

Investigations of Barnette's Graphs

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ABSTRACT. In this paper, we study Hamiltonicity of planar graphs, focusing on the conjecture of David Barnette that every graph that is 3-regular, 3-connected, planar, and bipartite has a Hamiltonian cycle. We review work that has been done on the conjecture, and study properties of the Barnette's graphs and the complexity of the Hamiltonian cycle problem on these graphs. Finally, using graph generation and visualization software, we study the graphs' diameters, and conjecture an upper bound for all such graphs.

Contents

Chapter 1. Background And Motivation	1
1.1. Basic Definitions	1
1.2. Graphs and Polytopes	3
1.3. Complexity Theory and Hamiltonian Cycles	3
1.4. Why Study Graphs of Polytopes?	6
Chapter 2. Studying Barnette Graphs	9
2.1. Holton-Manvel-McKay Construction	9
2.2. NP-Completeness and Barnette's Graphs	11
2.3. Diameters and Eigenvalues of Barnette Graphs	13
Appendix A. Tutte Embeddings of Barnette Graphs From Eight to Sixteen Vertices	17
Bibliography	33

CHAPTER 1

Background And Motivation

1.1. Basic Definitions

We begin with some basic definitions.

A set of points X in \mathbb{R}^n is **convex** if for any two points \mathbf{a} and \mathbf{b} , the line segment joining them, $\{\lambda\mathbf{a} + (1-\lambda)\mathbf{b} : 0 \leq \lambda \leq 1\}$ is in X . Figure 6 gives an example of convex and non-convex sets in the plane. The set on the left is not convex because the line segment shown in red between the two points lies partially outside the set. It is easy to see that the hexagon on the right is convex.

The **convex hull** of a set is the intersection of all convex sets containing the set. For intuition in the plane, consider a set of points and put a rubber band around the points. The set bounded by the rubber band and its interior is the convex hull of the set of points.

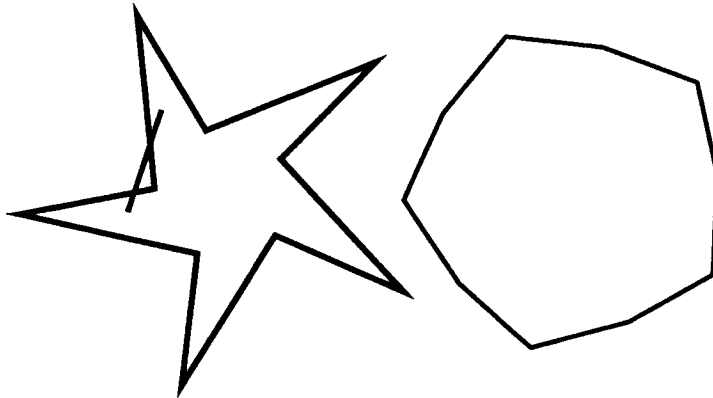


Figure 1: A non-convex (left) and convex set

A **hyperplane** in \mathbb{R}^n generalizes a line in \mathbb{R}^2 and a plane in \mathbb{R}^3 . It is the set $\{\mathbf{x} : a_1x_1 + a_2x_2 \dots a_nx_n = b\}$ where a_i and b are scalars. Replacing $=$ with \leq , we get a **halfspace**, a partition of \mathbb{R}^n into points above or below a hyperplane. The intersection of finitely many halfspaces defines a polyhedron. If a polyhedron is bounded, it is a **polytope**.

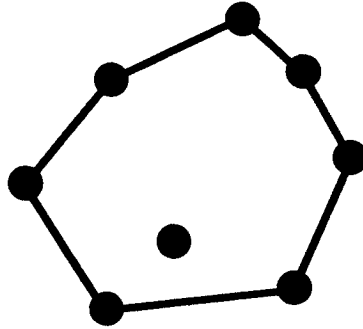


Figure 2: A Set of Points and Its Convex Hull

Writing all halfspaces with \leq (by multiplying through by -1 if necessary), we can write the halfspaces in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \end{aligned}$$

The polytope can then be written in the matrix form $\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leq \mathbf{b}$.

A d -dimensional polytope is simple if each vertex is adjacent to exactly d edges.

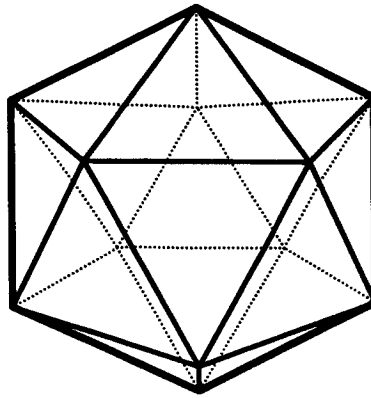


Figure 3: The icosahedron, a convex polytope

1.2. Graphs and Polytopes

A graph (V, E) is a set of vertices edges connecting the vertices. A graph is **simple** if there is at most one edge connecting each any pair of vertices and no self-loops, and **planar** if it can be drawn in the plane without any intersecting edges.

If a graph is planar, its planar embedding defines faces: regions of the graph bounded by sets of edges.

A graph is **connected** if there is a path along edges between any pair of vertices, and **k-connected** if at least k vertices need to be removed from the graph for it to become disconnected.

A graph is **k-regular** if each vertex is connected to k other vertices.

Taking the edges and vertices of a polytope as the edges of the graph, we can associate the **1-skeleton graph** to a polytope. Note that the notion of a simple graph and a simple polytope are distinct. Given a d -dimensional simple polytope P and its associated graph G , G is k -regular because P is simple, and G is simple because it has no self-loops or double edges. Conversely, the graph of a 3-octahedron is simple for the same reasons, but a 3-octahedron has vertices of degree 4.

A theorem of Steinitz [Ste22] says that a graph is the graph of a 3-polytope if and only if it is simple, 3-connected and planar. Many polytopes, such as a rectangular prism and a 3-cube have the same graphs, but different geometries. This thesis is primarily concerned with combinatorial properties, that is, properties of the graphs of polytopes.

A **Hamiltonian cycle** on a graph is a path along the edges that passes through each vertex exactly once and returns to its starting vertex. It is easy to find a Hamiltonian cycle on the graph in figure 4.

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into subsets $V_1 \cup V_2 = V$ such that any edge that has an end point in V_1 has its other end point in V_2 .

LEMMA 1. *A planar graph is bipartite if and only if all of its faces are of even degree.*

Proof: It is known that a graph is bipartite if and only if it contains no even cycles. Suppose a graph is bipartite. Then it cannot have a face of odd degree, because the face forms an odd cycle. Conversely, suppose that a graph is planar and has only even faces. Then a cycle on the graph has the set of faces $\{f_1, f_2, \dots, f_k\}$ in its interior, with a set X of edges strictly in the interior. Then

$$|C| = \sum_{i=1}^k |f_k| - 2|X|$$

. Since $|f_k|$ and $2|X|$ are both even, $|C|$ is even. \square

1.3. Complexity Theory and Hamiltonian Cycles

In addition to finding infinite families of graphs that are Hamiltonian, much research has been done in the problem of deciding whether a given graph is Hamiltonian. The

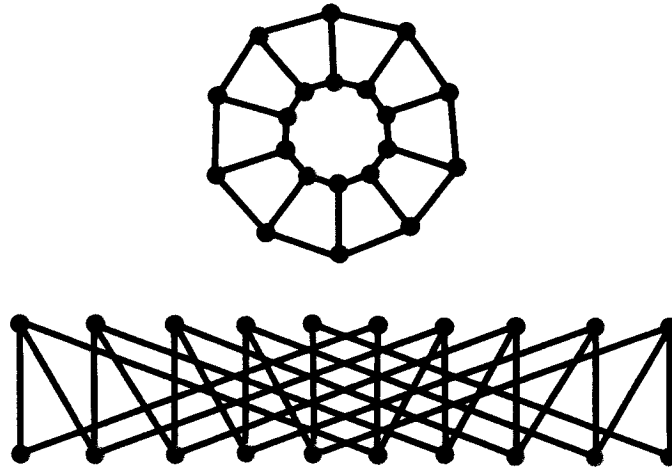


Figure 4: The graph of a decahedral prism, showing the two coloring and bipartition of the graph

Hamiltonian cycle decision problem, i.e., “given a graph G , is G Hamiltonian?” is an interesting problem in computational complexity, as it is computationally difficult for a general graph but may be easier for Barnette graphs, that graphs that we study and introduce in the next section.

