb{P}_k(E), \ (\Pi_k^{\nabla, E} v - v, m_\alpha) = 0 \ \forall m_\alpha \in \mathbb{P}_k(E) \setminus \mathbb{P}_{k-2}(E) \right\}.$$

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N_{v_f}}$ be the vertices of the polygonal face f where N_{v_f} is the number of vertices of f . To construct the elliptic face projection $\Pi_1^{\nabla, f} : W_k(f) \rightarrow \mathbb{P}_k(E)$, we define a local coordinate system over f with the origin placed at the centroid of f and basis $\{\mathbf{e}_1^f, \mathbf{e}_2^f\}$, where

$$(7.42) \quad \mathbf{e}_1^f = \frac{\mathbf{x}_1 - \mathbf{x}_0}{|\mathbf{x}_1 - \mathbf{x}_0|}, \quad \mathbf{e}_2^f = \mathbf{n}_f \times \mathbf{e}_1^f,$$

and $\mathbf{n}_f$ is the unit normal to the face. Then, to define a computable $\Pi_k^{\nabla, f}$, we proceed in a similar fashion as for $\Pi_k^{\nabla, E}$, and when we arrive at the two-dimensional analog of (7.19), we have

$$(7.43) \quad \int_f \nabla m_\alpha \cdot \nabla v \, d\mathbf{X} = \int_{\partial f} \nabla m_\alpha v \cdot \mathbf{n} \, dS - \int_f \Delta \mathbf{m}_\alpha v \, d\mathbf{X}.$$

We see that (7.43) is computable since on each edge, $v \in \mathbb{P}_k(E)$ may be explicitly constructed on each edge, and we know the $k-2$ moments of v over f . Thus, in three space dimensions, $\Pi_1^{\nabla, E}$ is computable from the DoFs. A similar procedure follows for the L^2 and energy projections.

7.4. Matrix Representation of Projection Operators

To build the element matrices, we use the N_{DoF} basis functions to compute the projection of the virtual element basis. Computing the projection of v onto $\mathbb{P}_k(E)$ reduces to solving the linear system

$$(7.44) \quad \mathbf{G}\mathbf{c}^{(i)} = \mathbf{b}^{(i)},$$

where

$$(7.45) \quad G_{\alpha\beta} = a^E(\mathbf{m}_\alpha, \mathbf{m}_\beta), \quad b_\alpha^{(i)} = a^E(m_\alpha, \varphi_i),$$

and $\mathbf{c}^{(i)}$ are the coefficients of the projection of the i -th VE basis function in the scaled monomial basis. Further, let $\mathbf{B}$ be the matrix whose columns are the $\mathbf{b}^{(i)}$ so that $\mathbf{\Pi}_* = \mathbf{G}^{-1}\mathbf{B}$ is the matrix

representation of the projection operator in the scaled monomial basis. Let

$$(7.46) \quad \mathbf{\Pi} \varphi_i = \pi_{ij} \varphi_j$$

be the projection of φ_i onto the canonical basis φ_i , where Einstein summation on repeated indices is implied. Since $\mathbb{P}_k(E)$ is a subset of the VE space, we may express $m_\alpha \in \mathbb{P}_k(E)$ in the VE basis,

$$(7.47) \quad m_\alpha = \text{dof}_j(m_\alpha) \varphi_j,$$

where $\text{dof}_i(m_\alpha)$ is the operator that extracts the i -th DoF of m_α . Then,

$$(7.48) \quad \mathbf{\Pi} \varphi_i = c_\alpha^{(i)} \mathbf{m}_\alpha = c_\alpha^{(i)} \text{dof}_j(\mathbf{m}_\alpha) \varphi_j,$$

implies $\pi_{ij} = \text{dof}_j(\mathbf{m}_\alpha) \cdot \mathbf{c}^{(i)}$. Letting $\mathbf{D}$ be the matrix of scaled monomials evaluated at the DoFs,

$$(7.49) \quad D_{\alpha\beta} = \text{dof}_\alpha(\mathbf{m}_\beta),$$

we find that the matrix representation of the projection operator in the VE basis is:

$$(7.50) \quad \mathbf{\Pi} = \mathbf{D} \mathbf{\Pi}_*.$$

7.5. Poisson

The development of the VEM for the Poisson equation was initially presented in [78], and the more practical aspects of the VEM are presented in [4]. Recall the weak formulation of the Poisson problem: find $u \in V = \{u \in H^1(\Omega) : u = u_D \text{ on } \partial\Omega_D\}$ such that

$$(7.51) \quad a(u, v) = \int_{\partial\Omega_N} u_N v \, dS + (f, v),$$

for all $v \in H_0^1(\Omega)$. Here,

$$(7.52) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\mathbf{X},$$

and

$$(7.53) \quad (u, v) = \int_{\Omega} uv \, d\mathbf{X}.$$

We have the element-wise splittings

$$(7.54) \quad a(u, v) = \sum_{E \in \mathcal{T}_h} a^E(u, v), \quad (u, v) = \sum_{E \in \mathcal{T}_h} (u, v).$$

Now, we take the VE space $V^h \subset V$ to be our finite dimensional subspace over which we will solve the associated discrete problem. Over V^h , we have the associated discrete bilinear form $a^h : V^h \times V^h \rightarrow \mathbb{R}$ so that for $u_h, v_h \in V^h$, we again have the element-wise splitting

$$(7.55) \quad a_h(u_h, v_h) = \sum_{E \in \mathcal{T}_h} a_h^E(u_h, v_h).$$

Associated with the DoFs of V^h , we have the canonical basis $\{\varphi_i\}$ which allows us to express $u_h, v_h \in V^h$ as

$$(7.56) \quad u_h = \sum_{i=1}^{N_{\text{DoF}}} u_i \varphi_i, \quad v_h = \sum_{i=1}^{N_{\text{DoF}}} v_i \varphi_i.$$

Using the expression of u_h and v_h in terms of the canonical basis $\{\varphi_i\}$ in the weak formulation (7.51), we have

$$(7.57) \quad \sum_{i=1}^{N_{\text{DoF}}} \sum_{j=1}^{N_{\text{DoF}}} u_j v_i \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, d\mathbf{X} = \sum_{i=1}^{N_{\text{DoF}}} v_i \int_{\partial\Omega_N} \varphi_i u_N \, dS + \int_{\Omega} \varphi_i f \, d\mathbf{X}$$

which yields the linear system

$$(7.58) \quad \mathbf{K} \mathbf{u} = \mathbf{b},$$

where $(\mathbf{u})_i = u_i$,

$$(7.59) \quad K_{ij} = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, d\mathbf{X}$$

is the global stiffness matrix, and

$$(7.60) \quad b_i = \int_{\partial\Omega_N} \varphi_i^{(i)} u_N \, dS + \int_{\Omega} \varphi_i^{(i)} f \, d\mathbf{X}$$

is the global load vector. As is the case for element-style methods, the global stiffness matrix is not constructed directly, but rather through element-wise assembly. In the VEM, the basis functions are not explicitly computed, so the polynomial projection of the virtual element basis plays a

fundamental role in constructing element matrices. The elliptic projection of the VE basis will be used to evaluate the suitably chosen discrete bilinear form.

Recall that the matrix representation of the elliptic projection in the scaled monomial basis is constructed by solving the linear system

$$(7.61) \quad \mathbf{G}\Pi_*^{\nabla,E} = \mathbf{B},$$

where $\mathbf{G}$ and the columns of $\mathbf{B}$ are given in (7.45). Additionally, the matrix representation in the VE basis is

$$(7.62) \quad \Pi^{\nabla,E} = \mathbf{D}\Pi_*^{\nabla,E},$$

where $\mathbf{D}$ is given in (7.49).

7.5.1. Element Stiffness Matrix Construction. The element stiffness matrix $\mathbf{K}_E$ is given by

$$(7.63) \quad (K_E)_{ij} = a_h^E(\varphi_i, \varphi_j),$$

where $a_h^E(\cdot, \cdot)$ is the discrete bilinear form. The key stipulation of the VEM is that the basis functions are never explicitly constructed, so rather than working with the VE basis directly, we use the elliptic projection $\Pi_k^{\nabla,E}$ to evaluate $a_h^E(\cdot, \cdot)$. Using the trick of adding zero in the element bilinear form, we have

$$(7.64) \quad \begin{aligned} a^E(\varphi_i, \varphi_j) &= a^E(\varphi_i + \Pi_k^{\nabla,E} \varphi_i - \Pi_k^{\nabla,E} \varphi_i, \varphi_j + \Pi_k^{\nabla,E} \varphi_j - \Pi_k^{\nabla,E} \varphi_j) \\ &= a^E(\Pi_k^{\nabla,E} \varphi_i, \Pi_k^{\nabla,E} \varphi_j) + a^E(\varphi_i - \Pi_k^{\nabla,E} \varphi_i, \Pi_k^{\nabla,E} \varphi_j) \\ &\quad + a^E(\Pi_k^{\nabla,E} \varphi_i, \varphi_j - \Pi_k^{\nabla,E} \varphi_j) + a^E(\varphi_i - \Pi_k^{\nabla,E} \varphi_i, \varphi_j - \Pi_k^{\nabla,E} \varphi_j) \\ &= a^E(\Pi_k^{\nabla,E} \varphi^{(i)}, \Pi_k^{\nabla,E} \varphi^{(j)}) + a^E(\varphi^{(i)} - \Pi_k^{\nabla,E} \varphi^{(i)}, \varphi^{(j)} - \Pi_k^{\nabla,E} \varphi^{(j)}), \end{aligned}$$

where the cross terms vanish on account of $\Pi_k^{\nabla,E} \varphi_i \in \mathbb{P}_k(E)$ and the orthogonality condition (7.14).

To ensure convergence, we require that the discrete bilinear form be consistent and stable. Consistency requires that for all $v_h \in V_k(E)$,

$$(7.65) \quad a_h^E(m_\alpha, v_h) = a^E(m_\alpha, v_h),$$

for all $m_\alpha \in \mathbb{P}_k(E)$, and stability requires that there exist constants α_* and α^* independent of h and E so that

$$(7.66) \quad \alpha_* a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h).$$

Choosing

$$(7.67) \quad a_h^E(u_h, v_h) = a^E(\Pi_k^{\nabla, E} u_h, \Pi_k^{\nabla, E} v_h) + S^E((I - \Pi_k^{\nabla, E}) u_h, (I - \Pi_k^{\nabla, E}) v_h),$$

ensures that the discrete bilinear form is consistent and stable. Here, $S^E(\cdot, \cdot)$ is a suitably chosen bilinear form which satisfies

$$(7.68) \quad c_0 a^E(v_h, v_h) \leq S^E(v_h, v_h) \leq c_1 a^E(v_h, v_h),$$

for constants c_0 and c_1 that are independent of h and E . The bilinear form S^E is called the stabilization term, and the choice of S^E is not precise, but it should scale like a^E on the nullspace of $\Pi_k^{\nabla, E}$. Additionally, the stabilization term should ensure that the element stiffness matrix is of sufficient rank.

Now, the element stiffness matrix is given by,

$$(7.69) \quad \mathbf{K}_E = \mathbf{K}_E^c + \mathbf{K}_E^s,$$

where

$$(7.70) \quad (K_E^c)_{ij} = a^E(\Pi_k^{\nabla, E} \varphi_i, \Pi_k^{\nabla, E} \varphi_j)$$

is the consistency term and the stabilization term is

$$(7.71) \quad (K_E^s)_{ij} = S^E((I - \Pi_k^{\nabla, E}) \varphi_i, (I - \Pi_k^{\nabla, E}) \varphi_j)$$

First, the consistency part of the element stiffness matrix is

$$\begin{aligned}
(K_E^c)_{ij} &= a_E(\Pi_k^{\nabla,E} \varphi_i, \Pi_k^{\nabla,E} \varphi_j) \\
&= a_E(c_\alpha^{(i)} m_\alpha, c_\beta^{(j)} m_\beta) \\
&= c_\alpha^{(i)} c_\beta^{(j)} a_E(m_\alpha, m_\beta) \\
&= (\mathbf{c}^{(i)})^T \tilde{\mathbf{G}} \mathbf{c}^{(j)},
\end{aligned}
\tag{7.72}$$

where $\tilde{\mathbf{G}}$ is the same as the $\mathbf{G}$ matrix, but with the first d rows being zero due to the nullspace of a_E . Thus, the consistency part of the element stiffness matrix is given by

$$\mathbf{K}_E^c = (\mathbf{\Pi}_*^{\nabla,E})^T \tilde{\mathbf{G}} \mathbf{\Pi}_*^{\nabla,E}.$$
\tag{7.73}

Next, the stabilization part of the stiffness matrix is

$$(K_E^s)_{ij} = S^E((I - \Pi_k^{\nabla,E}) \varphi_i, (I - \Pi_k^{\nabla,E}) \varphi_j)$$
\tag{7.74}

$$= S^E((I - \mathbf{\Pi}_k^{\nabla,E}) \varphi_i, (I - \mathbf{\Pi}_k^{\nabla,E}) \varphi_j)$$
\tag{7.75}

$$= (I - \mathbf{\Pi}_k^{\nabla,E})^T S^E(\varphi_i, \varphi_j) (I - \mathbf{\Pi}_k^{\nabla,E}).$$
\tag{7.76}

For the Poisson problem we are considering,

$$\mathbf{K}_E^s = (I - \mathbf{\Pi}_*^{\nabla,E})^T (I - \mathbf{\Pi}_*^{\nabla,E}).$$
\tag{7.77}

Thus, the element stiffness matrix is

$$\mathbf{K}_E = \mathbf{K}_E^c + \mathbf{K}_E^s$$
\tag{7.78}

$$= (\mathbf{\Pi}_*^{\nabla,E})^T \tilde{\mathbf{G}} \mathbf{\Pi}_*^{\nabla,E} + (I - \mathbf{\Pi}_*^{\nabla,E})^T (I - \mathbf{\Pi}_*^{\nabla,E}).$$
\tag{7.79}

7.5.2. Load Vector. The load vector is given by

$$b_i = \int_{\partial\Omega_N} u_N \varphi_i \, dS + (f, \varphi_i).$$
\tag{7.80}

In two space dimensions, φ_i can be explicitly constructed over $\partial\Omega_N$, and in three space dimensions, the elliptic face projection is used to approximate φ_i . The body forces term is computed by approximating φ_i using the elliptic projection.

7.6. Elastodynamics

We now formulate a VEM for solving the linear elastodynamic problem. We take ρ to be the density of the body Ω , $\mathbf{u} = \mathbf{x} - \mathbf{X}$ the displacement, $\mathbf{b}$ the body force,

$$(7.81) \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

the infinitesimal strain tensor, and the stress tensor to be

$$(7.82) \quad \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{u}),$$

with $\mathbf{C}$ the elasticity tensor for an isotropic material. The prescribed displacement and traction boundary conditions are

$$(7.83) \quad \mathbf{u} = \mathbf{u}_D \quad \text{on } \partial\Omega_D, \quad \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}_N \quad \text{on } \partial\Omega_N,$$

and the initial conditions are

$$(7.84) \quad \mathbf{u}(\mathbf{X}, 0) = \mathbf{u}_0(\mathbf{X}), \quad \dot{\mathbf{u}}(\mathbf{X}, 0) = \dot{\mathbf{u}}_0(\mathbf{X}).$$

The weak formulation is as follows: find $\mathbf{u} \in V = \{\mathbf{u} \in [H^1(\Omega)]^3 : \mathbf{u} = \mathbf{u}_D \text{ on } \partial\Omega_D\}$ such that

$$(7.85) \quad \rho(\ddot{\mathbf{u}}, \mathbf{v}) + a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\partial\Omega_N} \mathbf{t} \cdot \mathbf{v} \, dS + (\mathbf{b}, \mathbf{v}),$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$. Here,

$$(7.86) \quad a^\varepsilon(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{X}, \quad (\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{X}.$$

Again, taking the VE space to be our finite dimensional space and assuming the element-wise splitting of the discrete bilinear forms, we evaluate the bilinear forms using the L^2 and energy projections. Recall that the elliptic projection will be used during the construction of the L^2 projection, and that for $k = 1, 2$, the elliptic and L^2 projections are identical. The L^2 projection will be used in the construction of the mass matrix, and the energy projection will be used for the construction of the stiffness matrix.

7.6.1. Consistency and Stability of Bilinear Form. Convergence of the method requires that the discrete bilinear be consistent [78],

$$a_h^{\varepsilon,E}(\mathbf{m}, \mathbf{u}) = a_h^{\varepsilon,E}(\mathbf{m}, \Pi_k^{\varepsilon,E} \mathbf{u}),$$

and stable [78],

$$\alpha_* a^{\varepsilon,E}(\mathbf{u}, \mathbf{v}) \leq a_h^{\varepsilon,E}(\mathbf{u}, \mathbf{v}) \leq \alpha^* a^{\varepsilon,E}(\mathbf{u}, \mathbf{v}),$$

where α_*, α^* are constants independent of E . Consistency follows immediately from the definition of the energy projection, and stability follows from choosing a suitable bilinear form $S^{\varepsilon,E}$ so that

$$(7.87) \quad c_1 a^{\varepsilon,E}(\mathbf{v}, \mathbf{v}) \leq S^{\varepsilon,E}(\mathbf{v}, \mathbf{v}) \leq c_2 a^{\varepsilon,E}(\mathbf{v}, \mathbf{v}).$$

7.6.2. Stiffness Matrix Construction. The entries of the element stiffness matrix are given by

$$(7.88) \quad K_{ij}^E = a_h^{\varepsilon,E}(\varphi_i, \varphi_j),$$

and adding $\Pi_k^{\varepsilon,E} \varphi_i - \Pi_k^{\varepsilon,E} \varphi_i$ to each slot, we have

$$a^{\varepsilon,E}(\varphi_i, \varphi_j) = a^{\varepsilon,E}(\varphi_i + \Pi_k^{\varepsilon,E} \varphi_i - \Pi_k^{\varepsilon,E} \varphi_i, \varphi_j + \Pi_k^{\varepsilon,E} \varphi_j - \Pi_k^{\varepsilon,E} \varphi_j).$$

This reduces to

$$a^{\varepsilon,E}(\varphi_i, \varphi_j) = a^{\varepsilon,E}(\Pi_k^{\varepsilon,E} \varphi_i, \Pi_k^{\varepsilon,E} \varphi_j) + a^{\varepsilon,E}((I - \Pi_k^{\varepsilon,E}) \varphi_i, (I - \Pi_k^{\varepsilon,E}) \varphi_j),$$

since the cross terms drop out due to the orthogonality property of the projection. Then, choosing

$$a_h^{\varepsilon,E}(\mathbf{u}, \mathbf{v}) = a^{\varepsilon,E}(\Pi_k^{\varepsilon,E} \mathbf{u}, \Pi_k^{\varepsilon,E} \mathbf{v}) + S^{\varepsilon,E}((I - \Pi_k^{\varepsilon,E}) \mathbf{u}, (I - \Pi_k^{\varepsilon,E}) \mathbf{v})$$

ensures that the discrete bilinear form satisfies the consistency and stability conditions.

Now, the element stiffness matrix given by (7.88) is:

$$(7.89) \quad \mathbf{K}^E = \mathbf{K}_c^E + \mathbf{K}_s^E,$$

where

$$(7.90) \quad (\mathbf{K}_c^E)_{ij} = a^{\varepsilon,E}(\Pi_k^{\varepsilon,E} \boldsymbol{\varphi}_i, \Pi_k^{\varepsilon,E} \boldsymbol{\varphi}_j)$$

is the consistency part and

$$(7.91) \quad (\mathbf{K}_s^E)_{ij} = S^{\varepsilon,E}((I - \Pi_k^{\varepsilon,E}) \boldsymbol{\varphi}_i, (I - \Pi_k^{\varepsilon,E}) \boldsymbol{\varphi}_j)$$

is the stability part of the element stiffness matrix.

The consistency part of the stiffness matrix is:

$$\begin{aligned} (\mathbf{K}_E^c)_{ij} &= a^{\varepsilon,E}(\Pi_k^{\varepsilon,E} \boldsymbol{\varphi}_i, \Pi_k^{\varepsilon,E} \boldsymbol{\varphi}_j) = a^{\varepsilon,E}(c_\alpha^{(i)} \mathbf{m}_\alpha, c_\beta^{(j)} \mathbf{m}_\beta) \\ &= c_\alpha^{(i)} c_\beta^{(j)} a^{\varepsilon,E}(\mathbf{m}_\alpha, \mathbf{m}_\beta) = c_\alpha^{(i)} c_\beta^{(j)} \tilde{G}_{\alpha\beta}, \end{aligned}$$

where $\tilde{\mathbf{G}}$ is a matrix with the first six rows being zero due to the rigid-body modes, i.e., the nullspace of $a^{\varepsilon,E}$. Thus, the consistency part is given by

$$(7.92) \quad \mathbf{K}_E^c = (\boldsymbol{\Pi}_*^{\varepsilon,E})^T \tilde{\mathbf{G}} \boldsymbol{\Pi}_*^{\varepsilon,E}.$$

Next, for the stabilization part of the stiffness matrix, we have

$$\begin{aligned} (\mathbf{K}_E^s)_{ij} &= S^{\varepsilon,E}((I - \Pi_k^{\varepsilon,E}) \boldsymbol{\varphi}_i, (I - \Pi_k^{\varepsilon,E}) \boldsymbol{\varphi}_j) \\ (7.93) \quad &= S^{\varepsilon,E}((\mathbf{I} - \boldsymbol{\Pi}_k^{\varepsilon,E}) \boldsymbol{\varphi}_i, (\mathbf{I} - \boldsymbol{\Pi}_k^{\varepsilon,E}) \boldsymbol{\varphi}_j) \\ &= (\mathbf{I} - \boldsymbol{\Pi}_k^{\varepsilon,E})^T S^{\varepsilon,E}(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) (\mathbf{I} - \boldsymbol{\Pi}_k^{\varepsilon,E}), \end{aligned}$$

and letting $S_{ij}^E = S^{\varepsilon,E}(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \delta_{ij}$, the stabilization part is given by

$$(7.94) \quad \mathbf{K}_E^S = (\mathbf{I} - \boldsymbol{\Pi}_k^{\varepsilon,E})^T \mathbf{S}^E (\mathbf{I} - \boldsymbol{\Pi}_k^{\varepsilon,E}).$$

The choice of bilinear form $S^{\varepsilon,E}$ is not precise, but it should scale in a similar manner to $a^{\varepsilon,E}$ on the kernel of $\Pi_1^{\varepsilon,E}$. Additionally, the stabilization term should ensure that the element stiffness matrix is of sufficient rank. Approaches have been established in which the stabilization term is eliminated [13, 85], but the choice of stabilization for standard problems is readily available. For the energy projection, we use the stabilization term

$$(7.95) \quad S^{\varepsilon,E}(\mathbf{u}, \mathbf{v}) = \max \left\{ \frac{(3\mu + \lambda)h_E}{2}, a^{\varepsilon,E}(\Pi_k^{\varepsilon,E} \mathbf{u}, \Pi_k^{\varepsilon,E} \mathbf{v}) \right\}.$$

Thus, the element stiffness matrix is

$$(7.96) \quad \mathbf{K}_E = (\boldsymbol{\Pi}_*^{\varepsilon,E})^T \tilde{\mathbf{G}} \boldsymbol{\Pi}_*^{\varepsilon,E} + (\mathbf{I} - \boldsymbol{\Pi}^{\varepsilon,E})^T \mathbf{S}^E (\mathbf{I} - \boldsymbol{\Pi}^{\varepsilon,E}).$$

7.6.3. Mass Matrix Construction. Now, we construct the element mass matrix with entries given by

$$(7.97) \quad M_{ij} = \rho(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j)_h,$$

where ρ is the density of the body. The L^2 projection is used to construct the mass matrix, and for the case $k = 1, 2$, the L^2 and elliptic projection coincide, $\Pi_1^{0,E} \equiv \Pi_1^{\nabla,E}$. Proceeding as we did for the stiffness matrix, we choose

$$(7.98) \quad (\mathbf{u}, \mathbf{v})_h = \rho(\Pi_k^{0,E} \mathbf{u}, \Pi_k^{0,E} \mathbf{v})_E + \rho S^{0,E}((\mathbf{I} - \Pi_k^{0,E}) \mathbf{u}, (\mathbf{I} - \Pi_k^{0,E}) \mathbf{v}).$$

We see that the element mass matrix decomposes as

$$(7.99) \quad \mathbf{M}_E = \mathbf{M}_E^c + \mathbf{M}_E^s$$

where $\mathbf{M}_E^c$ and $\mathbf{M}_E^s$ are the consistency and stability terms, respectively, of the element mass matrix.

