Pattern Avoidance in Unimodal and V-unimodal Permutations

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Abstract

A characterization of unimodal, [321]-avoiding permutations and an enumeration shall be given.There is a bijection between unimodal [321]-avoiding permutations and v-unimodal permutations that exposes similarities between their structures and pattern avoidance properties.

1 Introduction

A permutation is a rearrangement of objects. One example is scrambling a word that consists of distinct letters such as "math".

Definition 1. A permutation is a bijection, a function that is both one-to-one and onto, from a finite set onto itself.

Here, the set $\{1, 2, \ldots n\}$, where n is a positive integer, will be considered. Whether or not the elements of the set being permuted are numbers, a permutation may be denoted using two-line notation, where the elements of the set are written on the first line and their images are below in the second line. For example, let p be a permutation of $\{m, a, t, h\}$, in two-line notation:

$$p = \begin{pmatrix} m & a & t & h \\ a & h & m & t \end{pmatrix}$$

This means that $p(m) = a$, $p(a) = h$, $p(t) = m$, $p(h) = t$. In general, a permutation, p, of a set $\{x_1, x_2, \ldots x_n\}$ in two-line notation looks like:

$$p = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ p(x_1) & p(x_2) & \ldots & p(x_n) \end{pmatrix}$$

Permutations of $\{1, 2, \ldots n\}$ are considered because numbers have a total ordering. In other words, there is a notion of "greater than" and "less than" and no possibility of confusion when comparing distinct numbers. In this case, permutations can be written in one-line notation, where the permutation is denoted by the second line of the two-line notation keeping the order in the first line fixed. For example, a permutation on $\{1, 2, 3, 4\}$ could be written:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

in two-line notation, and in one-line notation: [2413], which is the second line of the permutation's two-line notation. The element $p(x_i)$ will be denoted by p_{x_i}, so p can be written $[p_{x_1} p_{x_2} \ldots p_{x_n}]$ in one-line notation. The one-line notation will be used throughout.

Permutations can be drawn as graphs in the plane. The picture can be thought of as a matrix with a dot in the row corresponding to the image of the column number in the permutation and blanks everywhere else (Figure 1).

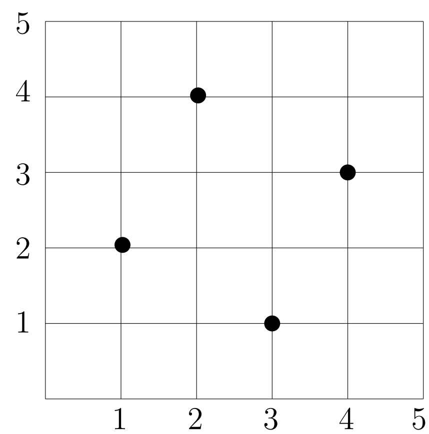

Figure 1: Graph of 2413

In several results, the number of elements in the set being permuted relates to some properties of permutations.

Definition 2. Let S be a set, and $p : S \to S$ be a permutation. The <u>size</u> of p is the number of elements in the set S.

A permutation of size n shall be denoted $p = [p_1 p_2 \dots p_n]$ in the one-line notation throughout. Ordering makes it possible to define what it means for a permutation to contain a smaller permutation.

Definition 3. Let $p = [p_1 p_2 \dots p_n]$ and $q = [q_1 q_2 \dots q_m]$ be permutations in the one-line notation. A <u>permutation p contains the permutation q</u> (or <u>p has an instance of q</u>) if there exist indices, $i_1, \dots, i_m$ such that the subsequence $p_{i_1} p_{i_2} \dots p_{i_m}$ has the property $p_{i_a} < p_{i_b}$ implies $q_a < q_b$ and $q_a < q_b$ implies $p_{i_a} < p_{i_b}$. Furthermore, if the indices do not exist, then <u>p avoids q</u>.

Consider the permutation [123654] (Figure 2). It has an instance of [321] which occurs as 654, and it has some instances of [132] as 264, 364, and 165. It avoids [213] and [312]. For more information about permutation avoidance and containment see [Bón04].

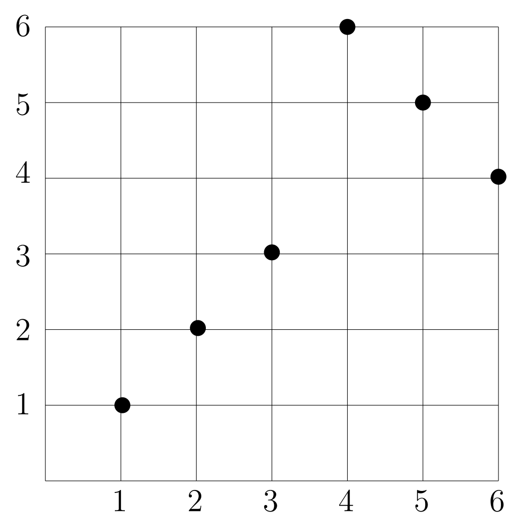

Figure 2: The graph of [123654]. An instance of [132] appears as the points $(2,2)$, $(4,6)$, $(6,4)$, corresponding to the occurrence 264. The graphs of [213] and [312] do not appear at all.

