The Spectra of Cyclic Elements in Simple Lie Algebras

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Abstract

In this thesis, we study cyclic elements in complex finite-dimensional simple Lie algebras. These are significant both in Lie theory and in physical applications such as affine Toda field theory and integrable hierarchies of PDEs. Rather than relying on explicit matrix realizations, this work employs a conceptual approach based on the Cartan-Coxeter correspondence.

Specifically, we focus on the adjoint representation and a fundamental representation of minimal dimension (namely, the fourth for $\mathfrak{f}_4$, the seventh for $\mathfrak{e}_7$, the eighth for $\mathfrak{e}_8$, and the first for the remaining types) of simple Lie algebras. We derive explicit trigonometric expressions for the spectra of cyclic elements. The obtained spectra are expressed as unions of rotationally symmetric configurations in the complex plane, with their radii given by trigonometric formulas. Furthermore, we show that, up to scaling, the cyclic spectrum of the minimal fundamental representation listed above is always contained in that of the adjoint representation.

Finally, we explore the behavior of these spectra with respect to the folding procedure of simple Lie algebras. We provide a conceptual proof demonstrating that the adjoint spectrum of a folded, non-simply laced Lie algebra is contained within the adjoint spectrum of the original, simply laced one.

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1 Introduction

The main focus of this thesis are cyclic elements in complex simple Lie algebras. They play an important role in many aspects of Lie theory, but their significance also extends beyond Lie theory itself. For example, cyclic elements are a basic ingredient in various quantum field theories, such as affine Toda field theory. In [12], Fring, Liao, and Olive use properties of cyclic elements to derive the affine Toda mass spectrum in terms of the Perron–Frobenius eigenvector of the Cartan matrix.

Furthermore, cyclic elements form a crucial part in the study of integrable hierarchies of PDEs. A key example is the work of Drinfeld and Sokolov [9] on generalizations of the KdV hierarchy. Their construction employs a cyclic element. For a more recent generalization, see [10], where Elashvili, Jibladze, and Kac study generalized cyclic elements in semisimple Lie algebras and give a classification of such elements. They also construct an associated integrable hierarchy of Drinfeld–Sokolov type.

In this thesis, we use the techniques introduced in [12] to study the spectra of cyclic elements with respect to suitable representations of simple Lie algebras. We focus on the fundamental representation of minimal dimension (namely, the fourth for $\mathfrak{f}_4$, the seventh for $\mathfrak{e}_7$, the eighth for $\mathfrak{e}_8$, and the first for the remaining types) and the adjoint representation. Using the Cartan–Coxeter correspondence, we obtain explicit trignometric expressions for these spectra. Moreover, we investigate how the spectra of cyclic elements behave with respect to the folding procedure of simple Lie algebras. In doing so, we find an intriguing containment relation between the cyclic spectra of the two simple Lie algebras involved.

To illustrate some of our results, Figures 1 and 2 show the cyclic spectra of the first fundamental representation and adjoint representation of $\mathfrak{e}_6$. Note that in [18], Kostant studies the Toda mass spectrum of $\mathfrak{e}_8$ from a geometric point of view and discusses the Gosset circles. In Figure 3, we draw the corresponding Gosset circles for $\mathfrak{e}_6$ and annotate the pairings with simple roots on the diagram.

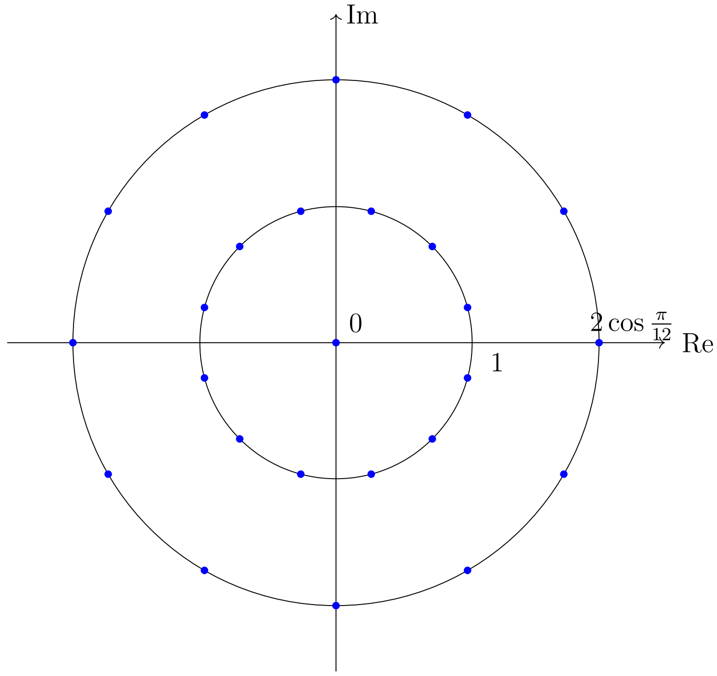


Figure 1: First fundamental cyclic spectrum for $\mathfrak{e}_6$

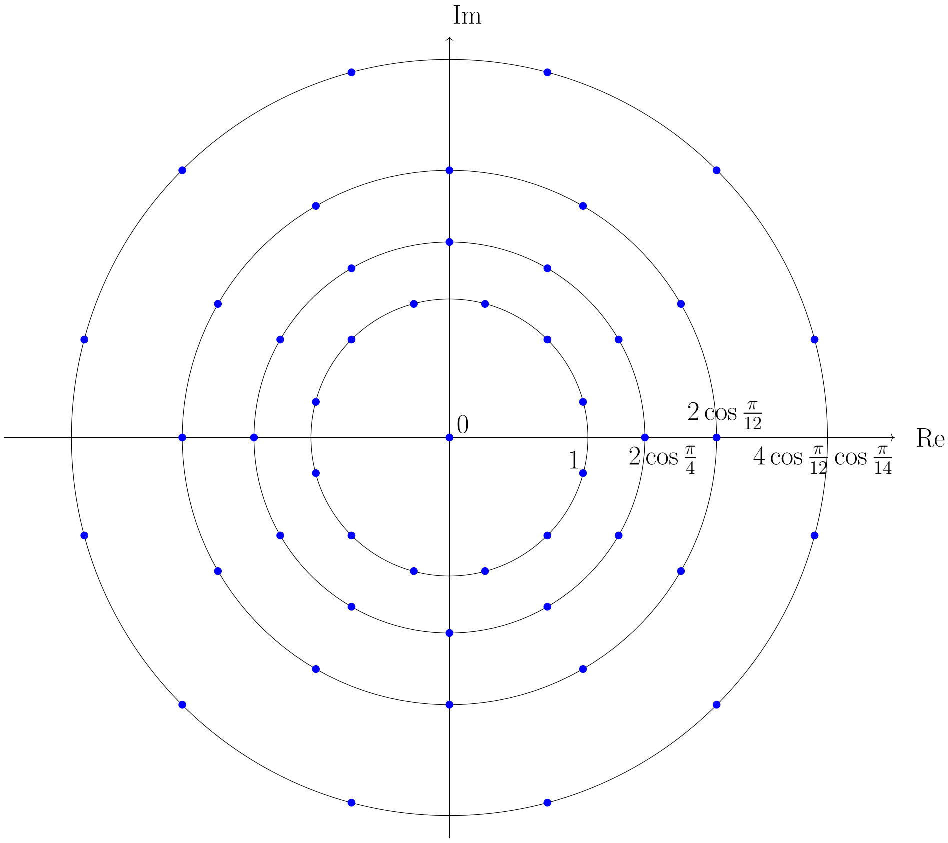


Figure 2: Adjoint cyclic spectrum for $\mathfrak{e}_6$

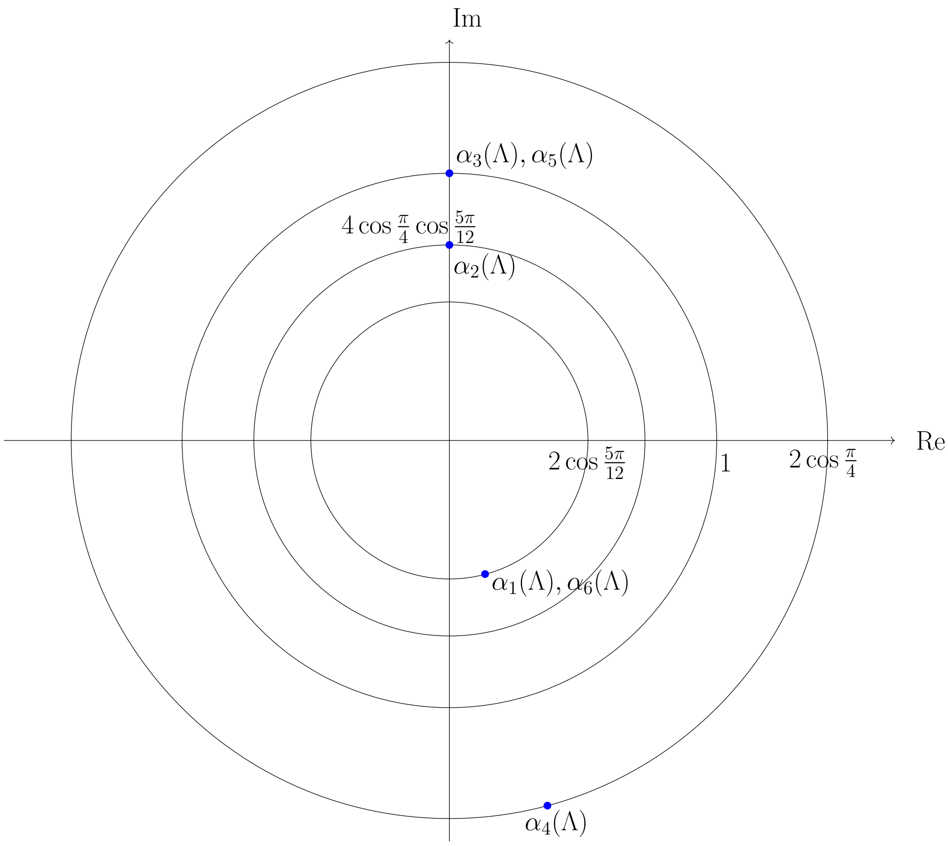


Figure 3: Gosset circles of $\mathfrak{e}_6$

1.1 Background

One of the fundamental building blocks in the theory of finite-dimensional complex Lie algebras is the class of simple Lie algebras. A Lie algebra is called simple if it is non-abelian and possesses no non-trivial proper ideals. Such Lie algebras play a crucial role in Lie theory, as they serve as the irreducible components in the decomposition of more general Lie algebras.

The classification of finite-dimensional complex simple Lie algebras was completed in the late 19th and early 20th centuries, in the work of Wilhelm Killing and Élie Cartan. In 1947, Eugene Dynkin developed the classification according to the geometry of the root systems and their associated Dynkin diagrams, which encode the angles and lengths between simple roots.

Their results reveal that every simple Lie algebra falls into one of two categories: the classical types or the exceptional types. The classical simple Lie algebras consist of four families A_n, B_n, C_n, D_n, which correspond to the special linear algebras, $\mathfrak{sl}_{n+1}\mathbb{C}$, the odd orthogonal algebras, $\mathfrak{so}_{2n+1}\mathbb{C}$, the symplectic algebras, $\mathfrak{sp}_{2n}\mathbb{C}$, and the even orthogonal algebras, $\mathfrak{so}_{2n}\mathbb{C}$. Additionally, there are five exceptional Lie algebras, denoted by $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$.

To understand the structure of these simple Lie algebras, we recall their root decompositions. For a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ with rank r, fix a Cartan subalgebra $\mathfrak{h}$. Let Γ be the corresponding root system, and let $\Gamma_+ \sqcup \Gamma_-$ be a root decomposition as a union of positive and negative roots. Denote the root subspace with respect to a root α by $\mathfrak{g}_\alpha$. Then:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{where} \quad \mathfrak{n}_\pm = \bigoplus_{\alpha\in\Gamma_\pm} \mathfrak{g}_\alpha$$

Choose non-zero elements $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}$, and $h_i = [e_i, f_i]$ for $i = 1, \cdots, r$. Then, $e_1, \cdots, e_r$ generate $\mathfrak{n}_+$, $f_1, \cdots, f_r$ generate $\mathfrak{n}_-$, $h_1, \cdots, h_r$ form a basis of $\mathfrak{h}$, so that $\{e_i, f_i, h_i\}_{i=1,\cdots,r}$ generate $\mathfrak{g}$.

Using these generators, we now turn to the main objects of this thesis, cyclic elements in simple Lie algebras. In the 1950s, Kostant in [17] defined cyclic elements in a simple Lie algebra and studied their important properties. The following is one way to define them.

Definition 1.1. *Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank r. Suppose $\{e_i, f_i, h_i\}_{1\le i\le r}$ is a set of generators of $\mathfrak{g}$, and suppose F is a non-zero element in the lowest root space. Then, the cyclic element Λ corresponding to this Lie theoretic choice is:*

$$\Lambda = \sum_{i=1}^{r} e_i + F$$

Now, we go over some examples of realizing Λ in a concrete matrix form.

Example 1.1. Consider an example of a cyclic element for a simple algebra of type A_{n-1}

in its standard representation, $\mathfrak{sl}_n$, the $n \times n$ trace-zero matrices. We can take $e_i = E_{i,i+1}$ for $1 \leq i \leq n-1$ (where $E_{i,j}$ is the matrix with 0's everywhere except a 1 at the (i,j) entry) and $f_i = E_{i+1,i}$ as well as $h_i = [e_i, f_i] = E_{i,i} - E_{i+1,i+1}$. It is known that F is non-zero multiple of $E_{n,1}$, therefore, one choice to realize a cyclic element is:

$$\Lambda = \sum_{i=1}^{n-1} E_{i,i+1} + E_{n,1} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix} \tag{1}$$

Scaling each e_i gives different generators of the root spaces, so the cyclic element is not unique.

Example 1.2. Now we consider another example of a cyclic element for a simple algebra of type D_n in its realization of even orthogonal matrices, $\mathfrak{so}_{2n}$. Consider the case when $n = 4$. From [16, Appendix A.4] by Kirillov, one can pick a set of generators as

$$\begin{aligned} e_1 &= E_{1,2} - E_{6,5} \quad & e_2 &= E_{2,3} - E_{7,6} \\ e_3 &= E_{3,4} - E_{8,7} \quad & e_4 &= E_{3,8} - E_{4,7} \end{aligned}$$

with

$$\begin{aligned} f_1 &= E_{2,1} - E_{5,6} \quad & f_2 &= E_{3,2} - E_{6,7} \\ f_3 &= E_{4,3} - E_{7,8} \quad & f_4 &= E_{7,4} - E_{8,3} \end{aligned}$$

such that $h_i = [e_i, f_i]$ are as follows

$$\begin{aligned} h_1 &= H_1 - H_2 \quad & h_2 &= H_2 - H_3 \\ h_3 &= H_3 - H_4 \quad & h_4 &= H_3 + H_4 \end{aligned}$$

where $H_i = E_{i,i} - E_{i+4,i+4}$ for $i = 1, \cdots, 4$. It is known that F is a non-zero multiple of $E_{5,2} - E_{6,1}$. Therefore, one choice to realize a cyclic element is:

$$\Lambda = \sum_{i=1}^{4} e_i + F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{2}$$

Similarly, we can also scale e_i's to obtain different cyclic elements.

As defined in Definition (1.1), Λ is not unique, but Kostant proves in [17, Theorem 6.2] that if Λ and Λ' are two different cyclic elements, then there exists a non-zero constant c such that Λ and $c \cdot \Lambda'$ are conjugate to each other. Hence, the spectrum of a cyclic element is well-defined in the following sense:

Corollary 1.1. *For $\mathfrak{g}$ as before, the spectrum of Λ is independent of the choice of the cyclic element up to a non-zero overall scaling.*

In this thesis, we study the spectra of cyclic elements with respect to various finite-dimensional representations of $\mathfrak{g}$. Specifically, with respect to the minimal fundamental representations listed before and the adjoint representations.

Naively, upon representing a cyclic element by a matrix as in Example 1.1 and Example 1.2, one can directly calculate its eigenvalues as in the following examples.

Example 1.3. Let us return to the discussion concerning about the Lie algebra of type A_{n-1} in Example 1.1. It is known that the first fundamental representation of type A_{n-1} is simply its standard representation. Therefore, from Equation (1), we can compute eigenvalues of Λ as follows:

$$|\Lambda - \lambda I| = \begin{vmatrix} -\lambda & 1 & & & & \\ 0 & -\lambda & 1 & & & \\ 0 & & -\lambda & 1 & & \\ \vdots & & & \ddots & \ddots & \\ 0 & \cdots & & 0 & -\lambda & 1 \\ 1 & 0 & \cdots & & 0 & -\lambda \end{vmatrix} = 0$$

Expanding the first column yields:

$$|\Lambda - \lambda I| = (-\lambda)^n + (-1)^{n+1} = 0$$

Hence, the spectrum is

$$\{1, \zeta_n, \cdots, \zeta_n^{n-1}\}$$

for $\zeta_n = e^{\frac{2\pi i}{n}}$.

Example 1.4. Now we continue the discussion concerning the Lie algebra of type D_4 in Example 1.2. Similar to the A_{n-1} example, the first fundamental representation of type D_4 is also its standard representation. Therefore, we can compute eigenvalues of Λ directly from Equation (2).

Observe that Λ has repeated rows, so zero is an eigenvalue. Now assume λ is a non-zero eigenvalue of Λ. Suppose $\vec{\boldsymbol{v}} = [v_1, v_2, \cdots, v_8]^\intercal$ is an eigenvector of λ. Then, by the equation:

$$\Lambda \vec{\boldsymbol{v}} = \lambda \vec{\boldsymbol{v}}$$

We obtain the following equations:

$$v_2 = \lambda v_1$$
$$v_3 = \lambda v_2$$

$$
\begin{aligned}
v_4 + v_8 &= \lambda v_3 \\
-v_7 &= \lambda v_4 \\
v_2 &= \lambda v_5 \\
-v_1 - v_5 &= \lambda v_6 \\
-v_6 &= \lambda v_7 \\
-v_7 &= \lambda v_8
\end{aligned}
$$

Solving these equations yields the consistency equation:

$$\frac{-4v_1}{\lambda^3} = \lambda^3 v_1$$

Since the eigenvector is non-zero, so $v_1 \neq 0$, therefore we have the characteristic polynomial for λ:

$$\lambda^6 = -4 \implies \lambda = \sqrt[6]{-4} \cdot e^{\frac{2k\pi i}{6}} \quad \text{for } k = 1, \cdots, 6$$

Now consider the case when the eigenvalue $\lambda = 0$. Similarly, solving the eigenvector equations yields:

$$
\begin{aligned}
v_2 = v_3 = v_6 = v_7 &= 0 \\
v_1 + v_5 &= 0 \\
v_4 + v_8 &= 0
\end{aligned}
$$

Hence, the eigenspace of the eigenvalue zero is spanned by vectors:

$$[1, 0, 0, 0, -1, 0, 0, 0]^\intercal; \quad [0, 0, 0, 1, 0, 0, 0, -1]^\intercal$$

So the multiplicity of the eigenvalue zero is 2. Let $\zeta_6 = e^{\frac{2\pi i}{6}}$, and scale all the eigenvalues to cancel the constant $\sqrt[6]{-4}$, we have the spectrum of Λ for type D_4 under its standard representation is:

$$\{1, \zeta_6, \zeta_6^2, \zeta_6^3, \zeta_6^4, \zeta_6^5, 0, 0\}$$

For the classical Lie algebras, one can obtain cyclic spectra via such explicit calculations, in principle. However, the matrix representations of Λ are harder to attain for the exceptional Lie algebras. Hence, a conceptual approach that can bypass the necessity to realize Λ as a matrix is more desirable.

Fortunately, the cyclic element has some useful Lie theoretical properties, which give us a chance to achieve that. The key is to relate the cyclic element to the Coxeter element in the Weyl group of $\mathfrak{g}$. Recall the definition of the Coxeter element:

Definition 1.2. *Let r denote the rank of $\mathfrak{g}$ and suppose $\mathcal{A} = \{\alpha_1, \cdots, \alpha_r\}$ is an ordered set of simple roots of $\mathfrak{g}$. Simple roots generate the root space V equipped with an inner product. Denote the reflection about the hyperplane orthogonal to the simple root α_j by S_j such that for a simple root α_i*

$$S_j(\alpha_i) = \alpha_i - \left(\frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}\right)\alpha_j$$

The group W generated by $\{S_i\}_{i=1,\cdots,r}$ is called the Weyl group. The element in the form

$$\sigma = \prod_{i=1}^{r} S_i \tag{3}$$

is called a Coxeter element.

While there are many different Coxeter elements, it follows from [8, p. 765] that different Coxeter elements are conjugates in the Weyl group.

To explore the relation between the Coxeter element and the cyclic element, we turn to the work of Kostant. In [17, Lemma 6.4B], Kostant shows that the centralizer of a cyclic element is a Cartan subalgebra. In Equation (6.7.1) of the same reference, he showed that for a Coxeter element acting on that Cartan subalgebra, the cyclic element is its eigenvector with the eigenvalue $e^{\frac{2\pi i}{h}}$ with h the Coxeter number. Recall the definition of the Cartan matrix:

Definition 1.3. *Let $\mathfrak{g}$ be a simple Lie algebra of rank r and let $\mathcal{A} = \{\alpha_1, \cdots, \alpha_r\}$ denote an ordered set of simple roots of $\mathfrak{g}$. Define the Cartan matrix C of $\mathfrak{g}$ by*

$$C_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2} \tag{4}$$

for $i, j = 1, \cdots, r$.

The linear algebra of a Cartan matrix is easy to study. So, if a correspondence between the eigenspaces of the Coxeter element and the Cartan matrix can be built, then it opens a window to perceive the eigen-properties of the cyclic element.

This correspondence is called the Cartan-Coxeter correspondence. In the 1950s, Coxeter in [8] studied the product of the generators of a finite group generated by reflections, and he showed that the linear algebra of the Coxeter element and Cartan matrices are related. Referring to Coxeter's research in the 1950s, Coleman in [7] used this correspondence to obtain the eigenvalues and eigenvectors of the Coxeter element, which helped to develop an easier way to compute the Betti numbers of the simple compact Lie groups.

Beyond the intrinsic algebraic interest, there are also exciting applications of this correspondence in physics. In 1991, Fring, Liao, and Olive in [12] showed that by using this correspondence, a cyclic element can be expressed in terms of the Perron-Frobenius eigenvector of the Cartan matrix. They used this property to build the relation between the masses of affine Toda theories and components of the Perron-Frobenius eigenvector of the Cartan matrix. Also, Freeman in [11] gave an alternative proof of the same results. Additionally, Brillon and Schechtman in [5] used this correspondence as a main tool to construct a family of Hermitian operators whose eigenvalues are equal to the coordinates of the eigenvectors of the Cartan matrix of a simple Lie algebra.

Now we introduce this Correspondence, which is the main tool in this thesis. Coxeter in [8, Equation (1.7)] and Coleman in [7, Lemma 1] represent the action of the Coxeter element on the root space by a matrix, which turns out to be closely related to the Cartan matrix.

Theorem 1.1 (Coxeter, Coleman)**.** *Let r be the rank of $\mathfrak{g}$, and let $\mathcal{A} = \{\alpha_1, \cdots, \alpha_r\}$ be an ordered set of simple roots of $\mathfrak{g}$. Let C be the Cartan matrix of $\mathfrak{g}$, and let the matrix U be defined as*

$$U_{ij} = \begin{cases} C_{ij} & \text{for } i < j \\ 1 & \text{for } i = j \\ 0 & \text{for } i > j \end{cases}$$

Let the matrix L be defined as

$$L_{ij} = \begin{cases} 0 & \text{for } i < j \\ 1 & \text{for } i = j \\ C_{ij} & \text{for } i > j \end{cases}$$

so that

$$L + U = C$$

Then, the matrix $R_{\mathcal{A},\mathcal{A}}$ representing the Coxeter element σ with respect to the basis $\mathcal{A}$ is:

$$R_{\mathcal{A},\mathcal{A}} = -(L^{\intercal})^{-1} \cdot U^{\intercal}$$

In Chapter 2, we revisit the proof of this theorem. It is the key ingredient in the proof of the Cartan-Coxeter correspondence.

Theorem 1.2 (Coxeter, Coleman)**.** *Let r be the rank of $\mathfrak{g}$, and let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_r\}$ be an ordered set of its simple roots. Denote the Coxeter number of $\mathfrak{g}$ by h, and let $R_{\mathcal{A},\mathcal{A}}$ be the matrix representing a Coxeter element σ, as defined earlier. The eigenvalues of $R_{\mathcal{A},\mathcal{A}}$ are the h'th roots of unity, given by ζ_h^k for $1 \leq k \leq h-1$. If λ_σ is an arbitrary eigenvalue* *of $R_{\mathcal{A},\mathcal{A}}$, then*

$$\lambda_C \;=\; 2 - (\lambda_\sigma^{1/2} + \lambda_\sigma^{-1/2})$$

is an eigenvalue of the Cartan matrix C. Furthermore, if $\vec{\boldsymbol{w}} = (w_1, \cdots, w_r)^\intercal$ is an eigenvector of λ_σ, then there exists a left eigenvector $\vec{\boldsymbol{v}} = (v_1, \cdots, v_r)^\intercal$ of λ_C and a function $\pi : \{1, \cdots, r\} \to \frac{\mathbb{Z}}{2}$, such that for every $i \in \{1, \cdots, r\}$,

$$w_i \;=\; \zeta_h^{\pi(i)} \cdot v_i$$

In particular, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ differ by the phases: $\zeta_h^{\pi(i)}$ for $i = 1, \cdots, r$.

Freeman proved this theorem (though some details are left to the reader) in [11] by utilizing the geometric property of the Dynkin diagram that it has no closed loop. We revisit his proof in Section 2.1. In Section 2.2, we revisit another proof given by Fring, Liao, and Olive in [12] by bi-coloring the Dynkin diagram such that adjacent nodes in the diagram are of different colors. Precisely, a bi-coloring of a Dynkin diagram can be defined as a function $p : \{1, \ldots, r\} \to \{0, 1\}$ such that for any two adjacent nodes i and j, $p(i) \neq p(j)$. The initial value $p(1)$ can be chosen as either 0 or 1, giving two possible bi-colorations.

Furthermore, Fring, Liao, and Olive provided a formula for pairings between simple roots and the cyclic element, which gives the main computational tool to calculate its spectra.

Theorem 1.3 (Fring, Liao, and Olive)**.** *For Λ and $\mathfrak{g}$ as before, let r be the rank of $\mathfrak{g}$. Consider the Cartan subalgebra:*

$$\mathfrak{h} \;=\; \ker_{\mathfrak{g}} \operatorname{ad} \Lambda$$

Let $\alpha_1, \cdots, \alpha_r$ be a choice of ordered simple roots, which are linear functionals on $\mathfrak{h}$. Suppose $x_1, \cdots, x_r$ are components of a left eigenvector of the Cartan matrix of $\mathfrak{g}$ corresponding to the smallest eigenvalue. Let h be the Coxeter number of $\mathfrak{g}$ and let $\zeta_h = e^{\frac{2\pi i}{h}}$.

Suppose p is a bi-coloring, then there exists a non-zero constant c such that

$$\alpha_j(\Lambda) \;=\; c \cdot \alpha_j^2 \cdot x_j \cdot \zeta_h^{(p(j)-p(1))/2} \cdot (1 - \zeta_h^{-2p(j)+1}) \tag{5}$$

for $j = 1, \cdots, r$.

This Theorem is essential to our work: for any finite-dimensional representation of $\mathfrak{g}$, after expressing its weights as linear combinations of simple roots, the cyclic spectrum can be easily calculated. We revisit the proof of this theorem in Section 2.3.

1.2 Results

Using the Cartan-Coxeter correspondence, we calculate the spectra of the cyclic element with respect to two representations. For convenience, we introduce the following notation

$$C_n(z) \;:=\; \{z \cdot \zeta_n^k \mid k = 1, \cdots, n\} \tag{6}$$

where $n \in \mathbb{Z}^{\geq 0}$ and $z \in \mathbb{C}$. Specifically, if $z = 0$, then $C_n(0) = \{0\}$. Also, denote:

$$\begin{aligned} C_n(z_1, \cdots, z_m) &= \bigsqcup_{i=1}^{m} C_n(z_i) \\ C_n(z^{(q)}) &= \underbrace{C_n(z) \bigsqcup \cdots \bigsqcup C_n(z)}_{q \text{ copies}} \end{aligned}$$

Additionally, let ν be an index depending on the type of the simple Lie algebra $\mathfrak{g}$, such that $\nu = 7$ for $\mathfrak{e}_7$, $\nu = 8$ for $\mathfrak{e}_8$, $\nu = 4$ for $\mathfrak{f}_4$, and $\nu = 1$ for all other types. We denote the corresponding minimal fundamental representation of $\mathfrak{g}$ by $\psi_{\nu(\mathfrak{g})}$.

Theorem 1.4. *Let Λ and $\mathfrak{g}$ be as before. Denote the spectra of Λ in the minimal fundamental representation listed before and the adjoint representation by* $\mathrm{Spec}(\psi_{\nu(\mathfrak{g})}, \mathfrak{g})$ *and* $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g})$*, respectively. All the spectra are in the form*

$$C_u(z_1^{(m_1)},\ \cdots,\ z_s^{(m_s)},\ 0^{(m_0)})$$

The values of u, s, m_i, and z_i for each type of simple Lie algebra are given in Tables 1, 2, and 3. Note that in Table 2, $p(i)$ and $q(i)$ are defined by

$$p(i) = \frac{1-(-1)^{i+\frac{n}{2}}}{2}; \quad q(i) = \frac{1-(-1)^{i+n}}{2}$$

Table 1: Data of $\mathrm{Spec}(\psi_{\nu(\mathfrak{g})}, \mathfrak{g})$

$\mathfrak{g}$	u	s	z_i	Numerical Approx. of $\lvert z_i\rvert$	m_i
$\mathfrak{sl}_n$	n	1	$z_1 = 1$	$\lvert z_1\rvert = 1$	$m_1 = 1;\ m_0 = 0$
$\mathfrak{so}_{2n+1}$	$2n$	1	$z_1 = 1$	$\lvert z_1\rvert = 1$	$m_1 = 1;\ m_0 = 1$
$\mathfrak{sp}_{2n}$	$2n$	1	$z_1 = 1$	$\lvert z_1\rvert = 1$	$m_1 = 1;\ m_0 = 1$
$\mathfrak{so}_{2n}$	$2n-2$	1	$z_1 = 1$	$\lvert z_1\rvert = 1$	$m_1 = 1;\ m_0 = 2$
$\mathfrak{e}_6$	12	2	$z_1 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{12}\right)$ $z_2 = 1$	$\lvert z_1\rvert \approx 1.9319$ $\lvert z_2\rvert = 1$	$m_1 = m_2 = 1;$ $m_0 = 3$
$\mathfrak{e}_7$	18	3	$z_1 = 2^2\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)$ $z_2 = 2\cos\left(\frac{\pi}{9}\right)$ $z_3 = 1$	$\lvert z_1\rvert \approx 2.8794$ $\lvert z_2\rvert \approx 1.8794$ $\lvert z_3\rvert = 1$	$m_1 = m_2 = m_3 = 1;$ $m_0 = 2$
$\mathfrak{e}_8$	30	8	$z_1 = 1$ $z_2 = \zeta_{60} \cdot 2\cos\left(\frac{11\pi}{30}\right)$ $z_3 = \zeta_{60} \cdot 2^2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{13\pi}{30}\right)$ $z_4 = \zeta_{60} \cdot 2\cos\left(\frac{2\pi}{5}\right)$ $z_5 = \zeta_{60} \cdot 2^2\cos\left(\frac{11\pi}{30}\right)\cos\left(\frac{2\pi}{5}\right)$ $z_6 = \zeta_{60} \cdot 2\cos\left(\frac{13\pi}{30}\right)$ $z_7 = 2^2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{7\pi}{15}\right)$ $z_8 = 2\cos\left(\frac{7\pi}{15}\right)$	$\lvert z_1\rvert = 1$ $\lvert z_2\rvert \approx 0.8135$ $\lvert z_3\rvert \approx 0.6728$ $\lvert z_4\rvert \approx 0.6180$ $\lvert z_5\rvert \approx 0.5028$ $\lvert z_6\rvert \approx 0.4258$ $\lvert z_7\rvert \approx 0.3383$ $\lvert z_8\rvert \approx 0.2091$	$m_i = 1$ for $i = 1, \cdots, 8;$ $m_0 = 8$
$\mathfrak{f}_4$	12	2	$z_1 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{12}\right)$ $z_2 = 1$	$\lvert z_1\rvert \approx 1.9319$ $\lvert z_2\rvert = 1$	$m_1 = m_2 = 1;$ $m_0 = 2$
$\mathfrak{g}_2$	6	1	$z_1 = 1$	$\lvert z_1\rvert = 1$	$m_1 = 1;\ m_0 = 1$

Table 2: Data of Spec(ad, $\mathfrak{g}$), $\mathfrak{g}$ classical

$\mathfrak{g}$	u	s	z_i	m_i
$\mathfrak{sl}_n$ (n odd)	$2n$	$\frac{n-1}{2}$	$z_i = \sin\big(\frac{i\pi}{n}\big)$	$m_i = 1$ for $i = 1, \cdots, \frac{n-1}{2}$; $m_0 = n-1$
$\mathfrak{sl}_n$ (n even)	n	$\frac{n}{2}$	$z_i = \zeta_{2n}^{p(i)} \cdot \sin\big(\frac{i\pi}{n}\big)$	$m_i = 2$ for $i = 1, \cdots, \frac{n}{2} - 1$; $m_{\frac{n}{2}} = 1$; $m_0 = n-1$
$\mathfrak{so}_{2n+1}$ ($n \not\equiv 0 \mod 3$)	$2n$	n	$z_i = \zeta_{4n}^{q(i)} \cdot \sin\big(\frac{i\pi}{2n}\big)$ for $i = 1, \cdots, n-1$; $z_n = \frac{1}{2}$	$m_i = 1$ for $i = 1, \cdots, n$; $m_0 = n$
$\mathfrak{so}_{2n+1}$ ($n \equiv 0 \mod 3$, $n = 3k$)	$2n$	$n-1$	$z_i = \zeta_{4n}^{q(i)} \cdot \sin\big(\frac{i\pi}{2n}\big)$	$m_i = 1$ for $i = 1, \cdots, n-1, i \neq k$; $m_k = 2$; $m_0 = n$
$\mathfrak{sp}_{2n}$	$2n$	n	$z_i = \zeta_{4n}^{q(i)} \cdot \sin\big(\frac{i\pi}{2n}\big)$	$m_i = 1$ for $i = 1, \cdots, n$; $m_0 = n$
$\mathfrak{so}_{2n}$ ($n \not\equiv 1 \mod 3$)	$2n-2$	$n-1$	$z_i = \zeta_{4n-4}^{1-q(i)} \cdot \sin\Big(\frac{i\pi}{2n-2}\Big)$	$m_i = 1$ for $i = 1, \cdots, n-2$; $m_{n-1} = 2$; $m_0 = n$
$\mathfrak{so}_{2n}$ ($n \equiv 1 \mod 3$, $n = 3k+1$)	$2n-2$	$n-2$	$z_i = \zeta_{4n-4}^{1-q(i)} \cdot \sin\Big(\frac{i\pi}{2n-2}\Big)$	$m_i = 1$ for $i = 1, \cdots, n-2, i \neq k$; $m_k = 3$; $m_0 = n$

Table 3: Data of $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g})$, $\mathfrak{g}$ exceptional

$\mathfrak{g}$	u	s	z_i	Numerical Approx. of $\lvert z_i\rvert$	m_i
$\mathfrak{e}_6$	12	4	$z_1 = 2^2\cos(\frac{\pi}{12})\cos(\frac{\pi}{4})$ $z_2 = \zeta_{24}\cdot 2\cos(\frac{\pi}{12})$ $z_3 = \zeta_{24}\cdot 2\cos(\frac{\pi}{4})$ $z_4 = 1$	$\lvert z_1\rvert \approx 2.7321$ $\lvert z_2\rvert \approx 1.9319$ $\lvert z_3\rvert \approx 1.4142$ $\lvert z_4\rvert = 1$	$m_1 = m_3 = 1$; $m_2 = m_4 = 2$; $m_0 = 6$
$\mathfrak{e}_7$	18	7	$z_1 = \zeta_{36}\cdot 2^2\cos(\frac{\pi}{18})\cos(\frac{\pi}{9})$ $z_2 = 2^2\cos(\frac{\pi}{9})\cos(\frac{2\pi}{9})$ $z_3 = 2^2\cos(\frac{\pi}{18})\cos(\frac{5\pi}{18})$ $z_4 = \zeta_{36}\cdot 2\cos(\frac{\pi}{18})$ $z_5 = 2\cos(\frac{\pi}{9})$ $z_6 = \zeta_{36}\cdot 2\cos(\frac{5\pi}{18})$ $z_7 = 1$	$\lvert z_1\rvert \approx 3.7017$ $\lvert z_2\rvert \approx 2.8794$ $\lvert z_3\rvert \approx 2.5321$ $\lvert z_4\rvert \approx 1.9696$ $\lvert z_5\rvert \approx 1.8794$ $\lvert z_6\rvert \approx 1.2856$ $\lvert z_7\rvert = 1$	$m_i = 1$ for $i = 1, \cdots, 7$; $m_0 = 7$
$\mathfrak{e}_8$	Same as in Data of $\mathrm{Spec}(\psi_{\nu(\mathfrak{g})}, \mathfrak{g})$				
$\mathfrak{f}_4$	12	4	$z_1 = 2^2\cos(\frac{\pi}{12})\cos(\frac{\pi}{4})$ $z_2 = \zeta_{24}\cdot 2\cos(\frac{\pi}{12})$ $z_3 = \zeta_{24}\cdot 2\cos(\frac{\pi}{4})$ $z_4 = 1$	$\lvert z_1\rvert \approx 2.7321$ $\lvert z_2\rvert \approx 1.9319$ $\lvert z_3\rvert \approx 1.4142$ $\lvert z_4\rvert = 1$	$m_i = 1$ for $i = 1, \cdots, 4$; $m_0 = 4$
$\mathfrak{g}_2$	6	2	$z_1 = 1$ $z_2 = \zeta_{12}\cdot 2\cos(\frac{\pi}{6})$	$\lvert z_1\rvert = 1$ $\lvert z_2\rvert \approx 1.7321$	$m_1 = m_2 = 1$; $m_0 = 2$

We prove this theorem in Chapter 3. Notice that for $\mathrm{Spec}(\psi_{\nu(\mathfrak{g})}, \mathfrak{g})$ and $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g})$, we have $u = h$, the Coxeter number, for all $\mathfrak{g}$ except $\mathfrak{sl}_n$ with n odd, in which case $u = 2n$ for the adjoint spectrum.

One can also observe, see [19, Equation (1.17)], that the cyclic spectra are invariant under multiplication by ζ_h. We state this special invariance precisely in the following lemma:

Lemma 1.1. *For Λ and $\mathfrak{g}$ as before, let h be the Coxeter number of $\mathfrak{g}$. Let the spectrum of Λ with respect to a finite-dimensional representation ψ be $\mathrm{Spec}(\psi, \mathfrak{g})$. Then,*

$$\mathrm{Spec}(\psi, \mathfrak{g}) = \zeta_h \cdot \mathrm{Spec}(\psi, \mathfrak{g})$$

where $\zeta_h = e^{\frac{2\pi i}{h}}$.

We will prove this lemma later in Remark 6. For convenience, we call this invariance the "ζ_h-symmetry" of the cyclic spectra. Referring to the results in Theorem 1.4, z_i's, or more essentially, r_i's, are key invariants in the cyclic spectra, because one eigenvalue in the form $z_i\zeta_h^k$ for some k can generate $C_h(z_i)$ by the ζ_h-symmetry.

To better understand the behavior of the cyclic spectra, we plot the eigenvalues in the complex plane to investigate how their heights, defined below, influence their positions. Let $\mu = a_1\alpha_1 + \cdots + a_r\alpha_r$ be an integral weight with respect to a representation of $\mathfrak{g}$. The height of μ is:

$$\mathrm{ht}(\mu) = \sum_i^r a_i$$

As previously discussed, each eigenvalue in the cyclic spectrum is computed via its corresponding weight pairing with Λ. This allows us to order them by the heights of their corresponding weights. For example, in Figure 4, we revisit the cyclic spectrum corresponding to the first fundamental representation of $\mathfrak{e}_6$. The heights of eigenvalues are colored in purple. Since the heights of origin-symmetric eigenvalues are always opposite, we only consider the eigenvalues with non-negative imaginary parts to connect them in descending order of their heights.

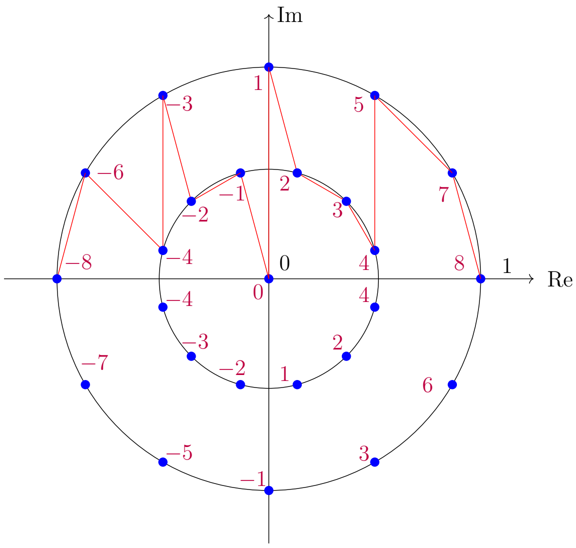


Figure 4: First fundamental cyclic spectrum for $\mathfrak{e}_6$

The graph clearly shows the ζ_{12}-symmetry discussed earlier. Additionally, we observe that the path exhibits a specific pattern, connecting eigenvalues in descending order of their real parts. Similar graphical patterns are observed in other cyclic spectra. Further details on this topic are provided in Chapter 3.

Now we turn to some interesting consequences of Theorem 1.4. They are not only numerical results but also provide deeper insights into the underlying structure of the cyclic spectra. By comparing the two spectra of $\mathfrak{g}$, we immediately obtain the first observation:

Corollary 1.2. *For Λ and $\mathfrak{g}$ as before, let $\mathrm{Spec}(\psi_{\nu(\mathfrak{g})}, \mathfrak{g})$ and $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g})$ denote the cyclic spectra with respect to the minimal fundamental representation listed before and the adjoint representation of $\mathfrak{g}$, respectively. Then, up to an overall scaling*

$$\mathrm{Spec}(\psi_{\nu(\mathfrak{g})}, \mathfrak{g}) \subseteq \mathrm{Spec}(\mathrm{ad}, \mathfrak{g}) \tag{7}$$

Remark 1. Note that it is not true that, for any $\mathfrak{g}$, the weight lattice of its minimal fundamental representation listed before contains a root. When $\mathfrak{g} = \mathfrak{e}_6$, it is not the case. However, the corollary still holds. For example, when $\mathfrak{g} = \mathfrak{e}_6$, the weight lattice of its first fundamental representation contains the following weights:

$$\begin{aligned} w_1 &= \frac{1}{3}(\alpha_1 - \alpha_3 + \alpha_5 + 2\alpha_6) \\ w_2 &= \frac{1}{3}(\alpha_1 + 2\alpha_3 + \alpha_5 - \alpha_6) \end{aligned}$$

As will be explained in more detail later, by the symmetry of the Perron–Frobenius eigenvector of the Cartan matrix of $\mathfrak{e}_6$, we have

$$\alpha_1(\Lambda) = \alpha_6(\Lambda); \quad \alpha_3(\Lambda) = \alpha_5(\Lambda)$$

Hence, it follows that

$$w_1(\Lambda) = \alpha_1(\Lambda) = \alpha_6(\Lambda); \quad w_2(\Lambda) = \alpha_3(\Lambda) = \alpha_5(\Lambda)$$

By "$\zeta_{12}-$symmetry", we obtain the containment between the corresponding spectra as stated in the corollary.

The second observation is deduced from the proof rather than the statement of Theorem 1.4. This result is also discussed in [12], although in a different form.

Theorem 1.5. *Let Λ and $\mathfrak{g}$ be as before. Let h be the Coxeter number of $\mathfrak{g}$. Suppose $\mathcal{A} = \{\alpha_1, \cdots, \alpha_r\}$ is a set of simple roots of $\mathfrak{g}$, and p is a bi-coloring on $\mathcal{A}$. Let*

$$sgn(j) = \begin{cases} 1 & \text{if } p(j) = 1 \\ -1 & \text{if } p(j) = 0 \end{cases}$$

for $j = 1, \cdots, r$. Then, up to an overall scaling,

$$\text{Spec}(\text{ad}, \mathfrak{g}) \; = \; \bigsqcup_{j=1}^{r} C_h(\; sgn(j) \cdot \alpha_j(\Lambda), \; 0 \;) \tag{8}$$

Remark 2. Note that sgn(j) can be chosen uniformly as 1 if $\mathfrak{g} \neq A_n$, n even. In contrast, when $\mathfrak{g} = A_n$ with n even, we have $h = n + 1$ odd, so that $C_h(\alpha_j(\Lambda)) \neq C_h(-\alpha_j(\Lambda))$.

This theorem shows that the pairings between the simple roots and the cyclic element encode all the information about the adjoint spectra for almost all simple Lie algebras. We prove this theorem in Chapter 3, using a method different from that of [12].

Then, for a multiset of n complex numbers, $S = \{Z_1, \cdots, Z_n\}$, we define its radii set $\mathsf{R}(S)$ by

$$\mathsf{R}(S) \; = \; \{|Z_1|, \cdots, |Z_n|\}$$

We call two radii sets $\mathsf{R}(S)$ and $\mathsf{R}(S')$ equivalent, denoted $\mathsf{R}(S) \sim \mathsf{R}(S')$, if they only differ by a non-zero overall scaling.

Now we apply this to the spectra of cyclic elements. This leads to the following corollary, which is immediately clear from Theorem 1.5.

Corollary 1.3. *Let $\mathfrak{g}$ be a simple Lie algebra with rank r. Suppose $\vec{\boldsymbol{x}} = [x_1, \cdots, x_r]$ is a left Perron-Frobenius eigenvector of the Cartan matrix of $\mathfrak{g}$, and suppose $\{\alpha_1, \cdots, \alpha_r\}$ is a set of simple roots of $\mathfrak{g}$. Then:*

$$\mathsf{R}(\mathrm{Spec}(\mathrm{ad}, \mathfrak{g})) \sim \{|\alpha_1(\Lambda)|, \cdots, |\alpha_r(\Lambda)|, 0\} \sim \{\alpha_1^2 \cdot x_1, \cdots, \alpha_r^2 \cdot x_r, 0\} \tag{9}$$

Furthermore, on the right-hand side of Equation (9), the vector with components $\alpha_1^2 \cdot x_1, \cdots, \alpha_r^2 \cdot x_r$ is actually a right eigenvector of the Cartan matrix. Because for any $j = 1, \cdots, r$, since $\vec{\boldsymbol{x}}$ is a left eigenvector, we have

$$\begin{aligned} & \sum_{i=1}^{r} \frac{2\alpha_i \alpha_j}{\alpha_j^2} \cdot x_i = \lambda \cdot x_j \\ \iff & \sum_{i=1}^{r} 2\alpha_i\alpha_j \cdot x_i = \lambda \cdot \alpha_j^2 x_j \\ \iff & \sum_{i=1}^{r} \frac{2\alpha_i\alpha_j}{\alpha_i^2} \cdot \alpha_i^2 x_i = \lambda \cdot \alpha_j^2 x_j \end{aligned}$$

Thus, $[\alpha_1^2 x_1, \cdots, \alpha_r^2 x_r]^\intercal$ is a right eigenvector of C with respect to λ. This result is also consistent with [12, Equation (24)].

Now we discuss the last consequence. For example, let us compare $\mathrm{Spec}(\mathrm{ad}, D_4)$ with $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}_2)$. By Theorem 1.4,

$$\mathrm{Spec}(\mathrm{ad}, D_4) = C_6(1^{(3)}, \zeta_{12} \cdot \sqrt{3}, 0^{(4)})$$

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}_2) = C_6(1, \zeta_{12} \cdot \sqrt{3}, 0^{(2)})$$

Then, it is clear that:

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}_2) \subset \mathrm{Spec}(\mathrm{ad}, D_4)$$

In fact, this is not a coincidence but an instance of a general phenomenon that connects the cyclic spectra with the **folding** procedure for Lie algebras. This procedure generates non-simply laced simple Lie algebras from certain simply-laced ones, namely A_{2n-1}, D_{n+1}, $\mathfrak{e}_6$, and D_4. For each such algebra $\mathfrak{g}$, there exists a Dynkin diagram automorphism τ, inducing a Lie algebra automorphism $\tilde{\tau}$, such that the fixed subalgebra $\mathfrak{g}^{\tilde{\tau}}$ is isomorphic to a non-simply laced simple Lie algebra. Specifically,

$$\begin{aligned} A_{2n-1}^{\tilde{\tau}} &\cong C_n \\ D_{n+1}^{\tilde{\tau}} &\cong B_n \\ \mathfrak{e}_6^{\tilde{\tau}} &\cong \mathfrak{f}_4 \\ D_4^{\tilde{\tau}} &\cong \mathfrak{g}_2 \end{aligned} \tag{10}$$

Referring to the numerical data in Theorem 1.4, we deduce the following result:

Theorem 1.6. *Let* $\mathfrak{g}, \mathfrak{g}^{\tilde{\tau}}$ *and* Λ *be as before, and let* $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}^{\tilde{\tau}})$ *and* $\mathrm{Spec}(\mathrm{ad}, \mathfrak{g})$ *be defined as before. Then, up to an overall scaling:*

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}^{\tilde{\tau}}) \subseteq \mathrm{Spec}(\mathrm{ad}, \mathfrak{g}) \tag{11}$$

In Chapter 4, we provide a more conceptual proof of this Theorem.

1.3 Appendix: Minimal Cosine Expression Finder

In Tables 1, 2, and 3, the expressions for the z_i, given as products of cosines, are determined only up to an overall scaling factor. We choose this scaling so that one of the z_i equals 1 and the total number of cosine factors appearing in all expressions is as small as possible. To carry out this simplification, we developed a `SageMath` program, `Simplify_Cosine`; its pseudocode is given below.

```
Require: N ∈ ℤ, V = {v_1, ..., v_m}                        ▷ Denominator and Input List
  𝓑 ← {cos(kπ/N) | 1 ≤ k ≤ ⌊N/2⌋}                          ▷ Define Search Basis
  S_min ← ∞
  P_best ← null
  L_best ← ∅                                               ▷ Store the optimal simplification log
  for p ∈ V do                                             ▷ Iterate through each pivot candidate
      if p = 0 then continue
      end if
      s ← 1/p
      S_curr ← 0
      L_curr ← []                                          ▷ Initialize log for this pivot
      for e ∈ V do
          t ← e · s                                        ▷ Compute scaled value in Cyclotomic Field
          found ← false
          for r ← 0 to 2N do
              if found then break
              end if
              for C ∈ 𝓑^r do                               ▷ Check all cosine products of length r
                  q ← t/∏(C)
                  if q ∈ ℚ then
                      S_curr ← S_curr + r
                      ω ← format(q, C)                     ▷ Record simplified form
                      append ω to L_curr
                      found ← true
                  end if
              end for
          end for
      end for
      if S_curr < S_min then
          S_min ← S_curr
          P_best ← p
          L_best ← L_curr                                  ▷ Save this log as the best candidate
      end if
  end for
  print "Best Strategy: Scale by ", P_best
  print "Simplified Forms: ", L_best
```

The actual `SageMath` code for this program is as follows:

```
import itertools


class CosineSolver:
    def __init__(self, N):
        self.N = N
        # Initialize Field for exact calculation
        self.K = CyclotomicField(2 * N)
        self.z = self.K.gen()
        # Pre-compute search indices (1 to N/2)
        self.search_indices = [k for k in range(1, N//2 + 1)
                               if (self.z^k + self.z^-k)/2 != 0]

    def c(self, k):
        """Internal cos(k*pi/N)"""
        return (self.z^k + self.z^-k)/2

    def _test_strategy(self, scaler, strategy_name, expressions):
        """
        Internal Helper: returns (score, log_string)
        Does NOT print anything immediately.
        """
        current_score = 0
        log_lines = []

        # Header for the final report
        log_lines.append(f"[ {strategy_name} ]")
        log_lines.append("-" * 60)
```

```
for j, expr_val in enumerate(expressions):
    scaled_val = expr_val * scaler

    found = False
    # Check complexity levels r = 0 to 2N
    for r in range(0, 2 * self.N + 1):

        # Level 0: Rational?
        if r == 0:
            if scaled_val in QQ:
                current_score += 0
                log_lines.append(f"  {j+1:<3} -> {QQ(scaled_val)}")
                found = True; break
            continue

        # Level r: Product of 'r' cosines?
        for k_vals in
        ↪ itertools.combinations_with_replacement(self.search_indices,
        ↪ r):
            cand = prod(self.c(k) for k in k_vals)
            if cand == 0: continue

            ratio = scaled_val / cand
            if ratio in QQ:
                s = QQ(ratio)
                t_str = "*".join([f"c({k})" for k in k_vals])
                if s == 1: fmt = t_str
                elif s == -1: fmt = f"-{t_str}"
                else: fmt = f"{s}*{t_str}"
```

```
                    current_score += r
                    log_lines.append(f"  {j+1:<3} -> {fmt}")
                    found = True; break
            if found: break

        if not found:
            current_score += 999
            log_lines.append(f"  {j+1:<3} -> (Complex)")

    log_lines.append(f"  >>> TOTAL TERMS: {current_score}")
    return (current_score, strategy_name, "\n".join(log_lines))

def solve(self, expressions):
    """Runs optimization and only prints the winner"""
    # 1. Convert inputs
    try:
        expressions = [self.K(expr) for expr in expressions]
    except TypeError:
        print("Error: Could not convert inputs to Cyclotomic Field
        ↪  elements.")
        return

    print(f"CASE (N={self.N}, List size={len(expressions)})")

    results = []

    # --- STRATEGY 0: Original List (No Scaling) ---
    results.append(self._test_strategy(1, "Original List (No Scaling)",
    ↪  expressions))
```

```
# Loop through every item to create a Base Scaler (Strategy 1)
expr_names = [f"Item {i+1}" for i in range(len(expressions))]

for i, val in enumerate(expressions):
    if val == 0: continue

    # The Base Scaler (Normalizes Item i to 1)
    base_scaler = 1 / val
    base_name = f"Scale by 1/{expr_names[i]}"

    # --- STRATEGY 1: Apply Base Scaler ---
    results.append(self._test_strategy(base_scaler, base_name,
    ↪  expressions))

    # --- STRATEGY 2: Base Scaler * c(k) ---
    for k in self.search_indices:
        scaler_2 = base_scaler * self.c(k)
        name_2 = f"{base_name} * c({k})"
        results.append(self._test_strategy(scaler_2, name_2,
        ↪  expressions))

    # --- STRATEGY 3: Base Scaler * c(j)*c(k) ---
    for j, k in
    ↪  itertools.combinations_with_replacement(self.search_indices,
    ↪  2):
        scaler_3 = base_scaler * self.c(j) * self.c(k)
        name_3 = f"{base_name} * c({j})*c({k})"
        results.append(self._test_strategy(scaler_3, name_3,
        ↪  expressions))
```

```
        # --- FILTER & PRINT BEST ---
        if not results:
            print("No valid strategy found.")
        else:
            # Find minimum score
            min_score = min(r[0] for r in results)
            winners = [r for r in results if r[0] == min_score]

            print(f"\nBest Result Found (Score: {min_score})")
            print("=" * 60)

            # Print details for the winner(s)
            for score, name, log_text in winners:
                print(log_text)
                print("\n")

# --- User-Friendly Input Wrapper ---

def c(k):
    if 'Coxeter_number' not in globals():
        raise NameError("Please define 'Coxeter_number = N' before using
        ↪  c(k)")
    return cos(k * pi / globals()['Coxeter_number'])

def Simplify_Cosine(expr_list):
    if 'Coxeter_number' not in globals():
        print("Error: 'Coxeter_number' is not defined.")
        return
    N = globals()['Coxeter_number']
    solver = CosineSolver(N)
```

```
    solver.solve(expr_list)
```

Note that in the program, after simplifying the list as described in `Strategy 1`, we further attempt to simplify the overall scaling of the results of `Strategy 1` by factors of the form $\cos\frac{j\pi}{h}$ and $\cos\frac{j\pi}{h}\cdot\cos\frac{k\pi}{h}$, for $j,k\in\{1,\cdots,h\}$. Those additional steps are annotated as `Strategy 2` and `Strategy 3`.

For example, given the list:

$$\frac{1}{2}\cos\frac{\pi}{5}\cos\frac{2\pi}{5},\ \cos^2\frac{3\pi}{5}\cos\frac{4\pi}{5}$$

We input this list and run the program as follows:

```
# 1. Set up the N
Coxeter_number = 5

# 2. User creates list, can use c(k) helper
list_of_expr = [
    c(1)*c(2)/2,  # Item 1
    c(4)*c(3)*c(3)   # Item 2
]

# 3. Run the simplifier
Simplify_Cosine(list_of_expr)
```

Then, this yields results:

```
CASE (N=5, List size=2)

Best Result Found (Score: 1)
============================================================
```

```
[ Original List (No Scaling) ]
------------------------------------------------------------
  1   -> 1/8
  2   -> -1/4*c(2)
  >>> TOTAL TERMS: 1


[ Scale by 1/Item 1 ]
------------------------------------------------------------
  1   -> 1
  2   -> -2*c(2)
  >>> TOTAL TERMS: 1


[ Scale by 1/Item 1 * c(1) ]
------------------------------------------------------------
  1   -> c(1)
  2   -> -1/2
  >>> TOTAL TERMS: 1


[ Scale by 1/Item 1 * c(1)*c(2) ]
------------------------------------------------------------
  1   -> 1/4
  2   -> -1/2*c(2)
  >>> TOTAL TERMS: 1


[ Scale by 1/Item 2 ]
------------------------------------------------------------
```

```
  1   -> -2*c(1)

  2   -> 1

  >>> TOTAL TERMS: 1


[ Scale by 1/Item 2 * c(2) ]
------------------------------------------------------------
  1   -> -1/2

  2   -> c(2)

  >>> TOTAL TERMS: 1


[ Scale by 1/Item 2 * c(1)*c(2) ]
------------------------------------------------------------
  1   -> -1/2*c(1)

  2   -> 1/4

  >>> TOTAL TERMS: 1
```

2 Cartan-Coxeter Correspondence

In this chapter, we introduce the Cartan-Coxeter correspondence as in Theorem 1.2, which establishes a relationship between the linear algebra of the Coxeter element and the Cartan matrix. This correspondence is largely attributed to Coxeter in [8].

We present two different proofs of this correspondence in Theorem 1.2. The first method, introduced by Freeman in [11], leverages a property of Dynkin diagrams that facilitates the direct computation of eigenvalues and eigenvectors of the Coxeter element. The second method, developed by Fring, Liao, and Olive in [12], computes the eigenvalues and eigenvectors of the Coxeter element through a bi-coloration of simple roots.

Lastly, Fring, Liao, and Olive also provided an explicit formula for the pairings between simple roots and a cyclic element arising from this correspondence. In Section 2.3, we firstly demonstrate the key fact that the centralizer of a cyclic element is a Cartan subalgebra through the example of the simple Lie algebra of type B_5. Then, we derive the pairing formula, which serves as a fundamental computational tool for analyzing the spectra of cyclic elements in the following chapter.

2.1 Method 1: No-loop Diagram Property

In this section, we review Freeman's proof in [11] of the Cartan-Coxeter correspondence as in Theorem 1.2. The proof relies on the geometric property of the Dynkin diagram that it contains no closed loops.