The consistency part of the element mass matrix is given by

$$(7.100) \quad \mathbf{M}_E^c = \rho(\boldsymbol{\Pi}_*^{0,E})^T \mathbf{H} \boldsymbol{\Pi}_*^{0,E},$$

where $\mathbf{H}$ is a matrix of the L^2 inner products of the scaled monomials:

$$(7.101) \quad H_{\alpha\beta} = (\mathbf{m}_\alpha, \mathbf{m}_\beta)_E.$$

For the stabilization term, we use

$$(7.102) \quad S^{0,E}(\mathbf{u}, \mathbf{v}) = |E|,$$

where $|E|$ is the volume of E . This choice of stabilization ensure that it scales like the consistency term. Therefore, the stabilization part of the element mass matrix is

$$(7.103) \quad \mathbf{M}_E^s = \rho|E|(\mathbf{I} - \boldsymbol{\Pi}^{0,E})^T (\mathbf{I} - \boldsymbol{\Pi}^{0,E}).$$

Thus, the element mass matrix is given by

$$(7.104) \quad \mathbf{M}_E = \rho(\boldsymbol{\Pi}_*^{0,E})^T \mathbf{H} \boldsymbol{\Pi}_*^{0,E} + \rho|E|(\mathbf{I} - \boldsymbol{\Pi}^{0,E})^T(\mathbf{I} - \boldsymbol{\Pi}^{0,E}).$$

7.6.4. Load Vector. Traction boundary conditions and body forces contribute to the load vector

$$(7.105) \quad f_i = \int_{\partial\Omega_N} \mathbf{t} \cdot \boldsymbol{\varphi}_i \, dS + (\mathbf{b}, \boldsymbol{\varphi}_i).$$

7.6.4.1. *2D.* In two space dimensions, $\boldsymbol{\varphi}_i$ may be explicitly constructed on element edges, so the boundary integral over $\partial\Omega_N$ is computable. For the body forces, we approximate $\boldsymbol{\varphi}_i$ using the L^2 projection,

$$(7.106) \quad (\mathbf{b}, \boldsymbol{\varphi}_i) = \sum_{E \in \mathcal{T}_h} (\mathbf{b}, \boldsymbol{\varphi}_i)_E$$

$$(7.107) \quad \approx (\mathbf{b}, \Pi_k^{0,E} \boldsymbol{\varphi}_i)_E.$$

7.6.4.2. *3D.* In three dimensions, the traction boundary conditions are computed over polygonal faces, so we use the L^2 face projection for the traction boundary conditions to obtain

$$f_i = \int_{\partial\Omega_N} \mathbf{t} \cdot \boldsymbol{\varphi}_i \, dS \approx \sum_f \int_f \mathbf{t} \cdot \Pi_1^{0,f} \boldsymbol{\varphi}_i \, dS,$$

and the body force contribution is computed in the same manner as in the two space dimension case by approximating $\boldsymbol{\varphi}_i \approx \Pi_k^{0,E} \boldsymbol{\varphi}_i$.

CHAPTER 8

Fracture Mechanics

Elastodynamic fracture mechanics governs how cracks propagate in our simulated bodies. While, other methods for modeling fracture such as peridynamics and phase field methods have been developed, we utilize the traditional Griffiths energy release approach. Starting from energy balance, we describe the crack propagation criterion of Griffith. Next, we construct the asymptotic crack tip fields of Williams which play a fundamental role in computing the SIFs of a crack tip. After constructing the original J -integral of Rice, we extend the J -integral to the elastodynamic regime by incorporating inertial effects. We then construct the interaction integral and establish the domain form of the J -integral, and finally we describe the crack model that governs how cracks propagate through the material.

8.1. Griffith Energy Balance

Griffith [28] approached the advancing of a crack in terms of the energy that is released by the creation of new crack surfaces. Let us consider a body Ω before and after crack advancement. Let state (0) be the state of the body with crack surface area A , and in state (1), the body has crack surface area $A + \Delta A$. The total strain energy of the body in state (0) is

$$(8.1) \quad U^{(0)} = \int_{\Omega} \psi(\mathbf{F}^{(0)}) d\mathbf{X},$$

where ψ is the strain energy density and $\mathbf{F}^{(0)}$ is the deformation gradient of the body in state (0). The work performed by going from state (0) to state (1) is

$$(8.2) \quad W^{(0,1)} = \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}^{(1)} - \mathbf{u}^{(0)}) d\mathbf{X} + \int_{\partial\Omega} \mathbf{t} \cdot (\mathbf{u}^{(1)} - \mathbf{u}^{(0)}) dS,$$

where $\mathbf{f}$ is the body force and $\mathbf{t}$ is the surface traction. The total energy of state (1) is

$$(8.3) \quad E^{(1)} = \int_{\Omega} \psi(\mathbf{F}^{(1)}) d\mathbf{X} + \int_{\Delta A} \gamma dS + \int_{\Omega} C(T^{(1)} - T^{(0)})\rho d\mathbf{X} + \int_{\Omega} \frac{1}{2}\rho(\dot{\mathbf{u}}^{(1)})^2 d\mathbf{X},$$

where γ is the free surface energy, $T^{(0)}$ and $T^{(1)}$ are the temperatures of the body in state (0) and state (1), C is the specific heat, and ρ is the mass density of the body. Working from left to right, the integrals represent the strain energy of state (1), the additional surface energy created from the extension of the crack, the increase in heat energy, and the kinetic energy of the body.

If we consider the kinetic and heat energy to be negligible, the Griffith criteria states that the strain energy and work performed going from state (0) to state (1) must exceed the total energy in state (1),

$$(8.4) \quad U^{(0)} + W^{(0,1)} \geq E^{(1)}.$$

Then, we have that

$$(8.5) \quad \begin{aligned} \int_{\Omega} \psi(\mathbf{F}^{(0)}) d\mathbf{X} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^{(0)} d\mathbf{X} - \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{u}^{(0)} dS \\ \geq \int_{\Omega} \psi(\mathbf{F}^{(1)}) d\mathbf{X} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^{(1)} d\mathbf{X} - \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{u}^{(1)} dS + \int_{\Delta A} \gamma dS. \end{aligned}$$

Now, the potential energy of the body is given by the difference between the strain energy and the work performed by external forces $W_{\text{ext.}}$,

$$(8.6) \quad \Pi = U - W_{\text{ext.}},$$

so the energy inequality states that

$$(8.7) \quad \Pi^{(0)} \geq \Pi^{(1)} + \int_{\Delta A} \gamma dS.$$

For small ΔA , we have

$$(8.8) \quad 0 \geq \frac{\Pi^{(1)} - \Pi^{(0)}}{\Delta A} + \gamma,$$

and as $\Delta A \rightarrow 0$,

$$(8.9) \quad 0 \geq \frac{d\Pi}{dA} + \gamma.$$

The inequality (8.9) states that the decrease in potential energy per area of the nearly formed surface must be greater than the free surface energy, and we call $G := -\frac{d\Pi}{dA}$ the energy release rate.

The free surface energy γ is replaced by the parameter G_c since the energy required to form new crack is much larger than free surface energy since there are other effects such as plastic dissipation, heat generation, and inertial effects that have been neglected. Thus, the Griffith criteria states that the crack advances if

$$(8.10) \quad G \geq G_c.$$

8.2. Asymptotic Crack-tip Fields

In 1957, Williams [82] presented an asymptotic solution for the crack tip stress fields. These asymptotic crack-tip fields play a fundamental role in determining the state of a loaded crack tip which in turn determines how the crack propagates. We go through the derivation of the fields, and we assume that the crack-tip fields live in a state of plane strain. To compute the crack-tip fields, we use a standard technique used to solve boundary value problems coming from linear elasticity, the Airy stress function.

Consider a body with a crack that is subjected to remote tractions. Let $\phi(x_0, x_1)$ be the Airy stress function so that the components of the stress tensor and the partial derivatives of ϕ are related by

$$(8.11) \quad \sigma_{11} = \frac{\partial^2 \phi}{\partial x_1^2}, \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}.$$

Now, the divergence of the stress is

$$(8.12) \quad \begin{aligned} \nabla \cdot \boldsymbol{\sigma} &= \sigma_{ij,j} \\ &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \\ &= \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2} - \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1} \right) \frac{\partial \phi}{\partial x_1 \partial x_2} + \frac{\partial \phi}{\partial x_2 \partial x_1^2} \\ &= 0, \end{aligned}$$

so the relation between the stress tensor and ϕ ensures that the stress tensor is divergence-free.

The stress is a function of the displacement field $\mathbf{u} = (u_1, u_2)$, and the two component functions of the displacement field express three components of the stress field, so the component functions of the displacement field are overdetermined. Therefore, there must be an additional constraint.

The components of the strain field are,

$$(8.13) \quad \varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right),$$

and upon differentiating the equations, we have

$$(8.14) \quad \begin{aligned} \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} &= \frac{\partial^3 u_1}{\partial x_2^2 \partial x_1} \\ \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} &= \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \\ \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} &= \frac{1}{2} \left(\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \right). \end{aligned}$$

We see that

$$(8.15) \quad \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2} \left(\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} \right),$$

and this relation is called the compatibility condition. The compatibility condition must be satisfied by the strain to ensure that the strain field is conservative. Using Hooke's law, we can express the compatibility condition in terms of the stress,

$$(8.16) \quad \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (\sigma_{11} + \sigma_{22}) = 0.$$

Now, applying the compatibility condition to the Airy stress function, we have

$$(8.17) \quad \begin{aligned} 0 &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (\sigma_{11} + \sigma_{22}) \\ &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \left(\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_1^2} \right) \\ &= \Delta^2 \phi, \end{aligned}$$

so the compatibility conditions requires ϕ to be biharmonic. Therefore, our problem reduces to solving the biharmonic equation subject to the far-field boundary conditions.

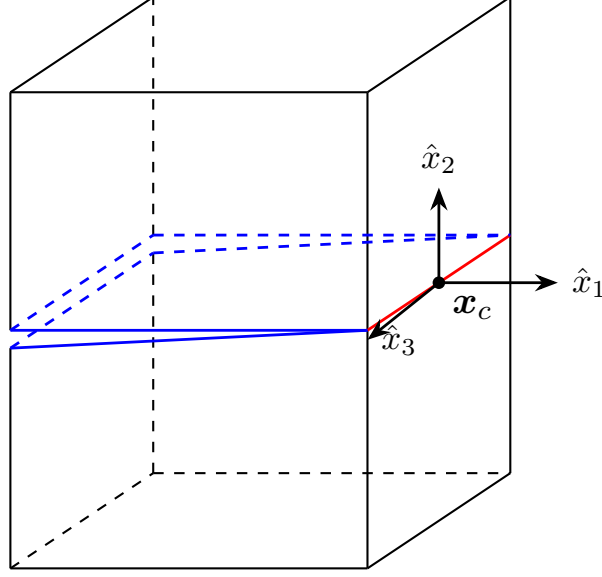


FIGURE 8.1. Local crack front coordinates. The red line is the crack front which runs through the thickness of the body. The crack front coordinates have $\hat{x}_1$ pointing in the advancing crack direction, $\hat{x}_3$ pointing along the crack front, and $\hat{x}_2$ orthogonal to $\hat{x}_1$ and $\hat{x}_3$.

The Williams solution will be obtained in the crack-tip coordinates which place the crack tip at the origin of the coordinate system with the x_1 axis pointing in the direction of crack advancement and the x_2 axis rotated 90° in the counterclockwise direction from the x_1 axis (see Figure 8.1). We change to polar coordinates, and the Airy stress function is now related to the stress by

$$\begin{aligned}
 \sigma_{rr} &= \frac{1}{r} \frac{\partial}{\partial r} \phi + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \phi \\
 \sigma_{\theta\theta} &= \frac{\partial}{\partial r^2} \phi \\
 \sigma_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \phi \right).
 \end{aligned}
 \tag{8.18}$$

The compatibility equation requires $\Delta^2 \phi = 0$, and in polar coordinates, this is given by

$$\left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \right) \left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \right) \phi = 0.$$

Williams proposed a solution of the form

$$\phi(r, \theta) = r^{\lambda+1} F(\theta),
 \tag{8.19}$$

which separates the radial and angular dependence of ϕ . Applying the Laplacian to (8.19), we have

$$(8.20) \quad \begin{aligned} \Delta \phi &= \left(\frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \right) r^{\lambda+1} F(\theta) \\ &= \lambda(\lambda+1) r^{\lambda-1} F(\theta) + (\lambda+1) r^{\lambda-1} F'(\theta) + r^{\lambda-1} F''(\theta), \end{aligned}$$

and applying the Laplacian once more, we have

$$(8.21) \quad \begin{aligned} \Delta^2 \phi &= \Delta \left(\lambda(\lambda+1) r^{\lambda-1} F(\theta) + (\lambda+1) r^{\lambda-1} F'(\theta) + r^{\lambda-1} F''(\theta) \right) \\ &= \left((\lambda+1)^2 (\lambda-1)^2 F(\theta) + ((\lambda-1)^2 + (\lambda+1)^2) F''(\theta) + F^{(4)}(\theta) \right) r^{\lambda-3}. \end{aligned}$$

Therefore, the requirement that (8.19) be biharmonic says

$$(8.22) \quad \left(\frac{d^2}{d\theta^2} + (\lambda+1)^2 \right) \left(\frac{d^2}{d\theta^2} + (\lambda-1)^2 \right) F(\theta) r^{\lambda-3} = 0,$$

so we must have that

$$(8.23) \quad \left(\frac{d^2}{d\theta^2} + (\lambda+1)^2 \right) \left(\frac{d^2}{d\theta^2} + (\lambda-1)^2 \right) F(\theta) = 0,$$

since $r^{\lambda-3} \neq 0$ for all $r \geq 0$. Now, F may be expressed as a linear combination of solutions to the two second order constant coefficient ordinary differential equations,

$$(8.24) \quad F(\theta) = F_1(\theta) + F_2(\theta).$$

Here, F_1 is a solution to

$$(8.25) \quad \left(\frac{d^2}{d\theta^2} + (\lambda-1)^2 \right) F_1(\theta) = 0,$$

and F_2 is a solution to

$$(8.26) \quad \left(\frac{d^2}{d\theta^2} + (\lambda+1)^2 \right) F_2(\theta) = 0.$$

Using standard techniques for solving second-order linear ODEs, we see that

$$(8.27) \quad F_1(\theta) = c_1 \cos((\lambda-1)\theta) + c_2 \sin((\lambda-1)\theta),$$

and

$$(8.28) \quad F_2(\theta) = c_3 \cos((\lambda+1)\theta) + c_4 \sin((\lambda+1)\theta).$$

Thus, the radial part of the solution is of the form

$$(8.29) \quad F(\theta) = c_1 \cos((\lambda - 1)\theta) + c_2 \sin((\lambda - 1)\theta) + c_3 \cos((\lambda + 1)\theta) + c_4 \sin((\lambda + 1)\theta),$$

which satisfies the equilibrium and compatibility equations.

Now, it remains to determine the values of λ for which (8.19) is a valid solution. Using (8.19) to write out the components of stress in terms of r and θ , we have

$$(8.30) \quad \begin{aligned} \sigma_{rr} &= ((\lambda + 1) F(\theta) + F''(\theta)) r^{\lambda-1} \\ \sigma_{\theta\theta} &= \lambda(\lambda + 1) r^{\lambda-1} F(\theta) \\ \sigma_{r\theta} &= -\lambda r^{\lambda-1} F'(\theta). \end{aligned}$$

Now, we assume that along the crack faces, the traction and shear stress is zero in the circumferential direction,

$$(8.31) \quad \sigma_{\theta\theta}(r, \pm\pi) = 0, \quad \sigma_{r\theta}(r, \pm\pi) = 0.$$

Then, in terms of (8.19), we see that

$$(8.32) \quad \lambda(\lambda + 1) r^{\lambda-1} F(\pm\pi) = 0,$$

and

$$(8.33) \quad \lambda r^{\lambda-1} F'(\pm\pi) = 0,$$

so we require that $F(\pm\pi) = 0$ and $F'(\pm\pi) = 0$ since λ is not necessarily zero. Writing these two conditions out explicitly, we have

$$(8.34) \quad c_1 \cos((\lambda - 1)\pi) \pm c_2 \sin((\lambda - 1)\pi) + c_3 \cos((\lambda + 1)\pi) \pm c_4 \sin((\lambda + 1)\pi) = 0,$$

and

$$(8.35) \quad \pm(\lambda - 1)c_1 \sin((\lambda - 1)\pi) + (\lambda - 1)c_2 \cos((\lambda - 1)\pi) \pm(\lambda + 1)c_3 \sin((\lambda + 1)\pi) \pm(\lambda + 1)c_4 \cos((\lambda + 1)\pi) = 0,$$

which results in the systems

$$(8.36) \quad \begin{bmatrix} \cos((\lambda-1)\pi) & \cos((\lambda+1)\pi) \\ (\lambda-1)\sin((\lambda-1)\pi) & (\lambda+1)\sin((\lambda+1)\pi) \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$(8.37) \quad \begin{bmatrix} \sin((\lambda-1)\pi) & \sin((\lambda+1)\pi) \\ (\lambda-1)\cos((\lambda-1)\pi) & (\lambda+1)\cos((\lambda+1)\pi) \end{bmatrix} \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If the matrices on the left-hand-sides of the two systems are invertible, we will arrive at the trivial solution, so to avoid the trivial solution, we require

$$(8.38) \quad (\lambda+1)\cos((\lambda-1)\pi)\sin((\lambda+1)\pi) - (\lambda-1)\sin((\lambda-1)\pi)\cos((\lambda+1)\pi) = 0,$$

and

$$(8.39) \quad (\lambda+1)\cos((\lambda+1)\pi)\sin((\lambda-1)\pi) - (\lambda-1)\cos((\lambda-1)\pi)\sin((\lambda+1)\pi) = 0.$$

Upon applying the trigonometric identity,

$$(8.40) \quad \cos \theta \sin \varphi = \frac{1}{2} (\sin(\theta + \varphi) + \sin(\theta - \varphi)),$$

we find that (8.38) and (8.2) reduce to

$$(8.41) \quad \sin(2\lambda\pi) = 0.$$

Thus, we find that we have countably many choices for λ ,

$$(8.42) \quad \lambda_n = \frac{n}{2},$$

for $n = 0, \pm 1, \pm 2, \dots$, so solutions are of the form,

$$(8.43) \quad \phi_n(r, \theta) = r^{\lambda_n+1} F_n(\theta),$$

or explicitly, we have

$$(8.44) \quad \phi_n(r, \theta) = r^{\frac{n}{2}+1} \left(c_{1n} \cos\left(\left(\frac{n}{2}-1\right)\theta\right) + c_{2n} \sin\left(\left(\frac{n}{2}-1\right)\theta\right) + c_{3n} \cos\left(\left(\frac{n}{2}+1\right)\theta\right) + c_{4n} \sin\left(\left(\frac{n}{2}+1\right)\theta\right) \right).$$

As we approach the crack tip, we see that the radial part of ϕ_n remains bounded, but depending on the value of λ_n , ϕ_n may be singular as $r \rightarrow 0^+$. We expect the stress to become singular as we approach the crack tip, and the stress is on the order of $r^{\lambda_n-1} = r^{\frac{n}{2}-1}$. While the stress being singular at the crack tip is physically admissible, we expect the displacement field to remain bounded, and the displacement field is on the order of $r^{\frac{n}{2}}$. Therefore, we require $n \geq 0$ to ensure that the displacement field is bounded at crack tip. Further, when $n = 0$, the displacement field becomes multivalued at the crack tip due to the radial part of ϕ_n , so we require $n > 0$. Thus, the admissible solutions are given by ϕ_n for $n = 1, 2, \dots$.