There are various applications of the idea of avoidance. The problem of avoiding permutations arose in computer scicence when the stack-sorting algorithm was studied by Donald Knuth, see [Knu75]. The stack-sorting algorithm sorts lists of numbers, putting them in increasing order. He proved that a permutation can be sorted using the stack-sorting algorithm if and only if the permutation is [231]-avoiding. For a proof see [Bón04], or for more information see [Bón03]. The stack sorting algorithm is as follows:

Let $p = [p_1 p_2 \dots p_n]$ be a permutation of size n.

Step 1: Place the first entry, p_1, in the stack, which is a box that holds the numbers by stacking them on top of each other.

Step 2: Take the i-th entry p_i and denote the top entry of the stack by s_t. If $p_i < s_t$ or the stack is empty, then p_i goes into the stack. If $p_i > s_t$ or no entries are left in p, then s_t is placed in the leftmost available space in the output permutation, $[q_1 q_2 \dots q_n]$.

Step 3: If p_i went into the stack, repeat Step 2 for p_{i+1}. If s_t moved out of the stack, repeat Step 2 for p_i.

Step 4: Once all entries of p are exhausted, the stack's remaining entries are put at the end of the output list in the order they are in the stack, starting from the top entry in the stack. The resulting permutation should be $[123 \dots n]$.

Here are two examples where a [231]-avoiding permutation and a permutation containing [231] are stack-sorted:

To sort [2134] using stack sorting do the following:

Step 1: 2 is the first element in the stack: $\begin{array}{|c|} 2 \\ \hline \end{array}$.

Step 2: $1 < 2$, so 1 goes on top of 2 in the stack, so the stack looks like: $\begin{array}{|c|} 1 \\ 2 \\ \hline \end{array}$.

Step 3: $3 > 1$ so 1 moves out of the stack and into the new permutation [1___].

Step 4: Now the stack is $\begin{array}{|c|} 2 \\ \hline \end{array}$.

Step 5: $3 > 2$, so 2 moves out of the stack and into the new permutaion [12__].

Step 6: Since the stack is empty, 3 goes into the stack, so the stack is now $\begin{array}{|c|} 3 \\ \hline \end{array}$

Step 7: $4 > 3$, so 3 moves out of the stack and into the permutation, so now it is [123_].

Step 8: Now, 4 goes into the stack, and it is the only entry in the stack, so 4 goes into the permutation, giving [1234].

When stack-sorting is applied to [4231], the algorithm produces a different permutation.

Step 1: 4 is the first entry in the stack: $\begin{array}{|c|} 4 \\ \hline \end{array}$.

Step 2: $2 < 4$, so 2 goes in the stack, so the stack looks like: $\begin{array}{|c|} 2 \\ 4 \\ \hline \end{array}$.

Step 3: $3 > 2$ so 2 moves into the new permutation [2___], so this does not produce [1234].

Julian West developed a method for generating permutations using rooted trees. The trees are called generating trees and they generate all permutations that avoid given patterns. They are rooted at the size 1 permutation, [1]. The children are formed by placing the next entry of the permutation in any space between entries or on the ends of the permutation as long as the children avoid the patterns. Using the generating tree and the idea of succession rules, permutations can be counted. For more information see [Wes96], and to see them used for counting [123]-avoiding permutations, and others, see [Wes95].

My partner, Cameron Alston, wrote a program based on Julian West's generating tree axioms. It generates signed permutations, which are permutations where entries can be negative. The program generates them avoiding given signed patterns. More algorithms for generating permutations can be found in [Knu05].

Here, the types of permutations that will be studied using permutation avoidance are unimodal permutations.

Definition 4. A permutation $p = [p_1 p_2 \dots p_n]$ is <u>unimodal</u> if and only if there exists an index i such that $p_1 < p_2 < \cdots < p_i > p_{i+1} > \cdots > p_n$.

Unimodal permutations have a hill shape; they rise to the top and then go down. For example, [1237654] and [346521] (Figure 3) are unimodal permutations. Note that in the definition i has no restriction forcing $1 < i$ nor $i < n$, so the identity permutation, which will be denoted $e = [123 \dots n]$, is a unimodal permutation ($i = n$). Also, $[n(n-1)\dots 321]$ is a unimodal permutation ($i = 1$).

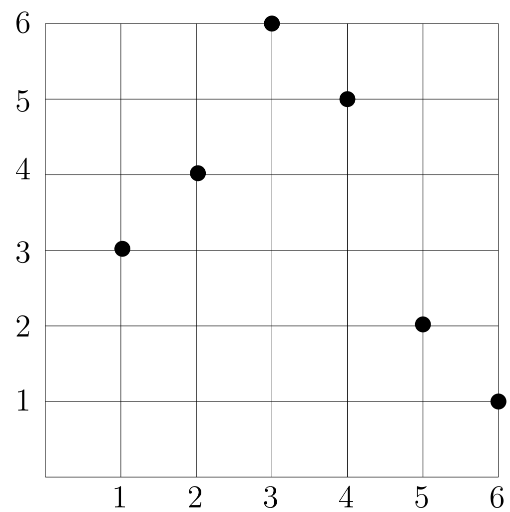

Figure 3: The graph of the unimodal permutation [346521]

Unimodal permutations share many properties with v-unimodal permutations, which are defined in Section 3. There is a bijection relating their structures. It will be shown that the number of unimodal permutations of size n is 2^{n-1} (for more information see [Slo04]). Also, it will be proven that the number of unimodal permutations of size n that avoid $[m(m-1)\dots 321]$ with $m \geq 2$ is

$$\sum_{k=2}^{m} \binom{n-1}{k-2} = \sum_{j=0}^{m-2} \binom{n-1}{j}.$$

The bijection proves the same for v-unimodal permutations.