2.1.1 Matrix Representing the Coxeter Element

The Coxeter element σ is a linear transformation on the root space V spanned by simple roots $\mathcal{A} = \{\alpha_1, \ldots, \alpha_r\}$. Its matrix representation is in the form

$$R_{\mathcal{A},\mathcal{A}} = \left([\sigma(\alpha_1)]_{\mathcal{A}} \middle| \quad \cdots \quad \middle| [\sigma(\alpha_r)]_{\mathcal{A}} \right) \tag{12}$$

where $[\sigma(\alpha_i)]_{\mathcal{A}}$ is the coefficient vector of $\sigma(\alpha_i)$ in terms of $\mathcal{A}$. Given the Cartan matrix of a simple Lie algebra, one can calculate $\sigma(\alpha_i)$ directly. For example, consider the simple Lie algebra of type $\mathfrak{g}_2$ having a Cartan matrix:

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Let S_1, S_2 be defined as in Definition 1.2. Then, the concerning Coxeter element is

$$\sigma = S_1 \cdot S_2 \tag{13}$$

It acts on $\mathcal{A} = \{\alpha_1, \alpha_2\}$ as:

$$\sigma(\alpha_1) = S_1S_2(\alpha_1) = 2\alpha_1 + \alpha_2$$
$$\sigma(\alpha_2) = S_1S_2(\alpha_2) = -3\alpha_1 - \alpha_2$$

Thus, the matrix representing σ in the basis $\mathcal{A}$ is:

$$R_{\mathcal{A},\mathcal{A}} = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$$

We observe that this matrix closely resembles the Cartan matrix. Indeed, they are related as illustrated in Theorem 1.1. Let

$$U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}; \quad L = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

Then, we have:

$$-(L^\intercal)^{-1} \cdot U^\intercal = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} = R_{\mathcal{A},\mathcal{A}}$$

Now we present Coxeter's proof of Theorem 1.1 as in [8, p. 766]. It is the essential part of the Cartan-Coxeter correspondence. Define the product of k simple reflections as

$$p_k = S_1 \cdots S_k$$

for $k = 1, \dots, r$. The key step in the proof is to relate the actions of p_k and σ on simple roots. For an arbitrary simple root α_j, $1 \leq j < r$, we have

$$\begin{aligned}
\sigma(\alpha_j) - p_j(\alpha_j) &= p_r(\alpha_j) - p_j(\alpha_j) \\
&= (p_{j+1}(\alpha_j) - p_j(\alpha_j)) + (p_{j+2}(\alpha_j) - p_{j+1}(\alpha_j)) + \cdots + (p_r(\alpha_j) - p_{r-1}(\alpha_j)) \\
&= \sum_{k=j+1}^{r} (p_k(\alpha_j) - p_{k-1}(\alpha_j)) \qquad (14)
\end{aligned}$$

Similarly, for $1 < j \leq r$, we have:

$$\begin{aligned}
p_j(\alpha_j) - \alpha_j &= (p_1(\alpha_j) - \alpha_j) + (p_2(\alpha_j) - p_1(\alpha_j)) + \cdots + (p_j(\alpha_j) - p_{j-1}(\alpha_j)) \\
&= \sum_{k=1}^{j} (p_k(\alpha_j) - p_{k-1}(\alpha_j)) \qquad (15)
\end{aligned}$$

Note that

$$\begin{aligned}
p_k(\alpha_j) - p_{k-1}(\alpha_j) &= p_k(\alpha_j) - p_k(S_k(\alpha_j)) \\
&= p_k(\alpha_j - S_k(\alpha_j)) \\
&= C_{jk} p_k(\alpha_k) \qquad (16)
\end{aligned}$$

where C_{jk} is the $(j,k)'$th entry of the Cartan matrix as in Definition 4. Therefore, Equation (14) and (15) can be rewritten as:

$$\sigma(\alpha_j) - p_j(\alpha_j) = \sum_{k=j+1}^{r} C_{jk} p_k(\alpha_k) \qquad (17)$$

$$p_j(\alpha_j) - \alpha_j = \sum_{k=1}^{j} C_{jk} p_k(\alpha_k) \qquad (18)$$

Note that Equation (17) can be expanded as

$$\sigma(\alpha_j) = p_j(\alpha_j) + C_{j,j+1}p_{j+1}(\alpha_{j+1}) + \cdots + C_{j,r}p_r(\alpha_r)$$

for $1 \leq j < r$. When $j = r$, it follows trivially that $\sigma(\alpha_r) = p_r(\alpha_r)$. Consequently, we have:

$$\begin{bmatrix} \sigma(\alpha_1) \\ \vdots \\ \sigma(\alpha_j) \\ \vdots \\ \sigma(\alpha_r) \end{bmatrix} = \underbrace{\begin{pmatrix} 1 & C_{12} & \cdots & & C_{1r} \\ & 1 & C_{23} & \cdots & C_{2r} \\ & & \ddots & & \vdots \\ & & & & 1 \end{pmatrix}}_{U} \cdot \begin{bmatrix} p_1(\alpha_1) \\ \vdots \\ p_j(\alpha_j) \\ \vdots \\ p_r(\alpha_r) \end{bmatrix} \tag{19}$$

Similarly, expanding Equation (15) yields

$$-\alpha_j = C_{j1}p_1(\alpha_1) + \cdots + C_{j,j-1}p_{j-1}(\alpha_{j-1}) + p_j(\alpha_j)$$

for $1 < j \leq r$. When $j = 1$, it follows trivially that $-\alpha_1 = p_1(\alpha_1)$. Consequently, we obtain:

$$-\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_r \end{bmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ C_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ C_{r1} & \cdots & C_{r,r-1} & 1 \end{pmatrix}}_{L} \cdot \begin{bmatrix} p_1(\alpha_1) \\ \vdots \\ p_j(\alpha_j) \\ \vdots \\ p_r(\alpha_r) \end{bmatrix} \tag{20}$$

Equation (19) and (20) imply that:

$$\begin{bmatrix} \sigma(\alpha_1) \\ \vdots \\ \sigma(\alpha_j) \\ \vdots \\ \sigma(\alpha_r) \end{bmatrix} = \left(-U \cdot L^{-1}\right) \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_r \end{bmatrix} \tag{21}$$

According to Equation (12), we have:

$$R_{\mathcal{A},\mathcal{A}} = -(L^{\intercal})^{-1} \cdot U^{\intercal}$$

The proof of Theorem 1.1 is now complete.

Example 2.1. If $\mathfrak{g}$ is of ADE type, a simply-laced simple Lie algebra, then the Cartan matrix C is symmetric. So $L^{\intercal} = U$, $U^{\intercal} = L$, and

$$R_{\mathcal{A},\mathcal{A}} = -U^{-1} \cdot L \tag{22}$$

2.1.2 Correspondence for Eigenvalues

Theorem 1.1 allows us to compute the eigenvalues and eigenvectors of the Coxeter element in terms of the Cartan matrix. The matrix $R_{\mathcal{A},\mathcal{A}}$ has finite order, since the Weyl group is finite. Let h denote the Coxeter number of $\mathfrak{g}$, the order of the Coxeter element. Then, any eigenvalue λ of $R_{\mathcal{A},\mathcal{A}}$ is an h'th root of unity, that is, $\lambda = (e^{\frac{2\pi i}{h}})^k$ for some integer $1 \leq k \leq h-1$.

We first show the relation between the eigenvalues of the Coxeter element and the Cartan matrix. The characteristic equations of $R_{\mathcal{A},\mathcal{A}}$ is given by

$$\det(R_{\mathcal{A},\mathcal{A}} - \lambda I) = \det\left(-(L^{\intercal})^{-1} \cdot U^{\intercal} - \lambda I\right) = 0$$
$$\Leftrightarrow \det\left((L^{\intercal})^{-1} \cdot U^{\intercal} + \lambda I\right) = \det\left(\lambda^{-1/2} U^{\intercal} + \lambda^{1/2} L^{\intercal}\right) = 0$$

where

$$\det\left(\lambda^{-1/2}U^{\intercal}+\lambda^{1/2}L^{\intercal}\right)=\begin{vmatrix}\lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{21} & \lambda^{1/2}C_{31} & \cdots & \lambda^{1/2}C_{r1}\\ \lambda^{-1/2}C_{12} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{32} & \cdots & \lambda^{1/2}C_{r2}\\ \vdots & \vdots & \ddots & & \vdots\\ \lambda^{-1/2}C_{1r} & \lambda^{-1/2}C_{2r} & \lambda^{-1/2}C_{3r} & \cdots & \lambda^{1/2}+\lambda^{-1/2}\end{vmatrix} \tag{23}$$

We now show that the determinant in Equation (23) can be simplified by using the no-loop property of the Dynkin diagram, so that it corresponds to the characteristic equation of the Cartan matrix.

Observe that all the Dynkin diagrams have no closed loops. Equivalently, there does not exist a subset of simple roots $\mathcal{S}=\{\alpha_{q_1},\alpha_{q_2},\cdots,\alpha_{q_m}\}$ such that the following entries of the Cartan matrix C

$$C_{q_1q_2},\quad C_{q_2q_3},\quad \cdots,\quad C_{q_{m-1}q_m},\quad C_{q_mq_1}$$

are all non-zero for $2<m\leq r$. Following Freeman, we refer to this as the no-loop property.

Now we apply this property to our argument. Let $\lambda^{-1/2}U^{\intercal}+\lambda^{1/2}L^{\intercal}=(a_{ij})$. Then, the determinant of this matrix is

$$\det\left(\lambda^{-1/2}U^{\intercal}+\lambda^{1/2}L^{\intercal}\right)=\sum_{\rho\in\mathcal{S}_r}\operatorname{sgn}(\rho)\cdot\prod_{i=1}^{r}a_{i,\rho(i)} \tag{24}$$

where $\mathcal{S}_r$ is the symmetric group of permutations. Assume the permutation $\rho\in\mathcal{S}_r$ is

$$\rho=(p_1,p_2,\cdots,p_l)$$

where $p_i\in\{1,\cdots,r\}$ for $i=1,\cdots,l$, $l>2$. Then, the corresponding term contributed by ρ in the determinant in Equation (24) contains the product:

$$C_{p_1p_2}C_{p_2p_3}C_{p_3p_4}\cdots C_{p_{l-1}p_l}C_{p_lp_1}$$

This product is zero since the Dynkin diagram of $\mathfrak{g}$ has the no-loop property. Therefore, all the terms in the determinant in Equation (24) contributed by a permutation containing a cycle longer than a transposition are zero.

Consequently, the determinant in Equation (24) only contains terms contributed by products of disjoint transpositions. Consider a permutation π equals a product of k disjoint transpositions, $\pi = (q_1, q_2) \cdot (q_3, q_4) \cdots (q_{2k-1}, q_{2k})$ where $q_i \in \{1, \cdots, r\}$ for $i = 1, \cdots, 2k$, then the term contributed by π is:

$$\begin{aligned}
&(-1)^k(\prod_{i=1}^{k} \lambda^{1/2} C_{q_{2i-1}q_{2i}} \cdot \lambda^{-1/2} C_{q_{2i}q_{2i-1}}) \cdot (\lambda^{1/2} + \lambda^{-1/2})^{n-2k} \\
&= (-1)^k(\prod_{i=1}^{k} C_{q_{2i-1}q_{2i}} \cdot C_{q_{2i}q_{2i-1}}) \cdot (\lambda^{1/2} + \lambda^{-1/2})^{n-2k}
\end{aligned}$$

Hence, Equation (24) equals:

$$\begin{aligned}
&\begin{vmatrix}
\lambda^{1/2}+\lambda^{-1/2} & C_{12} & C_{13} & \cdots & C_{1r} \\
C_{21} & \lambda^{1/2}+\lambda^{-1/2} & C_{23} & \cdots & C_{2r} \\
\vdots & \vdots & \ddots & & \vdots \\
C_{r1} & C_{r2} & C_{r3} & \cdots & \lambda^{1/2}+\lambda^{-1/2}
\end{vmatrix} \\
&= \det\big(C - (2 - \lambda^{1/2} - \lambda^{-1/2}) \cdot I\big)
\end{aligned}$$

This shows the bijection between the eigenvalues of the Coxeter element and the eigenvalues of the Cartan matrix. Therefore, the first statement in Theorem 1.2 is proved.

2.1.3 Correspondence for Eigenvectors

Building on the correspondence between the eigenvalues of the matrix $R_{\mathcal{A},\mathcal{A}}$ and the Cartan matrix, we now aim to establish the correspondence between their corresponding eigenvectors to complete the proof of Theorem 1.2.

We begin by recalling a key lemma from linear algebra to compute the eigenvectors of $R_{\mathcal{A},\mathcal{A}}$. Recall that for an $r \times r$ matrix M, its cofactor vector of the k'th row is

$$\vec{\mathrm{Cof}}_k \;=\; (\;(-1)^{k+i} \cdot \det(S_{ki})\;)^{\intercal}_{i=1,\cdots,r}$$

where S_{ki} is the submatrix of M deleting the k'th row and the i'th column.

Lemma 2.1. *Suppose $M = (m_{ij}), i, j \in \{1, \ldots, r\}$ is a $r \times r$ matrix with rank $r-1$. Then, there exists a row in M whose cofactor vector is non-zero and lies in the null space of M.*

Proof. First, we show that there exists a row of M such that the cofactor vector of that row is non-zero.

Since the rank of M is $r-1$, we can select $r-1$ linearly independent rows that span the row space. Let the k-th row be the row for which the remaining rows are linearly independent, and let S_k denote the submatrix obtained by deleting the k-th row from M. The matrix S_k is of dimension $(r-1) \times r$ and has rank $r-1$.

Next, consider the submatrices of S_k, denoted by S_{ki} for $i = 1, \ldots, r$, where S_{ki} is obtained by deleting the i-th column of S_k. Then, the cofactor of entry m_{ki} is:

$$\mathrm{Cof}_{ki} = (-1)^{k+i} \cdot \det(S_{ki})$$

Now, we claim that there exists an index j such that $\det(S_{kj}) \neq 0$. Since the rank of the submatrix S_k is $r-1$, it follows that S_k has $r-1$ linearly independent columns and one linearly dependent column. Let j be the index of a column such that the remaining columns are linearly independent. In this case, the submatrix S_{kj} is full rank, implying that $\det(S_{kj}) \neq 0$.

Finally, we show that the cofactor vector of the concerned k-th row lies in the null space of M. Note that for any index i, Cof_{ki} does not depend on the value of m_{ki}. For any other row of the matrix, say the l-th row with entries $m_{l1}, m_{l2}, \ldots, m_{lr}$, we have

$$\begin{bmatrix} m_{l1} \\ m_{l2} \\ \vdots \\ m_{lr} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Cof}_{k1} \\ \mathrm{Cof}_{k2} \\ \vdots \\ \mathrm{Cof}_{kn} \end{bmatrix} = \det\left(M_{\hat{l}}\right)$$

where $M_{\hat{l}}$ is the matrix obtained by replacing the k-th row of M with the l-th row of M. $M_{\hat{l}}$ is not full rank, so $\det(M_{\hat{l}}) = 0$. Thus, the chosen cofactor vector is orthogonal to the row space of M, implying that it lies in the null space of M. $\square$

Now, the matrix for which we aim to find a non-zero null space vector is:

$$\lambda^{-1/2}U^\intercal + \lambda^{1/2}L^\intercal = \begin{pmatrix} \lambda^{1/2} + \lambda^{-1/2} & \lambda^{1/2}C_{21} & \lambda^{1/2}C_{31} & \cdots & \lambda^{1/2}C_{r1} \\ \lambda^{-1/2}C_{12} & \lambda^{1/2} + \lambda^{-1/2} & \lambda^{1/2}C_{32} & \cdots & \lambda^{1/2}C_{r2} \\ \vdots & \vdots & \ddots & & \vdots \\ \lambda^{-1/2}C_{1r} & \lambda^{-1/2}C_{2r} & \lambda^{-1/2}C_{3r} & \cdots & \lambda^{1/2} + \lambda^{-1/2} \end{pmatrix} \tag{25}$$

Coleman proves in [7, Section 4(i)] that the eigenspace corresponding to λ has dimension 1, except in the case where $\mathfrak{g} = D_n$ with n even. In [11], Freeman assumes that the multiplicity of λ is 1. So in the following discussion, we restrict our attention to the case where $\mathfrak{g} \neq D_n$ with n even.

Therefore, by Lemma 2.1, we can compute a non-zero cofactor vector of the matrix $\lambda^{-1/2}U^\intercal + \lambda^{1/2}L^\intercal$, which is an eigenvector of $R_{\mathcal{A},\mathcal{A}}$. As noted by Freeman, in practice, the first row can always be chosen to compute a non-zero cofactor vector, as the submatrix S_{11} is always full rank, ensuring that $\mathrm{Cof}_{11} \neq 0$. Hence, we obtain an eigenvector of $R_{\mathcal{A},\mathcal{A}}$

corresponding to the eigenvalue λ, denoted by $\vec{\boldsymbol{x}} = (x_1, \cdots, x_r)^\intercal$, such that:

$$x_i = \mathrm{Cof}_{1i} \tag{26}$$

Similarly, we can derive an eigenvector of the Cartan matrix corresponding to the eigenvalue $(2 - \lambda^{1/2} - \lambda^{-1/2})$. This eigenvector, denoted by $\vec{\boldsymbol{y}} = (y_1, \cdots, y_r)^\intercal$, is given by

$$y_i = \mathrm{Cof}'_{1i} \tag{27}$$

where Cof'_{1i} is the cofactor of the $(1, i)$-entry of $C^\intercal - (2 - \lambda^{1/2} - \lambda^{-1/2}) \cdot I$ for $i = 1, \cdots, r$. Equations (26) and (27) are the key formulas to prove the second part of Theorem 1.2. Freeman proved this part in [11, p. 60], though certain details were left to the reader. The following theorem addresses and completes those omitted details.

Theorem 2.1. *For $\mathfrak{g}$ as before, let r denote its rank and h its Coxeter number. Suppose $\mathcal{A}$ is an ordered set of simple roots following the same indexing convention of simple roots as Bourbaki's in [3]. Let C be the Cartan matrix corresponding to $\mathcal{A}$, and let $R_{\mathcal{A},\mathcal{A}}$ be the matrix as before representing the Coxeter element. If $\vec{\boldsymbol{x}}$ is an eigenvector of $R_{\mathcal{A},\mathcal{A}}$ corresponding to the eigenvalue $\lambda = e^{\frac{2k\pi i}{h}}$ for some integer $1 \leq k \leq h-1$, then there exists a left eigenvector $\vec{\boldsymbol{y}}$ of C corresponding to the eigenvalue $2 - \lambda^{1/2} - \lambda^{-1/2}$, such that:*

1. *If $\mathfrak{g} \in \{A_n, B_n, C_n, \mathfrak{f}_4, \mathfrak{g}_2\}$, then*

$$x_i = (\lambda^{-1/2})^{i-1} \cdot y_i \quad \textit{for } i = 1, \cdots, r$$

2. *If $\mathfrak{g}$ is type D_n, n odd, then*

$$x_i = \begin{cases} (\lambda^{-1/2})^{i-1} \cdot y_i & \textit{for} \quad i = 1, \cdots, n-1 \\ (\lambda^{-1/2})^{n-2} \cdot y_i & \textit{for} \quad i = n \end{cases}$$

3. If $\mathfrak{g} \in \{\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$, *then*

$$x_i = \begin{cases} (\lambda^{-1/2})^{i-2} \cdot y_i & \textit{for} \quad i \geq 3 \\ (\lambda^{-1/2})^{i-1} \cdot y_i & \textit{for} \quad i \leq 2 \end{cases}$$

Proof. Consider the first case. In this case, the Dynkin diagram of $\mathfrak{g}$ takes the form of a straight chain connecting all the simple roots. This geometric characteristic determines that the matrix $\lambda^{-1/2}U^\intercal + \lambda^{1/2}L^\intercal$ for each type of simple Lie algebra in this case is in the form

$$\begin{pmatrix} \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{21} & & & & \\ \lambda^{-1/2}C_{12} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{32} & & & \\ & \lambda^{-1/2}C_{23} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{43} & & \\ & \ddots & \ddots & \ddots & & \\ & & \lambda^{-1/2}C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{n,n-1} \\ & & & \lambda^{-1/2}C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2} \end{pmatrix}$$

with all the unspecified entries zero. In other word, the (i,j)'th entry is:

$$\begin{cases} \lambda^{1/2}+\lambda^{-1/2} & \text{for } i=j \\ \lambda^{-1/2}C_{i,i+1} & \text{for } i=1,\cdots,r-1 \\ \lambda^{1/2}C_{i+1,i} & \text{for } i=1,\cdots,r-1 \\ 0 & \text{Otherwise} \end{cases}$$

Similarly, the matrix $C^\intercal - (2-\lambda^{1/2}-\lambda^{-1/2})I$ is

$$\begin{pmatrix} \lambda^{1/2}+\lambda^{-1/2} & C_{21} & & & & \\ C_{12} & \lambda^{1/2}+\lambda^{-1/2} & C_{32} & & & \\ & C_{23} & \lambda^{1/2}+\lambda^{-1/2} & C_{43} & & \\ & \ddots & \ddots & \ddots & & \\ & & C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & C_{n,n-1} \\ & & & C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2} \end{pmatrix}$$

with all the unspecified entries zero. In other word, the (i, j)'th entry of this matrix is:

$$\begin{cases} \lambda^{1/2} + \lambda^{-1/2} & \text{for } i = j \\ C_{i,i+1} & \text{for } i = 1, \cdots, n-1 \\ C_{i+1,i} & \text{for } i = 1, \cdots, n-1 \\ 0 & \text{Otherwise} \end{cases}$$

By Equation (26) and Equation (27):

$$x_1 = \begin{vmatrix} \lambda^{1/2} + \lambda^{-1/2} & \lambda^{1/2} C_{32} & & & \\ \lambda^{-1/2} C_{23} & \lambda^{1/2} + \lambda^{-1/2} & \lambda^{1/2} C_{43} & & \\ \ddots & \ddots & \ddots & & \\ & \lambda^{-1/2} C_{n-2,n-1} & \lambda^{1/2} + \lambda^{-1/2} & \lambda^{1/2} C_{n,n-1} \\ & & \lambda^{-1/2} C_{n-1,n} & \lambda^{1/2} + \lambda^{-1/2} \end{vmatrix}$$

$$y_1 = \begin{vmatrix} \lambda^{1/2} + \lambda^{-1/2} & C_{32} & & & \\ C_{23} & \lambda^{1/2} + \lambda^{-1/2} & C_{43} & & \\ \ddots & \ddots & \ddots & & \\ & C_{n-2,n-1} & \lambda^{1/2} + \lambda^{-1/2} & C_{n,n-1} \\ & & C_{n-1,n} & \lambda^{1/2} + \lambda^{-1/2} \end{vmatrix}$$

By applying the same argument based on the no-loop property as in the previous section, x_1 and y_1 consist only of terms contributed by products of disjoint transpositions. Let $\pi = (q_1, q_2) \cdot (q_3, q_4) \cdots (q_{2k-1}, q_{2k})$, where $q_i \in 1, \ldots, r$ for $i = 1, \ldots, 2k$, be an arbitrary such permutation. The term contributed by π in x_1 is:

$$(-1)^k (\prod_{i=1}^{k} \lambda^{-1/2} C_{q_{2i-1} q_{2i}} \cdot \lambda^{1/2} C_{q_{2i} q_{2i-1}}) \cdot (\lambda^{1/2} + \lambda^{-1/2})^{n-2k}$$

It equals the term contributed by π in y_1:

$$= (-1)^k (\prod_{i=1}^{k} C_{q_{2i-1}q_{2i}} \cdot C_{q_{2i}q_{2i-1}}) \cdot (\lambda^{1/2} + \lambda^{-1/2})^{n-2k}$$

This implies $x_1 = y_1$.

Now we check the second entry.

$$x_2 = (-1) \cdot \begin{vmatrix} \lambda^{-1/2}C_{12} & \lambda^{1/2}C_{32} & & & \\ & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{43} & & \\ & \ddots & \ddots & & \\ & \lambda^{-1/2}C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{n,n-1} & \\ & & \lambda^{-1/2}C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2} & \end{vmatrix}$$

$$= (-1) \cdot (\lambda^{-1/2}C_{12}) \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{43} & & & \\ & \ddots & & & \\ & \lambda^{-1/2}C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{n,n-1} & \\ & & \lambda^{-1/2}C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2} & \end{vmatrix}$$

$$y_2 = (-1) \cdot \begin{vmatrix} C_{12} & C_{32} & & & \\ & \lambda^{1/2}+\lambda^{-1/2} & C_{43} & & \\ & \ddots & \ddots & & \\ & C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & C_{n,n-1} & \\ & & C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2} & \end{vmatrix}$$

$$= (-1) \cdot (C_{12}) \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & C_{43} & & & \\ & \ddots & & & \\ & C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & C_{n,n-1} & \\ & & C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2} & \end{vmatrix}$$

Similar argument based on the no-loop property,

$$
\begin{vmatrix}
\lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{43} & & & \\
 & \ddots & & & \\
 & \lambda^{-1/2}C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & \lambda^{1/2}C_{n,n-1} \\
 & & \lambda^{-1/2}C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\lambda^{1/2}+\lambda^{-1/2} & C_{43} & & & \\
 & \ddots & & & \\
 & C_{n-2,n-1} & \lambda^{1/2}+\lambda^{-1/2} & C_{n,n-1} \\
 & & C_{n-1,n} & \lambda^{1/2}+\lambda^{-1/2}
\end{vmatrix}
$$

Therefore, $x_2 = \lambda^{-1/2}\cdot y_2$. Now, we generalize this for $2 \le j \le r$. The j'th entry of the vector $\vec{\boldsymbol{x}}$ is

$$
x_j = (-1)^{j+1}\cdot \begin{vmatrix} D & \star \\ 0 & N \end{vmatrix}
$$

where D is a $(j-1)\times(j-1)$ upper triangular matrix with diagonal entries

$$
\lambda^{-1/2}C_{12}, \lambda^{-1/2}C_{23}, \cdots, \lambda^{-1/2}C_{j-1,j}
$$

and N is a $(r-j)\times(r-j)$ matrix with the (m,n)'th entry is

$$
\begin{cases}
\lambda^{1/2}+\lambda^{-1/2} & \text{for } m=n \\
\lambda^{-1/2}C_{m,m+1} & \text{for } m=j+1,\cdots,r-j \\
\lambda^{1/2}C_{m+1,m} & \text{for } m=j+1,\cdots,r-j \\
0 & \text{Otherwise}
\end{cases}
$$

and $\star$ is a non-zero matrix. Then, we have:

$$\begin{aligned}
\begin{vmatrix} D & \star \\ 0 & N \end{vmatrix} &= \det(D) \cdot \det(N) \\
&= (\lambda^{-1/2})^{j-1} (\prod_{k=1}^{j-1} C_{k,k+1}) \cdot \det(N)
\end{aligned}$$

Similarly, we have the j'th entry for $\vec{\boldsymbol{y}}$ as

$$y_j = (-1)^{j+1} \cdot \begin{vmatrix} D' & \star \\ 0 & N' \end{vmatrix}$$

where D' is a $(j-1) \times (j-1)$ upper triangular matrix with diagonal entries

$$C_{12}, C_{23}, \cdots, C_{j-1,j}$$

and N' is a $(r-j) \times (r-j)$ matrix with the (m,n)'th entry is

$$\begin{cases} \lambda^{1/2} + \lambda^{-1/2} & \text{for } m = n \\ C_{m,m+1} & \text{for } m = j+1, \cdots, r-j \\ C_{m+1,m} & \text{for } m = j+1, \cdots, r-j \\ 0 & \text{Otherwise} \end{cases}$$

and $\star$ is a matrix with entries not all zero. Then:

$$\begin{aligned}
\begin{vmatrix} D' & \star \\ 0 & N' \end{vmatrix} &= \det(D') \cdot \det(N') \\
&= (\prod_{k=1}^{j-1} C_{k,k+1}) \cdot \det(N')
\end{aligned}$$

Again, by the same argument using the no-loop property,

$$\det(N) = \det(N')$$

Hence, $x_j = (\lambda^{-1/2})^{j-1} \cdot y_j$, the first case is proved.

Now we prove the second case when $\mathfrak{g} = D_n$, n odd. Then, the matrix $\lambda^{-1/2}U^\intercal + \lambda^{1/2}L^\intercal$ is:

$$\begin{pmatrix}
\lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & & & & \\
-\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & & & \\
 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & & \\
 & \ddots & \ddots & \ddots & & \\
 & & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & -\lambda^{1/2} \\
 & & & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & \\
 & & & -\lambda^{-1/2} & & \lambda^{1/2}+\lambda^{-1/2}
\end{pmatrix}$$

Indeed, this is very similar to the previous case. For $1 \leq j \leq n-1$. The j'th entry of the vector $\vec{\boldsymbol{x}}$ is

$$x_j = (-1)^{j+1} \cdot \begin{vmatrix} D & \star \\ 0 & N \end{vmatrix}$$

where D is a $(j-1) \times (j-1)$ upper triangular matrix with diagonal entries $-\lambda^{-1/2}$, and N is a $(n-j) \times (n-j)$ matrix with diagonal entries $\lambda^{1/2} + \lambda^{-1/2}$. Similarly, we have the j'th entry for $\vec{\boldsymbol{y}}$ as

$$y_j = (-1)^{j+1} \cdot \begin{vmatrix} D' & \star \\ 0 & N' \end{vmatrix}$$

where D' is a $(j-1) \times (j-1)$ upper triangular matrix with diagonal entries -1, and N' is a $(n-j) \times (n-j)$ matrix with diagonal entries $\lambda^{1/2} + \lambda^{-1/2}$.

Similar to the previous case, by applying the no-loop property, $\det(N)$ and $\det(N')$ are equal, so the only difference between x_j and y_j is the diagonal entries of D and D'.

Therefore, it is clear that $x_j = (\lambda^{-1/2})^{j-1} \cdot y_j$ for $1 \leq j \leq n-1$.

For the case when $j = n$, entry x_n is

$$x_n = (-1)^{n+1} \cdot \begin{vmatrix} D & \star \\ 0 & N \end{vmatrix}$$

where D is a $(n-2) \times (n-2)$ upper triangular matrix with diagonal entries $-\lambda^{-1/2}$, and N is a 2×2 matrix:

$$N = \begin{pmatrix} -\lambda^{-1/2} & \lambda^{1/2} + \lambda^{-1/2} \\ -\lambda^{-1/2} & 0 \end{pmatrix}$$

Similarly, we have the n'th entry for $\vec{\boldsymbol{y}}$ as

$$y_n = (-1)^{n+1} \cdot \begin{vmatrix} D' & \star \\ 0 & N' \end{vmatrix}$$

where D' is a $(n-2) \times (n-2)$ upper triangular matrix with diagonal entries -1, and N' is a 2×2 matrix:

$$N' = \begin{pmatrix} -1 & \lambda^{1/2} + \lambda^{-1/2} \\ -1 & 0 \end{pmatrix}$$

By expanding the lower-left entry of N and N' to compute its determinant, it is clear that: $x_n = -(\lambda^{-1/2})^{n-1} \cdot y_n$, and the second case is proved.

Now we prove for the last case when $\mathfrak{g} \in \{\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$. If $\mathfrak{g} = \mathfrak{e}_6$, then the matrix $\lambda^{-1/2}U^\intercal + \lambda^{1/2}L^\intercal$ is:

$$\begin{pmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 & 0 & 0 \\ 0 & \lambda^{1/2}+\lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 & 0 \\ 0 & -\lambda^{-1/2} & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{pmatrix}$$

The matrix $C^\intercal - (2-\lambda^{1/2}-\lambda^{-1/2})I$ is:

$$\begin{pmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda^{1/2}+\lambda^{-1/2} & 0 & -1 & 0 & 0 \\ -1 & 0 & \lambda^{1/2}+\lambda^{-1/2} & -1 & 0 & 0 \\ 0 & -1 & -1 & \lambda^{1/2}+\lambda^{-1/2} & -1 & 0 \\ 0 & 0 & 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} & -1 \\ 0 & 0 & 0 & 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} \end{pmatrix}$$

Again, by Equation (26) and Equation (27),

$$x_1 = \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 & 0 \\ 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$y_1 = \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & -1 & 0 & 0 \\ 0 & \lambda^{1/2}+\lambda^{-1/2} & -1 & 0 & 0 \\ -1 & -1 & \lambda^{1/2}+\lambda^{-1/2} & -1 & 0 \\ 0 & 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} & -1 \\ 0 & 0 & 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

By the same argument using the no-loop property, it is clear that $x_1 = y_1$. Now we check the second entry.

$$-x_2 = \begin{vmatrix} 0 & 0 & -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 & 0 \\ 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= \lambda^{-1/2} \cdot \begin{vmatrix} 0 & -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= (\lambda^{-1/2})^2 \cdot \begin{vmatrix} -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= -\lambda^{-1/2} \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= -\lambda^{-1/2} \cdot (\lambda^{1/2}+\lambda^{-1/2})^2$$

Similarly, we obtain

$$-y_2 = -1 \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & -1 \\ -1 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix} = (\lambda^{1/2}+\lambda^{-1/2})^2$$

Therefore, $x_2 = \lambda^{-1/2} \cdot y_2$. Now we check the third entry.

$$x_3 = \begin{vmatrix} 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 & 0 \\ 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= \lambda^{-1/2} \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 & 0 \\ -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

Similarly, we obtain

$$y_3 = \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & -1 & 0 & 0 \\ -1 & \lambda^{1/2}+\lambda^{-1/2} & -1 & 0 \\ 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} & -1 \\ 0 & 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

By the same reason, we have $x_3 = \lambda^{-1/2} \cdot y_3$. Now we check the fourth entry.

$$-x_4 = \begin{vmatrix} 0 & \lambda^{1/2}+\lambda^{-1/2} & 0 & 0 & 0 \\ -\lambda^{-1/2} & 0 & \lambda^{1/2}+\lambda^{-1/2} & 0 & 0 \\ 0 & -\lambda^{-1/2} & -\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & 0 & 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= \lambda^{-1/2} \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & 0 & 0 \\ -\lambda^{-1/2} & -\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= -(\lambda^{-1/2})^2 \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

Similarly, we obtain

$$-y_4 = -1 \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{1/2}+\lambda^{-1/2} & -1 \\ 0 & -1 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

By the same reason, we have $x_4 = (\lambda^{-1/2})^2 \cdot y_4$. Now we check the fifth entry.

$$x_5 = \begin{vmatrix} 0 & \lambda^{1/2}+\lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 \\ -\lambda^{-1/2} & 0 & \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & -\lambda^{-1/2} & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & 0 \\ 0 & 0 & 0 & -\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & 0 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= \lambda^{-1/2} \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 \\ -\lambda^{-1/2} & -\lambda^{-1/2} & \lambda^{1/2}+\lambda^{-1/2} & 0 \\ 0 & 0 & -\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= -(\lambda^{-1/2})^2 \cdot \begin{vmatrix} \lambda^{1/2}+\lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & -\lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & \lambda^{1/2}+\lambda^{-1/2} \end{vmatrix}$$

$$= -(\lambda^{-1/2})^2(\lambda^{1/2} + \lambda^{-1/2}) \cdot \begin{vmatrix} \lambda^{1/2} + \lambda^{-1/2} & -\lambda^{1/2} \\ 0 & -\lambda^{-1/2} \end{vmatrix}$$

$$= (\lambda^{-1/2})^3(\lambda^{1/2} + \lambda^{-1/2})^2$$

Similarly, we obtain

$$y_5 = (\lambda^{1/2} + \lambda^{-1/2})^2$$

Hence, we have $x_5 = (\lambda^{-1/2})^3 \cdot y_5$. Now we check the sixth entry.

$$-x_6 = \begin{vmatrix} 0 & \lambda^{1/2} + \lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 \\ -\lambda^{-1/2} & 0 & \lambda^{1/2} + \lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & -\lambda^{-1/2} & -\lambda^{-1/2} & \lambda^{1/2} + \lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2} + \lambda^{-1/2} \\ 0 & 0 & 0 & 0 & -\lambda^{-1/2} \end{vmatrix}$$

$$= \lambda^{-1/2} \cdot \begin{vmatrix} \lambda^{1/2} + \lambda^{-1/2} & 0 & -\lambda^{1/2} & 0 \\ -\lambda^{-1/2} & -\lambda^{-1/2} & \lambda^{1/2} + \lambda^{-1/2} & -\lambda^{1/2} \\ 0 & 0 & -\lambda^{-1/2} & \lambda^{1/2} + \lambda^{-1/2} \\ 0 & 0 & 0 & -\lambda^{-1/2} \end{vmatrix}$$

$$= -(\lambda^{-1/2})^2 \cdot \begin{vmatrix} \lambda^{1/2} + \lambda^{-1/2} & -\lambda^{1/2} & 0 \\ 0 & -\lambda^{-1/2} & \lambda^{1/2} + \lambda^{-1/2} \\ 0 & 0 & -\lambda^{-1/2} \end{vmatrix}$$

$$= (\lambda^{-1/2})^3 \cdot \begin{vmatrix} \lambda^{1/2} + \lambda^{-1/2} & -\lambda^{1/2} \\ 0 & -\lambda^{-1/2} \end{vmatrix}$$

$$= -(\lambda^{-1/2})^4 \cdot (\lambda^{1/2} + \lambda^{-1/2})$$

Similarly, we obtain

$$-y_6 = -1 \cdot (\lambda^{1/2} + \lambda^{-1/2})$$

Hence, we have $x_6 = (\lambda^{-1/2})^4 \cdot y_6$. The case of type $\mathfrak{e}_6$ is proved. The proofs for $\mathfrak{e}_7$ and $\mathfrak{e}_8$ are very similar. The key ingredient is always the no-loop property. $\square$

2.2 Method 2: Bi-coloration of the Dynkin Diagram

In this section, we review the proof of the Cartan-Coxeter correspondence in Theorem 1.2 presented by Fring, Liao, and Olive in [12]. Using a bi-coloration of simple roots on the Dynkin diagram, they explicitly construct an eigenvector of the Coxeter element corresponding to the eigenvalue λ, employing the Perron-Frobenius eigenvector of the Cartan matrix with eigenvalue $2 - (\lambda^{1/2} + \lambda^{-1/2})$. This construction establishes the Cartan-Coxeter correspondence.

At first, we define a bi-coloration of a Dynkin diagram.

Definition 2.1. *For $\mathfrak{g}$ as before, let r be the rank of $\mathfrak{g}$. Suppose $\mathcal{A} = \{\alpha_i\}_{i=1,\cdots,r}$ is a set of simple roots of $\mathfrak{g}$. A bi-coloration of $\mathcal{A}$ is a disjoint separation $\{\mathcal{A}_{\mathrm{B}}, \mathcal{A}_{\mathrm{W}}\}$ of $\mathcal{A}$ such that*

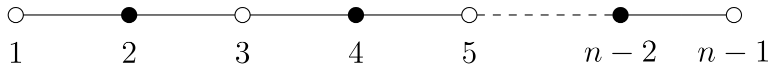
$$\mathcal{A} = \mathcal{A}_{\mathrm{B}} \sqcup \mathcal{A}_{\mathrm{W}}$$

and any pair of simple roots in $\mathcal{A}_{\mathrm{B}}$ or $\mathcal{A}_{\mathrm{W}}$ is not adjacent in the Dynkin diagram of $\mathfrak{g}$. Equivalently, if α_i and α_j belong to the same subset, then:

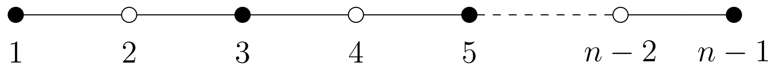
$$C_{ij} = C_{ji} = 0 \tag{28}$$

In practice, we treat simple roots in the subset $\mathcal{A}_{\mathrm{B}}$ as simple roots colored black, and simple roots in the subset $\mathcal{A}_{\mathrm{W}}$ as simple roots colored white.

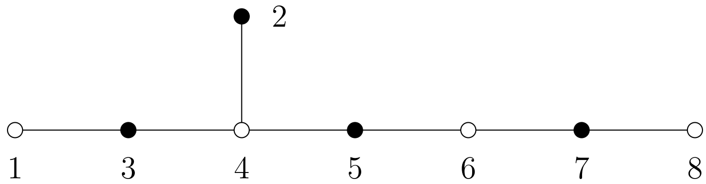
Example 2.2. For example, consider the Dynkin diagram of the type A_{n-1} simple Lie algebra. It can be bi-colored in a way as follows



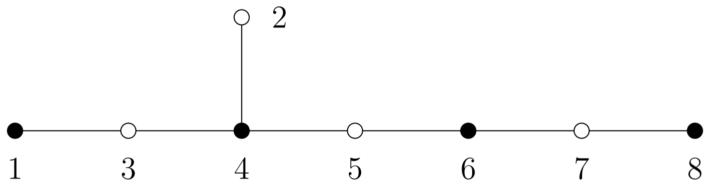
or the other way as follows



Similarly, for the $\mathfrak{e}_8$ simple Lie algebra, its Dynkin diagram can be bi-colored in a way as follows



or the other way as follows



Note that all the Dynkin diagrams of simple Lie algebras are bi-colorable since none of them has a loop. Furthermore, they each have exactly two bi-colorations coming from switching the two colors.

Next, we introduce a specific ordering of simple roots according to a bi-coloration, requiring that all the simple roots colored black precede those colored white. For instance, if we select the first type of bi-coloration for the simple Lie algebra $\mathfrak{e}_8$ in Example 2.2, one possible ordering of the simple roots is:

$$\mathcal{A} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_1, \alpha_4, \alpha_6, \alpha_8\}$$

Now we define two special products of simple reflections related to the Coxeter element, as presented by Fring, Liao, and Olive in [12, Equation (14)].

Definition 2.2. *For $\mathfrak{g}$ as before, let r be the rank of $\mathfrak{g}$. Suppose $\mathcal{A} = \{\alpha_i\}_{i=1,\cdots,r}$ is a set of simple roots. Choose a bi-coloration $\{\mathcal{A}_{\mathrm{B}}, \mathcal{A}_{\mathrm{W}}\}$ of $\mathfrak{g}$ and order the simple roots according to the bi-coloration. Let B and W be the index sets such that:*

$$\mathrm{B} = \{1 \leq i \leq r \mid \alpha_i \in \mathcal{A}_{\mathrm{B}}\}$$

$$\mathrm{W} = \{1 \leq j \leq r \mid \alpha_j \in \mathcal{A}_{\mathrm{W}}\}$$

Define the products of simple reflections σ_{B}, σ_{W} as

$$\sigma_{\mathrm{B}} = \prod_{i \in \mathrm{B}} S_i; \quad \sigma_{\mathrm{W}} = \prod_{j \in \mathrm{W}} S_j$$

where S_i and S_j are defined as before. Then, the Coxeter element σ is given by:

$$\sigma = \sigma_{\mathrm{B}} \cdot \sigma_{\mathrm{W}} \tag{29}$$

Remark 3. The ordering within σ_{B} or σ_{W} does not matter. By Equation (28), for any i, j belonging to the same index set,

$$S_i(\alpha_j) = \alpha_j \tag{30}$$

It implies that:

$$S_i \cdot S_j = S_j \cdot S_i \tag{31}$$

In fact, Equation (30) plays a crucial role in proving the correspondence between the eigenvalues of the Cartan matrix and the Coxeter element. It provides information on how $\sigma_{\mathrm{B}} + \sigma_{\mathrm{W}}$ acts on the simple roots in relation to the Cartan matrix. Suppose we arbitrarily choose a simple root α_i with $i \in \mathrm{B}$. Then, by Equations (30) and (31):

$$\sigma_{\mathrm{B}}(\alpha_i) = S_{\alpha_i}(\alpha_i) = -\alpha_i$$

$$\sigma_{\mathrm{W}}(\alpha_i) = \alpha_i - \sum_{j \in W} C_{ij}\alpha_j$$

Therefore:

$$\begin{aligned}(\sigma_{\mathrm{B}} + \sigma_{\mathrm{W}})\alpha_i &= -\sum_{j \in W} C_{ij}\alpha_j \\ &= 2\alpha_i - \sum_{j=1}^{r} C_{ij}\alpha_j \end{aligned} \tag{32}$$

The case when $i \in \mathrm{W}$ is analogous, with colors exchanged in the above argument. As in Section 2.1.1, $\sigma_{\mathrm{B}} + \sigma_{\mathrm{W}}$ can be represented by the matrix $R_{\mathrm{B}} + R_{\mathrm{W}}$. By Equation (32), this matrix is given by:

$$R_{\mathrm{B}} + R_{\mathrm{W}} = 2I - C^{\intercal} \tag{33}$$

Additionally, in the Weyl group, σ_{B} and σ_{W} have order 2. Therefore,

$$(\sigma_{\mathrm{B}} + \sigma_{\mathrm{W}})^2 = 2 + \sigma + \sigma^{-1} \tag{34}$$

where $\sigma^{-1} = \sigma_{\mathrm{W}} \cdot \sigma_{\mathrm{B}}$. Equivalently,

$$(2I - C^{\intercal})^2 = 2I + R_{\mathcal{A},\mathcal{A}} + R^{-1}_{\mathcal{A},\mathcal{A}} \tag{35}$$

This equation reveals the correspondence between the eigenvalues of the Cartan matrix and the Coxeter element. Suppose λ_σ is an eigenvalue of σ, then by Equation (35),

$$(2 - \lambda_C)^2 = 2 + \lambda_\sigma + \lambda_\sigma^{-1} \tag{36}$$

where λ_C is an eigenvalue of the Cartan matrix. Solving this equation, it yields:

$$\lambda_C = 2 \pm (\lambda_\sigma^{1/2} + \lambda_\sigma^{-1/2}) \tag{37}$$

Now we show that both roots yield eigenvalues of the Cartan matrix. As noted in Remark

3, simple roots are ordered according to the chosen bi-coloration. Suppose the chosen bi-coloration yields $k-1$ black simple roots. Then, by Equation (31), we can, without loss of generality, assume the ordered set of simple roots to be:

$$\mathcal{A} = \{\underbrace{\alpha_1, \cdots, \alpha_{k-1}}_{\mathcal{A}_{\mathrm{B}}}, \underbrace{\alpha_k, \cdots, \alpha_r}_{\mathcal{A}_{\mathrm{W}}}\} \tag{38}$$

Then, according to the definition of bi-coloration, the Cartan matrix is given by:

$$C = \begin{pmatrix} 2 & 0 & \cdots & 0 & C_{1,k} & \cdots & C_{1,r} \\ 0 & 2 \quad 0 & \cdots & 0 & C_{2,k} & \cdots & C_{2,r} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 2 & C_{k-1,k} & \cdots & C_{k-1,r} \\ C_{k,1} & \cdots & & C_{k,k-1} & 2 \quad 0 & \cdots & 0 \\ C_{k+1,1} & \cdots & & C_{k+1,k-1} & 0 \quad 2 \quad 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ C_{r,1} & \cdots & & C_{r,k-1} & 0 & \cdots & 2 \end{pmatrix} \tag{39}$$

Let $\vec{\boldsymbol{y}} = [y_1, \ldots, y_r]^\intercal$ be a left eigenvector corresponding to an arbitrary eigenvalue λ_C of C. Then, by Equation (39), we obtain:

$$\sum_{i=k}^{r} C_{ij} y_i = (\lambda_C - 2) \cdot y_j \quad \text{for } 1 \leq j \leq k-1 \tag{40}$$

$$\sum_{i=1}^{k-1} C_{ij} y_i = (\lambda_C - 2) \cdot y_j \quad \text{for } k \leq j \leq r \tag{41}$$

Next, consider a bi-coloration function p such that $p(j) = 1$ if $j \in \mathrm{B}$ and $p(j) = -1$ if $j \in \mathrm{W}$. Applying p to the components of $\vec{\boldsymbol{y}}$, we obtain the vector:

$$\vec{\boldsymbol{y}}^p = [p(j) \cdot y_j]^\intercal_{j=1,\cdots,r} = [y_1, \cdots, y_{k-1}, -y_k, \cdots, -y_r]^\intercal$$

By Equations (40) and (41), we have

$$\sum_{i=1}^{r} C_{ij}(p(j)\cdot y_i) = (4-\lambda_C)\cdot(p(j)\cdot y_j)$$

for $j = 1, \cdots, r$. Hence, $\vec{\boldsymbol{y}}^p$ is a left eigenvector of C of the eigenvalue $4-\lambda_C$.

This implies that both roots in Equation (37) are eigenvalues of C. Since the eigenvalues of σ are the h'th roots of unity, given by ζ_h^k for $1 \le k \le h-1$, it follows that

$$\mathrm{Spec}(C) = \{2-(\zeta_h^{k/2}+\zeta_h^{-k/2}) \mid 1 \le k \le h-1\} = \{2+(\zeta_h^{k/2}+\zeta_h^{-k/2}) \mid 1 \le k \le h-1\}$$

which proves the Cartan-Coxeter correspondence on the eigenvalues as in Theorem 1.2.

Remark 4. Equation (35) implies that a left eigenvector of C is also an eigenvector of $R_{\mathcal{A},\mathcal{A}} + R_{\mathcal{A},\mathcal{A}}^{-1}$ with the corresponding eigenvalues. However, an eigenvector of $R_{\mathcal{A},\mathcal{A}} + R_{\mathcal{A},\mathcal{A}}^{-1}$ is not necessarily an eigenvector $R_{\mathcal{A},\mathcal{A}}$, although $R_{\mathcal{A},\mathcal{A}}$ and $R_{\mathcal{A},\mathcal{A}} + R_{\mathcal{A},\mathcal{A}}^{-1}$ diagonalize simultaneously. The key issue is that the function: $f(\lambda_\sigma) = \lambda_\sigma + \lambda_\sigma^{-1}$ is not one-to-one, so this "collapse" makes $R_{\mathcal{A},\mathcal{A}} + R_{\mathcal{A},\mathcal{A}}^{-1}$ lose the original directions of the eigenvectors of $R_{\mathcal{A},\mathcal{A}}$.

Consider a simple example where $\mathfrak{g} = A_2$. Choose the bi-coloration

such that B $= \{2\}$ and W $= \{1\}$. This ordering arranges the set of simple roots as $\mathcal{A} = \{\alpha_2, \alpha_1\}$, leading to:

$$R_{\mathcal{A},\mathcal{A}} = R_{\mathrm{B}} \cdot R_{\mathrm{W}} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$R_{\mathcal{A},\mathcal{A}} + R_{\mathcal{A},\mathcal{A}}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Clearly, $R_{\mathcal{A},\mathcal{A}} + R_{\mathcal{A},\mathcal{A}}^{-1}$ has a single eigenvalue of -1, with the entire space $\mathbb{C}^2$ as its eigenspace. However, $R_{\mathcal{A},\mathcal{A}}$ has two distinct eigenvalues $\lambda_{\sigma,1} = \zeta_3$ and $\lambda_{\sigma,2} = \zeta_3^2$, with corresponding eigenvectors $\vec{\boldsymbol{v}}_1 = [\zeta_3^{\frac{1}{2}}, 1]^\intercal$ and $\vec{\boldsymbol{v}}_2 = [\zeta_3, -1]^\intercal$.

Fortunately, the left eigenvectors of C remain related to $R_{\mathcal{A},\mathcal{A}}$. In the example above, the Cartan matrix is given by

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with eigenvalues $\lambda_{C,1} = 1, \lambda_{C,2} = 3$ and corresponding eigenvectors $\vec{\boldsymbol{w}}_1 = [1,1]^\intercal$, and $\vec{\boldsymbol{w}}_2 = [1,-1]^\intercal$. For $i = 1,2$, let $\vec{\boldsymbol{w}}_i = [w_{i,1}, w_{i,2}]^\intercal$. Suppose $\vec{\boldsymbol{w}}_{i,\mathrm{B}} = [w_{i,1}, 0]^\intercal$ and $\vec{\boldsymbol{w}}_{i,\mathrm{W}} = [0, w_{i,2}]^\intercal$. Then, we observe that $\vec{\boldsymbol{v}}_i$ and $\vec{\boldsymbol{w}}_i$ are related in the following way:

$$\vec{\boldsymbol{v}}_i = \lambda_{\sigma,i}^{\frac{1}{2}} \cdot \vec{\boldsymbol{w}}_{i,\mathrm{B}} + \vec{\boldsymbol{w}}_{i,\mathrm{W}}$$

Indeed, $R_{\mathcal{A},\mathcal{A}}$ stabilizes the plane spanned by $\vec{\boldsymbol{w}}_{i,\mathrm{B}}$ and $\vec{\boldsymbol{w}}_{i,\mathrm{W}}$:

$$\begin{aligned} R_{\mathrm{B}}(\vec{\boldsymbol{w}}_{i,\mathrm{B}}) &= -\vec{\boldsymbol{w}}_{i,\mathrm{B}} \\ R_{\mathrm{B}}(\vec{\boldsymbol{w}}_{i,\mathrm{W}}) &= (2 - \lambda_{C,i}) \cdot \vec{\boldsymbol{w}}_{i,\mathrm{B}} + \vec{\boldsymbol{w}}_{i,\mathrm{W}} \\ R_{\mathrm{W}}(\vec{\boldsymbol{w}}_{i,\mathrm{W}}) &= -\vec{\boldsymbol{w}}_{i,\mathrm{W}} \\ R_{\mathrm{W}}(\vec{\boldsymbol{w}}_{i,\mathrm{B}}) &= (2 - \lambda_{C,i}) \cdot \vec{\boldsymbol{w}}_{i,\mathrm{W}} + \vec{\boldsymbol{w}}_{i,\mathrm{B}} \end{aligned}$$

It follows that $R_{\mathcal{A},\mathcal{A}}$ diagonalizes on this plane, with eigenvalues $\lambda_{\sigma,i}$ and the corresponding eigenvectors $\vec{\boldsymbol{v}}_i$ for $i = 1,2$.

The above example is a preview of the forthcoming proof of the Cartan-Coxeter correspondence between the eigenvectors of σ and $C^\intercal$. The key is to identify the plane that is stabilized by $R_{\mathcal{A},\mathcal{A}}$ and spanned by vectors related to the eigenvectors of $C^\intercal$.

Following the method of Fring, Liao, and Olive in [12], we start by explicitly expressing R_{B} and R_{W}. Let $\mathcal{A}$ and C be as in Equations (38) and (39). Then, by Equation (32), we obtain:

$$R_{\mathrm{B}} = \begin{pmatrix} -1 & 0 & & \cdots & 0 & -C_{k,1} & & \cdots & & -C_{r,1} \\ 0 & -1 & 0 & \cdots & 0 & -C_{k,2} & & \cdots & & -C_{r,2} \\ \vdots & & \ddots & & \vdots & \vdots & & & & \vdots \\ 0 & & \cdots & 0 & -1 & -C_{k,k-1} & & \cdots & & -C_{r,k-1} \\ 0 & & \cdots & & 0 & 1 & 0 & & \cdots & 0 \\ 0 & & \cdots & & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & 0 & & \cdots & & 1 \end{pmatrix} \tag{42}$$

$$R_{\mathrm{W}} = \begin{pmatrix} 1 & 0 & & \cdots & 0 & 0 & & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & & \cdots & & 0 \\ \vdots & & \ddots & & \vdots & \vdots & & & & \vdots \\ 0 & & \cdots & 0 & 1 & 0 & & \cdots & & 0 \\ -C_{1,k} & & \cdots & & -C_{k-1,k} & -1 & 0 & & \cdots & 0 \\ -C_{1,k+1} & & \cdots & & -C_{k-1,k+1} & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \ddots & & \vdots \\ -C_{1,r} & & \cdots & & -C_{k-1,r} & 0 & & \cdots & & -1 \end{pmatrix} \tag{43}$$

Now, we construct an eigenvector of $R_{\mathcal{A},\mathcal{A}} = R_{\mathrm{B}} \cdot R_{\mathrm{W}}$ in terms of a left eigenvector, $\vec{\boldsymbol{y}} = [y_1, \cdots, y_r]^\intercal$, of C. Similar to the example in Remark 4, consider the following vectors

$$\vec{\boldsymbol{y}}_{\mathrm{B}} = [y_1, \cdots, y_{k-1}, 0, \cdots, 0]^\intercal$$
$$\vec{\boldsymbol{y}}_{\mathrm{W}} = [0, \cdots, 0, y_k, \cdots, y_r]^\intercal$$

where $\vec{\boldsymbol{y}}_\mathrm{B} + \vec{\boldsymbol{y}}_\mathrm{W} = \vec{\boldsymbol{y}}$. By Equations (42) and (43), we can easily obtain:

$$R_\mathrm{B} \cdot \vec{\boldsymbol{y}}_\mathrm{B} = -\vec{\boldsymbol{y}}_\mathrm{B} \tag{44}$$

$$R_\mathrm{W} \cdot \vec{\boldsymbol{y}}_\mathrm{W} = -\vec{\boldsymbol{y}}_\mathrm{W} \tag{45}$$

By Equations (40) and (41), it follows that:

$$\begin{aligned} R_\mathrm{B} \cdot \vec{\boldsymbol{y}}_\mathrm{W} &= \left[-\sum_{i=k}^{r} C_{i,1} y_i, \quad -\sum_{i=k}^{r} C_{i,2} y_i, \quad \cdots, \quad -\sum_{i=k}^{r} C_{i,k-1} y_i, \quad y_k, \quad \cdots, \quad y_r\right]^\mathsf{T} \\ &= \left[(2-\lambda_C) y_1, \quad (2-\lambda_C) y_2, \quad \cdots, \quad (2-\lambda_C) y_{k-1}, \quad y_k, \quad \cdots, \quad y_r\right]^\mathsf{T} \\ &= (2-\lambda_C) \cdot \vec{\boldsymbol{y}}_\mathrm{B} + \vec{\boldsymbol{y}}_\mathrm{W} \end{aligned} \tag{46}$$

$$\begin{aligned} R_\mathrm{W} \cdot \vec{\boldsymbol{y}}_\mathrm{B} &= \left[y_1, \quad \cdots, \quad y_{k-1}, \quad -\sum_{i=1}^{k-1} C_{i,k} y_i, \quad -\sum_{i=1}^{k-1} C_{i,k+1} y_i, \quad \cdots, \quad -\sum_{i=1}^{k-1} C_{i,r} y_i\right]^\mathsf{T} \\ &= \left[y_1, \quad \cdots, \quad y_{k-1}, \quad (2-\lambda_C) y_k, \quad (2-\lambda_C) y_{k+1}, \quad \cdots, \quad (2-\lambda_C) y_r\right]^\mathsf{T} \\ &= \vec{\boldsymbol{y}}_\mathrm{B} + (2-\lambda_C) \cdot \vec{\boldsymbol{y}}_\mathrm{W} \end{aligned} \tag{47}$$

Therefore, on the subspace spanned by $\{\vec{\boldsymbol{y}}_\mathrm{B}, \vec{\boldsymbol{y}}_\mathrm{W}\}$, the action of R_B is represented by the matrix:

$$\begin{pmatrix} -1 & 2-\lambda_C \\ 0 & 1 \end{pmatrix}$$

The action of R_W is represented by the matrix:

$$\begin{pmatrix} 1 & 0 \\ 2-\lambda_C & -1 \end{pmatrix}$$

Hence, the action of $R_{\mathrm{B}} R_{\mathrm{W}}$ is represented by the matrix:

$$\begin{pmatrix} (2-\lambda_C)^2 - 1 & -(2-\lambda_C) \\ 2-\lambda_C & -1 \end{pmatrix}$$

One of its eigenvalues is λ_σ, with the corresponding eigenvector $[\lambda_\sigma^{\frac{1}{2}}, 1]^\intercal$, which corresponds to:

$$[\lambda_\sigma^{\frac{1}{2}} \cdot y_1, \cdots, \lambda_\sigma^{\frac{1}{2}} \cdot y_{k-1}, y_k, \cdots, y_r]^\intercal \tag{48}$$

This yields the Cartan-Coxeter correspondence on eigenvectors. The proof of Theorem 1.2 is now complete.

2.2.1 Comparison of Eigenvectors in Method 1 and 2

Note that the phases obtained from Method 2, see Equation (48), differ from those from Method 1, see in Theorem 2.1. The discrepancy arises from the change of basis obtained by permuting the simple roots so that all black roots precede the white ones, thereby altering the Coxeter element. In fact, the two resulting eigenvectors are related by a change of basis and a conjugation element in the Weyl group. We provide an example to illustrate this relation.

