An Approach to Bounding the Spectral Gap Above the Ground State of a Quantum Spin Chain

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Abstract

Using the representations of the Lie algebra $\mathfrak{su}_2$ to examine the decomposition of spin chains, we study the norm of a particular operator, C, on a system of three spins with magnitudes j_1, j_2, and j_3, described by the representation $\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$. The operator C is defined as a commutator: $C = [P_{12} \otimes I, I \otimes P_{23}]$, where P_{12} projects onto the subrepresentation with spin $j_1 + j_2$, contained in $\rho_{j_1} \otimes \rho_{j_2}$, and P_{23} projects onto the subrepresentation with spin $j_2 + j_3$, contained in $\rho_{j_2} \otimes \rho_{j_3}$. On the basis of partial results and numerical computation, we conjecture that $||C|| = \sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}/(j_1 + j_2)(j_2 + j_3)$. Estimating the magnitude of this norm is related to estimating the norm of the operator $G_{[2,3]}E_2$. Determining a sufficiently small upper bound ϵ for $||G_{[2,3]}E_2||$, guarantees a nonzero lower bound of the spectral gap above the ground state. We conjecture that this upper bound ϵ does exists, and when an additional condition is satisfied we calculate the bound to be $\sqrt{2j_1 j_2 j_3 (j_1 + j_2 + j_3)}/(j_1 + j_2)(j_2 + j_3)$. We are specifically interested in this result as it guarantees a nonzero spectral gap of a finite or infinite frustrated antiferromagnetic spin-3/2 chain.

1 Introduction

Using mathematical models to represent physical systems is commonly done by researchers in the mathematics field. From fluid dynamics, to neuron transmission, mathematical models help aid researchers in their quest to determine new properties of such systems. The physical system that we are interested in and analyze throughout this project is that

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of a one dimensional quantum spin chain. Quantum spin chains model the magnetic materials found in many electronic devices. One can imagine a one dimensional quantum spin chain as a line of particles held together by bonds between neighboring particles, see figure 1. Each particle has a number attached to it, called the spin of the particle. This determines the range of values of the angular momentum in the third component. These spin values characterize the each particle, its magnetic field, and the range of possible values of its intrinsic angular momentum. These systems, classically, are of particular interest to us as its energy is usually quantized, which means that the possible energy values for which the system can take on is not a continuous spectrum. Thus, there is a nontrivial difference between two consecutive energy states. Analyzing quantum spin chains brings new insight into properties of these systems.

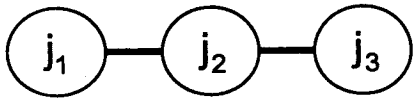

Figure 1: A quantum spin chain of three particles with spins j_1, j_2, j_3.

A spin particle is modeled mathematically by a representation of the group SU_2. These representations act as operators on a Hilbert space, which is a vector space where the inner product is well defined. The vectors in the Hilbert space represent the various states our system can take. It is not always convenient, however, to model the system using group representations of SU_2 as there is no finite basis. We solve this problem by using representations of the Lie algebra $\mathfrak{su}_2$. Note that for $\mathfrak{su}_2$ the Lie bracket $[\cdot,\cdot]$ is defined as $[A,B] = AB - BA$. Since there is a bijective correspondence between the group SU_2 and the Lie algebra $\mathfrak{su}_2$ via the exponential map, these two methods are equivalent ways of studying representations. The Lie algebra representation is a more convenient way of modeling these particles since it is spanned by the three different elements S^3, representing the third component of angular momentum, S^-, the lowering operator, and S^+, the raising operator. It is shown in [1] that for these operators the commutation relations are:

$$\begin{aligned} [S^+, S^-] &= 2S^3 \\ [S^3, S^\pm] &= \pm S^\pm \end{aligned}$$

Each particle has a spin, denoted usually by j, where $j = \frac{a}{2}$ with $a \in \mathbb{Z}_{\geq 0}$. Each particle also has an intrinsic angular momentum m with possible values ranging from $m = j, j-1, \ldots, -j$. As stated in [1], given a particle with spin j, there is a basis of $2j+1$ vectors for our Hilbert space $\mathbb{C}^{2j+1}$, one for each of the possible components of angular momentum, such that the irreducible representation is described by

$$\begin{aligned} &\rho_j(S^3)|m\rangle = m|m\rangle && -j \leq m \leq j \\ &\rho_j(S^\pm)|m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|m \pm 1\rangle && -j \leq m \leq j \end{aligned}$$

Since these spins are in positive half-integer increments and the dimension of the corresponding representation is $2j+1$, there will be an n-dimensional irreducible representation for all $n \in \mathbb{N}$. Notice that it is simple to create these representations such that the vectors $|m\rangle$ are the standard basis vectors. It is the set of representations using the standard

basis vectors as these eigenvectors that I will refer to throughout the rest of this paper. Furthermore, the notation of $|m\rangle$ will represent the standard vector that corresponds to the eigenvector of $\rho_j(S^3)$ with eigenvalue m.

In this specific paper, we are looking at chains of three particles, which are mathematically modeled by the tensor of their three corresponding irreducible representations of $\mathfrak{su}_2$, or $\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$. The actual representation for all $g \in \mathfrak{su}_2$ is given by

$$\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}(g) = \rho_{j_1}(g) \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes \rho_{j_2}(g) \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes \rho_{j_3}(g)$$

I will refer to this representation as ρ for the rest of the paper.

In general, the energy of a quantum spin chain is modeled by a Hamiltonian. A Hamiltonian is a sum of matrices where each specific term in the sum describes the interactions between pairs of particles in the system. The Hamiltonians of greatest interest describe interactions only between nearest neighbor pairs. Such is the case in the Hamiltonian used to describe our system, which we will discuss later. The eigenvalues of the Hamiltonian give the different energy states of the system. By ordering the eigenvalues from least to greatest, one can find a desired energy gap by subtracting successive pairs of energy values. Since the Hamiltonian is invariant under $SU(2)$ interactions, as rotations do not change how the particles interact, we used representations of this kind to model a chain with a specific set of spins. The classical example of a Hamiltonian is that of the Heisenberg model for the ferromagnetic chain of length N.

For a ferromagnetic chain of length N, the Heisenberg Hamiltonian describing the system is $H_N = -\sum_{x=1}^{N-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1}$, where $\mathbf{S}_x = (S_x^1, S_x^2, S_x^3)$ consists of the spin matrices for each component. One can show that the ground state energy (i.e. the lowest energy) is equal to $-\sum_{x=1}^{N-1} j_x j_{x+1}$ where j_x is the spin of the particle in the xth position in the chain. In the case where all the spins are equal, finding the first excited energy state (i.e. the second lowest energy) is equivalent to diagonalizing an $N-1 \times N-1$ banded matrix. The first excited energy state λ_1 in this example is of the order $1/N$. Hence, as $N \to \infty$ we see that $\lambda_1 \to 0$. Since we know the ground state energy is also 0, the gap between the two energy states tends to 0 as N gets large, and thus there is a vanishing spectral gap above the ground state. In this paper, we are interested in showing that a quantum spin chain with a specific Hamiltonian has a non-vanishing spectral gap.

The Casimir operator $C = (S^1)^2 + (S^2)^2 + (S^3)^2$ provides a means for determining an equivalent way of writing the Heisenberg and other similar Hamiltonians as a sum of projection matrices. This operator commutes locally with the Hamiltonian. This means that the Casimir operator commutes with each term inside the Hamiltonian. When the Casimir operator is diagonal, due to change in basis, each term in the Hamiltonian will be block diagonalized. This means that $H_N = \sum_{x=1}^{N-1} \sum_{i=|j_x - j_{x-1}|}^{j_x + j_{x-1}} \mu_i P_i$, where P_i is an orthogonal projector onto the subspace ρ_i acts on. There is a special class of quantum spin chains with Hamiltonians of the form

$$H_N = \sum_{i=1}^{N-1} P_{i,i+1}\,, \tag{1.1}$$

where $P_{i,i+1}$ is an orthogonal projection acting non-trivially on sites i and $i+1$. Assuming that H_N has a non-trivial null space: $\ker H_N \neq \{0\}$ means that the ground state energy of H_N is 0. This is true since having a non-trivial null space is equivalent to H_N having an eigenvector with eigenvalue 0.

This is the type of Hamiltonian we consider in this paper. Since we are analyzing a chain of three particles as a means to estimate a larger chain, our Hamiltonian is:

$$H = P_{12} \otimes \mathbb{I} + \mathbb{I} \otimes P_{23}$$

Inside the Hamiltonian, P_{12} and P_{23} represent the projectors, respectively, onto the largest irreducible representations in the decompositions of $\rho_{j_1} \otimes \rho_{j_2}$ and $\rho_{j_2} \otimes \rho_{j_3}$. These allow for the simultaneous examination of the each particle-interaction pair. We choose the largest irreducible representation in this decomposition to describe the neighboring particle interactions because it makes it possible to calculate the ground state energy. It is important to note that the calculation of the ground state is only possible when $j_x > 1/2$, or more precisely, when no two consecutive j_x's are $1/2$.