Now, we return to the two linear systems for c_{1n} , c_{2n} , c_{3n} , and c_{4n} and use the result that $\lambda_n = \frac{n}{2}$. We have

$$(8.45) \quad \begin{bmatrix} -\cos\left(\frac{n\pi}{2}\right) & -\cos\left(\frac{n\pi}{2}\right) \\ -\left(\frac{n}{2}-1\right)\sin\left(\frac{n\pi}{2}\right) & -\left(\frac{n}{2}+1\right)\sin\left(\frac{n\pi}{2}\right) \end{bmatrix} \begin{bmatrix} c_{1n} \\ c_{3n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$(8.46) \quad \begin{bmatrix} -\sin\left(\frac{n\pi}{2}\right) & -\sin\left(\frac{n\pi}{2}\right) \\ -\left(\frac{n}{2}-1\right)\cos\left(\frac{n\pi}{2}\right) & -\left(\frac{n}{2}+1\right)\cos\left(\frac{n\pi}{2}\right) \end{bmatrix} \begin{bmatrix} c_{2n} \\ c_{4n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

First, for even values of n , we have

$$(8.47) \quad \begin{bmatrix} (-1)^{m+1} & (-1)^{m+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{1n} \\ c_{3n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$(8.48) \quad \begin{bmatrix} 0 & 0 \\ (m-1)(-1)^{m+1} & (m+1)(-1)^{m+1} \end{bmatrix} \begin{bmatrix} c_{2n} \\ c_{4n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $n = 2m$ with $m = 1, 2, \dots$. We see that $c_{1n} = -c_{3n}$ and $c_{2n} = -\frac{m+1}{m-1}c_{4n}$. Note that for $m = 1$, c_{2n} is free and $c_{4n} = 0$.

Next, we consider odd values of n ,

$$(8.49) \quad \begin{bmatrix} 0 & 0 \\ \left(\frac{2k+1}{2}-1\right)(-1)^{k+1} & \left(\frac{2k+1}{2}+1\right)(-1)^{k+1} \end{bmatrix} \begin{bmatrix} c_{1n} \\ c_{3n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$(8.50) \quad \begin{bmatrix} (-1)^{k+1} & (-1)^{k+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{2n} \\ c_{4n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $n = 2k + 1$ for $k = 0, 1, \dots$. We see that so $c_{1n} = -\frac{2k+3}{2k-1}c_{3n}$ and $c_{2n} = -c_{4n}$.

Thus, the complete solution for ϕ is

$$(8.51) \quad \begin{aligned} \phi(r, \theta) = & \sum_{m=1}^{\infty} \left(c_{1n} (\cos((m-1)\theta) - \cos((m+1)\theta)) + c_{2n} (\sin((m-1)\theta) - \frac{m-1}{m+1} \sin((m+1)\theta)) \right) r^{m+1} \\ & + \sum_{k=0}^{\infty} \left(c_{1n} \left(\cos\left(\frac{2k-1}{2}\theta\right) - \frac{2k-1}{2k+3} \cos\left(\frac{2k+3}{2}\theta\right) \right) + c_{2n} \left(\sin\left(\frac{2k-1}{2}\theta\right) - \sin\left(\frac{2k+3}{2}\theta\right) \right) \right) r^{\frac{2k+3}{2}}. \end{aligned}$$

Near the crack tip ($r \ll 1$), the $k = 0$ term dominates, so we take

$$(8.52) \quad \phi_1(r, \theta) = \left(c_{11} \left(\cos\left(\frac{\theta}{2}\right) + \frac{1}{3} \cos\left(\frac{3}{2}\theta\right) \right) - c_{21} \left(-\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3}{2}\theta\right) \right) \right) r^{\frac{3}{2}}$$

to be our asymptotic crack-tip solution. Notice that ϕ_1 is composed of even terms and odd terms. Based on different far-field loading conditions, we may prescribe a physical quantity to c_{11} and c_{21} .

Suppose the body containing the crack tip is symmetrically loaded about the x_1 -axis. We expect the corresponding stress field to be symmetric, so $c_{21} = 0$. We define $\frac{K_I}{\sqrt{2\pi}} := c_{11}$ and call K_I the mode I stress intensity factor (SIF). Similarly, if the body is subjected to antisymmetric loading, we expect the stress to be antisymmetric so $c_{11} = 0$. We define $\frac{K_{II}}{\sqrt{2\pi}} := c_{21}$ and we call K_{II} the mode II SIF. Thus,

$$(8.53) \quad \phi_1(r, \theta) = \left(\frac{K_I}{\sqrt{2\pi}} \left(\cos\left(\frac{\theta}{2}\right) + \frac{1}{3} \cos\left(\frac{3}{2}\theta\right) \right) - \frac{K_{II}}{\sqrt{2\pi}} \left(-\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{3}{2}\theta\right) \right) \right) r^{\frac{3}{2}}.$$

Using this final expression for ϕ_1 , we may compute the stress, strain, and displacement fields near the crack tip.

We now consider the asymptotic solution when antiplane effects are present. Suppose the body containing the crack tip is subjected loading only in the x_3 direction. We assume that the solution is of the form $u_1 = u_2 = 0$, and u_3 only depends on x_1 and x_2 . Then, the equilibrium equation

reduces to

$$(8.54) \quad \Delta u_3 = 0.$$

In polar coordinates, we have

$$(8.55) \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u_3(r, \theta) = 0,$$

and upon supposing that the solution is of the form $u_3(r, \theta) = r^\lambda F(\theta)$, we find that

$$(8.56) \quad \left(\frac{\partial^2}{\partial \theta^2} + \lambda^2 \right) r^{\lambda-2} F(\theta) = 0.$$

Now, the solution to the ODE for $F(\theta)$ is

$$(8.57) \quad F(\theta) = c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta),$$

so

$$(8.58) \quad \sigma_{\theta 3} = \mu r^{\lambda-1} (-c_1 \lambda \sin(\lambda\theta) + c_2 \lambda \cos(\lambda\theta))$$

$$(8.59) \quad \sigma_{r 3} = \lambda r^{\lambda-1} (c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta)).$$

The crack faces are traction-free, so we require $\sigma_{\theta 3}(r, \pm\pi) = 0$,

$$(8.60) \quad -c_1 \lambda \sin(\lambda\pi) + c_2 \lambda \cos(\lambda\pi) = 0$$

$$(8.61) \quad c_1 \lambda \sin(\lambda\pi) + c_2 \lambda \cos(\lambda\pi) = 0,$$

or

$$(8.62) \quad \begin{bmatrix} -\lambda \sin(\lambda\pi) & \lambda \cos(\lambda\pi) \\ \lambda \sin(\lambda\pi) & \lambda \cos(\lambda\pi) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To avoid the trivial solution, we require

$$(8.63) \quad 2\lambda^2 \sin(\lambda\pi) \cos(\lambda\pi) = 0,$$

which reduces to

$$(8.64) \quad \lambda^2 \sin(2\lambda\pi) = 0.$$

We see that we have countably many solutions,

$$(8.65) \quad \lambda_n = \frac{n}{2}$$

for $n = 0, \pm 1, \pm 2, \dots$. For $n < 0$, we see that the displacement is unbounded as $r \rightarrow 0^+$, and for $n = 0$, the displacement is multi-valued. Therefore, the physically admissible solutions require $n > 0$,

$$(8.66) \quad u_3(r, \theta) = \sum_{n=1}^{\infty} r^{\frac{n}{2}} \left(c_{1_n} \cos\left(\frac{n}{2}\theta\right) + c_{2_n} \sin\left(\frac{n}{2}\theta\right) \right).$$

Now, returning to the system (8.62), we see that

$$(8.67) \quad c_{1_n} \sin\left(\frac{n}{2}\pi\right) + c_{2_n} \cos\left(\frac{n}{2}\pi\right) = 0.$$

First, for the even values of $n = 2m$ ($m = 1, 2, \dots$), we see that

$$(8.68) \quad c_{1_n} \sin(m\pi) + c_{2_n} \cos(m\pi) = 0,$$

so $c_{2_n} = 0$. For the odd values of $n = 2k + 1$ ($k = 0, 1, \dots$), we have

$$(8.69) \quad c_{1_n} \sin\left((2k+1)\frac{\pi}{2}\right) + c_{2_n} \cos\left((2k+1)\frac{\pi}{2}\right) = 0,$$

so $c_{1_n} = 0$. Therefore, u_3 decouples as

$$(8.70) \quad u_3(r, \theta) = \sum_{m=1}^{\infty} c_{1_n} r^m \cos(m\theta) + \sum_{k=0}^{\infty} c_{2_n} r^{\frac{2k+1}{2}} \sin\left(\frac{(2k+1)}{2}\theta\right).$$

Near the crack tip ($r \ll 1$), the $k = 0$ term dominates, so we take

$$(8.71) \quad u_3(r, \theta) = c_{2_1} r^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right)$$

to be the asymptotic crack-tip solution for the displacement. Finally, we define $\frac{2K_{III}}{\mu\sqrt{2\pi}} := c_{2_1}$ where K_{III} is the mode III SIF. Hence,

$$(8.72) \quad u_3(r, \theta) = \frac{K_{III}}{\mu} \sqrt{\frac{2r}{\pi}} \sin\left(\frac{\theta}{2}\right),$$

is the asymptotic solution for the crack tip displacement under antiplane loading.

8.3. Energy Release Rate and Stress Intensity Factor

Irwin [36] proposed the SIF K as a quantity that characterizes the state of a crack tip and established that the state of a crack tip is a superposition of three distinct states (K_I , K_{II} , and K_{III}). The three states are identified with how the body is loaded in the far-field. Mode I characterizes a crack tip subjected to tensile loading, mode II describes a crack tip under shear loading, and mode III corresponds to out-of-plane loading. In Williams's [82] asymptotic crack-tip stress field solution, we saw that the scaling constants of the leading order terms are identified as the SIFs, and by considering different far-field loading of the crack, the asymptotic stress fields decouple. The Griffith energy release rate may be decomposed into separate fracture mode contributions,

$$(8.73) \quad G = G_I + G_{II} + G_{III}.$$

The energy release rate and SIFs both characterize the state of the crack tip and are directly related. Consider a body of thickness t under tensile loading with a crack that has been extended by length Δa in the x_1 -direction. From the Williams asymptotic crack tip field, the tensile stress near the crack tip is

$$(8.74) \quad \sigma(x_1) = \frac{K_I}{\sqrt{2\pi x_1}},$$

and the displacement of the crack surface in the x_2 direction is

$$(8.75) \quad u_2(x_1) = \pm \frac{4(1 - \nu^2)K_I \sqrt{(\Delta a - x_1)}}{E\sqrt{2\pi}}.$$

Then, if the newly formed crack surface is closed, the strain energy stored in the body is

$$(8.76) \quad \Delta U = \frac{2(1 - \nu^2)tK_I^2}{\pi E} \int_0^{\Delta a} \frac{\sqrt{(\Delta a - x)}}{\sqrt{x}} dx$$

Upon computing the integral in (8.76), we find that

$$(8.77) \quad \frac{\Delta U}{t\Delta a} = \frac{1 - \nu^2}{E} K_I^2,$$

and as $\Delta A \rightarrow 0$,

$$(8.78) \quad \frac{dU}{dA} = \frac{1 - \nu^2}{E} K_I^2.$$

Therefore,

$$(8.79) \quad G = \frac{1 - \nu^2}{E} K_I^2,$$

and following the same reasoning for mode II and III loading, we find that

$$(8.80) \quad G = \frac{1 - \nu^2}{E} (K_I^2 + K_{II}^2) + \frac{1}{2\mu} K_{III}^2.$$

In the setting of elastodynamics, the SIFs and energy release rate are related by [25]

$$(8.81) \quad \begin{aligned} G_I &= \frac{(1 - \nu^2)}{E} A_I(\dot{a}) K_I^2, & A_I(\dot{a}) &= \frac{\dot{a}^2 \alpha_d}{(1 - \nu) c_s^2 D} \\ G_{II} &= \frac{(1 - \nu^2)}{E} A_{II}(\dot{a}) K_{II}^2, & A_{II}(\dot{a}) &= \frac{\dot{a}^2 \alpha_s}{(1 - \nu) c_s^2 D} \\ G_{III} &= \frac{1}{2\mu} A_{III}(\dot{a}) K_{III}^2, & A_{III}(\dot{a}) &= \frac{1}{\alpha_s}. \end{aligned}$$

The functions A_I , A_{II} , and A_{III} are scaling factors that depend on the crack speed $\dot{a}$ given by

$$(8.82) \quad A_I(\dot{a}) = \frac{\dot{a}^2 \alpha_d}{(1 - \nu) c_s^2 D}, \quad A_{II}(\dot{a}) = \frac{\dot{a}^2 \alpha_s}{(1 - \nu) c_s^2 D}, \quad A_{III}(\dot{a}) = \frac{1}{\alpha_s},$$

where

$$(8.83) \quad \alpha_d = \sqrt{1 - \frac{\dot{a}^2}{c_d^2}}, \quad \alpha_s = \sqrt{1 - \frac{\dot{a}^2}{c_s^2}}, \quad D = 4\alpha_d \alpha_s - (1 + \alpha_s^2)^2.$$

Here, $c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}$, $c_s = \sqrt{\frac{\mu}{\rho}}$, and c_R are the dilatational, shear, and Rayleigh wave speeds, respectively.

In our work, we approximate c_R as [68]

$$(8.84) \quad c_R \approx c_s \sqrt{\frac{30.9 - 14.9\nu - \sqrt{224.5\nu^2 - 93.1\nu + 124.6}}{26(1 - \nu)}}.$$

8.4. Computing the Energy Release Rate

We have seen that the energy release rate determines crack advance, so we require a method for computing the value of G for a loaded body. Consider a plate of thickness t and a curve Γ that surrounds the crack tip and has interior A . The potential energy inside the curve is

$$(8.85) \quad \Pi = t \int_A \psi(\mathbf{F}) d\mathbf{X} - \int_\Gamma \mathbf{t} \cdot \mathbf{u} dS.$$

Suppose the crack extends by an increment Δa , and let $\Gamma^{(\Delta a)}$ be the curve Γ translated by Δa in the increment direction, $A^{(\Delta a)}$ the region enclosed by $\Gamma^{(\Delta a)}$, and $\Pi^{(\Delta a)}$ the potential energy inside $\Gamma^{(\Delta a)}$. As the crack extends, new crack surfaces of area $t\Delta a$ are created, and the energy release rate is the difference in potential energy contained in $\Gamma^{(\Delta a)}$ and Γ ,

$$(8.86) \quad \frac{\partial \Pi}{\partial A} = \lim_{\Delta a \rightarrow 0} \frac{\Pi^{(\Delta a)} - \Pi}{t\Delta a}.$$

Now, when the crack extends, Γ and A translate with the crack tip, so

$$(8.87)$$

$$(8.88) \quad \begin{aligned} \lim_{\Delta a \rightarrow 0} \frac{\Pi^{(\Delta a)} - \Pi}{t\Delta a} &= \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \left(\int_{A^{(\Delta a)}} \psi(\mathbf{F}) d\mathbf{X} - \int_{\Gamma^{(\Delta a)}} \mathbf{t} \cdot \mathbf{u} dS - \left(\int_A \psi(\mathbf{F}) d\mathbf{X} - \int_{\Gamma} \mathbf{t} \cdot \mathbf{u} dS \right) \right) \\ &= \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \left(\int_{A^{(\Delta a)} - A} \psi(\mathbf{F}) d\mathbf{X} - \int_{\Gamma^{(\Delta a)}} \mathbf{t} \cdot \mathbf{u} dS + \int_{\Gamma} \mathbf{t} \cdot \mathbf{u} dS \right). \end{aligned}$$

We assume that the displacements in A and $A^{(\Delta a)}$ are related linearly,

$$(8.89) \quad u_i = u_i^{(\Delta a)} + \frac{\partial u_i}{\partial X_1} \Delta a,$$

and it follows that the stress, strain, and traction are constant between A and $A^{(\Delta a)}$. As $\Delta a \rightarrow 0$, we have

$$(8.90) \quad \lim_{\Delta a \rightarrow 0} \int_{\Gamma^{(\Delta a)}} \mathbf{t} \cdot \mathbf{u} dS - \int_{\Gamma} \mathbf{t} \cdot \mathbf{u} dS = \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \int_{\Gamma} \mathbf{t} \cdot (\mathbf{u}^{(\Delta a)} - \mathbf{u}) dS$$

$$(8.91) \quad = - \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \int_{\Gamma} \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial X_1} \Delta a dS$$

$$(8.92) \quad = - \int_{\Gamma} \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial X_1} dS.$$

Finally, as $\Delta a \rightarrow 0$, $d\mathbf{X} = \Delta a dX_2$, so

$$\begin{aligned} \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \int_{A^{(\Delta a)} - A} \psi(\mathbf{F}) d\mathbf{X} &= - \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \int_{\Gamma} \psi(\mathbf{F}) \Delta a dX_2 \\ &= - \int_{\Gamma} \psi(\mathbf{F}) dX_2. \end{aligned}$$

Thus, we have that

$$(8.93) \quad -\frac{\partial \Pi}{\partial A} = \int_{\Gamma} \psi(\mathbf{F}) dX_2 - \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial X_1} dS.$$

In 1968 Rice [71] introduced the J -integral which is exactly the expression we derived. For linear elastic materials, the Griffith energy release rate G and J integral coincide, $G = J$. Letting $W = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ be the strain energy density and expressing the traction in terms of the stress tensor and surface normal, we arrive at the conventional expression for the J -integral,

$$(8.94) \quad J = \int_{\Gamma} W \, dX_2 - \frac{\partial \mathbf{u}}{\partial x_1} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, dS.$$

8.5. Dynamic J-Integral

The J -integral in (8.94) does not consider the inertial effects that are present during rapid loading of the crack. In 1968, Atkinson [5] proposed incorporating inertial effects by replacing the elastic strain energy by the total energy. The integrand of (8.94) can be compactly written using the Eshelby energy-momentum tensor is

$$(8.95) \quad H_{ij} = (W + T)\delta_{ij} - \sigma_{ij} \frac{\partial u_i}{\partial x_j},$$

where $T = \frac{1}{2}\rho \mathbf{u} \cdot \mathbf{u}$ is the kinetic energy. Then, the dynamic J -integral becomes

$$(8.96) \quad J = \int_{\Gamma} H_{1j} n_j \, dS.$$

In 1972, Freund [24] derived the dynamic J integral from the elastodynamic fields, and we now construct the dynamic J -integral following Nakamura [61]. Starting from the pointwise balance of momentum,

$$(8.97) \quad \nabla \cdot \boldsymbol{\sigma} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2},$$

we take the dot product of (8.97) with the time derivative of the displacement and integrate over a subset Ω of the material body B ,

$$(8.98) \quad \int_{\Omega} (\nabla \cdot \boldsymbol{\sigma}) \cdot \frac{\partial \mathbf{u}}{\partial t} \, dV_{\mathbf{X}} = \int_{\Omega} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \frac{\partial \mathbf{u}}{\partial t} \, dV_{\mathbf{X}}.$$

Now,

$$(8.99) \quad \int_{\Omega} (\nabla \cdot \boldsymbol{\sigma}) \cdot \frac{\partial \mathbf{u}}{\partial t} \, dV_{\mathbf{X}} = \int_{\Omega} \nabla \cdot \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \, dV_{\mathbf{X}} - \int_{\Omega} \boldsymbol{\sigma} : \nabla \frac{\partial \mathbf{u}}{\partial t} \, dV_{\mathbf{X}},$$

so we have that

$$(8.100) \quad \int_{\Omega} \nabla \cdot \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) dV_{\mathbf{X}} = \int_{\Omega} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \frac{\partial \mathbf{u}}{\partial t} dV_{\mathbf{X}} + \int_{\Omega} \boldsymbol{\sigma} : \nabla \frac{\partial \mathbf{u}}{\partial t} dV_{\mathbf{X}}.$$

Now, the first integral on the RHS of (8.100) is the time rate of change of the kinetic energy,

$$(8.101) \quad \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t} \frac{1}{2} \rho \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2,$$

and the second integral on the RHS of (8.100) is the time rate of change of the stress work density,

$$(8.102) \quad \boldsymbol{\sigma} : \nabla \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial}{\partial t} \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}.$$

Letting

$$(8.103) \quad T = \frac{1}{2} \rho \left(\frac{\partial \mathbf{u}}{\partial t} \right)^2, \quad W = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon},$$

we have

$$(8.104) \quad \int_{\Omega} \nabla \cdot \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) dV_{\mathbf{X}} = \int_{\Omega} \frac{\partial}{\partial t} (W + T) dV_{\mathbf{X}},$$

which is the balance of mechanical energy. Since Ω is arbitrary, we have the pointwise result,

$$(8.105) \quad \nabla \cdot \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) = \frac{\partial}{\partial t} (W + T).$$

Now, consider a contour C which contains a crack tip extending in the x_1 -direction with crack faces that lie in the x_1x_3 -plane. We take the tip of the crack to be at the origin of the local coordinate system. Suppose the crack moves to the right with velocity $\dot{a}$ and that the crack faces are traction free. Let Γ be a small contour around the crack tip that translates with the crack tip, and consider the arbitrary time interval $[t_1, t_2]$. Now, consider the annular region $A(t)$ bounded by C , Γ and the crack faces in the x_1x_2 -plane, and consider the volume created by extruding the region in the time direction from $t = t_1$ to $t = t_2$. Integrating (8.105) over the volume gives

$$(8.106) \quad \int_{t_1}^{t_2} \int_{A(t)} \nabla \cdot \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) dA_{\mathbf{X}} dt = \int_{t_1}^{t_2} \int_{A(t)} \frac{\partial}{\partial t} (W + T) dA_{\mathbf{X}} dt.$$

Applying the divergence theorem to the LHS of (8.107) gives

$$(8.107) \quad \int_{t_1}^{t_2} \int_C \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \mathbf{n} dS dt = \int_{t_1}^{t_2} \int_{A(t)} \frac{\partial}{\partial t} (W + T) dA_{\mathbf{X}} dt + \int_{t_1}^{t_2} \int_{\Gamma} \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \mathbf{n} dS dt$$

since the crack faces are traction-free. Now, the Reynolds transport theorem allows the first integral on the RHS of (8.107) to be written as

$$(8.108) \quad \int_{t_1}^{t_2} \int_{A(t)} \frac{\partial}{\partial t} (W + T) dV_{\mathbf{X}} = \int_{t_1}^{t_2} \frac{\partial}{\partial t} \int_{A(t)} W + T dV_{\mathbf{X}} + \int_{\Gamma} (W + T) \mathbf{v} \cdot \mathbf{n} dS dt,$$

where $\mathbf{v}$ is the velocity on Γ . The integral over C does not appear since $\mathbf{C}$ remains fixed, i.e. $\mathbf{v} = \mathbf{0}$ on C . Since Γ translates with the crack tip and the crack extends in the x_1 direction, $\mathbf{v} = \dot{a} \mathbf{n}_1$. Therefore, we have that

$$(8.109) \quad \int_{t_1}^{t_2} \int_C \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \mathbf{n} dS dt = \left[\int_{A(t)} W + T dV_{\mathbf{X}} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Gamma} (W + T) \dot{a} \mathbf{n}_1 + \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \mathbf{n} dS dt$$

Now, the LHS of (8.109) is the work done on the body from t_1 to t_2 and the first term on the RHS of (8.109) is the increase in internal energy over the same interval. Therefore, the second integral on the RHS of (8.109) represents the rate at which energy is being lost across Γ ,

$$(8.110) \quad F(\Gamma) = \int_{\Gamma} (W + T) \dot{a} \mathbf{n}_1 + \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \mathbf{n} dS.$$

The dynamic energy release rate is defined to be

$$(8.111) \quad G = \lim_{\Gamma \rightarrow 0} \frac{F(\Gamma)}{\dot{a}},$$

which is the energy released per unit crack advance as the contour Γ shrinks down to the crack tip. Then, we see that

$$(8.112) \quad G = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} (W + T) \mathbf{n}_1 + \frac{1}{\dot{a}} \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \mathbf{n} dS.$$

Now, if we assume that as we shrink down onto the crack tip ($r \rightarrow 0^+$),

$$(8.113) \quad \frac{\partial \mathbf{u}}{\partial t} + \dot{a} \frac{\partial \mathbf{u}}{\partial x_1} = o\left(\frac{\partial \mathbf{u}}{\partial x_1}\right),$$

we may replace $\frac{\partial \mathbf{u}}{\partial t}$ with $-\dot{a} \frac{\partial \mathbf{u}}{\partial x_1}$ in (8.112). Thus, the dynamic J -integral is

$$(8.114) \quad G = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} (W + T) n_1 - \left(\boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial x_1} \right) \cdot \mathbf{n} dS.$$

8.6. Interaction Integral

The J -integral computes the energy release rate of a loaded crack tip, but under arbitrary loading conditions, the energy release rate does not directly reveal the state of the crack tip which we require to predict how the crack tip will propagate. To decompose the state of the crack tip into the SIFs, Yau et al. [86] developed the interaction integral.