2 Enumeration of $\{[321], [213], [312]\}$-avoiding Permutations

Unimodal permutations that avoid [321] have a specific form.

Proposition 5. *A unimodal* [321]*-avoiding permutation* $p = [p_1 p_2 \dots p_n]$ *has* $p_1 < p_2 < \cdots < p_{n-1}$.

Proof. Let $p = [p_1 p_2 \dots p_n]$ be a unimodal [321]-avoiding permutation with n fixed. Then, p is [321]-avoiding, so p has no descending subsequence of size larger than 2. Since n is the largest element of $\{1, \dots, n\}$ and p is unimodal, $p_{n-1} = n$ or $p_n = n$ are the only possible positions for n. Therefore, a unimodal [321]-avoiding permutation $p = [p_1 p_2 \dots p_n]$ has $p_1 < p_2 < \cdots < p_{n-1}$. □

This specific form for unimodal [321]-avoiding permutations helps count them.

Theorem 6. *For every* $k \in \{1, \dots, n\}$ *there is a unique unimodal* [321]*-avoiding permutation* $p = [p_1 \dots p_n]$ *such that* $p_n = k$.

Proof. Let n be fixed, $k \in \{1, \dots, n\}$ and $p = [p_1 p_2 \dots p_i \dots p_n]$ be a unimodal [321]-avoiding permutation with $p_n = k$. Since p is unimodal and [321]-avoiding, Proposition 5 implies the elements of $\{1, \dots, n\} - \{p_n\}$ are arranged in increasing order in $p_1 p_2 \dots p_{n-1}$. Since the increasing order on $\{1, \dots, n\} - \{p_n\}$ is unique, p is the unique unimodal [321]-avoiding permutation of size n with $p_n = k$. Therefore, for every $k \in \{1, \dots, n\}$ there is a unique unimodal [321]-avoiding permutation $p = [p_1 \dots p_n]$ such that $p_n = k$. □

This theorem implies the following corollary about the number of unimodal [321]-avoiding permutations.

Corollary 7. *The number of unimodal* [321]*-avoiding permutations of size* n *is* n*.*

Proof. Let n be fixed. By Theorem 6, a unimodal [321]-avoiding permutation is uniquely determined by $p_n \in \{1, \ldots, n\}$. Therefore, there are n unimodal, [321]-avoiding permutations of size n. □

The avoidance of permutations will be used to characterize unimodal [321]-avoiding permutations in the following result.

Theorem 8. *A permutation, p, is* $\{[321], [213], [312]\}$*-avoiding if and only if* p *is* [321]*-avoiding and unimodal.*

Proof. First, it will be shown that if p is $\{[321], [213], [312]\}$-avoiding, then p is [321]-avoiding and unimodal. This result is clear for e, so let $p \neq e$. Let $p = [p_1 p_2 \ldots p_i \ldots p_n]$ be a $\{[321], [213], [312]\}$-avoiding permutation of size n that is not the identity. If for some $i \in \{1, \ldots, n-2\}$, $p_i = n$, then $p_i > p_{i+1}$, and $p_i > p_{i+2}$. Since p is [321]-avoiding, $p_{i+1} < p_{i+2}$, but this implies that $p_i p_{i+1} p_{i+2}$ is a [312] instance, a contradiction to p avoiding [312]. If $p_n = n$, then it follows from $p \neq e$ that $p_1 p_2 \ldots p_{n-1}$ is not an increasing sequence. Hence, there exist j and k with $j < k \leq n-1$ such that $p_j > p_k$. Since $p_k < p_j < p_n$, $p_j p_k p_n$ forms a [213] instance which contradicts p avoiding [213]. Thus, $p_{n-1} = n$, and $n > p_n$.

Suppose for a contradiction that $p_1 p_2 \ldots p_{n-2}$ is not an increasing sequence. Then, there exist j and k with $j < k \leq n-2$ such that $p_j > p_k$. Since $p_k < p_j < p_{n-1}$, it follows $p_j p_k p_{n-1}$ forms a [213] instance contrary to p avoiding [213]. Hence, $p_1 p_2 \ldots p_{n-2}$ is an increasing subsequence, so $p_1 < p_2 < \ldots p_{n-2} < p_{n-1} = n > p_{i+1} > \cdots > p_n$, which implies that p is unimodal by definition. Furthermore, p is [321]-avoiding by assumption, so p is a unimodal [321]-avoiding permutation. Thus, if p, is $\{[321], [213], [312]\}$-avoiding, then p is a unimodal [321]-avoiding permutation.

Now consider when p is a unimodal [321]-avoiding permutation. Proposition 5 implies $p_1 < p_2 < \cdots < p_{n-2} < p_{n-1}$. It follows from $p_1 < p_2 < \cdots < p_{n-2} < p_{n-1}$ that no instances of [213] or [312] occur in $p_1 p_2 \ldots p_{n-1}$. The inequality $p_1 < p_2 < \cdots < p_{n-2} < p_{n-1}$ implies any pattern in p containing two entries in decreasing order will always end with p_n and the pattern will be a [123], [132], or [231] occurrence. Hence, p, is $\{[321], [213], [312]\}$-avoiding.