Computational complexity theory categorizes decision problems by asymptotic running time of algorithms used to solve them. Problems with algorithms that solve in polynomial time in the bit size of the input are said to be in P. Problems in P roughly correspond with those practically solvable on a computer in a general case.

An example of a graph theory problem in P is the path problem: “given a graph G and two vertices s and t , is there a path between s and t ?” For the Hamiltonian path (and the more difficult Hamiltonian cycle) problem, there is a naive algorithm: for each pair of vertices s and $t \in G$, find the path between them and check whether it traverses every vertex. This algorithm is exponential in the number of vertices. Such a brute-force enumeration can be used to solve many graph problems. For a graph in general, it is not known whether there is a polynomial time algorithm for finding a Hamiltonian cycle. Nevertheless, the problem is verifiable in polynomial time.

A problem is **verifiable in polynomial time** if, given a solution to the problem, the solution can be checked for correctness in polynomial time. A given cycle can be checked for Hamiltonicity in polynomial time by simply seeing if it traverses every vertex. Problems that can be verified in polynomial time are in the computational class NP. Although most

researchers believe that the classes P and NP are distinct, there is no problem in NP that has been proved to not be in P.

There are problems that are known to the “most difficult” problems in NP. These problems are **NP-complete**. A problem H is NP-complete if it is in NP and every problem in NP can be reduced to H in polynomial time. This means that if a polynomial algorithm for an NP-complete problem is found, all problems in NP can be solved in polynomial time.

Cook [Coo71] found the first NP-Complete problem, the Boolean satisfiability problem, or SAT.

SAT is the following: given a set of Boolean variables x_1, \dots, x_n and a sentence using \wedge, \vee, \neg and parenthesis, is there an assignment of true or false to each of the variables that makes the sentence true. While remaining NP-complete, the problem may be reduced to one in the form $(p_{1,i} \vee p_{1,j} \vee p_{1,k}) \wedge (p_{2,i} \vee p_{2,j} \vee p_{2,k}) \wedge \dots \wedge (p_{n,i} \vee p_{n,j} \vee p_{n,k})$ where each $p_{i,j}$ is some x_i or $\neg x_i$.

1.3.1. Tutte's Fragment and Hamiltonian Graphs. We are motivated by the problem of finding interesting families of Hamiltonian graphs. In 1886, P.G. Tait conjectured that all planar, 3-connected graphs have Hamiltonian cycles. In 1946, W.T. Tutte found a counterexample to Tait's conjecture.

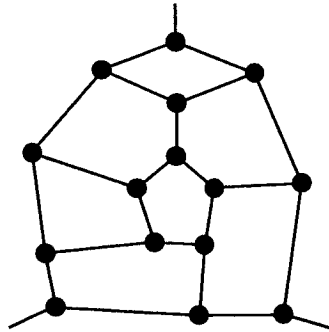


Figure 5: Tutte's Fragment

Tutte's fragment is the key to the counterexample to Tait's conjecture. It is 3-connected (except on the three edges in the corners) and planar. It is, as we will later explain, an "exclusive-or graph": any Hamiltonian path beginning at the bottom right or left-hand vertices must exit through the top vertex. We'll see later that if a graph with this property that is a Barnette graph exists, then Barnette's conjecture is not true and the Hamiltonian cycle problem is NP-complete for Barnette graphs. Similar graphs are used in the proofs of NP-completeness of the Hamiltonian cycle problem for planar, 3-connected, 3-regular graphs, and 3-connected, 3-regular, bipartite graphs. By adjoining three Tutte fragments, we get the counterexample:

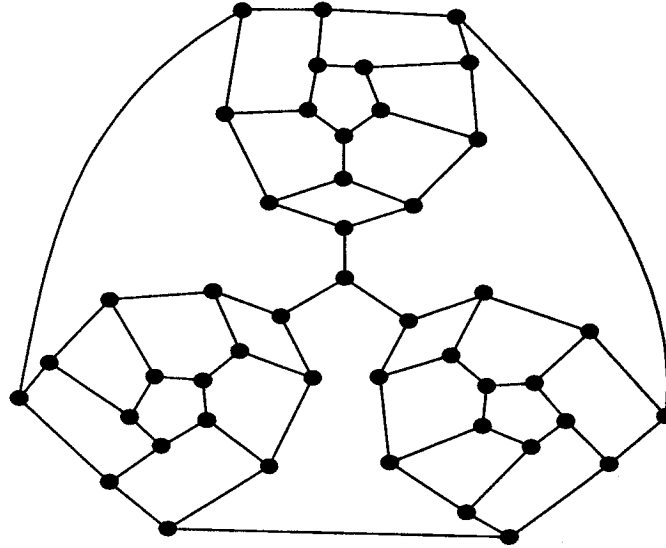


Figure 6: Tutte's Counterexample, Using Tutte's Fragment

Starting from any vertex, it is easy to see that there are no Hamiltonian cycles. At most you can go through two Tutte fragments before returning to the center, with no way of reaching the third to complete the Hamiltonian cycle.

Tutte later conjectured that 3-regular, 3-connected bipartite graphs were Hamiltonian, but many counterexamples, including a minimum example on 50 vertices were found.

This thesis is primarily concerned with the following conjecture of David Barnette:

[Bar69]: **Conjecture (D. Barnette)**

Every graph that is 3-connected, 3-regular, bipartite, and planar has a Hamiltonian cycle.

We call these graphs Barnette graphs and study their combinatorial properties. They are the graphs of 3-connected, simple, bipartite, 3-dimensional convex polytopes.

1.4. Why Study Graphs of Polytopes?

The study of graphs of polytopes has a strong connection to optimization and computational complexity theory.

1.4.1. Optimization. Linear programming problems are constrained optimization problems with linear objectives and constraints. It was created during World War II as an aid for military planning. An example of a linear programming problem is:

$$\begin{array}{ll}
\text{minimize} & 2x_1 + 3x_2 + x_3 \\
\text{subject to:} & 35x_1 + x_2 + 0.5x_3 \geq 0.5 \\
& 60x_1 + 300x_2 + 10x_3 \geq 15 \\
& 30x_1 + 20x_2 + 10x_3 \geq 4 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0
\end{array}$$

The inequalities of a linear programming problem describe halfspaces. Their intersection defines a polytope on which feasible solutions are found. Geometrically, the objective function is a vector whose direction is the direction of optimality. The optimal solution is found on a vertex or edge of a polytope, where a hyperplane normal to the objective function intersects the polytope on this vertex or edge. It is easy to see that any other solutions will be either less than optimal or infeasible. If the optimal solution is an edge, there are infinitely many optimal solutions, otherwise the solution is unique.

The first and most common method for solving linear programming problems is the simplex method, introduced by George Dantzig in 1950. The simplex method uses both the combinatorics and geometry of the problem. It begins at one vertex of the polytope and moves along the graph until reaching the optimal solution. At each step, it uses a fixed pivot rule to attempt to move in the optimal direction. Despite its great success in practice, in 1972 Klee and Minty showed that the simplex method has exponential complexity. They found that on the feasible region now called the n -dimensional Klee-Minty cube, the simplex method using Dantzig's original pivot rule traverses every vertex before arriving at the optimal solution. In 1979, Leonid Khachyan showed that the linear programming problem is solvable in polynomial time, but there are no known polynomial time simplex pivot rules, but it is not known whether any exist.