Suppose we have three elastic fields that we call states (0), (1), and (2) whose displacements are related by

$$(8.115) \quad \mathbf{u}^{(0)} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)},$$

and from which it follows that

$$(8.116) \quad \boldsymbol{\varepsilon}^{(0)} = \boldsymbol{\varepsilon}^{(1)} + \boldsymbol{\varepsilon}^{(2)}, \quad \boldsymbol{\sigma}^{(0)} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}.$$

Now, writing the state (0) fields in terms of the state (1) and (2) fields, we see that the J -integral of state (0) is

$$(8.117) \quad J^{(0)} = \int_{\Gamma} C_{ijkl} u_{ij}^{(0)} u_{kl}^{(0)} + \frac{1}{2} \rho \dot{u}_i^{(0)} \dot{u}_i^{(0)} dX_2 - \sigma_{ij}^{(0)} n_j \frac{\partial u_i^{(0)}}{\partial x} dS$$

$$(8.118) \quad = \int_{\Gamma} C_{ijkl} (u_{ij}^{(1)} + u_{ij}^{(2)}) (u_{kl}^{(1)} + u_{kl}^{(2)}) + \frac{1}{2} \rho (\dot{u}_i^{(1)} + \dot{u}_i^{(2)}) (\dot{u}_i^{(1)} + \dot{u}_i^{(2)}) dX_2$$

$$(8.119) \quad - (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) n_j \left(\frac{\partial u_i^{(1)}}{\partial x} + \frac{\partial u_i^{(2)}}{\partial x} \right) dS$$

$$(8.120) \quad = J^{(1)} + J^{(2)} + \int_{\Gamma} C_{ijkl} (u_{ij}^{(1)} u_{kl}^{(2)} + u_{ij}^{(2)} u_{kl}^{(1)}) + \rho \dot{u}_i^{(1)} \dot{u}_i^{(2)} dX_2 - \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x} \right) n_j dS.$$

We see that the J -integral of state (0) decomposes into

$$(8.121) \quad J^{(0)} = J^{(1)} + J^{(2)} + I^{(1,2)},$$

where $I^{(1,2)}$ involves terms from states (1) and (2) and is called the interaction integral between states (1) and (2). Notice that

$$(8.122) \quad C_{ijkl}(u_{ij}^{(1)}u_{kl}^{(2)} + u_{ij}^{(2)}u_{kl}^{(1)}) = C_{ijkl}u_{ij}^{(1)}u_{kl}^{(2)} = C_{ijkl}u_{ij}^{(2)}u_{kl}^{(1)},$$

and we let

$$(8.123) \quad W^{(1,2)} = C_{ijkl}u_{ij}^{(1)}u_{kl}^{(2)}$$

be the interaction strain work. Finally, letting

$$(8.124) \quad T^{(1,2)} = \rho \mathbf{u}^{(1)} \cdot \mathbf{u}^{(2)}$$

be the interaction kinetic energy, the interaction integral of states (1) and (2) is

$$(8.125) \quad I^{(1,2)} = \int_{\Gamma} W^{(1,2)} + T^{(1,2)} dX_2 - \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x} + \sigma_{ij}^{(2)} n_j \frac{\partial u_i^{(1)}}{\partial x} \right) n_j dS.$$

On the other hand, the state (0) SIFs and J -integral are related by

$$(8.126) \quad J^{(0)} = \frac{1-\nu^2}{E} \left(A_I(\dot{a})(K_I^{(0)})^2 + A_{II}(\dot{a})(K_{II}^{(0)})^2 \right) + \frac{A_{III}(\dot{a})}{2\mu} (K_{III}^{(0)})^2,$$

and the state (0) stress intensity factors may be decomposed as

$$(8.127) \quad K_I^{(0)} = K_I^{(1)} + K_I^{(2)}, \quad K_{II}^{(0)} = K_{II}^{(1)} + K_{II}^{(2)}, \quad K_{III}^{(0)} = K_{III}^{(1)} + K_{III}^{(2)}.$$

Then, we see that

$$(8.128) \quad J^{(0)} = \frac{1-\nu^2}{E} \left(A_I(\dot{a})(K_I^{(0)})^2 + A_{II}(\dot{a})(K_{II}^{(0)})^2 \right) + \frac{A_{III}(\dot{a})}{2\mu} (K_{III}^{(0)})^2$$

$$(8.129) \quad = \frac{1-\nu^2}{E} \left(A_I(\dot{a})(K_I^{(1)} + K_I^{(2)})^2 + A_{II}(\dot{a})(K_{II}^{(1)} + K_{II}^{(2)})^2 \right) + \frac{A_{III}(\dot{a})}{2\mu} (K_{III}^{(1)} + K_{III}^{(2)})^2$$

$$(8.130) \quad = J^{(1)} + J^{(2)} + \frac{2(1-\nu^2)}{E} \left(A_I(\dot{a})K_I^{(1)}K_I^{(2)} + A_{II}(\dot{a})K_{II}^{(1)}K_{II}^{(2)} \right) + \frac{A_{III}(\dot{a})}{\mu} K_{III}^{(1)}K_{III}^{(2)},$$

so in terms of the SIFs, the state (0) J -integral may be decomposed into the sum of the state (1) and state (2) J -integrals and a third term which depends on states (1) and (2). Equating the two expressions for the state (0) J -integral, we find that

$$(8.131) \quad I^{(1,2)} = \frac{2(1-\nu^2)}{E} \left(A_I(\dot{a}) K_I^{(1)} K_I^{(2)} + A_{II}(\dot{a}) K_{II}^{(1)} K_{II}^{(2)} \right) + \frac{A_{III}(\dot{a})}{\mu} K_{III}^{(1)} K_{III}^{(2)}.$$

This relation will allow us to extract the SIFs of a body under complex loading.

Typically, state (2) is some elastic state that we have computed, and the state (1) fields are taken to be the Williams asymptotic crack tip field solutions. After computing $I^{(1,2)}$, the SIFs of state (2) may be extracted. To determine the SIFs of state (2), we suppose state (1) is in a state of pure mode I, II, or III to extract K_I , K_{II} , or K_{III} , respectively. By supposing state (1) is in a pure mode I state, $K_{II}^{(1)} = K_{III}^{(1)} = 0$ and $K_I^{(1)} = 1$, so

$$(8.132) \quad K_I^{(2)} = \frac{E}{2(1-\nu^2)A_I(\dot{a})} I^{(1,2)}.$$

8.7. Domain Integral

To extract the SIFs from the computed state fields, we need to compute the interaction integral. The interaction integral requires computing a contour integral, but our computational approach uses an element-type method which does not mesh well with contour type integrals. Rather than computing the contour integral, Li [45] transformed the contour integral into an area integral. For simplicity, we demonstrate the technique used to convert the contour integral into an area integral using the J -integral.

Recall the J -integral is given by

$$(8.133) \quad J = \int_{\Gamma} H_{1j} dS,$$

where Γ is some contour that encircles the crack-tip and which begins at one crack face and ends at the other crack face. We will reformulate the line integral as an area integral over an annulus (see Figure 8.2).

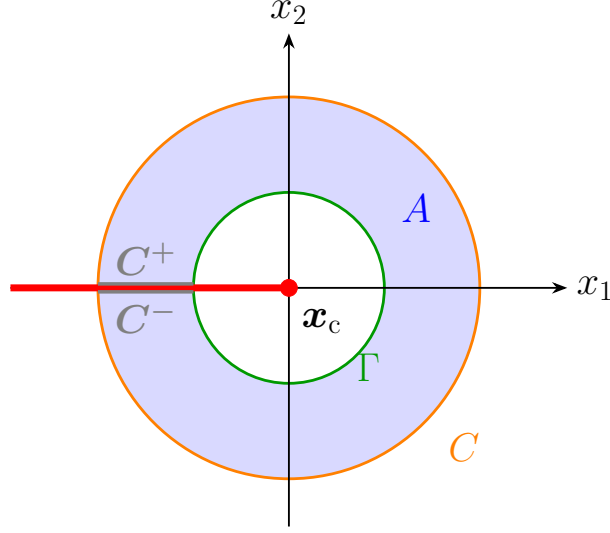


FIGURE 8.2. Domain interaction integral. By considering a suitable weighting function q so that the J -integral may be written as a contour integral over $\Gamma \cup C \cup C^+ \cup C^-$, the contour J -integral may be reformulated as a domain integral over the annulus A .

Let C be a region that surrounds Γ and that begins and ends at the crack face. Let C^+ and C^- lie on the crack faces and connect C and Γ . Let A be the annular region enclosed by $\Gamma \cup C \cup C^+ \cup C^-$. Let q be a sufficiently smooth weighting function that is zero on C and one on Γ so that

$$(8.134) \quad \int_{\Gamma} H_{1j} n_j dS = - \int_{\Gamma \cup C \cup C^+ \cup C^-} H_{1j} q n_j dS.$$

The integrals over C^+ and C^- drop out since the crack faces are traction-free and $n_1 = 0$. Now, using the divergence theorem, we have

$$(8.135) \quad \int_{\Gamma \cup C \cup C^+ \cup C^-} H_{1j} q n_j dS = \int_A \nabla \cdot (H_{1j} q) dA$$

$$(8.136) \quad = \int_A H_{1j,j} q + H_{1j} q_{,j} dA.$$

Thus, the J integral may be computed over an area A with the help of a suitable function q ,

$$(8.137) \quad J = - \int_A H_{1j,j} q + H_{1j} q_{,j} dA.$$

Following the same reasoning, the domain form of the interaction integral is given by

$$(8.138) \quad I^{(1,2)} = - \int_A H_{1j,j}^{(1,2)} q + H_{1j}^{(1,2)} q_{,j} dA,$$

where

$$(8.139) \quad H_{1j}^{(1,2)} = (W^{(1,2)} + T^{(1,2)})\delta_{1j} - \sigma_{kj}^{(1)} u_{k,1}^{(2)} + \sigma_{kj}^{(2)} u_{k,1}^{(1)}$$

is the interaction Eshelby energy-momentum tensor.

The interaction integral uses the Williams asymptotic crack tip fields, but the Williams fields are static fields. Take state (1) to be given by the Williams fields. To incorporate inertial effects into the static fields, we assume that

$$(8.140) \quad \dot{u}_i^{(1)} = -\dot{a}u_{i,1}^{(1)},$$

where $\dot{a}$ is the speed of the crack tip [5]. Then, the interaction kinetic energy (8.124) is now given by

$$(8.141) \quad T^{(1,2)} = -\rho\dot{a}\dot{u}_i^{(2)}u_{i,1}^{(1)}.$$

For the other inertial terms, we examine $H_{1j,j}^{(1,2)}$,

$$(8.142) \quad H_{1j,j}^{(1,2)} = \frac{\partial}{\partial x_j} \left((W^{(1,2)} + T^{(1,2)})\delta_{1j} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} \right)$$

$$(8.143) \quad = \frac{\partial}{\partial x_1} (\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} + \rho \dot{u}_i^{(1)} \dot{u}_i^{(2)}) - \sigma_{ij,j}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(1)} \frac{\partial^2 u_i^{(2)}}{\partial x_j \partial x_1} - \sigma_{ij,j}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial^2 u_i^{(1)}}{\partial x_j \partial x_1}$$

$$(8.144) \quad = \sigma_{ij,1}^{(1)} \varepsilon_{ij}^{(2)} + \sigma_{ij}^{(1)} \varepsilon_{ij,1}^{(2)} + \rho \dot{u}_{i,1}^{(1)} \dot{u}_i^{(2)} + \rho \dot{u}_i^{(1)} \dot{u}_{i,1}^{(2)} - \sigma_{ij,j}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(1)} \frac{\partial^2 u_i^{(2)}}{\partial x_j \partial x_1} - \sigma_{ij,j}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial^2 u_i^{(1)}}{\partial x_j \partial x_1}.$$

Now, we can express the gradient of the displacement as

$$(8.145) \quad \mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T)$. Now,

$$(8.146) \quad \sigma_{ij} u_{i,j1} = \sigma_{ij} (\varepsilon_{ij,1} + \omega_{ij,1})$$

$$(8.147) \quad = \sigma_{ij} \varepsilon_{ij,1}$$

since σ is symmetric and ω is antisymmetric, and $\sigma_{ij,1}^{(1)}\varepsilon_{ij}^{(2)} = \varepsilon_{ij,1}^{(1)}\sigma_{ij}^{(2)}$. Then, we find that the divergence of the interaction Eshelby energy-momentum tensor reduces to

$$(8.148) \quad H_{1j,j}^{(1,2)} = \rho \dot{u}_{i,1}^{(1)} \dot{u}_i^{(2)} + \rho \dot{u}_i^{(1)} \dot{u}_{i,1}^{(2)} - \sigma_{ij,j}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij,j}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1}$$

Now, since $\dot{u}_i^{(1)} = -\dot{a}u_{i,1}^{(1)}$, we see that

$$(8.149) \quad \dot{u}_{i,1}^{(1)} = -\dot{a}u_{i,11}^{(1)},$$

and

$$(8.150) \quad \sigma_{ij,j}^{(1)} = \rho \ddot{u}_i^{(1)}$$

$$(8.151) \quad = -\rho(\ddot{a}u_{i,1}^{(1)} - \dot{a}^2 u_{i,11}^{(1)}).$$

Therefore,

$$(8.152) \quad H_{1j,j}^{(1,2)} = -\rho \dot{a}u_{i,11}^{(1)} \dot{u}_i^{(2)} - \rho \dot{a}u_{i,1}^{(1)} \dot{u}_{i,1}^{(2)} + \rho(\ddot{a}u_{i,1}^{(1)} - \dot{a}^2 u_{i,11}^{(1)}) \frac{\partial u_i^{(2)}}{\partial x_1} - \rho \ddot{u}_i^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1},$$

and the the domain form of the interaction integral is

$$(8.153) \quad \begin{aligned} I^{(1,2)} &= \int_A (\sigma_{kj}^{(1)} u_{k,1}^{(2)} + \sigma_{kj}^{(2)} u_{k,1}^{(1)}) q_{,j} - (W^{(1,2)} + L^{(1,2)}) q_{,1} dA \\ &+ \int_A \left(-\rho(\ddot{a}u_{k,1}^{(1)} - \dot{a}^2 u_{k,11}^{(1)}) u_{k,1}^{(2)} + \rho \ddot{u}_k^{(2)} u_{k,1}^{(1)} + \rho \dot{a}u_{k,1}^{(1)} \dot{u}_{k,1}^{(2)} + \rho \dot{a} \dot{u}_k^{(2)} u_{k,11}^{(1)} \right) q dA. \end{aligned}$$

8.8. Crack Model

8.8.1. Angle of Propagation. There are several approaches to choosing how the crack propagates through the body, but they all agree to some degree. The maximal circumferential stress criterion introduced by. This hypothesis states that the crack should propagate in the direction of maximal circumferential stress and vanishing shear stress. That is, the crack should propagate at an angle θ such that

$$(8.154) \quad \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0, \quad \frac{\partial \sigma_{\theta\theta}}{\partial \theta^2} > 0, \quad \sigma_{r\theta} = 0.$$

We see that

$$(8.155) \quad \partial_\theta \sigma_{\theta\theta} = \partial_\theta \left(\frac{K_I}{\sqrt{2\pi r}} \cos^3\left(\frac{\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}} 3 \sin\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\theta}{2}\right) \right)$$

$$(8.156) \quad = -\frac{K_I}{\sqrt{2\pi r}} \frac{3}{2} \cos^2\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)$$

$$(8.157) \quad -\frac{K_{II}}{\sqrt{2\pi r}} 3 \left(\frac{1}{2} \cos^3\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \right)$$

$$(8.158) \quad = -\frac{3}{2\sqrt{2\pi r}} \cos^3\left(\frac{\theta}{2}\right) \left(K_I \tan\left(\frac{\theta}{2}\right) + K_{II} - 2K_{II} \tan^2\left(\frac{\theta}{2}\right) \right)$$

which is quadratic in $\tan\left(\frac{\theta}{2}\right)$, so

$$(8.159) \quad \tan\left(\frac{\theta}{2}\right) = \frac{-K_I \pm \sqrt{K_I^2 + 8(K_{II})^2}}{-4K_{II}}.$$

Note that $\cos\frac{\theta}{2} = 0$, for $\theta = \pm\pi$, but this angle is not admissible as this would have the crack advance back onto itself. Therefore, the critical angle of propagation is

$$(8.160) \quad \theta_c = 2 \tan^{-1} \left(\frac{-K_I \pm \sqrt{K_I^2 + 8(K_{II})^2}}{-4K_{II}} \right).$$

Thus, we see that the angle of crack propagation is determined by the stress intensity factors K_I and K_{II} . Note that choosing θ so that $\sigma_{r\theta} = 0$ yields the same result.

Suppose that the crack is in a pure mode I state where $K_I \neq 0$ and $K_{II} = 0$. Since $K_{II} = 0$, we start with the expression

$$(8.161) \quad K_I \tan\left(\frac{\theta}{2}\right) + K_{II} - 2K_{II} \tan^2\left(\frac{\theta}{2}\right) = 0,$$

we see that $\theta_c = 0$, so when a cracked body is subjected to uniaxial tension, we expect the crack to propagate along the x_0 axis. At the other extreme, suppose that our body is in a pure mode II state where $K_I = 0$ and $K_{II} \neq 0$. Then, we have that

$$(8.162) \quad \theta_c = 2 \tan^{-1} \left(\frac{\pm\sqrt{8}}{-4} \right) \approx \mp 70.5^\circ,$$

so when the cracked body is subjected to antisymmetric loading, we expect the crack to extend at an angle of approximately $\pm 70.5^\circ$ degrees from the x_0 axis. Therefore, when the body is in

a state of tension, we expect a crack to travel in a straight path, but when shear is introduced, the crack deviates from the x_0 axis and travels at some angle that is in the direction of maximal circumferential stress.

To incorporate mode III effects into the crack propagation angle, we adopt the criterion of Richard [72]. The crack propagates at angle angle

$$(8.163) \quad \theta = A \frac{|K_{II}|}{K_I + |K_{II}| + |K_{III}|} + B \left(\frac{|K_{II}|}{K_I + |K_{II}| + |K_{III}|} \right)^2,$$

with $\theta < 0$ when $K_{II} > 0$ and $\theta > 0$ when $K_{II} < 0$ and $K_I \geq 0$. Choosing $A = 140^\circ$ and $B = -70^\circ$ results in the propagation angle agreeing with the maximum principal stress criterion.

8.8.2. Crack extension Criterion. The fracture toughness of materials is often given in terms of the mode I critical SIF, so an equivalent mode I stress is defined and determines whether the crack extends. Again, we adopt the expression for the equivalent mode I stress given by Richard [72],

$$(8.164) \quad K_{eq} = \frac{K_I}{2} + \frac{1}{2} \sqrt{K_I^2 + 4(\alpha_1 K_{II})^2 + 4(\alpha_2 K_{III})^2},$$

where K_{Ic} is the mode I fracture toughness, and $\alpha_1 = \frac{K_{Ic}}{K_{IIc}}$ and $\alpha_2 = \frac{K_{Ic}}{K_{IIIc}}$. Choosing $\alpha_1 = 1.155$ and $\alpha_2 = 1$ closely matches the maximum principal stress criterion.

8.8.3. Crack Speed. If the equivalent mode I stress exceeds the mode I fracture toughness, the crack propagates at speed

$$(8.165) \quad \dot{a} = \begin{cases} c_R \left(1 - \frac{K_{Ic}}{K_{eq}} \right) & K_{Ic} \leq K_{eq} \\ 0 & \text{otherwise.} \end{cases}$$

which is bounded by the Rayleigh wave speed.

CHAPTER 9

Numerical Details

We gather a number of miscellaneous numerical details in this chapter. We first discuss our time stepping method for the elastodynamic method, the Newmark method. Next, we outline the mesh extrusion procedure we use to construct polyhedral volume meshes from polygonal surface meshes and the mesh splitting procedure used to represent the propagating cracks. After, we discuss the construction of the weighting function used in the domain interaction integral and the crack is discretely propagated. Finally, we outline the specialized quadrature methods we use for integrating over arbitrary polygons and polyhedra.