Example 2.3. Suppose $\mathfrak{g} = A_3$. Recall that if we order the simple root as Bourbaki's convention in [3] such that $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3\}$, then an eigenvector for minimal eigenvalue $2 - 2\cos\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$ of the Cartan matrix C is given by

$$\vec{\boldsymbol{y}} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

For the ordered basis $\mathcal{A}$, the Coxeter element is given by

$$\sigma = S_1 S_2 S_3$$

where S_i is the abstract simple reflection of α_i. Let $R_{\mathcal{A},\mathcal{A}}$ be as before, the matrix representing σ on the basis $\mathcal{A}$. Since Theorem 2.1 indicates that phases are increasing powers of $\lambda^{-1/2}$, an eigenvector of $R_{\mathcal{A},\mathcal{A}}$ with respect to eigenvalue $\lambda = e^{\frac{2\pi i}{4}} = i$ is given by

$$\vec{\boldsymbol{x}} = \begin{bmatrix} 1 \\ \lambda^{-1/2}\sqrt{2} \\ \lambda^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$$

Now we consider another basis $\mathcal{A}'$ following the order of bi-coloration as in Equation (38). Without loss of generality, we pick the bi-coloration where α_2 is white, and α_1, α_3 are black. Then, $\mathcal{A}' = \{\alpha_1, \alpha_3, \alpha_2\}$. For this basis, an eigenvector for the minimal eigenvalue $2-\sqrt{2}$ of the Cartan matrix C' is given by

$$\vec{\boldsymbol{y}}' = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

For the ordered basis $\mathcal{A}'$, by Equation (29), the Coxeter element is changed to

$$\sigma' = S_1 S_3 S_2$$

Let $R_{\mathcal{A}',\mathcal{A}'}$ be the matrix representing σ' on the basis $\mathcal{A}'$. By Equation (48), an eigenvector of $R_{\mathcal{A}',\mathcal{A}'}$ with respect to eigenvalue $\lambda = i$ is given by

$$\vec{\boldsymbol{x}}' = \begin{bmatrix} \lambda^{1/2} \\ \lambda^{1/2} \\ \sqrt{2} \end{bmatrix}$$

Now we illustrate the relation between $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$. Notice that in the Weyl group, σ and

σ' are conjugate. Let $\gamma = S_2 S_1$ be an element in the Weyl group. Then,

$$\gamma\sigma\gamma^{-1} = S_2 S_1 S_1 S_2 S_3 S_1 S_2 = S_3 S_1 S_2 = S_1 S_3 S_2 = \sigma'$$

Additionally, $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$ are derived from different basis, $\mathcal{A}$ and $\mathcal{A}'$, respectively. Let $\pi_{(2,3)}$ be the permutation matrix of the transposition $(2,3)$. Then, the matrix of the basis change from $\mathcal{A}$ to $\mathcal{A}'$ is $\pi_{(2,3)}$. Denote the matrix representing γ on the basis $\mathcal{A}$ by $\gamma_{\mathcal{A}}$. Then, we calculated that:

$$\pi_{(23)} \cdot \gamma_{\mathcal{A}} \cdot \vec{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix} = \begin{bmatrix} -i \\ -i \\ -1-i \end{bmatrix}$$

Notice that $\pi_{(23)} \cdot \gamma_{\mathcal{A}} \cdot \vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}}'$ only differ by an overall scaling.

Remark 5. In the previous example, when considering the permuted basis $\mathcal{A}'$, we have two methods to derive the matrix $R_{\mathcal{A}',\mathcal{A}'}$. By Method 1, as in Theorem 1.1, we have

$$R_{\mathcal{A}',\mathcal{A}'} = -(L'^{\intercal})^{-1} U'^{\intercal} = -\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

By Method 2, as in Equations (42) and (43), we have

$$R_{\mathcal{A}',\mathcal{A}'} = R_{\mathrm{B}} R_{\mathrm{W}} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

Those two different methods yield the same result.

To illustrate the relation between the results of Method 1 and Method 2, for example, in the cases of A_7 and $\mathfrak{e}_6$, we apply Method 1 to the simple Lie algebra with respect to the permuted basis $\mathcal{A}'$ and the Coxeter element σ'. In this setting, one expects the results to differ from those in Equation (48) only by an overall scaling.

Example 2.4. Suppose $\mathfrak{g} = A_7$. Without loss of generality, let simple roots with odd indices be black, and others be white. Then, $\mathcal{A} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_2, \alpha_4, \alpha_6\}$, and the Cartan matrix C is given by

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

For convenient notation, let s denote $\lambda^{1/2} + \lambda^{-1/2}$, p denote $\lambda^{1/2}$, and n denote $\lambda^{-1/2}$. Then,

$$\lambda^{-1/2} U^\intercal + \lambda^{1/2} L^\intercal = \begin{pmatrix} s & 0 & 0 & 0 & -p & 0 & 0 \\ 0 & s & 0 & 0 & -p & -p & 0 \\ 0 & 0 & s & 0 & 0 & -p & -p \\ 0 & 0 & 0 & s & 0 & 0 & -p \\ -n & -n & 0 & 0 & s & 0 & 0 \\ 0 & -n & -n & 0 & 0 & s & 0 \\ 0 & 0 & -n & -n & 0 & 0 & s \end{pmatrix};$$

$$C - (2 - \lambda^{1/2} - \lambda^{-1/2}) \cdot I = \begin{pmatrix} s & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & s & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & s & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & s & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & s & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & s & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & s \end{pmatrix}$$

Using the cofactor method on the first row of $\lambda^{-1/2}U^\intercal + \lambda^{1/2}L^\intercal$ and $C - (2 - \lambda^{1/2} - \lambda^{-1/2}) \cdot I$, for the first component of $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$, we have

$$x_1 = \begin{vmatrix} s & 0 & 0 & -p & -p & 0 \\ 0 & s & 0 & 0 & -p & -p \\ 0 & 0 & s & 0 & 0 & -p \\ -n & 0 & 0 & s & 0 & 0 \\ -n & -n & 0 & 0 & s & 0 \\ 0 & -n & -n & 0 & 0 & s \end{vmatrix} = \begin{vmatrix} s & 0 & 0 & -1 & -1 & 0 \\ 0 & s & 0 & 0 & -1 & -1 \\ 0 & 0 & s & 0 & 0 & -1 \\ -1 & 0 & 0 & s & 0 & 0 \\ -1 & -1 & 0 & 0 & s & 0 \\ 0 & -1 & -1 & 0 & 0 & s \end{vmatrix} = y_1$$

The two determinants of the above equation agree, following from the no-loop property as in the proof of Theorem 2.1.

Now we check the second component.

$$-x_2 = \begin{vmatrix} 0 & 0 & 0 & -p & -p & 0 \\ 0 & s & 0 & 0 & -p & -p \\ 0 & 0 & s & 0 & 0 & -p \\ -n & 0 & 0 & s & 0 & 0 \\ 0 & -n & 0 & 0 & s & 0 \\ 0 & -n & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} 0 & 0 & -p & -p & 0 \\ s & 0 & 0 & -p & -p \\ 0 & s & 0 & 0 & -p \\ -n & 0 & 0 & s & 0 \\ -n & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} s & 0 & -p & -p \\ 0 & s & 0 & -p \\ -n & 0 & s & 0 \\ -n & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot \begin{vmatrix} s & 0 & -1 & -1 \\ 0 & s & 0 & -1 \\ -1 & 0 & s & 0 \\ -1 & -1 & 0 & s \end{vmatrix} = -y_2$$

The very last line follows from the no-loop property again. Hence, $x_2 = y_2$.

Now we check the third component.

$$x_3 = \begin{vmatrix} 0 & s & 0 & -p & -p & 0 \\ 0 & 0 & 0 & 0 & -p & -p \\ 0 & 0 & s & 0 & 0 & -p \\ -n & -n & 0 & s & 0 & 0 \\ 0 & -n & 0 & 0 & s & 0 \\ 0 & 0 & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p & -p & 0 \\ 0 & 0 & 0 & -p & -p \\ 0 & s & 0 & 0 & -p \\ -n & 0 & 0 & s & 0 \\ 0 & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} 0 & 0 & -p & -p \\ 0 & s & 0 & -p \\ -n & 0 & s & 0 \\ 0 & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot \begin{vmatrix} 0 & -p & -p \\ s & 0 & -p \\ -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot (-1)^{1+2} \cdot (-p) \cdot \begin{vmatrix} s & -p \\ -n & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot (-1)^{3+1} \cdot (-1) \cdot (-1)^{1+2} \cdot (-1) \cdot \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}$$

$$= y_3$$

The very last line follows from the no-loop property again. Hence, $x_3 = y_3$.

Now we check the fourth component.

$$-x_4 = \begin{vmatrix} 0 & s & 0 & -p & -p & 0 \\ 0 & 0 & s & 0 & -p & -p \\ 0 & 0 & 0 & 0 & 0 & -p \\ -n & -n & 0 & s & 0 & 0 \\ 0 & -n & -n & 0 & s & 0 \\ 0 & 0 & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p & -p & 0 \\ 0 & s & 0 & -p & -p \\ 0 & 0 & 0 & 0 & -p \\ -n & -n & 0 & s & 0 \\ 0 & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} 0 & s & -p & -p \\ 0 & 0 & 0 & -p \\ -n & -n & s & 0 \\ 0 & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot \begin{vmatrix} s & -p & -p \\ 0 & 0 & -p \\ -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot (-1)^{1+2} \cdot (-p) \cdot \begin{vmatrix} 0 & -p \\ -n & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot (-1)^{1+2} \cdot (-p) \cdot$$
$$\cdot (-1)^{2+1} \cdot (-n) \cdot (-p)$$

$$= (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot (-1)^{3+1} \cdot (-1) \cdot (-1)^{1+2} \cdot (-1) \cdot$$
$$\cdot (-1)^{2+1} \cdot (-1) \cdot (-1)$$

$$= -y_4$$

Hence, $x_4 = y_4$.

Now we check the fifth component.

$$x_5 = \begin{vmatrix} 0 & s & 0 & 0 & -p & 0 \\ 0 & 0 & s & 0 & -p & -p \\ 0 & 0 & 0 & s & 0 & -p \\ -n & -n & 0 & 0 & 0 & 0 \\ 0 & -n & -n & 0 & s & 0 \\ 0 & 0 & -n & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & 0 & -p & 0 \\ 0 & s & 0 & -p & -p \\ 0 & 0 & s & 0 & -p \\ -n & -n & 0 & s & 0 \\ 0 & -n & -n & 0 & s \end{vmatrix}$$

$$= n \cdot (-1)^{4+1} \cdot (-1) \cdot \begin{vmatrix} s & 0 & 0 & -1 & 0 \\ 0 & s & 0 & -1 & -1 \\ 0 & 0 & s & 0 & -1 \\ -1 & -1 & 0 & s & 0 \\ 0 & -1 & -1 & 0 & s \end{vmatrix}$$

$$= ny_5$$

The very last line follows from the no-loop property again. Hence, $x_5 = ny_5$.

Now we check the sixth component.

$$-x_6 = \begin{vmatrix} 0 & s & 0 & 0 & -p & 0 \\ 0 & 0 & s & 0 & 0 & -p \\ 0 & 0 & 0 & s & 0 & -p \\ -n & -n & 0 & 0 & s & 0 \\ 0 & -n & -n & 0 & 0 & 0 \\ 0 & 0 & -n & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & 0 & -p & 0 \\ 0 & s & 0 & 0 & -p \\ 0 & 0 & s & 0 & -p \\ -n & -n & 0 & 0 & 0 \\ 0 & -n & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot \begin{vmatrix} 0 & s & 0 & -p \\ 0 & 0 & s & -p \\ -n & -n & 0 & 0 \\ 0 & -n & -n & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p \\ 0 & s & -p \\ -n & -n & s \end{vmatrix}$$

$$= n \cdot (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+4} \cdot (-1) \cdot (-1)^{3+1} \cdot (-1) \cdot \begin{vmatrix} s & 0 & -1 \\ 0 & s & -1 \\ -1 & -1 & s \end{vmatrix}$$

$$= n(-y_6)$$

The very last line follows from the no-loop property again. Hence, $x_6 = ny_6$.

Now we check the seventh component.

$$x_7 = \begin{vmatrix} 0 & s & 0 & 0 & -p & -p \\ 0 & 0 & s & 0 & 0 & -p \\ 0 & 0 & 0 & s & 0 & 0 \\ -n & -n & 0 & 0 & s & 0 \\ 0 & -n & -n & 0 & 0 & s \\ 0 & 0 & -n & -n & 0 & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & 0 & -p & -p \\ 0 & s & 0 & 0 & -p \\ 0 & 0 & s & 0 & 0 \\ -n & -n & 0 & 0 & s \\ 0 & -n & -n & 0 & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot \begin{vmatrix} 0 & s & 0 & -p \\ 0 & 0 & s & 0 \\ -n & -n & 0 & s \\ 0 & -n & -n & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p \\ 0 & s & 0 \\ -n & -n & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} 0 & s \\ -n & -n \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot (-1)^{3+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot$$
$$\cdot (-1)^{2+1} \cdot (-n) \cdot s$$

$$= n \cdot (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+4} \cdot (-1) \cdot (-1)^{3+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot$$
$$\cdot (-1)^{2+1} \cdot (-1) \cdot s$$

$$= y_7$$

Hence, $x_7 = ny_7$. In conclusion,

$$x_i = \begin{cases} y_i & \text{if } \alpha_i \text{ is black} \\ \lambda^{-1/2} \cdot y_i & \text{if } \alpha_i \text{ is white} \end{cases}$$

Notice that this result of $\vec{\boldsymbol{x}}$ only differs from an overall scaling of the result in Equation (48).

Example 2.5. Suppose $\mathfrak{g} = \mathfrak{e}_6$. Without loss of generality, let α_1 be black, then $\mathcal{A} =$

$\{\alpha_1, \alpha_4, \alpha_6, \alpha_2, \alpha_3, \alpha_5\}$, and the Cartan matrix C is given by

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Then

$$\lambda^{-1/2}U^{\intercal} + \lambda^{1/2}L^{\intercal} = \begin{pmatrix} s & 0 & 0 & 0 & -p & 0 \\ 0 & s & 0 & -p & -p & -p \\ 0 & 0 & s & 0 & 0 & -p \\ 0 & -n & 0 & s & 0 & 0 \\ -n & -n & 0 & 0 & s & 0 \\ 0 & -n & -n & 0 & 0 & s \end{pmatrix}$$

$$C - (2 - \lambda^{1/2} - \lambda^{-1/2}) \cdot I = \begin{pmatrix} s & 0 & 0 & 0 & -1 & 0 \\ 0 & s & 0 & -1 & -1 & -1 \\ 0 & 0 & s & 0 & 0 & -1 \\ 0 & -1 & 0 & s & 0 & 0 \\ -1 & -1 & 0 & 0 & s & 0 \\ 0 & -1 & -1 & 0 & 0 & s \end{pmatrix}$$

Using the cofactor method on the first row of $\lambda^{-1/2}U^{\intercal}+\lambda^{1/2}L^{\intercal}$ and $C-(2-\lambda^{1/2}-\lambda^{-1/2})\cdot I$, for the first component of $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$, we have

$$x_1 = \begin{vmatrix} s & 0 & -p & -p & -p \\ 0 & s & 0 & 0 & -p \\ -n & 0 & s & 0 & 0 \\ -n & 0 & 0 & s & 0 \\ -n & -n & 0 & 0 & s \end{vmatrix} = \begin{vmatrix} s & 0 & -1 & -1 & -1 \\ 0 & s & 0 & 0 & -1 \\ -1 & 0 & s & 0 & 0 \\ -1 & 0 & 0 & s & 0 \\ -1 & -1 & 0 & 0 & s \end{vmatrix} = y_1$$

The two determinants of the above equation agree, following from the no-loop property.

Now we check the second component.

$$\begin{aligned} -x_2 &= \begin{vmatrix} 0 & 0 & -p & -p & -p \\ 0 & s & 0 & 0 & -p \\ 0 & 0 & s & 0 & 0 \\ -n & 0 & 0 & s & 0 \\ 0 & -n & 0 & 0 & s \end{vmatrix} \\ &= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} 0 & -p & -p & -p \\ s & 0 & 0 & -p \\ 0 & s & 0 & 0 \\ -n & 0 & 0 & s \end{vmatrix} \\ &= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} s & 0 & -p \\ 0 & s & 0 \\ -n & 0 & s \end{vmatrix} \\ &= (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot \begin{vmatrix} s & 0 & -1 \\ 0 & s & 0 \\ -1 & 0 & s \end{vmatrix} \\ &= -y_2 \end{aligned}$$

The very last line follows from the no-loop property again. Hence, $x_2 = y_2$.

Now we check the third component.

$$x_3 = \begin{vmatrix} 0 & s & -p & -p & -p \\ 0 & 0 & 0 & 0 & -p \\ 0 & -n & s & 0 & 0 \\ -n & -n & 0 & s & 0 \\ 0 & -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & -p & -p & -p \\ 0 & 0 & 0 & -p \\ -n & s & 0 & 0 \\ -n & 0 & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} 0 & 0 & -p \\ -n & s & 0 \\ -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} -n & s \\ -n & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{2+1} \cdot (-n) \cdot s$$

$$= (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot (-1)^{2+1} \cdot (-1) \cdot s$$

$$= y_3$$

Hence, $x_3 = y_3$.

Now we check the fourth component.

$$-x_4 = \begin{vmatrix} 0 & s & 0 & -p & -p \\ 0 & 0 & s & 0 & -p \\ 0 & -n & 0 & 0 & 0 \\ -n & -n & 0 & s & 0 \\ 0 & -n & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p & -p \\ 0 & s & 0 & -p \\ -n & 0 & 0 & 0 \\ -n & -n & 0 & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot \begin{vmatrix} 0 & s & -p \\ -n & 0 & 0 \\ -n & -n & s \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+3} \cdot (-p) \cdot (-1)^{2+1} \cdot (-n) \cdot \begin{vmatrix} s & -p \\ -n & s \end{vmatrix}$$

$$= n \cdot (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+3} \cdot (-1) \cdot (-1)^{2+1} \cdot (-1) \cdot \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}$$

$$= n(-y_4)$$

The very last line follows from the no-loop property again. Hence, $x_4 = ny_4$.

Now we check the fifth component.

$$x_5 = \begin{pmatrix} 0 & s & 0 & -p & -p \\ 0 & 0 & s & 0 & -p \\ 0 & -n & 0 & s & 0 \\ -n & -n & 0 & 0 & 0 \\ 0 & -n & -n & 0 & s \end{pmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p & -p \\ 0 & s & 0 & -p \\ -n & 0 & s & 0 \\ -n & -n & 0 & s \end{vmatrix}$$

$$= n \cdot (-1)^{4+1} \cdot (-1) \cdot \begin{vmatrix} s & 0 & -1 & -1 \\ 0 & s & 0 & -1 \\ -1 & 0 & s & 0 \\ -1 & -1 & 0 & s \end{vmatrix}$$

$$= ny_5$$

The very last line follows from the no-loop property again. Hence, $x_5 = ny_5$.

Now we check the sixth component.

$$-x_6 = \begin{vmatrix} 0 & s & 0 & -p & -p \\ 0 & 0 & s & 0 & 0 \\ 0 & -n & 0 & s & 0 \\ -n & -n & 0 & 0 & s \\ 0 & -n & -n & 0 & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot \begin{vmatrix} s & 0 & -p & -p \\ 0 & s & 0 & 0 \\ -n & 0 & s & 0 \\ -n & -n & 0 & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot \begin{vmatrix} 0 & s & 0 \\ -n & 0 & s \\ -n & -n & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot (-1)^{1+2} \cdot s \cdot \begin{vmatrix} -n & s \\ -n & 0 \end{vmatrix}$$

$$= (-1)^{4+1} \cdot (-n) \cdot (-1)^{1+4} \cdot (-p) \cdot (-1)^{1+2} \cdot s \cdot (-1)^{2+1} \cdot (-n) \cdot s$$

$$= n \cdot (-1)^{4+1} \cdot (-1) \cdot (-1)^{1+4} \cdot (-1) \cdot (-1)^{1+2} \cdot s \cdot (-1)^{2+1} \cdot (-1) \cdot s$$

$$= n(-y_6)$$

Hence, $x_6 = ny_6$. In conclusion,

$$x_i = \begin{cases} y_i & \text{if } \alpha_i \text{ is black} \\ \lambda^{-1/2} \cdot y_i & \text{if } \alpha_i \text{ is white} \end{cases}$$

Notice that this result of $\vec{\boldsymbol{x}}$ only differs from an overall scaling of the result in Equation (48).

From the above two examples, we observe that by carefully selecting a column or row with a single nonzero entry to reduce the determinant of each component, one can repeatedly apply the no-loop property to deduce that $x_i = y_i$ when α_i is black, and $x_i = \lambda^{-1/2} \cdot y_i$ otherwise. This leads to the following proposition.

Proposition 1. *Suppose $\mathfrak{g}$ is a simple Lie algebra of any type but D_n with n even. Let r denote its rank and h its Coxeter number. Choose a bi-colored set of simple roots $\mathcal{A} = \mathcal{A}_{\mathrm{B}} \sqcup \mathcal{A}_{\mathrm{W}}$ of $\mathfrak{g}$ and order the simple roots such that the simple roots colored black always precede those colored white. Let C be the Cartan matrix corresponding to $\mathcal{A}$, and let $R_{\mathcal{A},\mathcal{A}}$ be the matrix as before representing the Coxeter element. If $\vec{\boldsymbol{x}}$ is an eigenvector of $R_{\mathcal{A},\mathcal{A}}$ corresponding to the eigenvalue $\lambda = e^{\frac{2k\pi i}{h}}$ for some integer $1 \leq k \leq h-1$, then there exists a left eigenvector $\vec{\boldsymbol{y}}$ of C corresponding to the eigenvalue $2 - \lambda^{1/2} - \lambda^{-1/2}$, such that:*

$$x_i = \begin{cases} y_i & \text{if } \alpha_i \in \mathcal{A}_{\mathrm{B}} \\ \lambda^{-\frac{1}{2}} \cdot y_i & \text{if } \alpha_i \in \mathcal{A}_{\mathrm{W}} \end{cases} \tag{49}$$

2.3 Derivation of the Formula in Theorem 1.3

In this section, we revisit the derivation of the pairing formula stated in Theorem 1.3, originally introduced by Fring, Liao, and Olive in [12]. As a prerequisite for this derivation, we begin with the key fact proved by Kostant in [17, Lemma 6.4B] that the centralizer of a cyclic element $\mathfrak{g}^{\Lambda}$ is a Cartan subalgebra, ensuring that the pairings between the simple roots and the cyclic element are well-defined. In [17, Theorem 8.6], Kostant proved that the element given by

$$P = e^{\frac{2\pi i \cdot \mathrm{ad} H_\rho}{h}}$$

is a Coxeter element on $\mathfrak{g}^\Lambda$, where ρ is the half sum of all the positive roots of $\mathfrak{g}$ and H_ρ is the corresponding element of ρ in $\mathfrak{h}$. Moreover, [17, Theorem 8.6] shows that the cyclic element Λ is an eigenvector of P with eigenvalue $\lambda = e^{\frac{2\pi i}{h}}$ and multiplicity 1. Therefore, the vector obtained from the bi-coloration method in Equation (48) is indeed a cyclic element, and we can discuss its pairings with the simple roots eventually.

In the following subsections, we illustrate these key results via the example $\mathfrak{g} = B_5$. We explicitly describe an eigen-basis of $\mathfrak{g}^\Lambda$ with respect to the action of P. To achieve these, a crucial ingredient is a basis of $\mathfrak{g}^E$, the centralizer of the principal nilpotent element,

$$E = \sum_{i=1}^{r} e_i$$

where e_i's are from a set of chosen Chevalley generators.

2.3.1 Centralizer of the Principal Nilpotent Element

Now we calculate a graded basis of $\mathfrak{g}^E$ when $\mathfrak{g} = B_5$. At first, we define a grading on $\mathfrak{g}$ over $\mathbb{Z}/h\mathbb{Z}$ by setting $\deg(e_i) = \deg(e_\gamma) = 1$ where γ is the lowest root of $\mathfrak{g}$. This grading is called the canonical grading, which decomposes $\mathfrak{g}$ as

$$\mathfrak{g} = \bigoplus_{j=0}^{h-1} \mathfrak{g}_j$$

where $\mathfrak{g}_j$ is the subspace of all homogeneous elements of degree j. Specifically, $\mathfrak{g}_j$ is generated by the direct sum of root space vectors with heights j and $j - h$.

Let $I_j = \mathfrak{g}^E \cap \mathfrak{g}_j$, the subspace of $\mathfrak{g}^E$ with degree j, then

$$\mathfrak{g}^E = \bigoplus_{j=1}^{h-1} I_j$$

Now, we calculate the generators of I_j's when $\mathfrak{g} = B_5$. Recall that if α and $\beta \neq -\alpha$ are two roots, then:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

This implies that the calculation relies on the successor diagram of roots of B_5. For any two roots α, β, if $\beta = \alpha + \alpha_i$ for some simple root α_i, then we call β a successor to α. The successor diagram draws each positive root as a vertex, and connects the vertices having successor relations.

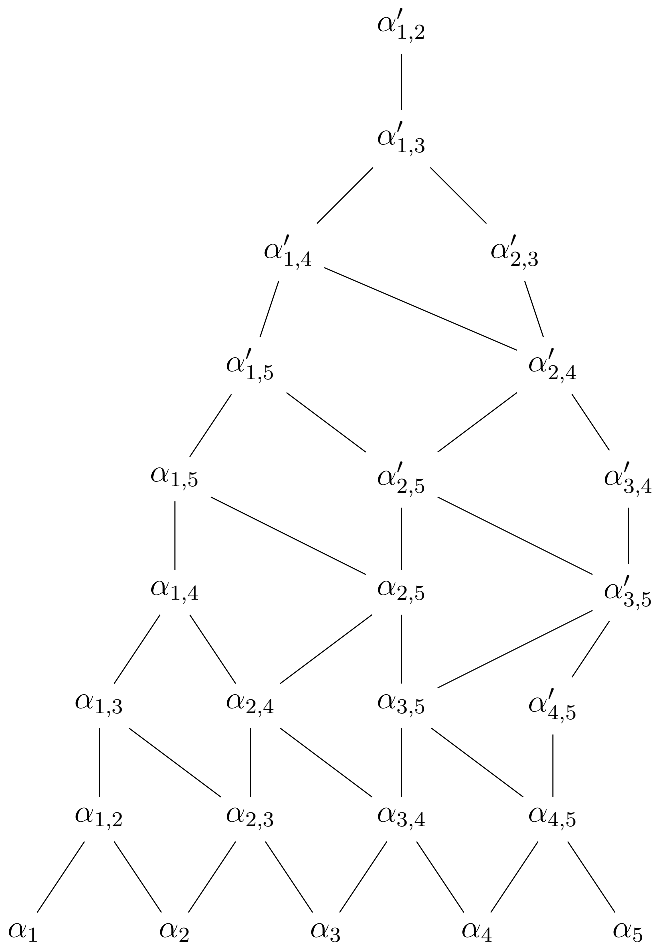
For B_5, besides simple roots $\alpha_1, \cdots, \alpha_5$, other positive roots are

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j$$

for $1 \leq i < j \leq 5$ with $\mathrm{ht}(\alpha_{ij}) = j - i + 1$, and

$$\alpha'_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_5$$

for $1 \leq i < j \leq 5$ with $\mathrm{ht}(\alpha'_{ij}) = j - i + 2(5 - j + 1) = 12 - j - i$. We draw the successor diagram of B_5 in the following graph. The roots on the same level are with the same heights, and the height increases as the level rises.



For a fixed Chevalley basis, let α and β be two different roots such that are both positive or negative. Let e_α and e_β be the generators of their corresponding root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$. If $\alpha + \beta$ is also a root, then there exists a vector $e_{\alpha+\beta}$, a generator of the root space $\mathfrak{g}_{\alpha+\beta}$, such that

$$[e_\alpha, e_\beta] = \pm(p+1)e_{\alpha+\beta} \tag{50}$$

where p is the greatest non-negative integer such that $\beta - p\alpha$ is a root. To determine the sign, we choose the lexicographic ordering of the roots which respects addition, so that

$$[e_\alpha, e_\beta] = (p+1)e_{\alpha+\beta}$$

if $\alpha < \beta$ in the lexicographic ordering.

To address the the case when α and β have different signs, the Chevalley relations listed in [4, p. 101] are necessary. If α, β are two positive roots, then:

$$[e_\alpha, e_{-\alpha}] = -h_\alpha = -\frac{2\alpha}{\alpha \cdot \alpha} \tag{51}$$

$$[h_\alpha, e_{-\beta}] = -\left(\frac{2\beta \cdot \alpha}{\alpha \cdot \alpha}\right) e_{-\beta} \tag{52}$$

Note that the Cartan matrix of $\mathfrak{g} = B_5$ is:

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Generators of I_1

Now, we calculate the generators of I_1. Suppose $E_1 + F_9$ is an arbitrary element of I_1, where

$$E_1 = \sum_{i=1}^{5} c_i e_i; \quad F_9 = c_6 e_{-\alpha'_{1,2}}$$

So that E_1 and F_9 are elements of the roots spaces of degree 1. Then,

$$[E, E_1 + F_9] = [E, E_1] + [E, F_9]$$

By Equations (50) and the successors of simple roots:

$$\begin{aligned}
[E, E_1] &= \sum_{1 \le i,j \le 5} c_i [e_i, e_j] \\
&= (c_1[e_2, e_1] + c_2[e_1, e_2]) + (c_2[e_3, e_2] + c_3[e_2, e_3]) + (c_3[e_4, e_3] + c_4[e_3, e_4]) \\
&\quad + (c_4[e_5, e_4] + c_5[e_4, e_5]) \\
&= (c_2 - c_1)e_{\alpha_{1,2}} + (c_3 - c_2)e_{\alpha_{2,3}} + (c_4 - c_3)e_{\alpha_{3,4}} + (c_5 - c_4)e_{\alpha_{4,5}}
\end{aligned}$$

By the successors diagram,

$$[E, F_9] = c_6[e_2, e_{-\alpha'_{1,2}}]$$

Using the Jacobi identity and the relations in Equations (51) and (52), we have

$$\begin{aligned}
[e_{\alpha_2}, e_{-\alpha'_{1,2}}] &= [e_{\alpha_2}, [e_{-\alpha_2}, e_{-\alpha'_{1,3}}]] \\
&= [[e_{\alpha_2}, e_{-\alpha_2}], e_{-\alpha'_{1,3}}] \\
&= -[h_2, e_{-\alpha'_{1,3}}] \\
&= \left(\frac{2\alpha_2 \cdot \alpha'_{1,3}}{\alpha_2 \cdot \alpha_2}\right) e_{-\alpha'_{1,3}} \\
&= (C_{1,2} + C_{2,2} + 2C_{3,2} + 2C_{4,2} + 2C_{5,2}) e_{-\alpha'_{1,3}} \\
&= -e_{-\alpha'_{1,3}} \qquad (53)
\end{aligned}$$

Hence,

$$[E, F_9] = -c_6 e_{-\alpha'_{1,3}} \qquad (54)$$

$$[E, E_1 + F_9] = (c_2 - c_1)e_{\alpha_{1,2}} + (c_3 - c_2)e_{\alpha_{2,3}} + (c_4 - c_3)e_{\alpha_{3,4}} + (c_5 - c_4)e_{\alpha_{4,5}} - c_6 e_{-\alpha'_{1,3}}$$

Since $E_1 + F_9 \in \mathfrak{g}^E$, therefore

$$c_1 = c_2 = c_3 = c_4 = c_5, \quad c_6 = 0$$

We can simply pick 1 as the values of $c_1, \cdots, c_5$, then the generator of I_1 for this choice is:

$$E_1 = \sum_{i=1}^{5} e_i = E$$

Generators of I_2

Now, we calculate the generators of I_2. Let $\Phi_2 = \{\alpha_{1,2}, \alpha_{2,3}, \alpha_{3,4}, \alpha_{4,5}\} = \{\beta_i\}_{i=1,\cdots,4}$. Suppose $E_2 + F_8$ is an arbitrary element of I_2, where

$$E_2 = \sum_{i=1}^{4} c_i e_{\beta_i}; \quad F_8 = c_5 e_{-\alpha'_{1,3}}$$

So that E_2 and F_8 are elements of the roots spaces of degree 2. Then

$$[E, E_2 + F_8] = [E, E_2] + [E, F_8]$$

By the successor diagram, $\alpha_{1,3}$ is the successor of β_1 and β_2, $\alpha_{2,4}$ is the successor of β_2 and β_3, $\alpha_{3,5}$ is the successor of β_3 and β_4, and $\alpha'_{4,5}$ is the successor of β_4. Thus, by the Chevalley formula:

$$\begin{aligned}
[E, E_2] &= \sum_{i=1}^{5} \sum_{j=1}^{4} c_j [e_i, e_{\beta_j}] \\
&= (c_1[e_3, e_{\beta_1}] + c_2[e_1, e_{\beta_2}]) + (c_2[e_4, e_{\beta_2}] + c_3[e_2, e_{\beta_3}]) + (c_3[e_5, e_{\beta_3}] + c_4[e_3, e_{\beta_4}]) \\
&\quad + c_4[e_5, e_{\beta_4}] \\
&= (c_2 - c_1)e_{\alpha_{1,3}} + (c_3 - c_2)e_{\alpha_{2,4}} + (c_4 - c_3)e_{\alpha_{3,5}} + (-2c_4)e_{\alpha'_{4,5}}
\end{aligned}$$

Note that the coefficient of $e_{\alpha'_{4,5}}$ comes from the fact that $\beta_4 - \alpha_5$ is also a root, and $\alpha_5 < \beta_4$ in lexicographic order. Next, by the successor diagram,

$$[E, F_8] = c_5[e_3, e_{-\alpha'_{1,3}}] + c_5[e_1, e_{-\alpha'_{1,3}}]$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned}
[e_3, e_{-\alpha'_{1,3}}] &= [e_{\alpha_3}, [e_{-\alpha_3}, e_{-\alpha'_{1,4}}]] \\
&= [[e_{\alpha_3}, e_{-\alpha_3}], e_{-\alpha'_{1,4}}]
\end{aligned}$$

$$
\begin{aligned}
&= -[h_3, e_{-\alpha'_{1,4}}] \\
&= (C_{1,3} + C_{2,3} + C_{3,3} + 2C_{4,3} + 2C_{5,3})e_{-\alpha'_{1,4}} \\
&= -e_{-\alpha'_{1,4}} && (55)
\end{aligned}
$$

$$
\begin{aligned}
[e_1, e_{-\alpha'_{1,3}}] &= [e_1, [e_{-\alpha'_{2,3}}, e_{-\alpha_1}]] \\
&= [e_{-\alpha'_{2,3}}, [e_1, e_{-\alpha_1}]] \\
&= [h_1, e_{-\alpha'_{2,3}}] \\
&= -(C_{2,1} + 2C_{3,1} + 2C_{4,1} + 2C_{5,1})e_{-\alpha'_{2,3}} \\
&= e_{-\alpha'_{2,3}} && (56)
\end{aligned}
$$

Hence,

$$[E, E_2+F_8] = (c_2-c_1)e_{\alpha_{1,3}} + (c_3-c_2)e_{\alpha_{2,4}} + (c_4-c_3)e_{\alpha_{3,5}} + (-2c_4)e_{\alpha'_{4,5}} - c_5e_{-\alpha'_{1,4}} + c_5e_{-\alpha'_{2,3}}$$

Since $E_2 + F_8 \in \mathfrak{g}^E$, therefore

$$c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

This implies that I_2 is trivial.

Generators of I_3

Now, we calculate the generators of I_3. Let $\Phi_3 = \{\alpha_{1,3}, \alpha_{2,4}, \alpha_{3,5}, \alpha'_{4,5}\} = \{\beta_i\}_{i=1,\cdots,4}$. Suppose $E_3 + F_7$ is an arbitrary element of I_3, where

$$E_3 = \sum_{i=1}^{4} c_i e_{\beta_i}; \quad F_7 = c_5e_{-\alpha'_{1,4}} + c_6e_{-\alpha'_{2,3}}$$

So that E_3 and F_7 are elements of the roots spaces of degree 3. Then

$$[E, E_3 + F_7] = [E, E_3] + [E, F_7]$$

By the successor diagram, $\alpha_{1,4}$ is the successor of β_1 and β_2, $\alpha_{2,5}$ is the successor of β_2 and β_3, $\alpha'_{3,5}$ is the successor of β_3 and β_4. Thus, by the Chevalley formula:

$$\begin{aligned}
[E, E_3] &= \sum_{i=1}^{5} \sum_{j=1}^{4} c_j [e_i, e_{\beta_j}] \\
&= (c_1[e_4, e_{\alpha_{1,3}}] + c_2[e_1, e_{\alpha_{2,4}}]) + (c_2[e_5, e_{\alpha_{2,4}}] + c_3[e_2, e_{\alpha_{3,5}}]) \\
&\quad + (c_3[e_5, e_{\alpha_{3,5}}] + c_4[e_3, e_{\alpha'_{4,5}}]) \\
&= (c_2 - c_1)e_{\alpha_{1,4}} + (c_3 - c_2)e_{\alpha_{2,5}} + (c_4 - 2c_3)e_{\alpha'_{3,5}}
\end{aligned}$$

Note that the coefficient of $e_{\alpha'_{3,5}}$ comes from the fact that $\beta_3 - \alpha_5$ is also a root, and $\alpha_5 < \beta_3$ in lexicographic order. Next, by the successor diagram,

$$\begin{aligned}
[E, F_7] &= c_5[E, e_{-\alpha'_{1,4}}] + c_6[E, e_{-\alpha'_{2,3}}] \\
&= c_5[e_1, e_{-\alpha'_{1,4}}] + c_5[e_4, e_{-\alpha'_{1,4}}] + c_6[e_3, e_{-\alpha'_{2,3}}]
\end{aligned}$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned}
[e_1, e_{-\alpha'_{1,4}}] &= [e_1, [e_{-\alpha'_{2,4}}, e_{-\alpha_1}]] \\
&= [e_{-\alpha'_{2,4}}, [e_1, e_{-\alpha_1}]] \\
&= [h_1, e_{-\alpha'_{2,4}}] \\
&= -(C_{2,1} + C_{3,1} + 2C_{4,1} + 2C_{5,1})e_{-\alpha'_{2,4}} \\
&= e_{-\alpha'_{2,4}} \qquad (57)
\end{aligned}$$

$$[e_4, e_{-\alpha'_{1,4}}] = [e_4, [e_{-\alpha_4}, e_{-\alpha'_{1,5}}]]$$

$$
\begin{aligned}
&= [[e_4, e_{-\alpha_4}], e_{-\alpha'_{1,5}}] \\
&= -[h_4, e_{-\alpha'_{1,5}}] \\
&= (C_{1,4} + C_{2,4} + C_{3,4} + C_{4,4} + 2C_{5,4}) e_{-\alpha'_{1,5}} \\
&= -e_{-\alpha'_{1,5}}
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
[e_3, e_{-\alpha'_{2,3}}] &= [e_3, [e_{-\alpha_3}, e_{-\alpha'_{2,4}}]] \\
&= [[e_3, e_{-\alpha_3}], e_{-\alpha'_{2,4}}] \\
&= -[h_3, e_{-\alpha'_{2,4}}] \\
&= (C_{2,3} + C_{3,3} + 2C_{4,3} + 2C_{5,3}) e_{-\alpha'_{2,4}} \\
&= -e_{-\alpha'_{2,4}}
\end{aligned} \tag{59}
$$

Hence

$$
[E, F_7] = -c_5 e_{-\alpha'_{1,5}} + (c_5 - c_6) e_{-\alpha'_{2,4}} \tag{60}
$$

$$
[E, E_3 + F_7] = (c_2 - c_1) e_{\alpha_{1,4}} + (c_3 - c_2) e_{\alpha_{2,5}} + (c_4 - 2c_3) e_{\alpha'_{3,5}} - c_5 e_{-\alpha'_{1,5}} + (c_5 - c_6) e_{-\alpha'_{2,4}}
$$

Since $E_3 + F_7 \in \mathfrak{g}^E$, therefore

$$
c_1 = c_2 = c_3 = \frac{c_4}{2}; \quad c_5 = c_6 = 0
$$

We can simply pick 1 as the values of c_1, c_2, c_3, hence, the generator of I_3 for this choice is

$$
E_3 = e_{\alpha_{1,3}} + e_{\alpha_{2,4}} + e_{\alpha_{3,5}} + 2e_{\alpha'_{4,5}} \tag{61}
$$

Generators of I_4

Now, we calculate the generators of I_4. Let $\Phi_4 = \{\alpha_{1,4}, \alpha_{2,5}, \alpha'_{3,5}\} = \{\beta_i\}_{i=1,\cdots,3}$. Suppose $E_4 + F_6$ is an arbitrary element of I_4, where

$$E_4 = \sum_{i=1}^{3} c_i e_{\beta_i}; \quad F_6 = c_4 e_{-\alpha'_{1,5}} + c_5 e_{-\alpha'_{2,4}}$$

So that E_4 and F_6 are elements of the roots spaces of degree 4. Then

$$[E, E_4 + F_6] = [E, E_4] + [E, F_6]$$

By the successor diagram, $\alpha_{1,5}$ is the successor of β_1 and β_2, $\alpha'_{2,5}$ is the successor of β_2 and β_3, and $\alpha'_{3,4}$ is the successor of β_3. Thus,

$$\begin{aligned}
[E, E_4] &= \sum_{i=1}^{5}\sum_{j=1}^{3} c_j [e_i, e_{\beta_j}] \\
&= (c_1[e_5, e_{\beta_1}] + c_2[e_1, e_{\beta_2}]) + (c_2[e_5, e_{\beta_2}] + c_3[e_2, e_{\beta_3}]) + c_3[e_4, e_{\beta_3}] \\
&= (c_2 - c_1)e_{\alpha_{1,5}} + (c_3 - 2c_2)e_{\alpha'_{2,5}} + (-c_3)e_{\alpha'_{3,4}}
\end{aligned}$$

Note that the coefficient of $e_{\alpha'_{2,5}}$ comes from the fact that $\beta_2 - \alpha_5$ is also a root, and $\alpha_5 < \beta_2$ in lexicographic order. Next, by the successor diagram,

$$\begin{aligned}
[E, F_6] &= c_4[E, e_{-\alpha'_{1,5}}] + c_5[E, e_{-\alpha'_{2,4}}] \\
&= c_4[e_5, e_{-\alpha'_{1,5}}] + c_4[e_1, e_{-\alpha'_{1,5}}] + c_5[e_4, e_{-\alpha'_{2,4}}] + c_5[e_2, e_{-\alpha'_{2,4}}]
\end{aligned}$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned}
[e_5, e_{-\alpha'_{1,5}}] &= \frac{1}{2}[e_5, [e_{-\alpha_5}, e_{-\alpha_{1,5}}]] \\
&= \frac{1}{2}[[e_5, e_{-\alpha_5}], e_{-\alpha_{1,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{1,5}}]] \\
&= -\frac{1}{2}[h_5, e_{-\alpha_{1,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{1,5}}]]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(C_{1,5} + C_{2,5} + C_{3,5} + C_{4,5} + C_{5,5})e_{-\alpha_{1,5}} + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{1,5}}]] \\
&= \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{1,5}}]]
\end{aligned}
$$

Note that

$$
\begin{aligned}
[e_5, e_{-\alpha_{1,5}}] &= [e_5, [e_{-\alpha_5}, e_{-\alpha_{1,4}}]] \\
&= [[e_5, e_{-\alpha_5}], e_{-\alpha_{1,4}}] \\
&= -[h_5, e_{-\alpha_{1,4}}] \\
&= (C_{1,5} + C_{2,5} + C_{3,5} + C_{4,5})e_{-\alpha_{1,4}} \\
&= -2e_{-\alpha_{1,4}} \qquad (62)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
[e_5, e_{-\alpha'_{1,5}}] &= -[e_{-\alpha_5}, e_{-\alpha_{1,4}}] \\
&= -e_{-\alpha_{1,5}} \qquad (63)
\end{aligned}
$$

Also,

$$
\begin{aligned}
[e_1, e_{-\alpha'_{1,5}}] &= [e_1, [e_{-\alpha'_{2,5}}, e_{-\alpha_1}]] \\
&= [e_{-\alpha'_{2,5}}, [e_1, e_{-\alpha_1}]] \\
&= [h_1, e_{-\alpha'_{2,5}}] \\
&= -(C_{2,1} + C_{3,1} + C_{4,1} + 2C_{5,1})e_{-\alpha'_{2,5}} \\
&= e_{-\alpha'_{2,5}} \qquad (64)
\end{aligned}
$$

$$
\begin{aligned}
[e_4, e_{-\alpha'_{2,4}}] &= [e_4, [e_{-\alpha_4}, e_{-\alpha'_{2,5}}]] \\
&= [[e_4, e_{-\alpha_4}], e_{-\alpha'_{2,5}}]
\end{aligned}
$$

$$= -[h_4, e_{-\alpha'_{2,5}}]$$
$$= (C_{2,4} + C_{3,4} + C_{4,4} + 2C_{5,4})e_{-\alpha'_{2,5}}$$
$$= -e_{-\alpha'_{2,5}} \tag{65}$$

$$\begin{aligned}[e_2, e_{-\alpha'_{2,4}}] &= [e_2, [e_{-\alpha'_{3,4}}, e_{-\alpha_2}]] \\ &= [e_{-\alpha'_{3,4}}, [e_2, e_{-\alpha_2}]] \\ &= [h_2, e_{-\alpha'_{3,4}}] \\ &= -(C_{3,2} + 2C_{4,2} + 2C_{5,2})e_{-\alpha'_{3,4}} \\ &= e_{-\alpha'_{3,4}} \end{aligned} \tag{66}$$

Hence,

$$[E, E_4 + F_6] = (c_2 - c_1)e_{\alpha_{1,5}} + (c_3 - 2c_2)e_{\alpha'_{2,5}} + (-c_3)e_{\alpha'_{3,4}} - c_4 e_{-\alpha_{1,5}} + (c_4 - c_5)e_{-\alpha'_{2,5}} + c_5 e_{-\alpha'_{3,4}}$$

Since $E_4 + F_6 \in \mathfrak{g}^E$, therefore,

$$c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

This implies that I_4 is trivial.

Generators of I_5

Now, we calculate the generators of I_5. Let $\Phi_5 = \{\alpha_{1,5}, \alpha'_{2,5}, \alpha'_{3,4}\} = \{\beta_i\}_{i=1,\cdots,3}$. Suppose $E_5 + F_5$ is an arbitrary element of I_5, where

$$E_5 = \sum_{i=1}^{3} c_i e_{\beta_i}; \quad F_5 = \sum_{i=4}^{6} c_i e_{-\beta_i}$$

By the successor diagram, $\alpha'_{1,5}$ is the successor of β_1 and β_2, $\alpha'_{2,4}$ is the successor of β_2 and β_3. Thus, by the Chevalley formula:

$$\begin{aligned}
[E, E_5] &= \sum_{i=1}^{5} \sum_{j=1}^{3} c_\beta [e_i, e_{\beta_j}] \\
&= (c_1[e_5, e_{\alpha_{1,5}}] + c_2[e_1, e_{\alpha'_{2,5}}]) + (c_2[e_4, e_{\alpha'_{2,5}}] + c_3[e_2, e_{\alpha'_{3,4}}]) \\
&= (c_2 - 2c_1)e_{\alpha'_{1,5}} + (c_3 - c_2)e_{\alpha'_{2,4}}
\end{aligned}$$

Note that the coefficient of $e_{\alpha'_{1,5}}$ comes from the fact that $\beta_1 - \alpha_5$ is also a root, and $\alpha_5 < \beta_1$ in lexicographic order. Next, by the successor diagram,

$$\begin{aligned}
[E, F_5] &= \sum_{i=4}^{6} c_i [E, e_{-\beta_i}] \\
&= c_4[e_5, e_{-\alpha_{1,5}}] + c_4[e_1, e_{-\alpha_{1,5}}] + c_5[e_5, e_{-\alpha'_{2,5}}] + c_5[e_2, e_{-\alpha'_{2,5}}] + c_6[e_4, e_{-\alpha'_{3,4}}]
\end{aligned}$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned}
[e_5, e_{-\alpha_{1,5}}] &= [e_5, [e_{-\alpha_5}, e_{-\alpha_{1,4}}]] \\
&= [[e_5, e_{-\alpha_5}], e_{-\alpha_{1,4}}] \\
&= -[h_5, e_{-\alpha_{1,4}}] \\
&= (C_{1,5} + C_{2,5} + C_{3,5} + C_{4,5})e_{-\alpha_{1,4}} \\
&= -2e_{-\alpha_{1,4}} \qquad (67)
\end{aligned}$$

$$\begin{aligned}
[e_1, e_{-\alpha_{1,5}}] &= [e_1, [e_{-\alpha_{2,5}}, e_{-\alpha_1}]] \\
&= [e_{-\alpha_{2,5}}, [e_1, e_{-\alpha_1}]] \\
&= [h_1, e_{-\alpha_{2,5}}] \\
&= -(C_{2,1} + C_{3,1} + C_{4,1} + C_{5,1})e_{-\alpha_{2,5}} \\
&= e_{-\alpha_{2,5}} \qquad (68)
\end{aligned}$$

$$
\begin{aligned}
[e_5, e_{-\alpha'_{2,5}}] &= \frac{1}{2}[e_5, [e_{-\alpha_5}, e_{-\alpha_{2,5}}]] \\
&= \frac{1}{2}[[e_5, e_{-\alpha_5}], e_{-\alpha_{2,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{2,5}}]] \\
&= -\frac{1}{2}[h_5, e_{-\alpha_{2,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, [e_{-\alpha_5}, e_{-\alpha_{2,4}}]]] \\
&= \frac{1}{2}(C_{2,5} + C_{3,5} + C_{4,5} + C_{5,5})e_{-\alpha_{2,5}} + \frac{1}{2}[e_{-\alpha_5}, [[e_5, e_{-\alpha_5}], e_{-\alpha_{2,4}}]] \\
&= \frac{1}{2}[e_{-\alpha_5}, -[h_5, e_{-\alpha_{2,4}}]] \\
&= \frac{1}{2}[e_{-\alpha_5}, (C_{2,5} + C_{3,5} + C_{4,5})e_{-\alpha_{2,4}}] \\
&= -[e_{-\alpha_5}, e_{-\alpha_{2,4}}] \\
&= -e_{-\alpha_{2,5}} \qquad (69)
\end{aligned}
$$

$$
\begin{aligned}
[e_2, e_{-\alpha'_{2,5}}] &= [e_2, [e_{-\alpha'_{3,5}}, e_{-\alpha_2}]] \\
&= [e_{-\alpha'_{3,5}}, [e_2, e_{-\alpha_2}]] \\
&= [h_2, e_{-\alpha'_{3,5}}] \\
&= -(C_{3,2} + C_{4,2} + 2C_{5,2})e_{-\alpha'_{3,5}} \\
&= e_{-\alpha'_{3,5}} \qquad (70)
\end{aligned}
$$

$$
\begin{aligned}
[e_4, e_{-\alpha'_{3,4}}] &= [e_4, [e_{-\alpha_4}, e_{-\alpha'_{3,5}}]] \\
&= [[e_4, e_{-\alpha_4}], e_{-\alpha'_{3,5}}] \\
&= -[h_4, e_{-\alpha'_{3,5}}] \\
&= -(C_{3,4} + C_{4,4} + 2C_{5,4})e_{-\alpha'_{3,5}} \\
&= e_{-\alpha'_{3,5}} \qquad (71)
\end{aligned}
$$

Therefore,

$$
[E, F_5] = -2c_4 e_{-\alpha_{1,4}} + (c_4 - c_5)e_{-\alpha_{2,5}} + (c_5 + c_6)e_{-\alpha'_{3,5}} \qquad (72)
$$

$$
\begin{aligned}
[E, E_5 + F_5] = & (c_2 - 2c_1)e_{\alpha'_{1,5}} + (c_3 - c_2)e_{\alpha'_{2,4}} - 2c_4 e_{-\alpha_{1,4}} + (c_4 - c_5)e_{-\alpha_{2,5}} \\
& + (c_5 + c_6)e_{-\alpha'_{3,5}}
\end{aligned}
$$

Since $E_5 + F_5 \in \mathfrak{g}^E$, therefore

$$
2c_1 = c_2 = c_3; \quad c_4 = c_5 = c_6 = 0
$$

We can simply pick 1 as the values of c_2, c_3, then the generator of I_5 for this choice is

$$
E_5 = 2e_{\alpha_{1,5}} + e_{\alpha'_{2,5}} + e_{\alpha'_{3,4}} \tag{73}
$$

Generators of I_6

Now, we calculate the generators of I_6. Let $\Phi_6 = \{\alpha'_{1,5}, \alpha'_{2,4}\} = \{\beta_i\}_{i=1,2}$ and $\Phi_{-4} = \{-\alpha_{1,4}, -\alpha_{2,5}, -\alpha'_{3,5}\} = \{\psi_i\}_{i=1,2,3}$. Suppose $E_6 + F_4$ is an arbitrary element of I_6, where

$$
E_6 = c_1 e_{\beta_1} + c_2 e_{\beta_2}; \quad F_4 = \sum_{i=3}^{5} c_i e_{\psi_i}
$$

By the successor diagram, $\alpha'_{1,4}$ is the successor of β_1 and β_2, and $\alpha'_{2,3}$ is the successor of β_2. Thus, by the Chevalley formula:

$$
\begin{aligned}
[E, E_6] &= \sum_{i=1}^{5} \sum_{j=1}^{2} c_j [e_i, e_{\beta_j}] \\
&= (c_1[e_4, e_{\beta_1}] + c_2[e_1, e_{\beta_2}]) + c_2[e_3, e_{\beta_2}] \\
&= (c_2 - c_1)e_{\alpha'_{1,4}} - c_2 e_{\alpha'_{2,3}}
\end{aligned}
$$

Next, by the successor diagram,

$$
\begin{aligned}
[E, F_4] &= c_3[E, e_{-\alpha_{1,4}}] + c_4[E, e_{-\alpha_{2,5}}] + c_5[E, e_{-\alpha'_{3,5}}] \\
&= c_3[e_4, e_{-\alpha_{1,4}}] + c_3[e_1, e_{-\alpha_{1,4}}] + c_4[e_5, e_{-\alpha_{2,5}}] + c_4[e_2, e_{-\alpha_{2,5}}] + c_5[e_5, e_{-\alpha'_{3,5}}]
\end{aligned}
$$

$$+ c_5[e_3, e_{-\alpha'_{3,5}}]$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned}
[e_4, e_{-\alpha_{1,4}}] &= [e_4, [e_{-\alpha_4}, e_{-\alpha_{1,3}}]] \\
&= [[e_4, e_{-\alpha_4}], e_{-\alpha_{1,3}}] \\
&= -[h_4, e_{-\alpha_{1,3}}] \\
&= (C_{1,4} + C_{2,4} + C_{3,4})e_{-\alpha_{1,3}} \\
&= -e_{-\alpha_{1,3}} \qquad (74)
\end{aligned}$$

$$\begin{aligned}
[e_1, e_{-\alpha_{1,4}}] &= [e_1, [e_{-\alpha_{2,4}}, e_{-\alpha_1}]] \\
&= [e_{-\alpha_{2,4}}, [e_1, e_{-\alpha_1}]] \\
&= [h_1, e_{-\alpha_{2,4}}] \\
&= -(C_{2,1} + C_{3,1} + C_{4,1})e_{-\alpha_{2,4}} \\
&= e_{-\alpha_{2,4}} \qquad (75)
\end{aligned}$$

$$\begin{aligned}
[e_5, e_{-\alpha_{2,5}}] &= [e_5, [e_{-\alpha_5}, e_{-\alpha_{2,4}}]] \\
&= [[e_5, e_{-\alpha_5}], e_{-\alpha_{2,4}}] \\
&= -[h_5, e_{-\alpha_{2,4}}] \\
&= (C_{2,5} + C_{3,5} + C_{4,5})e_{-\alpha_{2,4}} \\
&= -2e_{-\alpha_{2,4}} \qquad (76)
\end{aligned}$$

$$\begin{aligned}
[e_2, e_{-\alpha_{2,5}}] &= [e_2, [e_{-\alpha_{3,5}}, e_{-\alpha_2}]] \\
&= [e_{-\alpha_{3,5}}, [e_2, e_{-\alpha_2}]] \\
&= [h_2, e_{-\alpha_{3,5}}]
\end{aligned}$$

$$= -(C_{3,2} + C_{4,2} + C_{5,2})e_{-\alpha_{3,5}}$$

$$= e_{-\alpha_{3,5}} \tag{77}$$

$$\begin{aligned}
[e_5, e_{-\alpha'_{3,5}}] &= \frac{1}{2}[e_5, [e_{-\alpha_5}, e_{-\alpha_{3,5}}]] \\
&= \frac{1}{2}[[e_5, e_{-\alpha_5}], e_{-\alpha_{3,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{3,5}}]] \\
&= -\frac{1}{2}[h_5, e_{-\alpha_{3,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{3,5}}]] \\
&= \frac{1}{2}(C_{3,5} + C_{4,5} + C_{5,5})e_{-\alpha_{3,5}} + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{3,5}}]] \\
&= \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{3,5}}]]
\end{aligned}$$

Note that:

$$\begin{aligned}
[e_5, e_{-\alpha_{3,5}}] &= [e_5, [e_{-\alpha_5}, e_{-\alpha_{3,4}}]] \\
&= [[e_5, e_{-\alpha_5}], e_{-\alpha_{3,4}}] \\
&= -[h_5, e_{-\alpha_{3,4}}] \\
&= (C_{3,5} + C_{4,5})e_{-\alpha_{3,4}} \\
&= -2e_{-\alpha_{3,4}} \qquad (78)
\end{aligned}$$

Thus,

$$\begin{aligned}
[e_5, e_{-\alpha'_{3,5}}] &= -[e_{-\alpha_5}, e_{-\alpha_{3,4}}] \\
&= -e_{-\alpha_{3,5}} \qquad (79)
\end{aligned}$$

Also,

$$\begin{aligned}
[e_3, e_{-\alpha'_{3,5}}] &= [e_3, [e_{-\alpha'_{4,5}}, e_{-\alpha_3}]] \\
&= [e_{-\alpha'_{4,5}}, [e_3, e_{-\alpha_3}]] \\
&= [h_3, e_{-\alpha'_{4,5}}]
\end{aligned}$$

$$= -(C_{4,3} + 2C_{5,3})e_{-\alpha'_{4,5}}$$

$$= e_{-\alpha'_{4,5}} \tag{80}$$

Hence,

$$[E, E_6 + F_4] = (c_2 - c_1)e_{\alpha'_{1,4}} - c_2 e_{\alpha'_{2,3}} - c_3 e_{-\alpha_{1,3}} + (c_3 - 2c_4)e_{-\alpha_{2,4}} + (c_4 - c_5)e_{-\alpha_{3,5}} + c_5 e_{-\alpha'_{4,5}}$$

Since $E_6 + F_4 \in \mathfrak{g}^E$, therefore

$$c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

This implies that I_6 is trivial.

