Characterizing Automorphism and Permutation Polytopes

By

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ABSTRACT.

Permutation polytopes are polytopes whose vertices are determined by a representation of a permutation group, such as the cyclic group on n elements or the dihedral group on n elements. These polytopes appear in many different applications, yet little research has been done on any save the Birkhoff polytope, the permutation polytope whose vertices are based on the symmetric group on n elements. Our research focuses on determining the volume of some of these less-studied permutation polytopes. These volumes can be determined by finding the Ehrhart polynomial of a given polytope, a polynomial in t which counts the number of integer points contained within the t^{th} dilation of the polytope. We also consider the polytopes associated to the automorphism groups of trees and rigid graphs. Our results include Ehrhart polynomials and volumes of the polytopes corresponding to cyclic and dihedral groups, a method of calculating the Ehrhart polynomial of the set of permutations of a given binary tree, a formula for the normalized volume of polytopes of Frobenius groups, a proof that the polytopes of Frobenius groups are two-level, and a proof that the polytopes associated to the even permutation groups on n elements are not two-level for n greater than five, thus characterizing the complexity of the polytopes of Frobenius groups and even permutations.

Contents

CHAPTER 1

Convex Polytopes

Some basic knowledge of convex polytopes and group theory is required when researching permutation polytopes. This chapter briefly reviews the information needed from the study of convex polyhedra, while the next chapter deals with the group theory used in the research, and the final introductory chapter covers the graph theory concepts that were used. This chapter is largely paraphrased from the book *Actually Doing It* by Jesús de Loera; see [**5**] for a much more detailed treatment of the subject.

Definition 1.0.1. *A set $S \subset \mathbb{R}^n$ is called* convex *if and only if for every s_1 and $s_2 \in S$, the segment joining s_1 to s_2 is completely contained in S. In other words, S is convex if and only if for s_1 and $s_2 \in S$, $0 \leq \lambda \leq 1$, $\lambda s_1 + (1-\lambda)s_2 \in S$ for all choices of λ.*

The definition of convexity leads to the definition of a convex hull:

Definition 1.0.2. *The* convex hull *of a set S, conv(S), is the intersection of all convex sets containing S; equivalently, it is the smallest convex set which contains S.*

The convex hull of a set is thus the set itself combined with the interior of the set, so that a line segment joining any two points in the convex hull is completely contained within it. This is easily seen in Figure 1. The *affine hull* of a set S, aff(S), is very similar.

Definition 1.0.3. *A set $S \subset \mathbb{R}^n$ is called* affine *if and only if for every s_1 and $s_2 \in S$, the line passing through s_1 and $S-2$ is completely contained in S. In other words, S is affine if and only if for s_1 and $s_2 \in S$, $\alpha \in \mathbb{R}$, $\alpha s_1 + (1-\alpha)s_2 \in S$ for all choices of α. The* affine hull *of a set S, aff(S), is the smallest affine set containing S.*

Definition 1.0.4. *A* polytope *is the convex hull of a finite set of points in $\mathbb{R}^n$.*

A wide variety of objects fit this definition. conv(S) in Figure 1 is a polytope. A cube is a polytope, as is a tetrahedron or a dodecahedron. It is important to note the limitation on the size of the set - requiring the set of points to be finite does rule out curved objects, such as ovals or cylinders, from the set of polytopes.

The research presented in the later chapters of this thesis is focused on the *Ehrhart polynomials* associated to various polytopes. The definition of an Ehrhart polynomial relies on the definition of a *lattice point*.

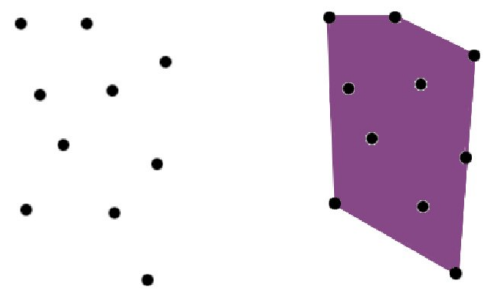


FIGURE 1. A set S on the left and its convex hull, $\mathrm{conv}(S)$, on the right.