9.1. Newmark Method

After the displacement field is expressed in terms of the time-independent basis functions, the elastodynamic problem reduces to solving a second order system of ODE,

$$(9.1) \quad \mathbf{M}\ddot{\mathbf{c}} + \mathbf{K}\mathbf{c} = \mathbf{f},$$

where $\mathbf{M}$ and $\mathbf{K}$ are the global mass and stiffness matrices, respectively, and $\mathbf{f}$ is the load vector. Here, $\mathbf{c}$ are the coefficients which express the displacement field in terms of the time-independent basis functions,

$$(9.2) \quad \mathbf{u}(\mathbf{X}, t) = \sum_{i=1}^{N_{\text{DoF}}} c_i(t) \boldsymbol{\varphi}_i(t).$$

We discretize in time with the Newmark-beta method [62]. We choose the trapezoidal rule member of the Newmark family due to its second-order accuracy, unconditional stability, and conservative properties. Using $\mathbf{a} = \ddot{\mathbf{c}}$ and $\mathbf{v} = \dot{\mathbf{c}}$, we first solve for the updated accelerations in

$$(9.3) \quad \left(\mathbf{M} + \frac{(dt)^2}{4} \mathbf{K} \right) \mathbf{a}^{n+1} = \mathbf{f} - \mathbf{K}\mathbf{c}^n + dt\mathbf{K}\mathbf{v} + \frac{(dt)^2}{4} \mathbf{K}\mathbf{a}^n,$$

and the position and velocity are subsequently updated using the new acceleration,

$$(9.4) \quad \begin{aligned} \mathbf{c}^{n+1} &= \mathbf{c}^n + dt\mathbf{v}^n + \frac{(dt)^2}{4}(\mathbf{a}^n + \mathbf{a}^{n+1}), \\ \mathbf{v}^{n+1} &= \mathbf{v}^n + \frac{dt}{2}(\mathbf{a}^n + \mathbf{a}^{n+1}). \end{aligned}$$

9.2. Mesh Construction

We represent our thin object domains in terms of thickened quadrilateral surfaces. Our mesh construction is composed of two primary components: extrusion and splitting. During mesh extrusion, the quadrilateral surface mesh is extruded into a volume mesh, and mesh splitting involves duplicating and moving nodes in the quadrilateral mesh to conform with the propagating crack.

9.2.1. Mesh Extrusion. Starting from a quadrilateral mesh, we thicken the quadrilaterals (out of surface) into N layers of thin, poorly conditioned, possibly non-convex hexahedral elements. If the quadrilateral mesh has N_v vertices, the extruded mesh will have $N * N_v$ vertices, and if the quadrilateral mesh has N_E quadrilateral elements, the extruded mesh will have $(N - 1) * N_E$ hexahedral elements. The elements resulting from the extrusion process will have high aspect ratio with elements having a thickness of roughly 1 – 5% the diameter of a quadrilateral from the quadrilateral mesh. In Figure 9.1, the extrusion process is shown. First, vertex normals are computed in the quadrilateral mesh, and the vertex normals are used to construct the extrusion layers. When extruding curved surface meshes, non-planar hexahedral faces may arise as shown in Figure 9.2. In this case, during the VEM construction, the faces of the hexahedron are triangulated, and the element is treated as a dodecahedron. Outside of the VEM construction, the element is still treated as a hexahedron.

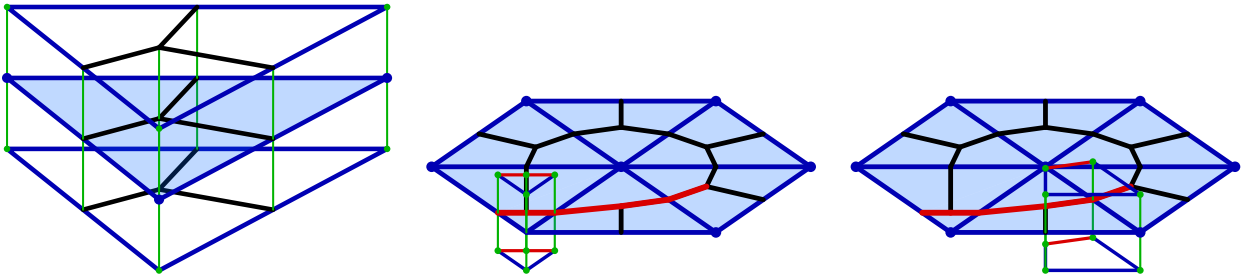


FIGURE 9.1. Overview of the mesh extrusion process.

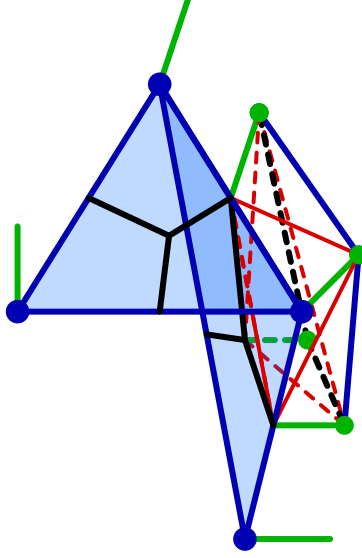


FIGURE 9.2. Triangulation of hexahedron face that is a nonplanar quadrilateral.

9.2.2. Mesh Splitting. We take all cracks to be through the thickness of the extruded mesh, and the path of the propagating crack is computed in the original quadrilateral mesh. We use the virtual node algorithm [58] to split triangles into quadrilaterals. Starting from a triangle mesh, we split each triangle into three quadrilaterals where vertices are inserted at triangle centroids and edge centers (topologically). Crack tips always live inside of a triangle from the original triangle mesh, and as the crack advances through the mesh, the crack tip moves around the triangle until the crack increment advances the crack tip outside of the triangle. When a crack tip exits a triangle, the triangle edge intersection is computed and the vertices that represented the crack tip in the original triangle and the intersected triangle edge node are duplicated. Figure 9.3 outlines the procedure for constructing the quadrilateral mesh from the triangle mesh, and adjusting the nodes of the quadrilateral mesh to conform with the crack path.

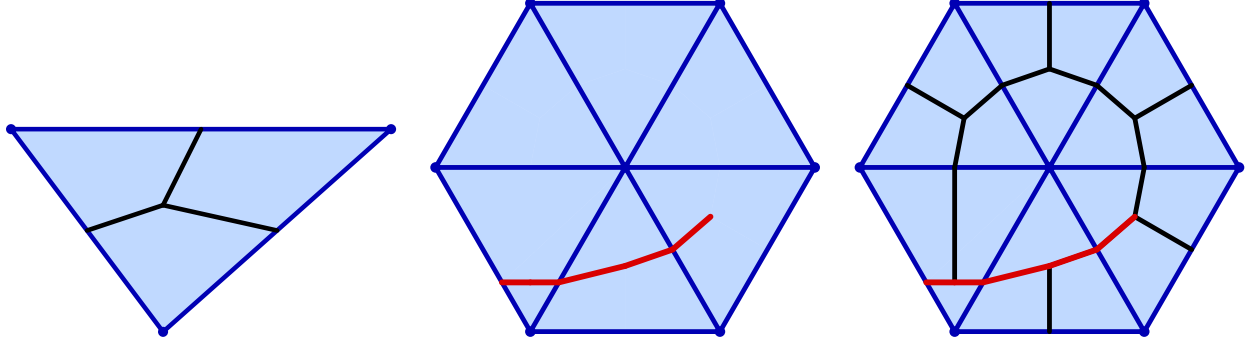


FIGURE 9.3. Construction of the quadrilateral mesh from the triangle mesh and the representation of the crack in the quadrilateral mesh.

After the mesh extrusion, each layer of vertices is in a one-to-one correspondence with the original quadrilateral mesh, and the propagation of the crack in the original quadrilateral mesh is duplicated in each extrusion layer.

In Figure 9.4, the quadrilateral mesh before and after a crack passes through the mesh is shown.

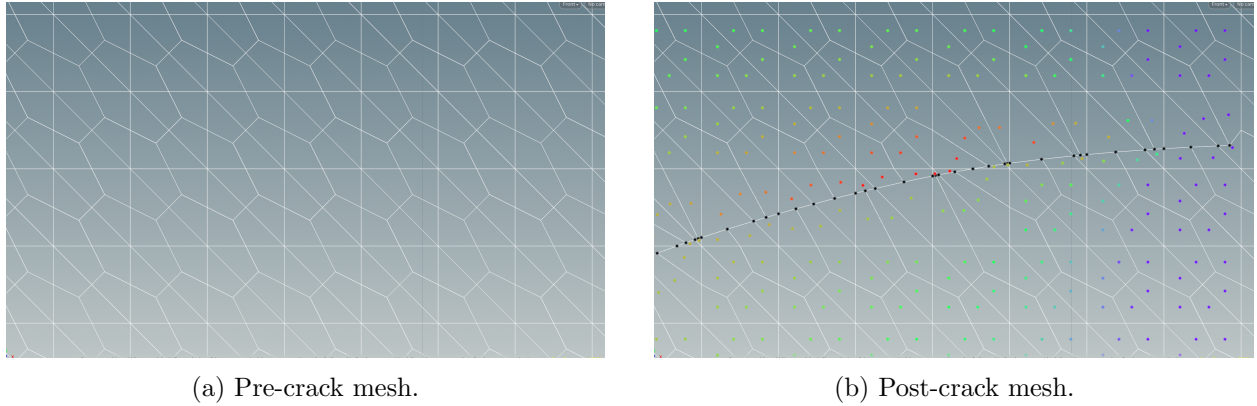


FIGURE 9.4. The quadrilateral mesh before and after a crack advancement.

The underlying triangle mesh is composed of right triangles, and the quadrilaterals are formed by adding nodes to the triangle edge midpoints and triangle centroids. Once the crack passes through the mesh, the triangle centroids and triangle edges are split and moved to conform with the advancing crack. At the very end of the crack path, we see that there is a nonconvex quadrilateral.

Figure 9.5 shows a crack propagating under mode I loading. The black dots are the vertices that make up the crack path which are composed of vertices inside of triangles and on edges of triangles.

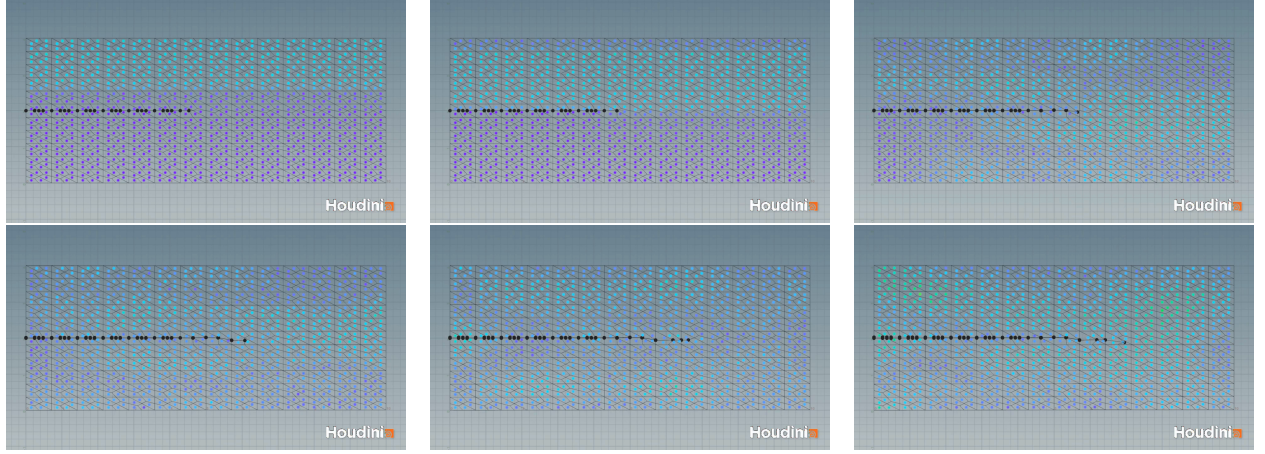


FIGURE 9.5. A crack propagating through a quadrilateral mesh under mode I loading.

9.3. Interaction Integral

The conversion of the contour integral into the domain integral required a sufficiently smooth function q which takes a value of 1 and 0 on inner and outer concentric contours, respectively. We construct the function q by choosing a radius r_D and collecting the elements that intersect a circle of radius r_D with center at the crack tip. That is, we query whether an element has at least one vertex inside and one vertex outside the circle, and we assign a nodal value of 0 to the vertices outside the circle and 1 to the vertices inside the circle. Then, we have that the domain over which the interaction integral is computed is given by $A = \cup_k E_k$, where E_k are the support elements. Over each support element, we construct q using the nodal values of q and interpolate over the element using the projection of the VE basis functions.

9.4. Discrete Crack Propagation

Before the crack is extended, suppose the crack tip is located at $s(t) = \mathbf{x}_c$. At time $t + dt$, we have the acceleration, velocity, and displacement fields. Using the interaction integral, we obtain the SIFs and compute the equivalent mode I stress. If the mode I fracture toughness is exceeded, the angle θ at which the crack extends from the crack tip is determined by Richard's criterion [72], and the speed of the crack tip $\dot{a}$ is determined by (8.165). At each time step, the crack path is

incremented by length $\Delta s = \dot{a}dt$ so that the updated crack tip is located at (see Fig. 9.6)

$$(9.5) \quad \mathbf{x}_c^{\text{new}} = \dot{a}dt \langle \cos \theta, \sin \theta, 0 \rangle + \mathbf{x}_c.$$

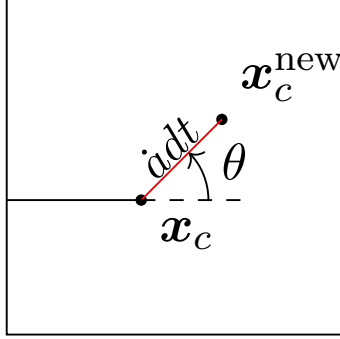


FIGURE 9.6. Extension of the crack.

9.5. Scaled Boundary Quadrature and Homogeneous Numerical Integration

Constructing the element matrices for the elastodynamic simulation and the extraction of SIFs from the interaction integral requires evaluation of integrals over elements. Typically, Gaussian quadrature is a suitable choice for computing integrals, but establishing a mapping from the reference domain proves to be cumbersome for arbitrary polygons and polyhedra. We use the scaled boundary cubature scheme for integrating over polygons and the homogeneous numerical integration technique for integrating monomials over polyhedra.

9.5.1. Scaled Boundary Cubature. To compute integrals over arbitrary polygons, we use the scaled boundary cubature scheme of Chin [19]. Let E be a polygon with edges e_i , $i = 0, 1, \dots, m$, and

$$(9.6) \quad \mathbf{c}_i(t) = \bar{\mathbf{e}}_i t + \mathbf{x}_i$$

a parameterization of edge e_i with $\bar{\mathbf{e}}_i$ the vector pointing from $\mathbf{x}_i$ (the starting point of e_i) to the ending point of e_i . The scaled boundary parameterization is given by

$$(9.7) \quad \mathbf{x} = \boldsymbol{\varphi}(\xi, t) = \mathbf{x}_0 + \xi(\mathbf{c}_i(t) - \mathbf{x}_0),$$

where $\mathbf{x}_0$ is a reference point that may be chosen as one of the vertices of E . Then,

$$(9.8) \quad \int_E f(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^m \int_0^1 \int_0^1 f(\boldsymbol{\varphi}) \det(\nabla \boldsymbol{\varphi}) d\xi dt,$$

and the integral on the RHS of (9.8) may be computed using Gaussian quadrature. Now, we see that

$$(9.9) \quad \nabla \boldsymbol{\varphi} = \begin{bmatrix} (\mathbf{c}_i(t) - \mathbf{x}_0) & \xi \mathbf{c}'_i(t) \end{bmatrix},$$

so

$$(9.10) \quad \det(\nabla \boldsymbol{\varphi}) = \xi (\mathbf{c}_i(t) - \mathbf{x}_0) \cdot \mathbf{n}_i |\bar{\mathbf{e}}_i|,$$

where $\mathbf{n}_i = \frac{\bar{\mathbf{e}}_i \times \mathbf{n}_E}{|\bar{\mathbf{e}}_i \times \mathbf{n}_E|}$.

9.5.2. Homogeneous Numerical Integration. During the construction of the VE polynomial projection operators, we require the computation of integrals over polyhedra of the scaled monomials. To compute these integrals, we use the homogeneous numerical integration approach proposed by Chin [18] which we now present. Let E be a polyhedron with faces F_i , $i = 1, \dots, m$, and suppose $f \in C^1(E)$. We say f is a homogeneous function of degree q if

$$(9.11) \quad f(\lambda \mathbf{x}) = \lambda^q f(\mathbf{x}),$$

where λ is some positive constant. The Euler homogeneous function theorem states that

$$(9.12) \quad qf(\mathbf{x}) = \mathbf{x} \cdot \nabla f(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^d$. Let $\boldsymbol{\Phi}$ be an arbitrary vector field. Then, generalized Stokes' theorem states that

$$(9.13) \quad \int_E (\nabla \cdot \boldsymbol{\Phi}) f(\mathbf{x}) d\mathbf{x} + \int_E \boldsymbol{\Phi} \cdot \nabla f(\mathbf{x}) d\mathbf{x} = \int_{\partial E} \boldsymbol{\Phi} \cdot \mathbf{n} f(\mathbf{x}) d\mu,$$

where $d\mu$ is the Lebesgue measure on ∂E . Applying the Euler homogeneous function theorem to the second integral on the LHS of (9.13) and taking $\boldsymbol{\Phi} := \mathbf{x}$, we have

$$(9.14) \quad \int_E df(\mathbf{x}) d\mathbf{x} + \int_E qf(\mathbf{x}) d\mathbf{x} = \int_{\partial E} \boldsymbol{\Phi} \cdot \mathbf{n} f(\mathbf{x}) d\mu.$$

Now, each face of E is a subset of a plane P_i , and letting the equation of the plane P_i be given by $\mathbf{a}_i \cdot \mathbf{x} = b_i$, we see that (9.14) becomes

$$(9.15) \quad (d+q) \int_E f(\mathbf{x}) d\mathbf{x} = \sum_{F_i} \frac{b_i}{|\mathbf{a}_i|} \int_{F_i} f(\mathbf{x}) d\mu.$$

Thus,

$$(9.16) \quad \int_E f(\mathbf{x}) d\mathbf{x} = \frac{1}{d+q} \sum_{F_i} \mathbf{n}_{F_i} \cdot \mathbf{x}_0 \int_{F_i} f(\mathbf{x}) d\mu,$$

where $\mathbf{n}_{F_i}$ is the unit normal to face F_i and $\mathbf{x}_0$ is a vertex of F_i . Note that the RHS of (9.16) is computed using the SBC.

CHAPTER 10

Validation Examples

We demonstrate the validity of our method using standard benchmark tests from the computational fracture mechanics literature. First, we consider a static tension problem and examine the behavior of the SIFs along the crack front as the number of extrusion layers is varied. Next, we use a dynamic mode I problem to demonstrate the validity of our extracted SIFS by comparing our results to the analytic expression for the mode I SIF by considering a stationary crack, delayed crack, and moving crack. Then, we consider a dynamic mixed mode (I+II) problem, and finally we consider a number of examples which demonstrate the desired qualitative results under different loading conditions.

10.1. Static Mode I Study

First, we examine a cracked specimen under static conditions that is subjected to mode I loading. We consider a through-cracked body of dimensions $10 \text{ m} \times 16.5 \text{ m} \times 15 \text{ m}$ shown in Figure 10.1 with material properties $E = 10^5 \text{ Pa}$ and $\nu = 0.3$, and uniform tensile tractions are applied on $x_2 = 0$ and $x_2 = 16.5$ surfaces,

$$(10.1) \quad \bar{\mathbf{t}}(\mathbf{x}) = \begin{cases} -\sigma_0 \mathbf{e}_2 & \text{if } x_2 = 0 \\ \sigma_0 \mathbf{e}_2 & \text{if } x_2 = 16.5 \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\sigma_0 = 1$. The plane strain solution for the mode I SIF is:

$$(10.2) \quad K_I = C \sigma_0 \sqrt{\pi a},$$

where σ_0 is the magnitude of the applied traction, and $C = 2.829125$ is a finite-geometry correction factor [75].

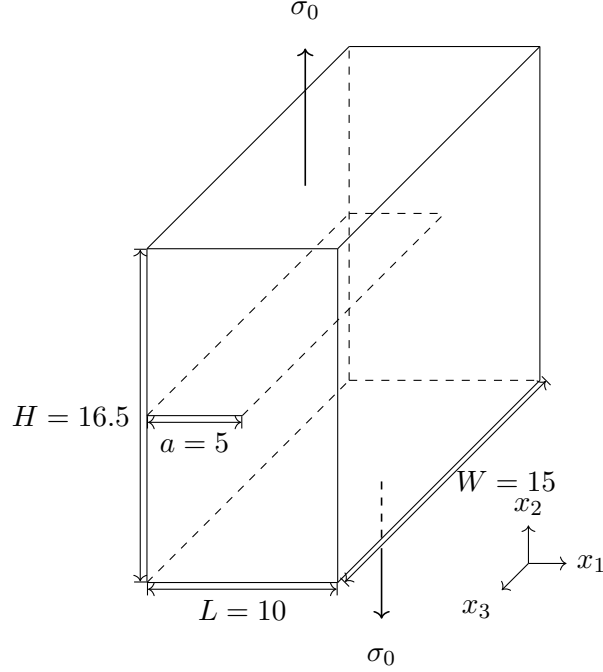


FIGURE 10.1. Static mode I setup.

This problem was studied by [8] which showed that the behavior of the stress state at the free surface is different than the plane strain solution. Further numerical studies in [7, 69] consistently demonstrate that the mode I SIF approaches the plane strain value in the center of the specimen and drops off to a nonzero value at the free surfaces.

Therefore, the SIFs vary through the thickness of the specimen, so a sufficient number of mesh extrusion layers are required to adequately resolve the SIFs. Let $s(t)$ be a parameterization of the crack front with $s(0)$ corresponding to the $x_3 = 0$ surface, and $s(1)$ corresponding to the $x_3 = 15$ surface. In Figure 10.2, we see that as $t \rightarrow 0^+, 1^-$, K_I rapidly drops off to a nonzero value, and as $t \rightarrow 0.5$, K_I approaches the plane strain solution.