In 1957, Hirsch conjectured that the diameter of any d -dimensional polyhedra with n facets is bounded by $n - d$. It is known to be false for unbounded polyhedra, and true for three-dimensional and 0-1 polytopes. The Hirsch conjecture implies a serious claim about linear programming methods like the simplex method. Consider the theoretical "oracle's rule" for pivoting: take the shortest path to the optimum. Assuming the Hirsch conjecture and using the oracle's rule, starting from any vertex, any linear programming problem is solvable by the simplex method in at most diameter many steps. [JB07]

Although Barnette's conjecture and the simplex method don't have a direct connection, they are connected in the sense that they are both problems in combinatorics of polytopes that lie in a gray area of complexity theory. By relaxing either the condition of planarity

or bipartiteness, the Hamiltonian cycle problem becomes NP-complete, while the simplex method is an algorithm that may be exponential for a problem that is polynomial.

1.4.2. The Four-Color Theorem. The following conjecture was posed by Frederick Guthrie, a student of DeMorgan, in 1852: every planar map, having no countries which are completely surrounded by others, can be colored such that no countries sharing a common border are the same color. By adjoining the capitals of adjacent countries by edges (i.e., taking the dual graph of the map), the four-color theorem can be restated in graph theoretical terms: Every planar graph has a vertex coloring such that no two adjacent vertices share the same color. The theorem was proved by Appel and Hakken in 1977.

Its relation to the Barnette conjecture comes from a condition that implies the four-color theorem. Tait showed that the four-color theorem is equivalent to the condition of finding a 3-edge coloring for every 3-regular planar graph, which in turn is equivalent to every 3-regular planar graph having a Hamiltonian cycle. [I. 06] As Tutte's counterexample shows, this condition is not true. Although Barnette graphs are four-colorable since they are bipartite, the work of Tait and others to find a Hamiltonian family of graphs eventually led to Barnette's conjecture.

For the remainder of the thesis, we study properties of Barnette's graphs and previous work that has been done on the conjecture.

CHAPTER 2

Studying Barnette Graphs

2.1. Holton-Manvel-McKay Construction

Barnette's conjecture remains an open problem. Holton, Manvel and McKay verified computationally in [HMM85] that the conjecture holds for all graphs with up to 66 vertices and have since extended the result to 84 vertices. Most significantly, they found a way to construct Barnette graphs. The smallest graph satisfying the conjecture is the graph of a cube. From this cube, we can construct Barnette graphs with iterations of the two operations. Figures 1 and 2 show the two operations. Given a planar embedding of a Barnette graph, the operations are as follows: for the first operation, given two edges that don't share a vertex, add two vertices to each edge, and connect them with two new edges. This results in a Barnette graph with four additional vertices. In Appendix A, we see that Barnette graph 12-1 (hexagonal prism) is a result of this operation being applied to Barnette graph 8-1 (the cube) on the two center edges. Every prism with even base is thus a Barnette graph by repeating this operation on the two center edges of the embedding of the cube shown. The second operation is applied to a vertex, by adding six vertices and five edges to the graph. Because Barnette graphs are 3-regular, any vertex looks like a "Y" with a planar embedding. In figure 2 we see how to do this operation: add two vertices to of the edges, and two new vertices, connecting them to make three new four-sided faces. In Appendix A, we see that Barnette graph 14-1 is applying this operation once on a vertex of a cube.

Theorem (Holton, Manvel, McKay)

Every Barnette graph can be generated up to isomorphism using the two operations shown.

Initially, it appears that graphs constructed in this way maintain Hamiltonicity. With the second operation, a cycle entering on any of the three original graphs and leaving on any other can be easily extended to the construction, while maintaining Hamiltonicity. We see that the addition from operation two looks similar to a small Tutte Fragment, except that a Hamiltonian cycle entering the subgraph can enter or exit on any vertex.

The problem in extending a Hamiltonian cycle comes from the first operation. If the cycle travels along the two original edges, then the cycle extends to the larger graph without a change. If instead the cycle on the smaller graph uses neither of the two original edges, it cannot get to the four middle vertices while maintaining Hamiltonicity. P.R. Goodey

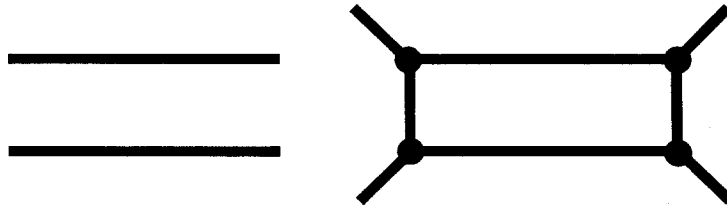


Figure 1: Barnette graph construction: add the vertices and edges in red

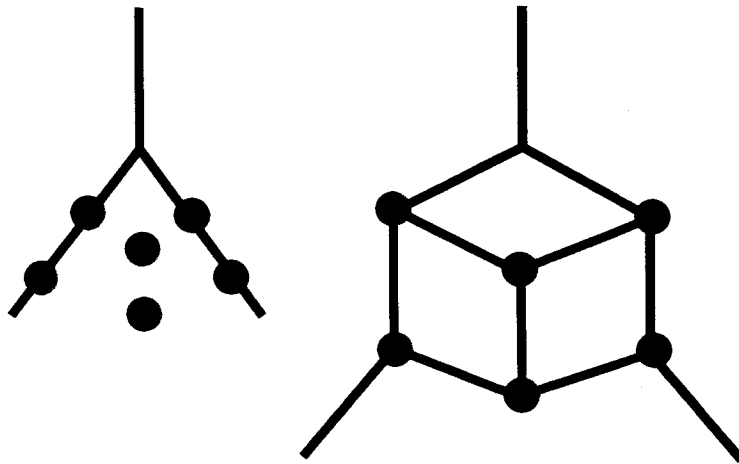


Figure 2: Barnette graph construction: add the vertices and edges in red

[P.R77] showed that Barnette's conjecture holds when all faces have four or six sides. Many conditions equivalent to Barnette's conjecture have been found: [Ale03] Note: a **facial cycle** in a planar graph is a cycle in the graph which forms the boundary of a face.

- For every bipartite, 3-regular, 3-connected, and planar graph G and for every two edges a, b of G , belonging to the same facial cycle of G , there is a Hamiltonian cycle in G , containing a and avoiding b .
- For every bipartite, 3-regular, 3-connected, and planar graph G and for every two edges a, b of G , belonging to the same facial cycle of G , there is a Hamiltonian cycle in G , containing both a and b .
- Every bipartite, cubic, cyclically 4-connected, and planar graph has a Hamiltonian cycle.

Although Barnette's conjecture could be approached through any one of many equivalent conditions, the Tutte Fragment gives rise to an obvious approach to finding a counterexample to Barnette's conjecture. If a Barnette graph exists with the Tutte Fragment's property that Hamiltonian cycles through it can only enter through certain vertices, it would create a quick counterexample to Barnette's conjecture. It turns out it also implies a stronger result about Barnette graphs.

2.2. NP-Completeness and Barnette's Graphs

Garey and Johnson [GJ90] showed that the Hamiltonian cycle problem is NP-Complete, and later along with Tarjan [GJT76] showed that the problem is NP-complete for 3-connected, 3-regular, and planar graphs. Akiyama, Nishizeki, and Saito [TTN80] showed that the Hamiltonian cycle problem is NP-complete on 3-connected, 3-regular, bipartite graphs. To prove this, both found a polynomial-time reduction of 3-SAT to the desired Hamiltonian cycle problem.

Using similar proofs, both cases begin with a logical statement, and build a polytope with the desired properties that corresponds to the statement. The statement determines the number of edges, vertices, and connectivity of the polytope, and they show that a statement is true if and only if the corresponding graph has a Hamiltonian cycle. Both instances are extremely relevant, as each class of graphs differs from Barnette graphs in only one property. In proving that this problem is NP-complete on the desired graphs, they also proved showed (although it was already known) that these classes of graphs are not Hamiltonian. The key to much of the proofs is the "required edge graph", a subgraph used in each construction that has the same properties as the Tutte Fragment.