Therefore, a permutation, p, is $\{[321],[213],[312]\}$-avoiding if and only if p is [321]-avoiding and unimodal. $\square$

Theorems 7 and 8 lead to the main result of this section; the enumeration of $\{[321],[213],[312]\}$-avoiding permutations using unimodal [321]-avoiding permutations.

Theorem 9. *The number of* $\{[321],[213],[312]\}$*-avoiding permutations of size* n *is* n*.*

Proof. By Theorem 8, a permutation that avoids [321], [213], and [312] is unimodal and [321]-avoiding. Theorem 7 implies the number of size n unimodal [321]-avoiding permutations is n. Hence, the number of size n $\{[321],[213],[312]\}$-avoiding permutations is n. $\square$

3 V-unimodal Permutations and an Enumeration of $\{[321],[231],[132]\}$-avoiding Permutations

There are similar results to the ones presented in section 2 for v-unimodal permutations.

Definition 10. A permutation $p = [p_1 p_2 \dots p_n]$ is v-unimodal if and only if there exists an index i such that $p_1 > p_2 > \cdots > p_i < p_{i+1} < \cdots < p_n$.

V-unimodal permutations have a "V" shape. They look like unimodal permutations, except they have a valley instead of a peak. One example is [642135] (Figure 4).

The permutations [213], and [312] are v-unimodal permutations, so Theorem 8 can be restated as: "A [321]-avoiding permutation is unimodal if and only if it avoids non-identity, v-unimodal permutations of size 3." Furthermore, there is a similar theorem for v-unimodal permutations. The theorems will be proved using a bijection between the unimodal [321]-avoiding permutations of size n and v-unimodal [321]-avoiding permutations of size n. Let U_n be the set of all unimodal [321]-avoiding permutations of size n and V_n be the set of all v-unimodal [321]-avoiding permutations of size n.

Definition 11. Let $f : U_n \to V_n$ be defined on $p = [p_1 p_2 \dots p_n]$ by $f(p) = [p_n p_1 p_2 \dots p_{n-1}]$.

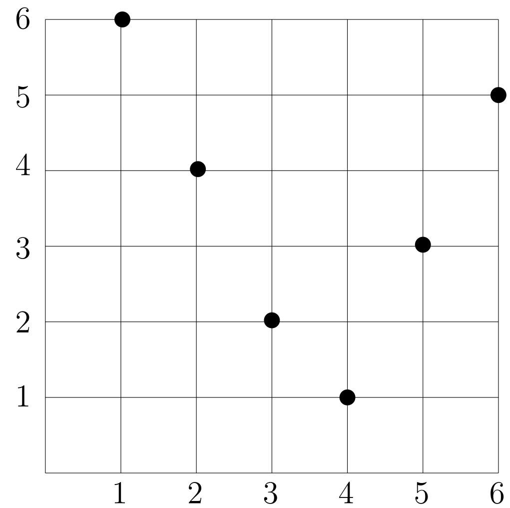

Figure 4: The graph of [642135]

Theorem 12. *$f : U_n \to V_n$ is a bijection.*

Proof. Let $p = [p_1p_2 \dots p_n] \in U_n$, so $f(p) = [p_np_1p_2 \dots p_{n-1}]$. By the unimodality of p and Proposition 5, $p_1 < p_2 < \cdots < p_{n-1}$. If $p_1 = 1$, then $p_n > p_1 < p_2 < \cdots < p_{n-1}$, so $f(p) \in V_n$. If $p_1 \neq 1$, then unimodality and Proposition 5 imply $1 < p_1 < p_2 < \cdots < p_{n-1} = n$, so $p_n = 1$. Since $p_n = 1$, $1 = p_n < p_1 < \cdots < p_{n-1}$, so $f(p)$ is the identity, implying $f(p) \in V_n$. Thus, $f(p) \in V_n$.

Take a permutation $q = [q_1q_2 \dots q_n]$ in V_n. and move q_1 to the end of the permutation resulting in $[q_2q_3 \dots q_nq_1]$. The permutation $[q_2q_3 \dots q_nq_1] \in U_n$ by definition because v-unimodality and [321]-avoiding imply $q_2 < q_3 < \dots q_n > q_1$. Denote by f^{-1} the action of placing q_1 in the last position. Then, $f^{-1} \circ f(p) = f^{-1}([p_np_1p_2 \dots p_{n-1}]) = [p_1p_2 \dots p_{n-1}p_n] = id_{U_n \to U_n}$ and $f \circ f^{-1}(q) = f([q_2q_3 \dots q_nq_1]) = [q_1q_2 \dots q_{n-1}q_n] = id_{V_n \to V_n}$. Therefore, $f : U_n \to V_n$ is a bijection. $\square$

Theorem 13. *The number of v-unimodal* [321]*-avoiding permutations of size n is n.*

This bijection takes the last entry of a unimodal [321]-avoiding permutation and moves it to the end of the permutation, and undoes that action by taking the first entry of a v-unimodal [321]-avoiding permutation and moving it to the beginning of the permutation. For example:

$$[2345\underline{1}] \mapsto [\underline{1}2345]$$

$$[1345\underline{2}] \mapsto [\underline{2}1345]$$

$$[1245\underline{3}] \mapsto [\underline{3}1245]$$

$$[1235\underline{4}] \mapsto [\underline{4}1235]$$

$$[1234\underline{5}] \mapsto [\underline{5}1234]$$

The bijection exposes the similarity between [321]-avoiding v-unimodal permutations and [321]-avoiding unimodal permutations. Just as the [321]-avoiding unimodal permutations are determined by their last element, the [321]-avoiding v-unimodal permutations are determined by their first.