The weighting function q that is used in the domain form of the interaction integral varies linearly over an element, so we require at least 3 extrusion layers to extract the SIFs across the thickness of the specimen. Additionally, in Figure 10.2, we see that 3 extrusion layers only captures the free surface behavior of the mode I SIF, and as the number of extrusion layers increases, the plane strain behavior dominates.

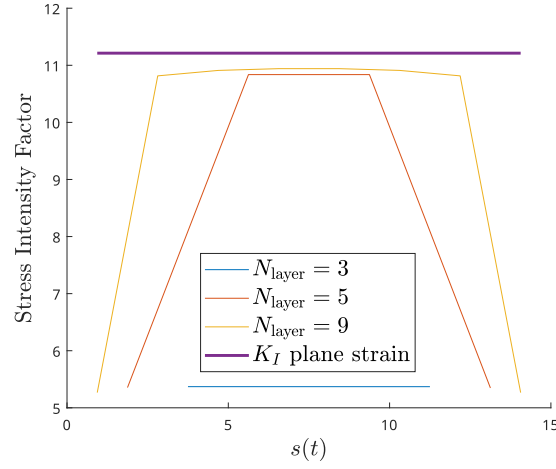


FIGURE 10.2. SIF for pure mode I edge crack problem.

10.2. Qualitative Quasistatic Crack Propagation Examples

10.2.1. Mode I. Our first qualitative quasistatic example is a crack propagating under mode I loading. We consider the prescribed displacement boundary condition

$$(10.3) \quad \mathbf{u}(\mathbf{X}) = \begin{cases} (0, 0.1, 0), & \mathbf{X} = (X_1, 1, X_3) \\ (0, -0.1, 0), & \mathbf{X} = (X_1, 0, X_3). \end{cases}$$

In Figure 10.3, we see that the crack advances with an angle of zero degrees which is exactly what we expect for mode I loading. Additionally, we see that the stress field ahead of the crack tip takes the familiar butterfly shape.

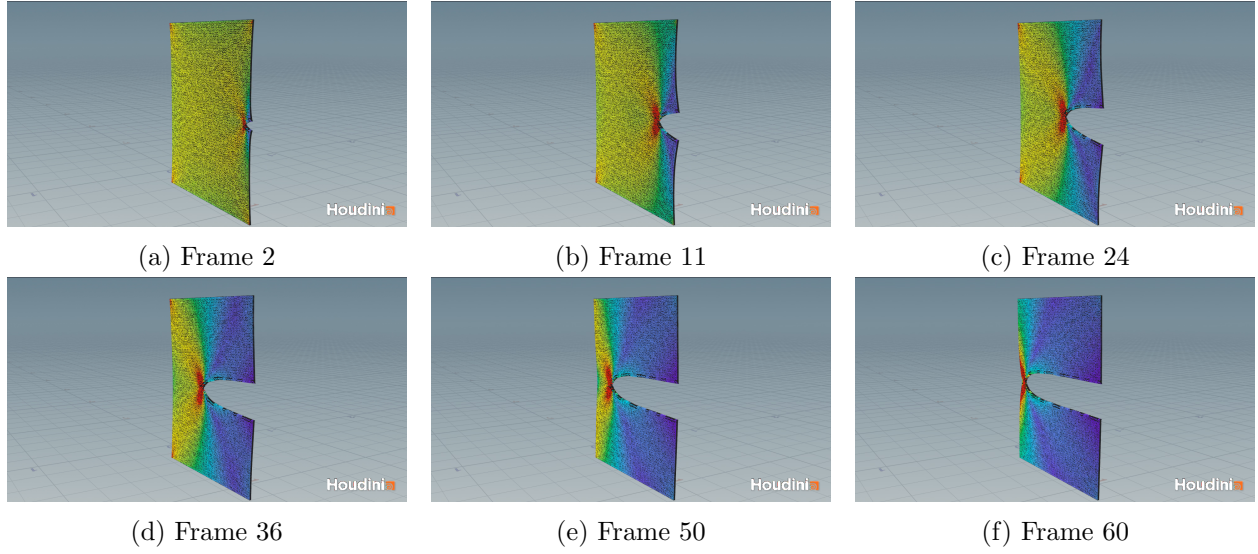


FIGURE 10.3. Quasistatic crack propagation under mode I loading.

10.2.2. Mode III. Under mode III loading, we expect the crack to advance in a straight line. We consider the out-of-plane prescribed boundary condition

$$(10.4) \quad \mathbf{u}(\mathbf{X}) = \begin{cases} (0, 0, 0.1), & \mathbf{X} = (1, X_2, X_3), X_2 \geq 0.5 \\ (0, 0, -0.1), & \mathbf{X} = (1, X_2, X_3), X_2 < 0.5. \end{cases}$$

In Figure 10.4, we see that the crack advances to the edge of body with an angle of zero degrees which is the behavior we expect. Additionally, Figure 10.4 shows that the region of high stress around the crack tip shrinks as the crack propagates away from the edge where the crack originates.

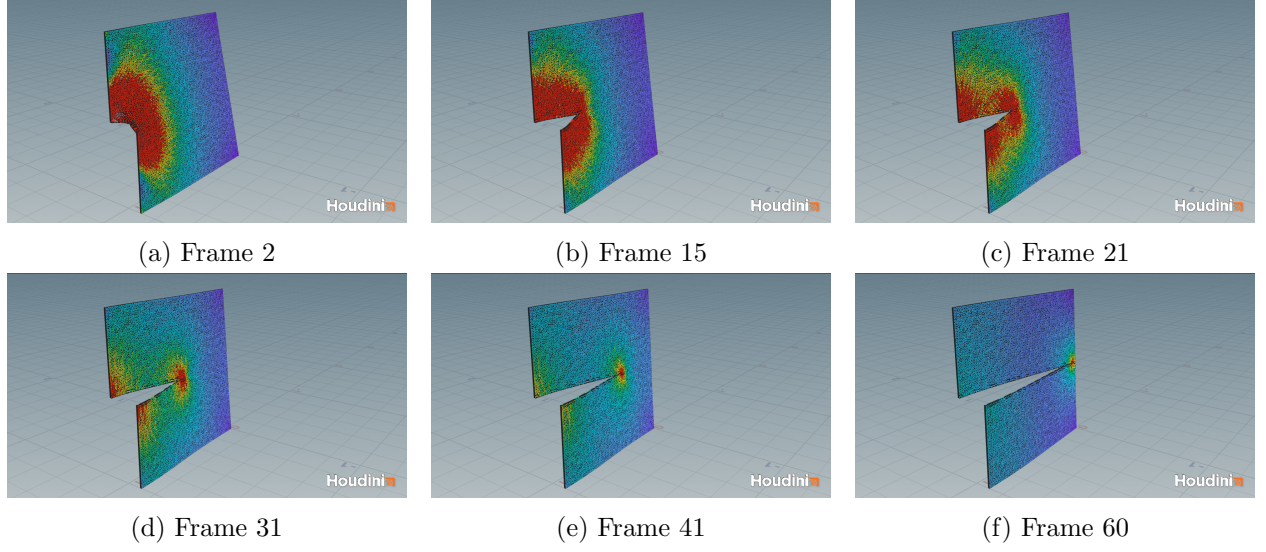


FIGURE 10.4. Quasistatic mode II crack propagation.

10.2.3. Mode II and III. We consider a mixed mode II and III loading scenario with the following prescribed displacement boundary condition

$$(10.5) \quad \mathbf{u}(\mathbf{X}) = \begin{cases} (0.1, 0, 0.1), & \mathbf{X} = (1, X_2, X_3), X_2 \geq 0.5 \\ (0.1, 0, 0), & \mathbf{X} = (X_1, 1, X_3), 0 \leq X_1 < 1 \\ \mathbf{0}, & \mathbf{X} = (0, X_2, X_3), X_2 < 0.5 \end{cases}$$

which combines shearing and antiplane effects. In figure 10.5 we see the crack travels upward with a curved trajectory. We expect that under mode II loading, the crack should travel either upward or downward depending on the shearing, and under mode III we expect the crack to travel straight forward. We see that the crack travels with a path that has the features of a crack propagating under mode II and mode III loading.

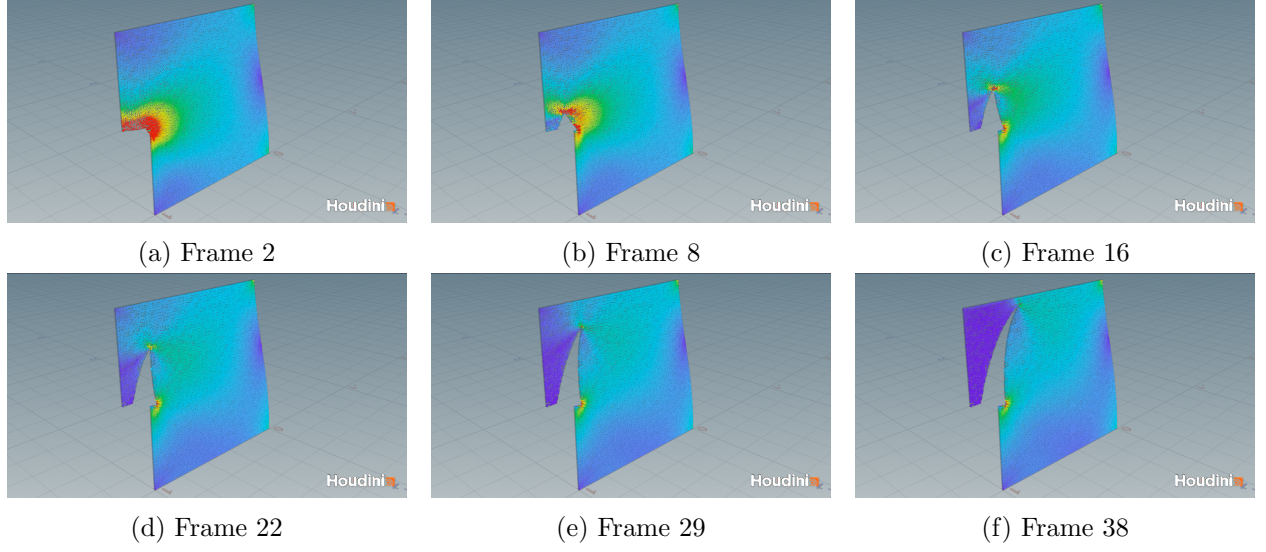


FIGURE 10.5. Quasistatic mode II and III crack propagation.

10.3. Dynamic Studies

10.3.1. Dynamic Mode I. We now consider a body subjected to mode I loading in a dynamic setting with the material parameters of glass: $E = 70$ GPa, $\nu = 0.2$, and $\rho = 2500$ kg/m³ that has zero initial displacement and is at rest. The dimensions of the body are $10 \text{ m} \times 4 \text{ m} \times 0.1 \text{ m}$, and the body has an initial edge crack of length 5 m as shown in Figure 10.6. At $t = 0$, an impulse load is applied to the $x_2 = 0$ and $x_2 = 4$ surfaces of the specimen for $t \geq 0$ and the impulse is modeled as a traction boundary condition,

$$(10.6) \quad \bar{\mathbf{t}}(t, \mathbf{x}) = \begin{cases} -\sigma_0 \mathbf{e}_2 & \text{if } x_2 = 0 \\ \sigma_0 \mathbf{e}_2 & \text{if } x_2 = 4 \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where we take $\sigma_0 = 50$ MPa.

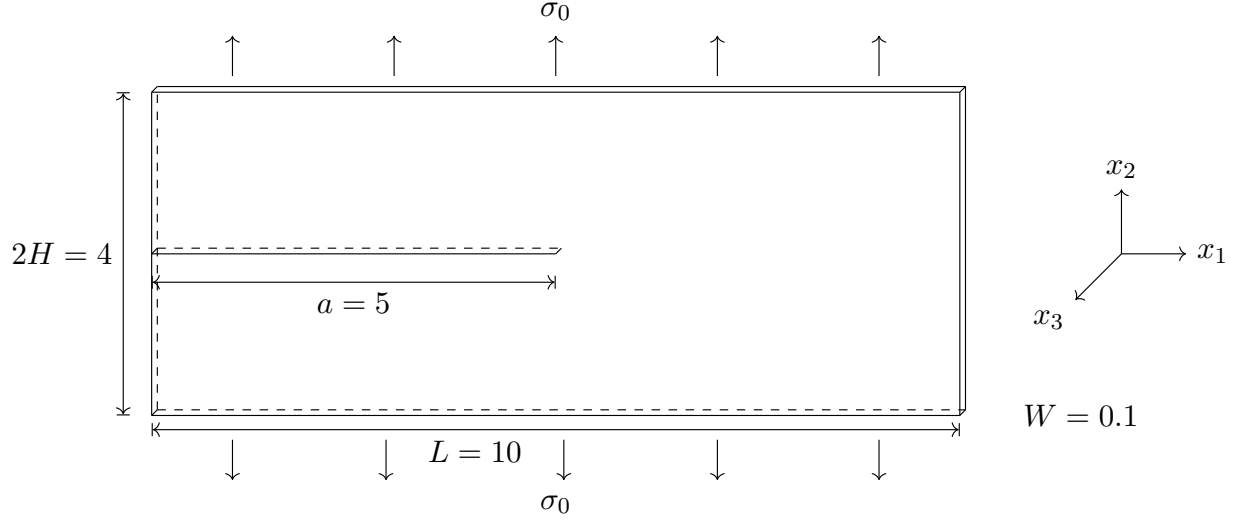


FIGURE 10.6. Dynamic mode I setup.

The elastic wave arrives at the crack tip at time $\hat{t} = \frac{H}{c_d}$ after which the mode I SIF begins to grow. The analytic expression for the two-dimensional plane strain solution [25] for the mode I SIF is given by

$$(10.7) \quad K_I(t, \dot{a}) = \frac{4\sigma k(\dot{a})}{1 - \nu} \sqrt{\frac{(1 - 2\nu)(t - \hat{t})c_d}{\pi}} H(t - \hat{t}),$$

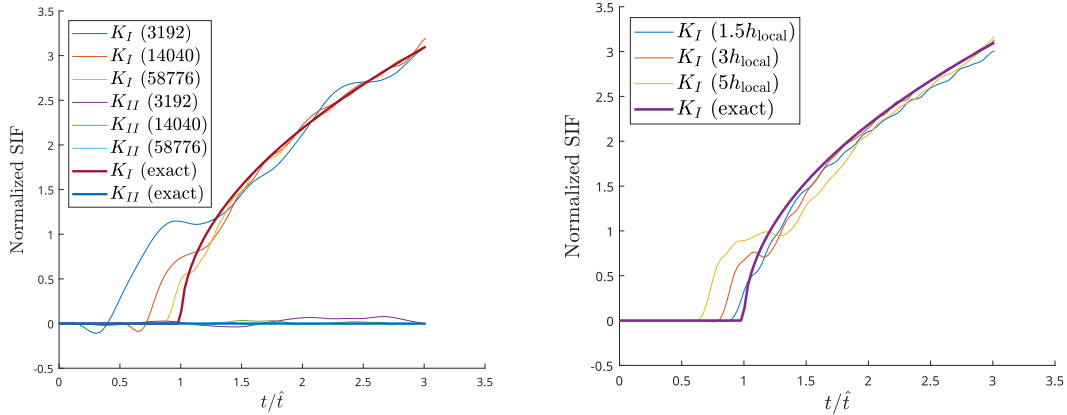
where $H(t)$ is the Heaviside function,

$$(10.8) \quad k(\dot{a}) = \frac{1 - \frac{\dot{a}}{c_R}}{1 - \frac{\dot{a}}{2c_R}},$$

and c_d and c_R are the dilatational and Rayleigh wave speeds, respectively. The function k is a scaling factor that takes into account the advancing crack front. The analytic expression for the mode I SIF is for the infinite-plane problem, so its applicability holds until the elastic wave traveling from the $x_2 = 4$ and $x_2 = 0$ surfaces reflects off of the $x_2 = 0$ and $x_2 = 4$ surfaces, respectively, and returns to the crack tip at time $\bar{t} = \frac{3H}{c_d}$.

In our discretization, we use five extrusion layers, and we report the average of the two interior SIF values since the plane strain state is rapidly approached in the interior elements as demonstrated in the previous example. The SIFs we report are normalized by $\sigma_0\sqrt{(H)}$ and plot the SIFs with respect to the normalized time $t/\hat{t}$.

10.3.1.1. *Stationary Crack.* First, we consider a stationary crack and study the effects of the size of the inner diameter of the annular region over which the interaction integral is computed. When considering elastostatic fracture mechanics, the contour over which the interaction integral is computed is path-independent, but under dynamic conditions, path-independence no longer holds. When the contour is too small, the discontinuous elastodynamic fields cannot be accurately resolved, but when the contour is too large local behavior of the crack tip cannot be captured. Therefore, a suitably sized contour must be chosen that balances resolution of the computed elastodynamic fields and the proximity of the contour to the crack tip. In Figure 10.7a, we see that K_I is in stronger agreement with the exact solution as the mesh is refined. Additionally, the time at which the computed K_I begins to increase approaches $\hat{t}$ as the mesh is refined. In Figure 10.7b, h_{local} is the local average element edge length near the crack tip, and the SIFs are computed for different interaction integral radii. We see that for a radius of $1.5h_{\text{local}}$, the growth of K_I begins closest to $\hat{t}$ when compared to $3h_{\text{local}}$ and $5h_{\text{local}}$, but $3h_{\text{local}}$ agrees more closely to the exact value of K_I for $t > 1.25\hat{t}$.

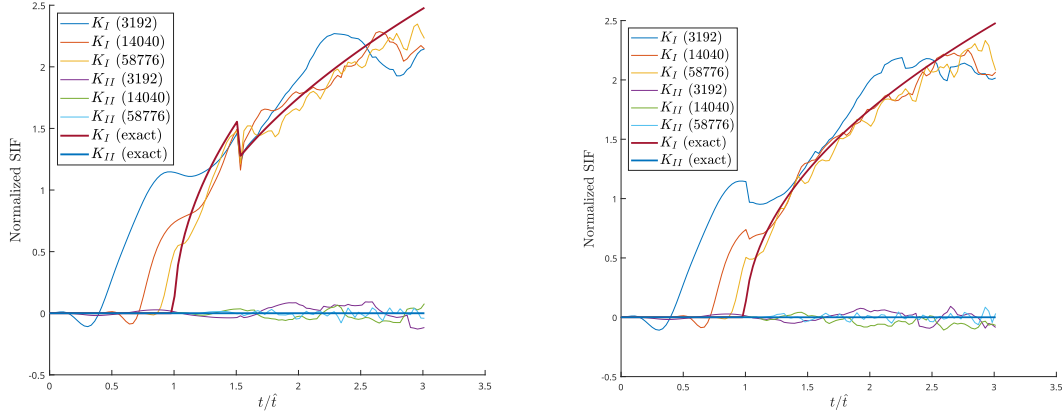


(a) Dynamic SIFs for pure mode I stationary crack problem. (b) Dynamic SIF for mode I problem with different radii for the interaction integral.

10.3.1.2. *Delayed Moving Crack.* Next, we consider a delayed crack which begins moving at time $t = 1.5\hat{t}$ with a prescribed speed of $\dot{a} = \frac{c_R}{3}$. In Figure 10.8a, we see that once the crack tip begins to move at $t = 1.5\hat{t}$, there is a drop in K_I with oscillations that decrease in amplitude and increase in frequency as the mesh is refined. As the number of elements is increased, the computed K_I approaches the exact value of K_I . Additionally, after the crack begins moving, K_I continues

growing at the lower rate due to the scaling factor in (10.8), and we see that the computed mode I SIFs capture this feature and grow at the same rate as the exact solution.

10.3.1.3. Moving Crack. As the last scenario, we consider a crack that begins moving when the elastic wave reaches the crack tip at $t = \hat{t}$ with a prescribed speed of $\dot{a} = \frac{c_R}{3}$. In Figure 10.8b, we see that the computed mode I SIFs have a drop at $t = \hat{t}$ when the crack begins to advance, and the growth rate of the computed values agree with the growth rate of the exact values as in the delayed moving crack problem. Again, the computed mode I SIFs grow in agreement with the exact mode I SIF as the number of elements is increased.



(a) Dynamic SIFs for pure mode I delayed crack problem. (b) Dynamic SIFs for pure mode I moving crack problem.

10.3.2. Kalthoff's Experiment. The impact of a cylinder into the side of a cracked specimen was experimentally studied by Kalthoff [38], and Lee [41] developed an analytic solution for the problem. The experiment considers a specimen with two parallel cracks, and a projectile impacts the specimen between the two cracks. From the symmetry of the problem, we only need to simulate the lower-half of the specimen which has dimensions $0.1 \text{ m} \times 0.1 \text{ m} \times 0.0165 \text{ m}$ as shown in Figure 10.9. We use the material parameters $E = 190 \text{ GPa}$, $\nu = 0.3$, $\rho = 8000 \text{ kg/m}^3$, and $K_{Ic} = 68 \text{ MPa}$, and the velocity of the projectile is $v_0 = 20 \text{ m/s}$. This problem involves Dirichlet boundary conditions due to the projectile impact and zero displacement boundary conditions arising from the symmetry

conditions,

$$(10.9) \quad \begin{aligned} \mathbf{u}_D(t, \mathbf{x}) &= \begin{cases} u_1 \mathbf{e}_1 + u_3 \mathbf{e}_3 & \text{if } x_2 = 0.2 \\ v_0 t \mathbf{e}_1 & \text{if } x_1 = 0, x_2 < 0.1, \end{cases} \\ \dot{\mathbf{u}}_D(t, \mathbf{x}) &= \begin{cases} v_0 \mathbf{e}_1 & \text{if } x_1 = 0, x_2 < 0.1. \end{cases} \end{aligned}$$

The analytic expressions for the mode I and II SIFs are available in Lee [41]. The elastic wave reaches the crack tip at time $\hat{t} = \frac{a}{c_d}$ at which point the analytic SIFs begin to grow, and the validity of the analytic expressions holds until $\bar{t} = \frac{3a}{2}$.