In the construction of Akiyama, et. al., the required-edge graph is the only non-planar component of the construction, and in Johnson, et. al., the required-edge graph is the only non-Bipartite component of the construction, as it contains pentagonal faces. Figure 3 shows the non-planar required edge graph. In the required-edge graph shown, a Hamiltonian cycle entering on B or C must leave the subgraph through A . The significance of this graph is that it is the only non-planar component of the graph constructed in the proof. Any graph that has the restricted Hamiltonian cycle property can replace the required-edge graph. If such a subgraph exists, it can be used in the same way as the Tutte counterexample, thus disproving Barnette's conjecture. Because it is the missing piece in a polynomial-time reduction of the Hamiltonian cycle problem to 3-SAT, its existence also proves that the Hamiltonian cycle problem is NP-complete on Barnette graphs. Of course, if Barnette's conjecture is true, then Hamiltonian cycles on Barnette graphs cannot be NP-complete. We now have the following conjecture:

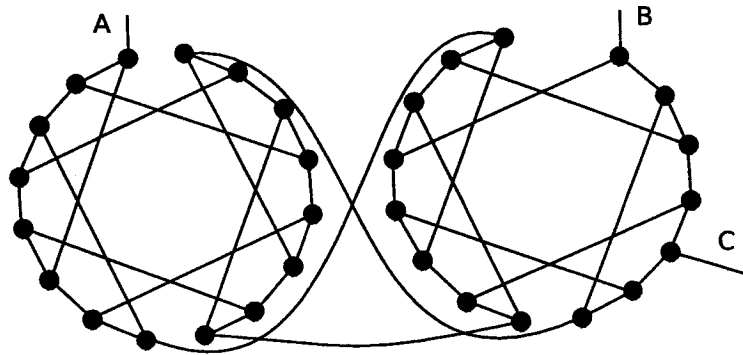


Figure 3: The required-edge graph and its two possible local states.

Conjecture

There is no graph with specified edges (a_1, a_2) , (b_1, b_2) , (c_1, c_2) that is bipartite, planar, 3-regular (except at a_1, b_1, c_2), and 3-connected (except at a_1, b_1 , and c_1) with Hamiltonian paths $b_1 \leftrightarrow c_1$, $a_1 \leftrightarrow b_1$, and $a_1 \leftrightarrow c_1$.

It is easy to see that the required-edge graph satisfies all properties except for planarity. Note that this conjecture is not equivalent to Barnette's conjecture. If such a graph exists, then the conjecture cannot be true and the Hamiltonian cycle problem is NP-complete. If such a graph does not exist, then we conclude nothing, the complexity remains undecided, and Barnette's conjecture may or may not be true.

The problem of finding such a counterexample is a very difficult computational problem. Checking whether a graph has the required Hamiltonian path properties falls in the computational category of **#P-Hard**, counting problems with an associated decision problem that is usually NP-hard. For a precise definition of #P-Hard, see [GJ90]. Checking the property can be rephrased as a counting problem: how many Hamiltonian paths on a possible required-edge graph that begin at a specified vertex and end at another? Simply checking the existence of this type of Hamiltonian path is more difficult than the general Hamiltonian path problem, let alone finding all such Hamiltonian paths. In addition to being difficult for any fixed graph, there are many more of these "possible required-edge" graphs for a given number of vertices than there are Barnette graphs themselves, as we can generate them with the following method:

Fix a Barnette graph G and a planar embedding. Choose any three vertices on the outer face of the graph, and add edges A , B , and C . These will connect this fragment to the rest of a larger Barnette graph. Figure 4 gives an example of this addition to a Barnette graph on 18 vertices. It is easy to see that here there are Hamiltonian paths beginning at each of A , B , or C that end at either of the other two. An exhaustive search thus requires an enormous number of attempts: For a given embedding, we have multiple choices of A , B ,

and C , and there are multiple embeddings, and for each graph there are multiple planar embeddings.

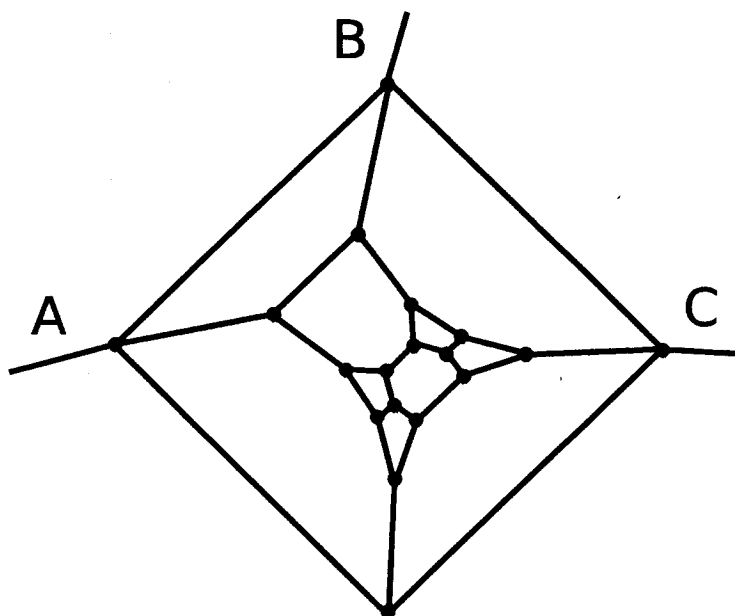


Figure 4: Barnette graph construction: add the vertices and edges in red

We now study the diameter of Barnette's graphs, including some upper bounds based on eigenvalues of the graphs' adjacency matrices.

2.3. Diameters and Eigenvalues of Barnette Graphs

2.3.1. Background on Diameter. In this section, we investigate properties of the diameter of Barnette graph. Although there is no obvious connection between Hamiltonicity and diameter of a graph, these graphs are nevertheless an interesting family to study.

Given a graph G , we can define the distance between vertices u and v as the length of the shortest path between them. The **diameter** of a graph is defined as the maximum shortest path over all pairs of vertices.

An **adjacency matrix** is a representation of a graph's connectivity. Given a graph with n vertices, an $n \times n$ adjacency matrix A is defined as

$$A_{i,j} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{if vertex } i \text{ is not adjacent to vertex } j \end{cases}$$

The *Laplacian matrix* of a graph is defined as

$$L_{i,j} = \begin{cases} -1 & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{if vertex } i \text{ is not adjacent to vertex } j \\ \text{deg}(i) & \text{if } i = j \end{cases}$$

For a k -regular graph, it is easy to see that the Laplacian of a graph is $kI - A$, where I is the $n \times n$ identity matrix.

There are several known upper bounds for the diameter of a graph. We investigate the diameter and the following upper bounds on the diameter for Barnette's graphs: We call $\lambda_1 \leq \lambda_2 \dots \leq \lambda_n$ the eigenvalues of the adjacency matrix.

- The number of unique eigenvalues of of the adjacency matrix of a graph
- $\left\lceil \left(\frac{2k - \lambda_{n-1}}{4(k - \lambda_{n-1})} \right) \ln(n - 1) \right\rceil$

The first bound holds for any graph, and the second holds for k -regular graphs. In [Moh91], a variation of the second bound is given in terms of the eigenvalues of the Laplacian. For k -regular graphs, the eigenvalues of the Laplacian are a shift by k of the eigenvalues of the adjacency matrix.

The software package Plantri implements Holton, Manvel and McKay's construction of Barnette graphs. Using this, we generate adjacency matrices for all graphs between 8 and 42 vertices. From the adjacency matrices we get the Laplacian matrix. We calculate the diameters using a well-known property of adjacency matrices: given an adjacency matrix A , $A^n_{i,j}$ is the number of paths of length n between vertex i and vertex j . Using this fact, the diameter can be defined as the smallest n such that the matrix $\sum_{i=1}^n A^n$ has no zero entries. This smallest n means that there at least one path between each vertex of length n , and for any $k < n$, there is no path of length k , thus the largest distance over all pairs of vertices is n .