DEFINITION 1.0.5. *A* lattice point *of a polytope is an integer point contained in the polytope. A* lattice polytope *is a polytope whose vertices are all integer points.*

If a polytope is a lattice polytope, each vertex is a lattice point, as well as any other integer points inside the polytope or along its edges. For a two-dimensional polytope placed on a grid, such as the polytope in Figure 2, the lattice points are the corners of the grid included in or on the edge of the polytope.

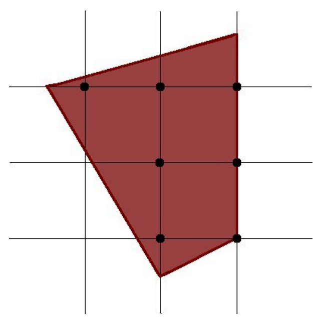


FIGURE 2. A polytope with lattice points marked in black.

It is possible to divide a d-dimensional grid into d-dimensional unit cubes, each one centered on a lattice point. Then the number of lattice points contained in a polytope becomes an estimate of the volume of the polytope. Of course, if the grid is made finer, this estimate becomes more and more precise. This process is very similar to the thinning of rectangles used to estimate area in Riemann integration.

The volume of a polytope is of great interest, so the number of lattice points within a given polytope is also important. Refining the grid is equivalent to dilating the polytope in relation to its grid. The *Ehrhart polynomial* assocated to a d-dimensional polytope is a polynomial of degree d which takes as input a dilation of the polytope and produces as output the number of lattice points contained in that dilation. So if we dilate a polytope by a factor of four, the associated Ehrhart polynomial takes four as an input and outputs the number of lattice points contained in the polytope when dilated by a factor of four. The Ehrhart polynomial has another highly useful property.

THEOREM 1.0.6. *The leading coefficient of the Ehrhart polynomial corresponds to the volume of the original polytope.*

So if we are able to find the Ehrhart polynomial of a polytope, we immediately know the volume, as well as the number of lattice points of any given dilation. Unfortunately, finding the Ehrhart polynomial of a polytope is not always easy, and some polytopes do not have proper Ehrhart polynomials at all, so we require a different approach to finding volume.

This alternate approach uses a method called *triangulation*. The idea is to split a given polytope into simplices (d-dimensional polytopes with $d + 1$ vertices) and then calculate the volume of those, as there are several formulas for finding the volume of a simplex, even a high-dimensional one. The sum of the simplices is the same as the volume of the original polytope.

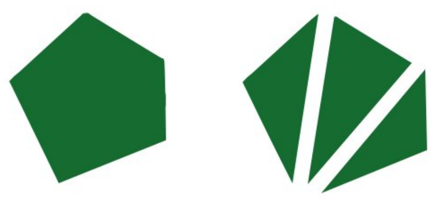


FIGURE 3. A two-dimensional triangulation.

These triangulations are an extremely important tool, and there is a great deal of research that has been done to further characterize various types of triangulations. A definition arising from this research regards the *unimodularity* of triangulations.

DEFINITION 1.0.7. *A lattice simplex S with vertices $v_1, \ldots, v_m$ is unimodular if the vectors $v_m - v_1,\ v_{m-1} - v_1, \ldots, v_2 - v_1$ form a basis for the lattice aff$(S) \cap \mathbb{Z}^d$. A triangulation of a lattice polytope is a unimodular triangulation if all its maximal dimensional simplices are unimodular.*

This definition comes directly from [**9**], a graduate text devoted to various methods of triangulation. There is a much more in-depth look at unimodularity in that text, but the basic definition is sufficient for the research presented in this thesis. The idea of unimodularity is that one vertex (v_1 in the definition) is fixed as the origin. Then the remaining vertices are recalculated using v_1 as the new origin, producing the vectors $v_m -$

$v_1,\ v_{m-1}-v_1,\ldots,v_2-v_1$. If these vectors form a basis for the lattice, then the simplex is unimodular. A *lattice simplex* is just a simplex with integer vertices.

Triangulations and Ehrhart polynomials are different approachs to finding the volume of a polytope, but they are actually very closely related, as the following theorem (also from [**9**]) demonstrates.