Finding the ground state of a quantum spin Hamiltonian of this type can be regarded as an eigenvalue problem. The classical case occurs when there is a tensor product basis $\{|\alpha_1, \ldots, \alpha_N\rangle \mid 1 \leq \alpha_i \leq d_i, 1 \leq i \leq N\}$ of simultaneous eigenvectors of the terms $P_{i,i+1}$. The set of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ is such so that $|\alpha\rangle$ is an eigenvalue of $P_{i,i+1}$ with eigenvalue either 0 or 1 for $1 \leq i \leq N-1$. The value of H_N considered as a function α, for arbitrary α, is then equal to the number of terms $P_{i,i+1}$ that have eigenvalue 1 when evaluated on a string α. In particular, the spectrum of H_N consist of non-negative integers.

In the general case, where the eigenvectors are not assumed to be of pure tensor product form but rather linear combinations of such vectors, the eigenvalues of H_N are usually not integers and different behaviors can occur. Typically, one is interested in Hamiltonians $\{H_N, N \geq 2\}$, and are interested in determining the the spectral gap above the ground state. Since we assume $\lambda_0^{(N)} = 0$, the spectral gap, $\gamma^{(N)}$ is simply the smallest non-zero eigenvalue of H_N. Note that in the classical case referred to in the previous paragraph it is clear that $\gamma^{(N)} \geq 1$. In the general case $\gamma^{(N)}$ may be bounded below by a non-zero constant, or may vanish at a certain rate as $N \to \infty$, implying a continuous spectrum of energy states. These different behaviors of the spectral gap have implications for the general properties of the dynamics of the quantum system modeled by H_N. For example, if H_N is the Hamiltonian of a quantum algorithm in the adiabatic model of quantum computation, the gap is related to estimates of the length of time (complexity) of the computation.

When the ground states are known, it is easy to obtain a reasonable upper bound for $\gamma^{(N)}$ using the variational principle. One can obtain the upper bound by finding a state orthogonal to the ground state and calculate the expectation value of H_N in this state. Non-trivial lower bounds are more difficult to discover. One approach which has been applied successfully in a number of cases is the so-called martingale method, which we now explain.

For an arbitrary subinterval $[a,b]$, with $1 \leq a < b \leq N$, let $G_{[a,b]}$ be the orthogonal projection onto

$$\ker \sum_{i=a}^{b-1} P_{i,i+1} \tag{1.2}$$

and let $G_{\{i\}} = \mathbb{1}$ for all $i = 0, \ldots, N$. It follows from these definitions that the orthogonal projections $G_{[a,b]}$ satisfy the following properties. For intervals $I_1, I_2 \subset [1,N]$ one has

$$\begin{aligned} G_{I_2} G_{I_1} &= G_{I_1} G_{I_2} = G_{I_2} \text{ if } I_1 \subset I_2 && \text{(1.3-a)} \\ G_{I_1} G_{I_2} &= G_{I_2} G_{I_1} \text{ if } I_1 \cap I_2 = \emptyset && \text{(1.3-b)} \\ P_{i,i+1} &= \mathbb{1} - G_{[i,i+1]} && \text{(1.3-c)} \end{aligned}$$

Define operators E_n, $1 \leq n \leq N$, by

$$E_n = \begin{cases} \mathbb{1} - G_{[1,2]} & \text{if } n = 1 \\ G_{[1,n]} - G_{[1,n+1]} & \text{if } 2 \leq n \leq N-1 \\ G_{[1,N]} & \text{if } n = N \end{cases} \tag{1.4}$$

One can then easily verify, using the properties (1.3-a)-(1.3-c), that $\{E_n \mid 1 \leq n \leq N\}$ is a family of mutually orthogonal projections summing up to $\mathbb{1}$, i.e.:

$$E_n^* = E_n, \qquad E_n E_m = \delta_{m,n} E_n, \qquad \sum_{n=1}^{N} E_n = \mathbb{1} \tag{1.5}$$

There is a non-trivial lower bound for the spectral gap using the martingale method if the following assumption is satisfied.

Assumption 1.1. *There exists a constant ϵ, $0 \leq \epsilon < 1/\sqrt{2}$, such that for all $1 \leq n \leq N-1$*

$$E_n G_{[n,n+1]} E_n \leq \epsilon^2 E_n \tag{1.6}$$

or, equivalently,

$$\| G_{[n,n+1]} E_n \| \leq \epsilon \tag{1.7}$$

Using (1.3-a), it follows that $G_{[n,n+1]} E_n = G_{[n,n+1]} G_{[1,n]} - G_{[1,n+1]}$. This relates Assumption 1.1 with Lemma 6.2 in [2], where an estimate for $\| G_{[n,n+1]} G_{[1,n]} - G_{[1,n+1]} \|$ is given for general Valence Bond Solid chains with a unique infinite volume ground state. Equation (1.3-a) also implies that $[G_{[n,n+1]}, G_{[1,n]}] = [G_{[n,n+1]}, E_n]$, which, if (1.7) holds, the norm is bounded above by 2ϵ.

The following theorem is a special case of Theorem 2.1 in [3].

Theorem 1.2. *With the definitions above and under Assumption 1.1 the following estimate holds for the smallest non-zero eigenvalue of H_N :*

$$\gamma^{(N)} \geq (1 - \sqrt{2}\epsilon)^2 \tag{1.8}$$

Thus, the spectral gap above the ground state H_N is at least $(1 - \sqrt{2}\epsilon)^2$.

Since for our case $n = 2$, theorem 1.2 relies on the norm of $G_{[2,3]}E_2$. This operator, however, is difficult to work with, as determining the intersection of the kernels quickly becomes complicated. To rectify this situation, we instead work with a commutator whose norm can be related to the norm of $G_{[2,3]}E_2$. We define this commutator as

$$C = [P_{12} \otimes \mathbb{I}, \mathbb{I} \otimes P_{23}]$$

Once this norm is calculated, we can look at how C and $G_{[2,3]}E_2$ intersect as projections to determine how the norms of the two operators correlate. The overall goal of this paper is to use $||C||$ to show that there is a sufficiently small upper bound of $||G_{[2,3]}E_2||$. This will, in return, guarantee that the nonzero spectral gap above the ground state that we desire does in fact exist.

As stated previously, in this work we are interested in investigating the Assumption 1.1 by estimating the chain using three particles. We will consider only a class of Valence Bond Solid models with nearest neighbor SU(2) invariant interactions. For these models the interaction terms $P_{i,i+1}$ can be characterized by their action on the irreducible representations of SU(2) found from the tensor product representation of two neighboring spins. In general, the different neighboring $P_{i,i+1}$ do not commute and one can suspect from the discussion above that to satisfy Assumption 1.1, the commutators $[P_{i,i+1}, P_{i+1,i+2}]$ will have to be sufficiently small. It turns out that this is a necessary but not a sufficient condition.

We start with a lemma that will help clarify the meaning of Assumption 1.1 in geometric terms.

Lemma 1.3. *Let E and F be two orthogonal projections on a Hilbert space, and let $E \wedge F$ denote the orthogonal projection onto the intersection of their ranges. Define $\tilde{E} = E - E \wedge F$ an $\tilde{F} = F - E \wedge F$. Then*

$$\begin{aligned} \|EF - E \wedge F\| &= \|\tilde{E}\tilde{F}\| && (1.9) \\ &= \sup\{|\langle \phi, \psi \rangle| \mid \phi \in \operatorname{ran} \tilde{E}, \psi \in \operatorname{ran} \tilde{F}, \|\phi\| = \|\psi\| = 1\} && (1.10) \end{aligned}$$

Proof. First, using the definition of $\tilde{E}$ and $\tilde{F}$ one easily verifies the first equality stated in the lemma:

$$\tilde{E}\tilde{F} = (E - E \wedge F)(F - E \wedge F) = EF - E(E \wedge F) - (E \wedge F)F + E \wedge F = EF - E \wedge F\,.$$

Here, we used that $E \wedge F$ projects onto a subspace of both the range of E and the range of F and hence $E(E \wedge F) = E \wedge F = (E \wedge F)F$.

When the Euclidean norm induces the operator norm, one can rewrite the norm as follows:

$$\|\tilde{E}\tilde{F}\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |y^* \tilde{E}\tilde{F}x|$$

Since $\tilde{E}^* = \tilde{E}$ we get

$$\begin{aligned}
\|\tilde{E}\tilde{F}\| &= \sup_{\|x\|=1} \sup_{\|y\|=1} |y^* \tilde{E}\tilde{F}x| \\
&= \sup_{\|x\|=1} \sup_{\|y\|=1} |(\tilde{E}y)^* \tilde{F}x| \\
&= \sup |\phi^* \psi| \text{ where } \phi = \tilde{E}y, \psi = \tilde{F}x \\
&= \sup\{|\langle \phi, \psi \rangle| \mid \phi \in \operatorname{ran} \tilde{E}, \psi \in \operatorname{ran} \tilde{F}, \|\phi\| = \|\psi\| = 1\}
\end{aligned}$$

□

Clearly, $\tilde{E}$ and $\tilde{F}$ project onto the orthogonal complement of the intersection of the ranges of E and F considered as subspaces of the range of E and the range of F, respectively. Formula (1.10) expresses $\|EF - E \wedge F\|$ as the maximum value of $\cos\theta$, where θ are the angles between these two orthogonal complements. Hence, the norm is small if the ranges of E and F are nearly orthogonal after subtraction of their intersection.