Table 1 shows that that for Barnette graphs between 8 and 42 vertices, the diameter ranges from 3 to 14. Mohar's bound depends completely on on the second largest eigenvalue of the adjacency matrix and the number of vertices. For the Barnette graphs we calculated, the second largest eigenvalue grows monotonically with diameter. For two graphs with the same diameter and distinct number of vertices, the graph with more vertices has a greater second largest eigenvalue. The rate of increase of the Mohar bound is much larger than the diameter, and it becomes worse with more vertices.

The number of distinct eigenvalues of the adjacency matrix is of course bounded above by the number of vertices of the graph. Table 2 is a small sample of data on the graphs. From this example and from the rest of the data, for Barnette graphs of a fixed number of vertices n , there is little connection between diameter and the number of unique eigenvalues. For graphs on each fixed number of vertices, we find graphs that have the minimum diameter and close to the maximum number of eigenvalues, and vice-versa.

In both cases, both the number of unique eigenvalues and the Mohar bound are nowhere near the diameter of the graph. In many cases the number of unique eigenvalues is the

Table 1: Number of Vertices & Values of Diameters in Barnette Graphs

8	3
12	4
14	5
16	5
18	5,6
20	5,6,7
22	6,7
24	6,7,8
26	6,7,8,9
28	6,7,8,9
30	7,8,9,10
32	7,8,9,10,11
34	7,8,9,10,11
36	7,8,9,10,11,12
38	7,8,9,10,11,12,13
40	7,8,9,10,11,12,13
42	7,8,9,10,11,12,13,14
44	7,8,9,10,11,12,13,14,15
46	8,9,10,11,12,13,14,15

maximum, and the Mohar bound is even worse. A simple bound turns out to be much better:

For any Barnette graph on n vertices, $\frac{n}{2}$ is an upper bound on the diameter, as the largest distance between any two vertices cannot be greater than traveling along half of the vertices of the graph.

For the 22,263 Barnette graphs with 36 or less vertices, the number of unique eigenvalues is a better bound for the diameter than $n/2$ for only six graphs, and the Mohar bound is better than $n/2$ for only five graphs. We conjecture an even better bound for Barnette graphs, apparent in table 1: Each odd numbered upper bound occurs twice, and each even numbered bound occurs twice. If this continues to be the case, then the $n/2$ bound for graphs should continue to perform well, at least for the graphs whose diameter is an upper bound for all others on the same number of vertices. From the pattern found in Table 2, we conclude with a conjecture of an even tighter bound on the diameter of the graph.

Conjecture

If G is a planar, 3-connected, 3-regular, and bipartite graph on n vertices, $d(G) \leq \lfloor \frac{2n+1}{3} \rfloor$

Table 2: Sample of Diameter Data For Barnette Graphs

Graph Name	Vertices	Edges	Diameter	Unique Eigenvalues	Mohar Bound
G_12_1	20	30	6	12	14
G_12_2	20	30	6	20	14
G_12_3	20	30	5	13	10
G_12_4	20	30	6	18	12
G_12_5	20	30	6	16	14
G_12_6	20	30	6	20	16
G_12_7	20	30	6	20	16
G_12_8	20	30	7	14	18
G_13_1	22	33	7	22	18
G_13_2	22	33	7	21	18
G_13_3	22	33	6	22	16
G_13_4	22	33	7	22	18
G_13_5	22	33	6	22	14
G_13_6	22	33	7	21	18
G_13_7	22	33	6	22	14
G_13_8	22	33	6	21	12
G_18_1323	32	48	7	31	24
G_18_1384	32	48	7	31	28
G_18_1400	32	48	7	32	22
G_18_1401	32	48	8	31	26
G_18_1402	32	48	8	32	34
G_18_1403	32	48	9	32	36
G_18_1404	32	48	9	32	36
G_18_1405	32	48	8	32	34
G_18_1406	32	48	9	32	38

[Slo07]

Barnette's Conjecture has been open for nearly 40 years at this point. Although much work has been done on the problem, there are many ways to approach the problem and make progress into understand the properties of the graphs. In future work, I hope to make progress on a proof of the diameter bound conjectured, and work on generating a Tutte Fragment-like counterexample, or at least give a lower bound on the size of a possible counterexample graph.

APPENDIX A

Tutte Embeddings of Barnette Graphs From Eight to Sixteen Vertices

In 1962, W.T. Tutte created the Tutte embedding, by proving that any planar, 3-connected graph could be embedded in the plane by defining a weight function the edges and minimizing this function to give embed vertices. Since Barnette graphs are planar and 3-connected, Tutte's embedding gives us nice planar visualizations, which may be useful in gaining insight into properties of Barnette graphs. The following are Tutte embeddings of the sixteen smallest graphs, using Plantri to produce the graphs and Tulip to visualize the Tutte embedding. Graphs are indexed $n - k$ where n is the number of vertices and k is the index assigned to the graph by the software Plantri, denoting only the order in which it was generated by the program.