Generators of I_7

Now, we calculate the generators of I_7. Let $\Phi_7 = \{\alpha'_{1,4}, \alpha'_{2,3}\} = \{\beta_i\}_{i=1,2}$, and $\Phi_{-3} = \{-\alpha_{1,3}, -\alpha_{2,4}, -\alpha_{3,5}, -\alpha'_{4,5}\} = \{\psi_i\}_{i=1,2,3,4}$. Suppose $E_7 + F_3$ is an arbitrary element of I_7, where

$$E_7 = c_1 e_{\beta_1} + c_2 e_{\beta_2}; \quad F_3 = \sum_{i=3}^{6} c_i e_{\psi_i}$$

By the successor diagram, $\alpha'_{1,3}$ is the successor of β_1 and β_2. Thus, by the Chevalley formula:

$$[E, E_7] = \sum_{i=1}^{5}\sum_{j=1}^{2} c_j [e_i, e_{\beta_j}]$$

$$= c_1[e_3, e_{\alpha'_{1,4}}] + c_2[e_1, e_{\alpha'_{2,3}}]$$

$$= (c_2 - c_1)e_{\alpha'_{1,3}}$$

Next, by the successor diagram,

$$[E, F_3] = c_3[E, e_{-\alpha_{1,3}}] + c_4[E, e_{-\alpha_{2,4}}] + c_5[E, e_{-\alpha_{3,5}}] + c_6[E, e_{-\alpha'_{4,5}}]$$

$$= c_3[e_3, e_{-\alpha_{1,3}}] + c_3[e_1, e_{-\alpha_{1,3}}] + c_4[e_4, e_{-\alpha_{2,4}}] + c_4[e_2, e_{-\alpha_{2,4}}] + c_5[e_5, e_{-\alpha_{3,5}}]$$
$$+ c_5[e_3, e_{-\alpha_{3,5}}] + c_6[e_5, e_{-\alpha'_{4,5}}]$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned}
[e_3, e_{-\alpha_{1,3}}] &= [e_3, [e_{-\alpha_3}, e_{-\alpha_{1,2}}]] \\
&= [[e_3, e_{-\alpha_3}], e_{-\alpha_{1,2}}] \\
&= -[h_3, e_{-\alpha_{1,2}}] \\
&= (C_{1,3} + C_{2,3}) e_{-\alpha_{1,2}} \\
&= -e_{-\alpha_{1,2}} \qquad (81)
\end{aligned}$$

$$\begin{aligned}
[e_1, e_{-\alpha_{1,3}}] &= [e_1, [e_{-\alpha_{2,3}}, e_{-\alpha_1}]] \\
&= [e_{-\alpha_{2,3}}, [e_1, e_{-\alpha_1}]] \\
&= [h_1, e_{-\alpha_{2,3}}] \\
&= -(C_{2,1} + C_{3,1}) e_{-\alpha_{2,3}} \\
&= e_{-\alpha_{2,3}} \qquad (82)
\end{aligned}$$

$$\begin{aligned}
[e_4, e_{-\alpha_{2,4}}] &= [e_4, [e_{-\alpha_4}, e_{-\alpha_{2,3}}]] \\
&= [[e_4, e_{-\alpha_4}], e_{-\alpha_{2,3}}] \\
&= -[h_4, e_{-\alpha_{2,3}}] \\
&= (C_{2,4} + C_{3,4}) e_{-\alpha_{2,3}} \\
&= -e_{-\alpha_{2,3}} \qquad (83)
\end{aligned}$$

$$\begin{aligned}
[e_2, e_{-\alpha_{2,4}}] &= [e_2, [e_{-\alpha_{3,4}}, e_{-\alpha_2}]] \\
&= [e_{-\alpha_{3,4}}, [e_2, e_{-\alpha_2}]]
\end{aligned}$$

$$\begin{aligned}
&= [h_2, e_{-\alpha_{3,4}}] \\
&= -(C_{3,2} + C_{4,2}) e_{-\alpha_{3,4}} \\
&= e_{-\alpha_{3,4}} \qquad (84)
\end{aligned}$$

$$\begin{aligned}
[e_5, e_{-\alpha_{3,5}}] &= [e_5, [e_{-\alpha_5}, e_{-\alpha_{3,4}}]] \\
&= [[e_5, e_{-\alpha_5}], e_{-\alpha_{3,4}}] \\
&= -[h_5, e_{-\alpha_{3,4}}] \\
&= (C_{3,5} + C_{4,5}) e_{-\alpha_{3,4}} \\
&= -2e_{-\alpha_{3,4}} \qquad (85)
\end{aligned}$$

$$\begin{aligned}
[e_3, e_{-\alpha_{3,5}}] &= [e_3, [e_{-\alpha_{4,5}}, e_{-\alpha_3}]] \\
&= [e_{-\alpha_{4,5}}, [e_3, e_{-\alpha_3}]] \\
&= [h_3, e_{-\alpha_{4,5}}] \\
&= -(C_{4,3} + C_{5,3}) e_{-\alpha_{4,5}} \\
&= e_{-\alpha_{4,5}} \qquad (86)
\end{aligned}$$

$$\begin{aligned}
[e_5, e_{-\alpha'_{4,5}}] &= \frac{1}{2}[e_5, [e_{-\alpha_5}, e_{-\alpha_{4,5}}]] \\
&= \frac{1}{2}[[e_5, e_{-\alpha_5}], e_{-\alpha_{4,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, e_{-\alpha_{4,5}}]] \\
&= -\frac{1}{2}[h_5, e_{-\alpha_{4,5}}] + \frac{1}{2}[e_{-\alpha_5}, [e_5, [e_{-\alpha_5}, e_{-\alpha_4}]]] \\
&= \frac{1}{2}(C_{4,5} + C_{5,5}) e_{-\alpha_{4,5}} + \frac{1}{2}[e_{-\alpha_5}, [[e_5, e_{-\alpha_5}], e_{-\alpha_4}]] \\
&= \frac{1}{2}[e_{-\alpha_5}, [-h_5, e_{-\alpha_4}]] \\
&= \frac{1}{2}[e_{-\alpha_5}, C_{4,5} e_{-\alpha_4}] \\
&= -[e_{-\alpha_5}, e_{-\alpha_4}] \\
&= -e_{-\alpha_{4,5}} \qquad (87)
\end{aligned}$$

Hence,

$$\begin{aligned}[E, F_3] &= -c_3 e_{-\alpha_{1,2}} + (c_3 - c_4)e_{-\alpha_{2,3}} + (c_4 - 2c_5)e_{-\alpha_{3,4}} + (c_5 - c_6)e_{-\alpha_{4,5}} \qquad (88)\\ [E, E_7 + F_3] &= (c_2 - c_1)e_{\alpha'_{1,3}} - c_3 e_{-\alpha_{1,2}} + (c_3 - c_4)e_{-\alpha_{2,3}} + (c_4 - 2c_5)e_{-\alpha_{3,4}}\\ &\quad + (c_5 - c_6)e_{-\alpha_{4,5}}\end{aligned}$$

Since $E_7 + F_3 \in \mathfrak{g}^E$, therefore

$$c_1 = c_2; \quad c_3 = c_4 = c_5 = c_6 = 0$$

We can simply pick 1 as values of c_1 and c_2. Then, the generator of I_7 for this choice is:

$$E_7 = e_{\alpha'_{1,4}} + e_{\alpha'_{2,3}}$$

Generators of I_8

Now, we calculate the generators of I_8. Let $\Phi_8 = \{\alpha'_{1,3}\}$, and let

$$\Phi_{-2} = \{-\alpha_{1,2}, -\alpha_{2,3}, -\alpha_{3,4}, -\alpha_{4,5}\} = \{\psi_i\}_{i=1,2,3,4}$$

Suppose $E_8 + F_2$ is an arbitrary element of I_8, where

$$E_8 = c_1 e_{\alpha'_{1,3}}; \quad F_2 = \sum_{i=2}^{5} c_i e_{\psi_i}$$

By the successor diagram, $\alpha'_{1,2}$ is the successor of $\alpha'_{1,3}$. Thus, by the Chevalley formula:

$$\begin{aligned}[E, E_8] &= c_1 \sum_{i=1}^{5} [e_i, e_{\alpha'_{1,3}}]\\ &= c_1 [e_2, e_{\alpha'_{1,3}}]\\ &= -c_1 e_{\alpha'_{1,2}}\end{aligned}$$

Next, by the successor diagram,

$$
\begin{aligned}
[E, F_2] &= c_2[E, e_{-\alpha_{1,2}}] + c_3[E, e_{-\alpha_{2,3}}] + c_4[E, e_{-\alpha_{3,4}}] + c_5[E, e_{-\alpha_{4,5}}] \\
&= c_2[e_1, e_{-\alpha_{1,2}}] + c_2[e_2, e_{-\alpha_{1,2}}] + c_3[e_2, e_{-\alpha_{2,3}}] + c_3[e_3, e_{-\alpha_{2,3}}] + c_4[e_3, e_{-\alpha_{3,4}}] \\
&\quad + c_4[e_4, e_{-\alpha_{3,4}}] + c_5[e_4, e_{-\alpha_{4,5}}] + c_5[e_5, e_{-\alpha_{4,5}}]
\end{aligned}
$$

Using the Jacobi identity and Chevalley relations, we have

$$
\begin{aligned}
[e_1, e_{-\alpha_{1,2}}] &= -[e_1, [e_{-\alpha_1}, e_{-\alpha_2}]] \\
&= -[[e_1, e_{-\alpha_1}], e_{-\alpha_2}]] \\
&= [h_1, e_{-\alpha_2}] \\
&= -C_{2,1} e_{-\alpha_2} \\
&= e_{-\alpha_2}
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
[e_2, e_{-\alpha_{1,2}}] &= [e_2, [e_{-\alpha_2}, e_{-\alpha_1}]] \\
&= [[e_2, e_{-\alpha_2}], e_{-\alpha_1}] \\
&= -[h_2, e_{-\alpha_1}] \\
&= C_{1,2} e_{-\alpha_1} \\
&= -e_{-\alpha_1}
\end{aligned} \tag{90}
$$

Similarly, it is easy to obtain:

$$
\begin{aligned}
[e_2, e_{-\alpha_{2,3}}] &= e_{-\alpha_3} \\
[e_3, e_{-\alpha_{2,3}}] &= -e_{-\alpha_2} \\
[e_3, e_{-\alpha_{3,4}}] &= e_{-\alpha_4} \\
[e_4, e_{-\alpha_{3,4}}] &= -e_{-\alpha_3} \\
[e_4, e_{-\alpha_{4,5}}] &= e_{-\alpha_5}
\end{aligned}
$$

$$[e_5, e_{-\alpha_{4,5}}] = -2e_{-\alpha_4}$$

Hence,

$$\begin{aligned}[E, E_8 + F_2] = &-c_1 e_{\alpha'_{1,2}} - c_2 e_{-\alpha_1} + (c_2 - c_3)e_{-\alpha_2} + (c_3 - c_4)e_{-\alpha_3} + (c_4 - 2c_5)e_{-\alpha_4} \\ &+ c_5 e_{-\alpha_5}\end{aligned}$$

Since $E_8 + F_2 \in \mathfrak{g}^E$, we obtain

$$c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

This implies that I_8 is trivial.

Generators of I_9

Now, we calculate the generators of I_9. Suppose $E_9 + F_1$ is an arbitrary element of I_9, where

$$E_9 = c_1 e_{\alpha'_{1,2}}; \quad F_1 = \sum_{i=1}^{5} c_{i+1} e_{-\alpha_i}$$

Then, since $\alpha'_{1,2}$ is already the highest root, so $[E, e_{\alpha'_{1,2}}] = 0$, c_1 is allowed to be any complex number. Next, by the successor diagram and Chevalley relations, we have

$$\begin{aligned}[E, F_1] &= [E, c_2 e_{-\alpha_1} + c_3 e_{-\alpha_2} + c_4 e_{-\alpha_3} + c_5 e_{-\alpha_4} + c_6 e_{-\alpha_5}] \\ &= c_2[e_1, e_{-\alpha_1}] + c_3[e_2, e_{-\alpha_2}] + c_4[e_3, e_{-\alpha_3}] + c_5[e_4, e_{-\alpha_4}] + c_6[e_5, e_{-\alpha_5}] \\ &= -c_2 h_1 - c_3 h_2 - c_4 h_3 - c_5 h_4 - c_6 h_5 \end{aligned} \tag{91}$$

Since $E_9 + F_1 \in \mathfrak{g}^E$, therefore

$$c_2 = \cdots = c_6 = 0$$

Hence, by picking $c_1 = 1$, we have the generator of I_9 for this choice is

$$E_9 = e_{\alpha'_{1,2}}$$

Now, we obtain a graded basis of $\mathfrak{g}^E$:

$$\mathfrak{g}^E = \bigoplus_{j=1,3,5,7,9} I_j = \bigoplus_{j=1,3,5,7,9} \mathbb{C}E_j$$

where

$$\begin{aligned}
&E_1 = E \\
&E_3 = e_{\alpha_{1,3}} + e_{\alpha_{2,4}} + e_{\alpha_{3,5}} + 2e_{\alpha'_{4,5}} \\
&E_5 = 2e_{\alpha_{1,5}} + e_{\alpha'_{2,5}} + e_{\alpha'_{3,4}} \\
&E_7 = e_{\alpha'_{1,4}} + e_{\alpha'_{2,3}} \\
&E_9 = e_{\alpha'_{1,2}}
\end{aligned}$$

2.3.2 Centralizer of the Cyclic Element

Now, we calculate a graded basis of $\mathfrak{g}^\Lambda$. As mentioned, in [17, Lemma 6.4A], Kostant proved that as vector spaces, $\mathfrak{g}^\Lambda$ is isomorphic to $\mathfrak{g}^E$ via a projection to the positive root space. Let us explain this lemma concretely for B_5. Let the cyclic element be $\Lambda = E + e_\gamma$ where γ is the lowest root of B_5, and let $c_1 x_j + c_2 x_{j-10}$ be an element of $\mathfrak{g}^\Lambda$ with degree j, specifically, x_j, x_{j-10} are with heights $j, j-10$, respectively. Then, we have

$$\begin{aligned}
[\Lambda, c_1 x_j + c_2 x_{j-10}] &= c_1[\Lambda, x_j] + c_2[\Lambda, x_{j-10}] \\
&= c_1 \underbrace{[E, x_j]}_{\text{degree } j+1} + c_1 \underbrace{[e_\gamma, x_j]}_{\text{degree } j-9} + c_2 \underbrace{[E, x_{j-10}]}_{\text{degree } j-9} + c_2 \underbrace{[e_\gamma, x_{j-10}]}_{=0} = 0
\end{aligned}$$

This implies that $[E, x_j] = 0$. Referring to the previous argument of $\mathfrak{g}^E$, we have $x_j \in \mathfrak{g}^E$ with $j = 1, 3, 5, 7, 9$. Hence, we only need to consider the odd degrees in the calculation of $\mathfrak{g}^\Lambda$.

Degree 1

When the degree $j = 1$, let $kE + F_9$ be an element of this component, where

$$F_9 = ce_{-\alpha'_{1,2}}$$

for $k, c \in \mathbb{C}, k \neq 0$. Then, by the successor diagram of B_5 and Equation (53),

$$\begin{aligned}
[\Lambda, kE + F_9] &= k[E + e_\gamma, E] + c[E + e_\gamma, e_{-\alpha'_{1,2}}] \\
&= k[E, E] + k[e_\gamma, E] + c[E, e_{-\alpha'_{1,2}}] + c[e_\gamma, e_{-\alpha'_{1,2}}] \\
&= k[e_\gamma, e_2] + c[e_2, e_{-\alpha'_{1,2}}] \\
&= (c - k)[e_2, e_{-\alpha'_{1,2}}] \\
&= -(c - k)e_{-\alpha'_{1,3}} \\
&= 0
\end{aligned}$$

Hence, this implies that $k = c$. Let $k = c = 1$, then a generator of this component fo this choice is simply $\Lambda = E + e_\gamma$.

Degree 3

When the degree $j = 3$, let $kE_3 + F_7$ be an element of this component, where

$$F_7 = c_1 e_{-\alpha'_{1,4}} + c_2 e_{-\alpha'_{2,3}}$$

for $k, c_1, c_2 \in \mathbb{C}, k \neq 0$. Then, by Equation (60),

$$\begin{aligned}
[\Lambda, kE_3 + F_7] &= [E + e_\gamma, kE_3 + F_7] \\
&= k[E, E_3] + [E, F_7] + k[e_\gamma, E_3] + [e_\gamma, F_7]
\end{aligned}$$

$$
\begin{aligned}
&= [E, F_7] + k[e_\gamma, E_3] \\
&= -c_1 e_{-\alpha'_{1,5}} + (c_1 - c_2) e_{-\alpha'_{2,4}} + k[e_\gamma, E_3] \qquad (92) \\
&= 0
\end{aligned}
$$

To calculate $k[e_\gamma, E_3]$, by Equation (61) and the successor diagram of B_5,

$$
\begin{aligned}
k[e_\gamma, E_3] &= k[e_{-\alpha'_{1,2}}, e_{\alpha_{1,3}} + e_{\alpha_{2,4}} + e_{\alpha_{3,5}} + 2e_{\alpha'_{4,5}}] \\
&= k[e_{-\alpha'_{1,2}}, e_{\alpha_{1,3}}] + k[e_{-\alpha'_{1,2}}, e_{\alpha_{2,4}}]
\end{aligned}
$$

Using the Jacobi identity and Chevalley relations, and by Equation (53), (55) , and (57), we have

$$
\begin{aligned}
[e_{-\alpha'_{1,2}}, e_{\alpha_{1,3}}] &= [e_{-\alpha'_{1,2}}, [e_{\alpha_1}, e_{\alpha_{2,3}}]] \\
&= [e_{\alpha_1}, [e_{-\alpha'_{1,2}}, e_{\alpha_{2,3}}]] \\
&= [e_{\alpha_1}, [e_{-\alpha'_{1,2}}, [e_{\alpha_2}, e_{\alpha_3}]]] \\
&= [e_{\alpha_1}, [[e_{-\alpha'_{1,2}}, e_{\alpha_2}], e_{\alpha_3}]] \\
&= [e_{\alpha_1}, [e_{-\alpha'_{1,3}}, e_{\alpha_3}]] \\
&= -[e_{\alpha_1}, [e_{\alpha_3}, e_{-\alpha'_{1,3}}]] \\
&= [e_{\alpha_1}, e_{-\alpha'_{1,4}}] \\
&= e_{-\alpha'_{2,4}} \qquad (93)
\end{aligned}
$$

By Equations (53), (55) and (58), we have

$$
\begin{aligned}
[e_{-\alpha'_{1,2}}, e_{\alpha_{2,4}}] &= [e_{-\alpha'_{1,2}}, [e_{\alpha_2}, e_{\alpha_{3,4}}]] \\
&= [[e_{-\alpha'_{1,2}}, e_{\alpha_2}], e_{\alpha_{3,4}}] \\
&= [e_{-\alpha'_{1,3}}, e_{\alpha_{3,4}}] \\
&= [e_{-\alpha'_{1,3}}, [e_{\alpha_3}, e_{\alpha_4}]]
\end{aligned}
$$

$$
\begin{aligned}
&= [[e_{-\alpha'_{1,3}}, e_{\alpha_3}], e_{\alpha_4}] \\
&= [e_{-\alpha'_{1,4}}, e_{\alpha_4}] \\
&= e_{-\alpha'_{1,5}} \qquad (94)
\end{aligned}
$$

Hence,

$$k[e_\gamma, E_3] = ke_{-\alpha'_{2,4}} + ke_{-\alpha'_{1,5}}$$

Referring to Equation (92),

$$
\begin{cases}
k - c_1 = 0 \\
c_1 - c_2 + k = 0
\end{cases}
$$

Solving this system, we have $c_1 = k, c_2 = 2k$. Let $k = 1$, then a generator of this component for this choice is $E_3 + F_7$ where $F_7 = e_{-\alpha'_{1,4}} + 2e_{-\alpha'_{2,3}}$.

Degree 5

When the degree $j = 5$, let $kE_5 + F_5$ be an element of this component, where

$$F_5 = c_1 e_{-\alpha_{1,5}} + c_2 e_{-\alpha'_{2,5}} + c_3 e_{-\alpha'_{3,4}}$$

for $k, c_1, c_2, c_3 \in \mathbb{C}, k \neq 0$. Then, by Equation (72),

$$
\begin{aligned}
[\Lambda, kE_5 + F_5] &= [E + e_\gamma, kE_5 + F_5] \\
&= k[E, E_5] + [E, F_5] + k[e_\gamma, E_5] + [e_\gamma, F_5] \\
&= [E, F_5] + k[e_\gamma, E_5] \\
&= -2c_1 e_{-\alpha_{1,4}} + (c_1 - c_2)e_{-\alpha_{2,5}} + (c_2 + c_3)e_{-\alpha'_{3,5}} + k[e_\gamma, E_5] \qquad (95) \\
&= 0
\end{aligned}
$$

To calculate $k[e_\gamma, E_5]$, by Equation (73) and the successor diagram of B_5,

$$\begin{aligned} k[e_\gamma, E_5] &= k[e_{-\alpha'_{1,2}}, 2e_{\alpha_{1,5}} + e_{\alpha'_{2,5}} + e_{\alpha'_{3,4}}] \\ &= 2k[e_{-\alpha'_{1,2}}, e_{\alpha_{1,5}}] + k[e_{-\alpha'_{1,2}}, e_{\alpha'_{2,5}}] \end{aligned}$$

Using the Jacobi identity and Chevalley relations, we have

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha_{1,5}}] &= [e_{-\alpha'_{1,2}}, [e_{\alpha_1}, e_{\alpha_{2,5}}]] \\ &= [e_{\alpha_1}, [e_{-\alpha'_{1,2}}, e_{\alpha_{2,5}}]] \end{aligned}$$

Note that by Equations (63) and (94),

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha_{2,5}}] &= -[e_{-\alpha'_{1,2}}, [e_{\alpha_5}, e_{\alpha_{2,4}}]] \\ &= -[e_{\alpha_5}, [e_{-\alpha'_{1,2}}, e_{\alpha_{2,4}}]] \\ &= -[e_{\alpha_5}, e_{-\alpha'_{1,5}}] \\ &= e_{-\alpha_{1,5}} \end{aligned} \tag{96}$$

Thus, by Equation (68),

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha_{1,5}}] &= [e_{\alpha_1}, e_{-\alpha_{1,5}}] \\ &= e_{-\alpha_{2,5}} \end{aligned} \tag{97}$$

By Equations (67) and (96), we have

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha'_{2,5}}] &= -\frac{1}{2}[e_{-\alpha'_{1,2}}, [e_{\alpha_5}, e_{\alpha_{2,5}}]] \\ &= -\frac{1}{2}[e_{\alpha_5}, [e_{-\alpha'_{1,2}}, e_{\alpha_{2,5}}]] \\ &= -\frac{1}{2}[e_{\alpha_5}, e_{-\alpha_{1,5}}] \\ &= e_{-\alpha_{1,4}} \end{aligned} \tag{98}$$

Hence,

$$k[e_\gamma, E_5] = 2ke_{-\alpha_{2,5}} + ke_{-\alpha_{1,4}}$$

Referring to Equation (95), we have

$$\begin{cases} -2c_1 + k = 0 \\ c_1 - c_2 + 2k = 0 \\ c_2 + c_3 = 0 \end{cases}$$

Solving this system yields $c_1 = \frac{k}{2}, c_2 = \frac{5k}{2}, c_3 = -\frac{5k}{2}$. Let $k = 1$, then $c_1 = \frac{1}{2}, c_2 = \frac{5}{2}, c_3 = -\frac{5}{2}$, and the generator of this component for this choice is $E_5 + F_5$ where $F_5 = \frac{1}{2}e_{-\alpha_{1,5}} + \frac{5}{2}e_{-\alpha'_{2,5}} - \frac{5}{2}e_{-\alpha'_{3,4}}$.

Degree 7

When the degree $j = 7$, let $kE_7 + F_3$ be an element of this component, where

$$F_3 = c_1e_{-\alpha_{1,3}} + c_2e_{-\alpha_{2,4}} + c_3e_{-\alpha_{3,5}} + c_4e_{-\alpha'_{4,5}}$$

for $k, c_1, c_2, c_3, c_4 \in \mathbb{C}, k \neq 0$. Then, by Equation (88),

$$\begin{aligned}
&[\Lambda, kE_7 + F_3] \\
&= [E + e_\gamma, kE_7 + F_3] \\
&= k[E, E_7] + [E, F_3] + k[e_\gamma, E_7] + [e_\gamma, F_3] \\
&= [E, F_3] + k[e_\gamma, E_7] \\
&= -c_1e_{-\alpha_{1,2}} + (c_1 - c_2)e_{-\alpha_{2,3}} + (c_2 - 2c_3)e_{-\alpha_{3,4}} + (c_3 - c_4)e_{-\alpha_{4,5}} + k[e_\gamma, E_7] \qquad (99) \\
&= 0
\end{aligned}$$

To calculate $k[e_\gamma, E_7]$,

$$\begin{aligned} k[e_\gamma, E_7] &= k[e_{-\alpha'_{1,2}}, e_{\alpha'_{1,4}} + e_{\alpha'_{2,3}}] \\ &= k[e_{-\alpha'_{1,2}}, e_{\alpha'_{1,4}}] + k[e_{-\alpha'_{1,2}}, e_{\alpha'_{2,3}}] \end{aligned} \tag{100}$$

By the Jacobi identity and Chevalley relations,

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha'_{1,4}}] &= [e_{-\alpha'_{1,2}}, [e_{\alpha'_{1,5}}, e_{\alpha_4}]] \\ &= [[e_{-\alpha'_{1,2}}, e_{\alpha'_{1,5}}], e_{\alpha_4}] \end{aligned}$$

By Equations (76) and (97),

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha'_{1,5}}] &= \frac{1}{2}[e_{-\alpha'_{1,2}}, [e_{\alpha_{1,5}}, e_{\alpha_5}]] \\ &= \frac{1}{2}[[e_{-\alpha'_{1,2}}, e_{\alpha_{1,5}}], e_{\alpha_5}] \\ &= \frac{1}{2}[e_{-\alpha_{2,5}}, e_{\alpha_5}] \\ &= e_{-\alpha_{2,4}} \end{aligned}$$

Thus, by Equation (83),

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha'_{1,4}}] &= [e_{-\alpha_{2,4}}, e_{\alpha_4}] \\ &= e_{-\alpha_{2,3}} \end{aligned} \tag{101}$$

Also, by the Jacobi identity and Chevalley relations,

$$\begin{aligned} [e_{-\alpha'_{1,2}}, e_{\alpha'_{2,3}}] &= [e_{-\alpha'_{1,2}}, [e_{\alpha'_{2,4}}, e_{\alpha_3}]] \\ &= [[e_{-\alpha'_{1,2}}, e_{\alpha'_{2,4}}], e_{\alpha_3}] \end{aligned}$$

By Equations (74) and (98),

$$\begin{aligned}
[e_{-\alpha'_{1,2}}, e_{\alpha'_{2,4}}] &= [e_{-\alpha'_{1,2}}, [e_{\alpha'_{2,5}}, e_{\alpha_4}]] \\
&= [[e_{-\alpha'_{1,2}}, e_{\alpha'_{2,5}}], e_{\alpha_4}] \\
&= [e_{-\alpha_{1,4}}, e_{\alpha_4}] \\
&= e_{-\alpha_{1,3}}
\end{aligned}$$

Thus, by Equation (81),

$$\begin{aligned}
[e_{-\alpha'_{1,2}}, e_{\alpha'_{2,3}}] &= [e_{-\alpha_{1,3}}, e_{\alpha_3}] \\
&= e_{-\alpha_{1,2}}
\end{aligned}$$

Hence,

$$k[e_\gamma, E_7] = ke_{-\alpha_{2,3}} + ke_{-\alpha_{1,2}} \tag{102}$$

Referring to Equation (99), we have

$$\begin{cases}
-c_1 + k = 0 \\
c_1 - c_2 + k = 0 \\
c_2 - 2c_3 = 0 \\
c_3 - c_4 = 0
\end{cases}$$

Solving this system yields $c_1 = c_3 = c_4 = k, c_2 = 2k$. Let $k = 1$, then $c_1 = c_3 = c_4 = 1, c_2 = 2$, the generator of this component for this choice is $E_7 + F_3$ where $F_3 = e_{-\alpha_{1,3}} + 2e_{-\alpha_{2,4}} + e_{-\alpha_{3,5}} + e_{-\alpha'_{4,5}}$.

Degree 9

When the degree $j = 9$, let $kE_9 + F_1$ be an element of this component, where

$$F_1 = c_1 e_{-\alpha_1} + c_2 e_{-\alpha_2} + c_3 e_{-\alpha_3} + c_4 e_{-\alpha_4} + c_5 e_{-\alpha_5}$$

for $k, c_1, c_2, c_3, c_4, c_5 \in \mathbb{C}, k \neq 0$. Then, by Equation (91),

$$\begin{aligned}
[\Lambda, kE_9 + F_1] &= [E + e_\gamma, kE_9 + F_1] \\
&= k[E, E_9] + [E, F_1] + k[e_\gamma, E_9] + [e_\gamma, F_1] \\
&= [E, F_1] + k[e_\gamma, E_9] \\
&= -c_2 h_1 - c_3 h_2 - c_4 h_3 - c_5 h_4 - c_6 h_5 + k[e_\gamma, E_9] && (103) \\
&= 0 && (104)
\end{aligned}$$

Also, note that:

$$\begin{aligned}
k[e_\gamma, E_9] &= k[e_{-\alpha'_{1,2}}, e_{\alpha'_{1,2}}] \\
&= kh_{\alpha'_{1,2}} && (105)
\end{aligned}$$

Since $\alpha'_{1,2}$ is only the successor of $\alpha'_{1,3}$, and they differ by an α_2, therefore

$$\alpha'_{1,2} \cdot \alpha'_{1,2} = \alpha'_{1,2} \cdot \alpha_2 = \frac{1}{2}\alpha_2 \cdot \alpha_2 \tag{106}$$

In the root system of B_5, all the simple roots are with equal length except α_5 is with half of the length, therefore

$$\alpha'_{1,2} \cdot \alpha'_{1,2} = \alpha_5 \cdot \alpha_5 = \frac{1}{2}\alpha_4 \cdot \alpha_4 = \frac{1}{2}\alpha_3 \cdot \alpha_3 = \frac{1}{2}\alpha_2 \cdot \alpha_2 = \frac{1}{2}\alpha_1 \cdot \alpha_1$$

It follows

$$
\begin{aligned}
h_{\alpha'_{1,2}} &= \frac{2\alpha'_{1,2}}{\alpha'_{1,2}\cdot\alpha'_{1,2}} \\
&= \frac{2\alpha_1}{\frac{1}{2}\alpha_1\cdot\alpha_1} + 2\cdot\frac{2\alpha_2}{\frac{1}{2}\alpha_2\cdot\alpha_2} + 2\cdot\frac{2\alpha_3}{\frac{1}{2}\alpha_3\cdot\alpha_3} + 2\cdot\frac{2\alpha_4}{\frac{1}{2}\alpha_4\cdot\alpha_4} + 2\cdot\frac{2\alpha_5}{\alpha_5\cdot\alpha_5} \\
&= 2kh_1 + 4kh_2 + 4kh_3 + 4kh_4 + 2kh_5
\end{aligned} \tag{107}
$$

Referring to Equation (103), we have $c_1 = c_5 = 2k$, and $c_2 = c_3 = c_4 = 4k$. Let $k = 1$, then $c_1 = c_5 = 2$, $c_2 = c_3 = c_4 = 4$, and the generator of this component for this choice is $E_9 + F_1$ where

$$F_1 = 2e_{-\alpha_1} + 4e_{-\alpha_2} + 4e_{-\alpha_3} + 4e_{-\alpha_4} + 2e_{-\alpha_5}$$

Finally, we summarize our results in the following proposition.

Proposition 2. *Let the cyclic element Λ of $\mathfrak{g} = B_5$ be defined as above. Then, its centralizer $\mathfrak{g}^\Lambda$ is given by*

$$\mathfrak{g}^\Lambda = \bigoplus_{j=1,3,5,7,9} \mathbb{C}(E_j + F_{10-j})$$

where

$$
\begin{aligned}
E_1 &= E & F_9 &= e_{-\alpha'_{1,2}} \\
E_3 &= e_{\alpha_{1,3}} + e_{\alpha_{2,4}} + e_{\alpha_{3,5}} + 2e_{\alpha'_{4,5}} & F_7 &= e_{-\alpha'_{1,4}} + 2e_{-\alpha'_{2,3}} \\
E_5 &= 2e_{\alpha_{1,5}} + e_{\alpha'_{2,5}} + e_{\alpha'_{3,4}} & F_5 &= \frac{1}{2}e_{-\alpha_{1,5}} + \frac{5}{2}e_{-\alpha'_{2,5}} - \frac{5}{2}e_{-\alpha'_{3,4}} \\
E_7 &= e_{\alpha'_{1,4}} + e_{\alpha'_{2,3}} & F_3 &= e_{-\alpha_{1,3}} + 2e_{-\alpha_{2,4}} + e_{-\alpha_{3,5}} + e_{-\alpha'_{4,5}} \\
E_9 &= e_{\alpha'_{1,2}} & F_1 &= 2e_{-\alpha_1} + 4e_{-\alpha_2} + 4e_{-\alpha_3} + 4e_{-\alpha_4} + 2e_{-\alpha_5}
\end{aligned}
$$

Now, we present how $P = e^{\frac{2\pi i\cdot \mathrm{ad} H_\rho}{h}}$ diagonalizes on $\mathfrak{g}^\Lambda$ to show that P is a Coxeter element on $\mathfrak{g}^\Lambda$. In [3, Proposition 29], Bourbaki showed that if α is a simple root, then

$$[H_\rho, e_\alpha] = 1\cdot e_\alpha = (H_\rho, \alpha)\cdot e_\alpha$$

Therefore, $(H_\rho, \alpha) = 1$ for any simple root α. For any root $\beta = \sum_{i=1}^{r} c_i\alpha_i$ where α_i's are simple roots, it follows that

$$(H_\rho, \beta) = (H_\rho, \sum_{i=1}^{r} c_i\alpha_i) = \sum_{i=1}^{r} c_i(H_\rho, \alpha_i) = \sum_{i=1}^{r} c_i = \mathrm{ht}(\beta)$$

Thus,

$$[H_\rho, e_\beta] = (H_\rho, \beta) \cdot e_\beta = \mathrm{ht}(\beta) \cdot e_\beta$$

It also follows that

$$P(e_\beta) = e^{\frac{2\pi i \cdot \mathrm{ad}_{H_\rho}}{h}}(e_\beta) = \lambda^{\mathrm{ht}(\beta)} \cdot e_\beta \tag{108}$$

where $\lambda = e^{\frac{2\pi i}{h}}$. Therefore, considering the graded basis of $\mathfrak{g}^\Lambda$ in Proposition 2, for $j = 1, 3, 5, 7, 9$,

$$P(E_j + F_{10-j}) = \lambda^j E_j + \lambda^{j-10} F_{10-j} = \lambda^j (E_j + F_{10-j})$$

Hence, each basis of degree j in $\mathfrak{g}^\Lambda$ is an eigenvector of P with respect to the eigenvalue λ^j, for $j = 1, 3, 5, 7, 9$. As Kostant proved, this implies that $\mathfrak{g}^\Lambda$ is a Cartan subalgebra. Moreover, in [8, Table 2], Coxeter proved that when $\mathfrak{g} = B_n$, the Coxeter element has a spectrum on a Cartan subalgebra as $\{e^{\frac{2k\pi i}{h}} : k = 1, 3, \cdots, 2n-1\}$. Hence, P is conjugate to a Coxeter element when $\mathfrak{g} = B_5$, which verifies the result of [17, Theorem 8.6] by Kostant when $\mathfrak{g} = B_5$. In general, one can apply the same style of explicit calculation to verify this theorem developed by Kostant for all types of simple Lie algebras.

Remark 6. Equation (108) provides us with the tool to prove Lemma 1.1. By Equation (108), it follows that:

$$\begin{aligned} e^{\frac{2\pi i \cdot \mathrm{ad}_{H_\rho}}{h}}(\Lambda) &= e^{\frac{2\pi i \cdot \mathrm{ad}_{H_\rho}}{h}}\left(\sum_{j=1}^{r} E_j\right) + e^{\frac{2\pi i \cdot \mathrm{ad}_{H_\rho}}{h}}(F_{h-1}) \\ &= e^{\frac{2\pi i}{h}} \cdot \left(\sum_{j=1}^{r} E_j\right) + e^{\frac{2\pi i(1-h)}{h}} \cdot F_{h-1} \end{aligned}$$

$$= \zeta_h \cdot \left(\sum_{j=1}^{r} E_j \ + \ F_{h-1} \right)$$

$$= \zeta_h \cdot \Lambda$$

where $\zeta_h = e^{\frac{2\pi i}{h}}$. Therefore, in any representation, $\zeta_h \cdot \Lambda$ is conjugate to Λ, so they have the same spectrum. Hence, for any finite-dimensional representation ψ of $\mathfrak{g}$,

$$\mathrm{Spec}(\psi, \mathfrak{g}) = \zeta_h \cdot \mathrm{Spec}(\psi, \mathfrak{g})$$

The proof of Lemma 1.1 is now complete.

2.3.3 Pairing Formula

Now we derive the pairing formula in Theorem 1.3. For the eigen-basis derived in Proposition 2, when the degree $j = 1$, the generator is simply the cyclic element Λ. Hence, Λ is an eigenvector of P, a conjugate of the Coxeter element, with eigenvalue λ. In fact, the multiplicity of this eigenvalue is 1. Since the order of the Coxeter element is h, the Coxeter number, the eigenvalues of the Coxeter element are given by

$$\lambda^{m_1}, \lambda^{m_2}, \cdots, \lambda^{m_r}$$

where $0 \leq m_i < h$. In [8, Table 2] by Coxeter, for each type of simple Lie algebra, $m_1, m_2, \cdots, m_r$ are given explicitly, and the power 1 shows up exactly once. This implies that the multiplicity of λ is 1. In conclusion, the eigenvector of the Coxeter element in Equation (48) is indeed a cyclic element.

Now we are ready to derive the pairing formula in Theorem 1.3. Choose the bi-coloring function $p : \{1, \ldots, r\} \to \{0, 1\}$ such that $p(1) = 0$. Suppose a simple root $\alpha_j \in \mathcal{A}_{\mathrm{B}}$ such that $p(j) = 1$. Then, by Equations (40) and (48):

$$\alpha_j(\Lambda) = \alpha_j \left(\lambda^{\frac{1}{2}} \sum_{i=1}^{k-1} y_i \alpha_i + \sum_{i=k}^{r} y_i \alpha_i \right)$$

$$
\begin{aligned}
&= \lambda^{\frac{1}{2}} \sum_{i=1}^{k-1} y_i \alpha_i \cdot \alpha_j + \sum_{i=k}^{r} y_i \alpha_i \cdot \alpha_j \\
&= \lambda^{\frac{1}{2}} y_j \alpha_j^2 + \frac{\alpha_j^2}{2} \cdot \sum_{i=k}^{r} C_{ij} y_i \\
&= \lambda^{\frac{1}{2}} y_j \alpha_j^2 + \frac{\alpha_j^2}{2} \cdot (-\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}) y_j \\
&= \frac{1}{2} \cdot \alpha_j^2 \cdot y_j \cdot (\lambda^{1/2} - \lambda^{-1/2})
\end{aligned}
$$

If $\alpha_j \in \mathcal{A}_{\mathrm{W}}$ such that $p(j) = 0$, then, by Equation (41):

$$
\begin{aligned}
\alpha_j(\Lambda) &= \lambda^{\frac{1}{2}} \sum_{i=1}^{k-1} y_i \alpha_i \cdot \alpha_j + \sum_{i=k}^{r} y_i \alpha_i \cdot \alpha_j \\
&= \frac{\lambda^{\frac{1}{2}} \cdot \alpha_j^2}{2} \cdot \sum_{i=1}^{k-1} C_{ij} y_i + y_j \alpha_j^2 \\
&= \frac{\lambda^{\frac{1}{2}} \cdot \alpha_j^2}{2} \cdot (-\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}) y_j + y_j \alpha_j^2 \\
&= \frac{1}{2} \cdot \alpha_j^2 \cdot y_j \cdot (1 - \lambda)
\end{aligned}
$$

They are equivalent to Equation (5) for the chosen bi-coloration. The case where the bi-coloring function p is chosen with $p(1) = 1$ follows similarly. The proof of Theorem 1.3 is now complete.

3 Spectra Calculation of Simple Lie Algebras

In this section, we calculate the cyclic spectra with respect to the first fundamental representation and the adjoint representation for all types of simple Lie algebras, thus proving Theorem 1.4. Theorem 1.3, especially Equation (5), is the key ingredient. Along the way, we also prove Theorem 1.5 for each type of simple Lie algebra.

3.1 Type A_{n-1}

To apply Theorem 1.3 to a simple Lie algebra of type A_{n-1}, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type A_{n-1} simple Lie algebra, $\mathfrak{sl}_n$, is of ADE type. Therefore, its Cartan matrix C is a symmetric with dimension $(n-1) \times (n-1)$ of the form:

$$C = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \tag{109}$$

The Coxeter number is $h = n$. From Theorem 1.2, the spectrum of the Cartan matrix is known to be $\{2 - 2\cos\left(\frac{k\pi}{n}\right)\}_{k=1,\ldots,n-1}$. The smallest eigenvalue occurs when $k = 1$.

Now, we solve for a left eigenvector corresponding to the eigenvalue $2 - 2\cos\left(\frac{\pi}{n}\right)$. Let $\vec{\boldsymbol{x}} = [x_1, \ldots, x_{n-1}]^\intercal$ be such an eigenvector. For computational convenience, we assume $x_1 = 1$. By expanding $\vec{\boldsymbol{x}}C = (2 - 2\cos\left(\frac{\pi}{n}\right))\vec{\boldsymbol{x}}$, we have:

$$2 - x_2 = 2 - 2\cos\left(\frac{\pi}{n}\right) \tag{110}$$

$$-1 + 2x_2 - x_3 = \left(2 - 2\cos\left(\frac{\pi}{n}\right)\right) x_2 \tag{111}$$

$$\vdots$$

$$-x_{n-3} + 2x_{n-2} - x_{n-1} = \left(2 - 2\cos\left(\frac{\pi}{n}\right)\right) x_{n-2} \tag{112}$$

$$-x_{n-2} + 2x_{n-1} = \left(2 - 2\cos\left(\frac{\pi}{n}\right)\right) x_{n-1} \tag{113}$$

From Equations (110)-(112), we observe a recursive relation between components of $\vec{\boldsymbol{x}}$

$$\begin{aligned}
x_1 &= 1; \\
x_2 &= 2\cos\left(\frac{\pi}{n}\right) \cdot x_1; \\
&\vdots \\
x_j &= -x_{j-2} + 2\cos\left(\frac{\pi}{n}\right) \cdot x_{j-1}
\end{aligned}$$

for $j = 3, \cdots, n-1$. We claim that for $j = 1, \cdots, n-1$,

$$x_j = \frac{\sin\left(\frac{j\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

satisfies the recursive relation. To prove this, for $j = 3, \cdots, n-1$:

$$-x_{j-2} + 2\cos\left(\frac{\pi}{n}\right) \cdot x_{j-1} = \frac{-\sin\left(\frac{j\pi}{n} - \frac{2\pi}{n}\right) + 2\cos\left(\frac{\pi}{n}\right) \cdot \sin\left(\frac{j\pi}{n} - \frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

Because $\sin(A - B) = \sin A \cos B - \cos A \sin B$, we have

$$-x_{j-2} + 2\cos\left(\frac{\pi}{n}\right) \cdot x_{j-1} = \frac{\sin\left(\frac{j\pi}{n}\right) \cdot 2\cos^2\left(\frac{\pi}{n}\right) - \cos\left(\frac{2\pi}{n}\right) \cdot \sin\left(\frac{j\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

Finally, by the identity $2\cos^2 A = 1 + \cos 2A$, we obtain

$$-x_{j-2} + 2\cos\left(\frac{\pi}{n}\right) \cdot x_{j-1} = \frac{\sin\left(\frac{j\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} = x_j$$

Additionally, this solution satisfies Equation (113) since

$$\begin{aligned} x_{n-1} \cdot 2\cos\left(\frac{\pi}{n}\right) &= \sin\left(\pi - \frac{\pi}{n}\right) \cdot 2\cos\left(\frac{\pi}{n}\right) \\ &= \sin\left(\frac{\pi}{n}\right) \cdot 2\cos\left(\frac{\pi}{n}\right) \\ &= \sin\left(\frac{2\pi}{n}\right) = \sin\left(\pi - \frac{2\pi}{n}\right) = x_{n-2} \end{aligned}$$

Hence, we obtain the desired components of $\vec{\boldsymbol{x}}$ as

$$x_j = \frac{\sin\left(\frac{j\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} = \frac{\zeta_n^{j/2} - \zeta_n^{-j/2}}{\zeta_n^{1/2} - \zeta_n^{-1/2}} \tag{114}$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$ for $j = 1, \cdots, n-1$.

3.1.1 The First Fundamental Cyclic Spectrum of Type A_{n-1}

In this section, we calculate the cyclic spectrum for the first fundamental representation of the type A_{n-1} simple Lie algebra using Theorem 1.3.

3.1.1.1 Weights

At first, we introduce functionals $\{\mu_i\}_{i=1,\cdots,n}$ on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n$ consisting of diagonal matrices with trace 0 such that:

$$\mu_i : \mathrm{diag}(a_1, \cdots, a_n) \mapsto a_i$$

These are the weights of the standard n-dimensional representation of $\mathfrak{sl}_n$, which also serves as its first fundamental representation, as noted in Example 1.3. It is known that simple roots $\{\alpha_i\}_{i=1,\cdots,n-1}$ can be chosen as:

$$\alpha_i = \mu_i - \mu_{i+1}$$

See for example [16, Appendix A.1]. Now, we express μ_i in terms of α_i. Clearly, for $i > 1$, there exists a recursive relation:

$$\mu_{i+1} = \mu_i - \alpha_i \tag{115}$$

Also, we claim that:

$$\mu_1 = \frac{1}{n} \cdot \sum_{k=1}^{n-1} (n-k) \cdot \alpha_k \tag{116}$$

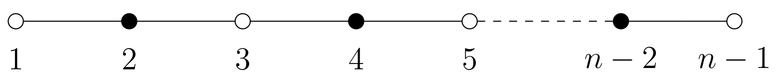
Because for any $\mathrm{diag}(a_1, \cdots, a_n) \in \mathfrak{h}$,

$$\begin{aligned}
\mu_1(\mathrm{diag}(a_1, \cdots, a_n)) &= \frac{1}{n} \cdot \sum_{k=1}^{n-1} (n-k) \cdot \alpha_k(\mathrm{diag}(a_1, \cdots, a_n)) \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n-1} (n-k) \cdot (a_k - a_{k+1}) \\
&= \frac{1}{n} \cdot ((n-1) \cdot a_1 - a_2 - \cdots - a_{n-1} - a_n) \\
&= \frac{1}{n} \cdot ((n-1) \cdot a_1 - a_2 - \cdots - a_{n-1} - (-a_1 - a_2 - \cdots - a_{n-1})) \\
&= a_1
\end{aligned}$$

Hence, Equations (115) and (116) express the weights $\{\mu_i\}_{i=1,\cdots,n}$ as linear combinations of simple roots.

3.1.1.2 Cyclic Spectrum

To apply Theorem 1.3, we also need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, $p(i) = 1$ if i is even and $p(i) = 0$ if i is odd. Since all simple roots have the same length, Equation (5) is equivalent to

$$\alpha_j(\Lambda) = x_j \cdot \zeta_n^{p(j)/2} \cdot (1 - \zeta_n^{-2p(j)+1}) \tag{117}$$

for $j = 1, \cdots, n-1$. Using Equation (114) for x_j, we have:

$$\alpha_j(\Lambda) = \zeta_n^{(-1)^j \lfloor \frac{j}{2} \rfloor} - \zeta_n^{(-1)^{j+1} \lfloor \frac{j+1}{2} \rfloor} \tag{118}$$

Specifically,

$$\alpha_j(\Lambda) = \begin{cases} \zeta_n^{\frac{j}{2}} - \zeta_n^{-\frac{j}{2}} & \text{if } j \text{ is even} \\ \zeta_n^{\frac{1}{2}} \cdot (\zeta_n^{-\frac{j}{2}} - \zeta_n^{\frac{j}{2}}) & \text{if } j \text{ is odd} \end{cases} \tag{119}$$

Then, using Equations (115) and (116), we obtain:

$$\begin{aligned}
\mu_1(\Lambda) &= \frac{1}{n} \cdot [(n-1) \cdot (1-\zeta_n) + (n-2) \cdot (\zeta_n - \zeta_n^{-1}) + (n-3) \cdot (\zeta_n^{-1} - \zeta_n^2) \\
&\quad + \cdots + (\zeta_n^{(-1)^{n-1} \lfloor \frac{n-1}{2} \rfloor} - \zeta_n^{(-1)^n \lfloor \frac{n}{2} \rfloor})] \\
&= \frac{1}{n} \cdot [(n-1-\sum_{j=1}^{n-1} (\zeta_n^j)] \\
&= 1 \\
\mu_2(\Lambda) &= \mu_1(\Lambda) - \alpha_1(\Lambda) = 1 - (1-\zeta_n) = \zeta_n \\
\mu_3(\Lambda) &= \mu_2(\Lambda) - \alpha_2(\Lambda) = \zeta_n - (\zeta_n - \zeta_n^{-1}) = \zeta_n^{-1} \\
\mu_4(\Lambda) &= \mu_3(\Lambda) - \alpha_3(\Lambda) = \zeta_n^{-1} - (\zeta_n^{-1} - \zeta_n^2) = \zeta_n^2 \\
\mu_5(\Lambda) &= \mu_4(\Lambda) - \alpha_4(\Lambda) = \zeta_n^2 - (\zeta_n^2 - \zeta_n^{-2}) = \zeta_n^{-2} \\
&\vdots \\
\mu_n(\Lambda) &= \zeta_n^{(-1)^n \lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

Thus, we obtain

$$\mu_j(\Lambda) = \zeta_n^{(-1)^j \lfloor \frac{j}{2} \rfloor} \tag{120}$$

for $i = 1, \cdots, n$. Hence,

$$\mathrm{Spec}(\psi_1, \mathfrak{sl}_n) = \{\mu_j(\Lambda)\}_{j=1,\cdots,n} = C_n(1)$$

The part of Theorem 1.4 concerning the first fundamental cyclic spectrum of $\mathfrak{sl}_n$ is now proved.

Remark 7. Similar to the example of $\mathfrak{e}_6$ in the first chapter, we visualize this spectrum in the complex plane. Figures 5 and 6 present the cyclic spectra for the first fundamental representation of type A_5 and A_6, respectively. Both paths begin with 1, followed by ζ_n^k and its conjugate ζ_n^{-k} for $k \leq \lfloor \frac{n}{2} \rfloor$. For $i < j$, $\mathrm{Re}(\mu_i(\Lambda)) \geq \mathrm{Re}(\mu_j(\Lambda))$.

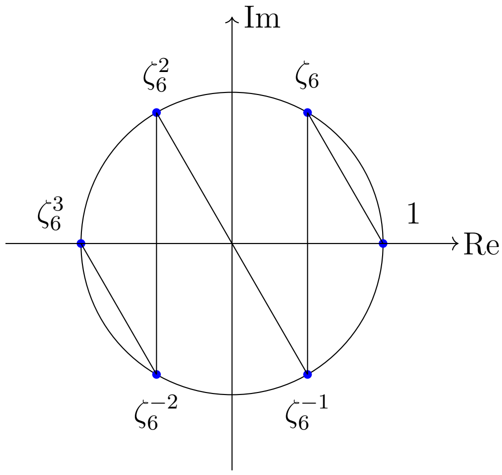


Figure 5: First fundamental cyclic spectrum for A_5

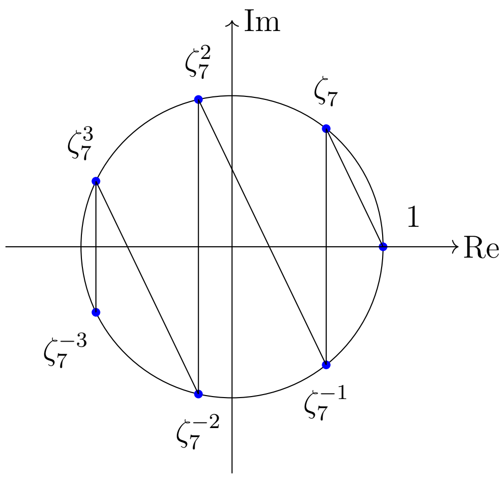


Figure 6: First fundamental cyclic spectrum for A_6

Remark 8. Recall that in the spectrum calculation, we chose the bi-coloration such that $p(i) = 1$ for even i and $p(i) = 0$ for odd i. However, as explained in Example 2.2, bi-coloration is not unique. The alternative choice comes from swapping the colors.

Now, we want to show that the spectrum remains invariant under different choices of bi-coloration. Numerically, the difference between bi-coloration choices lies in the value of $p(1)$. If $p(1) = 0$, as we previously assumed, then:

$$\begin{aligned} \alpha_j(\Lambda) &= x_j \cdot \zeta_h^{p(j)/2} \cdot (1 - \zeta_h^{-2p(j)+1}) \\ &= \begin{cases} x_j \cdot (1 - \zeta_h) & \text{if } j \in \mathrm{W} \\ x_j \cdot (\zeta_h^{1/2} - \zeta_h^{-1/2}) & \text{if } j \in \mathrm{B} \end{cases} \end{aligned}$$

If $p(1) = 1$, denote the pairing by $\alpha_j(\Lambda)'$, then:

$$\begin{aligned} \alpha_j(\Lambda)' &= x_j \cdot \zeta_h^{(p(j)-1)/2} \cdot (1 - \zeta_h^{-2p(j)+1}) \\ &= \begin{cases} x_j \cdot (1 - \zeta_h^{-1}) & \text{if } j \in \mathrm{B} \\ x_j \cdot (\zeta_h^{-1/2} - \zeta_h^{1/2}) & \text{if } j \in \mathrm{W} \end{cases} \end{aligned}$$

Since $\vec{\boldsymbol{x}}$ is a real vector, we have

$$\alpha_j(\Lambda)' = \overline{\alpha_j(\Lambda)}$$

for any $j = 1, \ldots, r$. Suppose μ is a weight of the first fundamental representation of $\mathfrak{sl}_n$. By Equations (115) and (116), μ is a linear combination of simple roots. So, it follows that

$$\mu(\Lambda)' = \overline{\mu(\Lambda)}$$

where $\mu(\Lambda)'$ is the eigenvalue corresponding to μ under the alternative bi-coloration. Therefore, choosing the alternative bi-coloration replaces each eigenvalue in $\mathrm{Spec}(\psi_1, \mathfrak{sl}_n)$ by its complex conjugate. However, the "ζ_h-symmetry" stated in Lemma 1.1 guarantees that both an eigenvalue and its conjugate appear in the spectrum. Moreover, each eigenvalue in $\mathrm{Spec}(\psi_1, \mathfrak{sl}_n)$ has the same multiplicity as its conjugate. Hence, $\mathrm{Spec}(\psi_1, \mathfrak{sl}_n)$

remains invariant under the alternative choice of bi-coloration.

In fact, this result is not specific to $\mathrm{Spec}(\psi_1, \mathfrak{sl}_n)$ but holds for any simple Lie algebra, both in the first representation and in the adjoint representation. The same argument applies.

3.1.2 The Adjoint Cyclic Spectrum of Type A_{n-1}

In this section, we adopt the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{sl}_n$. In Section 2.3, we discussed that the centralizer of Λ is a Cartan subalgebra, so the adjoint spectrum contains at least $r = n-1$ zero eigenvalues. Then, using Equation (118), it suffices to express the weights - which, in this case, are the roots of $\mathfrak{sl}_n$ - in terms of simple roots. It is known that the positive roots are

$$\mu_i - \mu_j = \sum_{k=i}^{j-1} \alpha_k \tag{121}$$

for $i < j$, $i, j \in \{1, \cdots, n\}$. The total number of roots is $n(n-1)$.

Now, we go through several examples to understand the patterns in the eigenvalues obtained from the pairing of roots with Λ. Interestingly, the pattern of non-zero eigenvalues is closely related to the modulo classes of $n \pmod 2$ and $n \pmod 4$.