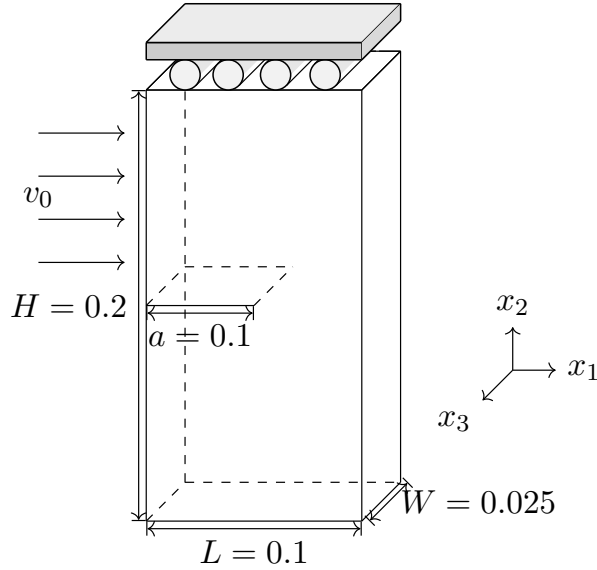


FIGURE 10.9. Kalthoff experiment setup.

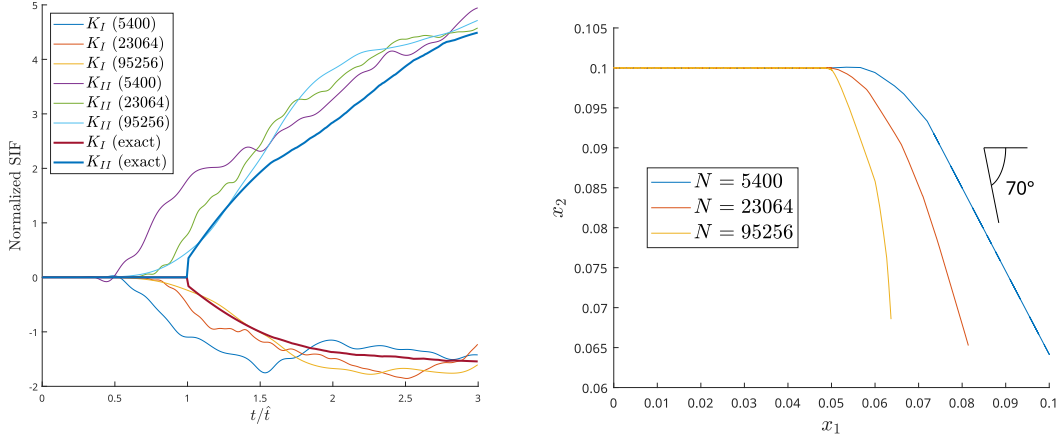
We use five extrusion layers and report the average of the normalized SIFs over the two interior elements where

$$(10.10) \quad \frac{E v_0 \sqrt{\frac{a}{\pi}}}{2 c_d (1 - \nu^2)}$$

is the normalization factor, and time is normalized with respect to $\hat{t}$.

The crack is expected to propagate at an angle of approximately 70° from the initial crack surface, and since we simulate the lower-half of the specimen, we expect the crack to travel downward. In Figure 10.10b, the crack paths at $t = \bar{t}$ are plotted for three different mesh refinements. We see that the cracks travel downward, and as the number of elements increases, the 70° angle is

approached by the crack paths. In Figure 10.10a, we see that the mode I and mode II SIFs capture the same behavior as the exact solution, and as the mesh is refined, the SIFs approach the analytic solution. Again, we see the influence of the local mesh size on the growth of the SIFs before the elastic wave arrives at the crack tip.



(a) Dynamic SIFs for mode I and II Kalthoff experiment.

(b) Crack path for mode I and II problem.

10.4. Qualitative Dynamic Examples

10.4.1. Mode I. As our first qualitative example under elastodynamics, we consider an internal crack subjected to mode I loading. We consider a time dependent prescribed displacement boundary condition

$$(10.11) \quad \mathbf{u}(\mathbf{X}, t) = \begin{cases} (0, 0.05t, 0), & \mathbf{X} = (X_1, 1, X_3), t \geq 0 \\ (0, -0.05t, 0), & \mathbf{X} = (X_1, 0, X_3), t \geq 0. \end{cases}$$

In Figure 10.11, we see two crack tips propagate in a straight line to the boundary. This is exactly the behavior we expect of a propagating crack under mode I loading. Additionally, we see that as the crack tip propagate toward the boundary, the stress reacts to the propagating crack tips.

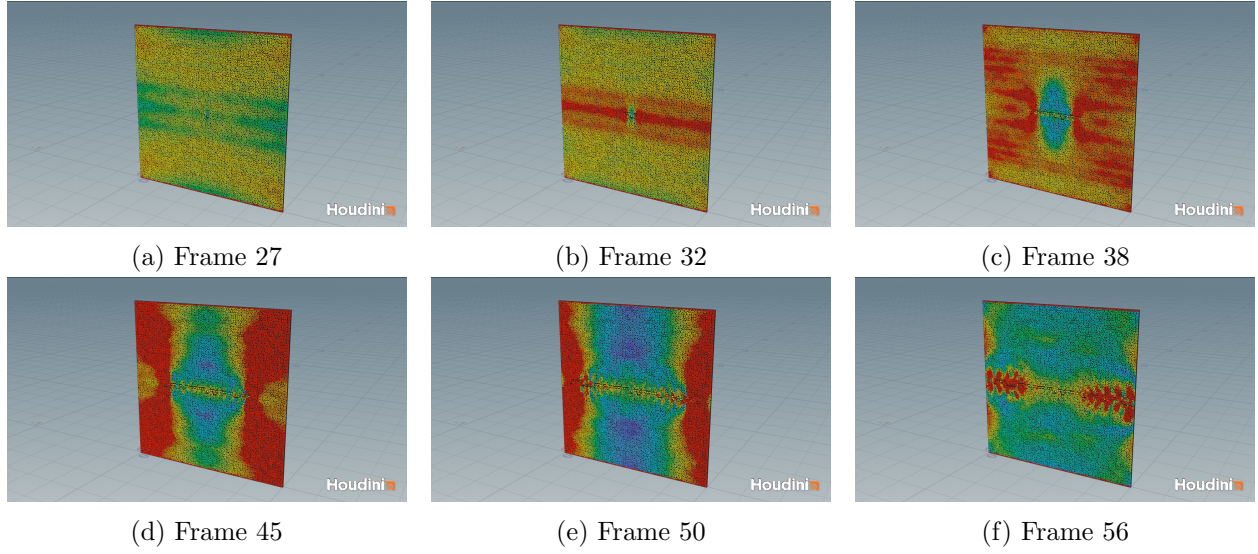


FIGURE 10.11. Dynamic mode I crack propagation.

10.4.2. Mode II. We consider an internal crack in the center of the body which we subject to mode II loading. We consider the time dependent prescribed displacement boundary condition

$$(10.12) \quad \mathbf{u}(\mathbf{X}, t) = \begin{cases} (0.05t, 0, 0), & \mathbf{X} = (X_1, 1, X_3), t \geq 0 \\ (-0.05t, 0, 0), & \mathbf{X} = (X_1, 0, X_3), t \geq 0. \end{cases}$$

In Figure 10.12, we see the two propagating crack tips with one tip traveling upward and the other tip traveling downward. Qualitatively, we see that the crack tips propagate according to the expected mode II path by traveling either upward or downward depending on the orientation of the crack tip with respect to the shear loading.

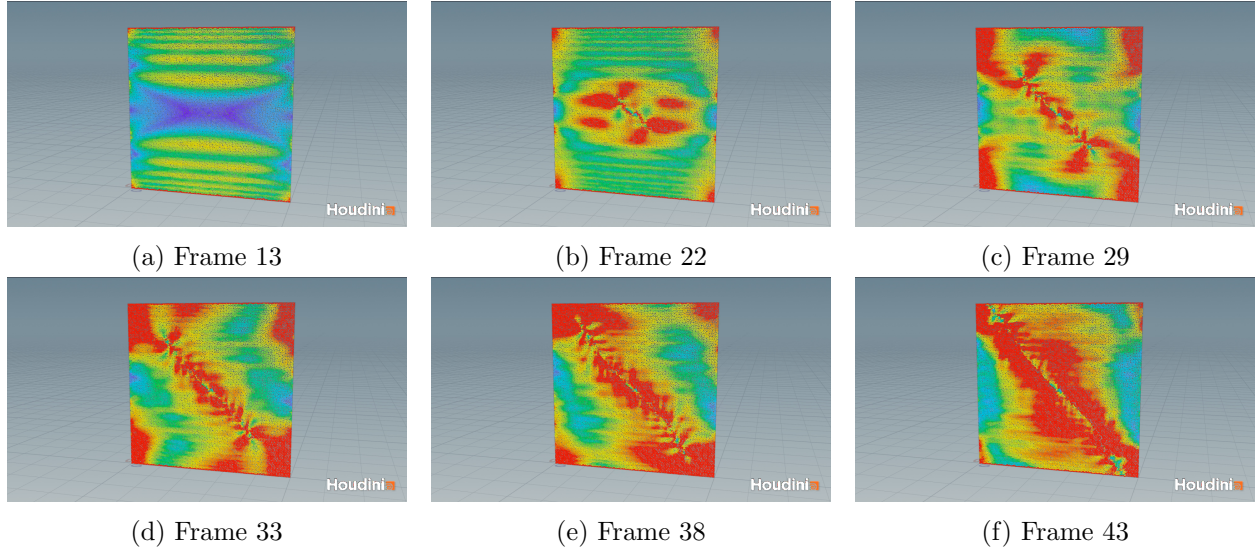


FIGURE 10.12. Dynamic mode II crack propagation.

10.4.3. Antiplane Traction. When collisions occur during the rigid body phase of the simulation, traction boundary conditions are built from the collision impulses. These traction boundary conditions are primarily out-of-plane loading conditions. We do not have any prescribed boundary conditions, and the traction boundary condition is of the form

$$(10.13) \quad \mathbf{t}(\mathbf{X}, t) = (0, 0, \tau),$$

where τ is some constant. The traction is applied over a small circular region at a number of different points, and four cracks are initially seeded in the body. In Figure 10.13, we see multiple cracks propagating under the antiplane loading conditions. We see that as the cracks propagate, the elastic waves are cutoff from other regions of the body, and this adds to the complexity of simulating the propagation of multiple crack tips.

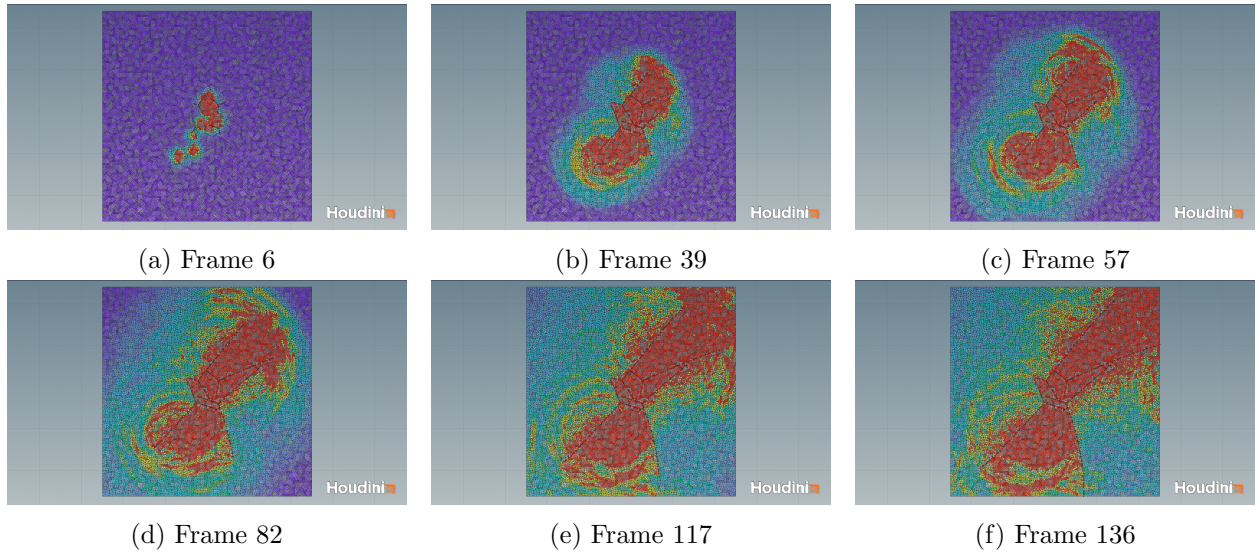


FIGURE 10.13. Dynamic crack propagation under antiplane loading with multiple propagating cracks.

CHAPTER 11

Rendered Examples

11.1. Dropped Bead

In Figure 11.1, a glass bead collides with the ground. The cracks originate from the impact location at the bottom of the bead, and travel around the surface of the bead. From the resulting pieces, we see that cracks intersected, and the bead breaks into a variety of different sized debris.

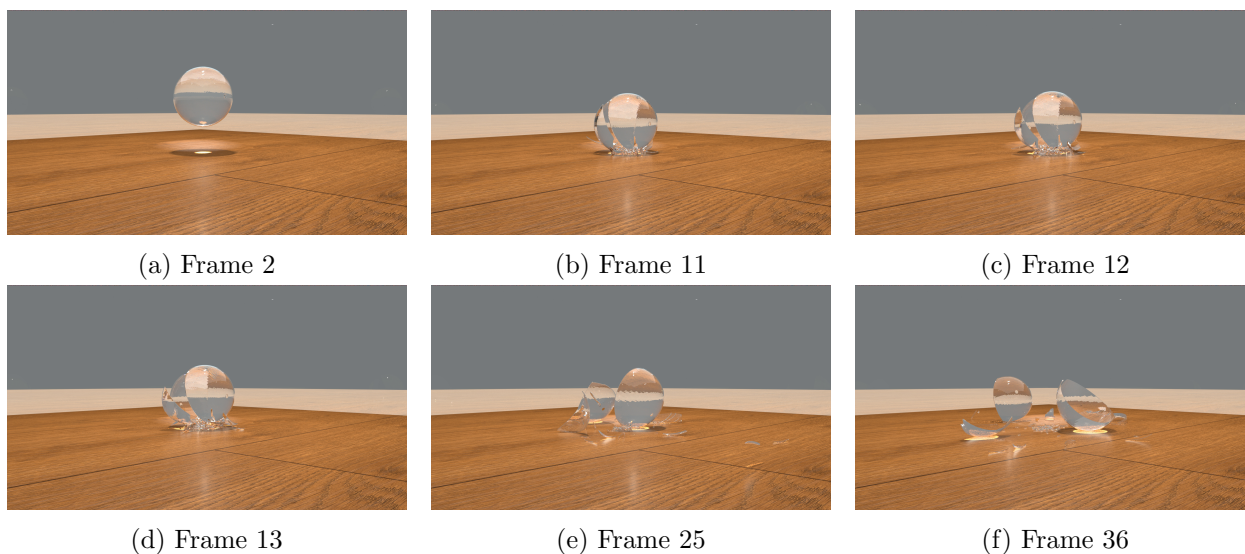


FIGURE 11.1. A glass bead dropping onto the ground.

11.2. Dropped Decanter

In Figure 11.2, a turquoise decanter hits the ground. Cracks originate from impact location at the bottom of the decanter and travel around the surface of the decanter. We see that several cracks traveled up the sides of the decanter and circled around the neck of the decanter.

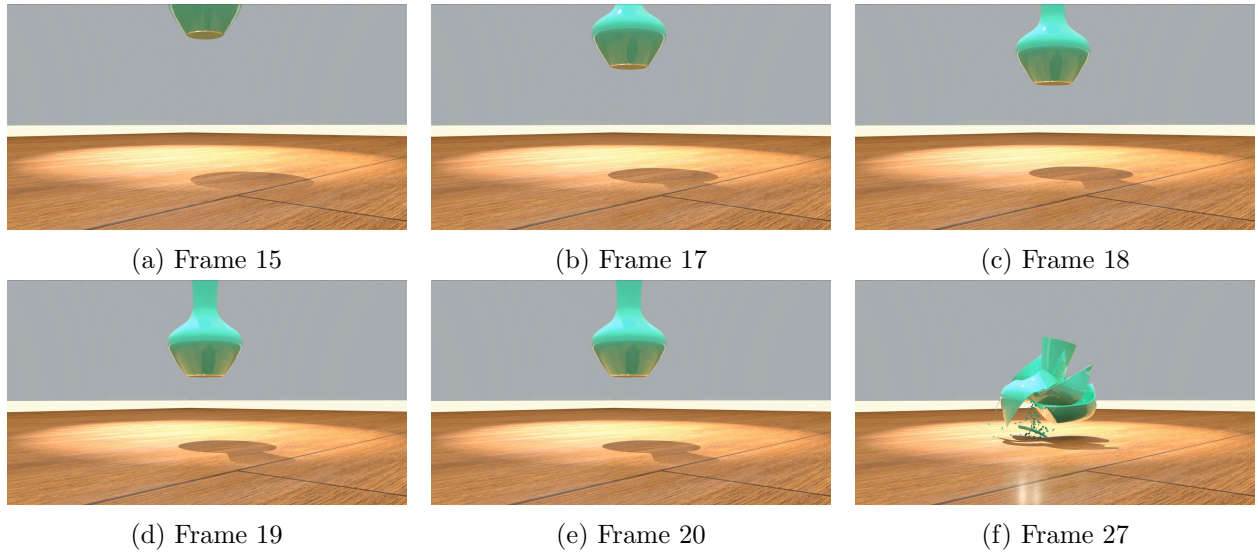


FIGURE 11.2. A decanter dropping onto the ground.

11.3. Dropped SIGGRAPH Letters

In Figure 11.3, we drop colorful SIGGRAPH letters, and each letter breaks in a unique fashion. The cracks originate from the bottom of the letters, and we see a variety of debris generated by the propagating cracks. Notice that the two identical letter G's have slightly different post-impact pieces which highlights the sensitive dependence of the initial conditions on the resulting fractured pieces.

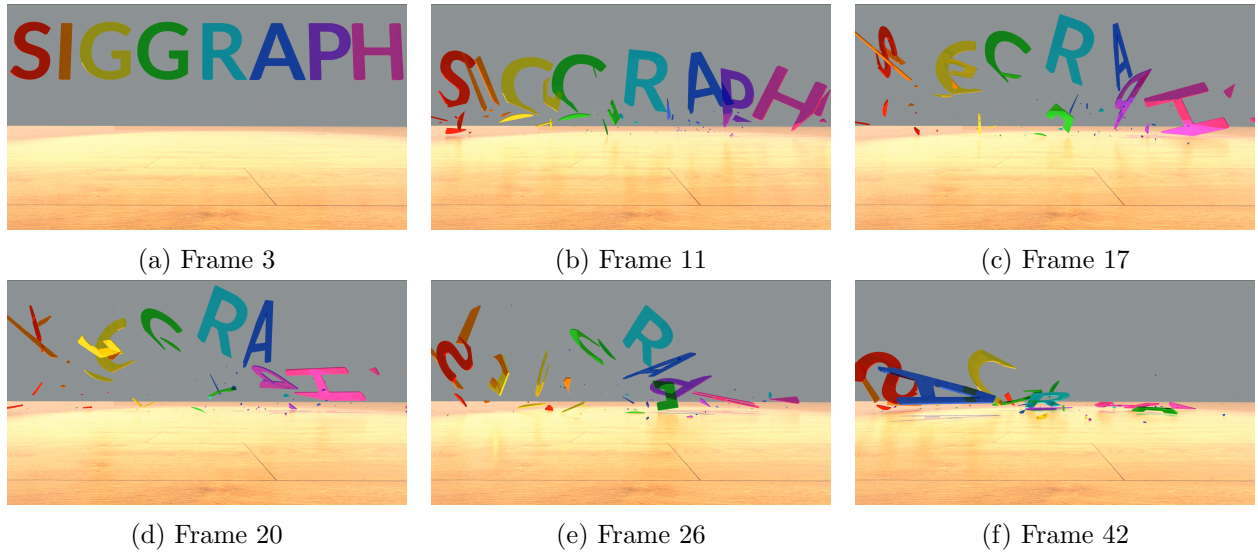


FIGURE 11.3. SIGGRAPH letters dropping onto the ground.

11.4. Projectile Through Window

In Figure 11.4, we show a ball breaking through a glass window. Fine debris is generated near the impact location, and we see that the cracks propagate radially outward from the impact location. This example shows that mesh resolution limits the size of the generated debris, but without prior knowledge of the impact location, guaranteeing fine dust debris requires a uniformly fine mesh.

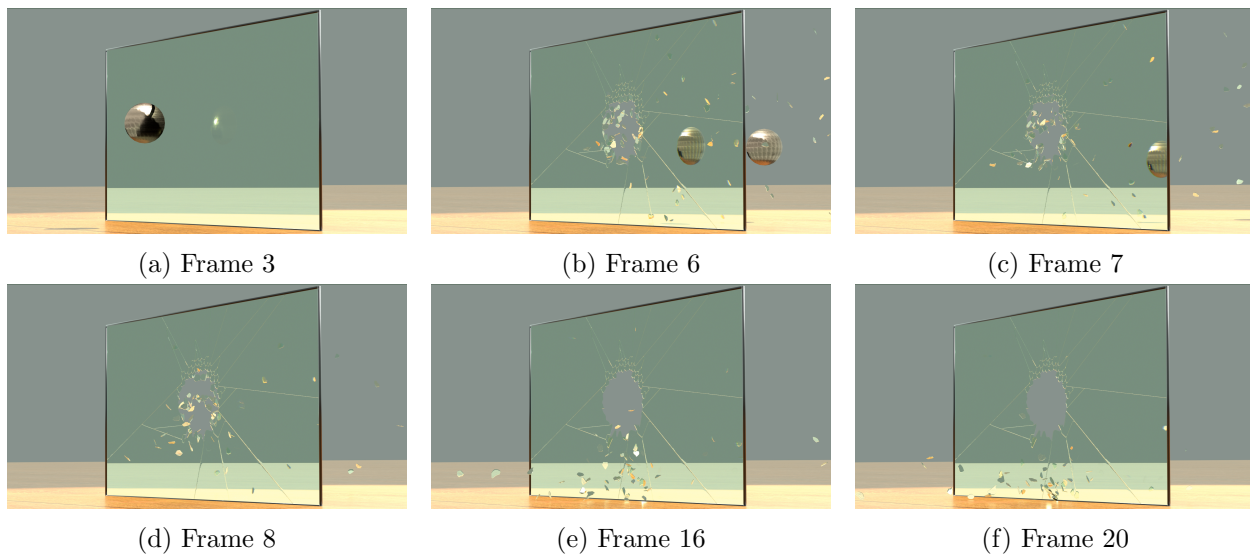


FIGURE 11.4. A ball hitting a window.