Recall that in this work we are dealing with a chain of three particles. Thus, the orthogonal projectors E and F in this case correspond to the terms $G_{[1,2]}$ and $G_{[2,3]}$, where these terms are related to the two terms in our Hamiltonian by (1.3-c). Also, we have that $G_{[2,3]}E_3 = G_{[2,3]}G_{[1,2]} - G_{[1,3]}$ is equivalent to $EF - E \wedge F$, and C is equivalent to $[E, F]$ in this vocabulary. Since $E \wedge F$ commutes with both E and F, we have that $[E, F] = [\tilde{E}, \tilde{F}]$. Since $\|\tilde{E}\tilde{F}\| = \|(\tilde{E}\tilde{F})^*\| = \|\tilde{F}\tilde{E}\|$, we therefore immediately have the inequality

$$\|[E, F]\| \leq 2\|\tilde{E}\tilde{F}\| = 2\|EF - E \wedge F\|. \tag{1.11}$$

It would be tempting to try to replace the condition on $\|EF - E \wedge F\|$ by one on $\|[E, F]\|$, because the latter quantity is somewhat easier to compute. Unfortunately, in general there is no inequality that bounds $\|\tilde{E}\tilde{F}\|$ by a multiple of $\|[E, F]\|$. In our research, we discovered that such a counterexample occurs for $\|G_{[2,3]}E_2\|$ and $\|C\|$ when we use the spins $j_1 = 3/2, j_2 = 1/2, j_3 = 3$ to create our spin chain.

However, Section 4 we will obtain such an inequality under an additional condition determined by the form of $G_{[2,3]}E_2$ and $[G_{[1,2]}, G_{[2,3]}]$ under a specific change of basis. To obtain this result, we will begin by look at some of the important background information needed to analyze quantum spin chains. Brief discussions of the form of the orthogonal projectors, the type of eigenvalues of the commutator, and the relation between these eigenvalues and the irreducible decomposition of ρ will be given. Also, proofs that in the decomposition of ρ the second largest irreducible appears exactly twice, and that the operator norm of C corresponds to the magnitude of its largest eigenvalue will be presented. It is with this information and some numerical results that we conjecture that the operator norm of C corresponds to the magnitude of the eigenvalue related to the second largest irreducible in the decomposition of ρ. We will calculate this value to be $||C|| = \sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}/(j_1 + j_2)(j_2 + j_3)$. Finally, looking at how C and $G_{[2,3]}E_2$ interact as projections onto subspaces, we will end the paper with the calculation of an upper bound of $G_{[2,3]}E_2$ when the additional assumption is satisfied.

2 Background

This section will be used to present the information needed to understand the thought process that led to the conjecture for the operator norm of C. The structure of the projectors P_{12} and P_{23} will be discussed, along with some properties of ρ and C. Then, connections between ρ and C will be made to understand how the commutator norm depends on ρ.

2.1 The Projectors

To make the projectors for the commutator we must find orthonormal bases that span the vector spaces for the largest irreducible representations of both $\rho_{j_1} \otimes \rho_{j_2}$ and $\rho_{j_2} \otimes \rho_{j_3}$. Using Clebsch-Gordan coefficients, we determine that the largest irreducible representation for the tensor of two irreducibles $\rho_{j_1} \otimes \rho_{j_2}$, where j_1 and j_2 are the spins, is $\rho_{j_1+j_2}$. This representation is of dimension $2(j_1+j_2)+1$. To find the required basis for this representation, one only needs to notice that the highest irreducible representation is the only irreducible representation that will have $j_1 + j_2$ as an eigenvalue of $\rho_{j_1} \otimes \rho_{j_2}(S^3)$. Observe that the corresponding eigenvector is $|j_1\rangle \otimes |j_2\rangle$. Using the lowering operator repeatedly on this eigenvector will produce a series of eigenvectors of $\rho(S^3)$ each with eigenvalue one less than the previous vector. The lowering operator will will produce the zero vector after the eigenvector with corresponding eigenvalue $-(j_1 + j_2)$ of $\rho_{j_1} \otimes \rho_{j_2}(S^3)$ is reached. This set of non-zero vectors are guaranteed to be orthogonal since they have distinct eigenvalues of $\rho_{j_1} \otimes \rho_{j_2}(S^3)$. By normalizing the vectors, we create an orthonormal basis. Now that we understand the process used to create these vectors, we wish to know more about the structure of any given vector as it will only have nonzero entries in specific coordinates.

Proposition 2.1. *For $i = 0, \ldots, 2(j_1 + j_2)$ the orthogonal vectors that create the orthogonal projectors are of the form*

$$\vec{u}_i = \sum_{k=0}^{i} \binom{i}{k} C(j_1 - i + k, j_2 - k)|j_1 - i + k\rangle \otimes |j_2 - k\rangle$$

where $C(j_1 - i + k, j_2 - k) = \prod_{k_1=1}^{i-k} \prod_{k_2=1}^{k} \sqrt{(2j_1 - k_1 + 1)k_1}\sqrt{(2j_2 - k_2 + 1)k_2}$.

Proof. This is a proof by induction. Consider the case when $i = 0$. Then,

$$\begin{aligned} \vec{u}_0 &= \sum_{k=0}^{0} \binom{0}{k} C(j_1 + k, j_2 - k)|j_1 + k\rangle \otimes |j_2 - k\rangle \\ \vec{u}_0 &= |j_1\rangle \otimes |j_2\rangle \end{aligned}$$

Now, consider this true for some arbitrary i within the given bounds. Then,

$$\begin{aligned}
\vec{u}_{i+1} &= \rho_{j_1} \otimes \rho_{j_2}(S^-)\vec{u}_i \\
&= \sum_{k=0}^{i} \binom{i}{k} C(j_1 - i + k, j_2 - k)\rho_{j_1} \otimes \rho_{j_2}(S^-)|j_1 - i + k\rangle \otimes |j_2 - k\rangle \\
&= \sum_{k=0}^{i} \binom{i}{k} C(j_1 - i + k, j_2 - k)(\sqrt{2j_1 - (i-k+1)(i-k+1)}|j_1 - i + k\rangle \otimes |j_2 - k\rangle \\
&\quad + \sqrt{(2j_2 - k)(k+1)}|j_1 - i + k\rangle \otimes |j_2 - k\rangle)
\end{aligned}$$

Reindexing k will produce

$$\vec{u}_{i+1} = \sum_{k=0}^{i+1} \binom{i+1}{k} C(j_1 - (i+1) + k, j_2 - k)|j_1 - i + k\rangle \otimes |j_2 - k\rangle$$

Notice that when $i = 2(j_1 + j_2) + 1$ the coefficient $C(j_1 - i + k, j_2 - k) = 0$ for all terms in the sum, giving the zero vector as expected. □

The desired orthonormal basis is $\mathcal{B} = \{\vec{v}_i\}_{i=0}^{2(j_1+j_2)}$

$$\vec{v}_i = \frac{\vec{u}_i}{\|\vec{u}_i\|}$$

Hence, the orthogonal projector is

$$\sum_{i=0}^{2(j_1+j_2)} |\vec{v}_i\rangle\langle\vec{v}_i| \tag{2.12}$$

2.2 The Second Largest Irreducible Representation

Key to the calculation of $\|C\|$ will be knowing how many times the second largest irreducible representations appears in the decomposition of ρ. This is important as it is conjectured later in the paper that $\|C\|$ is related to two copies of the second largest irreducible representation. Knowing that there are exactly two copies in the decomposition allows use to know that these must be the two copies that relate to $\|C\|$.

Lemma 2.2. *In the irreducible decomposition of $\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$, where j_1, j_2, and j_3 are the respective nonzero spins, the second largest irreducible representation appears exactly twice.*

Proof. Consider $\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$ where ρ_{j_i} is an irreducible representation of $\mathfrak{su}_2$ and $\dim \rho_{j_i} = 2j_i + 1$. Then,

$$\rho_{j_1} \otimes \rho_{j_2} = \rho_{j_1+j_2} \oplus \rho_{j_1+j_2-1} \oplus \ldots \oplus \rho_{|j_1-j_2|}$$

and

$$\begin{aligned}
\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3} &= (\rho_{j_1+j_2} \oplus \rho_{j_1+j_2-1} \oplus \cdots \oplus \rho_{|j_1-j_2|}) \otimes \rho_{j_3} \\
&= \rho_{j_1+j_2} \otimes \rho_{j_3} \oplus \rho_{j_1+j_2-1} \otimes \rho_{j_3} \oplus \cdots \oplus \rho_{|j_1-j_2|} \otimes \rho_{j_3} \\
&= (\rho_{j_1+j_2+j_3} \oplus \rho_{j_1+j_2+j_3-1} \oplus \cdots \oplus \rho_{|j_1+j_2-j_3|}) \\
&\qquad \oplus (\rho_{j_1+j_2+j_3-1} \oplus \rho_{j_1+j_2+j_3-2} \oplus \cdots \oplus \rho_{|j_1+j_2-j_3-1|}) \\
&\qquad \cdots \oplus (\rho_{|j_1-j_2|+j_3} \oplus \rho_{|j_1-j_2|+j_3-1} \oplus \cdots \oplus \rho_{||j_1-j_2|-j_3|})
\end{aligned}$$