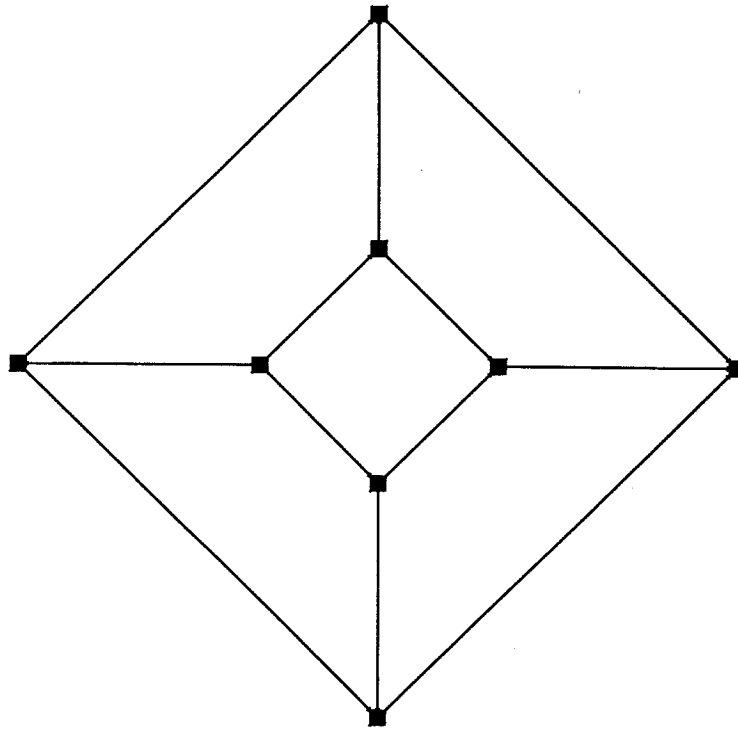


Figure 1: Barnette Graph 8-1

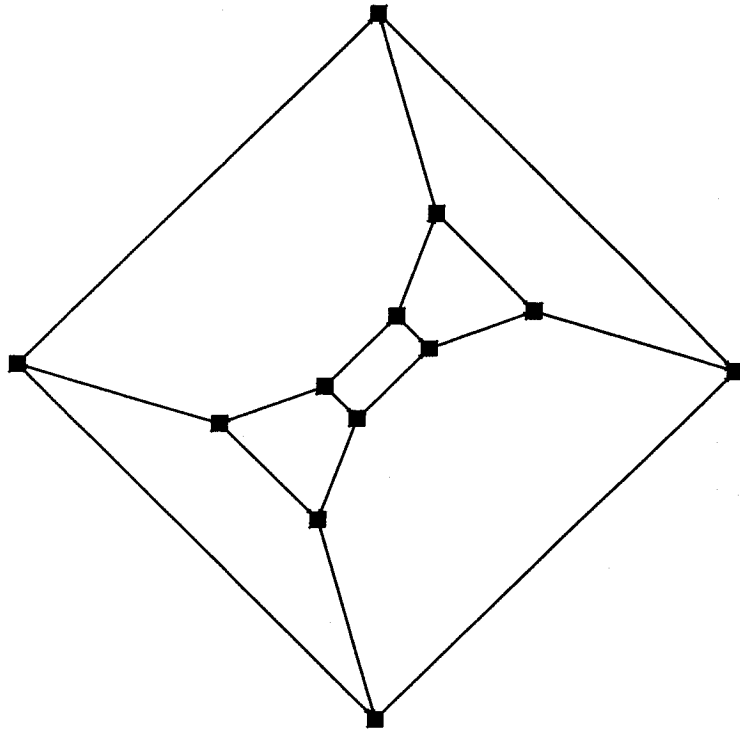


Figure 2: Barnette Graph 12-1

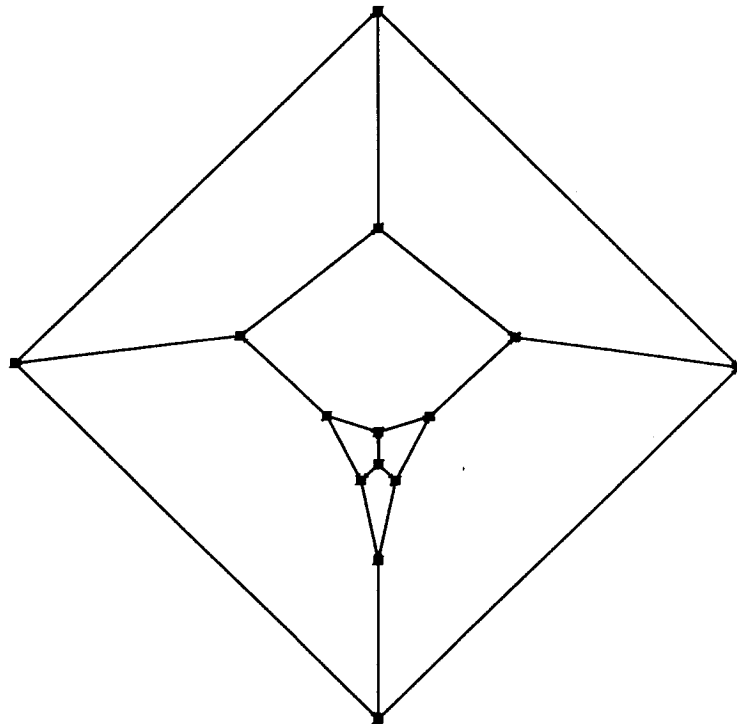


Figure 3: Barnette Graph 14-1

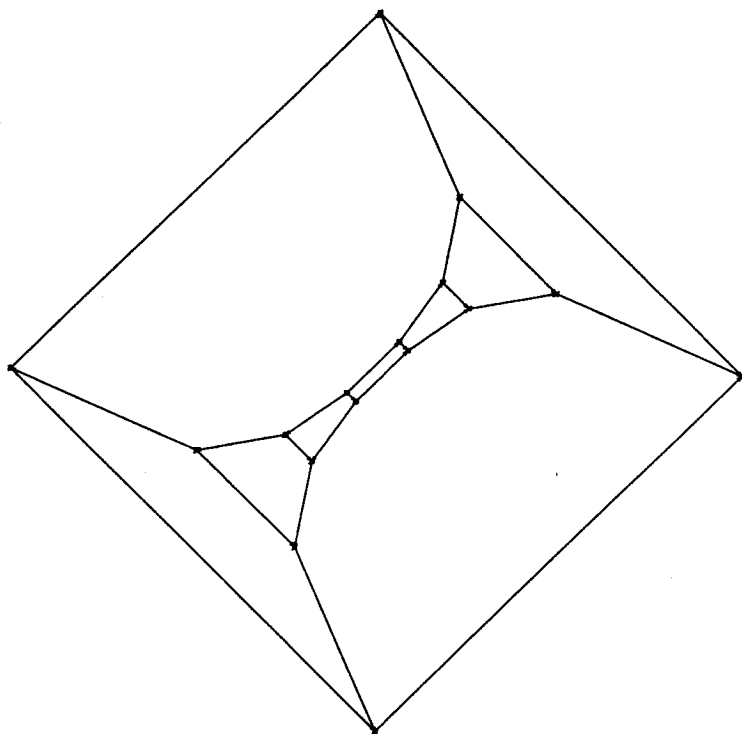


Figure 4: Barnette Graph 16-1

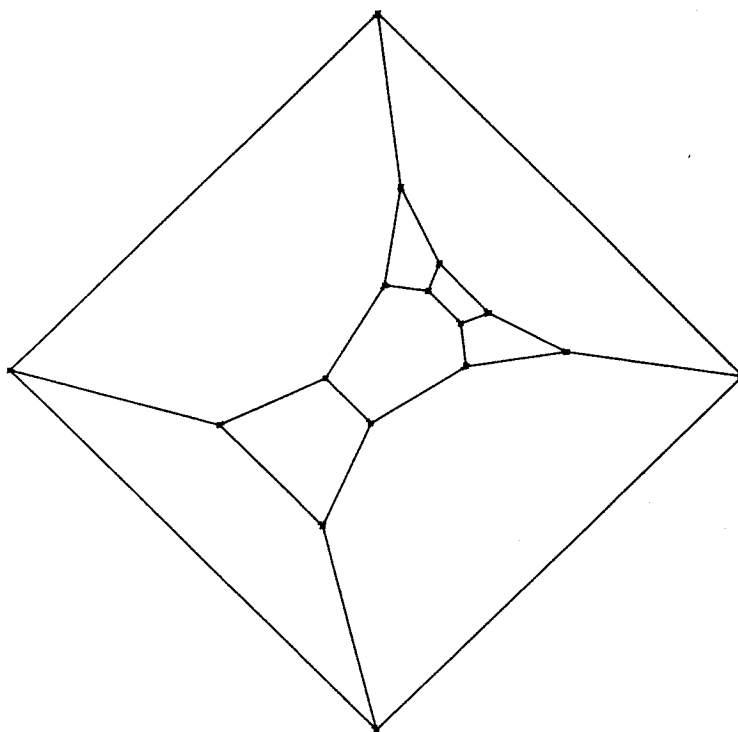


Figure 5: Barnette Graph 16-2