11.5. Dropped Cup

In Figure 11.5, we show a low resolution cup that is dropped. We see that several pieces are created by the impact with cracks traveling around the side of the cup. This example shows that even with a relatively low resolution mesh, visually compelling pieces are created, and finer debris can be included in a post-processing step.

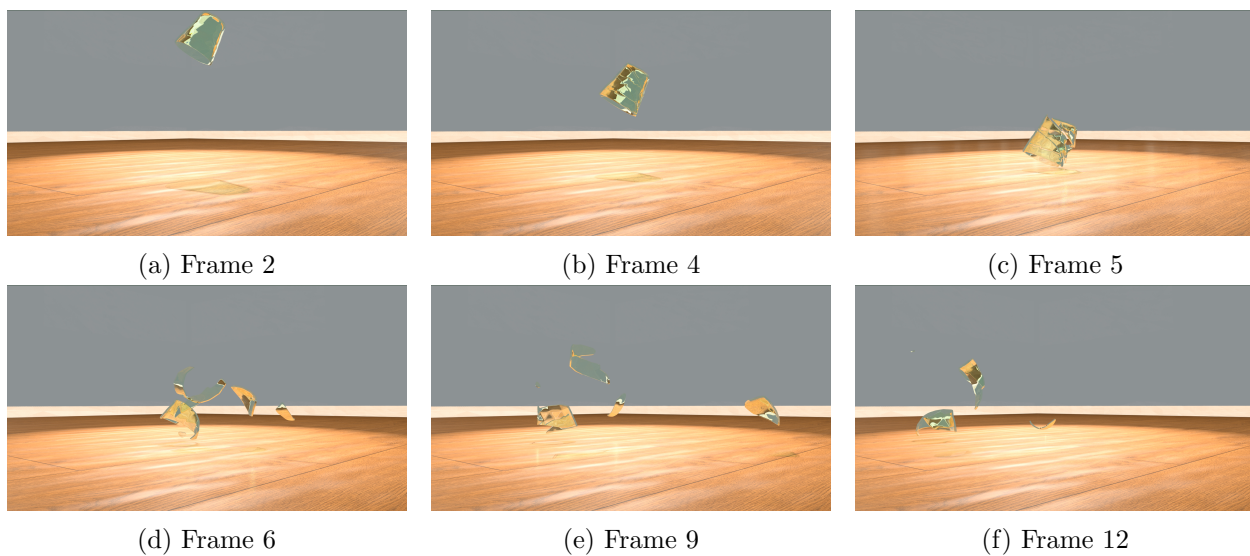


FIGURE 11.5. A cup dropping onto the ground.

CHAPTER 12

Conclusions

12.1. Conclusions

We presented a virtual element method for simulating brittle dynamic fracture of thin solids, which was verified against established LEFM benchmarks. In contrast to current state-of-the-art methods that use thin-shell theory, we adopted a 3D linear elasticity formulation and leveraged the flexibility of virtual elements on thin structures that were composed of poorly-shaped polyhedra (convex and nonconvex elements) to realize visually compelling fracture scenes. A drawback of our method is the limits of generating fine debris, but this can be overcome using an adaptive mesh refinement scheme which VEM naturally lends itself to. Cracks travel in an inherently 2D fashion, but extending this to thicker objects where out-of-plane crack propagation becomes necessary is naturally attainable with our approach by extending our splitting to tetrahedron and adopting crack models with out-of-plane propagation angles. An outstanding issue is that in areas of high curvature, the mesh extrusion process can potentially lead to self-intersections. Future avenues of study include the addition of an agglomeration technique, other material models (hyperelasticity, plasticity, strain-rate effects and thermodynamic effects), additions to the crack model (crack branching and nucleation criteria), and multiphysics couplings (fluid-structure interaction, phase changes, and electromagnetism).

Bibliography

- [1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev Spaces*, Pure and applied mathematics ; v. 140, Academic Press, Amsterdam, 2nd ed. ed., 2003.
- [2] B. AHMAD, A. ALSAEDI, F. BREZZI, L. MARINI, AND A. RUSSO, *Equivalent projectors for virtual element methods*, Comput. Math. Appl., 66 (2013), pp. 376–391.
- [3] L. B. AO DA VEIGA, F. BREZZI, AND L. MARINI, *Virtual elements for linear elasticity problems*, SIAM J. Numer. Anal., 51 (2013), pp. 794–812.
- [4] L. B. AO DA VEIGA, F. BREZZI, L. MARINI, AND A. RUSSO, *The hitchhiker’s guide to the virtual element method*, Math. Models Methods Appl. Sci., 24 (2014), pp. 1541–1573.
- [5] C. ATKINSON AND J. ESHELBY, *The flow of energy into the tip of a moving crack*, Int. J. Fract. Mech., 4 (1968), pp. 3–8.
- [6] Z. BAO, J. HONG, J. TERAN, AND R. FEDKIW, *Fracturing rigid materials*, IEEE Trans. Visual Comput. Graphics, 13 (2007), p. 370–378.
- [7] Z. BAŽANT AND L. ESTENSSORO, *Surface singularity and crack propagation*, Int. J. Solids Struct., 15 (1979), pp. 405–426.
- [8] J. BENTHEM, *State of stress at the vertex of a quarter-infinite crack in a half-space*, Int. J. Solids Struct., 13 (1977), pp. 479–492.
- [9] E. BENVENUTI, A. CHIOZZI, G. MANZINI, AND N. SUKUMAR, *Extended virtual element method for two-dimensional linear elastic fracture*, Comput. Methods Appl. Mech. Eng., 390 (2022), p. 114352.
- [10] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, Texts in applied mathematics ; 15, Springer, New York, NY, 3rd ed. ed., 2008.
- [11] O. BUSARYEV, T. DEY, AND H. WANG, *Adaptive fracture simulation of multi-layered thin plates*, ACM Trans. Graphics, 32 (2013).
- [12] W. CAO, L. LYU, X. REN, B. ZHANG, Z. YANG, AND E. WU, *Fracture patterns design for anisotropic models with the material point method*, Comput. Graphics Forum, 39 (2020), pp. 93–104.
- [13] A. CHEN AND N. SUKUMAR, *Stabilization-free virtual element method for plane elasticity*, Comput. Math. Appl., 138 (2023), pp. 88–105.
- [14] W. CHEN, F. ZHU, J. ZHAO, S. LI, AND G. WANG, *Peridynamics-based fracture animation for elastoplastic solids*, Comput. Graphics Forum, 37 (2018), pp. 112–124.

- [15] Z. CHEN, M. YAO, R. FENG, AND H. WANG, *Physics-inspired adaptive fracture refinement*, ACM Trans. Graphics, 33 (2014).
- [16] H. CHI, L. B. DA VEIGA, AND G. PAULINO, *Some basic formulations of the virtual element method (vem) for finite deformations*, Comput. Methods Appl. Mech. Eng., 318 (2017), pp. 148–192.
- [17] E. CHIN, F. DASSI, G. MANZINI, AND N. SUKUMAR, *Numerical integration in the virtual element method with the scaled boundary cubature scheme*, Int. J. Numer. Methods Eng., 125 (2024), p. e7549.
- [18] E. CHIN, J. LASSERRE, AND N. SUKUMAR, *Numerical integration of homogeneous functions on convex and nonconvex polygons and polyhedra*, Comput. Mech., 56 (2015), pp. 967 – 981.
- [19] E. CHIN AND N. SUKUMAR, *Scaled boundary cubature scheme for numerical integration over planar regions with affine and curved boundaries*, Comput. Methods Appl. Mech. Eng., 380 (2021), p. 113796.
- [20] F. M. CHITALU, Q. MIAO, K. SUBR, AND T. KOMURA, *Displacement-correlated xfem for simulating brittle fracture*, Comput. Graphics Forum, 39 (2020), pp. 569–583.
- [21] C. DICK, J. GEORGH, AND R. WESTERMANN, *A hexahedral multigrid approach for simulating cuts in deformable objects*, IEEE Trans. Visual Comput. Graphics, 17 (2011), pp. 1663–1675.
- [22] L. FAN, F. M. CHITALU, AND T. KOMURA, *Simulating brittle fracture with material points*, ACM Trans. Graph., 41 (2022).
- [23] ———, *A hybrid lagrangian–eulerian formulation of thin-shell fracture*, Comput. Graphics Forum, 44 (2025), p. e15273.
- [24] L. FREUND, *Energy flux into the tip of an extending crack in an elastic solid*, J. Elast., 2 (1972), pp. 341–349.
- [25] ———, *Dynamic Fracture Mechanics*, Cambridge Monographs on Mechanics, Cambridge University Press, Cambridge, 1990.
- [26] A. GAIN, C. TALISCHI, AND G. PAULINO, *On the virtual element method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes*, Comput. Methods Appl. Mech. Eng., 282 (2014), pp. 132–160.
- [27] O. GONZALEZ AND A. STUART, *A First Course in Continuum Mechanics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2008.
- [28] A. GRIFFITH, *Vi. the phenomena of rupture and flow in solids*, Philos. Trans. R. Soc. London, Ser. A, 221 (1921), pp. 163–198.
- [29] E. GUENDELMAN, R. BRIDSON, AND R. FEDKIW, *Nonconvex rigid bodies with stacking*, 22 (2003), p. 871–878.
- [30] D. HAHN AND C. WOJTAN, *High-resolution brittle fracture simulation with boundary elements*, ACM Trans. Graph., 34 (2015).
- [31] ———, *Fast approximations for boundary element based brittle fracture simulation*, ACM Trans. Graphics, 35 (2016).
- [32] C. HAN, T. XUE, AND M. AANJANEYA, *A lagrangian particle-based formulation for coupled simulation of fracture and diffusion in thin membranes*, Comput. Graphics Forum, 40 (2021), pp. 97–108.

- [33] J. HEGEMANN, C. JIANG, C. SCHROEDER, AND J. TERAN, *A level set method for ductile fracture*, in Proceedings of the 12th ACM SIGGRAPH/Eurographics Symposium on Computer Animation, SCA '13, New York, NY, USA, 2013, Association for Computing Machinery, p. 193–201.
- [34] T. HUGHES, *The Finite Element Method : Linear Static and Dynamic Finite Element Analysis*, Dover Publications, Mineola, NY, 2000.
- [35] A. HUSSEIN, F. ALDAKHEEL, B. HUDOBIVNIK, P. WRIGGERS, P.-A. GUIDAULT, AND O. ALLIX, *A computational framework for brittle crack-propagation based on efficient virtual element method*, Finite Elem. Anal. Des., 159 (2019), pp. 15–32.
- [36] G. IRWIN, *Analysis of stresses and strains near the end of a crack traversing a plate*, J. Appl. Mech., 24 (1957), pp. 361–364.
- [37] A. ISHITOBI, M. NAKAYAMA, AND I. FUJISHIRO, *Visual simulation of crack and bend generation in deteriorated films coated on metal objects: Combination of static fracture and position-based deformation*, Visual Comput., 39 (2023), pp. 3403–3415.
- [38] J. KALTHOFF, *Shadow optical analysis of dynamic shear fracture*, Opt. Eng., 27 (1988), p. 271035.
- [39] P. KAUFMANN, S. MARTIN, M. BOTSCH, E. GRINSPUN, AND M. GROSS, *Enrichment textures for detailed cutting of shells*, in ACM SIGGRAPH 2009 Papers, SIGGRAPH '09, New York, NY, USA, 2009, Association for Computing Machinery.
- [40] D. KOSCHIER, S. LIPPONER, AND J. BENDER, *Adaptive tetrahedral meshes for brittle fracture simulation*, in Proceedings of the ACM SIGGRAPH/Eurographics Symposium on Computer Animation, SCA '14, Goslar, DEU, 2015, Eurographics Association, p. 57–66.
- [41] Y. J. LEE AND L. B. FREUND, *Fracture initiation due to asymmetric impact loading of an edge cracked plate*, J. Appl. Mech., 57 (1990), pp. 104–111.
- [42] T. LEJEMBLE, A. FONDEVILLA, N. DURIN, T. BLANC-BEYNE, C. SCHRECK, P.-L. MANTEAUX, P. KRY, AND M. CANI, *Interactive procedural simulation of paper tearing with sound*, in Proceedings of the 8th ACM SIGGRAPH Conference on Motion in Games, MIG '15, New York, NY, USA, 2015, Association for Computing Machinery, p. 143–149.
- [43] J. LEVINE, A. BARGTEIL, C. CORSI, J. TESSENDORF, AND R. GEIST, *A peridynamic perspective on spring-mass fracture*, in Proceedings of the ACM SIGGRAPH/Eurographics Symposium on Computer Animation, SCA '14, Goslar, DEU, 2015, Eurographics Association, p. 47–55.
- [44] C. LI, C. WANG, AND H. QIN, *Novel adaptive sph with geometric subdivision for brittle fracture animation of anisotropic materials*, Visual Comput., 31 (2015), pp. 937–946.
- [45] F. LI, C. SHIH, AND A. NEEDLEMAN, *A comparison of methods for calculating energy release rates*, Eng. Fract. Mech., 21 (1985), pp. 405–421.
- [46] X. LI, S. ANDREWS, B. JONES, AND A. BARGTEIL, *Energized rigid body fracture*, Proc. ACM Comput. Graph. Interact. Tech., 1 (2018).

- [47] S. LIU, M. WU, C. JIANG, AND Y. ZHANG, *An incompressible crack model for volume preserving mpm fracture*, Proc. ACM Comput. Graph. Interact. Tech., 8 (2025).
- [48] A. LOVE, *On the small free vibrations and deformations of elastic shells*, Philos. Trans. R. Soc. Lond., 179 (1888), pp. 491–546.
- [49] J. LU, C. LI, G. CAO, AND S. HU, *Simulating fractures with bonded discrete element method*, IEEE Trans. Visual Comput. Graphics, 28 (2022), pp. 4810–4824.
- [50] J.-M. LU, G. CAO, C. LI, AND S.-M. HU, *Implicit bonded discrete element method with manifold optimization*, ACM Trans. Graphics, 44 (2025).
- [51] A. MANDAL, *Fast remeshing-free methods for complex cutting and fracture simulation*, in SIGGRAPH Asia 2023 Doctoral Consortium, SA '23, New York, NY, USA, 2023, Association for Computing Machinery.
- [52] A. MANDAL, P. CHAUDHURI, AND S. CHAUDHURI, *Simulating fracture in anisotropic materials containing impurities*, in Proceedings of the 15th ACM SIGGRAPH Conference on Motion, Interaction and Games, MIG '22, New York, NY, USA, 2022, Association for Computing Machinery.
- [53] S. MARTIN, P. KAUFMANNR, M. BOTSCH, M. WICKE, AND M. GROSS, *Polyhedral finite elements using harmonic basis functions*, Comput. Graphics Forum, 27 (2008), pp. 1521–1529.
- [54] T. MCGRAW, *Gram-schmidt voxel constraints for real-time destructible soft bodies*, in Proceedings of the 17th ACM SIGGRAPH Conference on Motion, Interaction, and Games, MIG '24, New York, NY, USA, 2024, Association for Computing Machinery.
- [55] T. MCGRAW AND X. ZHOU, *Real-time voxelized mesh fracture with gram-schmidt constraints*, Comput. Graphics, 132 (2025), p. 104382.
- [56] P. MESEURE, X. SKAPIN, E. DARLES, AND G. DELAITRE, *Accelerated simulation of brittle objects for interactive applications*, in VISIGRAPP Conference proceedings, vol. Volume 1 : GRAPP of VISIGRAPP Conference proceedings, Funchal (Madère), Portugal, Jan. 2018, pp. 28–39. GRAPP2018 Best Paper award.
- [57] R. MINDLIN, *Influence of rotary inertia and shear on flexural motions of isotropic elastic plates*, J. Appl. Mech., 18 (1951), pp. 31–38.
- [58] N. MOLINO, Z. BAO, AND R. FEDKIW, *A virtual node algorithm for changing mesh topology during simulation*, ACM Trans. Graph., 23 (2004), pp. 385–392.
- [59] M. MÜLLER, N. CHENTANEZ, AND T.-Y. KIM, *Real time dynamic fracture with volumetric approximate convex decompositions*, ACM Trans. Graphics, 32 (2013).
- [60] Y. NAJIMN, G. TRIANTAFYLIDIS, AND G. PALAMAS, *Dynamic fracturing of 3d models for real time computer graphics*, in 2018 - 3DTV-Conference: The True Vision - Capture, Transmission and Display of 3D Video (3DTV-CON), 2018, pp. 1–4.
- [61] T. NAKAMURA, C. SHIH, AND L. FREUND, *Computational methods based on an energy integral in dynamic fracture*, Int. J. Fract., 27 (1985), pp. 229–243.

- [62] N. M. NEWMARK, *A method of computation for structural dynamics*, Journal of the Engineering Mechanics Division, 85 (1959), pp. 67–94.
- [63] V. NGUYEN-THANH, X. ZHUANG, H. NGUYEN-XUAN, T. RABCZUK, AND P. WRIGGERS, *A virtual element method for 2d linear elastic fracture analysis*, Comput. Methods Appl. Mech. Eng., 340 (2018), pp. 366–395.
- [64] J. O’BRIEN, A. BARGTEIL, AND J. HODGINS, *Graphical modeling and animation of ductile fracture*, ACM Trans. Graph., 21 (2002), p. 291–294.
- [65] J. O’BRIEN AND J. HODGINS, *Graphical modeling and animation of brittle fracture*, in Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques, SIGGRAPH ’99, USA, 1999, ACM Press/Addison-Wesley Publishing Co., p. 137–146.
- [66] S. OH, S. SHIN, AND H. JUN, *Practical simulation of hierarchical brittle fracture*, Comput. Anim. Virtual Worlds, 23 (2012), pp. 291–300.
- [67] T. PFAFF, R. NARAIN, J. DE JOYA, AND J. O’BRIEN, *Adaptive tearing and cracking of thin sheets*, ACM Trans. Graphics, 33 (2014).
- [68] M. RAHMAN AND T. MICHELITSCH, *A note on the formula for the rayleigh wave speed*, Wave Motion, 43 (2006), pp. 272–276.
- [69] I. RAJU AND J. NEWMAN, *Three dimensional finite-element analysis of finite-thickness fracture specimens*, Technical report NASA-TN-D-8414, NASA Langley Research Center, Hampton, VA 23665, June 1977.
- [70] E. REISSNER, *The effect of transverse shear deformation on the bending of elastic plates*, J. Appl. Mech., 12 (1945), pp. A69–A77.
- [71] J. RICE, *A path independent integral and the approximate analysis of strain concentration by notches and cracks*, Journal of Applied Mechanics, 35 (1968), pp. 379–386.
- [72] H. RICHARD, M. FULLAND, AND M. SANDER, *Theoretical crack path prediction*, Fatigue Fract. Eng. Mater. Struct., 28 (2005), pp. 3–12.
- [73] C. SCHRECK, D. ROHMER, AND S. HAHMANN, *Interactive paper tearing*, Comput. Graphics Forum, 36 (2017), pp. 95–106.
- [74] S. SELLÁN, J. LUONG, L. M. D. SILVA, A. RAMAKRISHNAN, Y. YANG, AND A. JACOBSON, *Breaking good: Fracture modes for realtime destruction*, ACM Trans. Graph., 42 (2023).
- [75] H. TADA, P. PARIS, AND G. IRWIN, *The Stress Analysis of Cracks Handbook*, ASME Press, New York, 3rd ed. ed., 2000.
- [76] Q. TON-THAT, P. KRY, AND S. ANDREWS, *Generalized extended finite element method for deformable cutting via boolean operations*, Comput. Graphics Forum, 43 (2024), p. e15184.
- [77] C. TRUESDELL, *A First Course in Rational Continuum Mechanics*, Pure and applied mathematics, a series of monographs and textbooks ; v. 71 pt. 1, Academic Press, New York, 1977.
- [78] L. B. A. D. VEIGA, F. BREZZI, A. CANGIANI, G. MANZINI, L. MARINI, AND A. RUSSO, *Basic principles of virtual element methods*, Math. Models Methods Appl. Sci., 23 (2013), pp. 199–214.

- [79] S. WANG, M. DING, T. GAST, L. ZHU, S. GAGNIERE, C. JIANG, AND J. TERAN, *Simulation and visualization of ductile fracture with the material point method*, Proc. ACM Comput. Graph. Interact. Tech., 2 (2019).
- [80] P. WAPPLER, K. SCHMITZ, AND A. RICOEUR, *The virtual element method for dynamic crack analysis*, Int. J. Fract., 250 (2026), p. 19.
- [81] M. WICKE, M. BOTSCH, AND M. GROSS, *A finite element method on convex polyhedra*, Comput. Graphics Forum, 26 (2007), pp. 355–364.
- [82] M. WILLIAMS, *On the stress distribution at the base of a stationary crack*, Journal of Applied Mechanics, 24 (1957), pp. 109–114.
- [83] J. WOLPER, Y. FANG, M. LI, J. LU, M. GAO, AND C. JIANG, *Cd-mpm: continuum damage material point methods for dynamic fracture animation*, ACM Trans. Graphics, 38 (2019).
- [84] J. WRETBORN, R. ARMIENTO, AND K. MUSETH, *Animation of crack propagation by means of an extended multi-body solver for the material point method*, Comput. Graphics, 69 (2017), pp. 131–139.
- [85] B.-B. XU AND P. WRIGGERS, *3d stabilization-free virtual element method for linear elastic analysis*, 2024.
- [86] J. YAU, S. WANG, AND H. CORTEN, *A mixed-mode crack analysis of isotropic solids using conservation laws of elasticity*, Journal of Applied Mechanics, 47 (1980), pp. 335–341.
- [87] J. ZHANG, F. DUAN, M. ZHOU, D. JIANG, X. WANG, Z. WU, Y. HUANG, G. DU, S. LIU, P. ZHOU, AND X. SHANG, *Stable and realistic crack pattern generation using a cracking node method*, Front. Comput. Sci., 12 (2018), pp. 777–797.
- [88] Z. ZHAO, K. HUANG, C. LI, C. WANG, AND H. QIN, *A novel plastic phase-field method for ductile fracture with gpu optimization*, Comput. Graphics Forum, 39 (2020), pp. 105–117.
- [89] Y. ZHU, R. BRIDSON, AND C. GREIF, *Simulating rigid body fracture with surface meshes*, ACM Trans. Graphics, 34 (2015).