The only way for the second largest irreducible representation to appear more than twice is if $|j_1 - j_2| + j_3 = j_1 + j_2 + j_3$ or $|j_1 - j_2| + j_3 = j_1 + j_2 + j_3 - 1$. For $|j_1 - j_2| + j_3 = j_1 + j_2 + j_3$ we need either $j_1 = 0$ or $j_2 = 0$, a contradiction to one of our assumptions. For, the second case, let $j_1 \geq j_2$. Then $j_1 - j_2 \geq 0$ and $|j_1 - j_2| = j_1 - j_2$. Thus, if $|j_1 - j_2| + j_3 = j_1 + j_2 + j_3 - 1$ we must have that $j_2 = \frac{1}{2}$. Similarly, if $j_2 \geq j_1$, then we must have $j_1 = \frac{1}{2}$. Without loss of generality, let $j_1 = \frac{1}{2}$. Then, $\rho_{j_1} \otimes \rho_{j_2} = \rho_{j_2+1/2} \oplus \rho_{j_2-1/2}$, and thus

$$\begin{aligned}
\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3} &= (\rho_{j_2+j_3+1/2} \oplus \rho_{j_2+j_3-1/2} \oplus \cdots \oplus \rho_{|j_2-j_3+1/2|}) \\
&\qquad \oplus (\rho_{j_2+j_3-1/2} \oplus \cdots \oplus \rho_{|j_2-j_3-1/2|}) \\
&= \rho_{j_2+j_3+1/2} \oplus 2\rho_{j_2+j_3-1/2} \oplus \cdots \oplus \rho_{|j_2-j_3-1/2|}
\end{aligned}$$

Once again, the second largest irreducible representation occurs twice. Thus, in all cases, the second largest irreducible representation occurs exactly two times. □

2.3 The Eigenvalues of the Commutator

The operator C for the system $\rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$ has been defined to be

$$C = [P_{12} \otimes \mathbb{I}, \mathbb{I} \otimes P_{23}]$$

where P_{12} and P_{23} are the orthogonal projectors onto the vector spaces acted on by the largest irreducibles in the decompositions of $\rho_{j_1} \otimes \rho_{j_2}$ and $\rho_{j_2} \otimes \rho_{j_3}$, respectively. Since P_{12} and P_{23} are both hermitian (real-symmetric) matrices, so are $P_{12} \otimes \mathbb{I}$ and $\mathbb{I} \otimes P_{23}$. Now we will look at the possible kinds of eigenvalues for such a commutator.

Proposition 2.3. *The eigenvalues of C are purely imaginary. For any non-zero eigenvalue λ of C, $-\lambda$ is also an eigenvalue of C.*

Proof. Consider any two hermitian matrices A and B. It is easily calculated that:

$$\begin{aligned}
[A, B]^* &= (AB)^* - (BA)^* \\
&= B^*A^* - A^*B^* \\
&= BA - AB \\
&= -[A, B]
\end{aligned}$$

Therefore, since both $P_{12} \otimes \mathbb{I}$ and $\mathbb{I} \otimes P_{23}$ are hermitian matrices, the commutator C is anti-hermitian and can be written as iA for some hermitian matrix A. we know from the Spectral Theorem there exists a unitary matrix U composed of eigenvectors of A such that

$$U^*HU = \Lambda$$

where Λ is a real, diagonal matrix. Since C is only a scalar multiple of A, the Spectral Theorem can be applied to find

$$\begin{aligned} U^*CU &= U^*iAU \\ &= iU^*AU \\ &= i\Lambda \end{aligned}$$

Hence, C is diagonalizable with all eigenvalues either imaginary or zero. Furthermore, since C is composed of projectors with real entries, this implies that its characteristic polynomial $p(\lambda)$ has real coefficients. Since the roots of this polynomial are the eigenvalues of C, if any imaginary eigenvalue λ of C exists, λ^* must also be an eigenvalue of C. □

2.4 The Operator Norm of C

Lemma 2.4. *For all anti-hermitian matrices C, $\|C\| = \max_{\|x\|=1} \|Cx\| = \max|\lambda|$ where λ is an eigenvalue of C.*

Proof. It has been shown that for any anti-hermitian matrix C there exists a unitary matrix U such that $U^*CU = D$ is diagonalized. Since the operator norm is invariant under change of bases, $\|C\| = \|U^*CU\| = \|D\|$. Now, choose U such that $D = diag(\mu_1, \mu_2, \ldots, \mu_n)$ with $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_n|$. Then $|\mu_1| = \max|\lambda|$. Let $\vec{v}$ be any unit vector. Hence,

$$\|D\vec{v}\| = \sqrt{(\mu_1 v_1)^2 + \ldots + (\mu_n v_n)^2} \leq \sqrt{\mu_1^2(x_1^2 + \ldots + x_n^2)} = |\mu_1|$$

Since $\|De_1\| = \|\mu_1 e_1\| = |\mu_1|$ we see that in fact, $\|D\| = |\mu_1|$. Hence, we have that

$$\|C\| = \max_{\|x\|=1} \|Dx\| = |\mu_1| = \max|\lambda|$$

□

2.5 Relating the Eigenvalues of C and the Invariant Subspaces

As shown in the previous section, we wish to calculate the largest eigenvalue of C in order to determine its operator norm. The final piece of information we use in developing our conjecture for $\|C\|$ comes from relating the eigenvalues of C to the invariant subspaces on which ρ acts. We will conclude this section with a proof detailing that the degeneracy of

a given eigenvalue of C is at least equal to the dimension of an irreducible representation of ρ. Furthermore, we will show that there exists a basis for each irreducible subspace ρ acts on consisting of eigenvectors of C. In order to prove the aforementioned theorem, we first start with a proof that $\rho(g)$ commutes with C for all g.

Lemma 2.5. *For any $g \in \mathfrak{su}_2$, $\rho(g)C = C\rho(g)$.*

Proof. Consider the tensor $\rho_{j_1} \otimes \rho_{j_2}$ of any two irreducible representations of $\mathfrak{su}_2$ which acts on the vector space $\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1}$. It is possible to find an orthonormal basis $\mathcal{B}$ of the vector space $\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1}$ such that $\mathcal{B}$ can be partitioned so each partition is a basis of one of the irreducible subspaces. Using this basis we can block diagonalize $\rho_{j_1} \otimes \rho_{j_2}(g)$ $\forall g \in \mathfrak{su}_2$ such that

$$\rho_{j_1} \otimes \rho_{j_2}(g) = \begin{pmatrix} \rho_{j_1+j_2}(g) & 0 & \cdots & 0 \\ 0 & \rho_{j_1+j_2-1}(g) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \rho_{|j_1-j_2|}(g) \end{pmatrix}$$

Define M to be the partition of $\mathcal{B}$ described above correlating to $\rho_{j_1+j_2}$. Let N be the remaining ordered subset of $\mathcal{B}$ that produces the block diagonalized form described above. We can write the orthogonal projector P_{12} onto the highest irreducible representation of $\rho_{j_1} \otimes \rho_{j_2}$ as

$$P_{12} = (M|N) \begin{pmatrix} \mathbb{I}_{j_1+j_2} & 0 \\ 0 & 0 \end{pmatrix} (M|N)^*$$

Using the change of basis $(M|N)^* A (M|N)$ on this space will keep all the linear transformations $(M|N)^* \rho_{j_1} \otimes \rho_{j_2}(g)(M|N)$ in a block diagonal form, but will produce the new orthogonal projector

$$P'_{12} = \begin{pmatrix} \mathbb{I}_{j_1+j_2} & 0 \\ 0 & 0 \end{pmatrix}$$

In this basis it is easy to see that $\rho_{j_1} \otimes \rho_{j_2}(g) P'_{12} = P'_{12} \rho_{j_1} \otimes \rho_{j_2}(g)$. Note that this process will work for any block-diagonalized set of matrices with any orthogonal projector of this form. Define $\rho = \rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$ acting on the vector space $\mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j_2+1} \otimes \mathbb{C}^{2j_3+1}$. Then, $\forall g \in \mathfrak{su}_2$

$$\begin{aligned} [\rho(g), P'_{12} \otimes \mathbb{I}] &= \rho(g) P'_{12} \otimes \mathbb{I} - P'_{12} \otimes \mathbb{I} \rho(g) \\ &= \rho(g) P'_{12} \otimes \mathbb{I} - \rho(g) P'_{12} \otimes \mathbb{I} \\ &= 0 \end{aligned}$$

This shows that $[\rho(g), P_{12} \otimes \mathbb{I}] = 0$ Similarly, $[\rho(g), \mathbb{I} \otimes P_{23}] = 0$. Recall that $C = [P_{12} \otimes \mathbb{I}, \mathbb{I} \otimes P_{23}]$. From the properties proven above and the linearity of $[\cdot, \cdot]$ it immediately follows that $\forall g \in \mathfrak{su}_2$

$$\begin{aligned} [\rho(g), C] &= [\rho(g), [P_{12} \otimes \mathbb{I}, \mathbb{I} \otimes P_{23}]] \\ &= 0 \end{aligned}$$

□

Knowing that C commutes with all representations $\rho(g)$ makes it possible to prove the main theorem of this subsection.

Theorem 2.6. *Given any eigenvalue λ of C, the degeneracy of λ is at least equal to the dimension of a specific irreducible representation in the decomposition of ρ. Furthermore, there exists orthonormal bases for each subspace acted on by an irreducible representation of ρ consisting of eigenvectors of C.*

Proof. Consider any $C\rho(g)\psi$ given that $\forall g \in \mathfrak{su}_2, C\rho(g) = \rho(g)C$ and ψ is an eigenvector of C with eigenvalue λ. Then,

$$C\rho(g)\psi = \rho(g)C\psi = \lambda\rho(g)\psi$$

This implies that $\rho(g)\psi$ is also an eigenvector of C with eigenvalue λ. Therefore one can see that $span\{\rho(g)\psi : g \in \mathfrak{su}_2\}$ is an invariant subspace of ρ. Using this process repeatedly and the fact that $\dim(C) = \dim(\rho)$, we can find that all the invariant subspaces of ρ are of this form. Since there is a basis for each irreducible representation of ρ that is a set of linearly independent eigenvectors of C this implies that the degeneracy of each eigenvalue of C is at least equal to the dimension of the irreducible representation of ρ to which its eigenvectors correspond. □

It should be noted that the degeneracy of an eigenvalue may increase if the invariant subspaces of two or more irreducibles have the same corresponding eigenvalue.