Example 3.1. Consider the case of A_2, $\mathfrak{sl}_3$. As before, we choose the bi-coloration:

Then, by Equation (118), we obtain

$$\alpha_1(\Lambda) = 1 - \zeta_3 = \zeta_{12}^3 \cdot \mathsf{r} \cdot \zeta_3^2$$

$$\alpha_2(\Lambda) = \zeta_3 - \zeta_3^{-1} = \zeta_{12}^3 \cdot \mathsf{r} \cdot \zeta_3^3$$

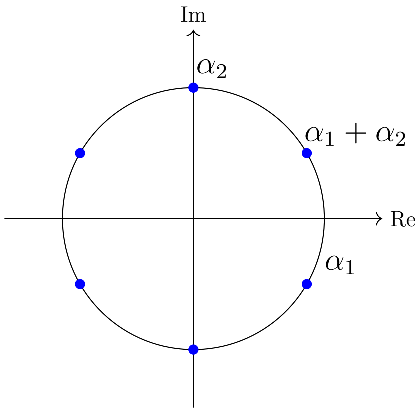
where $\mathsf{r} = 2\sin\left(\frac{\pi}{3}\right)$. The only non-simple positive root is $\alpha_1 + \alpha_2$. So:

$$(\alpha_1 + \alpha_2)(\Lambda) = 1 - \zeta_3^{-1} = \zeta_{12} \cdot \mathsf{r} \cdot \zeta_3$$

Therefore, the pairings of roots with Λ yield the following eigenvalues:

$$C_3(\zeta_{12}\cdot\mathsf{r},\ \zeta_{12}^3\cdot\mathsf{r}) = C_6(\zeta_{12}\cdot\mathsf{r})$$

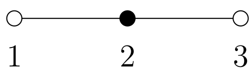
Graphically, they are presented as follows:



The corresponding positive roots are annotated on the graph. The spectrum is equivalent up to an overall scaling, so by dividing all eigenvalues by ζ_{12}, we have

$$\mathrm{Spec}(\mathrm{ad}, A_2) = C_6(\mathsf{r}, 0^{(2)})$$

Example 3.2. Consider the case of A_3, $\mathfrak{sl}_4$. As before, we choose the bi-coloration:



By Equation (118), we obtain

$$\begin{aligned}
\alpha_1(\Lambda) &= 1-\zeta_4 = \zeta_8\cdot\mathsf{r}_1\cdot\zeta_4^3\\
\alpha_2(\Lambda) &= \zeta_4-\zeta_4^{-1} = \mathsf{r}_2\cdot\zeta_4\\
\alpha_3(\Lambda) &= \zeta_4^{-1}-\zeta_4^2 = \zeta_8\cdot\mathsf{r}_1\cdot\zeta_4^3
\end{aligned}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{4}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{2}\right)$. Other positive roots pair with Λ as:

$$(\alpha_1+\alpha_2)(\Lambda) = 1-\zeta_4^{-1} = \zeta_8\cdot\mathsf{r}_1\cdot\zeta_4$$

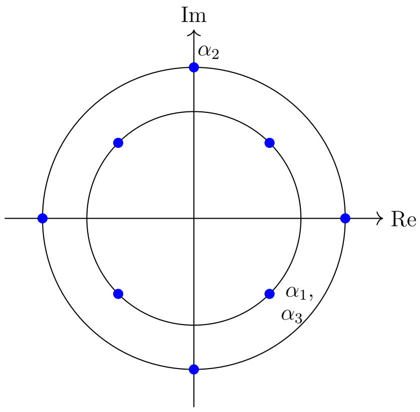
$$(\alpha_1 + \alpha_2 + \alpha_3)(\Lambda) = 1 - \zeta_4^2 = \mathsf{r}_2 \cdot \zeta_4^4$$

$$(\alpha_2 + \alpha_3)(\Lambda) = \zeta_4 - \zeta_4^2 = \zeta_8 \cdot \mathsf{r}_1 \cdot \zeta_4$$

Therefore, the pairings of roots with Λ yield the following eigenvalues:

$$C_4(\zeta_8 \cdot \mathsf{r}_1^{(2)}, \mathsf{r}_2)$$

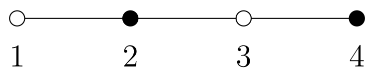
Graphically, they are presented as follows:



The corresponding simple roots are annotated on the graph. Hence:

$$\mathrm{Spec}(\mathrm{ad}, A_3) = C_4(\zeta_8 \cdot \mathsf{r}_1^{(2)},\ \mathsf{r}_2,\ 0^{(3)})$$

Example 3.3. Consider the case of A_4, $\mathfrak{sl}_5$. As before, we choose the bi-coloration:



By Equation (118), we obtain

$$\alpha_1(\Lambda) = 1 - \zeta_5 = \zeta_{20} \cdot \mathsf{r}_1 \cdot \zeta_5^4$$

$$\alpha_2(\Lambda) = \zeta_5 - \zeta_5^{-1} = \zeta_{20} \cdot \mathsf{r}_2 \cdot \zeta_5$$

$$\alpha_3(\Lambda) = \zeta_5^{-1} - \zeta_5^2 = \zeta_{20} \cdot \mathsf{r}_2 \cdot \zeta_5^4$$

$$\alpha_4(\Lambda) = \zeta_5^2 - \zeta_5^{-2} = \zeta_{20} \cdot \mathsf{r}_1 \cdot \zeta_5$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{5}\right), \mathsf{r}_2 = 2\sin\left(\frac{2\pi}{5}\right)$. Other positive roots pair with Λ as:

$$(\alpha_1 + \alpha_2)(\Lambda) = 1 - \zeta_5^{-1} = \zeta_{20}^3 \cdot \mathsf{r}_1 \cdot \zeta_5^5$$

$$(\alpha_1 + \alpha_2 + \alpha_3)(\Lambda) = 1 - \zeta_5^2 = \zeta_{20}^3 \cdot \mathsf{r}_2 \cdot \zeta_5^4$$

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(\Lambda) = 1 - \zeta_5^{-2} = \zeta_{20} \cdot \mathsf{r}_2 \cdot \zeta_5^5$$

$$(\alpha_2 + \alpha_3)(\Lambda) = \zeta_5 - \zeta_5^2 = \zeta_{20} \cdot \mathsf{r}_1 \cdot \zeta_5^5$$

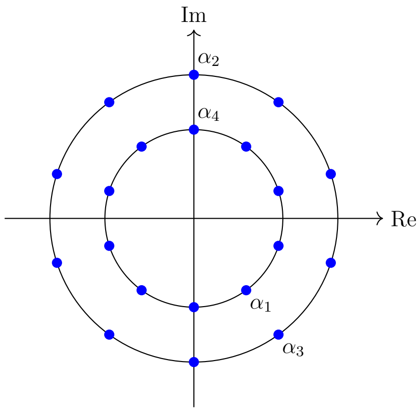
$$(\alpha_2 + \alpha_3 + \alpha_4)(\Lambda) = \zeta_5 - \zeta_5^{-2} = \zeta_{20}^3 \cdot \mathsf{r}_2 \cdot \zeta_5^5$$

$$(\alpha_3 + \alpha_4)(\Lambda) = \zeta_5^{-1} - \zeta_5^{-2} = \zeta_{20}^3 \cdot \mathsf{r}_1 \cdot \zeta_5^4$$

Therefore, the pairings of roots with Λ yield the following eigenvalues:

$$C_5(\zeta_{20} \cdot \mathsf{r}_1, \zeta_{20}^3 \cdot \mathsf{r}_1, \zeta_{20} \cdot \mathsf{r}_2, \zeta_{20}^3 \cdot \mathsf{r}_2) = C_{10}(\zeta_{20} \cdot \mathsf{r}_1,\ \zeta_{20} \cdot \mathsf{r}_2)$$

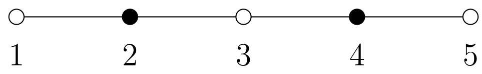
Graphically, they are presented as follows:



The corresponding simple roots are annotated on the graph. The spectrum is equivalent up to an overall scaling, so by an overall scaling of ζ_{20}^{-1}, we obtain

$$\mathrm{Spec}(\mathrm{ad}, A_4) = C_{10}(\mathsf{r}_1,\ \mathsf{r}_2, 0^{(4)})$$

Example 3.4. Consider the case of A_5, $\mathfrak{sl}_6$. As before, we choose the bi-coloration:



By Equation (118), we obtain

$$\alpha_1(\Lambda) = 1 - \zeta_6 = \mathsf{r}_1 \cdot \zeta_6^5$$

$$\alpha_2(\Lambda) = \zeta_6 - \zeta_6^{-1} = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6$$

$$\alpha_3(\Lambda) = \zeta_6^{-1} - \zeta_6^2 = \mathsf{r}_3 \cdot \zeta_6^5$$

$$\alpha_4(\Lambda) = \zeta_6^2 - \zeta_6^{-2} = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6$$

$$\alpha_5(\Lambda) = \zeta_6^{-2} - \zeta_6^3 = \mathsf{r}_1 \cdot \zeta_6^5$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{6}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{3}\right), \mathsf{r}_3 = 2\sin\left(\frac{\pi}{2}\right)$. Other positive roots pair with Λ as:

$$(\alpha_1 + \alpha_2)(\Lambda) = 1 - \zeta_6^{-1} = \mathsf{r}_1 \cdot \zeta_6$$

$$(\alpha_1 + \alpha_2 + \alpha_3)(\Lambda) = 1 - \zeta_6^2 = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6^5$$

$$\sum_{k=1}^{4} \alpha_k(\Lambda) = 1 - \zeta_6^{-2} = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6^6$$

$$\sum_{k=1}^{5} \alpha_k(\Lambda) = 1 - \zeta_6^3 = \mathsf{r}_3 \cdot \zeta_6^6$$

$$(\alpha_2 + \alpha_3)(\Lambda) = \zeta_6 - \zeta_6^2 = \mathsf{r}_1 \cdot \zeta_6^6$$

$$(\alpha_2 + \alpha_3 + \alpha_4)(\Lambda) = \zeta_6 - \zeta_6^{-2} = \mathsf{r}_3 \cdot \zeta_6$$

$$\sum_{k=2}^{5} \alpha_k(\Lambda) = \zeta_6 - \zeta_6^3 = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6^6$$

$$(\alpha_3 + \alpha_4)(\Lambda) = \zeta_6^{-1} - \zeta_6^{-2} = \mathsf{r}_1 \cdot \zeta_6^6$$

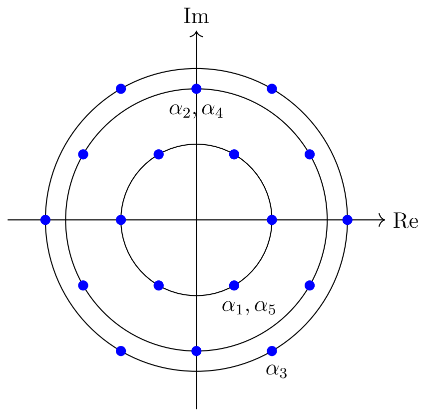
$$(\alpha_3 + \alpha_4 + \alpha_5)(\Lambda) = \zeta_6^{-1} - \zeta_6^3 = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6^5$$

$$(\alpha_4 + \alpha_5)(\Lambda) = \zeta_6^2 - \zeta_6^3 = \mathsf{r}_1 \cdot \zeta_6$$

Therefore, the pairings of roots with Λ yield the following eigenvalues:

$$C_6(\mathsf{r}_1^{(2)},\ \zeta_{12}\cdot\mathsf{r}_2^{(2)},\ \mathsf{r}_3)$$

Graphically, they are presented as follows:



The corresponding simple roots are annotated on the graph. Hence:

$$\mathrm{Spec}(\mathrm{ad}, A_5) = C_6(\mathsf{r}_1^{(2)},\ \zeta_{12}\cdot\mathsf{r}_2^{(2)},\ \mathsf{r}_3,\ 0^{(5)})$$

The above examples suggest certain patterns in the eigenvalues of $\mathrm{Spec}(\mathrm{ad}, A_{n-1})$. As stated in Corollary 1.3, the first pattern concerns the radii set of $\mathrm{Spec}(\mathrm{ad}, A_{n-1})$. We now prove the corresponding part of Corollary 1.3, namely:

$$\begin{aligned}\mathsf{R}(\mathrm{Spec}(\mathrm{ad}, A_{n-1})) &\sim \{|\alpha_1(\Lambda)|, \cdots, |\alpha_{n-1}(\Lambda)|, 0\} \\ &\sim \{2\sin\left(\frac{j\pi}{n}\right) \mid j = 1, \cdots, \lfloor\frac{n}{2}\rfloor\} \cup \{0\}\end{aligned} \tag{122}$$

By Equation (117), we obtain that for any $j = 1, \ldots, n-1$,

$$\begin{aligned}|\alpha_j(\Lambda)| &= |x_j|\cdot|\zeta_n^{1/2} - \zeta_n^{-1/2}| \\ &= 2\sin\left(\frac{j\pi}{n}\right)\end{aligned}$$

Since $\sin(\pi-\theta)=\sin(\theta)$, so $|\alpha_j(\Lambda)|=|\alpha_{n-j}(\Lambda)|$ for $j=1,\cdots,\lfloor\frac{n}{2}\rfloor$, hence

$$\{|\alpha_1(\Lambda)|,\cdots,|\alpha_{n-1}(\Lambda)|\}\sim\{2\sin\left(\frac{j\pi}{n}\right)\mid j=1,\cdots,\lfloor\frac{n}{2}\rfloor\}$$

Next, by Equation (120), we have

$$\begin{aligned}|\mu_i(\Lambda)-\mu_j(\Lambda)|&=|\zeta_n^{(-1)^i\lfloor\frac{i}{2}\rfloor}-\zeta_n^{(-1)^j\lfloor\frac{j}{2}\rfloor}|\\&=\sqrt{2-(\zeta_n^{|(-1)^i\lfloor\frac{i}{2}\rfloor-(-1)^j\lfloor\frac{j}{2}\rfloor|}+\zeta_n^{-|(-1)^i\lfloor\frac{i}{2}\rfloor-(-1)^j\lfloor\frac{j}{2}\rfloor|})}\\&=\sqrt{2-2\cos\left(|(-1)^i\lfloor\frac{i}{2}\rfloor-(-1)^j\lfloor\frac{j}{2}\rfloor|\cdot\frac{2\pi}{n}\right)}\\&=2\sin\left(|(-1)^i\lfloor\frac{i}{2}\rfloor-(-1)^j\lfloor\frac{j}{2}\rfloor|\cdot\frac{\pi}{n}\right)\end{aligned}\tag{123}$$

for any $i\neq j$, $i,j\in\{1,\cdots,n\}$. It is clear that $|(-1)^i\lfloor\frac{i}{2}\rfloor-(-1)^j\lfloor\frac{j}{2}\rfloor|$ is an integer bounded between 1 and $n-1$. Hence, for any $i\neq j$,

$$|\mu_i(\Lambda)-\mu_j(\Lambda)|\in\{2\sin\left(\frac{j\pi}{n}\right)\mid j=1,\cdots,n-1\}$$

Equation (122) is now proved, so is the corresponding part of Corollary 1.3.

The second pattern reveals the general form of $\mathrm{Spec}(\mathrm{ad},A_{n-1})$, corresponding to Theorem 1.4.

1. If $n\equiv 1\mod 2$, then we obtain

$$\mathrm{Spec}(\mathrm{ad},\mathfrak{sl}_\mathfrak{n})=C_{2n}(\mathsf{r}_1,\cdots,\mathsf{r}_{\frac{n-1}{2}},0^{(n-1)})\tag{124}$$

   for $\mathsf{r}_j=2\sin\left(\frac{j\pi}{n}\right)$, $j=1,\cdots,\frac{n-1}{2}$.

2. If $n\equiv 0\mod 2$, then we obtain

$$\mathrm{Spec}(\mathrm{ad},\mathfrak{sl}_\mathfrak{n})=C_n(z_1^{(2)},\cdots,z_{\frac{n}{2}-1}^{(2)},z_{\frac{n}{2}},0^{(n-1)})\tag{125}$$

where $z_j = \zeta_{2n}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{n}\right)$ and

$$p(j) = \begin{cases} 0 & \text{if } j, \frac{n}{2} \text{ are with the same parity} \\ 1 & \text{Otherwise} \end{cases}$$

Now, we prove the two above equations to prove the corresponding part of Theorem 1.4. The key ingredient in the proof is the "ζ_h-symmetry" in Lemma 1.1. Combining with the dimension of the adjoint representation, we will show that $\{\pm\alpha_j(\Lambda)\}_{j=1,\dots,n-1}$ generates all the eigenvalues in the spectrum, allowing us to bypass the direct computation of all root pairings with Λ.

By Equation (119), we can rewrite $\pm\alpha_j(\Lambda)$ for $j = 1, \cdots, n-1$ as

$$\alpha_j(\Lambda) = \begin{cases} \zeta_{2n}^{p_1} \cdot 2\sin\left(\frac{j\pi}{n}\right) \cdot \zeta_n^{m_1} & \text{if } j \text{ is odd} \\ \zeta_{2n}^{p_2} \cdot 2\sin\left(\frac{j\pi}{n}\right) \cdot \zeta_n^{m_2} & \text{if } j \text{ is even} \end{cases}$$

$$-\alpha_j(\Lambda) = \begin{cases} \zeta_{2n}^{p_3} \cdot 2\sin\left(\frac{j\pi}{n}\right) \cdot \zeta_n^{m_3} & \text{if } j \text{ is odd} \\ \zeta_{2n}^{p_4} \cdot 2\sin\left(\frac{j\pi}{n}\right) \cdot \zeta_n^{m_4} & \text{if } j \text{ is even} \end{cases}$$

where $p_1, \cdots, p_4 \in \frac{\mathbb{Z}^{\geq 0}}{2}$, and $m_1, \cdots, m_4 \in \mathbb{Z}^{\geq 0}$. The values of p_i and m_i are not unique. In our discussion, we consistently choose them to be the smallest possible. These quantities encode essential information about the spectrum. Our argument proceeds by analyzing how their values vary according to the congruence class of $n \pmod 2$ and $n \pmod 4$.

When n is odd, we split our argument by $n \pmod 4$. When $n \equiv 1 \pmod 4$, for instance, type A_4 in Example 3.3, we have

$$p_1 = p_2 = \frac{1}{2}, p_3 = p_4 = \frac{3}{2}$$

$$m_1 = \frac{3n+1}{4}, m_2 = m_3 = \frac{n-1}{4}, m_4 = \frac{3(n-1)}{4}$$

Since n is odd, for $j = 1, \cdots, n-1$, $C_n(\alpha_j(\Lambda)) \neq C_n(-\alpha_j(\Lambda))$. By "$\zeta_n$–symmetry",

$\alpha_j(\Lambda)$ generate $C_n(\zeta_{4n} \cdot 2\sin\left(\frac{j\pi}{n}\right))$, and $-\alpha_j(\Lambda)$ generate $C_n(\zeta_{4n}^3 \cdot 2\sin\left(\frac{j\pi}{n}\right))$. Combining these, for $j = 1, \cdots, n-1$, we have

$$C_n(\pm\alpha_j(\Lambda)) = C_{2n}\left(\zeta_{4n} \cdot 2\sin\left(\frac{j\pi}{n}\right)\right)$$

Furthermore, notice that for $j = 1, \ldots, \frac{n-1}{2}$,

$$C_n(\alpha_j(\Lambda)) = C_n(\alpha_{n-j}(\Lambda)) = C_n\left(\zeta_{4n} \cdot 2\sin\left(\frac{j\pi}{n}\right)\right)$$

Thus,

$$C_n(\pm\alpha_1(\Lambda), \cdots, \pm\alpha_{\frac{n-1}{2}}(\Lambda)) = \bigsqcup_{j=1}^{\frac{n-1}{2}} C_{2n}\left(\zeta_{4n} \cdot \mathsf{r}_j\right)$$

where $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{n}\right)$. By an overall scaling, it is equivalent to

$$C_{2n}(\mathsf{r}_1, \cdots, \mathsf{r}_{\frac{n-1}{2}})$$

which consists of $n(n-1)$ distinct non-zero eigenvalues. Combining these with the $n-1$ zero eigenvalues contributed by the Cartan subalgebra, we obtain the complete adjoint spectrum. Each non-zero value only appears once with multiplicity 1.

When $n \equiv 3 \pmod 4$, for instance, type A_2 in Example 3.1, we have

$$p_1 = p_2 = \frac{3}{2},\ p_3 = p_4 = \frac{1}{2}$$

$$m_1 = \frac{3n-1}{4}, m_2 = \frac{n-3}{4}, m_3 = \frac{n+1}{4}, m_4 = \frac{3n-1}{4}$$

Similar to the previous case, by "ζ_n−symmetry", $\alpha_j(\Lambda)$ generate $C_n(\zeta_{4n}^3 \cdot 2\sin\left(\frac{j\pi}{n}\right))$ and $-\alpha_j(\Lambda)$ generate $C_n(\zeta_{4n} \cdot 2\sin\left(\frac{j\pi}{n}\right))$. Combining these, for $j = 1, \cdots, n-1$, we have

$$C_n(\pm\alpha_j(\Lambda)) = C_{2n}\left(\zeta_{4n} \cdot 2\sin\left(\frac{j\pi}{n}\right)\right)$$

Also, $C_n(\alpha_j(\Lambda)) = C_n(\alpha_{n-j}(\Lambda))$ for $j = 1, \cdots, n-1$. Thus,

$$C_n(\pm\alpha_1(\Lambda), \cdots, \pm\alpha_{\frac{n-1}{2}}(\Lambda)) = \bigsqcup_{j=1}^{\frac{n-1}{2}} C_{2n}\left(\zeta_{4n} \cdot \mathsf{r}_j\right)$$

where $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{n}\right)$. By an overall scaling, it is equivalent to

$$C_{2n}(\mathsf{r}_1, \cdots, \mathsf{r}_{\frac{n-1}{2}})$$

which consists of $n(n-1)$ distinct non-zero eigenvalues. Using the same dimension argument as in the previous case, combined with the $n-1$ zero eigenvalues, they yield the complete adjoint spectrum, where every non-zero eigenvalue has multiplicity 1. The proof of Equation (124) is now complete.

Now we prove Equation (125). When n is even, for instance, type A_3 and A_5 in Examples 3.2 and 3.4, let $p(j)$ be defined as above and let

$$q = \begin{cases} 1 & \text{if } n \pmod 4 = 2 \\ 0 & \text{if } n \pmod 4 = 0 \end{cases}$$

Then, we have

$$p_i = p(i) \quad \text{for } i = 1, 2, 3, 4$$

$$m_1 = \frac{3n+2q}{4}, m_2 = \frac{n-2q}{4}, m_3 = \frac{n+2q}{4}, m_4 = \frac{3n-2q}{4}$$

Since n is even, by "ζ_n–symmetry", for $j = 1, \cdots, n-1$, $C_n(\alpha_j(\Lambda)) = C_n(-\alpha_j(\Lambda)) = C_n(\zeta_{2n}^{p(j)} \cdot \mathsf{r}_j)$ where $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{n}\right)$. Furthermore, for $j = 1, \ldots, \frac{n}{2}$, $\pm\alpha_j(\Lambda) = \pm\alpha_{n-j}(\Lambda)$. Thus

$$C_n(\pm\alpha_1(\Lambda), \cdots, \pm\alpha_{\frac{n}{2}}(\Lambda)) \; = \; C_n(\zeta_{2n}^{p(1)} \cdot \mathsf{r}_1, \cdots, \zeta_{2n}^{p(\frac{n}{2}-1)} \cdot \mathsf{r}_{\frac{n}{2}-1}, \mathsf{r}_{\frac{n}{2}})$$

Now we show that the eigenvalues in $C_n(\mathsf{r}_{\frac{n}{2}})$ have multiplicity 1. By Equation (123), only when $i+j=n+1$, we have

$$|(-1)^i\lfloor\frac{i}{2}\rfloor - (-1)^j\lfloor\frac{j}{2}\rfloor| = \frac{n}{2}$$

Therefore, only the eigenvalues corresponding to the roots

$$\pm(\mu_i - \mu_{n-i+1}) = \pm\sum_{k=i}^{n-i}\alpha_k \tag{126}$$

have radius $\mathsf{r}_{\frac{n}{2}} = 2$. There are exactly n such eigenvalues with multiplicity 1.

Next, we show that all the other eigenvalues have multiplicity 2. Observe that

$$\pm(\mu_i - \mu_j)(\Lambda) = \pm\sum_{k=i}^{j-1}\alpha_k(\Lambda) = \pm\sum_{k=n+1-j}^{n-i}\alpha_k(\Lambda) = \pm(\mu_{n+1-j} - \mu_{n+1-i})(\Lambda) \tag{127}$$

for $j \neq n-i+1$. There are $\frac{n(n-2)}{2}$ distinct pairs of such i, j, which implies the eigenvalues in

$$C_n(\zeta_{2n}^{p(1)} \cdot \mathsf{r}_1, \cdots, \zeta_{2n}^{p(\frac{n}{2}-1)} \cdot \mathsf{r}_{\frac{n}{2}-1})$$

have multiplicity 2. Using the same dimension argument as in the previous case, combined with the $n-1$ zero eigenvalues, they yield the complete adjoint spectrum. The proof of Equation (125) is now complete, so is the proof of Theorem 1.4 for $\mathfrak{sl}_n$. Based on the above analysis, this also completes the proof of Theorem 1.5 for $\mathfrak{sl}_n$.

3.2 Type B_n

To apply Theorem 1.3 to a simple Lie algebra of type B_n, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type B_n simple Lie algebra, $\mathfrak{so}_{2n+1}$, is not of ADE type. Its Cartan matrix C is not symmetric and with dimension $n \times n$ in the form:

$$C = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$$

As in the case of $\mathfrak{sl}_n$, the Cartan matrix of $\mathfrak{so}_{2n+1}$ has real eigenvalues, with the smallest being $2 - 2\cos\left(\frac{\pi}{h}\right)$, where $h = 2n$ is the Coxeter number.

Now we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{2n}\right)$. Let $\vec{\boldsymbol{x}} = [x_1, \ldots, x_n]^\intercal$ be such an eigenvector. For computational convenience, we assume $x_1 = 1$. By expanding $\vec{\boldsymbol{x}}C = (2 - 2\cos\left(\frac{\pi}{2n}\right))\vec{\boldsymbol{x}}$, we have:

$$\begin{aligned} 2 - x_2 &= 2 - 2\cos\left(\frac{\pi}{2n}\right) && (128) \\ -1 + 2x_2 - x_3 &= \left(2 - 2\cos\left(\frac{\pi}{2n}\right)\right) x_2 && (129) \\ &\vdots \\ -x_{n-2} + 2x_{n-1} - x_n &= \left(2 - 2\cos\left(\frac{\pi}{2n}\right)\right) x_{n-1} && (130) \\ -2x_{n-1} + 2x_n &= \left(2 - 2\cos\left(\frac{\pi}{2n}\right)\right) x_n && (131) \end{aligned}$$

From Equations (128)-(130), we observe a recursive relation between components of $\vec{\boldsymbol{x}}$

$$\begin{aligned} & x_1 = 1; \\ & x_2 = 2\cos\left(\frac{\pi}{2n}\right) \cdot x_1; \\ & \quad\vdots \\ & x_j = -x_{j-2} + 2\cos\left(\frac{\pi}{2n}\right) \cdot x_{j-1} \end{aligned}$$

for $j = 3, \cdots, n$. Note that this relation is identical to the relation in the $\mathfrak{sl}_n$ case.

Therefore, using the same argument, $\vec{\boldsymbol{x}}$ has components

$$x_j = \frac{\sin\left(\frac{j\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} = \frac{\zeta_{2n}^{j/2} - \zeta_{2n}^{-j/2}}{\zeta_{2n}^{1/2} - \zeta_{2n}^{-1/2}} \tag{132}$$

for $j = 1, \cdots, n$. This solution also satisfies Equation (131). Since $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$, we have

$$\begin{aligned} x_n \cdot \cos\left(\frac{\pi}{2n}\right) &= \frac{\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} \\ &= \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} \\ &= \frac{\sin\left(\frac{(n-1)\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} = x_{n-1} \end{aligned}$$

3.2.1 The First Fundamental Representation of Type B_n

In this section, we calculate the cyclic spectrum for the first fundamental representation of the type B_n simple Lie algebra using Theorem 1.3.

3.2.1.1 Weights

At first, we introduce functionals $\{\mu_i\}_{i=1,\cdots,n}$ on the Cartan subalgebra of $\mathfrak{so}_{2n+1}$ consisting of matrices in the form $\mathrm{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n, 0)$ such that:

$$\mu_i : \mathrm{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n, 0) \mapsto x_i$$

In particular, $\{\mu_1, \cdots, \mu_n, -\mu_1, \cdots, -\mu_n, 0\}$ are the $2n+1$ weights in the standard $2n+1$-dimensional representation of $\mathfrak{so}_{2n+1}$, which is also the first fundamental representation of $\mathfrak{so}_{2n+1}$. It is known that the simple roots $\{\alpha_i\}_{i=1,\cdots,n}$ for $\mathfrak{so}_{2n+1}$ are

$$\alpha_i = \begin{cases} \mu_i - \mu_{i+1} & \text{for } i = 1, \cdots, n-1 \\ \mu_n & \text{for } i = n \end{cases}$$

See for example [16, Appendix A.2]. One can obtain from simple calculations that the weights for the standard representation of $\mathfrak{so}_{2n+1}$ in terms with simple roots are:

$$\begin{cases} \pm\mu_j = \pm\sum_{i=j}^{n} \alpha_i & \text{for } j = 1, \cdots n \\ 0 \end{cases} \tag{133}$$

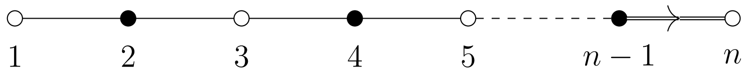
Observe that there is a recursive relation between weights as

$$\mu_{j+1} = \mu_j - \alpha_j \tag{134}$$

for $j = 1, \cdots, n-1$.

3.2.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, $p(i) = 1$ if i is even and $p(i) = 0$ if i is odd.

Unlike the Dynkin diagram of $\mathfrak{sl}_n$, not all simple roots of $\mathfrak{so}_{2n+1}$ have the same length. Specifically, for $j = 1, \ldots, n-1$, we have $\alpha_j^2 = 2\alpha_n^2$. As a result, a factor of $\frac{1}{2}$ appears in $\alpha_n(\Lambda)$. Therefore, Equation (5) becomes equivalent to:

$$\alpha_j(\Lambda) = \begin{cases} x_j \cdot \zeta_{2n}^{p(j)/2} \cdot (1 - \zeta_{2n}^{-2p(j)+1}) & \text{for } j = 1, \cdots, n-1 \\ (\frac{1}{2}) \cdot x_n \cdot \zeta_{2n}^{p(n)/2} \cdot (1 - \zeta_{2n}^{-2p(n)+1}) & \text{for } j = n \end{cases}$$

By Equation (132) for x_j, it yields

$$\alpha_j(\Lambda) = \left(\frac{1}{2}\right)^{\delta_{j,n}} \left(\zeta_{2n}^{(-1)^j \lfloor \frac{j}{2} \rfloor} - \zeta_{2n}^{(-1)^{j+1} \lfloor \frac{j+1}{2} \rfloor}\right) \tag{135}$$

where

$$\delta_{j,n} = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n \end{cases}$$

Equivalently,

$$\alpha_j(\Lambda) = \begin{cases} \zeta_{2n}^{j/2} - \zeta_{2n}^{-j/2} & \text{if } j < n \text{ is even} \\ \zeta_{2n}^{\frac{1}{2}} \cdot (\zeta_{2n}^{-j/2} - \zeta_{2n}^{j/2}) & \text{if } j < n \text{ is odd} \\ \frac{1}{2} \cdot (\zeta_{2n}^{n/2} - \zeta_{2n}^{-n/2}) & \text{if } j = n \text{ is even} \\ \frac{1}{2} \cdot \zeta_{2n}^{\frac{1}{2}} \cdot (\zeta_{2n}^{-n/2} - \zeta_{2n}^{n/2}) & \text{if } j = n \text{ is odd} \end{cases} \tag{136}$$

By Equations (133), we obtain

$$\begin{aligned} \mu_1(\Lambda) &= \sum_{i=1}^{n} \alpha_i(\Lambda) \\ &= (1 - \zeta_{2n}) + (\zeta_{2n} - \zeta_{2n}^{-1}) + (\zeta_{2n}^{-1} - \zeta_{2n}^{2}) + \cdots + (\zeta_{2n}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} - \zeta_{2n}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor}) \\ &\quad + (\frac{1}{2})(\zeta_{2n}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor} - \zeta_{2n}^{(-1)^{n+1}\lfloor\frac{n+1}{2}\rfloor}) \\ &= 1 - (\frac{1}{2})(\zeta_{2n}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor} + \zeta_{2n}^{(-1)^{n+1}\lfloor\frac{n+1}{2}\rfloor}) \end{aligned}$$

When n is odd,

$$\left|(-1)^n\lfloor\frac{n}{2}\rfloor - (-1)^{n+1}\lfloor\frac{n+1}{2}\rfloor\right| = \left|-\frac{n-1}{2} - \frac{n+1}{2}\right| = n$$

When n is even,

$$\left|(-1)^n\lfloor\frac{n}{2}\rfloor - (-1)^{n+1}\lfloor\frac{n+1}{2}\rfloor\right| = \left|\frac{n}{2} + \frac{n}{2}\right| = n$$

Thus, $\zeta_{2n}^{(-1)^n\lfloor\frac{n}{2}\rfloor}$ and $\zeta_{2n}^{(-1)^{n+1}\lfloor\frac{n+1}{2}\rfloor}$ are symmetric about the origin. Hence,

$$\zeta_{2n}^{(-1)^n\lfloor\frac{n}{2}\rfloor} + \zeta_{2n}^{(-1)^{n+1}\lfloor\frac{n+1}{2}\rfloor} = 0 \implies \mu_1(\Lambda) = 1$$

Then, by the relation in Equation (134), we have:

$$\begin{aligned}
&\mu_2(\Lambda) = \mu_1(\Lambda) - \alpha_1(\Lambda) = 1 - (1 - \zeta_{2n}) = \zeta_{2n} \\
&\mu_3(\Lambda) = \mu_2(\Lambda) - \alpha_2(\Lambda) = \zeta_{2n} - (\zeta_{2n} - \zeta_{2n}^{-1}) = \zeta_{2n}^{-1} \\
&\mu_4(\Lambda) = \mu_3(\Lambda) - \alpha_3(\Lambda) = \zeta_{2n}^{-1} - (\zeta_{2n}^{-1} - \zeta_{2n}^{2}) = \zeta_{2n}^{2} \\
&\qquad\quad \cdots \\
&\mu_n(\Lambda) = \mu_{n-1}(\Lambda) - \alpha_{n-1}(\Lambda) = \zeta_{2n}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} - (\zeta_{2n}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} - \zeta_{2n}^{(-1)^n\lfloor\frac{n}{2}\rfloor}) = \zeta_{2n}^{(-1)^n\lfloor\frac{n}{2}\rfloor}
\end{aligned}$$

Therefore,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n} = \{\zeta_{2n}^{(-1)^j\lfloor\frac{j}{2}\rfloor}\}_{1,\cdots,n} \tag{137}$$

which contains n different $2n$-th roots of unity. Note that for any $1 \leq j < k \leq n$,

$$\left|(-1)^j\lfloor\frac{j}{2}\rfloor - (-1)^k\lfloor\frac{k}{2}\rfloor\right| \neq n \implies \mu_j(\Lambda) \neq -\mu_k(\Lambda)$$

Thus,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n} \cap \{-\mu_i(\Lambda)\}_{i=1,\cdots,n} = \emptyset$$

Since $\{-\mu_i(\Lambda)\}_{i=1,\cdots,n}$ also contains n different $2n$-th roots of unity, therefore,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n} \cup \{-\mu_i(\Lambda)\}_{i=1,\cdots,n} = \{\zeta_{2n}^j \mid j = 1, \cdots, 2n\}$$

It follows that

$$\mathrm{Spec}(\psi_1, \mathfrak{so}_{2n+1}) = C_{2n}(1, 0)$$

The part of Theorem 1.4 concerning the first fundamental cyclic spectrum of $\mathfrak{so}_{2n+1}$ is now proved.

Remark 9. Similar to Remark 7, we visualize this spectrum in the complex plane. The weights in Equation (133) are arranged in ascending order of height as follows:

$$\mu_1(\Lambda), \mu_2(\Lambda), \cdots, \mu_n(\Lambda), 0, -\mu_n(\Lambda), \cdots, -\mu_2(\Lambda), -\mu_1(\Lambda)$$

By connecting eigenvalues in this order, Figures 7 and 8 present the cyclic spectra for the first fundamental representation of type B_3 and B_4, respectively. The path consistently begins with 1, followed by ζ_{2n}^k and its conjugate ζ_{2n}^{-k} for $k \leq n-1$. For $i < j$, $\mathrm{Re}(\mu_i(\Lambda)) \geq \mathrm{Re}(\mu_j(\Lambda))$.

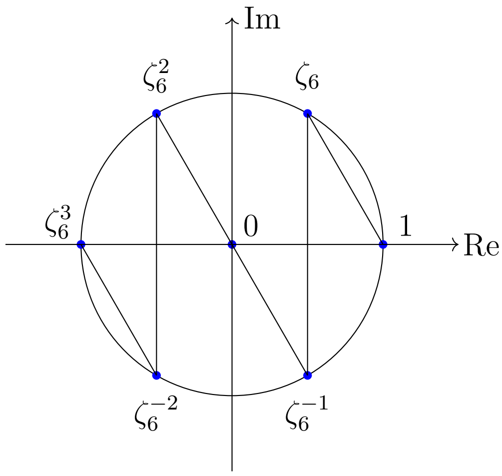


Figure 7: First fundamental cyclic spectrum for B_3

3.2.2 The Adjoint Representation of Type B_n

In this section, we apply the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{so}_{2n+1}$. The dimension of this representation is $2n^2+n$, so there are at most $2n^2$ non-zero eigenvalues. As mentioned in Section 2.3.1, in addition to the n simple roots, the remaining positive roots of $\mathfrak{so}_{2n+1}$ are given by

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j$$

$$\alpha'_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$$

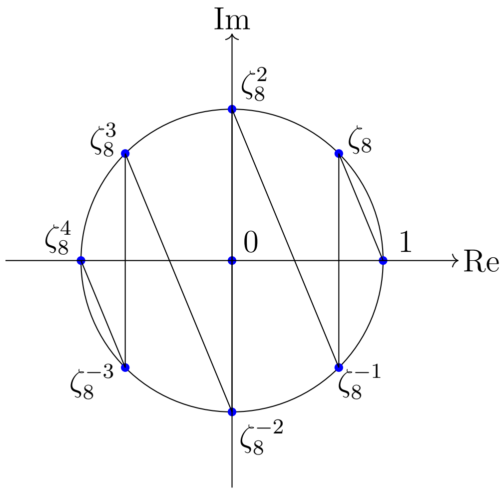


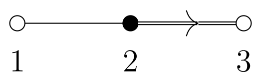
Figure 8: First fundamental cyclic spectrum for B_4

for $1 \leq i < j \leq n$. As in [3, Plate II], the remaining positive roots of $\mathfrak{so}_{2n+1}$ can be written as linear combinations of the first fundamental weight. By Equation (137), the remaining eigenvalues are as follows:

$$\begin{aligned}
\alpha_{in}(\Lambda) &= \mu_i(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} \quad 1 \leq i \leq n \\
\alpha_{ij}(\Lambda) &= \mu_i(\Lambda) - \mu_{j+1}(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} - \zeta_{2n}^{(-1)^{j+1} \lfloor \frac{j+1}{2} \rfloor} \quad 1 \leq i < j \leq n \\
\alpha'_{ij}(\Lambda) &= \mu_i(\Lambda) + \mu_j(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} + \zeta_{2n}^{(-1)^j \lfloor \frac{j}{2} \rfloor} \quad 1 \leq i < j \leq n
\end{aligned}$$

But the information about radii and multiplicities in the spectrum is not explicit. To understand the patterns in the adjoint spectrum, we first go through several examples.

Example 3.5. Consider the case of B_3, $\mathfrak{so}_7$. As before, we choose the bi-coloration:



By Equation (136), we obtain

$$\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{12} \cdot (\zeta_6^{-1/2} - \zeta_6^{1/2}) = \mathsf{r}_1 \cdot \zeta_6^5 \\
\alpha_2(\Lambda) &= \zeta_6 - \zeta_6^{-1} = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6 \\
\alpha_3(\Lambda) &= \frac{1}{2}\zeta_{16} \cdot (\zeta_8^{-3/2} - \zeta_8^{3/2}) = \mathsf{r}_3 \cdot \zeta_6^5
\end{aligned}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{6}\right) = 1 = \mathsf{r}_3, \mathsf{r}_2 = 2\sin\left(\frac{\pi}{3}\right)$. Thus, $\alpha_1(\Lambda) = \alpha_3(\Lambda) = \zeta_6^5$. It also follows that:

$$\alpha_{1,2}(\Lambda) = \alpha_{2,3}(\Lambda) = \zeta_6$$
$$\alpha_{1,3}(\Lambda) = \alpha'_{2,3}(\Lambda) = 1$$

Therefore, the circle $C_6(\mathsf{r}_1)$ generated by $\alpha_1(\Lambda)$ has multiplicity 2. By the "ζ_6-symmetry", the pairings of roots and Λ yields

$$C_6(\mathsf{r}_1^{(2)},\ \zeta_{12}\cdot \mathsf{r}_2)$$

It follows that

$$\mathrm{Spec}(\mathrm{ad}, B_3) = C_6(\mathsf{r}_1^{(2)},\ \zeta_{12}\cdot \mathsf{r}_2,\ 0^{(3)})$$

Example 3.6. Consider the case of B_4, $\mathfrak{so}_9$. As before, we choose the bi-coloration:

By Equation (136), we obtain:

$$\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{16}\cdot(\zeta_8^{-1/2} - \zeta_8^{1/2}) = \zeta_{16}\cdot \mathsf{r}_1 \cdot \zeta_8^6 \\
\alpha_2(\Lambda) &= \zeta_8 - \zeta_8^{-1} = \mathsf{r}_2\cdot\zeta_8^2 \\
\alpha_3(\Lambda) &= \zeta_{16}\cdot(\zeta_8^{-3/2} - \zeta_8^{3/2}) = \zeta_{16}\cdot \mathsf{r}_3 \cdot \zeta_8^6 \\
\alpha_4(\Lambda) &= \frac{1}{2}(\zeta_8^2 - \zeta_8^{-2}) = \mathsf{r}_4\cdot\zeta_8^2
\end{aligned}$$

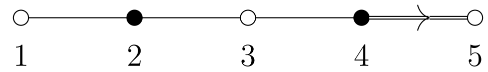
where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{8}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{4}\right), \mathsf{r}_3 = 2\sin\left(\frac{3\pi}{8}\right), \mathsf{r}_4 = 1$. By "$\zeta_8$-symmetry", $\alpha_j(\Lambda)$'s generate:

$$C_8(\zeta_{16}\cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{16}\cdot \mathsf{r}_3,\ \mathsf{r}_4)$$

There are at most 32 non-zero eigenvalues, so we have obtained all of them. Hence,

$$\mathrm{Spec}(\mathrm{ad}, B_4) = C_8(\zeta_{16} \cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{16} \cdot \mathsf{r}_3,\ \mathsf{r}_4, 0^{(4)})$$

Example 3.7. Consider the case of B_5, $\mathfrak{so}_{11}$. As before, we choose the bi-coloration:



By Equation (136), we obtain

$$\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{20} \cdot (\zeta_{10}^{-1/2} - \zeta_{10}^{1/2}) = \mathsf{r}_1 \cdot \zeta_{10}^{8} \\
\alpha_2(\Lambda) &= \zeta_{10} - \zeta_{10}^{-1} = \zeta_{20} \cdot \mathsf{r}_2 \cdot \zeta_{10}^{2} \\
\alpha_3(\Lambda) &= \zeta_{20} \cdot (\zeta_{10}^{-3/2} - \zeta_{10}^{3/2}) = \mathsf{r}_3 \cdot \zeta_{10}^{8} \\
\alpha_4(\Lambda) &= \zeta_{10}^{2} - \zeta_{10}^{-2} = \zeta_{20} \cdot \mathsf{r}_4 \cdot \zeta_{10}^{2} \\
\alpha_5(\Lambda) &= \frac{1}{2}\zeta_{20} \cdot (\zeta_{10}^{-5/2} - \zeta_{10}^{5/2}) = \mathsf{r}_5 \cdot \zeta_{10}^{8}
\end{aligned}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{10}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{5}\right), \mathsf{r}_3 = 2\sin\left(\frac{3\pi}{10}\right), \mathsf{r}_4 = 2\sin\left(\frac{2\pi}{5}\right), \mathsf{r}_5 = 1$. By "$\zeta_{10}$-symmetry", $\alpha_j(\Lambda)$'s generate:

$$C_{10}(\mathsf{r}_1,\ \zeta_{20} \cdot \mathsf{r}_2,\ \mathsf{r}_3,\ \zeta_{20} \cdot \mathsf{r}_4,\ \mathsf{r}_5)$$

There are at most 50 non-zero eigenvalues, so we have obtained all of them. Hence,

$$\mathrm{Spec}(\mathrm{ad}, B_5) = C_{10}(\mathsf{r}_1,\ \zeta_{20} \cdot \mathsf{r}_2,\ \mathsf{r}_3,\ \zeta_{20} \cdot \mathsf{r}_4,\ \mathsf{r}_5, 0^{(5)})$$

Example 3.8. Consider the case of B_6, $\mathfrak{so}_{13}$. As before, we choose the bi-coloration:

By Equation (136), we obtain

$$\alpha_1(\Lambda) = \zeta_{24} \cdot (\zeta_{12}^{-1/2} - \zeta_{12}^{1/2}) = \zeta_{24} \cdot \mathsf{r}_1 \cdot \zeta_{12}^9$$
$$\alpha_2(\Lambda) = \zeta_{12} - \zeta_{12}^{-1} = \mathsf{r}_2 \cdot \zeta_{12}^3$$
$$\alpha_3(\Lambda) = \zeta_{24} \cdot (\zeta_{12}^{-3/2} - \zeta_{12}^{3/2}) = \zeta_{24} \cdot \mathsf{r}_3 \cdot \zeta_{12}^8$$
$$\alpha_4(\Lambda) = \zeta_{12}^2 - \zeta_{12}^{-2} = \mathsf{r}_4 \cdot \zeta_{12}^3$$
$$\alpha_5(\Lambda) = \zeta_{24} \cdot (\zeta_{12}^{-5/2} - \zeta_{12}^{5/2}) = \zeta_{24} \cdot \mathsf{r}_5 \cdot \zeta_{12}^9$$
$$\alpha_6(\Lambda) = \frac{1}{2}(\zeta_{12}^3 - \zeta_{12}^{-3}) = \mathsf{r}_6 \cdot \zeta_{12}^3$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{12}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{6}\right) = 1 = \mathsf{r}_6, \mathsf{r}_3 = 2\sin\left(\frac{\pi}{4}\right), \mathsf{r}_4 = 2\sin\left(\frac{\pi}{3}\right), \mathsf{r}_5 = 2\sin\left(\frac{5\pi}{12}\right)$. Thus, $\alpha_2(\Lambda) = \alpha_6(\Lambda) = \zeta_{12}^3$. It implies that:

$$\alpha_{2,5}(\Lambda) = \alpha_{3,6}(\Lambda) = \zeta_{12}^{-1}$$
$$\alpha_{2,6}(\Lambda) = \alpha'_{3,6}(\Lambda) = \zeta_{12}$$

Moreover,

$$\zeta_{12}^2 + \zeta_{12}^{-2} = 1$$

Thus,

$$\alpha_{1,3}(\Lambda) = \alpha_{5,6}(\Lambda) = \zeta_{12}^{-2}$$

It implies that:

$$\alpha_{1,4}(\Lambda) = \alpha_{4,6}(\Lambda) = \zeta_{12}^2$$
$$\alpha_{1,6}(\Lambda) = \alpha'_{4,5}(\Lambda) = 1$$

Therefore, the circle $C_{12}(\mathsf{r}_2)$ generated by $\alpha_2(\Lambda)$ has multiplicity 2. By "ζ_{12}-symmetry", the pairings of roots and Λ yields

$$C_{12}(\zeta_{24}\cdot \mathsf{r}_1,\ \mathsf{r}_2^{(2)},\ \zeta_{24}\cdot \mathsf{r}_3,\ \mathsf{r}_4,\ \zeta_{24}\cdot \mathsf{r}_5)$$

It follows that

$$\mathrm{Spec}(\mathrm{ad}, B_6) = C_{12}(\zeta_{24}\cdot \mathsf{r}_1,\ \mathsf{r}_2^{(2)},\ \zeta_{24}\cdot \mathsf{r}_3,\ \mathsf{r}_4,\ \zeta_{24}\cdot \mathsf{r}_5,\ 0^{(6)})$$

The above examples suggest the following patterns in $\mathrm{Spec}(\mathrm{ad}, B_n)$.

1. If $n \not\equiv 0 \mod 3$, let

$$p(j) = \begin{cases} 0 & \text{if } j, n \text{ are with the same parity} \\ 1 & \text{otherwise} \end{cases}$$

for $j = 1, \cdots, n$. Then

$$\mathrm{Spec}(\mathrm{ad}, B_n) = C_{2n}(\zeta_{4n}^{p(1)}\cdot \mathsf{r}_1,\ \cdots,\ \zeta_{4n}^{p(n)}\cdot \mathsf{r}_n,\ 0^{(n)}) \tag{138}$$

where $\mathsf{r}_j = |\alpha_j(\Lambda)| = 2\sin\left(\frac{j\pi}{2n}\right)$ for $j = 1, \cdots, n-1$, and $\mathsf{r}_n = 1$.

2. If $n \equiv 0 \mod 3$, such that $n = 3k$ for some positive integer k, let $p(j)$ be defined as in the previous case for $j = 1, \cdots, n-1$. Then

$$\mathrm{Spec}(\mathrm{ad}, B_n) = C_{2n}(z_1, \cdots, z_{k-1}, z_k^{(2)}, z_{k+1}, \cdots, z_{n-1}, 0^{(n)}) \tag{139}$$

where $z_j = \zeta_{4n}^{p(j)}\cdot \mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ for $j = 1, \cdots, n-1$.

Now, we prove the above two equations to prove the corresponding part of Theorem 1.4. Consider the first case when n is not a multiple of 3. By Equation (136) and the calculation of $\mu_j(\Lambda)$ in the previous section, we obtain:

$$\alpha_j(\Lambda) = \begin{cases} 2\sin\left(\frac{j\pi}{2n}\right) \cdot i & \text{if } j < n \text{ even} \\ 2\sin\left(\frac{j\pi}{2n}\right) \cdot \zeta_{4n} \cdot (-i) & \text{if } j < n \text{ odd} \\ \zeta_{2n}^{(-1)^n \lfloor \frac{n}{2} \rfloor} & \text{if } j = n \end{cases} \tag{140}$$

Note that when n is even, $i = \zeta_{2n}^{\frac{n}{2}}$, and $-i = \zeta_{2n}^{\frac{3n}{2}}$. When n is odd, $i = \zeta_{4n} \cdot \zeta_{2n}^{\frac{n-1}{2}}$, and $\zeta_{4n} \cdot (-i) = \zeta_{2n}^{\frac{3n-1}{2}}$. Thus, $p(j)$ in Equation (138) equals 0 if j, n are with the same parity, and 1 otherwise. Also, since n is not a multiple of 3, thus

$$\{2\sin\left(\frac{j\pi}{2n}\right) \mid j = 1, \cdots, n-1\} \cap \{1\} = \emptyset$$

In other words, each $\alpha_j(\Lambda)$ yields distinct eigenvalue with distinct radius for $j = 1, \cdots, n$. Therefore, by the "ζ_{2n}-symmetry", it follows that $\{\alpha_j(\Lambda)\}_{j=1,\ldots,n}$ generates Equation (138).

Now we prove the other case when $n = 3k$ for some positive integer k. The argument is almost the same as the previous case, except when $j = k$, $\mathsf{r}_j = 1$, then $C_{2n}(1)$ always has double multiplicities. We now prove this special case by listing n pairs of distinct positive roots whose pairings with Λ yield the same eigenvalue in that circle. The key identity to show

$$\alpha_k(\Lambda) = \alpha_n(\Lambda) = \zeta_{2n}^{(-1)^n \lfloor \frac{n}{2} \rfloor} = \mu_n(\Lambda) \tag{141}$$

and for $j = 1, \cdots, k-1$,

$$\alpha_{k-j,k+j}(\Lambda) = \alpha_{n-j,n}(\Lambda) = \zeta_{2n}^{(-1)^{n-j} \lfloor \frac{n-j}{2} \rfloor} = \mu_{n-j}(\Lambda) \tag{142}$$

We draw this equivalence on the Dynkin diagram in Figure 9 to help visualize it.

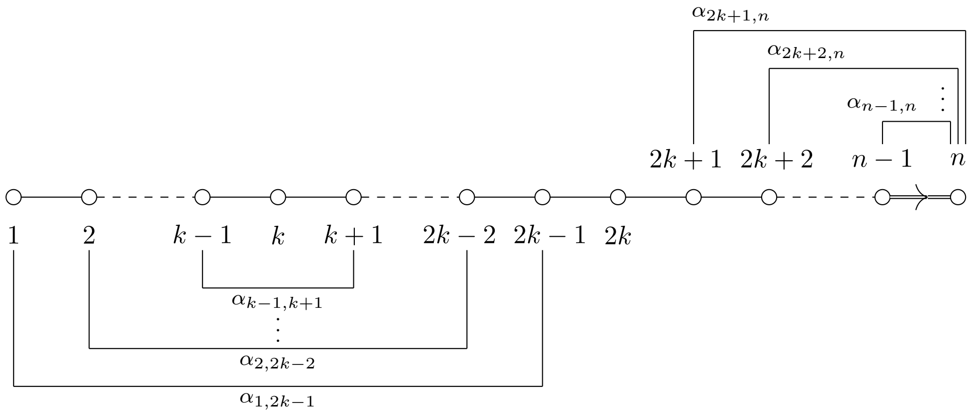


Figure 9: Visualization of Roots in Equations (141) and (142)

Equation (141) follows from Equations (137), (140) and $2\sin\left(\frac{k\pi}{2n}\right) = 2\sin\left(\frac{\pi}{6}\right) = 1$. For Equation (142), by the calculation of Equation (137), it follows that $\alpha_{n-j,n}(\Lambda) = \mu_{n-j}(\Lambda)$. To prove that $\alpha_{k-j,k+j}(\Lambda) = \mu_{n-j}(\Lambda)$, the key identity is:

$$\zeta_{2n}^{k} + \zeta_{2n}^{-k} = 2\sin\left(\frac{\pi}{6}\right) = 1 \tag{143}$$

For $1 \leq j \leq k-1$,

$$\begin{aligned}
\alpha_{k-j,k+j}(\Lambda) &= \zeta_{2n}^{(-1)^{k-j}\lfloor\frac{k-j}{2}\rfloor} - \zeta_{2n}^{(-1)^{k+j+1}\lfloor\frac{k+j+1}{2}\rfloor} \\
&= \begin{cases} \zeta_{2n}^{-\left(\frac{k-j-1}{2}\right)} - \zeta_{2n}^{\frac{k+j+1}{2}} & \text{if } k \pm j \text{ are odd} \\ \zeta_{2n}^{\frac{k-j}{2}} - \zeta_{2n}^{-\left(\frac{k+j}{2}\right)} & \text{if } k \pm j \text{ are even} \end{cases}
\end{aligned}$$

When $k \pm j$ are odd, by Equation (143),

$$\begin{aligned}
\zeta_{2n}^{-\left(\frac{k-j-1}{2}\right)} - \zeta_{2n}^{\frac{k+j+1}{2}} &= \zeta_{2n}^{-\left(\frac{k-j-1}{2}\right)} + \zeta_{2n}^{\frac{k+j+1}{2}-n} \\
&= \zeta_{2n}^{-\left(\frac{n-j-1}{2}\right)} \cdot \left(\zeta_{2n}^{k} + \zeta_{2n}^{-k}\right) \\
&= \zeta_{2n}^{-\left(\frac{n-j-1}{2}\right)} \\
&= \zeta_{2n}^{(-1)^{n-j}\lfloor\frac{n-j}{2}\rfloor}
\end{aligned}$$

When $k \pm j$ are even, by Equation (143),

$$\begin{aligned}
\zeta_{2n}^{\frac{k-j}{2}} - \zeta_{2n}^{-\left(\frac{k+j}{2}\right)} &= \zeta_{2n}^{\frac{k-j}{2}} + \zeta_{2n}^{-\left(\frac{k+j}{2}\right)+n} \\
&= \zeta_{2n}^{\frac{n-j}{2}} \cdot (\zeta_{2n}^{-k} + \zeta_{2n}^{k}) \\
&= \zeta_{2n}^{\frac{n-j}{2}} \\
&= \zeta_{2n}^{(-1)^{n-j}\lfloor\frac{n-j}{2}\rfloor}
\end{aligned}$$

Equation (142) is now proved. Based on these two equations and Figure 9, it also follows that:

$$\begin{aligned}
\alpha_{k,n-1}(\Lambda) = \alpha_{k+1,n}(\Lambda) = \zeta_{2n}^{(-1)^{k+1}\lfloor\frac{k+1}{2}\rfloor} = \mu_{k+1}(\Lambda) \\
\alpha_{k,n}(\Lambda) = \alpha'_{k+1,n}(\Lambda) = \zeta_{2n}^{(-1)^{k}\lfloor\frac{k}{2}\rfloor} = \mu_{k}(\Lambda)
\end{aligned}$$

Also, for $j = 1, \cdots, k-1$,

$$\begin{aligned}
\alpha_{k-j,n-j-1}(\Lambda) = \alpha_{k+j+1,n}(\Lambda) = \zeta_{2n}^{(-1)^{k+j+1}\lfloor\frac{k+j+1}{2}\rfloor} = \mu_{k+j+1}(\Lambda) \\
\alpha_{k-j,n}(\Lambda) = \alpha'_{k+j+1,n-j}(\Lambda) = \zeta_{2n}^{(-1)^{k-j}\lfloor\frac{k-j}{2}\rfloor} = \mu_{k-j}(\Lambda)
\end{aligned}$$

Hence, we obtained n pairs of distinct positive roots whose pairings with Λ yield eigenvalues:

$$\{\zeta_{2n}^{(-1)^j\lfloor\frac{j}{2}\rfloor}\}_{j=1,\cdots,n} = \{\mu_j(\Lambda)\}_{j=1,\cdots,n}$$

Therefore, the circle $C_{2n}(1)$ has multiplicity 2, which proves Equation (139). The proof of Theorem 1.4 for $\mathfrak{so}_{2n+1}$ is now complete. In view of Equation (140), this also completes the proof of Theorem 1.5 for $\mathfrak{so}_{2n+1}$.

3.3 Type C_n

To apply Theorem 1.3 to a simple Lie algebra of type C_n, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type C_n simple Lie algebra, $\mathfrak{sp}_{2n}$, is not of ADE type. Its Cartan matrix C is not symmetric and is with dimension $n \times n$ in the form:

$$C = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$$

As in the case of $\mathfrak{so}_{2n+1}$, the Cartan matrix has real eigenvalues, with the smallest being $2 - 2\cos\left(\frac{\pi}{h}\right)$, where $h = 2n$ is the Coxeter number.

Now we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{2n}\right)$. Let $\vec{\boldsymbol{x}} = [x_1, \ldots, x_n]^\intercal$ be such an eigenvector. For computational convenience, we assume $x_1 = 1$. By expanding $\vec{\boldsymbol{x}}C = (2 - 2\cos\left(\frac{\pi}{2n}\right))\vec{\boldsymbol{x}}$, we have:

$$\begin{aligned} 2 - x_2 &= 2 - 2\cos\left(\frac{\pi}{2n}\right) && (144) \\ -1 + 2x_2 - x_3 &= \left(2 - 2\cos\left(\frac{\pi}{2n}\right)\right) x_2 && (145) \\ &\vdots \\ -x_{n-2} + 2x_{n-1} - 2x_n &= \left(2 - 2\cos\left(\frac{\pi}{2n}\right)\right) x_{n-1} && (146) \\ -x_{n-1} + 2x_n &= \left(2 - 2\cos\left(\frac{\pi}{2n}\right)\right) x_n && (147) \end{aligned}$$

From Equations (144)-(146), we observe that for $j = 1, \cdots, n-1$, the recursive relation between components of $\vec{\boldsymbol{x}}$ is identical to those in the $\mathfrak{sl}_n$ case:

$$x_1 = 1;$$

$$x_2 = 2\cos\left(\frac{\pi}{2n}\right) \cdot x_1;$$

$$\vdots$$

$$x_j = -x_{j-2} + 2\cos\left(\frac{\pi}{2n}\right) \cdot x_{j-1}$$

When $j = n$, x_n is halved:

$$x_n = \frac{-x_{n-2} + 2\cos\left(\frac{\pi}{2n}\right) \cdot x_{n-1}}{2}$$

Therefore, using the same argument, $\vec{\boldsymbol{x}}$ has components:

$$x_j = \begin{cases} \dfrac{\sin\left(\dfrac{j\pi}{2n}\right)}{\sin\left(\dfrac{\pi}{2n}\right)} = \dfrac{\zeta_{2n}^{j/2} - \zeta_{2n}^{-j/2}}{\zeta_{2n}^{1/2} - \zeta_{2n}^{-1/2}} & \text{for } j = 1, \cdots, n-1 \\ \\ \dfrac{1}{2\sin\left(\dfrac{\pi}{2n}\right)} = \dfrac{\zeta_{2n}^{n/2} - \zeta_{2n}^{-n/2}}{2\zeta_{2n}^{1/2} - 2\zeta_{2n}^{-1/2}} & \text{for } j = n \end{cases} \tag{148}$$

This solution also satisfies Equation (147). It follows from the same argument as in $\mathfrak{so}_{2n+1}$.

3.3.1 The First Fundamental Representation of Type C_n

In this section, we calculate the cyclic spectrum for the first fundamental representation of the type C_n simple Lie algebra using Theorem 1.3.

3.3.1.1 Weights

At first, we introduce functionals $\{\mu_i\}_{i=1,\cdots,n}$ on the Cartan subalgebra of $\mathfrak{sp}_{2n}$ consisting of matrices in the form $\mathrm{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n)$ such that:

$$\mu_i : \mathrm{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n) \mapsto x_i$$

In particular, $\{\mu_1, \cdots, \mu_n, -\mu_1, \cdots, -\mu_n\}$ are the $2n$ weights in the standard $2n$-dimen--sional representation of $\mathfrak{sp}_{2n}$, which is also the first fundamental representation of $\mathfrak{sp}_{2n}$. It is known that the simple roots $\{\alpha_i\}_{i=1,\cdots,n}$ for $\mathfrak{sp}_{2n}$ are

$$\alpha_i = \begin{cases} \mu_i - \mu_{i+1} & \text{for } i = 1, \cdots, n-1 \\ 2\mu_n & \text{for } i = n \end{cases}$$

See for example [16, Appendix A.2]. One can obtain from simple calculations that the weights for the standard representation of $\mathfrak{sp}_{2n}$ in terms of simple roots are:

$$\begin{cases} \pm\mu_j = \pm\left(\sum_{i=j}^{n-1} \alpha_i + \frac{\alpha_n}{2}\right) & \text{for } j = 1, \cdots n-1 \\ \pm\mu_n = \pm\frac{\alpha_n}{2} \end{cases} \tag{149}$$

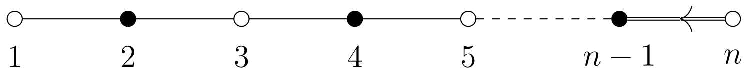
Observe that there is a recursive relation between weights as

$$\mu_{j+1} = \mu_j - \alpha_j \tag{150}$$

for $j = 1, \cdots, n-1$.

3.3.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, $p(i) = 1$ if i is even and $p(i) = 0$ if i is odd.

Unlike the Dynkin diagram of $\mathfrak{sl}_n$, not all simple roots of $\mathfrak{sp}_{2n}$ have the same length. Specifically, for $j = 1, \ldots, n-1$, we have $2\alpha_j^2 = \alpha_n^2$. As a result, a factor of 2 appears in

$\alpha_n(\Lambda)$. Therefore, Equation (5) is equivalent to:

$$\alpha_j(\Lambda) = \begin{cases} x_j \cdot \zeta_{2n}^{p(j)/2} \cdot (1 - \zeta_{2n}^{-2p(j)+1}) & \text{for } j = 1, \cdots, n-1 \\ 2 \cdot x_n \cdot \zeta_{2n}^{p(n)/2} \cdot (1 - \zeta_{2n}^{-2p(n)+1}) & \text{for } j = n \end{cases}$$

By Equation (148) for x_j, it yields

$$\alpha_j(\Lambda) = \zeta_{2n}^{(-1)^j \lfloor \frac{j}{2} \rfloor} - \zeta_{2n}^{(-1)^{j+1} \lfloor \frac{j+1}{2} \rfloor} \tag{151}$$

Equivalently,

$$\alpha_j(\Lambda) = \begin{cases} \zeta_{2n}^{j/2} - \zeta_{2n}^{-j/2} & \text{if } j \text{ is even} \\ \zeta_{2n}^{\frac{1}{2}} \cdot (\zeta_{2n}^{-j/2} - \zeta_{2n}^{j/2}) & \text{if } j \text{ is odd} \end{cases} \tag{152}$$

By Equations (149), we obtain:

$$\begin{aligned} \mu_1(\Lambda) &= \sum_{i=1}^{n-1} \alpha_i(\Lambda) + \frac{\alpha_n(\Lambda)}{2} \\ &= (1 - \zeta_{2n}) + (\zeta_{2n} - \zeta_{2n}^{-1}) + (\zeta_{2n}^{-1} - \zeta_{2n}^{2}) + \cdots + (\zeta_{2n}^{(-1)^{n-1} \lfloor \frac{n-1}{2} \rfloor} - \zeta_{2n}^{(-1)^n \lfloor \frac{n}{2} \rfloor}) \\ &\quad + (\frac{1}{2})(\zeta_{2n}^{(-1)^n \lfloor \frac{n}{2} \rfloor} - \zeta_{2n}^{(-1)^{n+1} \lfloor \frac{n+1}{2} \rfloor}) \\ &= 1 - (\frac{1}{2})(\zeta_{2n}^{(-1)^n \lfloor \frac{n}{2} \rfloor} + \zeta_{2n}^{(-1)^{n+1} \lfloor \frac{n+1}{2} \rfloor}) \end{aligned}$$

By the same argument as in $\mathfrak{so}_{2n+1}$, $\mu_1(\Lambda) = 1$. Then, by the relation in Equation (150), we have:

$$\begin{aligned} \mu_2(\Lambda) &= \mu_1(\Lambda) - \alpha_1(\Lambda) = 1 - (1 - \zeta_{2n}) = \zeta_{2n} \\ \mu_3(\Lambda) &= \mu_2(\Lambda) - \alpha_2(\Lambda) = \zeta_{2n} - (\zeta_{2n} - \zeta_{2n}^{-1}) = \zeta_{2n}^{-1} \\ \mu_4(\Lambda) &= \mu_3(\Lambda) - \alpha_3(\Lambda) = \zeta_{2n}^{-1} - (\zeta_{2n}^{-1} - \zeta_{2n}^{2}) = \zeta_{2n}^{2} \\ &\cdots \end{aligned}$$

$$\mu_n(\Lambda) = \mu_{n-1}(\Lambda) - \alpha_{n-1}(\Lambda) = \zeta_{2n}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} - (\zeta_{2n}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} - \zeta_{2n}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor}) = \zeta_{2n}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor}$$

Therefore,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n} = \{\zeta_{2n}^{(-1)^j\lfloor\frac{j}{2}\rfloor}\}_{1,\cdots,n} \tag{153}$$

which contains n different $2n$-th roots of unity. By the same argument as in $\mathfrak{so}_{2n+1}$, $\{-\mu_i(\Lambda)\}_{i=1,\cdots,n}$ also contains n different $2n$-th roots of unity, therefore,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n} \cup \{-\mu_i(\Lambda)\}_{i=1,\cdots,n} = \{\zeta_{2n}^j \mid j = 1,\cdots,2n-1\}$$

It follows that

$$\mathrm{Spec}(\psi_1, \mathfrak{sp}_{2n}) = C_{2n}(1,0)$$

The part of Theorem 1.4 concerning the first fundamental cyclic spectrum of $\mathfrak{sp}_{2n}$ is now proved.