Proposition 14. *If $q = [q_1 \dots q_n]$ is a v-unimodal [321]-avoiding permutation, then it is uniquely determined by q_1.*

Proof. Let $q = [q_1 \dots q_n]$ be a v-unimodal [321]-avoiding permutation. Then, $f^{-1}(q) = [q_2 \dots q_n q_1]$ is a [321]-avoiding unimodal permutation, and by Theorem 6 it is uniquely determined by q_1, so $f \circ f^{-1}(q) = q$ is uniquely determined by q_1. □

Proposition 15. *A v-unimodal [321]-avoiding permutation $q = [q_1 q_2 \dots q_n]$ has $q_2 < q_3 < \cdots < q_n$.*

Proof. Let $q = [q_1 \dots q_n]$ be a v-unimodal [321]-avoiding permutation. Then, $f^{-1}(q) = [q_2 \dots q_n q_1]$ is a [321]-avoiding unimodal permutation, and $[q_2 \dots q_n q_1]$ has $q_2 < q_3 < \cdots < q_n$ by Proposition 5. □

V-unimodal permutations avoid the size 3 non-identity unimodal [321]-avoiding permutations.

Theorem 16. *A permutation, p, is $\{[321], [231], [132]\}$-avoiding if and only if p is [321]-avoiding and v-unimodal.*

Proof. Suppose that $p = [p_1 p_2 \dots p_n]$ is a [321]-avoiding permutation. First, notice that if $p_2 < p_3 < \dots p_n$, then $p_1 < p_2$ or $p_1 > p_2$, so in either case p is v-unimodal by definition. Now, suppose p also avoids $\{[231], [132]\}$. Also, suppose to the contrary that p is not v-unimodal. Then, there are $i, j \in \{2, \dots, n\}$, $i < j$, such that $q_i > q_j$. Consider the pattern $q_1 q i q_j$. Then, the pattern either has $q_1 < q_i > q_j$ which is a [231] or [132] occurrence, or $q_1 > q_i > q_j$ which is a [321]-occurrence. Thus, p has $p_1 > p_2 < p_3 < \dots p_n$, implying p is v-unimodal.

Now suppose p is a v-unimodal [321]-avoiding permutation. Then, Proposition 15 implies $p_2 < p_3 < \dots p_n$, so $p_2 p_3 \dots p_n$ avoids [231] and [132]. Furthermore, in $p = f \circ f^{-1}(p)$, $p_1 > p_2 < p_3 < \dots p_n$ or $p_1 < p_2 < p_3 < \dots p_n$, so [231] nor [132] occur in p. Thus, p avoids [231] and [132]. Therefore, a permutation, p, is $\{[321], [231], [132]\}$-avoiding if and only if p is [321]-avoiding and v-unimodal. $\square$

4 A Generalization of Unimodal Permutations that Avoid [321]

The total number of unimodal permutations of size n can be found by counting how many avoid patterns of the type $[m(m-1 \dots 321)]$ for each m. The structure of the [321]-avoiding unimodal permutations generalizes so it can help count how many of them avoid $[m(m-1 \dots 321)]$ for each m. Knowing how many unimodal permutations of size n avoid $[m(m-1) \dots 321]$ helps count the total number of unimodal permutations of size n, which is 2^{n-1}.

Definition 17. Let $p = [p_1 p_2 \dots p_n]$. The p_i-decrease of p is the maximal decreasing subsequence $p_i p_{i+1} p_{i+2} \dots p_d$.

For example, the 3-decrease of [123] is 3 and the 6-decrease of [632451] is 632. The n-decrease of a unimodal permutation helps check if a unimodal permutation contains an instance of $[m(m-1) \dots 321]$.

Proposition 18. *Let p be a unimodal permutation of size n, and fix $m \leq n$. Then p avoids $[m(m-1) \dots 321]$ if and only if the n-decrease of p has size less than m.*

Proof. Let $p = [p_1 p_2 \dots p_n]$ be a unimodal permutation of size n that has an n-decrease of size $l < m$, with $n \geq m$, and m fixed. The result is clear for e, so let $p \neq e$. Since p is unimodal and $p \neq e$, the n-decrease of p is $n \dots p_n$ by definition. Therefore, $p_{n-l+1} = n$. Hence, $p_1 < p_2 < \dots < p_{n-l+1} = n$. It follows that a decreasing subsequence has at most one element of $\{p_1, \dots, p_{n-l+1}\}$. By unimodality, $n = p_{n-l+1} > p_{n-l+2} > \dots > p_n$, so all other elements of a decreasing subsequence are in $\{p_{n-l+2}, \dots, p_n\}$.

Furthermore, there are $l-1$ elements in $\{p_{n-l+2}, \dots, p_n\}$, so a decreasing subsequence in p has size at most l. Thus, p avoids $[m(m-1)\dots 321]$.