3 Calculating $||C||$

Since we know that the second largest representation occurs twice, and that all eigenvalues of C are either 0 or come in purely imaginary complex-conjugate pairs, we can determine that the two corresponding eigenvalues of C for the second largest irrep will always be complex-conjugate pairs (or both zero). Using the information about how to make the orthogonal projectors, and how each eigenvalue of C relates to a specific irreducible representation, we used numerical methods to look at the eigenvalues of the commutator C for some small cases and with the results from 2 we could determine which irreducible representation corresponded to $\max|\lambda|$. For a copy of the Maple code we used to obtain these results, see Appendix A. In all cases, $\max|\lambda|$ corresponded to the second largest irreducible representation (irrep). A table of our numerical results is given below.

Table 1: For a Specific System, $||C||$ and The Corresponding Irreducible Representation

ρ	Irrep corresponding to max $\lvert\lambda\rvert$	max $\lvert\lambda\rvert$
$\rho_{1/2} \otimes \rho_{1/2} \otimes \rho_{1/2}$	$\rho_{1/2}$	$\frac{\sqrt{3}}{4}$
$\rho_1 \otimes \rho_{1/2} \otimes \rho_{1/2}$	ρ_1	$\frac{\sqrt{2}}{3}$
$\rho_{1/2} \otimes \rho_1 \otimes \rho_{1/2}$	ρ_1	$\frac{\sqrt{8}}{9}$
$\rho_1 \otimes \rho_1 \otimes \rho_{1/2}$	$\rho_{3/2}$	$\frac{\sqrt{5}}{6}$
$\rho_1 \otimes \rho_{1/2} \otimes \rho_1$	$\rho_{3/2}$	$\frac{2\sqrt{5}}{9}$
$\rho_1 \otimes \rho_1 \otimes \rho_1$	ρ_2	$\frac{\sqrt{3}}{4}$
$\rho_{1/2} \otimes \rho_{3/2} \otimes \rho_{3/2}$	$\rho_{5/2}$	$\frac{\sqrt{7}}{8}$
$\rho_{3/2} \otimes \rho_{1/2} \otimes \rho_{3/2}$	$\rho_{5/2}$	$\frac{\sqrt{37}}{16}$
$\rho_{3/2} \otimes \rho_{3/2} \otimes \rho_1$	ρ_3	$\frac{\sqrt{4}}{5}$
$\rho_{3/2} \otimes \rho_{3/2} \otimes \rho_{3/2}$	$\rho_{7/2}$	$\frac{\sqrt{3}}{4}$

Conjecture 3.1. *The eigenvalues $\pm\lambda'$ corresponding to the two copies of the second largest irreducible representation satisfy the equation $|\pm\lambda'| \geq |\lambda|$ for all eigenvalues λ of C.*

Since there are bases of the invariant subspaces of $\rho_{j_1+j_2+j_3-1}$ that consist of the desired eigenvectors of C, we only need one eigenvector to determine the corresponding eigenvalues of C. Since we know that the two copies of $\rho_{j_1+j_2+j_3-1}(S^3)$ are the only other two representations in the decomposition other than $\rho_{j_1+j_2+j_3}(S^3)$ to have the eigenvalue $j_1+j_2+j_3-1$, finding a orthonormal basis for the space spanned by these two corresponding eigenvectors will create a two-dimensional space containing two eigenvectors of C. Next we can determine how C acts on this basis to make a 2×2 matrix that has the same eigenvalues as these two eigenvectors of C. It is important to note that the eigenvectors of $\rho_{j_1+j_2+j_3-1}(S^3)$ are some linear combination of the vectors

$$|j_1, j_2, j_3 - 1\rangle = |j_1\rangle \otimes |j_2\rangle \otimes |j_3 - 1\rangle \tag{3.13}$$

$$|j_1, j_2 - 1, j_3\rangle = |j_1\rangle \otimes |j_2 - 1\rangle \otimes |j_3\rangle \tag{3.14}$$

$$|j_1 - 1, j_2, j_3\rangle = |j_1 - 1\rangle \otimes |j_2\rangle \otimes |j_3\rangle \tag{3.15}$$

It is also important to notice that each of these vectors as tensors is a standard basis vector. Hence, these proposed eigenvectors of $\rho_{j_1+j_2+j_3-1}$ will have at most three non-zero entries. The rest of this section will be dedicated to calculating the conjectured norm of C using the method described above.

Theorem 3.2. *The norm of the eigenvalues of C related to $\rho_{j_1+j_2+j_3-1}$ and consequently, the conjecture for $||C||$ is $\frac{\sqrt{j_1 j_2 j_3 (j_1+j_2+j_3)}}{(j_1+j_2)(j_2+j_3)}$.*

Proof. The lowering operator can be applied to $|j_1, j_2, j_3\rangle$ to find the eigenvector $\vec{u}$ of $\rho_{j_1+j_2+j_3}(S^3)$ with eigenvalue $j_1 + j_2 + j_3 - 1$. Using $\langle \vec{u}, \vec{v} \rangle = 0$ where $\vec{v}$ is an arbitrary linear combination of the three vectors mentioned above, and then using the Gram-Schmidt

process, we find an orthonormal basis for the two dimensional space spanned by the eigenvectors of $\rho_{j_1+j_2+j_3-1}(S^3)$ with eigenvalue $j_1+j_2+j_3-1$. These vectors are

$$\vec{v}_1 = \sqrt{\frac{j_1}{j_1+j_3}}|j_1,j_2,j_3-1\rangle - \sqrt{\frac{j_3}{j_1+j_3}}|j_1-1,j_2,j_3\rangle$$

$$\vec{v}_2 = \sqrt{\frac{j_2j_3}{(j_1+j_3)(j_1+j_2+j_3)}}|j_1,j_2,j_3-1\rangle - \sqrt{\frac{j_1+j_3}{(j_1+j_3)(j_1+j_2+j_3)}}|j_1,j_2-1,j_3\rangle$$
$$+\sqrt{\frac{j_1j_2}{(j_1+j_3)(j_1+j_2+j_3)}}|j_1-1,j_2,j_3\rangle$$

To know how C acts on these vectors is equivalent to knowing how $P_{12}\otimes\mathbb{I}$ and $\mathbb{I}\otimes P_{23}$ act on 3.13-3.15. This is given by:

$$\begin{aligned}
P_{12}\otimes\mathbb{I}|j_1,j_2,j_3-1\rangle &= |j_1,j_2,j_3-1\rangle \\
P_{12}\otimes\mathbb{I}|j_1,j_2-1,j_3\rangle &= \frac{j_2}{j_1+j_2}|j_1,j_2-1,j_3\rangle + \frac{\sqrt{j_1j_2}}{j_1+j_2}|j_1-1,j_2,j_3\rangle \\
P_{12}\otimes\mathbb{I}|j_1-1,j_2,j_3\rangle &= \frac{\sqrt{j_1j_2}}{j_1+j_2}|j_1,j_2-1,j_3\rangle + \frac{j_1}{j_1+j_2}|j_1-1,j_2,j_3\rangle \\
\mathbb{I}\otimes P_{23}|j_1,j_2,j_3-1\rangle &= \frac{j_3}{j_2+j_3}|j_1,j_2,j_3-1\rangle + \frac{\sqrt{j_2j_3}}{j_2+j_3}|j_1,j_2-1,j_3\rangle \\
\mathbb{I}\otimes P_{23}|j_1,j_2-1,j_3\rangle &= \frac{\sqrt{j_2j_3}}{j_2+j_3}|j_1,j_2,j_3-1\rangle + \frac{j_2}{j_2+j_3}|j_1,j_2-1,j_3\rangle \\
\mathbb{I}\otimes P_{23}|j_1-1,j_2,j_3\rangle &= |j_1-1,j_2,j_3\rangle
\end{aligned}$$

By acting on $\vec{v}_1$ first by $(P_{12}\otimes\mathbb{I})$ and then by $(\mathbb{I}\otimes P_{23})$ and vice-versa, then subtracting the two results gives $C\vec{v}_1$. The same processes can be used to also find $C\vec{v}_2$. These calculations will result in finding:

$$C\vec{v}_1 = \frac{\sqrt{j_1}j_2j_3}{\sqrt{j_1+j_3}(j_2+j_3)(j_1+j_2)}|j_1,j_2,j_3-1\rangle - \frac{(j_1+j_3)\sqrt{j_1j_2j_3}}{\sqrt{j_1+j_3}(j_2+j_3)(j_1+j_2)}|j_1,j_2-1,j_3\rangle$$
$$+\frac{j_1j_2\sqrt{j_3}}{\sqrt{j_1+j_3}(j_2+j_3)(j_1+j_2)}|j_1-1,j_2,j_3\rangle$$

$$C\vec{v}_2 = \frac{-j_1\sqrt{j_2j_3(j_1+j_2+j_3)}}{\sqrt{j_1+j_3}(j_2+j_3)(j_1+j_2)}|j_1,j_2,j_3-1\rangle + \frac{j_3\sqrt{j_1j_2(j_1+j_2+j_3)}}{\sqrt{j_1+j_3}(j_2+j_3)(j_1+j_2)}|j_1-1,j_2,j_3\rangle$$