THEOREM 1.0.8. *If a convex d-dimensional lattice polytope P has a unimodular triangulation T with f-vector $(f_0,\ldots,f_d)$, where f_i indicates the number of i-dimensional simplices in the triangulation, then the Ehrhart polynomial $i_P(t)$, with $t\in\mathbb{Z}$ the dilation factor, is given by $\sum_{k=0}^{d}\binom{t-1}{k}f_k$.*

This theorem makes it possible to calculate the Ehrhart polynomial of a polytope with just its unimodular triangulation, which greatly simplifies the process. It also makes it very easy to find the volume of unimodular simplices, since the simplex itself is its own unimodular triangulation. We use this theorem later in this thesis to find the volumes of a series of high-dimensional unimodular simplices.

The proofs of Theorem 1.0.8 can be found in [**9**], along with a much more detailed look at the process of triangulation.

There is an unfortunate complication to these volume calculations; volumes are dependent on the lattice in which they originate. A polytope may have a certain volume in the lattice spanned by the standard unit vectors and a different volume in a different lattice, in particular the lattice spanned by its vertices. Figure 4 shows an example of this.

In order to deal with this discrepancy, we specify the lattice we use to calculate the volumes of our polytopes. Since the differences hinge upon the use of different lattices, the type of volume found through the Ehrhart polynomial or by finding a unimodular triangulation depends on the lattice used in the Ehrhart polynomial or the one used to determine unimodularity. If we find the volume of a polytope in the lattice generated by its own vertices, we refer to it as the polytope's *normalized volume*. If instead we use the standard lattice to calculate the volume, we refer to the result as the *volume* of the polytope, as that is the more standard definition of volume.

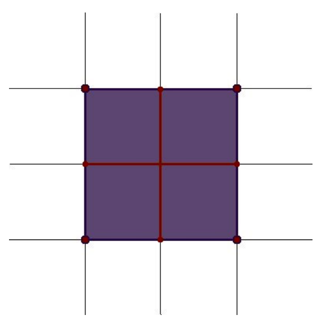


FIGURE 4. A polytope with volume 4 in the standard lattice and volume 1 in its own lattice.

Some of the convex polytopes studied in this thesis proved to be extremely complex. Rather than determining a specific Ehrhart polynomial for these polytopes, we have results relating to their complexity. One of the ways to describe the complexity of a polytope is to determine whether it is *two-level* or not.

DEFINITION 1.0.9. *A polytope is* two-level *if all of its vertices can be contained within two translations of a plane. In other words, P is two-level if all of its vertices are contained in* $\{x : a \cdot x - b = 0\} \cup \{x : a \cdot x - c = 0\}$.

If a polytope is two-level, there are several methods that can approximate it and be used to gain an idea of its structure. If it is not, then these methods are not available, and its structure is often much more difficult to determine.

CHAPTER 2

Group Theory

The research work collected in this thesis requires the information on convexity in the previous chapter, but it also involves a working knowledge of group theory, as covered in this chapter.

Group theory is the study of *groups*, which are defined below. For a more thorough discussion of the topic, see [**10**].

DEFINITION 2.0.10. *A* group G *is a set with some operation* $\circ$ *such that:*

i. There exists some element $I \in G$ *such that, for any other* $g \in G$, $I \circ g = g \circ I = g$.
ii. For every $g \in G$, *there is some other element* $g^{-1} \in G$ *such that* $g \circ g^{-1} = g^{-1} \circ g = I$.
iii. For any two elements $g \in G$ *and* $h \in G$, $h \circ g \in G$ *and* $g \circ h \in G$.

The operator $\circ$ can be almost anything, although it is most commonly addition, multiplication, or, in the case where G is a set of functions, composition. All that is required is that once an operation is determined, the set meets the three requirements above. The first requirement is the "identity" requirement. Note that the actual value of I is based on the operation $\circ$. For example, if $\circ$ is addition, I is 0. If $\circ$ is multiplication, I is 1, and if $\circ$ denotes function composition, I is $f(x) = x$.