Suppose that p avoids $[m(m-1)\dots 321]$. The definition of n-decrease implies it is the maximal decreasing subsequence of p. Since $[m(m-1)\dots 321]$ is size m, p avoiding $[m(m-1)\dots 321]$ implies the n-decrease has size less than m. Therefore, p avoids $[m(m-1)\dots 321]$ if and only if the n-decrease of p has size less than m. □

Theorem 19. *Let $n \geq m$. The number of unimodal permutations of size n that have an n-decrease of size m is $\binom{n-1}{m-1}$.*

Proof. Let n and m be fixed, $n \geq m$, and $p = [p_1 p_2 \dots p_n]$ be a unimodal permutation of size n with an n-decrease of size m. Since the n-decrease of p is $n \dots p_n$ and has size m it follows $p_{n-m+1} = n$, so $m-1$ elements of $\{1, \dots, n-1\}$ need to be chosen and arranged in decreasing order to form $p_{n-m+2} \dots p_n$. There are $\binom{n-1}{m-1}$ ways to do so. The remaining $n-m$ elements must be arranged in increasing order to form $p_1 p_2 \dots p_{n-m}$. By the construction, both $p_1 p_2 \dots p_{n-m}$ and $p_{n-m+2} \dots p_n$ are unique. Therefore, the number of unimodal permutations of size n that have an n-decrease of size m is $\binom{n-1}{m-1}$. □

Theorem 20. *The number of unimodal permutations of size n that avoid $[m(m-1)\dots 321]$ with $m \geq 2$ is*

$$\sum_{k=2}^{m} \binom{n-1}{k-2} = \sum_{j=0}^{m-2} \binom{n-1}{j}.$$

Proof. Fix n and m with $m \geq 2$. Consider k with $2 \leq k \leq m$. If a permutation avoids $[k \dots 21]$, Proposition 18 implies its n-decrease has size less than k and $k \leq m$, so Proposition 18 implies the permutation also avoids $[m(m-1)\dots 321]$. Hence, by Theorem 19, there are $\binom{n-1}{k-2}$ unimodal, $[m(m-1)\dots 321]$-avoiding permutations with n-decrease of size $k-1$. When $k = 2$, the permutation that avoids $[21]$ is the identity permutation. Therefore, the number of unimodal permutations of size n that avoid $[m(m-1)\dots 321]$ with $m \geq 2$ is

$$\sum_{k=2}^{m} \binom{n-1}{k-2}$$

and by reindexing with $j = k-2$,

$$\sum_{k=2}^{m} \binom{n-1}{k-2} = \sum_{j=0}^{m-2} \binom{n-1}{j}.$$

$\square$

Corollary 21. *The number of unimodal permutations of size n is 2^{n-1}.*

Proof. Theorem 20 implies the number of unimodal permutations of size n that avoid $[(n+1)n\dots321]$ is

$$\sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}.$$

$\square$

5 A Bijection from Unimodal Permutations onto V-unimodal Permutations

Corollary 21 implies there is a bijection between unimodal permutations of size n and the subsets of $\{1,\dots,n-1\}$. There is also a bijection from $\{2,\dots,n\}$ onto v-unimodal permutations of size n.

Definition 22. Let $\mathcal{U}_n$ be the set of all unimodal permutations of size n and $\mathscr{P}(\{1,\dots,n-1\})$ be the power set of $\{1,\dots,n-1\}$. Define $f : \mathcal{U}_n \to \mathscr{P}(\{1,\dots,n-1\})$ as follows: If p is a unimodal permutation with n-decrease $np_dp_{d+1}\dots p_n$, then $f(p) = \{p_d, p_{d+1},\dots,p_n\}$.

Definition 23. Let $f_1 : \mathscr{P}(\{1,\dots,n-1\}) \to \mathcal{U}_n$ be defined as follows: Let $S \in \mathscr{P}(\{1,\dots,n-1\})$, and denote the elements of S by $p_{n-|S|+1},\dots,p_n$ so $p_{n-|S|+1} > p_{n-|S|+2} > \dots > p_n$. Then, $f_1(S) = [p_1p_2\dots np_{n-|S|+1}\dots p_n]$ where $p_1 < p_2 < \dots < n$, and $\{p_1,p_2,\dots,p_{n-|S|-1}\} = \{1,\dots,n-1\} - S$.

The function f is a bijection with f_1 being f's inverse. This bijection takes a unimodal permutation of size n to the subset of $\{1,\dots,n-1\}$ consisting of all entries in the permutation that are to the right of n. Also, given a subset S of $\{1,\dots,n-1\}$, the bijection takes S to the unique unimodal permutation of size n whose n-decrease consists of all elements in $\{n\} \cup S$. For example:

$$\text{Unimodal} \to \text{subsets of } \{1,2,3\}$$

$$[1234] \mapsto \emptyset$$

$$[2341] \mapsto \{1\}$$

$$[1342] \mapsto \{2\}$$

$$[1243] \mapsto \{3\}$$

$$[3421] \mapsto \{2,1\}$$

$$[2431] \mapsto \{3,1\}$$

$$[1432] \mapsto \{3,2\}$$

$$[4321] \mapsto \{3,2,1\}$$

Theorem 24. *$f : \mathcal{U}_n \to \mathscr{P}(\{1,\dots,n-1\})$ is a bijection.*

Proof. Let $p \in \mathcal{U}_n$, and $np_dp_{d+1}\dots p_n$ be the n-decrease of p.