From here, the hermitian product is used to determine that

$$C\vec{v}_1 = \frac{\sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}}{(j_1 + j_2)(j_2 + j_3)} \vec{v}_2$$
$$C\vec{v}_2 = \frac{-\sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}}{(j_1 + j_2)(j_2 + j_3)} \vec{v}_1$$

Thus, C can be viewed as a two-dimensional linear transformation described by the matrix:

$$\begin{pmatrix} 0 & \frac{-\sqrt{j_1 j_2 j_3 (j_1+j_2+j_3)}}{(j_1+j_2)(j_2+j_3)} \\ \frac{\sqrt{j_1 j_2 j_3 (j_1+j_2+j_3)}}{(j_1+j_2)(j_2+j_3)} & 0 \end{pmatrix}$$

The desired eigenvalues of C are determined by finding the eigenvalues of this matrix. It is fairly apparent that these eigenvalues are:

$$\lambda = \pm \frac{\sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}}{(j_1 + j_2)(j_2 + j_3)} i$$

These are the two eigenvalues of C that correspond to the irreducible representation $\rho_{x+y+z-1}$, which is the second largest irreducible in the decomposition. Calculating the specific results of λ using the spins from the table agreed with the numerical results of $\|C\|$ for these cases. Hence, it is the conjecture of this paper that $\|C\| = |\lambda|$ or

$$\|C\| = \frac{\sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}}{(j_1 + j_2)(j_2 + j_3)} \tag{3.16}$$

□

4 Bounding $||G_{[2,3]}E_2||$ Via $||C||$

Calculating $||C||$ does not directly give us an any information related to the spectral gap above the ground state of our system. The operator in question that does have a corresponding relationship with this gap is $G_{[2,3]}E_2$. We know that if we are able to bound this operator from above by a sufficiently small number, the spectral gap will be non-zero, see 1.1. Recall that $G_{[2,3]}E_2 = G_{[2,3]}(G_{[1,2]} - G_{[1,3]})$ where $G_{[1,3]} = ker(G_{[1,2]}) \cap ker(G_{[2,3]})$. Working with this operator is rather difficult, as the term $G_{[1,3]}$ becomes increasingly more complex as the size of our Hilbert space increases. Instead, we use the fact that these projections have a nonorthogonal intersection to determine a relationship between their norms.

We were not able to bound the norm for all cases. We found that with our approach, whether the norm of $G_{[2,3]}E_2$ can be approximated depends on the angles between the subspaces projected on by $G_{[2,3]}E_2$ and C. This is because the projection matrices corresponding to these operators will, in general, have a nontrivial dependence on one another

as they usually have a nonorthogonal intersection. The rest of this section is dedicated to determining a sufficient condition for which $\|C\|$ bounds $\|G_{[2,3]}E_2\|$.

To simplify notation, we will be using $G_{[1,2]}$ and $G_{[2,3]}$ instead of $\mathbb{I} \otimes P_{12}$ and $\mathbb{I} \otimes P_{23}$ since a simple calculation can show

$$[G_{[1,2]}, G_{[2,3]}] = [\mathbb{I} \otimes P_{12}, \mathbb{I} \otimes P_{23}]$$

Hence, the norm of the former is equal to the norm of C which we have already conjectured a calculation. We begin first by defining some constants and describing a loose process for finding a desirable basis for the Hilbert space.

Definition 4.1. *Define three particular subspaces of our Hilbert space by*

$$\begin{aligned} \mathcal{G}_{[1,2]} &= \mathrm{Ran}(G_{[1,2]}) \\ \mathcal{G}_{[2,3]} &= \mathrm{Ran}(G_{[2,3]}) \\ \mathcal{G}_{[1,3]} &= \mathrm{Ran}(G_{[1,3]}) \end{aligned}$$

Let m_1 and m_2 be the dimensions of $\mathcal{G}_{[1,2]}$ and $\mathcal{G}_{[2,3]}$, respectively. Let n be the dimension of the space spanned by the intersection of $\mathcal{G}_{[1,2]}$ and $\mathcal{G}_{[2,3]}$. Finally, define $m = m_2 - n$ and $p = m_1 + m_2 - n$. The constants n, m, and p give the dimensions of the non-zero square blocks in the projectors $G_{[1,2]}$, $G_{[1,3]}$ and $G_{[1,3]}$.

Before we can create these projectors, however, we must first pick a basis for the Hilbert space. We begin by first choosing an orthonormal basis $\mathcal{B}'$ of n vectors for the intersection of $\mathcal{G}_{[1,2]}$ and $\mathcal{G}_{[2,3]}$. Note that this is also a basis for $\mathcal{G}_{[1,3]}$. Next, extend this basis to include another m orthonormal vectors to complete a basis $\mathcal{G}_{[2,3]}$. It is imperative that these m vectors are also orthonormal to the original n vectors. Append these vectors onto $\mathcal{B}'$. Next, create another m_1-n orthonormal vectors that are also orthonormal to $\mathcal{B}'$ to complete a basis for $\mathcal{G}_{[1,2]}$. Once again, append these vectors onto $\mathcal{B}'$. By adding additional orthonormal vectors until we span the whole space, we create an orthonormal basis for the Hilbert space that will make nice projectors.

By partitioning the vectors into the sections in which we constructed them, we will get that the projectors $G_{[1,2]}$, $G_{[2,3]}$ and $G_{[1,3]}$ are block diagonalized. Moreover, the majority of these blocks are either 0's or identity blocks, and structurally each one looks like:

$$\begin{aligned} G_{[2,3]} &= \begin{pmatrix} \mathbb{I}_n & & \\ & \mathbb{I}_m & \\ & & 0 \end{pmatrix} \\ G_{[1,2]} &= \begin{pmatrix} \mathbb{I}_n & & \\ & A & \\ & & 0 \end{pmatrix} \\ G_{[1,3]} &= \begin{pmatrix} \mathbb{I}_n & & \\ & 0 & \\ & & 0 \end{pmatrix} \end{aligned}$$

The matrix A in $G_{[1,2]}$ is a p x p matrix. This is not an identity matrix due to the fact that the intersection of $\mathcal{G}_{[1,2]}$ and $\mathcal{G}_{[2,3]}$ is not necessarily orthogonal, implying that there may be some non-trivial dependence on the basis vectors corresponding to $G_{[1,2]}$. We further break down A into a block of four matrices: A_1, A_2, A_3, and A_4 such that the size of A_1 is m x m, the size of A_4 is $p-m$ x $p-m$, and finally, that A_2 and A_3 are of size m x $p-m$, and $p-m$ x m, respectively. This break down of A is of the following manner:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

Lemma 4.2. *A is an orthogonal projection, and thus satisfies that $A^* = A$ and $A^2 = A$.*

Proof. $G_{[1,2]}$ is an orthogonal projector. So $G_{[1,2]}$ satisfies $G^*_{[1,2]} = G_{[1,2]}$, and $G^2_{[1,2]} = G_{[1,2]}$. Since $G_{[1,2]}$ is a block diagonal matrix, and A is one of the blocks, it immediately follows that $A^* = A$, and $A^2 = A$. Hence, by definition, A is an orthogonal projection. □

The previously lemma gives that $A_3 = A_2^*$, and we can replace this for A_3 in our block matrix.

The set of the G matrices gives us all the information needed to analyze how $||C||$ and $||G_{[2,3]}E_2||$ relate. Using the above described matrices, one can easily determine the following:

$$||C|| = \left|\left|\begin{pmatrix} 0 & & & \\ & 0 & A_2 & \\ & -A_2^* & 0 & \\ & & & 0 \end{pmatrix}\right|\right|$$

$$||G_{[2,3]}E_2|| = \left|\left|\begin{pmatrix} 0 & & & \\ & A_1 & A_2 & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix}\right|\right|$$

From lemma 4.2, we can also determine the following relations.

$$A_1^2 + A_2A_2^* = A_1 \tag{4.17}$$
$$A_1A_2 + A_2A_4 = A_2 \tag{4.18}$$
$$A_2^*A_1 + A_4A_2^* = A_2^* \tag{4.19}$$
$$A_2^*A_2 + A_4^2 = A_4 \tag{4.20}$$

Proposition 4.3. *Given that the matrix A_1 satisfies $||A_1|| \leq \frac{1}{2}$, we will have $||G_{[2,3]}E_2|| \leq \sqrt{2}||C||$.*

The proof of 4.3 is given directly after the proof of lemma 4.4.

Now, for convenience, define X_C and X_{GE} to be the following two matrices:

$$X_C = \begin{pmatrix} 0 & A_2 \\ -A_2^* & 0 \end{pmatrix} \tag{4.21}$$

$$X_{GE} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} \tag{4.22}$$

Lemma 4.4. *Defining X_C and X_{GE} from above, it follows that $||C|| = ||X_C||$ and $||G_{[2,3]}E_2|| = ||X_{GE}||$.*

Proof. Since X_C and X_{GE} are the only nonzero components of these representations of C and $G_{[2,3]}E_2$, and they are on the main diagonal, we know that $||C|| = ||X_C||$ and $||G_{[2,3]}E_2|| = ||X_{GE}||$. □

We now finish with the proof of lemma 4.3.