The second and third requirements allow an entire group to be produced from a single element or a set of elements, known as the *generator* or *generators* of the group. If a single element g and a group operation $\circ$ are given, the group must include an identity element and g^n for all n, since $g \circ g$ is a composition within the group. Once those are included, the inverse of every power of g must be added, which is the same as adding the powers of g^{-1}. If g has finite *order* - that is, if $g^n = I$ for some finite n - then the resulting group is finite. If not, the group is infinite. If more than one element is given, the inverse and powers of every element must be included, as well as the composition of elements with each other and the inverses of those compositions.

In order to complete the presentation of our research, we need to include a description of subgroups as well.

DEFINITION 2.0.11. *A* subgroup *of a group* G *is a set* $H \subset G$ *such that* H *is itself a group.*

Every group contains the trivial subgroup $\{I\}$, where I is the group's identity element. Every group is also a subgroup of itself. If a group has a subgroup that is not one of these trivial cases, it is called a *proper* subgroup.

Subgroups can be used with their group to produce other sets.

DEFINITION 2.0.12. *A* conjugate *of a subgroup H of a group G is a set of the form gHg^{-1}, where $g \in G$. A* normal subgroup *is a subgroup such that $gHg^{-1} = H$ for all $g \in G$.*

DEFINITION 2.0.13. *A* permutation group *is a group G such that every $g \in G$ is a permutation.*

Permutation groups could be permutations of anything, but they are generally written as permutations on the numbers 1 through n, with n the number of elements being permuted. The set of all permutations of n elements, S_n, is a permutation group.

All permutations can be written in *cycle notation*. Take some given permutation g. g can then be written as $(x_1\ x_2\ \ldots x_m)(x_{m+1} \ldots x_n)(\ldots)$, where g takes x_1 to x_2, x_2 to x_3, ..., x_{m-1} to x_m, x_m to x_1, and so on. *Disjoint cycle notation* is the standard form of cycle notation, where a permutation g is written as the product of cycles that do not overlap - that is, that are disjoint. $g = (1\ 2)(2\ 3)$ is not written in disjoint cycles, but $g = (1\ 3)(2\ 4)$ is.

Permutations can also be written in *matrix notation*. This consists of creating an $n \times n$ matrix and numbering the rows and columns 1 through n. Every entry is either a 1 or a 0, and each row and column has exactly one 1 and $n - 1$ 0's. The first column has a 1 in the ith row - that indicates that 1 is mapped to i. The second column's 1 determines where 2 goes, and so on. This notation is the form used to prove some of our results. Matrix form can also be easily converted into *vector notation*, which consists of converting the matrix into a vector by writing each row of the matrix in order, so $V = (R_1\ R_2\ \ldots\ R_n)$. This form was used in several of the programs we used to collect our data, and it is essential to the concept of permutation polytopes.

DEFINITION 2.0.14. *A* permutation polytope *is a polytope associated to some permutation group, where the vertices of the polytope are the elements of the permutation group in vector notation.*

These polytopes are precisely the objects we have focused our research on.

Groups can be combined by taking various sorts of group products. We use three of these products in our research: the *direct product*, the *semidirect product*, and the *wreath product*.

DEFINITION 2.0.15. *The* direct product *of two groups G and H is the group $G \times H = \{(g, h) : g \in G,\ h \in H\}$, with $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ h_2)$ for g_1 and g_2 in G, h_1 and h_2 in H.*

For example, consider the two groups $G = \{1, -1\}$ and $H = \{(), (1,\ 2)\}$, with group composition multiplication on G and permutation composition on H. Then $G \times H = \{(1, ()), (1, (1\ 2)), (-1, ()), (-1, (1\ 2))\}$, with $(g_1, h_2) \circ (g_2, h_2) = (g_1 \cdot g_2, h_1 \circ h_2)$ and group identity $(1, ())$.

In order to define the semidirect product of groups, we must introduce the idea of a *group homomorphism.*

Definition 2.0.16. *A* group homomorphism *is a function Φ mapping a group H onto a group G such that $\Phi(h_1 \circ h_2) = \Phi(h_1) \circ \Phi(h_2)$ for all h_1 and h_2 in H.*

We can define a group homomorphism between our earlier groups $H = \{(), (1\ 2)\}$ and $G = \{1, -1\}$ by declaring $\Phi(()) = 1$ and $\Phi((1\ 2)) = -1$. Then we have $\Phi(() \circ (1\ 2)) = \Phi((1\ 2)) = -1 = 1 \cdot -1 = \Phi(()) \cdot \Phi((1\ 2))$.