Remark 10. Similar to Remark 7, we visualize this spectrum in the complex plane. The weights in Equation (149) are arranged in ascending order of height as follows:

$$\{\mu_1(\Lambda), \mu_2(\Lambda), \cdots, \mu_n(\Lambda), -\mu_n(\Lambda), \cdots, -\mu_2(\Lambda), -\mu_1(\Lambda)\}$$

By connecting eigenvalues in this order, Figures 10 and 11 present the cyclic spectra for the first fundamental representation of type C_3 and C_4, respectively. The path consistently begins with 1, followed by ζ_{2n}^k and its conjugate ζ_{2n}^{-k} for $k \leq n-1$. For $i < j$, $\mathrm{Re}(\mu_i(\Lambda)) \geq \mathrm{Re}(\mu_j(\Lambda))$.

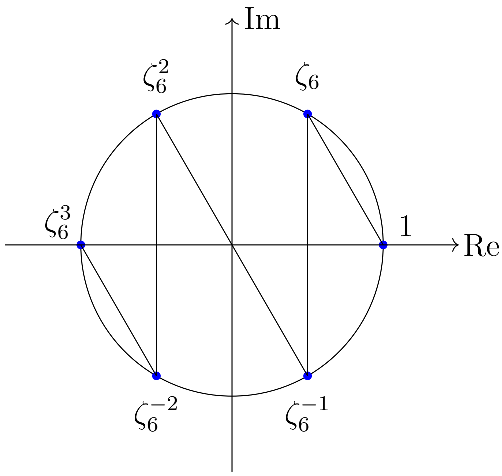


Figure 10: First fundamental cyclic spectrum for C_3

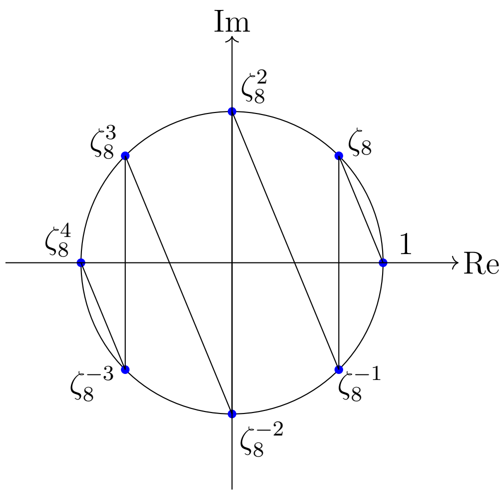


Figure 11: First fundamental cyclic spectrum for C_4

3.3.2 The Adjoint Representation of Type C_n

In this section, we apply the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{sp}_{2n}$. The dimension of this representation is $2n^2 + n$, so there are at most $2n^2$ non-zero eigenvalues. It is known that apart from n simple roots, the remaining positive roots of $\mathfrak{sp}_{2n}$ are as follows:

$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j \quad 1 \leq i \leq j \leq n-1$$

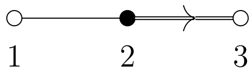
$$\alpha'_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n \quad 1 \leq i \leq j \leq n$$

As in [3, Plate III], the remaining positive roots of $\mathfrak{sp}_{2n}$ can be written as linear combinations of the first fundamental weight. By Equation (153), the remaining eigenvalues are as follows:

$$\begin{aligned}
&\alpha_{ij}(\Lambda) = \mu_i(\Lambda) - \mu_{j+1}(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} - \zeta_{2n}^{(-1)^{j+1} \lfloor \frac{j+1}{2} \rfloor} \quad 1 \le i < j \le n-1 \\
&\alpha'_{ij}(\Lambda) = \mu_i(\Lambda) + \mu_j(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} + \zeta_{2n}^{(-1)^j \lfloor \frac{j}{2} \rfloor} \quad 1 \le i < j \le n \\
&\alpha'_{ii}(\Lambda) = 2\mu_i(\Lambda) = 2\zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} \quad 1 \le i \le n
\end{aligned}$$

But the information about radii and multiplicities in the spectrum is not explicit. To understand the patterns in the adjoint spectrum, we first go through several examples.

Example 3.9. Consider the case of C_3, $\mathfrak{sp}_3$. As before, we choose the bi-coloration:



By Equation (152), we obtain

$$\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{12} \cdot (\zeta_6^{-1/2} - \zeta_6^{1/2}) = \mathsf{r}_1 \cdot \zeta_6^5 \\
\alpha_2(\Lambda) &= \zeta_6 - \zeta_6^{-1} = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6 \\
\alpha_3(\Lambda) &= \zeta_{16} \cdot (\zeta_8^{-3/2} - \zeta_8^{3/2}) = \mathsf{r}_3 \cdot \zeta_6^5
\end{aligned}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{6}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{3}\right), \mathsf{r}_3 = 2$. By the "$\zeta_6$-symmetry", the pairings of roots and Λ yields

$$C_6(\mathsf{r}_1,\ \zeta_{12} \cdot \mathsf{r}_2,\ \mathsf{r}_3)$$

There are at most 18 non-zero eigenvalues, so we have obtained all of them. Hence,

$$\mathrm{Spec}(\mathrm{ad}, C_3) = C_6(\mathsf{r}_1,\ \zeta_{12} \cdot \mathsf{r}_2,\ \mathsf{r}_3,\ 0^{(3)})$$

Example 3.10. Consider the case of C_4, $\mathfrak{sp}_4$. As before, we choose the bi-coloration:

By Equation (152), we obtain

$$\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{16} \cdot (\zeta_8^{-1/2} - \zeta_8^{1/2}) = \zeta_{16} \cdot \mathsf{r}_1 \cdot \zeta_8^6 \\
\alpha_2(\Lambda) &= \zeta_8 - \zeta_8^{-1} = \mathsf{r}_2 \cdot \zeta_8^2 \\
\alpha_3(\Lambda) &= \zeta_{16} \cdot (\zeta_8^{-3/2} - \zeta_8^{3/2}) = \zeta_{16} \cdot \mathsf{r}_3 \cdot \zeta_8^6 \\
\alpha_4(\Lambda) &= \zeta_8^2 - \zeta_8^{-2} = \mathsf{r}_4 \cdot \zeta_8^2
\end{aligned}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{8}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{4}\right), \mathsf{r}_3 = 2\sin\left(\frac{3\pi}{8}\right), \mathsf{r}_4 = 2$. By "$\zeta_8$-symmetry", $\alpha_j(\Lambda)$'s generate:

$$C_8(\zeta_{16} \cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{16} \cdot \mathsf{r}_3,\ \mathsf{r}_4)$$

There are at most 32 non-zero eigenvalues, so we have obtained all of them. Hence,

$$\mathrm{Spec}(\mathrm{ad}, C_4) = C_8(\zeta_{16} \cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{16} \cdot \mathsf{r}_3,\ \mathsf{r}_4,\ 0^{(4)})$$

The above examples suggest that the pattern in $\mathrm{Spec}(\mathrm{ad}, C_n)$ is somewhat simpler, as the value of each $\alpha_j(\Lambda)$ is unique. Indeed, let

$$p(j) = \begin{cases} 0 & \text{if } j, n \text{ are with the same parity} \\ 1 & \text{otherwise} \end{cases}$$

for $j = 1, \cdots, n$. Then

$$\mathrm{Spec}(\mathrm{ad}, C_n) = C_{2n}(\zeta_{4n}^{p(1)} \cdot \mathsf{r}_1,\ \cdots,\ \zeta_{4n}^{p(n)} \cdot \mathsf{r}_n,\ 0^{(n)}) \tag{154}$$

where $\mathsf{r}_j = |\alpha_j(\Lambda)| = 2\sin\left(\frac{j\pi}{2n}\right)$ for $j = 1, \cdots, n$.

Now we prove Equation (154). By Equation (152), we obtain

$$\alpha_j(\Lambda) = \begin{cases} 2\sin\left(\frac{j\pi}{2n}\right)\cdot i & \text{if } j \text{ even} \\ 2\sin\left(\frac{j\pi}{2n}\right)\cdot \zeta_{4n}\cdot(-i) & \text{if } j \text{ odd} \end{cases} \tag{155}$$

for $j = 1, \cdots, n$. Note that when n is even, $i = \zeta_{2n}^{\frac{n}{2}}$, and $-i = \zeta_{2n}^{\frac{3n}{2}}$. When n is odd, $i = \zeta_{4n}\cdot\zeta_{2n}^{\frac{n-1}{2}}$, and $\zeta_{4n}\cdot(-i) = \zeta_{2n}^{\frac{3n+1}{2}}$. Thus, $p(j)$ in Equation (154) equals 0 if j, n are with the same parity, and 1 otherwise. Also, note that

$$\{|\alpha_j(\Lambda)|\}_{j=1,\cdots,n} = \{2\sin\left(\frac{j\pi}{2n}\right) \mid j = 1, \cdots, n\}$$

There are n different values in the set $\{|\alpha_j(\Lambda)|\}_{j=1,\cdots,n}$. Hence, by "$\zeta_{2n}$−symmetry", $\{\alpha_j(\Lambda)\}$ generates $2n^2$ different non-zero eigenvalues, which exhausts all. The proof of Equation (154) is now complete, so is the proof Theorem 1.4 for $\mathfrak{sp}_{2n}$. In view of Equation (155), this also completes the proof of Theorem 1.5 for $\mathfrak{sp}_{2n}$.

3.4 Type D_n

To apply Theorem 1.3 to a simple Lie algebra of type D_n, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type D_n simple Lie algebra, $\mathfrak{so}_{2n}$, is of ADE type. Therefore, its Cartan matrix C is symmetric with dimension $n \times n$ in the form:

$$C = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}$$

As in the case of $\mathfrak{sl}_n$, the Cartan matrix of $\mathfrak{so}_{2n}$ has real eigenvalues, with the smallest being $2 - 2\cos\left(\frac{\pi}{h}\right)$, where $h = 2n - 2$ is the Coxeter number.

Now we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{2n-2}\right)$. Let $\vec{\boldsymbol{x}} = [x_1, \ldots, x_n]^\intercal$ be such an eigenvector. For computational convenience, we assume $x_1 = 1$. By expanding $\vec{\boldsymbol{x}}C = (2 - 2\cos\left(\frac{\pi}{2n-2}\right))\vec{\boldsymbol{x}}$, we have:

$$
\begin{aligned}
2 - x_2 &= 2 - 2\cos\left(\frac{\pi}{2n-2}\right) && (156)\\
-1 + 2x_2 - x_3 &= \left(2 - 2\cos\left(\frac{\pi}{2n-2}\right)\right) x_2 && (157)\\
&\vdots \\
-x_{n-4} + 2x_{n-3} - x_{n-2} &= \left(2 - 2\cos\left(\frac{\pi}{2n-2}\right)\right) x_{n-3} && (158)\\
-x_{n-3} + 2x_{n-2} - x_{n-1} - x_n &= \left(2 - 2\cos\left(\frac{\pi}{2n-2}\right)\right) x_{n-2} && (159)\\
-x_{n-2} + 2x_{n-1} &= \left(2 - 2\cos\left(\frac{\pi}{2n-2}\right)\right) x_{n-1} && (160)\\
-x_{n-2} + 2x_n &= \left(2 - 2\cos\left(\frac{\pi}{2n-2}\right)\right) x_n && (161)
\end{aligned}
$$

From Equations (156)-(158), we observe a recursive relation between components of $\vec{\boldsymbol{x}}$:

$$
\begin{aligned}
x_1 &= 1;\\
x_2 &= 2\cos\left(\frac{\pi}{2n-2}\right) \cdot x_1;\\
&\vdots\\
x_j &= -x_{j-2} + 2\cos\left(\frac{\pi}{2n-2}\right) \cdot x_{j-1}\\
&\vdots\\
x_{n-2} &= -x_{n-4} + 2\cos\left(\frac{\pi}{2n-2}\right) \cdot x_{n-3}
\end{aligned}
$$

Note that this relation is identical to the relation in the $\mathfrak{sl}_n$ case. Using the same argument, for $j = 1, \cdots, n-2$, $\vec{\boldsymbol{x}}$ has components

$$x_j = \frac{\sin\left(\dfrac{j\pi}{2n-2}\right)}{\sin\left(\dfrac{\pi}{2n-2}\right)} = \frac{\zeta_{2n-2}^{j/2} - \zeta_{2n-2}^{-j/2}}{\zeta_{2n-2}^{1/2} - \zeta_{2n-2}^{-1/2}}$$

From Equations (159)-(161), we observe the relation between the remaining components of $\vec{\boldsymbol{x}}$:

$$\begin{aligned}
x_{n-1} &= -x_{n-3} + 2\cos\left(\frac{\pi}{2n-2}\right) \cdot x_{n-2} - x_n \\
x_{n-1} &= \frac{x_{n-2}}{2\cos\left(\frac{\pi}{2n-2}\right)} \\
x_n &= \frac{x_{n-2}}{2\cos\left(\frac{\pi}{2n-2}\right)}
\end{aligned}$$

We claim that the remaining components of $\vec{\boldsymbol{x}}$ are

$$x_{n-1} = x_n = \frac{1}{2\sin\left(\dfrac{\pi}{2n-2}\right)} = \frac{\zeta_{2n-2}^{(n-1)/2} - \zeta_{2n-2}^{-(n-1)/2}}{2\zeta_{2n-2}^{1/2} - 2\zeta_{2n-2}^{-1/2}}$$

The entire solution satisfies Equations (159)-(161) since

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2} &= -\cos\left(\frac{2\pi}{2n-2}\right) + 2\cos\left(\frac{\pi}{2n-2}\right)\cos\left(\frac{\pi}{2n-2}\right) \\
\frac{1}{2} &= -\sin\left(\frac{\pi}{2} - \frac{2\pi}{2n-2}\right) + 2\cos\left(\frac{\pi}{2n-2}\right)\sin\left(\frac{\pi}{2} - \frac{\pi}{2n-2}\right) - \frac{1}{2} \\
\frac{1}{2} &= -\sin\left(\frac{(n-3)\pi}{2n-2}\right) + 2\cos\left(\frac{\pi}{2n-2}\right)\sin\left(\frac{(n-2)\pi}{2n-2}\right) - \frac{1}{2} \\
x_{n-1} &= -x_{n-3} + 2\cos\left(\frac{\pi}{2n-2}\right) \cdot x_{n-2} - x_n
\end{aligned}$$

It follows that:

$$\frac{1}{2} = \frac{\cos\left(\frac{\pi}{2n-2}\right)}{2\cos\left(\frac{\pi}{2n-2}\right)}$$

$$\frac{1}{2} = \frac{\sin\left(\frac{(n-2)\pi}{2n-2}\right)}{2\cos\left(\frac{\pi}{2n-2}\right)}$$

$$x_{n-1} = \frac{x_{n-2}}{2\cos\left(\frac{\pi}{2n-2}\right)} = x_n$$

Therefore, we obtain

$$x_j = \begin{cases} \dfrac{\sin\left(\dfrac{j\pi}{2n-2}\right)}{\sin\left(\dfrac{\pi}{2n-2}\right)} = \dfrac{\zeta_{2n-2}^{j/2} - \zeta_{2n-2}^{-j/2}}{\zeta_{2n-2}^{1/2} - \zeta_{2n-2}^{-1/2}} & \text{for } j = 1, \cdots, n-2 \\ \dfrac{1}{2\sin\left(\dfrac{\pi}{2n-2}\right)} = \dfrac{\zeta_{2n-2}^{(n-1)/2} - \zeta_{2n-2}^{-(n-1)/2}}{2\zeta_{2n-2}^{1/2} - 2\zeta_{2n-2}^{-1/2}} & \text{for } j = n-1 \text{ or } n \end{cases} \tag{162}$$

3.4.1 The First Fundamental Representation of Type D_n

In this section, we calculate the cyclic spectrum for the first fundamental representation of the type D_n simple Lie algebra using Theorem 1.3.

3.4.1.1 Weights

At first, we introduce functionals $\{\mu_i\}_{i=1,\cdots,n}$ on the Cartan subalgebra of $\mathfrak{sl}_n$ consisting of matrices in the form $\mathrm{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n)$ such that:

$$\mu_i : \mathrm{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n) \mapsto x_i$$

In particular, $\{\mu_1, \cdots, \mu_n, -\mu_1, \cdots, -\mu_n\}$ are the $2n$ weights in the standard $2n$-dimen--sional representation of $\mathfrak{so}_{2n}$, which is also the first fundamental representation of $\mathfrak{so}_{2n}$. It is known that the simple roots $\{\alpha_i\}_{i=1,\cdots,n}$ for $\mathfrak{so}_{2n}$ are

$$\alpha_i = \begin{cases} \mu_i - \mu_{i+1} & \text{for } i = 1, \cdots, n-1 \\ \mu_{n-1} + \mu_n & \text{for } i = n \end{cases}$$

See for example [16, Appendix A.2]. One can see from simple calculations that the weights for the standard representation of $\mathfrak{so}_{2n}$ in terms of simple roots are:

$$\begin{cases} \mu_j = \sum_{i=j}^{n-1} \alpha_i + \dfrac{\alpha_n - \alpha_{n-1}}{2} & \text{for } j = 1, \cdots n-1 \\ \mu_n = \frac{\alpha_n - \alpha_{n-1}}{2} & \\ -\mu_j = -\sum_{i=j}^{n-1} \alpha_i - \dfrac{\alpha_n - \alpha_{n-1}}{2} & \text{for } j = 1, \cdots n-1 \\ -\mu_n = -\frac{\alpha_n - n-1}{2} & \end{cases} \tag{163}$$

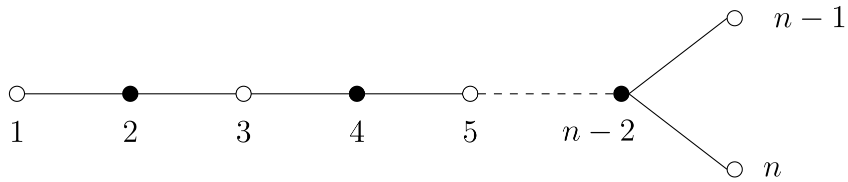
Furthermore, there is a recursive relation between weights as

$$\mu_{j+1} = \mu_j - \alpha_j \tag{164}$$

for $j = 1, \cdots, n-1$.

3.4.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, p is chosen as:

$$p(i) = \begin{cases} 1 & \text{for } i \leq n-2 \text{ even} \\ 0 & \text{for } i \leq n-2 \text{ odd} \\ 0 & \text{for } i = n-1 \text{ or } n, \text{ if } n \text{ is even} \\ 1 & \text{for } i = n-1 \text{ or } n, \text{ if } n \text{ is odd} \end{cases}$$

All the simple roots are with the same length, so Equation (5) is equivalent to:

$$\alpha_j(\Lambda) = x_j \cdot \zeta_{2n-2}^{p(j)/2} \cdot (1 - \zeta_{2n-2}^{-2p(j)+1})$$

By Equation (162) for x_j, it yields

$$\alpha_j(\Lambda) = \left(\frac{1}{2}\right)^{\delta_{j\in\{n-1,n\}}} \left(\zeta_{2n-2}^{(-1)^j\lfloor\frac{j}{2}\rfloor} - \zeta_{2n-2}^{(-1)^{j+1}\lfloor\frac{j+1}{2}\rfloor}\right) \tag{165}$$

where

$$\delta_{j\in\{n-1,n\}} = \begin{cases} 1 & \text{if } j \in \{n-1, n\} \\ 0 & \text{Otherwise} \end{cases}$$

Equivalently,

$$\alpha_j(\Lambda) = \begin{cases} \zeta_{2n-2}^{j/2} - \zeta_{2n-2}^{-j/2} & \text{for } j \le n-2 \text{ is even} \\ \zeta_{2n-2}^{\frac{1}{2}} \cdot (\zeta_{2n-2}^{-j/2} - \zeta_{2n-2}^{j/2}) & \text{for } j \le n-2 \text{ is odd} \\ (\frac{1}{2}) \cdot \zeta_{2n-2}^{\frac{1}{2}} \cdot (\zeta_{2n-2}^{-(n-1)/2} - \zeta_{2n-2}^{(n-1)/2}) & \text{for } j = n-1 \text{ or } n, \text{ if } n \text{ is even} \\ (\frac{1}{2}) \cdot (\zeta_{2n-2}^{(n-1)/2} - \zeta_{2n-2}^{-(n-1)/2}) & \text{for } j = n-1 \text{ or } n, \text{ if } n \text{ is odd} \end{cases} \tag{166}$$

By Equations (163), we obtain:

$$\begin{aligned} \mu_1(\Lambda) = {} & (1 - \zeta_{2n-2}) + (\zeta_{2n-2} - \zeta_{2n-2}^{-1}) + (\zeta_{2n-2}^{-1} - \zeta_{2n-2}^{2}) \\ & + \cdots + (\zeta_{2n-2}^{(-1)^{n-2}\lfloor\frac{n-2}{2}\rfloor} - \zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor}) + \frac{1}{2} \cdot (\zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} - \zeta_{2n-2}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor}) \\ = {} & 1 - \frac{1}{2} \cdot (\zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} + \zeta_{2n-2}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor}) \end{aligned}$$

When n is odd,

$$\left|(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor - (-1)^{n}\lfloor\frac{n}{2}\rfloor\right| = \left|\frac{n-1}{2} + \frac{n-1}{2}\right| = n-1$$

When n is even,

$$\left|(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor - (-1)^{n}\lfloor\frac{n}{2}\rfloor\right| = \left|-\frac{n-2}{2} - \frac{n}{2}\right| = n-1$$

Thus, $\zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor}$ and $\zeta_{2n-2}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor}$ are symmetric about the origin. Hence,

$$\zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} + \zeta_{2n-2}^{(-1)^{n}\lfloor\frac{n}{2}\rfloor} = 0 \implies \mu_1(\Lambda) = 1$$

Then, by the relation in Equation (164), we have:

$$\begin{aligned}
\mu_2(\Lambda) &= \mu_1(\Lambda) - \alpha_1(\Lambda) = 1 - (1 - \zeta_{2n-2}) = \zeta_{2n-2} \\
\mu_3(\Lambda) &= \mu_2(\Lambda) - \alpha_2(\Lambda) = \zeta_{2n-2} - (\zeta_{2n-2} - \zeta_{2n-2}^{-1}) = \zeta_{2n-2}^{-1} \\
\mu_4(\Lambda) &= \mu_3(\Lambda) - \alpha_3(\Lambda) = \zeta_{2n-2}^{-1} - (\zeta_{2n-2}^{-1} - \zeta_{2n-2}^{2}) = \zeta_{2n-2}^{2} \\
&\cdots \\
\mu_{n-1}(\Lambda) &= \mu_{n-2}(\Lambda) - \alpha_{n-2}(\Lambda) = \zeta_{2n-2}^{(-1)^{n-2}\lfloor\frac{n-2}{2}\rfloor} - (\zeta_{2n-2}^{(-1)^{n-2}\lfloor\frac{n-2}{2}\rfloor} - \zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor}) \\
&= \zeta_{2n-2}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} \\
\mu_n(\Lambda) &= 0
\end{aligned}$$

Therefore,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n} = \{\zeta_{2n-2}^{(-1)^{j}\lfloor\frac{j}{2}\rfloor}, 0\}_{1,\cdots,n-1} \tag{167}$$

which contains $n-1$ different $(2n-2)$-th roots of unity. Note that for any $1 \leq j < k \leq n-1$,

$$\left|(-1)^{j}\lfloor\frac{j}{2}\rfloor - (-1)^{k}\lfloor\frac{k}{2}\rfloor\right| \neq n-1 \implies \mu_j(\Lambda) \neq -\mu_k(\Lambda)$$

Thus,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n-1} \cap \{-\mu_i(\Lambda)\}_{i=1,\cdots,n-1} = \emptyset$$

Since $\{-\mu_i(\Lambda)\}_{i=1,\cdots,n-1}$ also contains $n-1$ different $(2n-2)$-th roots of unity, therefore,

$$\{\mu_i(\Lambda)\}_{i=1,\cdots,n-1} \cup \{-\mu_i(\Lambda)\}_{i=1,\cdots,n-1} = \{\zeta_{2n-2}^j \mid j = 1,\cdots,2n-2\} \tag{168}$$

It follows that

$$\mathrm{Spec}(\psi_1, \mathfrak{so}_{2n}) = C_{2n-2}(1, 0^{(2)})$$

The part of Theorem 1.4 concerning the first fundamental cyclic spectrum of $\mathfrak{so}_{2n}$ is now proved.

Remark 11. Similar to Remark 7, we visualize this spectrum in the complex plane. The weights in Equation (163) are arranged in ascending order of height as follows:

$$\{\mu_1(\Lambda), \mu_2(\Lambda), \cdots, \mu_n(\Lambda) = -\mu_n(\Lambda) = 0, \cdots, -\mu_2(\Lambda), -\mu_1(\Lambda)\}$$

By connecting eigenvalues in this order, Figures 12 and 13 present the cyclic spectra for the first fundamental representation of type D_4 and D_5, respectively. The path consistently begins with 1, followed by ζ_{2n-2}^k and its conjugate ζ_{2n-2}^{-k} for $k \leq n-1$. For $i < j$, $\mathrm{Re}(\mu_i(\Lambda)) \geq \mathrm{Re}(\mu_j(\Lambda))$.

3.4.2 The Adjoint Representation of Type D_n

In this section, we apply the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{so}_{2n}$. The dimension of this representation is $2n^2 - n$, so there are at most $2n^2 - 2n$ non-zero eigenvalues. In addition to the n simple roots, the remaining positive roots of $\mathfrak{so}_{2n}$ are as follows:

$$\begin{aligned} &\alpha_{ij} = \alpha_i + \cdots + \alpha_j && 1 \leq i < j \leq n-1 \\ &\alpha_i' = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n && 1 \leq i \leq n-1 \end{aligned} \tag{169}$$

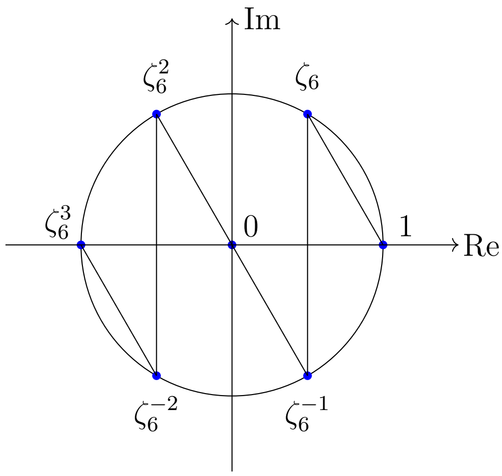


Figure 12: First fundamental cyclic spectrum for D_4

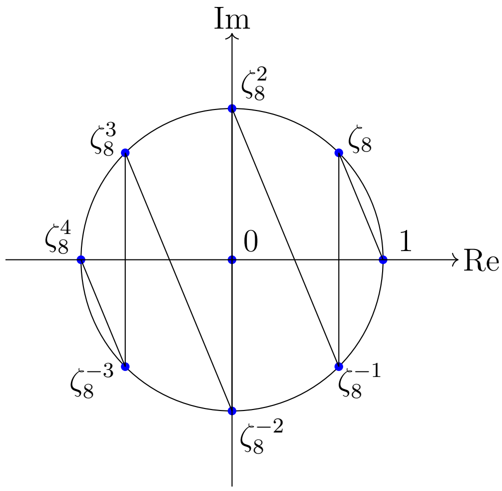


Figure 13: First fundamental cyclic spectrum for D_5

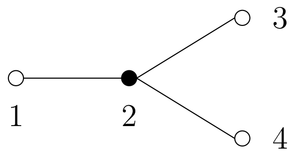
$$\alpha'_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \qquad 1 \leq i < j \leq n-1$$

As in [3, Plate IV], the remaining positive roots of $\mathfrak{so}_{2n}$ can be written as linear combinations of the first fundamental weight. By Equation (167), the remaining eigenvalues are as follows:

$$\alpha_{ij}(\Lambda) = \mu_i(\Lambda) - \mu_{j+1}(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} - \zeta_{2n}^{(-1)^{j+1} \lfloor \frac{j+1}{2} \rfloor} \quad 1 \leq i < j \leq n-1$$
$$\alpha'_i(\Lambda) = \mu_i(\Lambda) + \mu_n(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} \quad 1 \leq i \leq n-1$$
$$\alpha'_{ij}(\Lambda) = \mu_i(\Lambda) + \mu_j(\Lambda) = \zeta_{2n}^{(-1)^i \lfloor \frac{i}{2} \rfloor} + \zeta_{2n}^{(-1)^j \lfloor \frac{j}{2} \rfloor} \quad 1 \leq i < j \leq n-1$$

But the information about radii and multiplicities in the spectrum is not explicit. To understand the patterns in the adjoint spectrum, we first go through several examples.

Example 3.11. Consider the case of D_4, $\mathfrak{so}_8$. As before, we choose the bi-coloration:



By Equation (136), we obtain

$$\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{12} \cdot (\zeta_6^{-1/2} - \zeta_6^{1/2}) = \mathsf{r}_1 \cdot \zeta_6^5 \\
\alpha_2(\Lambda) &= \zeta_6 - \zeta_6^{-1} = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6 \\
\alpha_3(\Lambda) &= \alpha_4(\Lambda) = \frac{1}{2}\zeta_{16} \cdot (\zeta_8^{-3/2} - \zeta_8^{3/2}) = \mathsf{r}_3 \cdot \zeta_6^5
\end{aligned}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{6}\right) = 1 = \mathsf{r}_3, \mathsf{r}_2 = 2\sin\left(\frac{\pi}{3}\right)$. Thus, $\alpha_1(\Lambda) = \alpha_3(\Lambda) = \alpha_4(\Lambda) = \zeta_6^5$. Upon checking the Dynkin diagram, it also follows that:

$$\begin{aligned}
\alpha_{2,3}(\Lambda) &= \alpha_2'(\Lambda) = \alpha_{1,2}(\Lambda) = \zeta_6 \\
\alpha_{1,3}(\Lambda) &= \alpha_1'(\Lambda) = \alpha_3'(\Lambda) = 1
\end{aligned}$$

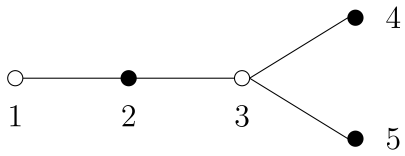
Therefore, by "ζ_8-symmetry", the circle $C_6(1)$ is with multiplicity 3. Hence, $\alpha_j(\Lambda)$'s generate:

$$C_6(1^{(3)},\ \zeta_{12} \cdot \mathsf{r}_2)$$

There are 24 roots in total, so we have obtained all non-zero eigenvalues. Hence,

$$\mathrm{Spec}(\mathrm{ad}, D_4) = C_6(1^{(3)},\ \zeta_{12} \cdot \mathsf{r}_2,\ 0^{(4)})$$

Example 3.12. Consider the case of D_5, $\mathfrak{so}_{10}$. As before, we choose the bi-coloration:



By Equation (166), we obtain

$$\alpha_1(\Lambda) = \zeta_{16} \cdot (\zeta_8^{-1/2} - \zeta_8^{1/2}) = \zeta_{16} \cdot \mathsf{r}_1 \cdot \zeta_8^6$$
$$\alpha_2(\Lambda) = \zeta_8 - \zeta_8^{-1} = \mathsf{r}_2 \cdot \zeta_8^2$$
$$\alpha_3(\Lambda) = \zeta_{16} \cdot (\zeta_8^{-3/2} - \zeta_8^{3/2}) = \zeta_{16} \cdot \mathsf{r}_3 \cdot \zeta_8^6$$
$$\alpha_4(\Lambda) = \alpha_5(\Lambda) = \frac{1}{2}(\zeta_8^2 - \zeta_8^{-2}) = \mathsf{r}_4 \cdot \zeta_8^2$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{8}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{4}\right), \mathsf{r}_3 = 2\sin\left(\frac{3\pi}{8}\right), \mathsf{r}_4 = 1$. In addition,

$$\alpha_{3,4}(\Lambda) = \alpha_3'(\Lambda) = \mathsf{r}_4 \cdot \zeta_8^{-1}$$
$$\alpha_{2,4}(\Lambda) = \alpha_2'(\Lambda) = \mathsf{r}_4 \cdot \zeta_8$$
$$\alpha_{1,4}(\Lambda) = \alpha_1'(\Lambda) = \mathsf{r}_4$$

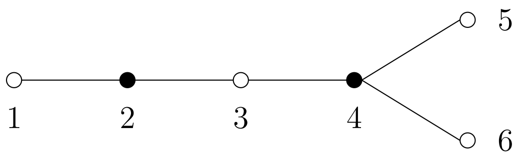
Therefore, by "ζ_8-symmetry", the circle $C_8(\mathsf{r}_4)$ is with multiplicity 2. Hence, $\alpha_j(\Lambda)$'s generate:

$$C_8(\zeta_{16} \cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{16} \cdot \mathsf{r}_3,\ \mathsf{r}_4^{(2)})$$

There are 40 roots in total, so we have obtained all non-zero eigenvalues. Hence,

$$\mathrm{Spec}(\mathrm{ad}, D_5) = C_8(\zeta_{16} \cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{16} \cdot \mathsf{r}_3,\ \mathsf{r}_4^{(2)},\ 0^{(5)})$$

Example 3.13. Consider the case of D_6, $\mathfrak{so}_{12}$. As before, we choose the bi-coloration:



By Equation (166), we obtain

$$\alpha_1(\Lambda) = \zeta_{20}\cdot(\zeta_{10}^{-1/2} - \zeta_{10}^{1/2}) = \mathsf{r}_1\cdot\zeta_{10}^{8}$$
$$\alpha_2(\Lambda) = \zeta_{10} - \zeta_{10}^{-1} = \zeta_{20}\cdot\mathsf{r}_2\cdot\zeta_{10}^{2}$$
$$\alpha_3(\Lambda) = \zeta_{20}\cdot(\zeta_{10}^{-3/2} - \zeta_{10}^{3/2}) = \mathsf{r}_3\cdot\zeta_{10}^{8}$$
$$\alpha_4(\Lambda) = \zeta_{10}^{2} - \zeta_{10}^{-2} = \zeta_{20}\cdot\mathsf{r}_4\cdot\zeta_{10}^{2}$$
$$\alpha_5(\Lambda) = \alpha_6(\Lambda) = \frac{1}{2}\zeta_{20}\cdot(\zeta_{10}^{-5/2} - \zeta_{10}^{5/2}) = \mathsf{r}_5\cdot\zeta_{10}^{8}$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{10}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{5}\right), \mathsf{r}_3 = 2\sin\left(\frac{3\pi}{10}\right), \mathsf{r}_4 = 2\sin\left(\frac{2\pi}{5}\right), \mathsf{r}_5 = 1$. In addition,

$$\alpha_{4,5}(\Lambda) = \alpha_4'(\Lambda) = \zeta_{10}^{2}$$
$$\alpha_{3,5}(\Lambda) = \alpha_3'(\Lambda) = \zeta_{10}^{-1}$$
$$\alpha_{2,5}(\Lambda) = \alpha_2'(\Lambda) = \zeta_{10}$$
$$\alpha_{1,5}(\Lambda) = \alpha_1'(\Lambda) = 1$$

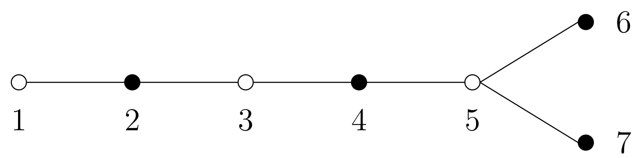
Therefore, by "ζ_{10}-symmetry", the circle $C_{10}(\mathsf{r}_5)$ is with multiplicity 2. Hence, $\alpha_j(\Lambda)$'s generate:

$$C_{10}(\mathsf{r}_1,\ \zeta_{20}\cdot\mathsf{r}_2,\ \mathsf{r}_3,\ \zeta_{20}\cdot\mathsf{r}_4,\ \mathsf{r}_5^{(2)})$$

There are 60 roots in total, so we have obtained all non-zero eigenvalues. Hence,

$$\mathrm{Spec}(\mathrm{ad}, D_6) = C_{10}(\mathsf{r}_1,\ \zeta_{20}\cdot\mathsf{r}_2,\ \mathsf{r}_3,\ \zeta_{20}\cdot\mathsf{r}_4,\ \mathsf{r}_5^{(2)},\ 0^{(6)})$$

Example 3.14. Consider the case of D_7, $\mathfrak{so}_{14}$. As before, we choose the bi-coloration:



By Equation (166), we obtain

$$
\begin{aligned}
\alpha_1(\Lambda) &= \zeta_{24}\cdot(\zeta_{12}^{-1/2}-\zeta_{12}^{1/2}) = \zeta_{24}\cdot \mathsf{r}_1\cdot\zeta_{12}^{9}\\
\alpha_2(\Lambda) &= \zeta_{12}-\zeta_{12}^{-1} = \mathsf{r}_2\cdot\zeta_{12}^{3}\\
\alpha_3(\Lambda) &= \zeta_{24}\cdot(\zeta_{12}^{-3/2}-\zeta_{12}^{3/2}) = \zeta_{24}\cdot \mathsf{r}_3\cdot\zeta_{12}^{8}\\
\alpha_4(\Lambda) &= \zeta_{12}^{2}-\zeta_{12}^{-2} = \mathsf{r}_4\cdot\zeta_{12}^{3}\\
\alpha_5(\Lambda) &= \zeta_{24}\cdot(\zeta_{12}^{-5/2}-\zeta_{12}^{5/2}) = \zeta_{24}\cdot \mathsf{r}_5\cdot\zeta_{12}^{9}\\
\alpha_6(\Lambda) &= \alpha_7(\Lambda) = \frac{1}{2}(\zeta_{12}^{3}-\zeta_{12}^{-3}) = \mathsf{r}_6\cdot\zeta_{12}^{3}
\end{aligned}
$$

where $\mathsf{r}_1 = 2\sin\left(\frac{\pi}{12}\right), \mathsf{r}_2 = 2\sin\left(\frac{\pi}{6}\right) = 1 = \mathsf{r}_6, \mathsf{r}_3 = 2\sin\left(\frac{\pi}{4}\right), \mathsf{r}_4 = 2\sin\left(\frac{\pi}{3}\right), \mathsf{r}_5 = 2\sin\left(\frac{5\pi}{12}\right)$. Thus, $\alpha_2(\Lambda) = \alpha_6(\Lambda) = \alpha_7(\Lambda) = \zeta_{12}^{3}$. Upon checking the Dynkin diagram, it also follows that:

$$
\begin{aligned}
\alpha_{2,5}(\Lambda) = \alpha_{3,6}(\Lambda) = \alpha'_3(\Lambda) = \zeta_{12}^{-1}\\
\alpha'_{3,6}(\Lambda) = \alpha_{2,6}(\Lambda) = \alpha'_2(\Lambda) = \zeta_{12}
\end{aligned}
$$

Additionally, similar to Example 3.8, since

$$\zeta_{12}^{2}+\zeta_{12}^{-2} = 1$$

Hence, we obtain:

$$\alpha_{1,3}(\Lambda) = \alpha_{5,6}(\Lambda) = \alpha'_5(\Lambda) = \zeta_{12}^{-2}$$

Upon checking the Dynkin diagram, it also follows that:

$$
\begin{aligned}
\alpha_{1,4}(\Lambda) = \alpha_{4,6}(\Lambda) = \alpha'_4(\Lambda) = \zeta_{12}^{2}\\
\alpha_{2,5}(\Lambda) = \alpha_{1,6}(\Lambda) = \alpha'_{4,5}(\Lambda) = 1
\end{aligned}
$$

Therefore, by "ζ_{12}-symmetry", the circle $C_{12}(1)$ is with multiplicity 3. Hence, $\alpha_j(\Lambda)$'s generate:

$$C_{12}(\zeta_{24} \cdot \mathsf{r}_1,\ 1^{(3)},\ \zeta_{24} \cdot \mathsf{r}_3,\ \mathsf{r}_4,\ \zeta_{24} \cdot \mathsf{r}_5)$$

There are 72 roots in total, so we have obtained all non-zero eigenvalues. Hence,

$$\mathrm{Spec}(\mathrm{ad}, D_7) = C_{12}(\zeta_{24} \cdot \mathsf{r}_1,\ 1^{(3)},\ \zeta_{24} \cdot \mathsf{r}_3,\ \mathsf{r}_4,\ \zeta_{24} \cdot \mathsf{r}_5,\ 0^{(7)})$$

The above examples suggest the following patterns in $\mathrm{Spec}(\mathrm{ad}, D_n)$.

1. If $n \not\equiv 1 \mod 3$, let

$$p(j) = \begin{cases} 1 & \text{if } j, n \text{ are with the same parity} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \cdots, n-1$. Then

$$\mathrm{Spec}(\mathrm{ad}, D_n) = C_{2n-2}(z_1, \cdots, z_{n-2}, z_{n-1}^{(2)}, 0^{(n)}) \tag{170}$$

where $z_j = \zeta_{4n-4}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = |\alpha_j(\Lambda)| = 2\sin\left(\frac{j\pi}{2n-2}\right)$ for $j = 1, \cdots, n-1$.

2. If $n \equiv 1 \mod 3$, such that $n = 3k+1$ for some positive integer k, let $p(j)$ be defined as in the previous case for $j = 1, \cdots, n-2$. Then

$$\mathrm{Spec}(\mathrm{ad}, D_n) = C_{2n-2}(z_1, \cdots, z_{k-1}, z_k^{(3)}, z_{k+1}, \cdots, z_{n-2}, 0^{(n)}) \tag{171}$$

where $z_j = \zeta_{4n-4}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = |\alpha_j(\Lambda)| = 2\sin\left(\frac{j\pi}{2n-2}\right)$ for $j = 1, \cdots, n-2$.

Now, we prove the above two equations to prove the corresponding part of Theorem 1.4. Consider the first case when $n \not\equiv 1 \mod 3$. By Equation (166) and the calculation of $\mu_j(\Lambda)$ in the previous section, we obtain:

$$\alpha_j(\Lambda) = \begin{cases} 2\sin\left(\frac{j\pi}{2n-2}\right) \cdot i & \text{if } j < n-1 \text{ even} \\ 2\sin\left(\frac{j\pi}{2n-2}\right) \cdot \zeta_{4n-4} \cdot (-i) & \text{if } j < n-1 \text{ odd} \\ \zeta_{2n-2}^{(-1)^n \lfloor \frac{n-1}{2} \rfloor} & \text{if } j = n-1, n \end{cases} \tag{172}$$

Note that when $n-1$ is even, $i = \zeta_{2n-2}^{\frac{n-1}{2}}$, and $-i = \zeta_{2n}^{\frac{3(n-1)}{2}}$. When $n-1$ is odd, $i = \zeta_{4n-4} \cdot \zeta_{2n-2}^{\frac{n-2}{2}}$, and $\zeta_{4n-4} \cdot (-i) = \zeta_{2n-2}^{\frac{3n-4}{2}}$. Thus, $p(j)$ in Equation (170) equals 1 if j, n are with the same parity, and 1 otherwise. Also, since $n \not\equiv 1 \mod 3$, thus

$$\{2\sin\left(\frac{j\pi}{2n-2}\right) \mid j = 1, \cdots, n-2\} \cap \{1\} = \emptyset$$

In other words, each $\alpha_j(\Lambda)$ when $j = 1, \cdots, n-1$ yields a distinct eigenvalue with a distinct radius. By the "ζ_{2n-2}-symmetry", they generate

$$C_{2n-2}(\zeta_{4n-4}^{p(1)} \cdot \mathsf{r}_1, \ \cdots, \ \zeta_{4n-4}^{p(n-1)} \cdot \mathsf{r}_{n-1})$$

as in Equation (170). Additionally, since:

$$\alpha_{n-1}(\Lambda) = \alpha_n(\Lambda) = \zeta_{2n-2}^{(-1)^{n-1} \lfloor \frac{n-1}{2} \rfloor} = \mu_{n-1}(\Lambda) \tag{173}$$

Upon checking the Dynkin diagram, it also follows that

$$\alpha_{j,n-1}(\Lambda) = \alpha_j'(\Lambda) = \zeta_{2n-2}^{(-1)^j \lfloor \frac{j}{2} \rfloor} = \mu_j(\Lambda) \tag{174}$$

for $j = 1, \cdots, n-2$. By Equation (168), it implies that the circle $C_{2n-2}(1)$ has multiplicity 2. There are $n(2n-2)$ roots in total, so we have exhausted all non-zero eigenvalues. The proof of Equation (170) is now complete.

Now we prove the other case when $n = 3k+1$ for some positive integer k. The argument is almost the same as the previous case, except when $j = k$, $\mathsf{r}_j = 1$, the circle $C_{2n-2}(1)$ has triple multiplicities. By Equations (173) and (174), the multiplicity of $C_{2n-2}(1)$ is doubled. The next argument is very similar to the argument of the double multiplicity in type B_n. Since $n = 3k+1$, it follows that:

$$\zeta_{2n-2}^{k} + \zeta_{2n-2}^{-k} = 2\sin\left(\frac{\pi}{6}\right) = 1 \tag{175}$$

Referring to the proof of Equation (142) in type B_n, we obtain

$$\alpha_k(\Lambda) = \alpha_{n-1}(\Lambda) = \alpha_n(\Lambda) = \zeta_{2n-1}^{(-1)^{n-1}\lfloor\frac{n-1}{2}\rfloor} = \mu_{n-1}(\Lambda) \tag{176}$$

$$\alpha_{k-j,k+j}(\Lambda) = \alpha_{n-1-j,n-1}(\Lambda) = \alpha'_{n-1-j}(\Lambda) = \zeta_{2n-2}^{(-1)^{n-1-j}\lfloor\frac{n-1-j}{2}\rfloor} = \mu_{n-1-j}(\Lambda) \tag{177}$$

for $j = 1, \cdots, k-1$. Upon checking the Dynkin diagram, Equation (176) implies that:

$$\alpha_{k,n-2}(\Lambda) = \alpha_{k+1,n-1}(\Lambda) = \alpha'_{k+1}(\Lambda) = \zeta_{2n-2}^{(-1)^{k+1}\lfloor\frac{k+1}{2}\rfloor} = \mu_{k+1}(\Lambda) \tag{178}$$

$$\alpha'_{k+1,n-1}(\Lambda) = \alpha_{k,n-1}(\Lambda) = \alpha'_k(\Lambda) = \zeta_{2n-2}^{(-1)^{k}\lfloor\frac{k}{2}\rfloor} = \mu_k(\Lambda) \tag{179}$$

For any $j = 1, \cdots, k-1$, Equation (177) implies that:

$$\alpha_{k-j,n-2-j}(\Lambda) = \alpha_{k+j+1,n-1}(\Lambda) = \alpha'_{k+j+1}(\Lambda) = \zeta_{2n-2}^{(-1)^{k+j+1}\lfloor\frac{k+j+1}{2}\rfloor} = \mu_{k+j+1}(\Lambda) \tag{180}$$

$$\alpha'_{k+j+1,n-j-1}(\Lambda) = \alpha_{k-j,n-1}(\Lambda) = \alpha'_{k-j}(\Lambda) = \zeta_{2n-2}^{(-1)^{k-j}\lfloor\frac{k-j}{2}\rfloor} = \mu_{k-j}(\Lambda) \tag{181}$$

Combine the eigenvalue in Equations (176)-(181), they form $\{\mu_i(\Lambda)\}_{i=1,\cdots,n-1}$. Hence, it implies that the circle $C_{2n-2}(1)$ is with multiplicity 3. The proof of Equation (171) is now complete. The proof of Theorem 1.4 for $\mathfrak{so}_{2n}$ is also complete. In view of Equations (170), (171), and (172), this also completes the proof of Theorem 1.5 for $\mathfrak{so}_{2n}$.

3.5 Type $\mathfrak{e}_6$

To apply Theorem 1.3 to a simple Lie algebra of type $\mathfrak{e}_6$, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type $\mathfrak{e}_6$ simple Lie algebra is of ADE type. Therefore, its Cartan matrix C is symmetric in the form:

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The smallest eigenvalue of C is $2 - 2\cos\left(\frac{\pi}{h}\right)$, where the Coxeter number $h = 12$. Then, we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{12}\right)$ as:

$$\vec{\boldsymbol{x}} = \left(1,\ \frac{\zeta_{12} + \zeta_{12}^{-1} + 1}{\zeta_{12}^{1/2} + \zeta_{12}^{-1/2}},\ \zeta_{12}^{1/2} + \zeta_{12}^{-1/2},\ \zeta_{12} + \zeta_{12}^{-1} + 1,\ \zeta_{12}^{1/2} + \zeta_{12}^{-1/2},\ 1\right)^{\mathsf{T}} \tag{182}$$

3.5.1 The First Fundamental Representation of $\mathfrak{e}_6$

In this section, we calculate the cyclic spectrum for the first fundamental representation of the type $\mathfrak{e}_6$ simple Lie algebra using Theorem 1.3.

3.5.1.1 Weights

Let $\{\epsilon_i\}_{i=1,\cdots,8}$ be the canonical basis for $\mathbb{R}^8$. Referring to [3, PLATE V], the highest weight of the first fundamental representation of $\mathfrak{e}_6$ is given by

$$\mu = (2/3)(\epsilon_8 - \epsilon_7 - \epsilon_6) = \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \tag{183}$$

where $\alpha_1, \cdots, \alpha_6$ are simple roots of $\mathfrak{e}_6$. They can be written in terms of ϵ_i's as:

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7) \\
\alpha_2 &= \epsilon_1 + \epsilon_2 \\
\alpha_3 &= \epsilon_2 - \epsilon_1 \\
\alpha_4 &= \epsilon_3 - \epsilon_2 \\
\alpha_5 &= \epsilon_4 - \epsilon_3 \\
\alpha_6 &= \epsilon_5 - \epsilon_4
\end{aligned} \tag{184}$$

The dimension of the first fundamental representation of $\mathfrak{e}_6$ is 27. Next, we show the remaining 26 weights are:

$$-\frac{1}{2}\mu \pm \epsilon_i \qquad \text{for } 1 \leq i \leq 5 \tag{185}$$

$$\frac{1}{4}\mu - \frac{1}{2}\sum_{i=1}^{5} c_i\epsilon_i \qquad \text{where the } c_i\text{'s are } \pm 1 \text{ such that } \prod c_i = 1 \tag{186}$$

For convenience, we denote weights $-\frac{1}{2}\mu \pm \epsilon_i$ as $\epsilon_i^{\pm}$. Also, let $\mathcal{C}^{i_1 \cdots i_k,}$ denote the weights $\frac{1}{4}\mu - \frac{1}{2}\sum_{i=1}^{5} c_i\epsilon_i$ where $c_{i_1} = \cdots = c_{i_k} = -1$ for $k \leq 5$. When $c_1 = \cdots = c_5 = 1$, denote it by $\mathcal{C}^0$.

We use Python to find the remaining 26 weights by repeatedly applying simple reflections to μ until obtaining all 26 of them. Details are included in Appendix 3.5.3. Similar to the successor diagram between positive roots, we draw a successor diagram of these weights. Two weights are connected if their difference is a simple root.

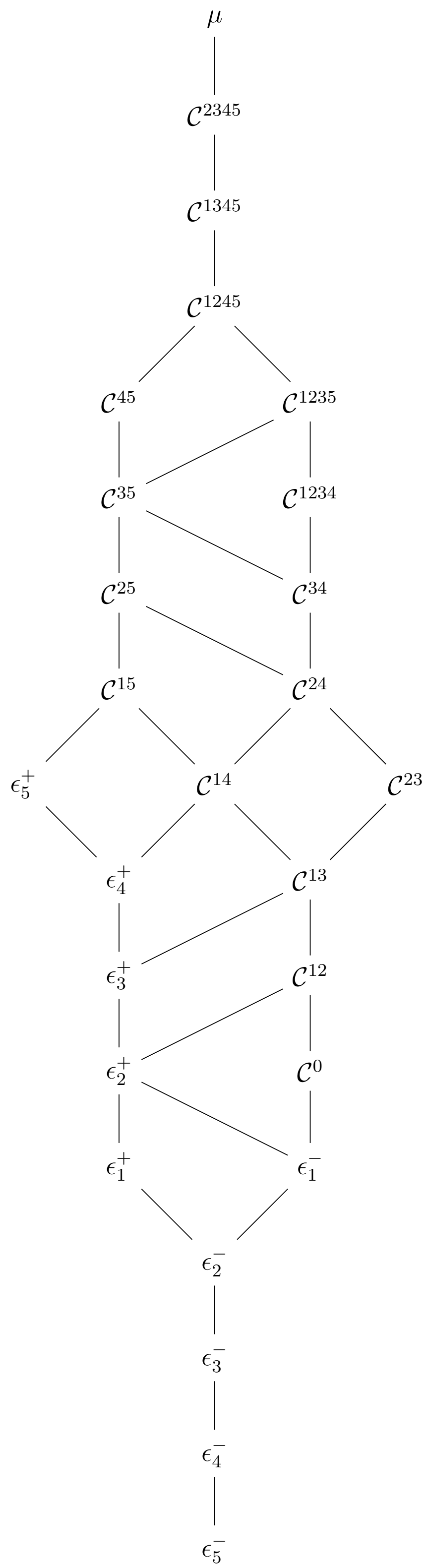
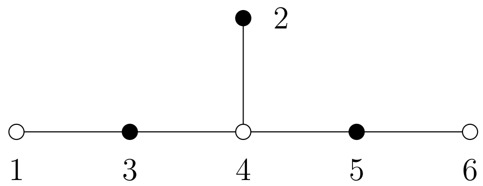


Figure 14: The Successor Diagram of the First Fundamental Weights of $\mathfrak{e}_6$

3.5.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, p is chosen as:

$$p(i) = \begin{cases} 0 & \text{if } i = 1, 4, 6 \\ 1 & \text{if } i = 2, 3, 5 \end{cases}$$

All the simple roots are with the same length, so Equation (5) is equivalent to:

$$\alpha_j(\Lambda) = x_j \cdot \zeta_{12}^{p(j)/2} \cdot (1 - \zeta_{12}^{-2p(j)+1})$$

By Equation (182) for x_j, it yields

$$\alpha_j(\Lambda) = \begin{cases} -\zeta_{12} + 1 & \text{if } j = 1, 6 \\ -\zeta_{12}^3 + 2\zeta_{12}^2 - 1 & \text{if } j = 2 \\ \zeta_{12}^3 & \text{if } j = 3, 5 \\ -\zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12} & \text{if } j = 4 \end{cases} \tag{187}$$

By Equation (183), we obtain:

$$\begin{aligned} \mu(\Lambda) &= \frac{1}{3}(4\alpha_1(\Lambda) + 3\alpha_2(\Lambda) + 5\alpha_3(\Lambda) + 6\alpha_4(\Lambda) + 4\alpha_5(\Lambda) + 2\alpha_6(\Lambda)) \\ &= 1 \end{aligned}$$

By Equation (184), we obtain:

$$\epsilon_1 = \alpha_2/2 - \alpha_3/2$$
$$\epsilon_2 = \alpha_2/2 + \alpha_3/2$$
$$\epsilon_3 = \alpha_2/2 + \alpha_3/2 + \alpha_4$$
$$\epsilon_4 = \alpha_2/2 + \alpha_3/2 + \alpha_4 + \alpha_5$$
$$\epsilon_5 = \alpha_2/2 + \alpha_3/2 + \alpha_4 + \alpha_5 + \alpha_6$$

Then, we obtain:

$$\epsilon_1(\Lambda) = \alpha_2(\Lambda)/2 - \alpha_3(\Lambda)/2 = -\zeta_{12}^3 + \zeta_{12}^2 - \frac{1}{2}$$
$$\epsilon_2(\Lambda) = \alpha_2(\Lambda)/2 + \alpha_3(\Lambda)/2 = \zeta_{12}^2 - \frac{1}{2}$$
$$\epsilon_3(\Lambda) = \alpha_2(\Lambda)/2 + \alpha_3(\Lambda)/2 + \alpha_4(\Lambda) = -\zeta_{12}^3 + \zeta_{12} - \frac{1}{2}$$
$$\epsilon_4(\Lambda) = \alpha_2(\Lambda)/2 + \alpha_3(\Lambda)/2 + \alpha_4(\Lambda) + \alpha_5(\Lambda) = \zeta_{12} - \frac{1}{2}$$
$$\epsilon_5(\Lambda) = \alpha_2(\Lambda)/2 + \alpha_3(\Lambda)/2 + \alpha_4(\Lambda) + \alpha_5(\Lambda) + \alpha_6(\Lambda) = \frac{1}{2}$$

These $\zeta_{12}-$polynomials can be rewritten in the form given in Theorem 1.4. We will give an example in the case of $\mathfrak{e}_8$ later on in Remark 15. Let $\mathsf{r}_1 = 1$, and by the Cyclotomic polynomial $\Phi_{12} = x^4 - x^2 + 1$, we have equations:

$$\begin{aligned}
\zeta_{12}^4 &= \zeta_{12}^2 - 1 = -\zeta_{12}^{10} \\
\zeta_{12}^5 &= \zeta_{12}^3 - \zeta_{12} = -\zeta_{12}^{11} \\
\zeta_{12}^6 &= -1 \\
\zeta_{12}^7 &= -\zeta_{12} \\
\zeta_{12}^8 &= -\zeta_{12}^2 \\
\zeta_{12}^9 &= -\zeta_{12}^3
\end{aligned} \tag{188}$$

Let:

$$\mathsf{r}_2 = 2\cos\left(\frac{5\pi}{12}\right) = \zeta_{12}^{\frac{5}{2}} - \zeta_{12}^{-\frac{5}{2}}$$

Then, by Equations (185) and (186), it follows that:

$$\epsilon_1^+(\Lambda) = -\zeta_{12}^3 + \zeta_{12}^2 - 1 = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^6$$

$$\epsilon_1^-(\Lambda) = \zeta_{12}^3 - \zeta_{12}^2 = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^5$$

$$\epsilon_2^+(\Lambda) = \zeta_{12}^2 - 1 = \zeta_{12}^4$$

$$\epsilon_2^-(\Lambda) = -\zeta_{12}^2 = \zeta_{12}^8$$

$$\epsilon_3^+(\Lambda) = -\zeta_{12}^3 + \zeta_{12} - 1 = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^8$$

$$\epsilon_3^-(\Lambda) = \zeta_{12}^3 - \zeta_{12} = \zeta_{12}^5$$

$$\epsilon_4^+(\Lambda) = \zeta_{12} - 1 = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^3$$

$$\epsilon_4^-(\Lambda) = -\zeta_{12} = \zeta_{12}^7$$

$$\epsilon_5^+(\Lambda) = 0$$

$$\epsilon_5^-(\Lambda) = -1 = \zeta_{12}^6$$

$$\mathcal{C}^0(\Lambda) = \zeta_{12}^3 - \zeta_{12}^2 - \zeta_{12} + 1 = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^7$$

$$\mathcal{C}^{12}(\Lambda) = \zeta_{12}^2 - \zeta_{12} = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^4$$

$$\mathcal{C}^{13}(\Lambda) = -\zeta_{12}^3 = \zeta_{12}^9$$

$$\mathcal{C}^{14}(\Lambda) = 0$$

$$\mathcal{C}^{15}(\Lambda) = 1 - \zeta_{12} = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^9$$

$$\mathcal{C}^{23}(\Lambda) = 0$$

$$\mathcal{C}^{24}(\Lambda) = \zeta_{12}^3$$

$$\mathcal{C}^{25}(\Lambda) = \zeta_{12}^3 - \zeta_{12} + 1 = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^2$$

$$\mathcal{C}^{34}(\Lambda) = -\zeta_{12}^2 + \zeta_{12} = \zeta_{24}\cdot \mathsf{r}_2\zeta_{12}^{10}$$

$$\mathcal{C}^{35}(\Lambda) = -\zeta_{12}^2 + 1 = \zeta_{12}^{10}$$

$$\mathcal{C}^{45}(\Lambda) = \zeta_{12}^3 - \zeta_{12}^2 + 1 = \zeta_{24}\cdot \mathsf{r}_2$$

$$\mathcal{C}^{2345}(\Lambda) = \zeta_{12}$$

$$\mathcal{C}^{1345}(\Lambda) = -\zeta_{12}^3 + \zeta_{12} = \zeta_{12}^{11}$$

$$\mathcal{C}^{1245}(\Lambda) = \zeta_{12}^2$$

$$\mathcal{C}^{1235}(\Lambda) = -\zeta_{12}^3 + \zeta_{12}^2 = \zeta_{24} \cdot \mathsf{r}_2\zeta_{12}^{11}$$

$$\mathcal{C}^{1234}(\Lambda) = -\zeta_{12}^3 + \zeta_{12}^2 + \zeta_{12} - 1 = \zeta_{24} \cdot \mathsf{r}_2\zeta_{12}$$

Hence,

$$\mathrm{Spec}(\psi_1, \mathfrak{e}_6) = C_{12}(1,\ \zeta_{24} \cdot \mathsf{r}_2,\ 0^{(3)})$$

Applying the `Simplify_Cosine` program from Appendix 1.3 to choose an appropriate overall scaling, we obtain the equivalent expression

$$\mathrm{Spec}(\psi_1, \mathfrak{e}_6) = C_{12}(z_1,\ z_2,\ 0^{(3)})$$

where z_i's are as listed in Table 1. The part of Theorem 1.4 concerning the first fundamental cyclic spectrum of $\mathfrak{e}_6$ is now proved.