Proof. We also know that the norms of X_C and X_{GE} are the square roots of their, respective, largest singular values. From the singular value decomposition of matrices, we know that for any matrix A, A^*A and AA^* have the same set of nonzero eigenvalues. We also know that $||A||^2 = ||A^*A|| = ||AA^*||$ when the Euclidean norm induces the operator norm. Since it does not matter which way we multiply the matrices when determining the norm, we choose to multiply X_C as $X_C^*X_C$, and X_{GE} as $X_{GE}X_{GE}^*$. From these multiplications, we get:

$$X_C^*X_C = \begin{pmatrix} A_2A_2^* & 0 \\ 0 & A_2^*A_2 \end{pmatrix}$$

$$X_{GE}X_{GE}^* = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$$

Using equations 4.17, we can rewrite the first of these two matrices as follows:

$$X_C^*X_C = \begin{pmatrix} A_1(\mathbb{I} - A_1) & 0 \\ 0 & A_3(\mathbb{I} - A_3) \end{pmatrix}$$

The original structure of $X_C^*X_C$ shows that the two new blocks on the main diagonal will have the same set of nonzero eigenvalues. This means we can look at either block to determine how the eigenvalues of $X_C^*X_C$ and $X_{GE}^*X_{GE}^*$ relate. Thus, we choose the first block in $X_C^*X_C$ since it depends only on A_1.

Lemma 4.5. *Given any eigenvalue λ of A_1, λ is also an eigenvalue of $X_{GE}X_{GE}^*$. Furthermore, $\lambda(1-\lambda)$ is an eigenvalue of $X_C^*X_C$.*

Proof. Trivially, λ is an eigenvalue of $X_{GE}X_{GE}^*$. Let $\vec{v}$ be an associated eigenvector of λ. Then,

$$\begin{aligned} A_1(\mathbb{I} - A_1)\vec{v} &= A_1(\mathbb{I}\vec{v} - A_1\vec{v}) \\ &= A_1(1-\lambda)\vec{v} \\ &= (1-\lambda)A_1\vec{v} \\ &= \lambda(1-\lambda)\vec{v} \end{aligned}$$

□

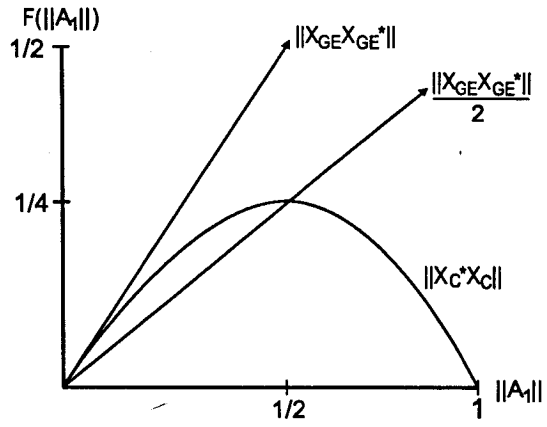

Figure 2: A graphic representation of $||X_C^*X_C||$, $||X_{GE}X_{GE}^*||$, *and* $\frac{||X_{GE}X_{GE}^*||}{2}$ *as functions of* $||A_1||$.

Therefore, the eigenvalues of $X_C^*X_C$ are quadratically related to the eigenvalues of $X_{GE}X_{GE}^*$. It directly follows that if and only if the largest eigenvalue of A_1 has a norm less than or equal to a half, that the following relation holds, see figure 2.

$$||X_C^*X_C|| \leq ||X_{GE}X_{GE}^*|| \leq 2||X_C^*X_C||$$

Recall that $||A||^2 = ||A^*A|| = ||A^*A||$. From here we can find:

$$\begin{aligned} & ||X_C^*X_C|| \leq ||X_{GE}X_{GE}^*|| \leq 2||X_C^*X_C|| \\ \Leftrightarrow \quad & ||X_C||^2 \leq ||X_{GE}||^2 \leq 2||X_C||^2 \\ \Leftrightarrow \quad & ||X_C|| \leq ||X_{GE}|| \leq \sqrt{2}||X_C|| \end{aligned}$$

Since $||C|| = ||X_C||$ and $||G_{[2,3]}E_2|| = ||X_{GE}||$, we know that:

$$||A_1|| \leq \frac{1}{2} \Leftrightarrow ||C|| \leq ||G_{[2,3]}E_2|| \leq \sqrt{2}||C||$$

□

Table 2: The Corresponding $||A_1||$ Given the Spins of a System

Spins	$\|\|A_1\|\|$
1/2, 1/2, 1/2	0.2500
1, 1/2, 1	0.4444
3/2, 3, 3/2	0.1111
1/2, 1/2, 3	0.4285
4, 2, 2	0.3333
1, 4, 2	0.0666
7/2, 2, 5/2	0.3535
1, 3/2, 2	0.2285
3, 2, 2	0.3000
2, 1, 2	0.4444
3/2, 1/2, 3	0.6428
3, 3, 3	0.2500
2, 2, 2	0.2500
3/2, 3/2, 3/2	0.2500
4, 3/2, 2	0.4155
1/2, 2, 3	0.1200
3/2, 4, 7/2	0.1272
3, 1/2, 3/2	0.6428
5/2, 1, 1/2	0.2380
5/2, 1, 3/2	0.4285

The most natural question to now address is, "What is this matrix A_1 and is its norm always less than a half?" Determining A_1 depends on how one follows the choice of basis described above. However, we know that $||A_1|| = ||(G_{[2,3]}E_2)^* G_{[2,3]}E_2||$ and while in general $G_{[2,3]}E_2$ is difficult to calculate, numerically, it is only as difficult as calculating C. Hence, we can use the calculation of $(G_{[2,3]}E_2)^* G_{[2,3]}E_2$ to our advantage to obtain some numerical results. Note that using a similar argument to that of section 2.5, we can determine that the singular values of $G_{[2,3]}E_2$ also have a correlation with the irreducible representations of a given system. However, $G_{[2,3]}E_2$ is neither Hermitian or anti-Hermitian, and thus the method we used to determine which irreducible representation corresponded to $||C||$ does not apply for all $G_{[2,3]}E_2$. Hence, further research is required to conjecture which irreducible representation corresponds to $||G_{[2,3]}E_2||$. A summary of the numerical results obtained through a Matlab program, code given in Appendix B, is given in the table 4.

These results show that there are indeed cases where $||A_1|| > 1/2$. This shows that the process described above does not always work as a means to bound $G_{[2,3]}E_2$. What is interesting, though, is there does seem to be some pattern to $||A_1||$ as the norm, in the cases above, is the same when $j_1 = j_2 = j_3 = j$. Since $||G_{[2,3]}E_2|| = \sqrt{||A_1||}$, this implies there could be another approach to calculate the explicit value of $||G_{[2,3]}E_2||$ given a set of

three spins where all spins are the same value.

5 Conclusion

Determining the energy gap above the ground state of a given particle system is of great interest in the physics and engineering realms. One such system of interest is quantum spin chains. While in actuality these chains tend towards the infinite limit, we hope to obtain a good estimate for the spectral gap above the ground state by looking at a small subsection of a patterned chain (i.e. one in which the spins are repeated in a specific order). For this specific project, we examined a subsystem of three particles, which we mathematically represented by taking the tensor product of the irreducible representations of the Lie algebra of $\mathfrak{su}2$. The final representation $\rho = \rho_{j_1} \otimes \rho_{j_2} \otimes \rho_{j_3}$ was not itself irreducible, but by using Clebsch-Gordon coefficients, we could break ρ into its irreducible parts. We know that the spectral gap above the ground state as an inverse relation with $||G_{[2,3]}E_2||$. This means, that if we are able to bound $G_{[2,3]}E_2$ from above by a sufficiently small positive number, that the spectral gap, call it γ is bounded below by a nonzero positive value. This insures that the specified spectral gap is non-vanishing.

The operator $G_{[2,3]}E_2$, however, is difficult to work with as it requires knowing the zero eigenspace of P_{12} and P_{23}, which project orthogonally onto the subspaces corresponding to the highest irreducible representations of $\rho_{j_1} \otimes \rho_{j_2}$, and $\rho_{j_2} \otimes \rho_{j_3}$, respectively. Instead, we defined and analyzed the operator $C = [P_{12} \otimes \mathbb{I}, \mathbb{I} \otimes P_{23}]$, in attempts to relate norm of this operator to the norm of $G_{[2,3]}E_2$. Looking at properties of both ρ and C determined a relationship between $||C|| = \max|\lambda|$ and an irreducible representation in the decomposition of ρ. Furthermore, numerical analysis led to the conjecture that this $\max|\lambda|$ is associated with the second largest irreducible representations of ρ. Since the form of the projectors used to create C was known, we calculated the conjectured value by looking at how C acted on a pair of vectors where each vector was known to belong the span of one of the two copies of the second largest irreducible representation. We found the conjectured value $\max|\lambda|$ to be

$$\max|\lambda| = \frac{\sqrt{j_1 j_2 j_3 (j_1 + j_2 + j_3)}}{(j_1 + j_2)(j_2 + j_3)}$$

After determining the proposed value of $||C||$, we worked toward finding a relationship between $||C||$, and $||G_{[2,3]}E_2||$. By choosing a nice orthonormal basis, we determined that the norms of these operators depend only on a single matrix A, and because the operators were orthogonal projectors, depend only on a single block of A labeled A_1. Finally, we showed that as long as $||A_1|| \leq 1/2$ that $||C|| \leq ||G_{[2,3]}E_2|| \leq \sqrt{2}||C||$. Using the results from calculating $||C||$, this implies that

$$||G_{[2,3]}E_2|| \leq \frac{\sqrt{2 j_1 j_2 j_3 (j_1 + j_2 + j_3)}}{(j_1 + j_2)(j_2 + j_3)}$$

Numerical solutions, however, showed that there are indeed combinations of spins that result in $||A_1|| > 1/2$. These results also indicated, however, that there may be some nice pattern to $||A_1||$ and thus to $||G_{[2,3]}E_2||$. Therefore, new questions regarding what set of spins do and do not imply $||A_1|| \leq 1/2$ and other possible approaches to solve the problem arise. Possible future research includes determining conditions on the spin that will insure that $||A_1|| \leq 1/2$, and possibly analyzing the structure and properties of $G_{[2,3]}E_2$ in attempts to calculate its norm explicitly. Also, finding a proof or counterexample to the conjectured value of $||C||$ will provide further information into the direction one might take in approaching the problem in the future.