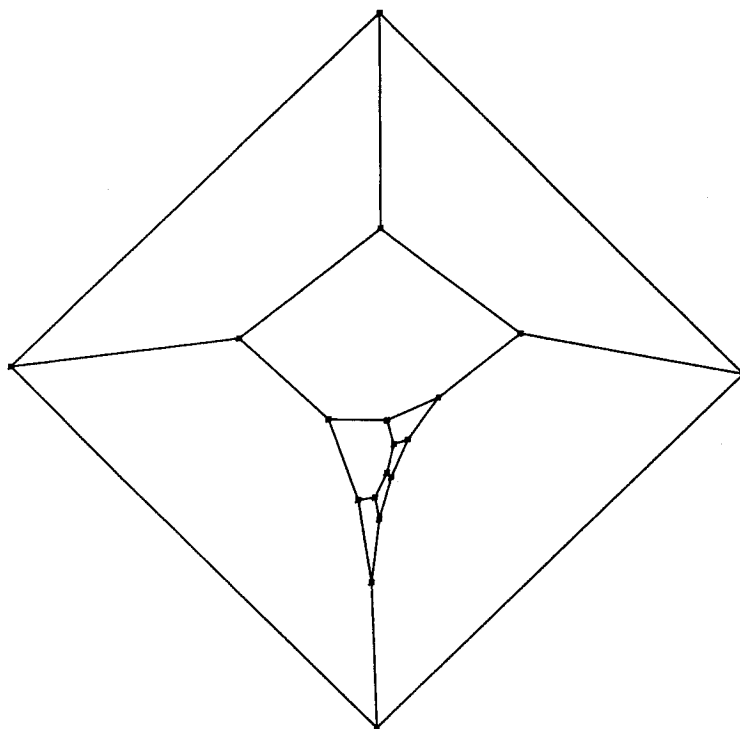


Figure 6: Barnette Graph 18-1

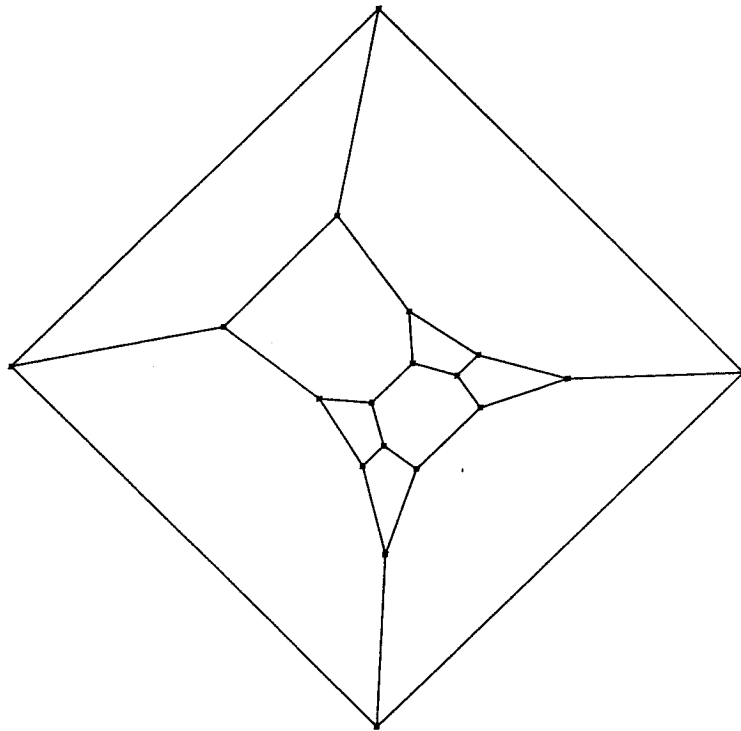


Figure 7: Barnette Graph 18-2

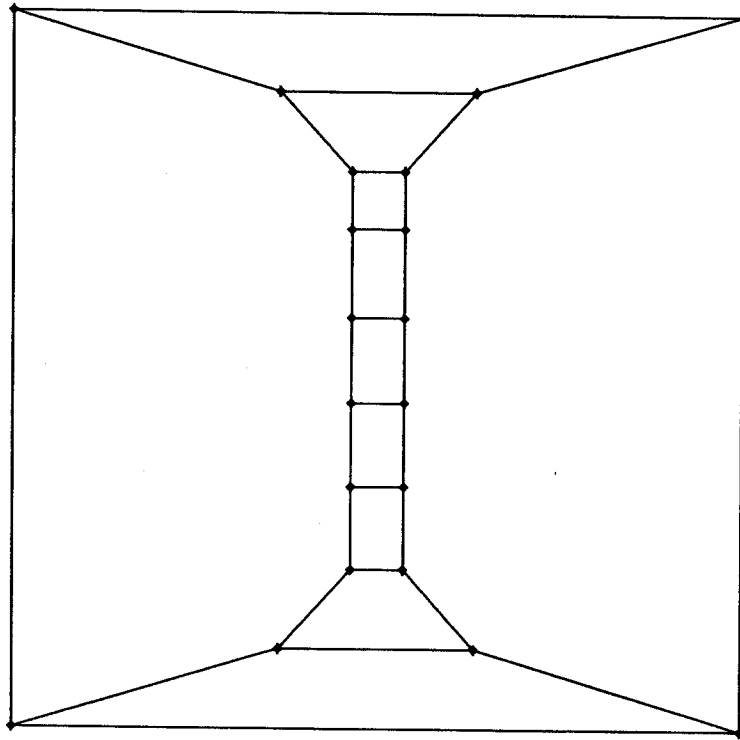


Figure 8: Barnette Graph 20-1

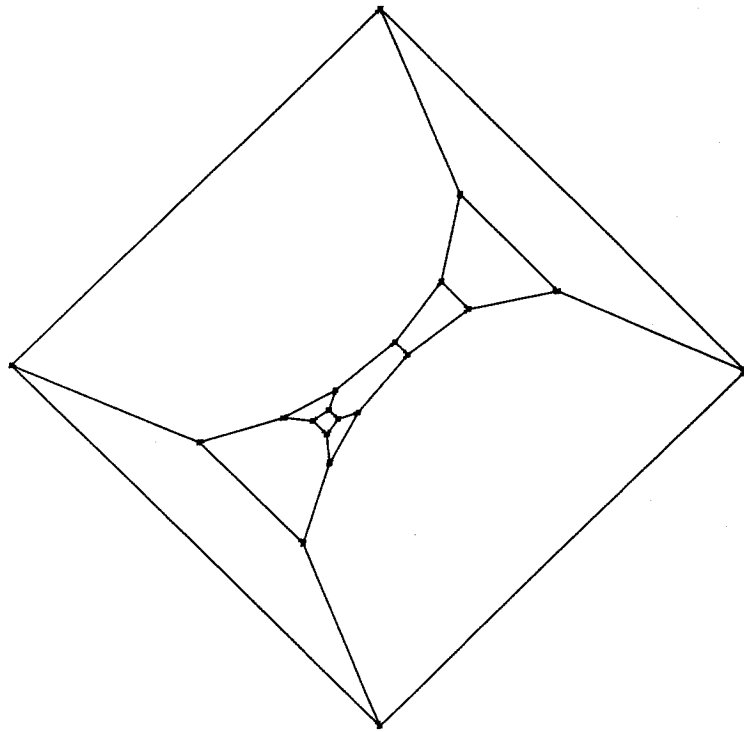


Figure 9: Barnette Graph 20-2

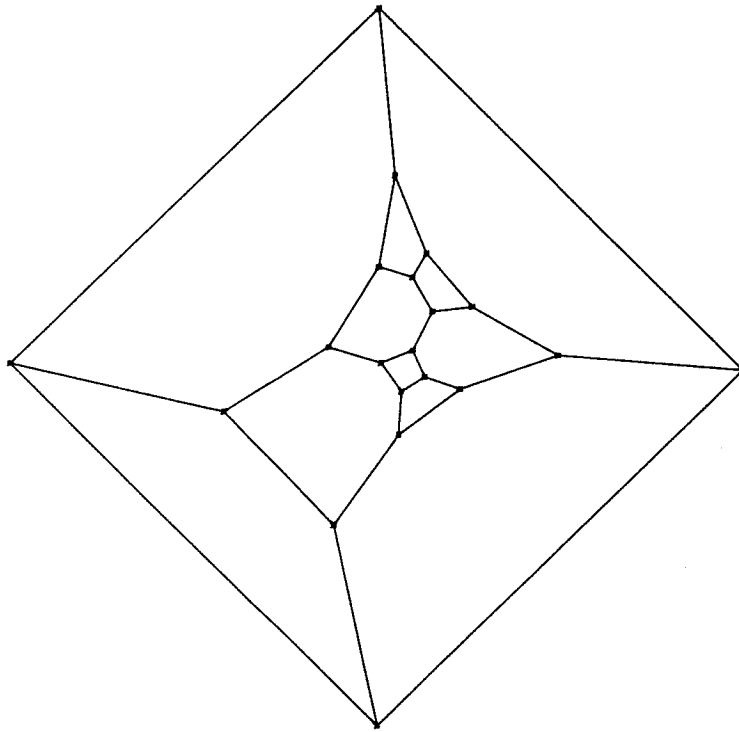


Figure 10: Barnette Graph 20-3

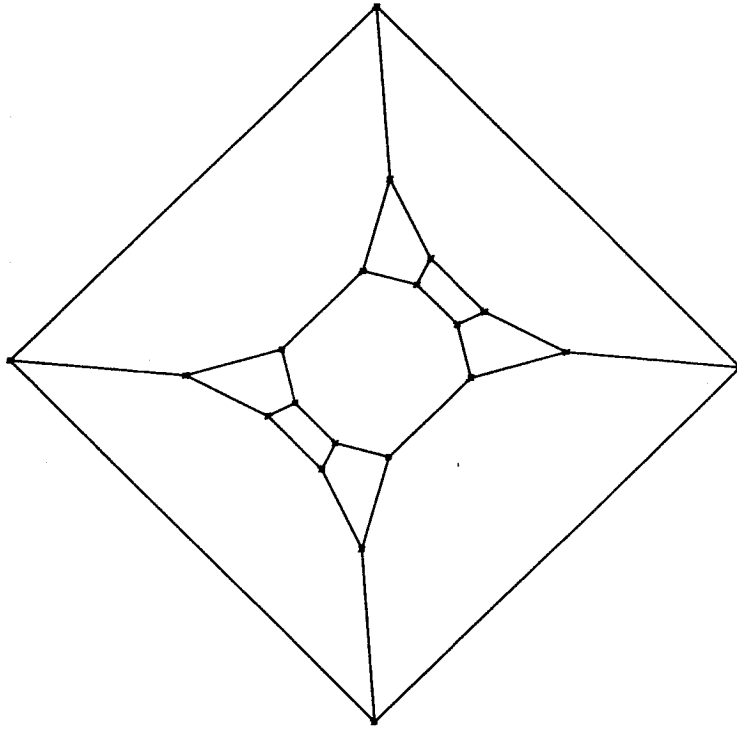