$$\begin{aligned} f_1 \circ f(p) &= f_1(\{p_d, p_{d+1}, \dots, p_n\}) && \text{(by definition of } f\text{)} \\ &= [p_1 \dots p_{d-2} n p_d p_{d+1} \dots p_n] && \text{(by definition of } f_1\text{)} \\ &= p && (1) \end{aligned}$$

because p is the only unimodal permutation with $np_dp_{d+1}\dots p_n$ as its n-decrease by the proof of Theorem 19.
Let $S \in \mathscr{P}(\{1,\dots,n-1\})$. As in the definition of f_1, $S = \{p_{n-|S|+1}, \dots, p_n\}$, so

$$\begin{aligned} f \circ f_1(S) &= f([p_1p_2 \dots np_{n-|S|+1} \dots p_n]) && \text{(by definition of } f_1\text{)} \\ &= \{p_{n-|S|+1}, \dots, p_n\} && \text{(by definition of } f\text{)} \\ &= S && (2) \end{aligned}$$

Thus, $f_1 \circ f = id_{\mathcal{U}_n \to \mathcal{U}_n}$ and $f \circ f_1 = id_{\mathscr{P}(\{1,\dots,n-1\}) \to \mathscr{P}(\{1,\dots,n-1\})}$, so $f_1 = f^{-1}$. Therefore, f is a bijection. □

There is a similar bijection for v-unimodal permutations. Note that with v-unimodal permutations, the decrease taken from the first element will be used. The definition of v-unimodal permutation implies that if $q = [q_1 q_2 \dots 1 \dots q_n]$ is a v-unimodal permutation then the q_1-decrease is $q_1 q_2 \dots 1$.

Definition 25. Let $\mathcal{V}_n$ be the set of all v-unimodal permutations of size n and $\mathscr{P}(\{2,\dots,n\})$ be the power set of $\{2,\dots,n\}$. Define $g : \mathcal{V}_n \to \mathscr{P}(\{2,\dots,n\})$ as follows: If q is a v-unimodal permutation with q_1-decrease $q_1 q_2 \dots q_d 1$, then $g(q) = \{q_1, q_2, \dots, q_d\}$.

Definition 26. Let
$g_1 : \mathscr{P}(\{2,\dots,n\}) \to \mathcal{V}_n$ be defined as follows: Let $S \in \mathscr{P}(\{2,\dots,n\})$, and arrange the elements of S in decreasing order, and in that order label them $q_1, \dots, q_{|S|}$ respectively, so $q_1 > q_2 > \dots > q_{|S|}$. Then, $S = \{q_1, \dots, q_{|S|}\}$ and $g_1(S) = [q_1 q_2 \dots q_{|S|} 1 q_{|S|+2} \dots q_n]$ where $q_{|S|+2} < q_{|S|+3} < \dots < q_n$, and $\{q_{|S|+2}, \dots, q_n\} = \{2,\dots,n\} - S$.

Lemma 27. *If q is a v-unimodal permutation of size n, then q is the only v-unimodal permutation with $q_1 q_2 \dots 1$ as its q_1-decrease.*

Proof. Let q be a v-unimodal permutation of size n, with n fixed. Let $q_d = 1$. Then, $d-1$ elements of $\{2,\dots,n\}$ must be chosen and arranged in decreasing order to form $q_1 q_2 \dots q_{d-1}$. Arrange the remaining $n-d$ elements of $\{2,\dots,n\}$ in increasing order to form $q_{d+1} \dots q_n$. The decreasing order on $\{q_1, q_2, \dots q_{d-1}, 1\}$ is unique, and the increasing order on $\{q_{d+1}, \dots, q_n\}$ is unique, so by construction, q is unique. Therefore, if q is a v-unimodal permutation of size n, then q is the only v-unimodal permutation with $q_1 q_2 \dots 1$ as its q_1-decrease. $\square$

The function g is a bijection which works similarly to f. The bijection, g, takes a v-unimodal permutation of size n to the subset of $\{2,\dots,n\}$ that consists of all entries in the permutation that are to the left of 1. The proof that g is a bijection uses arguments similar to the ones used in the proofs of Theorem 24 and Lemma 27 except that instead of forming subsets using the n-decrease, it forms them using the q_1-decrease. For example:

$$\text{V-unimodal} \to \text{subsets of } \{2,3,4\}$$

$$[1234] \mapsto \emptyset$$

$$[2134] \mapsto \{2\}$$

$$[3124] \mapsto \{3\}$$

$$[4123] \mapsto \{4\}$$

$$[3214] \mapsto \{2, 3\}$$

$$[4213] \mapsto \{2, 4\}$$

$$[4312] \mapsto \{3, 4\}$$

$$[4321] \mapsto \{4, 3, 2\}$$

In order to give a bijection from $\mathcal{U}_n$ onto $\mathcal{V}_n$ that corresponds the n-decrease with the q_1-decrease, a bijection from $\mathscr{P}(\{1, \ldots, n-1\})$ onto $\mathscr{P}(\{2, \ldots, n\})$ is needed. This bijection will give a bridge between $\mathscr{P}(\{1, \ldots, n-1\})$ and $\mathscr{P}(\{2, \ldots, n\})$, which helps relate f and g to form a bijection from $\mathcal{U}_n$ onto $\mathcal{V}_n$.