Remark 12. Similar to Remark 7, we visualize this spectrum in the complex plane. The graph is as shown in Figure 4. By connecting eigenvalues in ascending order of the heights of their weights, we can see that the real parts of the eigenvalues are non-increasing.

3.5.2 The Adjoint Representation of Type $\mathfrak{e}_6$

In this section, we apply the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{e}_6$. The dimension of this representation is 78, so there are at most 72 non-zero eigenvalues. For convenience, we denote the root

$$\alpha_{k_1} + \cdots + \alpha_{k_n} = (k_1 \cdots k_n)$$

for $1 \leq n \leq 6$. If any simple root in the linear combination appears m times for $m \geq 2$, then we put $m-1$ bars on top of the index of the simple roots. Then, given the highest root as:

$$(1\bar{2}\bar{3}\bar{\bar{4}}\bar{5}6) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

Similar to the first representation, by the programming results as in Appendix 3.5.3, we obtain the successor diagram of $\mathfrak{e}_6$ as such:

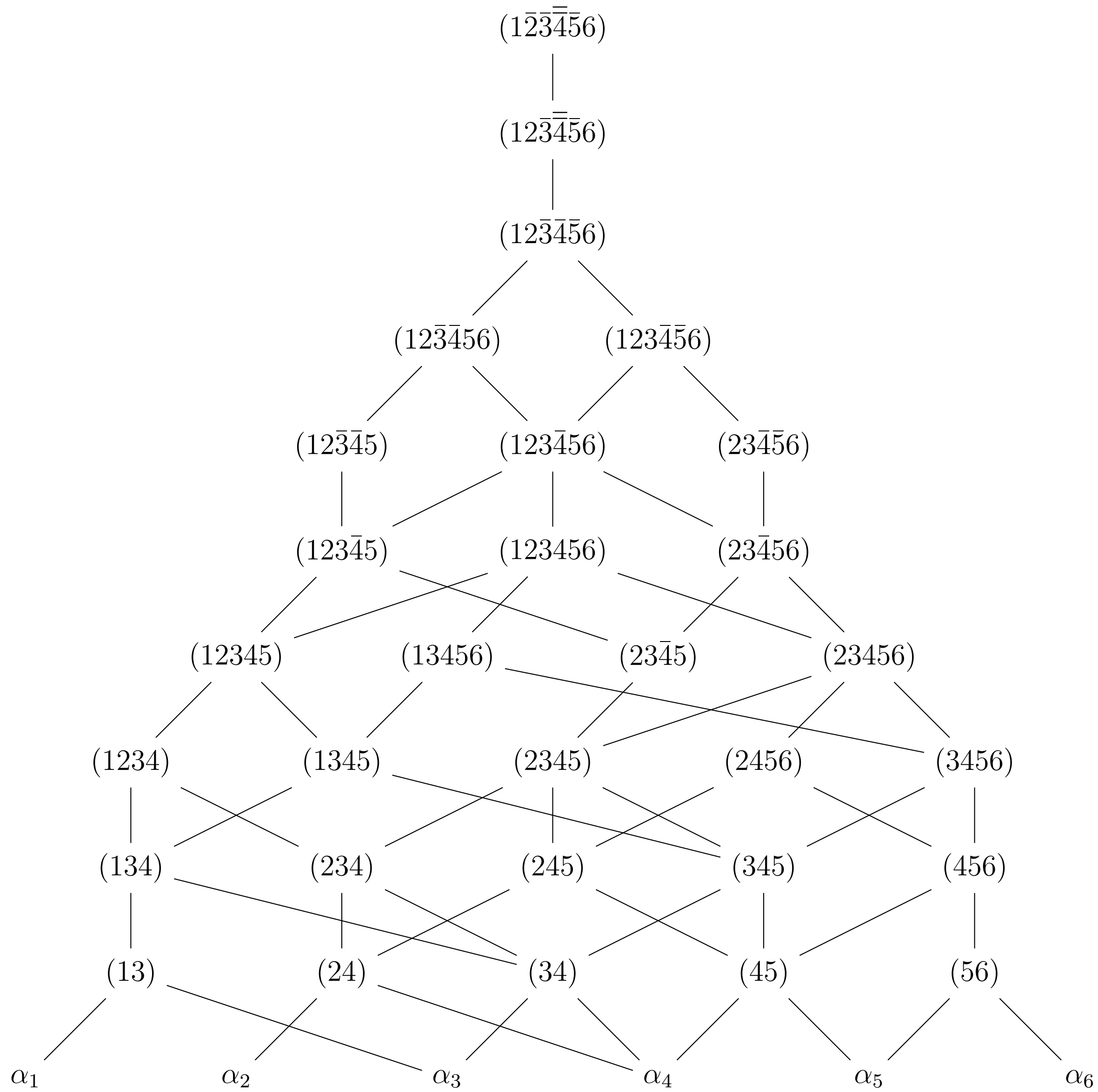


Figure 15: The Successor Diagram of the Positive Roots of $\mathfrak{e}_6$

Now we calculate the pairings between roots and Λ. Similar to Remark 15 later on, by Equation (188), let

$$
\begin{aligned}
\mathsf{r}_1 &= 2\cos\left(\frac{\pi}{4}\right) = \zeta_{12}^{\frac{3}{2}} + \zeta_{12}^{-\frac{3}{2}} \\
\mathsf{r}_2 &= 1 \\
\mathsf{r}_3 &= 2^2\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{12}\right) = -\zeta_{12}^3 + 2\zeta_{12} - 1 \\
\mathsf{r}_4 &= 2\cos\left(\frac{5\pi}{12}\right) = \zeta_{12}^{\frac{5}{2}} + \zeta_{12}^{-\frac{5}{2}}
\end{aligned}
$$

Then, by Equation (188), we obtain:

$$
\begin{aligned}
(1\bar{2}\bar{3}\bar{\bar{4}}\bar{5}6)(\Lambda) &= -\zeta_{12}^3 + \zeta_{12}^2 + \zeta_{12} = \zeta_{24} \cdot \mathsf{r}_1 \\
(12\bar{3}\bar{\bar{4}}\bar{5}6)(\Lambda) &= -\zeta_{12}^2 + \zeta_{12} + 1 = \zeta_{24} \cdot \mathsf{r}_1\zeta_{12}^{11} \\
(12\bar{3}\bar{4}\bar{5}6)(\Lambda) &= \zeta_{12}^3 + 1 = \zeta_{24} \cdot \mathsf{r}_1\zeta_{12} \\
(12\bar{3}\bar{4}56)(\Lambda) &= (123\bar{4}\bar{5}6)(\Lambda) = \mu(\Lambda) = 1 \\
(12\bar{3}\bar{4}5)(\Lambda) &= (23\bar{4}\bar{5}6)(\Lambda) = \mathcal{C}^{2345}(\Lambda) = \zeta_{12} \\
(123\bar{4}56)(\Lambda) &= -\zeta_{12}^3 + 1 = \zeta_{24} \cdot \mathsf{r}_1\zeta_{12}^{10} \\
(123\bar{4}5)(\Lambda) &= (23\bar{4}56)(\Lambda) = \mathcal{C}^{1345}(\Lambda) = -\zeta_{12}^3 + \zeta_{12} = \zeta_{12}^{11} \\
(123456)(\Lambda) &= \zeta_{12}^2 - \zeta_{12} + 1 = \mathsf{r}_3\zeta_{12} \\
(12345)(\Lambda) &= (23456)(\Lambda) = \mathcal{C}^{1245}(\Lambda) = \zeta_{12}^2 \\
(13456)(\Lambda) &= \zeta_{12}^3 - \zeta_{12}^2 - \zeta_{12} + 1 = \mathsf{r}_3\zeta_{12}^{11} \\
(23\bar{4}5)(\Lambda) &= -\zeta_{12}^3 + 2\zeta_{12} - 1 = \mathsf{r}_3 \\
(1234)(\Lambda) &= (2456)(\Lambda) = \mathcal{C}^{1235}(\Lambda) = -\zeta_{12}^3 + \zeta_{12}^2 = \zeta_{24} \cdot \mathsf{r}_4\zeta_{12}^{11} \\
(1345)(\Lambda) &= (3456)(\Lambda) = \mathcal{C}^{45}(\Lambda) = \zeta_{12}^3 - \zeta_{12}^2 + 1 = \zeta_{24} \cdot \mathsf{r}_4 \\
(2345)(\Lambda) &= \zeta_{12}^3 + \zeta_{12} - 1 = \zeta_{24} \cdot \mathsf{r}_1\zeta_{12}^2 \\
(134)(\Lambda) &= (456)(\Lambda) = \mathcal{C}^{35}(\Lambda) = -\zeta_{12}^2 + 1 = \zeta_{12}^{10} \\
(234)(\Lambda) &= (245)(\Lambda) = \mathcal{C}^{1234}(\Lambda) = -\zeta_{12}^3 + \zeta_{12}^2 + \zeta_{12} - 1 = \zeta_{24} \cdot \mathsf{r}_4\zeta_{12} \\
(345)(\Lambda) &= \zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12} = \mathsf{r}_3\zeta_{12}^2
\end{aligned}
$$

$$
\begin{aligned}
(13)(\Lambda) = (56)(\Lambda) = \mathcal{C}^{25}(\Lambda) &= \zeta_{12}^3 - \zeta_{12} + 1 = \zeta_{24} \cdot \mathsf{r}_4 \zeta_{12}^2 \\
(24)(\Lambda) &= -2\zeta_{12}^3 + \zeta_{12}^2 + \zeta_{12} - 1 = \mathsf{r}_3 \zeta_{12}^{10} \\
(34)(\Lambda) = (45)(\Lambda) = \mathcal{C}^{34}(\Lambda) &= -\zeta_{12}^2 + \zeta_{12} = \zeta_{24} \cdot \mathsf{r}_4 \zeta_{12}^{10} \\
\alpha_1(\Lambda) = \alpha_6(\Lambda) = \mathcal{C}^{15}(\Lambda) &= 1 - \zeta_{12} = \zeta_{24} \cdot \mathsf{r}_4 \zeta_{12}^9 \\
\alpha_2(\Lambda) &= -\zeta_{12}^3 + 2\zeta_{12}^2 - 1 = \mathsf{r}_3 \zeta_{12}^3 \\
\alpha_3(\Lambda) = \alpha_5(\Lambda) = \mathcal{C}^{24}(\Lambda) &= \zeta_{12}^3 \\
\alpha_4(\Lambda) &= -\zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12} = \zeta_{24} \cdot \mathsf{r}_1 \zeta_{12}^9
\end{aligned}
$$

Hence,

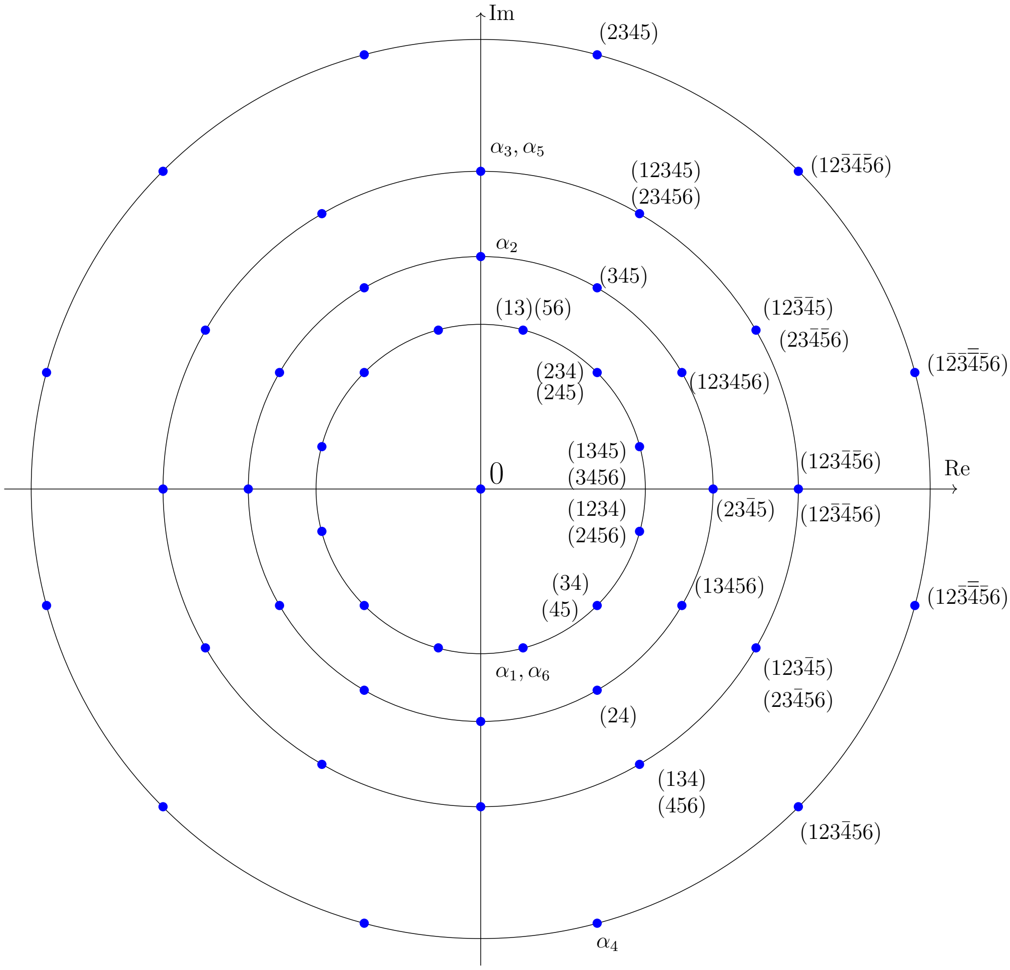
$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_6) = C_{12}(\zeta_{24} \cdot \mathsf{r}_1,\ 1^{(2)},\ \mathsf{r}_3,\ \zeta_{24} \cdot \mathsf{r}_4^{(2)},\ 0^{(6)})$$

In view of the values of $\alpha_i(\Lambda)$ given in the above equations, this also completes the proof of Theorem 1.5 for $\mathfrak{e}_6$. Applying the `Simplify_Cosine` program again, we obtain the equivalent expression

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_6) = C_{12}(z_1,\ z_2^{(2)},\ z_3,\ z_4^{(2)},\ 0^{(6)})$$

where z_i's are as listed in Table 2. The proof of Theorem 1.4 for $\mathfrak{e}_6$ is now complete.

Remark 13. Similar to Figure 4, we plot the eigenvalues in $\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_6)$ in the complex plane, and annotate corresponding positive roots.



3.5.3 Appendix: Weights Finder

We now provide a Python implementation for calculating all the weights of the first fundamental representation of $\mathfrak{e}_6$. The algorithm needs the Cartan matrix C and the highest weight μ to initialize. Then, the algorithm contains three steps:

Step one: generate all the simple reflection matrices based on the given Cartan matrix. The matrix of a simple reflection S_i of a simple root α_i on chosen set of simple roots $\mathcal{A} = \{\alpha_1, \cdots, \alpha_r\}$ is defined as

$$S_i = \left([S_i(\alpha_1)]_{\mathcal{A}} \;\middle|\; [S_i(\alpha_2)]_{\mathcal{A}} \;\middle|\; \cdots \;\middle|\; [S_i(\alpha_r)]_{\mathcal{A}} \right)$$

where $[S_i(\alpha_j)]_{\mathcal{A}}$ is the coefficient column vector of $S_i(\alpha_j)$ based on the set $\mathcal{A}$. Recall in

Definition 1.2, we know the simple reflection S_i is defined as:

$$S_i(\alpha_j) = \alpha_j - \left(\frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2}\right)\alpha_i = \alpha_j - C_{ji}\alpha_i$$

Therefore, we deduce that the (kj)'th entry of the matrix S_i is:

$$\delta_{kj} - C_{ji}\delta_{ki}$$

Notice that for any fixed simple reflection S_i, if $k \neq i$, then

$$(S_i)_{kj} = \delta_{kj}$$

If $k = i$,

$$(S_i)_{ij} = \delta_{ij} - C_{ji}$$

Therefore, we deduce that

$$S_i = I - E_{ii}C^{\intercal}$$

where E_{ii} is the matrix having 0 at every entry except the (ii)'th entry being 1. Based on this formula, we generate all the simple reflection matrices and collect them in the set R. The following is the pseudo-code of this step:

Algorithm 1 Generate the matrices of simple reflections given the Cartan matrix

```
Require: C                                   ▷ Cartan matrix
  for i ← 1 to r do
      S_i ← I − E_ii C^T
  end for
  R ← { S_1, . . . , S_r }                   ▷ set of reflection matrices
```

Step two: apply simple reflections repeatedly to the highest weight μ following the breadth-first-search (BFS) algorithm. Firstly, the highest weight μ can be represented as a coefficient vector based on the set $\mathcal{A}$. Specifically, for $\mathfrak{e}_6$, based on the set of simple roots $\{\alpha_1, \cdots, \alpha_6\}$, the highest weight μ of the first representation can be represented as the vector:

$$\begin{bmatrix} \frac{4}{3} & 1 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \end{bmatrix}^\intercal$$

Then, we perform BFS to generate all the weights. Concretely:

1. Initialize a queue with μ and mark μ as seen.

2. While the queue is nonempty, pop a weight v, apply each simple reflection in R to obtain $w = S_i v$, and if w has not been seen, enqueue w, mark it as seen, and record it.

3. When no new weight appears, all weights have been generated. Count the number of weights in the record, and print out the statistics. If the number of weights agrees with the dimension, then we obtain all the weights.

The BFS will eventually terminate because the representation is finite-dimensional (here 27 dimensional), and the Weyl group also with finite order. This algorithm also will not yield repeated weights, thanks to the "seen" set, so that we never revisit a weight. The following is the pseudo-code of this step:

Algorithm 2 Enumerate all weights of a minusicle representation given the highest weight

---

Require: μ ▷ highest-weight vector
  `to_explore` ← $[\mu]$
  `all_weights` ← $\{\mu\}$
  while `to_explore` $\neq \varnothing$ **do**
    v ← first element of `to_explore`
    remove that element from `to_explore`
    for $M \in R$ **do**
      $w \leftarrow M\, v$
      if $w \notin$ `all_weights` **then**
        add w to `all_weights`
        append w to `to_explore`
      end if
    end for
  end while
  let total ← |`all_weights`|
  print "Total weights: ", total

Final step: calculate the height of each weight, group weights by height, and list them in descending order. This step is helpful to visualize the weights in the successor diagram shown in Figure 14. Overall, the code of this algorithm is as follows:

```
import numpy as np
from fractions import Fraction
from collections import deque
from collections import defaultdict


def frac(vec):
    return [Fraction(x) for x in vec]            #coding convenience


def simple_reflections(C):
    n  = C.shape[0]              #rank
    CT = C.T                            # transpose once
    I  = np.eye(n, dtype=int)            # identity matrix
    S_list = []
    for i in range(n):
        # E_{ii} C^T  =  outer product of e_i and column i of C
        row_update = np.outer(I[i], CT[i])
        S_list.append(I - row_update)
    return S_list


def list_weights_from_cartan(C, mu):
    # 1) build reflection matrices
    R = simple_reflections(C)

    # 2) breadth-first search
    queue   = deque([mu])
    seen    = {tuple(mu)}
```

```
weights = [mu.copy()]

while queue:
    v = queue.popleft()
    for M in R:
        w = M @ v
        key = tuple(w)
        if key not in seen:
            seen.add(key)
            queue.append(w)
            weights.append(w.copy())

# 3) report the total number
total = len(weights)
print(f"Total weights found: {total}\n")

# 4) group by height = sum of coordinates
by_height = defaultdict(list)
for w in weights:
    h = sum(w)
    by_height[h].append(w)

# 5) print from largest to smallest height
for h in sorted(by_height.keys(), reverse=True):
    line = "  ".join("[" + ", ".join(str(x) for x in w) + "]"
                     for w in by_height[h])
    print(f"Height {h}:  {line}")

return weights, total
```

Now, we input the Cartan matrix and the highest weight μ:

```
C = np.array([[2,0,-1,0,0,0],
              [0,2,0,-1,0,0],
              [-1,0,2,-1,0,0],
              [0,-1,-1,2,-1,0],
              [0,0,0,-1,2,-1],
              [0,0,0,0,-1,2]
             ])
mu =  frac(['4/3', '1', '5/3', '2', '4/3', '2/3'])
```

Plugging into the function `list_weights_from_cartan`, we obtain the following output:

```
weights, total = list_weights_from_cartan(C, mu)
```

```
Total weights found: 27

Height 8:  [4/3, 1, 5/3, 2, 4/3, 2/3]
Height 7:  [1/3, 1, 5/3, 2, 4/3, 2/3]
Height 6:  [1/3, 1, 2/3, 2, 4/3, 2/3]
Height 5:  [1/3, 1, 2/3, 1, 4/3, 2/3]
Height 4:  [1/3, 0, 2/3, 1, 4/3, 2/3]  [1/3, 1, 2/3, 1, 1/3, 2/3]
Height 3:  [1/3, 0, 2/3, 1, 1/3, 2/3]  [1/3, 1, 2/3, 1, 1/3, -1/3]
Height 2:  [1/3, 0, 2/3, 0, 1/3, 2/3]  [1/3, 0, 2/3, 1, 1/3, -1/3]
Height 1:  [1/3, 0, -1/3, 0, 1/3, 2/3]  [1/3, 0, 2/3, 0, 1/3, -1/3]
Height 0:  [-2/3, 0, -1/3, 0, 1/3, 2/3]  [1/3, 0, -1/3, 0, 1/3, -1/3]  [1/3,
↪  0, 2/3, 0, -2/3, -1/3]
Height -1:  [-2/3, 0, -1/3, 0, 1/3, -1/3]  [1/3, 0, -1/3, 0, -2/3, -1/3]
Height -2:  [-2/3, 0, -1/3, 0, -2/3, -1/3]  [1/3, 0, -1/3, -1, -2/3, -1/3]
Height -3:  [-2/3, 0, -1/3, -1, -2/3, -1/3]  [1/3, -1, -1/3, -1, -2/3, -1/3]
```

```
Height -4:  [-2/3, -1, -1/3, -1, -2/3, -1/3]  [-2/3, 0, -4/3, -1, -2/3, -1/3]
Height -5:  [-2/3, -1, -4/3, -1, -2/3, -1/3]
Height -6:  [-2/3, -1, -4/3, -2, -2/3, -1/3]
Height -7:  [-2/3, -1, -4/3, -2, -5/3, -1/3]
Height -8:  [-2/3, -1, -4/3, -2, -5/3, -4/3]
```

It is known that the first representation of $\mathfrak{e}_6$ is 27-dimensional, so we have obtained all the weights for this representation.

3.6 Type $\mathfrak{e}_7$

To apply Theorem 1.3 to a simple Lie algebra of type $\mathfrak{e}_7$, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type $\mathfrak{e}_7$ simple Lie algebra is of ADE type. Therefore, its Cartan matrix C is symmetric in the form:

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The smallest eigenvalue of C is $2 - 2\cos\left(\frac{\pi}{h}\right)$, where the Coxeter number $h = 18$. Then, we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{18}\right)$ as:

$$\vec{\boldsymbol{x}} = \Bigg(1,\ \frac{\zeta_{18} + 1 + \zeta_{18}^{-1}}{\zeta_{18}^{1/2} + \zeta_{18}^{-1/2}},\ \zeta_{18}^{1/2} + \zeta_{18}^{-1/2},\ \zeta_{18} + 1 + \zeta_{18}^{-1},\ \frac{\zeta_{18}^{2} + \zeta_{18} + 1 + \zeta_{18}^{-1} + \zeta_{18}^{-2}}{\zeta_{18}^{1/2} + \zeta_{18}^{-1/2}},\ \zeta_{18}^{2} + \zeta_{18}^{-2},\ \frac{\zeta_{18}^{2} + \zeta_{18}^{-2}}{\zeta_{18}^{1/2} + \zeta_{18}^{-1/2}}\Bigg)^{\intercal} \tag{189}$$

3.6.1 The Seventh Fundamental Representation of $\mathfrak{e}_7$

In this section, we calculate the cyclic spectrum for the seventh fundamental representation of the type $\mathfrak{e}_7$ simple Lie algebra using Theorem 1.3.

3.6.1.1 Weights

Let $\{\epsilon_i\}_{i=1,\cdots,8}$ be the canonical basis for $\mathbb{R}^8$. Referring to [3, PLATE VI], the highest weight of the seventh fundamental representation of $\mathfrak{e}_7$ is

$$\begin{aligned}\mu &= \frac{1}{2}(\epsilon_8 - \epsilon_7) + \epsilon_6 \\ &= \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)\end{aligned} \tag{190}$$

where $\alpha_1, \cdots, \alpha_6$ are simple roots of $\mathfrak{e}_6$. They can be written in terms of ϵ_i's as:

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7) \\ \alpha_2 &= \epsilon_1 + \epsilon_2 \\ \alpha_3 &= \epsilon_2 - \epsilon_1 \\ \alpha_4 &= \epsilon_3 - \epsilon_2 \\ \alpha_5 &= \epsilon_4 - \epsilon_3 \\ \alpha_6 &= \epsilon_5 - \epsilon_4 \\ \alpha_7 &= \epsilon_6 - \epsilon_5\end{aligned}$$

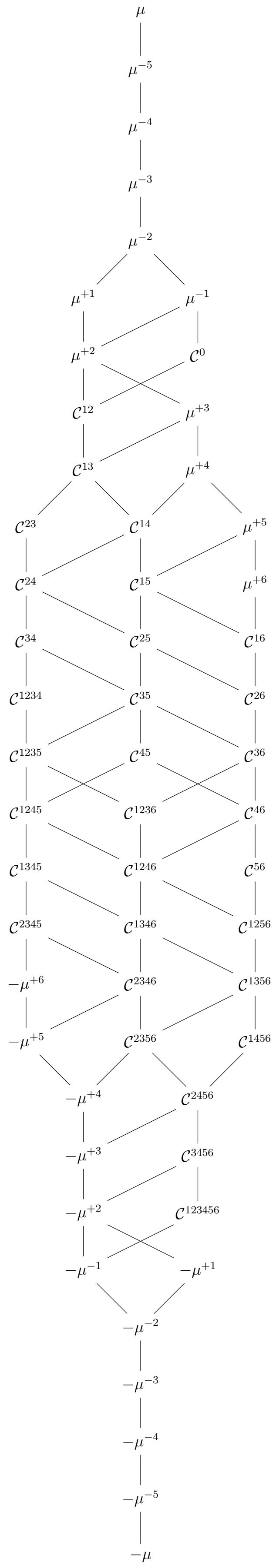
The dimension of the seventh fundamental representation of $\mathfrak{e}_6$ is 56. Next, we show the weights are:

$$\pm\left(\mu - (\epsilon_6 \pm \epsilon_i)\right) \qquad \text{for } 1 \le i \le 6 \tag{191}$$

$$\frac{1}{2}\sum_{i=1}^{6} c_i\epsilon_i \qquad \text{where the } c_i\text{'s are } \pm 1 \text{ such that } \prod c_i = 1 \tag{192}$$

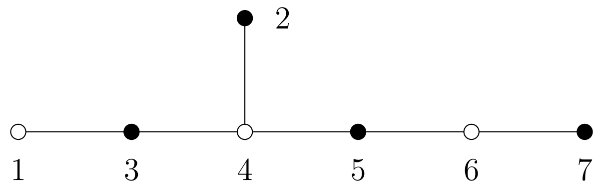
For convenience, we denote weights $\pm(\mu - (\epsilon_6 \pm \epsilon_i))$ as $\pm\mu^{\pm i}$. Also, let $\mathcal{C}^{i_1 \cdots i_k}$, denote the weights $\frac{1}{2}\sum_{i=1}^{6} c_i\epsilon_i$ where $c_{i_1} = \cdots = c_{i_k} = -1$ for $k \leq 6$. When $c_1 = \cdots = c_6 = 1$, denote it by $\mathcal{C}^0$.

Similar to the proof in the previous section of $\mathfrak{e}_6$, we use Python to generate the remaining 55 weights by repeatedly applying simple reflections to μ until obtaining all 56 weights. The code is the same as in Appendix 3.5.3, except that the matrices and vectors representing simple reflection and simple roots are customized for $\mathfrak{e}_7$. Based on the results from the programming, we draw a successor diagram of these weights.


μ
μ^{-5}
μ^{-4}
μ^{-3}
μ^{-2}
μ^{+1} μ^{-1}
μ^{+2} $\mathcal{C}^{0}$
$\mathcal{C}^{12}$ μ^{+3}
$\mathcal{C}^{13}$ μ^{+4}
$\mathcal{C}^{23}$ $\mathcal{C}^{14}$ μ^{+5}
$\mathcal{C}^{24}$ $\mathcal{C}^{15}$ μ^{+6}
$\mathcal{C}^{34}$ $\mathcal{C}^{25}$ $\mathcal{C}^{16}$
$\mathcal{C}^{1234}$ $\mathcal{C}^{35}$ $\mathcal{C}^{26}$
$\mathcal{C}^{1235}$ $\mathcal{C}^{45}$ $\mathcal{C}^{36}$
$\mathcal{C}^{1245}$ $\mathcal{C}^{1236}$ $\mathcal{C}^{46}$
$\mathcal{C}^{1345}$ $\mathcal{C}^{1246}$ $\mathcal{C}^{56}$
$\mathcal{C}^{2345}$ $\mathcal{C}^{1346}$ $\mathcal{C}^{1256}$
$-\mu^{+6}$ $\mathcal{C}^{2346}$ $\mathcal{C}^{1356}$
$-\mu^{+5}$ $\mathcal{C}^{2356}$ $\mathcal{C}^{1456}$
$-\mu^{+4}$ $\mathcal{C}^{2456}$
$-\mu^{+3}$ $\mathcal{C}^{3456}$
$-\mu^{+2}$ $\mathcal{C}^{123456}$
$-\mu^{-1}$ $-\mu^{+1}$
$-\mu^{-2}$
$-\mu^{-3}$
$-\mu^{-4}$
$-\mu^{-5}$
$-\mu$

3.6.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, p is chosen as:

$$p(i) = \begin{cases} 0 & \text{if } i = 1, 4, 6 \\ 1 & \text{if } i = 2, 3, 5, 7 \end{cases}$$

All the simple roots are with the same length, so Equation (5) is equivalent to:

$$\alpha_j(\Lambda) = x_j \cdot \zeta_{18}^{p(j)/2} \cdot (1 - \zeta_{18}^{-2p(j)+1})$$

Recall the Cyclotomic polynomial: $\Phi_{18} = x^6 - x^3 + 1$. As a consequence, we obtain the following identities:

$$\begin{aligned} \zeta_{18}^6 &= \zeta_{18}^3 - 1 = -\zeta_{18}^{15} = -\zeta_{18}^{-3} \\ \zeta_{18}^7 &= \zeta_{18}^4 - \zeta_{18} = -\zeta_{18}^{16} = -\zeta_{18}^{-2} \\ \zeta_{18}^8 &= \zeta_{18}^5 - \zeta_{18}^2 = -\zeta_{18}^{17} = -\zeta_{18}^{-1} \\ \zeta_{18}^9 &= -1 \end{aligned} \tag{193}$$

Referring to these identities, by Equation (189) for x_j, we obtain and simplify the values of $\alpha_j(\Lambda)$ as:

$$\alpha_j(\Lambda) = \begin{cases} -\zeta_{18} + 1 \quad \text{for } j = 1 \\ \frac{1}{3}\zeta_{18}^5 + \frac{2}{3}\zeta_{18}^4 - \frac{2}{3}\zeta_{18}^3 + \frac{1}{3}\zeta_{18}^2 - \frac{1}{3}\zeta_{18} + \frac{1}{3} \quad \text{for } j = 2 \\ \zeta_{18}^5 - \zeta_{18}^2 + \zeta_{18} \quad \text{for } j = 3 \\ -\zeta_{18}^5 \quad \text{for } j = 4 \\ -\frac{1}{3}\zeta_{18}^5 + \frac{1}{3}\zeta_{18}^4 + \frac{2}{3}\zeta_{18}^3 + \frac{2}{3}\zeta_{18}^2 - \frac{2}{3}\zeta_{18} - \frac{1}{3} \quad \text{for } j = 5 \\ \zeta_{18}^5 - \zeta_{18}^4 - \zeta_{18}^3 + \zeta_{18} \quad \text{for } j = 6 \\ -\frac{2}{3}\zeta_{18}^5 - \frac{1}{3}\zeta_{18}^4 + \frac{4}{3}\zeta_{18}^3 + \frac{1}{3}\zeta_{18}^2 - \frac{1}{3}\zeta_{18} - \frac{2}{3} \quad \text{for } j = 7 \end{cases} \tag{194}$$

By Equation (190), we obtain:

$$\begin{aligned} \mu(\Lambda) &= \frac{1}{2}(2\alpha_1(\Lambda) + 3\alpha_2(\Lambda) + 4\alpha_3(\Lambda) + 6\alpha_4(\Lambda) + 5\alpha_5(\Lambda) + 4\alpha_6(\Lambda) + 3\alpha_7(\Lambda)) \\ &= -\frac{1}{3}\zeta_{18}^5 - \frac{2}{3}\zeta_{18}^4 + \frac{2}{3}\zeta_{18}^3 + \frac{2}{3}\zeta_{18}^2 + \frac{1}{3}\zeta_{18} - \frac{1}{3} \end{aligned}$$

Now we show $\mu(\Lambda) = \zeta_{36}/(2\cos(\frac{5\pi}{18}))$. Using

$$\cos\left(\frac{k\pi}{18}\right) = \frac{\zeta_{18}^{\frac{k}{2}} + \zeta_{18}^{-\frac{k}{2}}}{2} \tag{195}$$

for $k = 1, \cdots, 18$, we obtain that

$$\zeta_{36} \cdot \frac{1}{2\cos(\frac{5\pi}{18})} = \frac{\zeta_{18}^{1/2}}{2\cos(\frac{5\pi}{18})} = \frac{\zeta_{18}^{1/2}}{\zeta_{18}^{5/2} + \zeta_{18}^{-5/2}} = \frac{1}{\zeta_{18}^2 + \zeta_{18}^{-3}}$$

We also have that:

$$\begin{aligned} \mu(\Lambda) \cdot (\zeta_{18}^2 + \zeta_{18}^{-3}) = \frac{1}{3}(&-\zeta_{18}^7 - \zeta_{18}^2 - 2\zeta_{18}^6 - 2\zeta_{18} + 2\zeta_{18}^5 + 2 + 2\zeta_{18}^4 + 2\zeta_{18}^{-1} + \zeta_{18}^3 + \zeta_{18}^{-2} \\ &- \zeta_{18}^2 - \zeta_{18}^{-3}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}(\zeta_{18} - \zeta_{18}^4 - \zeta_{18}^2 - 2\zeta_{18}^3 + 2 - 2\zeta_{18} + 2\zeta_{18}^5 + 2 + 2\zeta_{18}^4 + 2\zeta_{18}^2 - 2\zeta_{18}^5 + \\
&\quad + \zeta_{18}^3 + \zeta_{18} - \zeta_{18}^4 - \zeta_{18}^2 + \zeta_{18}^3 - 1) \\
&= 1
\end{aligned}
$$

Hence, $\mu(\Lambda) = \zeta_{36}/(2\cos\left(\frac{5\pi}{18}\right))$. Next, we perform analogous computations for all the seventh fundamental weights of $\mathfrak{e}_7$, using the weights generated by the Python functions described earlier. Following the same procedure as in the calculation for $\mu(\Lambda)$, and employing `SageMath` to carry out computations in the algebra of ζ_{18}-polynomials, we obtain the following eigenvalues. Let

$$
\mathsf{r}_1 = \frac{1}{2\cos\left(\frac{5\pi}{18}\right)}, \ \mathsf{r}_2 = \frac{1}{2\cos\left(\frac{\pi}{18}\right)}, \ \mathsf{r}_3 = \frac{\cos\left(\frac{7\pi}{18}\right)}{2\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{5\pi}{18}\right)} \tag{196}
$$

Then:

$$
\begin{aligned}
\mu^{-5}(\Lambda) &= \mu(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{17} \\
\mu^{-4}(\Lambda) &= \mu(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18} \\
\mu^{-3}(\Lambda) &= \mu(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{16} \\
\mu^{-2}(\Lambda) &= \mu(\Lambda) - \alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{2} \\
\mu^{+1}(\Lambda) &= \mu(\Lambda) - \alpha_2(\Lambda) - \alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2 \\
\mu^{-1}(\Lambda) &= \mu(\Lambda) - \alpha_3(\Lambda) - \alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{17} \\
\mu^{+2}(\Lambda) &= \mu(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - \alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{15} \\
\mathcal{C}^{0}(\Lambda) &= \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_3(\Lambda) - \alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18} \\
\mathcal{C}^{12}(\Lambda) &= \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - \alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{16} \\
\mu^{+3}(\Lambda) &= \mu(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{2} \\
\mathcal{C}^{13}(\Lambda) &= \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{3} \\
\mu^{+4}(\Lambda) &= \mu(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{15} \\
\mathcal{C}^{23}(\Lambda) &= \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 2\alpha_4(\Lambda) - \alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3 \\
\mathcal{C}^{14}(\Lambda) &= \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{17} \\
\mu^{+5}(\Lambda) &= \mu(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}
\end{aligned}
$$

$$\mathcal{C}^{24}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{14}$$

$$\mathcal{C}^{15}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{3}$$

$$\mu^{+6}(\Lambda) = \mu(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{16}$$

$$\mathcal{C}^{34}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{3}$$

$$\mathcal{C}^{25}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{15}$$

$$\mathcal{C}^{16}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - \alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{2}$$

$$\mathcal{C}^{1234}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - \alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{14}$$

$$\mathcal{C}^{35}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{4}$$

$$\mathcal{C}^{26}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 2\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{14}$$

$$\mathcal{C}^{1235}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{4}$$

$$\mathcal{C}^{45}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = 0$$

$$\mathcal{C}^{36}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{4}$$

$$\mathcal{C}^{1245}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{13}$$

$$\mathcal{C}^{1236}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 2\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = 0$$

$$\mathcal{C}^{46}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{13}$$

$$\mathcal{C}^{1345}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{5}$$

$$\mathcal{C}^{1246}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{13}$$

$$\mathcal{C}^{56}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - \alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{5}$$

$$\mathcal{C}^{2345}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 3\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - \alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{11}$$

$$\mathcal{C}^{1346}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{6}$$

$$\mathcal{C}^{1256}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 3\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{12}$$

$$-\mu^{+6}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{7}$$

$$\mathcal{C}^{2346}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 3\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 2\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{12}$$

$$\mathcal{C}^{1356}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{5}$$

$$-\mu^{+5}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{10}$$

$$\mathcal{C}^{2356}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 3\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 3\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_3\zeta_{18}^{8}$$

$$\mathcal{C}^{1456}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 2\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 4\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = -\zeta_{36} \cdot \mathsf{r}_3$$

$$-\mu^{+4}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2\zeta_{18}^{6}$$

$$\mathcal{C}^{2456}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 3\alpha_3(\Lambda) - 4\alpha_4(\Lambda) - 4\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1\zeta_{18}^{12}$$

$$-\mu^{+3}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2 \zeta_{18}^{11}$$

$$\mathcal{C}^{3456}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 2\alpha_2(\Lambda) - 3\alpha_3(\Lambda) - 5\alpha_4(\Lambda) - 4\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2 \zeta_{18}^{7}$$

$$-\mu^{+2}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1 \zeta_{18}^{6}$$

$$\mathcal{C}^{123456}(\Lambda) = \mu(\Lambda) - \alpha_1(\Lambda) - 3\alpha_2(\Lambda) - 3\alpha_3(\Lambda) - 5\alpha_4(\Lambda) - 4\alpha_5(\Lambda) - 3\alpha_6(\Lambda) - 2\alpha_7(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2 \zeta_{18}^{10}$$

$$-\mu^{-1}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_2 \zeta_{18}^{8}$$

$$-\mu^{+1}(\Lambda) = -\zeta_{36} \cdot \mathsf{r}_2$$

$$-\mu^{-2}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1 \zeta_{18}^{11}$$

$$-\mu^{-3}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1 \zeta_{18}^{7}$$

$$-\mu^{-4}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1 \zeta_{18}^{10}$$

$$-\mu^{-5}(\Lambda) = \zeta_{36} \cdot \mathsf{r}_1 \zeta_{18}^{8}$$

$$-\mu(\Lambda) = -\zeta_{36} \cdot \mathsf{r}_1$$

Hence,

$$\mathrm{Spec}(\psi_7, \mathfrak{e}_7) = C_{18}(\zeta_{36} \cdot \mathsf{r}_1,\ \zeta_{36} \cdot \mathsf{r}_2,\ \zeta_{36} \cdot \mathsf{r}_3,\ 0^{(2)})$$

It is equivalent to:

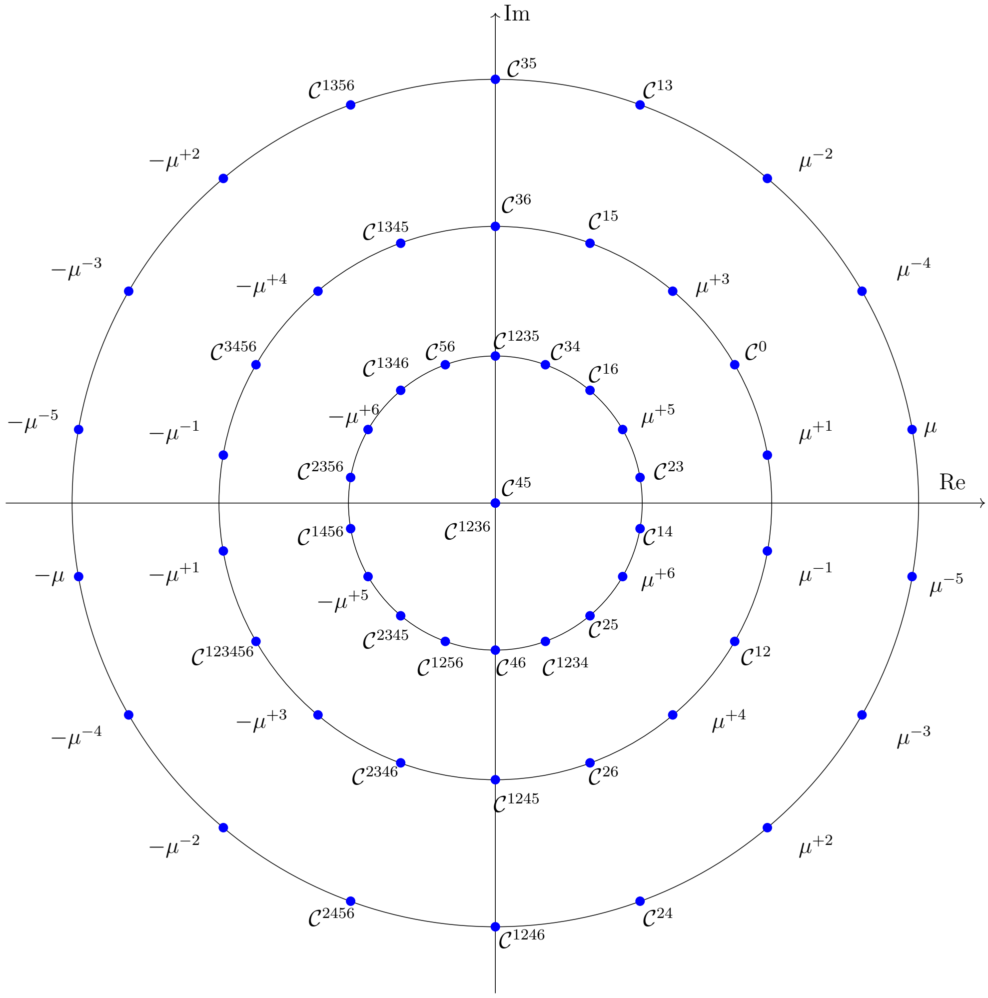
$$C_{18}(\mathsf{r}_1,\ \mathsf{r}_2,\ \mathsf{r}_3,\ 0^{(2)})$$

Applying the `Simplify_Cosine` program from Appendix 1.3 to choose an appropriate overall scaling, we obtain the equivalent expression

$$\mathrm{Spec}(\psi_7, \mathfrak{e}_7) = C_{18}(z_1,\ z_2,\ z_3,\ 0^{(2)})$$

where z_i's are as listed in Table 1. The part of Theorem 1.4 concerning the seventh fundamental cyclic spectrum of $\mathfrak{e}_7$ is now proved.

Remark 14. Similar to Remark 7, we visualize this spectrum (without overall scalings) in the complex plane. The graph is as shown in the following graph:



By connecting eigenvalues in ascending order of the heights of their weights, we can see that the real parts of the eigenvalues are non-increasing.

3.6.2 The Adjoint Representation of $\mathfrak{e}_7$

Let

$$\mathsf{r}_1 = \frac{\cos\left(\frac{7\pi}{18}\right)}{\cos\left(\frac{\pi}{18}\right)}$$

$$\mathsf{r}_2 = \frac{1}{\cos\left(\frac{\pi}{18}\right)}$$

$$\mathsf{r}_3 = 2\cos\left(\frac{7\pi}{18}\right)$$

$$\mathsf{r}_4 = 1$$

$$\mathsf{r}_5 = \frac{1}{\cos\left(\frac{5\pi}{18}\right)}$$
$$\mathsf{r}_6 = \frac{\cos\left(\frac{7\pi}{18}\right)}{\cos\left(\frac{5\pi}{18}\right)}$$
$$\mathsf{r}_7 = \frac{\cos\left(\frac{7\pi}{18}\right)}{2\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{5\pi}{18}\right)}$$

By Equations (195) and (193), it follows that:

$$\alpha_j(\Lambda) = \begin{cases} \mathsf{r}_j\zeta_{18}^{14} & \text{for } j = 1, 4, 6 \\ \zeta_{36} \cdot \mathsf{r}_j\zeta_{18}^{4} & \text{for } j = 2, 3, 5, 7 \end{cases}$$

By "ζ_{18}-symmetry", the pairings of simple roots and Λ generates:

$$C_{18}(\mathsf{r}_1,\ \zeta_{36}\cdot\mathsf{r}_2,\ \zeta_{36}\cdot\mathsf{r}_3,\ \mathsf{r}_4,\ \zeta_{36}\cdot\mathsf{r}_5,\ \mathsf{r}_6,\ \zeta_{36}\cdot\mathsf{r}_7)$$

The dimension of the adjoint representation is 133, so there are at most 126 non-zero eigenvalues in $\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_7)$. Therefore, by "$\zeta_{18}-$symmetry", $\{\alpha_j(\Lambda)\}_{j=1}^{7}$ generates all non-zero eigenvalues. This completes the proof of Theorem 1.5 for $\mathfrak{e}_7$. Applying the `Simplify_Cosine` program again, we obtain the equivalent expression

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_7) = C_{18}(z_1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_6,\ z_7,\ 0^{(7)})$$

where z_i's are as listed in Table 2. The proof of Theorem 1.4 for $\mathfrak{e}_7$ is now complete.

3.7 Type $\mathfrak{e}_8$

To apply Theorem 1.3 to a simple Lie algebra of type $\mathfrak{e}_8$, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type $\mathfrak{e}_8$ simple Lie algebra is of ADE type. Therefore, its Cartan matrix C is symmetric in the form:

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

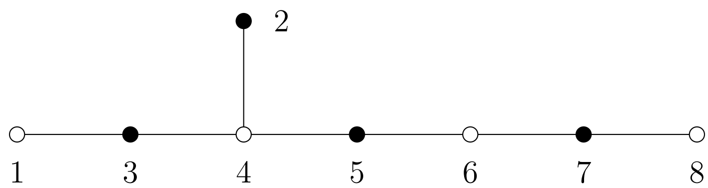
The smallest eigenvalue of C is $2 - 2\cos\left(\frac{\pi}{h}\right)$, where the Coxeter number $h = 30$. Then, we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{30}\right)$ as:

$$\begin{aligned} \vec{\boldsymbol{x}} = \Bigg(1,\ & \frac{\zeta_{30} + 1 + \zeta_{30}^{-1}}{\zeta_{30}^{1/2} + \zeta_{30}^{-1/2}},\ \zeta_{30}^{1/2} + \zeta_{30}^{-1/2},\ \zeta_{30} + 1 + \zeta_{30}^{-1},\ \frac{\zeta_{30}^{2} + \zeta_{30} + 1 + \zeta_{30}^{-1} + \zeta_{30}^{-2}}{\zeta_{30}^{1/2} + \zeta_{30}^{-1/2}}, \\ & \zeta_{30}^{2} + \zeta_{30}^{-2},\ \frac{\zeta_{30}^{3} + \zeta_{30}^{2} - 1 + \zeta_{30}^{-2} + \zeta_{30}^{-3}}{\zeta_{30}^{1/2} + \zeta_{30}^{-1/2}},\ \zeta_{30}^{3} - 1 + \zeta_{30}^{-3}\Bigg)^{\intercal} \end{aligned} \tag{197}$$

3.7.1 The Eighth Fundamental Representation of $\mathfrak{e}_8$

In this section, we calculate the cyclic spectrum for the eighth fundamental representation of the type $\mathfrak{e}_8$ simple Lie algebra using Theorem 1.3. The eighth fundamental representation of $\mathfrak{e}_8$ is also its adjoint representation, with the highest root $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, p is chosen as:

$$p(i) = \begin{cases} 0 & \text{if } i = 1, 4, 6, 8 \\ 1 & \text{if } i = 2, 3, 5, 7 \end{cases}$$

All the simple roots are with the same length, so Equation (5) is equivalent to:

$$\alpha_j(\Lambda) = x_j \cdot \zeta_{30}^{p(j)/2} \cdot (1 - \zeta_{30}^{-2p(j)+1})$$

By the Cyclotomic polynomial $\Phi_{30} = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$, it implies:

$$\begin{aligned}
\zeta_{30}^{8} &= -\zeta_{30}^{7} + \zeta_{30}^{5} + \zeta_{30}^{4} + \zeta_{30}^{3} - \zeta_{30} - 1 = -\zeta_{30}^{23} \\
\zeta_{30}^{9} &= \zeta_{30}^{7} + \zeta_{30}^{6} - \zeta_{30}^{3} - \zeta_{30}^{2} + 1 = -\zeta_{30}^{24} \\
\zeta_{30}^{10} &= \zeta_{30}^{5} - 1 = -\zeta_{30}^{25} \\
\zeta_{30}^{11} &= \zeta_{30}^{6} - \zeta_{30} = -\zeta_{30}^{26} \\
\zeta_{30}^{12} &= \zeta_{30}^{7} - \zeta_{30}^{2} = -\zeta_{30}^{27} \\
\zeta_{30}^{13} &= -\zeta_{30}^{7} + \zeta_{30}^{5} + \zeta_{30}^{4} - \zeta_{30} - 1 = -\zeta_{30}^{28} \\
\zeta_{30}^{14} &= \zeta_{30}^{7} + \zeta_{30}^{6} - \zeta_{30}^{4} - \zeta_{30}^{3} - \zeta_{30}^{2} + 1 = -\zeta_{30}^{29} \\
\zeta_{30}^{15} &= -1
\end{aligned} \tag{198}$$

Combine with Equation (197) for x_j's, we obtain:

$$\begin{aligned}
\alpha_1(\Lambda) &= -\zeta_{30} + 1 \\
\alpha_2(\Lambda) &= -\zeta_{30}^{7} + \zeta_{30}^{6} + \zeta_{30}^{4} - \zeta_{30}^{3} + \zeta_{30}^{2} - \zeta_{30} \\
\alpha_3(\Lambda) &= \zeta_{30}^{7} + \zeta_{30}^{6} - \zeta_{30}^{4} - \zeta_{30}^{3} - \zeta_{30}^{2} + \zeta_{30} + 1 \\
\alpha_4(\Lambda) &= -\zeta_{30}^{7} - \zeta_{30}^{6} + \zeta_{30}^{4} + \zeta_{30}^{3} - 1 \\
\alpha_5(\Lambda) &= -\zeta_{30}^{6} + \zeta_{30}^{5} + \zeta_{30}^{3} - 1 \\
\alpha_6(\Lambda) &= 2\zeta_{30}^{7} + \zeta_{30}^{6} - \zeta_{30}^{5} - 2\zeta_{30}^{4} - 2\zeta_{30}^{3} + \zeta_{30} + 2
\end{aligned}$$

$$\alpha_7(\Lambda) = -\zeta_{30}^5 + \zeta_{30}^4 + \zeta_{30}^3 - \zeta_{30}$$

$$\alpha_8(\Lambda) = -2\zeta_{30}^7 + \zeta_{30}^5 + \zeta_{30}^3 + \zeta_{30}^2 - 2$$

Remark 15. In Theorem 1.4, all r_i's are expressed in trigonometric values. To obtain those values, for example, consider one of the expressions for the case of $\mathfrak{e}_8$ in these tables. Using Equation (5), we obtain

$$\alpha_5(\Lambda) \;=\; \frac{\zeta_{30}^2 + \zeta_{30} + 1 + \zeta_{30}^{-1} + \zeta_{30}^{-2}}{\zeta_{30}^{1/2} + \zeta_{30}^{-1/2}} \cdot (\zeta_{30}^{1/2} - \zeta_{30}^{-1/2})$$

where $\zeta_{30} = e^{\frac{2\pi i}{30}}$. Now we compute $|\alpha_5(\Lambda)|$. Note that

$$|\zeta_{30}^{1/2} - \zeta_{30}^{-1/2}| \;=\; 2\cos\left(\frac{7\pi}{15}\right)$$

Also, for $k = 1, \cdots, 30$,

$$\cos\left(\frac{k\pi}{30}\right) \;=\; \frac{\zeta_{30}^{\frac{k}{2}} + \zeta_{30}^{-\frac{k}{2}}}{2} \tag{199}$$

Thus,

$$\frac{\zeta_{30}^2 + \zeta_{30} + 1 + \zeta_{30}^{-1} + \zeta_{30}^{-2}}{\zeta_{30}^{1/2} + \zeta_{30}^{-1/2}} \;=\; \frac{1 + 2\cos\left(\frac{\pi}{15}\right) + 2\cos\left(\frac{2\pi}{15}\right)}{2\cos\left(\frac{\pi}{30}\right)}$$

By the Dirichlet kernel identity:

$$\frac{\sin\left((n+\frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)} \;=\; 1 + 2\cos(\theta) + 2\cos(2\theta) + 2\cos(3\theta) + \cdots + 2\cos(n\theta)$$

We obtain:

$$\frac{1 + 2\cos\left(\frac{\pi}{15}\right) + 2\cos\left(\frac{2\pi}{15}\right)}{2\cos\left(\frac{\pi}{30}\right)} \;=\; \frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{15}\right)2\cos\left(\frac{\pi}{30}\right)} \;=\; \frac{1}{2^2\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{\pi}{30}\right)}$$

Hence:

$$|\alpha_5(\Lambda)| \;=\; \frac{1}{2\cos\left(\frac{\pi}{30}\right)}$$

By applying the same approach as in Remark 15, we obtain

$$\begin{aligned}
\mathsf{r}_1 &= 2\cos\left(\frac{7\pi}{15}\right) \\
\mathsf{r}_2 &= \frac{2^2\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{15}\right)}{\cos\left(\frac{\pi}{30}\right)} \\
\mathsf{r}_3 &= 2^2\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{\pi}{30}\right) \\
\mathsf{r}_4 &= 2^3\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{15}\right) \\
\mathsf{r}_5 &= \frac{1}{2\cos\left(\frac{\pi}{30}\right)} \\
\mathsf{r}_6 &= 2^2\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{2\pi}{15}\right) \\
\mathsf{r}_7 &= 2^4\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{13\pi}{30}\right)\cos\left(\frac{7\pi}{30}\right)\cos\left(\frac{\pi}{30}\right) \\
\mathsf{r}_8 &= 2^3\cos\left(\frac{7\pi}{15}\right)\cos\left(\frac{13\pi}{30}\right)\cos\left(\frac{7\pi}{30}\right)
\end{aligned} \tag{200}$$

By Equations (198) and (199), it follows that:

$$\alpha_j(\Lambda) = \begin{cases} \mathsf{r}_j\zeta_{30}^{23} & \text{for } j = 1,4,6,8 \\ \zeta_{60}\cdot\mathsf{r}_j\zeta_{30}^{7} & \text{for } j = 2,3,5,7 \end{cases}$$

By "ζ_{30}-symmetry", the pairings of simple roots and Λ generates:

$$\bigsqcup_{j=1}^{8} C_{30}(\alpha_j(\Lambda))$$

The dimension of the adjoint representation is 248, so there are at most 240 non-zero eigenvalues in $\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_8)$. Therefore, $\{\alpha_j(\Lambda)\}_{j=1}^{8}$ generates all non-zero eigenvalues. It follows that:

$$\mathrm{Spec}(\psi_8, \mathfrak{e}_8) = \mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_8) = \bigsqcup_{j=1}^{8} C_{30}(\alpha_j(\Lambda),\ 0^{(8)})$$

This completes the proof of Theorem 1.5 for $\mathfrak{e}_8$. Applying the `Simplify_Cosine` program from Appendix 1.3 to choose an appropriate overall scaling, we obtain the equivalent expression

$$\mathrm{Spec}(\psi_8, \mathfrak{e}_8) = \mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_8) = C_{30}(z_1, \cdots, z_8,\ 0^{(8)})$$

where z_i's are as listed in Table 1. The proof of Theorem 1.4 for $\mathfrak{e}_8$ is now complete.

3.8 Type $\mathfrak{f}_4$

To apply Theorem 1.3 to a simple Lie algebra of type $\mathfrak{f}_4$, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type $\mathfrak{f}_4$ simple Lie algebra is not of ADE type. Therefore, its Cartan matrix C is not symmetric in the form:

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The smallest eigenvalue of C is $2 - 2\cos\left(\frac{\pi}{h}\right)$, where the Coxeter number $h = 12$. Then, we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{12}\right)$ as:

$$\vec{\boldsymbol{x}} = \left(1\ ,\ \zeta_{12}^{1/2} + \zeta_{12}^{-1/2}\ ,\ \zeta_{12} + \zeta_{12}^{-1} + 1\ ,\ \zeta_{12}^{3/2} + \zeta_{12}^{-3/2}\right)^{\intercal} \tag{201}$$

3.8.1 The Fourth Fundamental Representation of $\mathfrak{f}_4$

In this section, we calculate the cyclic spectrum for the fourth fundamental representation of the type $\mathfrak{f}_4$ simple Lie algebra using Theorem 1.3.