6 Appendix A: Maple Code

```
with (LinearAlgebra); interface(rtablesize = infinity);

CommutatorNorm :=proc(dim) local S,Id,i,

    #Defining local variables
    a,b,x,y,z,
    u,v,
    Sxy,Syz,
    P12, P23, Q, lambda;

    #Defining the three dimensions, one fore each particle in the
    chain.
    x:=dim[1];
    y:=dim[2];
    z:=dim[3];

    #Creating the lowering operator and identity matrices associated
    with each particle in the chain.
    for i from 1 to 3 do
        S[i]:=LowOp(dim[i]);
    Id[i]:=IdentityMatrix(dim[i]);
od;

#Defining the final lowering operators used to create the
projections P12 and P23
Sxy:=Kron(S[1],Id[2])+Kron(Id[1],S[2]);\\
Syz:=Kron(S[2],Id[3])+Kron(Id[2],S[3]);\\

# The dimensions of the highest irreducible representations from the
neighboring pairs.
a:=x+y-1; b:=y+z-1;

#Creating the first vectors for the highest irreps for the
```

```
neighboring pairs. Then creating the two projections.

u[1]:=StdVec(x*y); v[1]:=StdVec(y*z);

P12:=basis(a, u[1],Sxy); P23:=basis(b, v[1],Syz);

# The final operator C and the printing out the matrix lambda
consisting of the eigenvalues of C.

C:=Kron(P12,Id[3]).Kron(Id[1],P23)-Kron(Id[1],P23).Kron(P12,Id[3]);
lamba:=Eigenvalues(C);

end;

# The function LowOP makes the individual lowering operators of each
particle. It puts the values j*(x-j) in the jth column. Here, x=2s+1
where s is the spin of the particle.

LowOp:=proc(x) local i,j,f;
  f:=(i,j)->piecewise(j=i-1,sqrt(j*(x-j)));
  Matrix(x,f);
end;

#The function StdVec makes the std basis vector e_1 for the given
dimension d.

StdVec:=proc(d)
  local s;
  s:={1=1};
  Vector(d,s);
end;

#The function Kron defines the tensor product of two matrices. This
code is not found in the version of Maple I used. The source code I
used was found at
http://www.mapleprimes.com/forum/kronecker/tensor-products

Kron:=proc(A::Matrix,B::Matrix)
  local M,P,i,j;
  M:=Matrix(RowDimension(A)*RowDimension(B),
            ColumnDimension(A)*ColumnDimension(B)):
  P:=Matrix(RowDimension(B),ColumnDimension(B)):
  for i to RowDimension(A) do
     for j to ColumnDimension(A) do
        P:=ScalarMultiply(B,A[i,j]):

        M[1+(i-1)*RowDimension(B)..(i-1)*RowDimension(B)+RowDimension(B),1
```

```
        +(j-1)*ColumnDimension(B)..(j-1)*ColumnDimension(B)
        +ColumnDimension(B)]:=P:
    od;
  od;
  M;
end;

#The function basis creates the basis for the subspace acted on by
the highest irreducible representation of a pair of particles. It
uses the lowering operator to get a new independent vector in the
span, and then normalizes the vector. The result is an orthonormal
basis. Finally, the projector is made by taking the sum of the
tensor product of each vector with itself.

basis:=proc(d,x,S)
  local w, norm, A, R, Rt, P;

  w[1]:=x;
  for i from 1 to d-1 do
     norm:=VectorNorm(w[i],2,conjugate=true);
     w[i]:=w[i]/norm;
     w[i+1]:=S.w[i];
  od;

  norm:=VectorNorm(w[d],2,conjugate=true);
  w[d]:=w[d]/norm;

  A:=[];
  for i from 1 to d do
     A:=[op(A), w[i]];
  od;
  R:=Matrix(A);
  Rt:=HermitianTranspose(R);
  P:=R.Rt;
end;
```

7 Appendix B: Matlab Code

```
%Program: Energy  Values of G_23E_2
%Input: A matrix A of spins. Each row consists of the spins for a chain of
%three particles.
%Output: The eigenvalues (in decreasing order) of the operator A1, which
%satisfies ||G_23E_2||^2=||A1|| and the dimensions of all irreps of the 3
%particle spin chain. Both are sent out of the program as arrays.
```

```
function [enval, irreps]=energyvalues(A)

A

for i=1:3
    dim(i)=2*A(i)+1;
end

%The next section of code creates the lowering operators for each
%individual particle. The number at the end of the S denotes its position
%in the chain of three particles.
%-------------------------------------------------------

S1=zeros(dim(1),dim(1)); for j=1:dim(1)-1
    S1(j+1,j)=sqrt(j*(dim(1)-j));
end

S2=zeros(dim(2),dim(2)); for j=1:dim(2)-1
    S2(j+1,j)=sqrt(j*(dim(2)-j));
end

S3=zeros(dim(3),dim(3)); for j=1:dim(3)-1
    S3(j+1,j)=sqrt(j*(dim(3)-j));
end

%-------------------------------------------------------

%Creating the lowering operators that act on the two neighboring pairs of
%particles.

S12=kron(S1,eye(dim(2)))+kron(eye(dim(1)), S2);
S23=kron(S2,eye(dim(3)))+kron(eye(dim(2)), S3);

%u and v are the eigenvectors of the S_3 operator on each neighboring pair
%with the highest eigenvalue. These are also known to be two vectors
%correlating to the highest irreducible representation.

u=zeros(dim(1)*dim(2),1); u(1)=1;

v=zeros(dim(2)*dim(3),1); v(1)=1;

%Creates the projectors by sending to a projector making function.

P12=projector(dim(1)+dim(2)-1, u, S12);
P23=projector(dim(2)+dim(3)-1, v, S23);
```

```
%P is the sum of the two projectors, which acts on the whole Hilbert space.
%The zero eigenvectors of this operator are used to create G13, which
%projects onto the intersection of the kernel of these two projections.

P=kron(P12, eye(dim(3)))+kron(eye(dim(1)), P23); Q=null(P);
G13=Q*Q';

%Creates the projection onto the null space of the projectors of the
%highest irreps.

G12=eye(dim(1)*dim(2)*dim(3))-kron(P12, eye(dim(3)));
G23=eye(dim(1)*dim(2)*dim(3))-kron(eye(dim(1)), P23);

%The final operator G_23E_2, and the matrix A1.

G23E2=G23*G12-G13; A1=G23E2'*G23E2;

%swap(eig(G23E2))
%norm(G23E2)
%swap(eig(A1))
%norm(A1)


enval=eig(A1);
%enval=eig(G23E2);

enval=swap(enval);

%The next protion of code determines the spins associated with the
%irreducible representations for the given system of three particles.
%-----------------------------------------------------------------------
k=0; for s=A(1)+A(2):-1:abs(A(1)-A(2))
    k=k+1;
    irreps12(k,1)=s;
end %final k is the final number of rows.


l=0; for s=irreps12(1,1)+A(3):-1:abs(irreps12(1,1)-A(3))
    l=l+1;
    I(1,l)=s;
end %final l is the final number of cols.

for i=2:k
    j=0;
    for s=irreps12(i,1)+A(3):-1:abs(irreps12(i,1)-A(3))
        j=j+1;
        I(i,j)=s;
```

```
    end
    if j<l
        for m=j+1:l
            I(i,m)=-1; %Since I know -1 will not be a size of an irrep,
                       %I use this to fill in space to complete the matrix.
                       %Once I have the matrix filled in, I will read off
                       %the values that are not -1 into an array for the
                       %final irrep vector.
        end
    end
end

%Creating final irrep vector by throwing out -1 entries in the matrix.
m=0; for i=1:k
    for j=1:l
        if I(i,j)~=-1
            m=m+1;
            irreps(m)=I(i,j);
        end
    end
end

%Sorting the irrep vector into descending order like the enval vector.
irreps=irreps'; irreps=swap(irreps); irreps=irreps';

%The next functions create the Projectors for the program.
%-----------------------------------------------------------------------
function P=projector(n, w, S)
    %n=number of vectors in largest irrep
    %w=initial vector in largest irrep
    %S=lowering operator used to create other vectors.
    %Sends back the projector P.

    P=w*w';
    for i=2:n
        w=S*w;
        w=w/norm(w);
        P=P+w*w';
    end

end

function v=swap(u)
    %Takes in a vector u.
    %Sends back vector with its entries ordered from largest to smallest
    %in value.
```

```
    v=-u;
    v=sortrows(v);
    v=-v;
end

end
```

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