Definition 28. Define $t : \mathscr{P}(\{1, \ldots, n-1\}) \to \mathscr{P}(\{2, \ldots, n\})$ as follows:
If $S \in \mathscr{P}(\{1, \ldots, n-1\}) \cap \mathscr{P}(\{2, \ldots, n\})$, then $t(S) = S$
If $1 \in S \in \mathscr{P}(\{1, \ldots, n-1\})$, then $t(S) = (S - \{1\}) \cup \{n\}$

The function t is also a bijection with an inverse that replaces 1 with n and leaves the other elements in $\{2, \ldots, n\}$ alone. For example:

$$\text{subsets of } \{1, 2, 3\} \to \text{subsets of } \{2, 3, 4\}$$

$$\emptyset \mapsto \emptyset$$

$$\{1\} \mapsto \{4\}$$

$$\{2\} \mapsto \{2\}$$

$$\{3\} \mapsto \{3\}$$

$$\{2, 1\} \mapsto \{2, 4\}$$

$$\{3, 1\} \mapsto \{3, 4\}$$

$$\{2,3\} \mapsto \{2,3\}$$

$$\{1,2,3\} \mapsto \{2,3,4\}$$

It follows that $g^{-1} \circ t \circ f : \mathcal{U}_n \to \mathcal{V}_n$ is a composition of bijections, which is a bijection from the size n unimodal permutations onto the size n v-unimodal permutations. For example, the bijection $g^{-1} \circ t \circ f$ does the following:

$$\text{Unimodal} \mapsto \text{subsets of } \{1,2,3\} \mapsto \text{subsets of } \{2,3,4\} \mapsto \text{V-unimodal}$$

$$[1234] \mapsto \emptyset \mapsto \emptyset \mapsto [1234]$$

$$[2341] \mapsto \{1\} \mapsto \{4\} \mapsto [4123]$$

$$[1342] \mapsto \{2\} \mapsto \{2\} \mapsto [2134]$$

$$[1243] \mapsto \{3\} \mapsto \{3\} \mapsto [3124]$$

$$[3421] \mapsto \{2,1\} \mapsto \{4,2\} \mapsto [4213]$$

$$[2431] \mapsto \{3,1\} \mapsto \{4,3\} \mapsto [4312]$$

$$[1432] \mapsto \{3,2\} \mapsto \{3,2\} \mapsto [3214]$$

$$[4321] \mapsto \{3,2,1\} \mapsto \{4,3,2\} \mapsto [4321]$$

Since $g^{-1} \circ t \circ f$ is a bijection, Corollary 21 implies,

Theorem 29. *The number of v-unimodal permutations of size n is 2^{n-1}.*

Proof. By Corollary 21 the number of unimodal permutations of size n is 2^{n-1}. Since $g^{-1} \circ t \circ f : \mathcal{U}_n \to \mathcal{V}_n$ is a bijection from the unimodal permutations of size n onto the v-unimodal permutations of size n, the number of v-unimodal permutations of size n is 2^{n-1}. □

The bijection $g^{-1} \circ t \circ f$ preserves some of the structure in unimodal permutations that made it possible to count how many avoid $[m(m-1)\dots 321]$. Hence, the v-unimodal permutations have similar avoidance properties.

Lemma 30. *Let $\mathcal{U}_{n,m}$ be the set of all unimodal permutations of size n with n-decrease of size m, and $\mathcal{V}_{n,m}$ be the set of all v-unimodal permutations $q = [q_1 \dots q_n]$ with q_1-decrease of size m. Then, $g^{-1} \circ t \circ f(\mathcal{U}_{n,m}) = \mathcal{V}_{n,m}$*

Proof. Let n and m be fixed. Let $p \in \mathcal{U}_{n,m}$, and $q = g^{-1} \circ t \circ f(p)$. Then, $f(p)$ is the subset of $\{1, \ldots, n-1\}$ consisting of the entries in p that are to the right of n. Since the n-decrease has size m, the cardinality of $f(p)$ is $m-1$. From the definition of t, it follows $t \circ f(p)$ also has cardinality $m-1$. Hence, the q_1-decrease of q is also size m, so $g^{-1} \circ t \circ f(\mathcal{U}_{n,m}) \subseteq \mathcal{V}_{n,m}$. The proof that $\mathcal{V}_{n,m} \subseteq g^{-1} \circ t \circ f(\mathcal{U}_{n,m})$ is a similar argument using the inverse of $g^{-1} \circ t \circ f)$. Therefore, $g^{-1} \circ t \circ f(\mathcal{U}_{n,m}) \subseteq \mathcal{V}_{n,m}$ and $\mathcal{V}_{n,m} \subseteq g^{-1} \circ t \circ f(\mathcal{U}_{n,m})$ imply $g^{-1} \circ t \circ f(\mathcal{U}_{n,m}) = \mathcal{V}_{n,m}$. □

Theorem 31. *Let $n \geq m$. The number of v-unimodal permutations, $q = [q_1 \ldots q_n]$ of size n with q_1 decrease of size m is $\binom{n-1}{m-1}$.*

Proof. The result is implied by Lemma 30 and Theorem 19.

□

Proposition 32. *Let q be a v-unimodal permutation of size n and fix $m \leq n$. Then $q = [q_1 \ldots q_n]$ avoids $[m(m-1)\ldots 321]$ if and only if the q_1-decrease of q has size less than m.*

Proof. The proof is similar to the proof of Proposition 18.

□

Theorem 33. *The number of v-unimodal permutations of size n that avoid $[m(m-1)\ldots 321]$, $m \geq 2$, is*

$$\sum_{k=2}^{m} \binom{n-1}{k-2} = \sum_{j=0}^{m-2} \binom{n-1}{j}.$$

Proof. The proof is similar to the proof of Theorem 20.

□

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