3.8.1.1 Weights

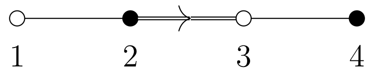
The fourth fundamental representation of $\mathfrak{f}_4$ is 26-dimensional with the highest weight $\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. After applying the function `list_weights_from_cartan` by using Python, we obtain 24 weights:

$$\pm(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$$
$$\pm(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)$$
$$\pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$$
$$\pm(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)$$
$$\pm(\alpha_2 + 2\alpha_3 + \alpha_4), \pm(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$$
$$\pm(\alpha_2 + \alpha_3 + \alpha_4), \pm(\alpha_1 + \alpha_2 + \alpha_3)$$
$$\pm(\alpha_3 + \alpha_4), \pm(\alpha_2 + \alpha_3)$$
$$\pm\alpha_4, \pm\alpha_3$$

It implies that the fourth fundamental representation of $\mathfrak{f}_4$ is not minusicle. In [1, p. 56], Adams states that the fourth fundamental representation of $\mathfrak{f}_4$ contains the zero weight twice; hence, the missing two weights are both zero weights.

3.8.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, p is chosen as:

$$p(i) = \begin{cases} 0 & \text{if } i = 1, 3 \\ 1 & \text{if } i = 2, 4 \end{cases}$$

Note that the squared length of α_1 and α_2 is twice the squared length of α_3 and α_4. Therefore, for $j = 1, \cdots, 4$, Equation (5) yields:

$$\alpha_j(\Lambda) = \begin{cases} x_j \cdot \zeta_{12}^{(p(j)-p(1))/2} \cdot (1 - \zeta_{12}^{-2p(j)+1}) & \text{for } j = 1, 2 \\ \frac{1}{2} \cdot x_j \cdot \zeta_{12}^{(p(j)-p(1))/2} \cdot (1 - \zeta_{12}^{-2p(j)+1}) & \text{for } j = 3, 4 \end{cases}$$

By Equation (201) for x_j and Equations (188), it yields:

$$\alpha_j(\Lambda) = \begin{cases} -\zeta_{12} + 1 & \text{for } j = 1 \\ \zeta_{12}^3 & \text{for } j = 2 \\ \frac{1}{2}(-\zeta_{12}^3 - \zeta_{12}^2 + \zeta_{12}) & \text{for } j = 3 \\ \frac{1}{2}(-\zeta_{12}^3 + 2\zeta_{12}^2 - 1) & \text{for } j = 4 \end{cases}$$

Using `SageMath` to carry out computations in the algebra of ζ_{12}-polynomials, we rewrite those values as follows. Let

$$\begin{aligned} \mathsf{r}_1 &= 2\cos\left(\frac{5\pi}{12}\right) \\ \mathsf{r}_2 &= 1 \\ \mathsf{r}_3 &= \cos\left(\frac{\pi}{4}\right) \\ \mathsf{r}_4 &= 2\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{5\pi}{12}\right) \end{aligned}$$

Then:

$$\alpha_j(\Lambda) = \begin{cases} \zeta_{24} \cdot \mathsf{r}_1 \cdot \zeta_{12}^9 & \text{for } j = 1 \\ \mathsf{r}_2 \cdot \zeta_{12}^3 & \text{for } j = 2 \\ \zeta_{24} \cdot \mathsf{r}_3 \cdot \zeta_{12}^9 & \text{for } j = 3 \\ \mathsf{r}_4 \cdot \zeta_{12}^3 & \text{for } j = 4 \end{cases} \tag{202}$$

Notice that the weights of the fourth fundamental representation contain simple roots α_3 and α_4. By "$\zeta_{12}-$symmetry", they generate eigenvalues:

$$C_{12}(\zeta_{24} \cdot \mathsf{r}_3, \mathsf{r}_4)$$

These exhaust all non-zero eigenvalues since there are only 24 non-zero weights. Applying the `Simplify_Cosine` program from Appendix 1.3 to choose an appropriate overall scaling, we obtain the equivalent expression

$$C_{12}(z_1,\ z_2)$$

where $z_1 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{12}\right),\ z_2 = 1$. Hence,

$$\mathrm{Spec}(\psi_4, \mathfrak{f}_4) = C_{12}(z_1,\ z_2,\ 0^{(2)})$$

The part of Theorem 1.4 concerning the fourth fundamental cyclic spectrum of $\mathfrak{f}_4$ is now proved.

3.8.2 The Adjoint Representation of $\mathfrak{f}_4$

In this section, we apply the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{f}_4$ whose dimension is 52. Notice that by "$\zeta_{12}-$symmetry", $\alpha_j(\Lambda)$'s generate:

$$C_{12}(\zeta_{24} \cdot \mathsf{r}_1,\ \mathsf{r}_2,\ \zeta_{24} \cdot \mathsf{r}_3,\ \mathsf{r}_4)$$

These exhaust all non-zero eigenvalues since there are only 48 non-zero roots. This completes the proof of Theorem 1.5 for $\mathfrak{f}_4$. Applying the `Simplify_Cosine` program again, we obtain the equivalent expression

$$C_{12}(z_1,\ z_2,\ z_3,\ z_4)$$

where $z_1 = 2^2 \cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right),\ z_2 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{12}\right),\ z_3 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{4}\right),\ z_4 = 1.$ Hence,

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{f}_4) = C_{12}(z_1,\ z_2,\ z_3,\ z_4,\ 0^{(4)})$$

The proof of Theorem 1.4 for $\mathfrak{f}_4$ is now complete.

3.9 Type $\mathfrak{g}_2$

To apply Theorem 1.3 to a simple Lie algebra of type $\mathfrak{g}_2$, we first calculate the components of a left eigenvector of the Cartan matrix corresponding to the smallest eigenvalue. The type $\mathfrak{g}_2$ simple Lie algebra is not of ADE type. Therefore, its Cartan matrix C is not symmetric in the form:

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

The smallest eigenvalue of C is $2 - 2\cos\left(\frac{\pi}{h}\right)$, where the Coxeter number $h = 6$. Then, we calculate a left eigenvector $\vec{\boldsymbol{x}}$ of the eigenvalue $2 - 2\cos\left(\frac{\pi}{6}\right)$ as:

$$\vec{\boldsymbol{x}} = \left(1, \frac{\zeta_6^{1/2} + \zeta_6^{-1/2}}{3}\right)^{\mathsf{T}} \tag{203}$$

3.9.1 The First Fundamental Representation of $\mathfrak{g}_2$

In this section, we calculate the cyclic spectrum for the first fundamental representation of the type $\mathfrak{g}_2$ simple Lie algebra using Theorem 1.3.

3.9.1.1 Weights

The first fundamental representation of $\mathfrak{g}_2$ is 7-dimensional with the highest weight $\mu = 2\alpha_1 + \alpha_2$. After applying the function `list_weights_from_cartan` by using Python, we obtain 6 weights:

$$\pm\,(2\alpha_1 + \alpha_2)$$

$$\pm(\alpha_1+\alpha_2)$$

$$\pm\alpha_1$$

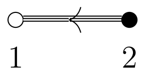
It implies that the first fundamental representation of $\mathfrak{g}_2$ is not minusicle. Now we show the only missing weight is the zero weight. Let the weight be $a\alpha_1 + b\alpha_2$. Since its orbit under the Weyl group action only contains itself, then:

$$\begin{cases} S_1(a\alpha_1+b\alpha_2) = (3b-a)\alpha_1 + b\alpha_2 = a\alpha_1 + b\alpha_2 \Rightarrow 3b = 2a \\ S_2(a\alpha_1+b\alpha_2) = a\alpha_1 + (a-b)\alpha_2 = a\alpha_1 + b\alpha_2 \Rightarrow a = 2b \end{cases}$$

Solving this linear system yields $a = b = 0$, hence the missing weight is just the zero weight.

3.9.1.2 Cyclic Spectrum

To apply Theorem 1.3, we need a bi-coloration of the Dynkin diagram, which is chosen as follows:



Equivalently, p is chosen as $p(1) = 0$ and $p(2) = 1$. Note that the squared length of α_2 is three times the squared length of α_1. Therefore, by Equation (203), for $j = 1, 2$, Equation (5) yields:

$$\alpha_j(\Lambda) = \begin{cases} x_j \cdot \zeta_6^{(p(j)-p(1))/2} \cdot (1-\zeta_6^{-2p(j)+1}) = 1-\zeta_6 & \text{for } j = 1 \\ 3 \cdot x_j \cdot \zeta_6^{(p(j)-p(1))/2} \cdot (1-\zeta_6^{-2p(j)+1}) = 2\zeta_6 - 1 & \text{for } j = 2 \end{cases}$$

Using `SageMath` to carry out computations in the algebra of ζ_6-polynomials, we rewrite those values as follows. Let

$$\mathsf{r}_1 = 1$$

$$\mathsf{r}_2 = 2\cos\left(\frac{\pi}{6}\right)$$

Then:

$$\alpha_1(\Lambda) = \mathsf{r}_1 \cdot \zeta_6^5 \tag{204}$$

$$\alpha_2(\Lambda) = \zeta_{12} \cdot \mathsf{r}_2 \cdot \zeta_6 \tag{205}$$

Notice that the weights of the first fundamental representation contain simple roots α_1. By "ζ_6-symmetry", it generate eigenvalues: $C_6(\mathsf{r}_1)$. These exhaust all non-zero eigenvalues since there are only 6 non-zero weights. Hence,

$$\mathrm{Spec}(\psi_1, \mathfrak{g}_2) = C_6(\mathsf{r}_1, 0)$$

The part of Theorem 1.4 concerning the first fundamental cyclic spectrum of $\mathfrak{g}_2$ is now proved.

3.9.2 The Adjoint Representation of $\mathfrak{g}_2$

In this section, we apply the same approach to compute the cyclic spectrum for the adjoint representation of $\mathfrak{g}_2$ whose dimension is 14. Notice that by "ζ_6-symmetry", $\alpha_j(\Lambda)$'s generate:

$$C_6(\mathsf{r}_1,\ \zeta_{12} \cdot \mathsf{r}_2)$$

These exhaust all non-zero eigenvalues since there are only 12 non-zero roots. This completes the proof of Theorem 1.5 for $\mathfrak{g}_2$. Hence,

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}_2) = C_6(\mathsf{r}_1,\ \zeta_{12} \cdot \mathsf{r}_2,\ 0^{(2)})$$

The proof of Theorem 1.4 for $\mathfrak{g}_2$ is now complete.

4 Adjoint Spectra and Folding of Simple Lie Algebras

In the previous chapter, we observed certain interesting coincidences when comparing the adjoint spectra across different types of simple Lie algebras.

Observation 1: We calculated that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{sl}_{2n}) = C_{2n}(z_1^{(2)}, \cdots, z_{n-1}^{(2)}, z_n, 0^{(2n-1)})$$

where $z_j = \zeta_{4n}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ and

$$p(j) = \begin{cases} 0 & \text{if } j, n \text{ are with the same parity} \\ 1 & \text{Otherwise} \end{cases} \tag{206}$$

for $j = 1, \cdots, n$. We also calculated that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{sp}_{2n}) = C_{2n}(z_1, \cdots, z_n, 0^{(n)})$$

where $z_j = \zeta_{4n}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ and the same $p(j)$ as in Equation (206) for $j = 1, \cdots, n$. It is clear that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{sp}_{2n}) \subset \mathrm{Spec}(\mathrm{ad}, \mathfrak{sl}_{2n})$$

Observation 2: We calculated that: if $n + 1 \equiv 1 \mod 3$, such that $n = 3k$ for some positive integer k, then

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+2}) = C_{2n}(z_1, \cdots, z_{k-1}, z_k^{(3)}, z_{k+1}, \cdots, z_{n-1}, 0^{(n+1)})$$

where $z_j = \zeta_{4n}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ and

$$p(j) = \begin{cases} 0 & \text{if } j, n+1 \text{ are with the same parity} \\ 1 & \text{Otherwise} \end{cases} \tag{207}$$

for $j = 1, \cdots, n-1$. We also calculated that given $n = 3k$, then

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+1}) = C_{2n}(z_1, \cdots, z_{k-1}, z_k^{(2)}, z_{k+1}, \cdots, z_{n-1}, 0^{(n)})$$

where $z_j = \zeta_{4n}^{p(j)} \cdot \mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ and the same $p(j)$ as in Equation (206) for $j = 1, \cdots, n-1$. It is clear that up to an overall scaling,

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+1}) \subset \mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+2})$$

Similarly, if $n+1 \not\equiv 1 \mod 3$, i.e. $n \not\equiv 0 \mod 3$, then

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+2}) = C_{2n}(z_1, \cdots, z_{n-1}, z_n^{(2)}, 0^{(n+1)})$$

where $z_j = \zeta_{4n}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ and the same $p(j)$ as in Equation (207) for $j = 1, \cdots, n$. Given such n, we also have

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+1}) = C_{2n}(z_1, \cdots, z_n, 0^{(n)})$$

where $z_j = \zeta_{4n}^{p(j)} \cdot \mathsf{r}_j$ for $\mathsf{r}_j = 2\sin\left(\frac{j\pi}{2n}\right)$ for $j = 1, \cdots, n-1$, $\mathsf{r}_n = 1$, and the same $p(j)$ as in Equation (206). It is clear that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+1}) \subset \mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_{2n+2})$$

Observation 3: We calculated that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_6) = C_{12}(z_1, z_2^{(2)}, z_3, z_4^{(2)}, 0^{(6)})$$

where $z_1 = 2^2 \cos\left(\frac{\pi}{12}\right) \cos\left(\frac{\pi}{4}\right), z_2 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{12}\right), z_3 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{4}\right), z_4 = 1$. We also calculated that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{f}_4) = C_{12}(z_1,\ z_2,\ z_3,\ z_4,\ 0^{(4)})$$

where $z_1 = 2^2 \cos\left(\frac{\pi}{12}\right) \cos\left(\frac{\pi}{4}\right), z_2 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{12}\right), z_3 = \zeta_{24} \cdot 2\cos\left(\frac{\pi}{4}\right), z_4 = 1$. It is clear that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{f}_4) \subset \mathrm{Spec}(\mathrm{ad}, \mathfrak{e}_6)$$

Observation 4: We calculated that:

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_8) = C_6(1^{(3)},\ \zeta_{12} \cdot \sqrt{3},\ 0^{(4)})$$

We also calculated that:

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}_2) = C_6(1,\ \zeta_{12} \cdot \sqrt{3},\ 0^{(2)})$$

It is clear that

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}_2) \subset \mathrm{Spec}(\mathrm{ad}, \mathfrak{so}_8)$$

In this chapter, we demonstrate that these are not coincidences but arise from the structural relations between different simple Lie algebras. Specifically, non-simply laced algebras can be obtained from simply-laced ones by taking an appropriate quotient by a diagram automorphism. This process is known as folding. By analyzing this folding procedure, we gain insight into why the adjoint spectra of cyclic elements exhibit such coincidences across distinct Lie algebras.

In Section 4.1, we recall the concept of folding. Then, in Section 4.2, we present a non-computational proof of the observed coincidences. The argument relies on two parts; one part is about the embedding of cyclic elements of the relevant simple Lie algebras into their folded counterparts, and the other part is about the symmetries of the Perron–Frobenius eigenvectors of the relevant Cartan matrices.

4.1 Introduction to Folding

In this section, we briefly review some basic facts about the folding of simple Lie algebras, following closely [6, Section 9.5].

Definition 4.1. *Let $\mathfrak{g}$ be the simple Lie algebra of rank r with Cartan matrix C. Let τ be a permutation of $\{1, \cdots, r\}$ such that for all i, j,*

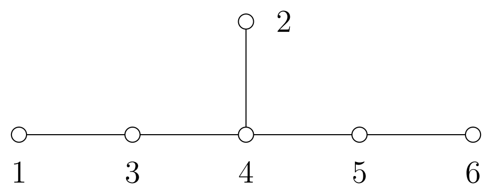
$$C_{\tau(i),\tau(j)} = C_{i,j} \tag{208}$$

Then, τ is called a Dynkin diagram automorphism of $\mathfrak{g}$.

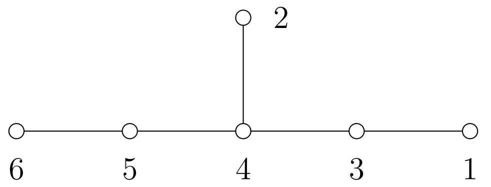
Example 4.1. The Cartan matrix of a simple Lie algebra of type $\mathfrak{e}_6$ is

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

with Dynkin diagram



If a permutation τ fixes indices 2 and 4, and swaps 1 and 6, 3 and 5, then the Dynkin diagram is changed to:



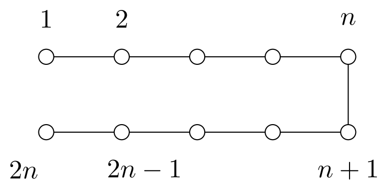
But the Cartan matrix is unchanged.

In [6, p. 165], all the non-trivial Dynkin diagram automorphisms have been classified into five types:

1. For type A_{2n} simple Lie algebra,

$$\tau(i) \;=\; 2n+1-i$$

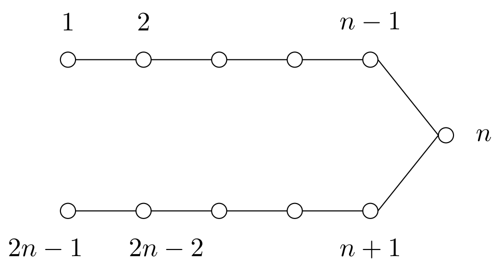
   for $i = 1, \cdots, 2n$. Graphically,



2. For type A_{2n-1} simple Lie algebra,

$$\tau(i) \;=\; 2n-i$$

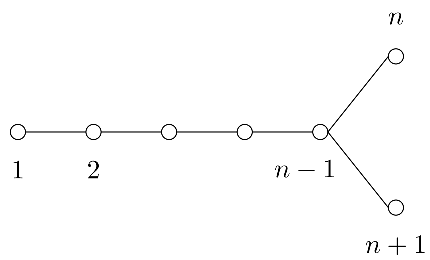
   for $i = 1, \cdots, 2n-1$. Graphically,



3. For type D_{n+1} simple Lie algebra,

$$\tau(i) = \begin{cases} i & \text{if } i = 1, \cdots, n-1 \\ n+1 & \text{if } i = n \\ n & \text{if } i = n+1 \end{cases}$$

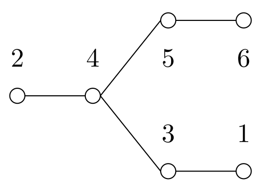
Graphically,



4. For type $\mathfrak{e}_6$ simple Lie algebra,

$$\tau(i) = \begin{cases} 6 & \text{if } i = 1 \\ i & \text{if } i = 2, 4 \\ 5 & \text{if } i = 3 \\ 3 & \text{if } i = 5 \\ 1 & \text{if } i = 6 \end{cases}$$

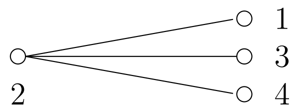
Graphically,



5. For type D_4 simple Lie algebra,

$$\tau(i) = \begin{cases} 3 & \text{if } i = 1 \\ 2 & \text{if } i = 2 \\ 4 & \text{if } i = 3 \\ 1 & \text{if } i = 4 \end{cases}$$

Graphically,



Now, we show how the permutations of the classified Dynkin diagram automorphisms give rise to a Lie algebra automorphism, and we show how a simply-laced simple Lie algebra is "quotiented" by automorphism to yield a new multiply-laced simple Lie algebra.

Given a non-trivial permutation τ, it gives rise to a Lie algebra automorphism of a free Lie algebra $F(\mathcal{S})$ generated by a set of chosen Chevalley generators $\mathcal{S} = \{e_i, f_i, h_i\}_{i=1,\cdots,r}$. This Lie algebra automorphism $\tilde{\tau}^{\#}$ acts on $\mathcal{S}$ as:

$$\tilde{\tau}^{\#}(e_i) = e_{\tau(i)},\ \tilde{\tau}^{\#}(f_i) = f_{\tau(i)},\ \tilde{\tau}^{\#}(h_i) = h_{\tau(i)}$$

By the universal property of the free Lie algebra, this permutation extends uniquely to a Lie algebra homomorphism on the Lie algebra $\mathfrak{g} = \dfrac{F(\mathcal{S})}{\langle R \rangle}$, where $\langle R \rangle$ is the ideal generated by Serre relations. Denote this Lie algebra homomorphism by $\tilde{\tau}$. Then, since

$$\tilde{\tau}[x, y] = [\tilde{\tau}(x), \tilde{\tau}(y)]$$

We obtain that $\tilde{\tau}$ respects all the Serre relations:

$$[h_i, h_j] = 0 \implies [h_{\tau(i)},\, h_{\tau(j)}] = 0$$

$$[h_i, e_j] = a_{ij}\, e_j \implies [h_{\tau(i)},\, e_{\tau(j)}] = C_{ij}\, e_{\tau(j)} = C_{\tau(i)\tau(j)}\, e_{\tau(j)}$$
$$[h_i, f_j] = -\,C_{ij}\, f_j \implies [h_{\tau(i)},\, f_{\tau(j)}] = -\,C_{ij}\, f_{\tau(j)} = -\,C_{\tau(i)\tau(j)}\, f_{\tau(j)}.$$
$$[e_i, f_j] = \delta_{ij}\, h_i \implies [e_{\tau(i)},\, f_{\tau(j)}] = \delta_{\tau(i)\tau(j)}\, h_{\tau(i)} = \delta_{ij}\, h_{\tau(i)}.$$
$$(\mathrm{ad}\, e_i)^{\,1-C_{ij}}(e_j) = 0 \quad (i \neq j) \implies (\mathrm{ad}\, e_{\tau(i)})^{\,1-C_{ij}}(e_{\tau(j)}) = (\mathrm{ad}\, e_{\tau(i)})^{\,1-C_{\tau(i)\tau(j)}}(e_{\tau(j)}) = 0$$
$$(\mathrm{ad}\, f_i)^{\,1-C_{ij}}(f_j) = 0 \quad (i \neq j) \implies (\mathrm{ad}\, f_{\tau(i)})^{\,1-C_{ij}}(f_{\tau(j)}) = (\mathrm{ad}\, f_{\tau(i)})^{\,1-C_{\tau(i)\tau(j)}}(f_{\tau(j)}) = 0$$

Permutations are invertible, so $\tilde{\tau}$ is in fact a Lie algebra automorphism.

Next, we describe the folding process of a simply-laced simple Lie algebra $\mathfrak{g}$ based on Lie algebra automorphism $\tilde{\tau}$. First, consider the fixed point subalgebra $\mathfrak{g}^{\tilde{\tau}}$ under $\tilde{\tau}$, defined as:

$$\mathfrak{g}^{\tilde{\tau}} = \{x \in \mathfrak{g} \mid \tilde{\tau}(x) = x\}$$

According to [6, Corollary 9.15], this subalgebra of $\mathfrak{g}$ turns out to be a simple Lie algebra as well. To determine its root system, $\tilde{\tau}$ acts on the simple roots of $\mathfrak{g}$ as

$$\tilde{\tau}(\alpha_i) = \alpha_{\tau(i)}$$

for $i = 1, \cdots, r$, and its actions on other roots are extended by linearity. For each $\tau-$orbit J, there is a corresponding $\tilde{\tau}-$orbit $\mathcal{O}_J$ of simple roots under the action of $\tilde{\tau}$. Define:

$$\beta_J = \frac{1}{|J|} \sum_{j \in J} \alpha_j \tag{209}$$

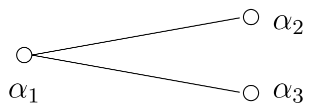
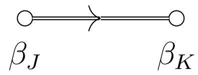
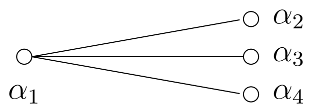
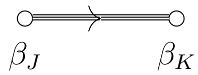
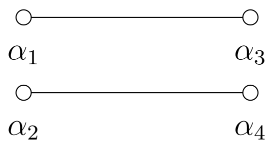
By the diagrams of the five non-trivial Dynkin diagram automorphisms, there are three possible $\tilde{\tau}-$orbit:

1. $\mathcal{O}_J = \{\alpha_j\}$ for $\tau(j) = j$, then $\beta_J = \alpha_j$.
2. $\mathcal{O}_J = \{\alpha_j, \alpha_k\}$ for $\tau(j) = k$ and $\tau(k) = j$. Then, $\beta_J = \dfrac{\alpha_j + \alpha_k}{2}$.
3. $\mathcal{O}_J = \{\alpha_j, \alpha_k, \alpha_l\}$ for $\tau(j) = k$, $\tau(k) = l$ and $\tau(l) = j$. Then, $\beta_J = \dfrac{\alpha_j + \alpha_k + \alpha_l}{3}$.

Note that simple roots in the same orbit must be orthogonal; otherwise, the quotient diagram will have a loop, from identifying two simple roots but having an edge between them. Loops are not allowed in Dynkin diagrams; thus, the diagram automorphism in Case 1 is not considered.

Let $\mathcal{O}_J$ and $\mathcal{O}_K$ be two distinct $\tilde{\tau}$-orbits, with corresponding β_J and β_K as defined in Equation (209). All possible positions of $\mathcal{O}_J$ and $\mathcal{O}_K$ on the Dynkin diagram have been classified in [6, Lemma 9.14]. We list them in Table 4, together with the corresponding diagrams and inner products of β_J and β_K. In the following table, simple roots are represented by dots in the diagram, placed on the same side for $\mathcal{O}_J$ and $\mathcal{O}_K$ when they belong to the same orbit.

Table 4: Different Types of β_J, β_K with Inner Products

	$\mathcal{O}_J$ $\quad$ $\mathcal{O}_K$	Diagram of β_J, β_K	Inner Product of β_J, β_K
(i)			$\beta_J \cdot \beta_K = -\frac{1}{2}\alpha_1^2$
(ii)			$\beta_J \cdot \beta_K = -\frac{1}{2}\alpha_1^2$
(iii)			$\beta_J \cdot \beta_K = -\frac{1}{2}\alpha_1^2$
(iv)			$\beta_J \cdot \beta_K = -\frac{1}{8}\alpha_1^2 - \frac{1}{8}\alpha_2^2$
(v)	If $\mathcal{O}_J, \mathcal{O}_K$ are disjoint, so are β_J, β_K, and $\beta_J \cdot \beta_K = 0$		

We briefly prove the inner product and the diagram of β_J, β_K for each case in the table. In case (i), $\beta_J = \alpha_1, \beta_K = \alpha_2$. Referring to the Dynkin diagram of α_1 and α_2, it is clear that:

$$\frac{2\beta_J \cdot \beta_K}{\beta_K^2} = \frac{2\alpha_1 \cdot \alpha_2}{\alpha_1^2} = -1 \qquad \Longrightarrow \beta_J \cdot \beta_K = -\frac{1}{2}\alpha_1^2$$

In this case, we also have $\beta_J^2 = \beta_K^2 = \alpha_1^2$, so the diagram of β_J and β_K takes the following form.

In case (ii), $\beta_J = \alpha_1, \beta_K = \dfrac{\alpha_2 + \alpha_3}{2}$. Referring to the Dynkin diagram of α_1, α_2, and α_3, we have:

$$\beta_J \cdot \beta_K = \alpha_1 \cdot \frac{\alpha_2 + \alpha_3}{2} = \frac{\alpha_1 \cdot \alpha_2}{2} + \frac{\alpha_1 \cdot \alpha_3}{2} = \frac{-\frac{1}{2}\alpha_1^2}{2} + \left(\frac{-\frac{1}{2}\alpha_1^2}{2}\right) = -\frac{1}{2}\alpha_1^2$$

In this case, we also have $\beta_J^2 = 2\beta_K^2 = \alpha_1^2$, so the diagram of β_J and β_K takes the following form.

In case (iii), $\beta_J = \alpha_1, \beta_K = \dfrac{\alpha_2 + \alpha_3 + \alpha_4}{3}$. Referring to the Dynkin diagram of α_1, α_2, α_3, and α_4, we have:

$$\begin{aligned}\beta_J \cdot \beta_K = \alpha_1 \cdot \frac{\alpha_2 + \alpha_3 + \alpha_4}{3} &= \frac{\alpha_1 \cdot \alpha_2}{3} + \frac{\alpha_1 \cdot \alpha_3}{3} + \frac{\alpha_1 \cdot \alpha_4}{3} \\ &= \frac{-\frac{1}{2}\alpha_1^2}{3} + \left(\frac{-\frac{1}{2}\alpha_1^2}{3}\right) + \left(\frac{-\frac{1}{2}\alpha_1^2}{3}\right) = -\frac{1}{2}\alpha_1^2\end{aligned}$$

In this case, we also have $\beta_J^2 = 3\beta_K^2 = \alpha_1^2$, so the diagram of β_J and β_K takes the following form.

In case (iv), $\beta_J = \dfrac{\alpha_1 + \alpha_2}{2}, \beta_K = \dfrac{\alpha_3 + \alpha_4}{2}$. Referring to the Dynkin diagrams of α_1 and α_2, α_3, and α_4, we have:

$$\begin{aligned}\beta_J \cdot \beta_K &= \frac{\alpha_1 + \alpha_2}{2} \cdot \frac{\alpha_3 + \alpha_4}{2} = \frac{\alpha_1 \cdot \alpha_3}{4} + \frac{\alpha_2 \cdot \alpha_4}{4} = \frac{-\frac{1}{2}\alpha_1^2}{4} + \left(\frac{-\frac{1}{2}\alpha_2^2}{4}\right) \\ &= -\frac{1}{8}\alpha_1^2 - \frac{1}{8}\alpha_2^2\end{aligned}$$

In this case, we also have $\beta_J^2 = \beta_K^2 = \dfrac{1}{2}\alpha_1^2$, so the diagram of β_J and β_K takes the following form. The proof of all cases in Table 4 is now complete.

By [6, Corollary 9.15], it turns out that the collection Π' of all β_J across $\tilde{\tau}$-orbits forms a fundamental root system, i.e., the set of simple roots of a simple Lie algebra $\mathfrak{g}'$. By [6, Theorem 9.19], by choosing Chevalley generators of $\mathfrak{g}'$ by

$$e_J = \sum_{j \in J} e_j, \; f_J = \sum_{j \in J} f_j, \; h_J = \sum_{j \in J} h_j \tag{210}$$

for all τ-orbits, we obtain

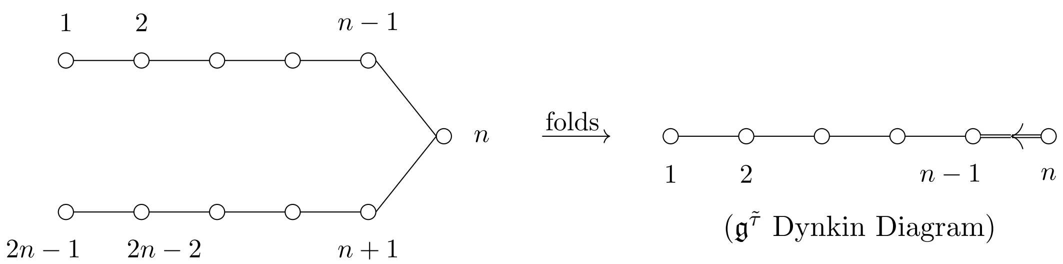
$$\mathfrak{g}^{\tilde{\tau}} \; \cong \; \mathfrak{g}'$$

Based on Table 4, we describe the fixed-point subalgebra $\mathfrak{g}^{\tau}$ from the folding process of each type simple Lie algebra in Equation (10).

4.1.1 $A_{2n-1} \to C_n$

As mentioned in Case 2, the Dynkin diagram automorphism τ and the corresponding folding of type A_{2n-1} simple Lie algebra is:

$$\tau(i) = 2n - i \quad \text{for } i = 1, \cdots, 2n-1$$



(Type A_{2n-1} Dynkin Diagram)

($\mathfrak{g}^{\tilde{\tau}}$ Dynkin Diagram)

The orbits of $\tilde{\tau}$ are: $\mathcal{O}_i = \{\alpha_i, \alpha_{2n-i}\}$ for $i = 1, \cdots, n$. Let:

$$\begin{aligned} \beta_i &= \frac{1}{2}(\alpha_i + \alpha_{2n-i}) \quad \text{for } i = 1, \cdots, n-1 \\ \beta_n &= \alpha_n \end{aligned}$$

Then, based on the proofs of cases (ii) and (iv) in Table 4, we have the inner products between β's as:

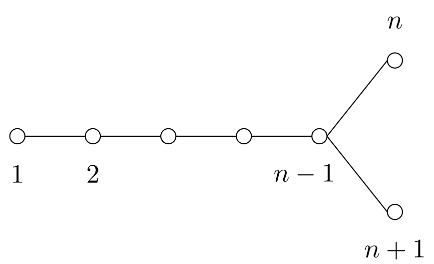
$$\beta_i \cdot \beta_{i+1} = -\frac{1}{4}\alpha_1^2 \quad \text{for } i = 1, \cdots, n-2$$
$$\beta_{n-1} \cdot \beta_n = -\frac{1}{2}\alpha_1^2$$
$$\beta_1^2 = \cdots = \beta_{n-1}^2 = \frac{1}{2}\beta_n^2 = \frac{1}{2}\alpha_n^2$$

By the inner products above, the Dynkin diagram of the folded algebra $\mathfrak{g}^{\tilde{\tau}}$ is of type C_n simple Lie algebra.

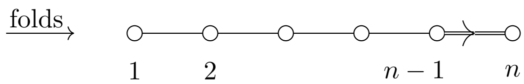
4.1.2 $D_{n+1} \to B_n$

As mentioned in Case 3, the Dynkin diagram automorphism τ and the corresponding folding of type D_{n+1} simple Lie algebra is:

$$\tau(i) = \begin{cases} i & \text{for } i = 1, \cdots, n-1 \\ n+1 & i = n \\ n & i = n+1 \end{cases}$$



(Type D_{n+1} Dynkin Diagram)



($\mathfrak{g}^{\tilde{\tau}}$ Dynkin Diagram)

The orbits of $\tilde{\tau}$ are $\mathcal{O}_i = \{\alpha_i\}$ for $i = 1, \cdots, n-1$, and $\mathcal{O}_n = \{\alpha_n, \alpha_{n+1}\}$. Let:

$$\beta_i = \alpha_i \quad \text{for } i = 1, \cdots, n-1$$
$$\beta_n = \frac{1}{2}(\alpha_n + \alpha_{n+1})$$

Then, based on the proofs of cases (i) and (ii) in Table 4, we have the inner products between β's as:

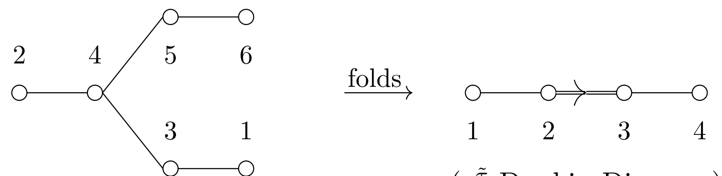
$$\beta_i \cdot \beta_{i+1} = -\frac{1}{2}\alpha_1^2 \quad \text{for } i = 1, \cdots, n-1$$
$$\beta_1^2 = \cdots = \beta_{n-1}^2 = 2\beta_n^2 = \alpha_1^2$$

By the inner products above, the Dynkin diagram of the folded algebra $\mathfrak{g}^{\tilde{\tau}}$ is of type C_n simple Lie algebra.

4.1.3 $\mathfrak{e}_6 \to \mathfrak{f}_4$

As mentioned in Case 4, the Dynkin diagram automorphism τ and the corresponding folding of type $\mathfrak{e}_6$ simple Lie algebra is:

$$\tau(i) = \begin{cases} 6 & \text{if } i = 1 \\ i & \text{if } i = 2, 4 \\ 5 & \text{if } i = 3 \\ 3 & \text{if } i = 5 \\ 1 & \text{if } i = 6 \end{cases}$$



(Type $\mathfrak{e}_6$ Dynkin Diagram) (${\mathfrak{g}}^{\tilde{\tau}}$ Dynkin Diagram)

Te orbits of $\tilde{\tau}$ are $\mathcal{O}_1 = \{\alpha_2\}, \mathcal{O}_2 = \{\alpha_4\}, \mathcal{O}_3 = \{\alpha_3, \alpha_5\}, \mathcal{O}_4 = \{\alpha_1, \alpha_6\}$. Let:

$$\beta_1 = \alpha_2$$

$$\beta_2 = \alpha_4$$
$$\beta_3 = \frac{1}{2}(\alpha_3 + \alpha_5)$$
$$\beta_4 = \frac{1}{2}(\alpha_1 + \alpha_6)$$

Then, based on the proofs of cases (i), (ii), and (iv) in Table 4, we have inner products between β's as:

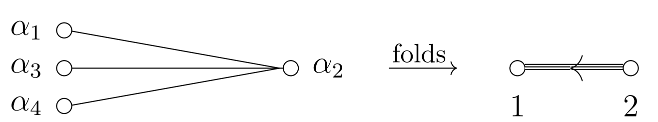
$$\beta_1 \cdot \beta_2 = \beta_2 \cdot \beta_3 = 2\beta_3 \cdot \beta_4 = -\frac{1}{2}\alpha_1^2$$
$$\beta_1^2 = \beta_2^2 = 2\beta_3^2 = 2\beta_4^2 = \alpha_1^2$$

By the inner products above, the Dynkin diagram of the folded algebra $\mathfrak{g}^{\tilde{\tau}}$ is of type $\mathfrak{f}_4$ simple Lie algebra.

4.1.4 $D_4 \to \mathfrak{g}_2$

As mentioned in Case 5, the Dynkin diagram automorphism τ and the corresponding folding of type D_4 simple Lie algebra is: As shown in the following diagrams, the Dynkin diagram automorphism τ of type D_4 simple Lie algebra is:

$$\tau(i) = \begin{cases} 3 & \text{for } i = 1 \\ 2 & i = 2 \\ 4 & i = 3 \\ 1 & i = 4 \end{cases}$$



(Type D_4 Dynkin Diagram) (${\mathfrak{g}}^{\tilde{\tau}}$ Dynkin Diagram)

The orbits of $\tilde{\tau}$ are $\mathcal{O}_1 = \{\alpha_1, \alpha_3, \alpha_4\}\}, \{\mathcal{O}_2 = \{\alpha_2\}$. Let:

$$\beta_1 = \frac{1}{3}(\alpha_1 + \alpha_3 + \alpha_4)$$
$$\beta_2 = \alpha_2$$

Then, by the proof of case (iii) in Table 4, we have the inner products between β's as:

$$\beta_1 \cdot \beta_2 = -\frac{1}{2}\alpha_1^2; \quad 3\beta_1^2 = \beta_2^2 = \alpha_1^2$$

By the inner products above, the Dynkin diagram of the folded algebra $\mathfrak{g}^{\tilde{\tau}}$ is of type $\mathfrak{g}_2$ simple Lie algebra.

4.2 Proof of Theorem 1.6

For a simply-laced Lie algebra $\mathfrak{g}$ and an associated automorphism $\tilde{\tau}$ as before, recall in Theorem 1.6, we obtain equation

$$\mathrm{Spec}(\mathrm{ad}, \mathfrak{g}^{\tilde{\tau}}) \subseteq \mathrm{Spec}(\mathrm{ad}, \mathfrak{g}) \tag{211}$$

It follows from comparing of the numerical results of Theorem 1.4. In this section, we provide a more conceptual proof of Equation (211). The key ingredient in the proof is the following proposition.

Proposition 3. *For $\mathfrak{g}$ and $\mathfrak{g}^{\tilde{\tau}}$ as before, let r and s be the rank of $\mathfrak{g}$ and $\mathfrak{g}^{\tilde{\tau}}$, respectively. Choose Λ to be a cyclic element in $\mathfrak{g}$ that can be embedded in $\mathfrak{g}^{\tilde{\tau}}$. Consider the following Cartan subalgebra*

$$\mathfrak{h} = \ker_{\mathfrak{g}} \mathrm{ad}\ \Lambda$$

and the Cartan subalgebra of $\mathfrak{g}^{\tilde{\tau}}$

$$\mathfrak{h}^{\tilde{\tau}} = \ker_{\mathfrak{g}^{\tilde{\tau}}} \mathrm{ad}\ \Lambda$$

Then, there exist a set of simple roots $\{\alpha_1, \cdots, \alpha_r\}$ for $\mathfrak{g}$ and a set of simple roots $\{\beta_1, \cdots, \beta_s\}$ for $\mathfrak{g}^{\tilde{\tau}}$ such that the following relation holds:

$$\{\beta_1(\Lambda), \cdots, \beta_s(\Lambda)\} \subseteq \{\alpha_1(\Lambda), \cdots, \alpha_r(\Lambda)\}$$

The proof of this proposition has two parts. In Section 4.2.1, we show that there exists a cyclic element $\Lambda \in \mathfrak{g}$ invariant under $\tilde{\tau}$, and that this Λ is also a cyclic element in $\mathfrak{g}^{\tilde{\tau}}$. We call this phenomenon the embedding of the cyclic element. In Section 4.2.2, we finish the proof by using the symmetries of the Perron–Frobenius eigenvector of the Cartan matrix. Finally, with this proposition and Theorem 1.5, we establish Equation (211).

4.2.1 Embedding of Cyclic Elements

Recall from Definition 1.1, a cyclic element is defined as

$$\Lambda = \sum_{i=1}^{r} e_i + F \tag{212}$$

where F is a non-zero element in the lowest root space, and $e_i = e_{\alpha_i} \in \mathbb{C}E_{\alpha_i}$ are from the chosen Chevalley generators.

To show that a cyclic element Λ is invariant under $\tilde{\tau}$, we first show that $\sum e_i$ is invariant under $\tilde{\tau}$. From the action of $\tilde{\tau}$ on the Chevalley generators, it follows that:

$$\{\tilde{\tau}(e_1), \cdots, \tilde{\tau}(e_r)\} \;=\; \{e_{\tau(1)}, \cdots, e_{\tau(r)}\} \;=\; \{e_1, \cdots, e_r\}$$

Therefore,

$$\tilde{\tau}\left(\sum_{i=1}^{r} e_i\right) \;=\; \sum_{i=1}^{r} e_{\tau(i)} \;=\; \sum_{i=1}^{r} e_i \tag{213}$$

Now, we prove that a non-zero lowest root space vector F is also invariant under $\tilde{\tau}$. Equivalently, we prove that a non-zero highest root space vector for each type of folding simple Lie algebra is invariant under the corresponding $\tilde{\tau}$.

For the type A_{2n-1}, it is known that the highest root is $\tilde{\alpha} = \alpha_1 + \cdots + \alpha_{2n-1}$. Let e_i be a non-zero element in $\mathfrak{g}_{\alpha_i}$. Define

$$X_0 := e_n$$
$$X_k := \big[\,[\,e_{n-k},\, X_{k-1}\,],\, e_{n+k}\,\big]$$

for $k = 1, 2, \cdots, n-1$. We claim $X = X_{n-1}$ is a non-zero element in $\mathfrak{g}_{\tilde{\alpha}}$, and X is invariant under $\tilde{\tau}$. Let us demonstrate this via a short example.

Example 4.2. When $n = 2$, $\mathfrak{g}$ is of type A_3, we construct:

$$X_0 = e_2, \quad X_1 = [[e_1, e_2], e_3]$$

Recall a well-known Lie theory lemma:

Lemma 4.1. *Suppose $\mathfrak{g}$ is a simple Lie algebra. Let α, β be two roots of $\mathfrak{g}$ such that $\beta \neq \pm\alpha$ and $\alpha + \beta$ is also a root, then:*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

If $\alpha + \beta$ is not a root, then:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$$

By this lemma, since $\alpha_1 + \alpha_2$ is a root of $\mathfrak{g}$, $[e_1, e_2] \in \mathfrak{g}_{\alpha_1+\alpha_2}$ is non-zero. Similarly, it follows that $X = [[e_1, e_2], e_3] \in \mathfrak{g}_{\tilde{\alpha}}$ is also non-zero. Next, referring to the action of $\tilde{\tau}$, we have:

$$\tilde{\tau}(X_1) = [[e_3, e_2], e_1]$$

Using the Jacobi identity and the fact that $\alpha_1 + \alpha_3$ is not a root of $\mathfrak{g}$, we have

$$X_1 = [e_1, [e_2, e_3]] = [[e_3, e_2], e_1] = \tilde{\tau}(X_1)$$

Hence, X is a non-zero element in $\mathfrak{g}_{\tilde{\alpha}}$ and is invariant under $\tilde{\tau}$.

Now we prove the claim for any n. At first, by Lemma 4.1, for each k in $\{1, \cdots, n-1\}$,

$$X_k \in \mathfrak{g}_{\alpha_{n-k}+\alpha_{n-k+1}+\cdots+\alpha_{n+k}}$$

since an adjacent root space vector is included for each step, and the sum of any consecutive adjacent simple roots is a root of $\mathfrak{g}$. Thus, for any k, X_k does not vanish, so $X \in \mathfrak{g}_{\tilde{\alpha}}$ is non-zero.

Next, we prove the invariance of X by induction on the step sequence k. For the base case when $k = 0$ and $X_0 = e_n$, since $\tau(n) = n$, it is clear that X_0 is fixed under $\tilde{\tau}$. Then, assume $\tilde{\tau}(X_{k-1}) = X_{k-1}$. By the Jacobi identity and the fact that $[e_{n-k}, e_{n+k}] = 0$ since being nonadjacent on the Dynkin diagram, we have

$$\tilde{\tau}(X_k) = \big[\,[e_{n+k}, X_{k-1}], e_{n-k}\,\big] = \big[\,e_{n-k},\, [X_{k-1}, e_{n+k}]\,\big] = \big[\,[e_{n-k}, X_{k-1}], e_{n+k}\,\big] = X_k$$

Hence, $\tilde{\tau}(X_k) = X_k$ for all k, it proves that $X \in \mathfrak{g}^{\tilde{\tau}}$.

For the type D_{n+1}, it is known that the highest root is

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}$$

Pick the highest root space vector to be:

$$X = \big[\cdots [[[\,[\cdots [[[e_1, e_2], e_3] \cdots, e_{n-1}], e_n], e_{n+1}], e_{n-1}], e_{n-2}], \ldots, e_2\big]$$

Same reason as in the previous type, an adjacent root space vector is included for each layer of bracket, so $X \in \mathfrak{g}_{\tilde{\alpha}}$ non-zero.

To prove the invariance of X, since τ only changes between n and $n+1$, we only need to show for any root space vector Y,

$$[[Y, e_n], e_{n+1}] = [[Y, e_{n+1}], e_n]$$

By the Jacobi identity and $[e_n, e_{n+1}] = 0$ since being nonadjacent on the Dynkin diagram, the equality holds. It follows that $\tilde{\tau}(X) = X$, thus $X \in \mathfrak{g}^{\tilde{\tau}}$.

For the type $\mathfrak{e}_6$, the highest root is

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

Pick the highest root space vector to be:

$$X = [[[[[[[[[e_4, e_3], e_5], e_1], e_6], e_2], e_4], e_3], e_5], e_4], e_2]$$

Referring to the successor diagram in Figure 15, $X \in \mathfrak{g}_{\tilde{\alpha}}$ is non-zero.

To prove the invariance of X, since τ only swaps pairs $3, 5$, and $1, 6$, we need to show for any root space vector Y,

$$[[Y, e_3], e_5] = [[Y, e_5], e_3]$$
$$[[Y, e_1], e_6] = [[Y, e_6], e_1]$$

By the Jacobi identity, these equalities hold by a similar argument to the previous type, given that $[e_1, e_6] = [e_3, e_5] = 0$. Hence, $\tilde{\tau}(X) = X$, so $X \in \mathfrak{g}^{\tilde{\tau}}$.

For the type D_4, the highest root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. Pick the highest root space vector to be:

$$X = [[[[e_1, e_2], e_3], e_4], e_2]$$

Same argument as in the type D_{n+1}, $X \in \mathfrak{g}_{\tilde{\alpha}}$ is non-zero. To prove the invariance of X, since τ cycles the triplet $1,3,4$, we need to show

$$[[[[e_1,e_2],e_3],e_4],e_2] \;=\; \tilde{\tau}([[[[e_1,e_2],e_3],e_4],e_2]) \;=\; [[[[e_3,e_2],e_4],e_1],e_2]$$

By the Jacobi identity and the facts that $[e_1,e_3]=[e_1,e_4]=0$:

$$[[e_1,e_2],e_3]=[e_1,[e_2,e_3]] \implies [\,[e_1,[e_2,e_3]]\,,e_4]=[\,e_1,\,[\,[e_2,e_3],e_4]\,]=[[[e_3,e_2],e_4],e_1]$$

Hence, it follows that $\tilde{\tau}(X)=X$, so $X\in\mathfrak{g}^{\tilde{\tau}}$.

Since the highest root space is one-dimensional, the above arguments imply that $\mathfrak{g}_{\tilde{\alpha}}=\mathbb{C}X$. By the linearity of $\tilde{\tau}$, it follows that any non-zero highest root space vector is invariant under $\tilde{\tau}$. Equivalently, so is any non-zero lowest root space vector. The proof of the invariance of Λ under $\tilde{\tau}$ is now complete.

Now we show that the invariant cyclic element $\Lambda=\sum e_i+F$ in $\mathfrak{g}$ is still a cyclic element in $\mathfrak{g}^{\tilde{\tau}}$. By the construction of Chevalley generators in Equation (210), it is clear that:

$$\sum_{i=1}^{r} e_i=\sum_{\text{all }\tau-\text{orbits }J} e_J$$

The remaining part is to show that given $F\in\mathfrak{g}_{-\tilde{\alpha}}$ non-zero,

$$F\in\mathfrak{g}^{\tilde{\tau}}_{-\tilde{\beta}}$$

where $\tilde{\alpha},\tilde{\beta}$ are the highest roots of $\mathfrak{g}$ and $\mathfrak{g}^{\tilde{\tau}}$, respectively. Equivalently, we prove this for a non-zero highest root space vector $X\in\mathfrak{g}_{\tilde{\alpha}}$. For any $h_J\in\mathfrak{h}^{\tilde{\tau}}$, the Cartan subalgebra of $\mathfrak{g}^{\tilde{\tau}}$, we have:

$$\begin{aligned}[h_J,\ X]&=\left[\sum_{j\in J}h_j,\ X\right]\\ &=\sum_{j\in J}[h_j,\ X]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j \in J} \tilde{\alpha}(h_j) X \\
&= \tilde{\alpha}(\sum_{j \in J} h_j) X \\
&= \tilde{\beta}(h_J) X
\end{aligned}
$$

The very last step follows because for any type of folding,

$$\tilde{\alpha} = \tilde{\beta}$$

To verify this, for type C_n, its highest root is

$$\tilde{\beta} = 2\beta_1 + \cdots + 2\beta_{n-1} + \beta_n = \alpha_1 + \cdots + \alpha_{2n-1} = \tilde{\alpha}$$

where $\tilde{\alpha}$ is the highest root of type A_{2n-1}.

For type B_n, its highest root is

$$\tilde{\beta} = \beta_1 + 2\beta_2 + \cdots + 2\beta_n = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1} = \tilde{\alpha}$$

where $\tilde{\alpha}$ is the highest root of type D_{n+1}.

For type $\mathfrak{f}_4$, its highest root is

$$\tilde{\beta} = 2\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \tilde{\alpha}$$

where $\tilde{\alpha}$ is the highest root of type $\mathfrak{e}_6$.

For type $\mathfrak{g}_2$, its highest root is

$$\tilde{\beta} = 3\beta_1 + 2\beta_2 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = \tilde{\alpha}$$

where $\tilde{\alpha}$ is the highest root of type D_4. Therefore, X is also a non-zero element in the highest root space of $\mathfrak{g}^{\tilde{\tau}}$. Equivalently, F is also a non-zero element in the lowest root space of $\mathfrak{g}^{\tilde{\tau}}$. Hence, Λ is a cyclic element in $\mathfrak{g}^{\tilde{\tau}}$, the proof of its embedding is now complete.

4.2.2 Proof of Proposition 3

Using the embedding of the cyclic element Λ, we now finish the proof of Proposition 3. The key ingredient in the proof is to show

$$\tilde{\tau}(\alpha_i)(\Lambda) \;=\; \alpha_i(\Lambda) \tag{214}$$

where α_i's are simple roots of the folding simple Lie algebra $\mathfrak{g}$ for $i = 1, \cdots, r$. Let us go over a brief example to present this result.

Example 4.3. For example, consider $\mathfrak{g}$ of type A_{2n-1} simple Lie algebra with the Coxeter number $2n$. Observe that for any $k \in \mathbb{Q}$, we have

$$\zeta_{2n}^{k} \;=\; -\zeta_{2n}^{k+n} \;=\; -\zeta_{2n}^{k-n}$$

Then, it follows that

$$\zeta_{2n}^{\frac{j}{2}} - \zeta_{2n}^{-\frac{j}{2}} \;=\; \zeta_{2n}^{\frac{2n-j}{2}} - \zeta_{2n}^{-\frac{2n-j}{2}}$$

for $j = 1, \cdots, 2n-1$. By Equation (119), we obtain

$$\alpha_j(\Lambda) \;=\; \alpha_{2n-j}(\Lambda)$$

for $j = 1, \cdots, 2n-1$. This shows $\tilde{\tau}(\alpha_j)(\Lambda) \;=\; \alpha_j(\Lambda)$ for any $j = 1, \cdots, 2n-1$.

Now we start to prove Equation (214) in full generality. By Equation (5), the pairings between simple roots and a cyclic element are determined by two factors: components of a left Perron-Frobenius eigenvector, x_j for $j = 1, \cdots, r$, and the color-value of the simple roots, $p(j)$ for $j = 1, \cdots, r$. In the following proof, we first show that the relevant left

Perron-Frobenius eigenvectors have the symmetry such that

$$x_j \;=\; x_{\tau(j)} \tag{215}$$

for $j = 1, \cdots, r$. Then, we show that the bi-coloring allows us to assign colors to the simple roots such that

$$p(j) \;=\; p(\tau(j)) \tag{216}$$

for $j = 1, \cdots, r$.

To prove Equation (215), we first derive an equivalent equation of Equation (208) in Definition 4.1. Let the associated permutation matrix P of τ be the matrix such that

$$P_{i,j} \;=\; \delta_{i,\tau(j)}$$

for $i, j \in \{1, \cdots, r\}$. For example, for type D_4, $\tau = (1\ 3\ 4)$, hence the associated permutation matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It is clear that the permutation matrix of τ^{-1} is $P^{-1} = P^{\intercal}$. Then, we show that Equation (208) holds if and only if:

$$P^{-1}CP \;=\; C$$

To prove this, we expand the entries of $P^{-1}CP$:

$$\begin{aligned}(P^{-1}CP)_{ij} &= \sum_{k}^{r}\sum_{l}^{r} P^{-1}_{ik}\; C_{kl}\; P_{lj} \\ &= \sum_{k}^{r}\sum_{l}^{r} \delta_{i,\tau^{-1}(k)}\; C_{kl}\; \delta_{l,\tau(j)}\end{aligned}$$

$$= \sum_{k}^{r} \delta_{i,\tau^{-1}(k)} \left(\sum_{l}^{r} C_{kl}\ \delta_{l,\tau(j)} \right)$$
$$= \sum_{k}^{r} \delta_{i,\tau^{-1}(k)}\ C_{k,\tau(j)}$$
$$= C_{\tau(i),\ \tau(j)}$$

Therefore, Equation (208) is equivalent to $P^{-1}CP = C$. Based on this result, the Cartan matrix of $\mathfrak{g}$ satisfies the equation

$$P^{-1}CP \ = \ C$$

with respect to the corresponding permutation matrix P. Then, we relate this equation to the Perron-Frobenius eigenvector of C. Recall the Perron-Frobenius Theorem of non-negative matrices:

Theorem 4.1 (Perron-Frobenius Theorem)**.** *Let A be an irreducible non-negative square matrix with the maximal eigenvalue λ. Then, the following statements hold.*

1. *The eigenvalue λ is a positive real number. It is called Perron–Frobenius eigenvalue.*
2. *The Perron–Frobenius eigenvalue is simple. Both right and left eigenspaces associated with λ are one-dimensional.*
3. *A has both right and left eigenvectors with eigenvalue λ and whose components are all positive. Moreover, the only eigenvectors whose components are all positive are those associated with the eigenvalue λ.*

We apply this Theorem to the matrix $2I - C$. One can see that Cartan matrices are irreducible, meaning there does not exist a permutation matrix Q such that $Q^{-1}CQ$ is triangular. It follows that the matrix $2I - C$ is also irreducible. Let $\vec{\boldsymbol{x}} = (x_1, \cdots, x_r)^\intercal$ be a left eigenvector of $2I - C$ with respect to its maximal eigenvalue λ. Then, $\vec{\boldsymbol{x}}$ is also a left eigenvector of the minimal eigenvalue of C. By the Perron-Frobenius Theorem, we have

1. $\vec{\boldsymbol{x}}$ is simple, i.e. the eigenspace of λ is one-dimensional;

2. We can further assume that all the components of $\vec{\boldsymbol{x}}$ are positive.

Since $P^{-1}CP = C$, it follows that:

$$(\vec{\boldsymbol{x}}P) \cdot C \;=\; (\vec{\boldsymbol{x}}P) \cdot P^{-1}CP \;=\; (\vec{\boldsymbol{x}} \cdot C)P \;=\; \lambda(\vec{\boldsymbol{x}}P)$$

Thus, $\vec{\boldsymbol{x}}P$ is also a left eigenvector of C with respect to λ. By Statement (1), it follows that

$$\vec{\boldsymbol{x}}P \;=\; c \cdot \vec{\boldsymbol{x}}$$

for some non-zero scalar c. By Statement (2) and the fact that P is a non-negative matrix, it follows that $c = 1$. Notice that

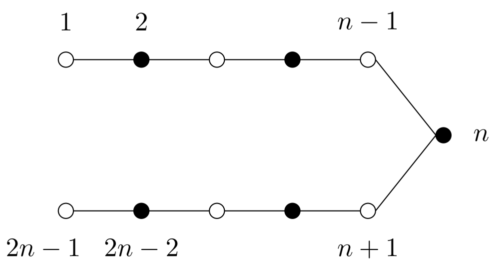
$$\vec{\boldsymbol{x}}P \;=\; (x_{\tau(j)})_{j=1,\cdots,r}$$

Therefore,

$$(x_{\tau(j)})_{j=1,\cdots,r} \;=\; \vec{\boldsymbol{x}}$$

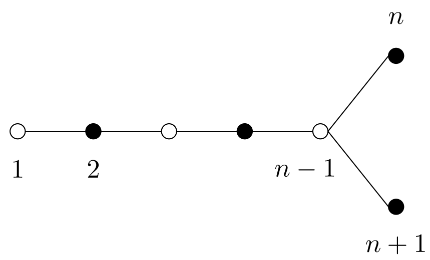
Since the eigenspace is one-dimensional, by linearity, this result holds for any Perron-Frobenius eigenvector of C. The proof of Equation (215) is now complete.

Now we prove Equation (216) case by case. For type A_{2n-1}, we can bi-color its Dynkin diagram as follows:



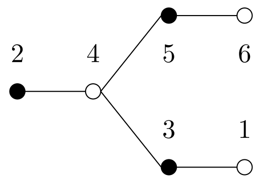
It is clear that $p(i) = p(\tau(i))$ for all $i = 1, \cdots, 2n-1$.

For type D_{n+1}, we can bi-color its Dynkin diagram as follows:



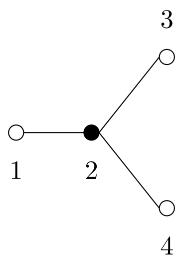
It is clear that $p(i) = p(\tau(i))$ for all $i = 1, \cdots, n+1$.

For type $\mathfrak{e}_6$, we can bi-color its Dynkin diagram as follows:



It is clear that $p(i) = p(\tau(i))$ for all $i = 1, \cdots, 6$.

For type D_4, we can bi-color its Dynkin diagram as follows:



It is clear that $p(i) = p(\tau(i))$ for all $i = 1, \cdots, 4$. The proof of Equation (216) is now complete.

Equations (215) and (216) imply Equation (214). Recall from Equation (209), for a $\tau-$orbit J, we have:

$$\beta_J(\Lambda) \;=\; \frac{1}{|J|} \sum_{j \in J} \alpha_j(\Lambda) \;=\; \frac{1}{|J|} \cdot |J| \cdot \alpha_j(\Lambda) \;=\; \alpha_j(\Lambda)$$

Therefore,

$$\{\beta_J(\Lambda)\}_{J=1\cdots,s} \;\subseteq\; \{\alpha_i(\Lambda)\}_{i=1\cdots,r}$$

where s and r are the ranks of $\mathfrak{g}^{\tilde{\tau}}$ and $\mathfrak{g}$, respectively. This proves Proposition 3. Finally, combining Proposition 3 with Theorem 1.5, we obtain Equation (211).

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