Figure 11: Barnette Graph 20-4

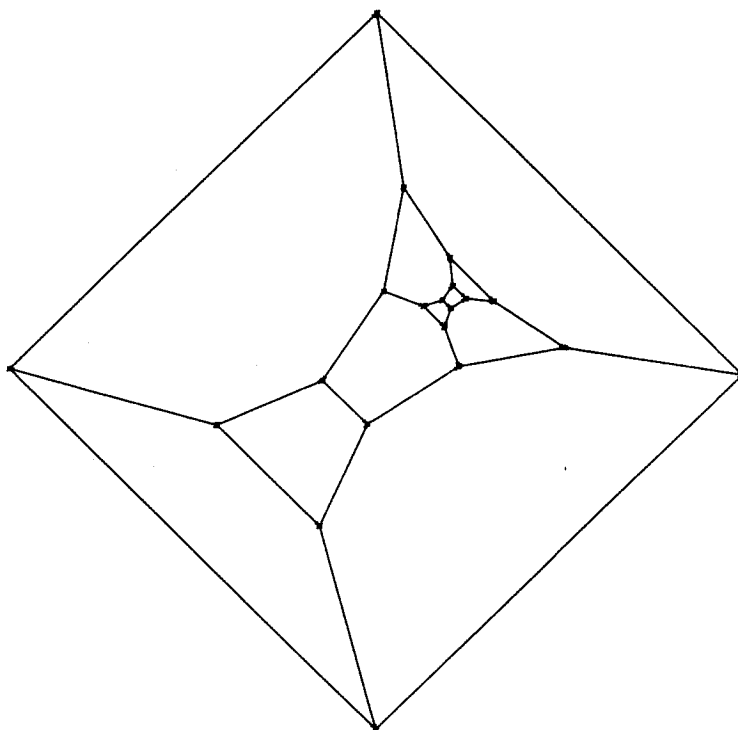


Figure 12: Barnette Graph 20-5

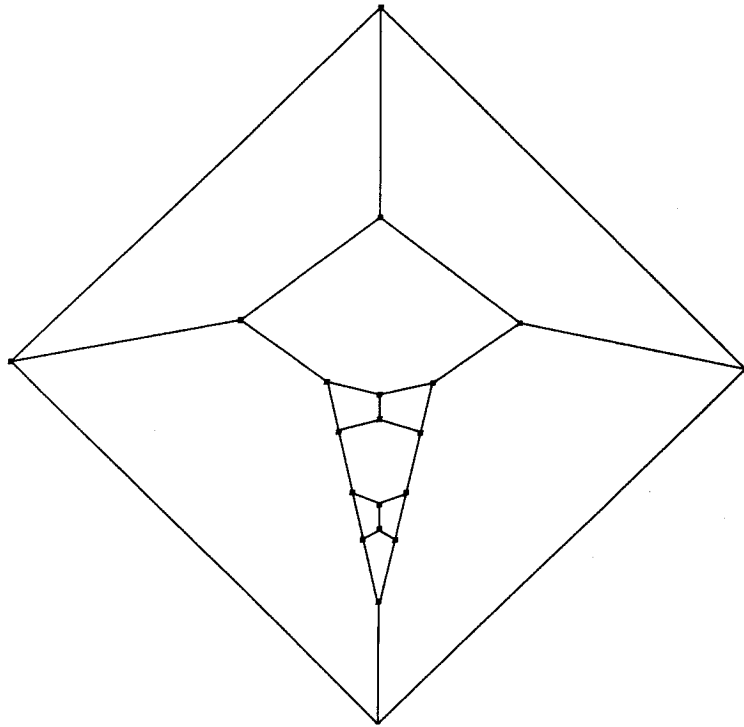


Figure 13: Barnette Graph 20-6

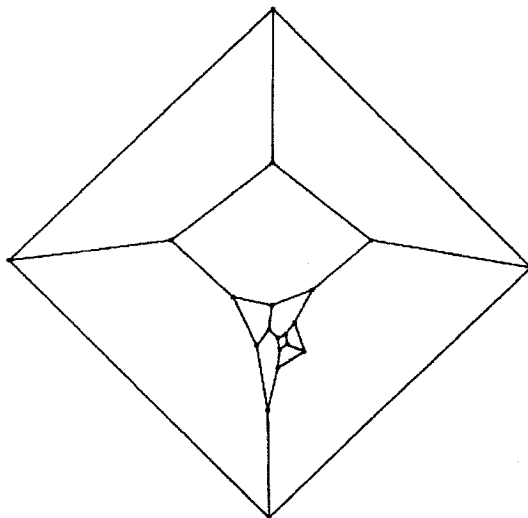


Figure 14: Barnette Graph 20-7

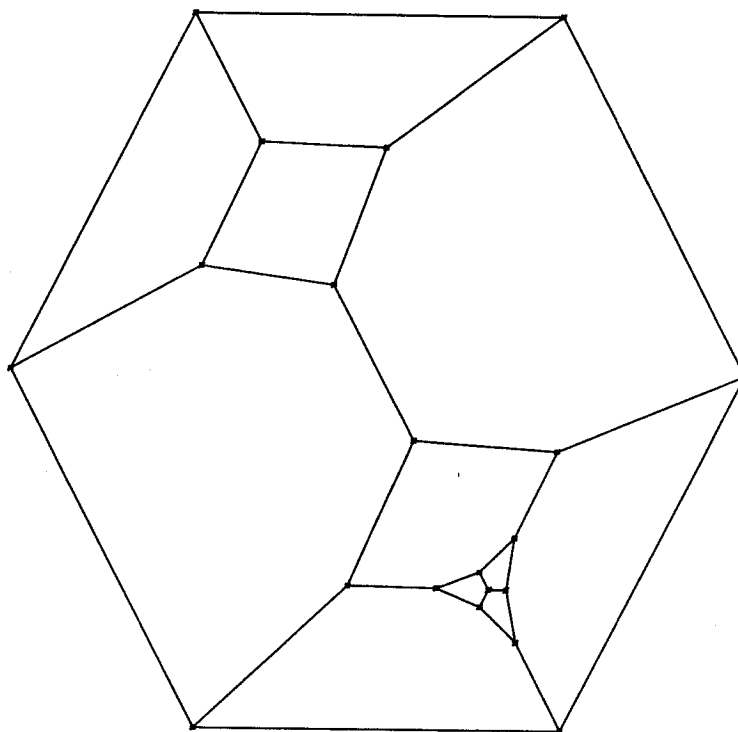


Figure 15: Barnette Graph 20-8

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