Definition 2.0.17. *Consider some group homomorphism Φ from a group H to a group G. The* semidirect product *of G by H is the group $G \rtimes H = \{(g, h) : g \in G,\ h \in H\}$ with group composition defined as $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ \Phi(h_1) \circ g_2, h_1 \circ h_2)$.*

So for our example homomorphism Φ and our groups G and H, $G \rtimes H = G \times H$, but whereas in $G \times H$ $(1, (1\ 2)) \circ (-1, ()) = (1 \cdot -1, (1\ 2) \circ ()) = (-1, (1\ 2))$, in $G \rtimes H$ $(1, (1\ 2)) \circ (-1, ()) = (1 \cdot \Phi((1\ 2)) \cdot -1, (1\ 2) \circ ()) = (1, (1\ 2))$.

Definition 2.0.18. *Consider a permutation group H on n elements. The wreath product $G \wr H$ of a group G by H is defined as the semidirect product of the direct product of n copies of G by H; that is, $(G \times G \times \ldots \times G) \rtimes H$.*

Again consider our groups G and H with the group homomorphism Φ from H onto G. The wreath product of G by H is then $(G \times G) \rtimes H$, which is $\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ $\rtimes H$.

Every permutation can be decomposed into *transpositions*, which are permutations that exchange exactly two terms. The permutation that takes 1 to 5 and vice versa ((1 5)) is a transposition, as is the permutation (4 17). These transpositions (and any permutations) can be composed, so the transposition (1 2), which takes 1 to 2 and 2 to 1, composed with the transposition (2 3), which takes 2 to 3 and 3 to 2, produces (1 3 2), which takes 1 to 3, 2 to 1, and 3 to 2.

Definition 2.0.19. *The set of all even permutations on n elements, A_n, is the set of permutations that are products of an even number of transpositions.*

We used A_n for part of our research, but there are three other very famous permutation groups that are central to the research for this thesis: the *cyclic group*, the *dihedral group*, and *Frobenius groups.*

Definition 2.0.20. *The* cyclic group *on n elements is the group generated by the permutation $(1\ 2\ \ldots\ n)$. It can be thought of as the set of clockwise rotations of a regular n-gon with vertices numbered 1 through n clockwise.*

Definition 2.0.21. *The* dihedral group *on n elements is the group generated by* $(1\ 2\ \ldots\ n)$ *together with* $(1\ n)(2\ n-1)(\ldots)(\lfloor\frac{n+1}{2}\rfloor\ \lceil\frac{n+1}{2}\rceil)$. *It can be thought of as the set of clockwise rotations of a regular n-gon with vertices numbered 1 through n clockwise together with the reflections of that n-gon onto itself across axes through the n-gon. There is an axis through each vertex of the n-gon and one through the center of each side.*

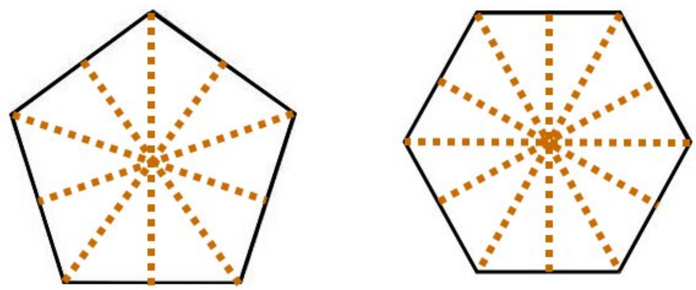


Figure 1. The axes of reflection of a pentagon on the left and a hexagon on the right.

There are two important concepts used when researching permutation groups - the *fix* of a permutation, and its *support*.

Definition 2.0.22. *The* fix *of a permutation g, denoted fix(g), is the set of values fixed by g. For example, if g is the permutation on n elements* $(3\ 6\ 8)$, *fix*$(g) = \{1,\ 2,\ 4,\ 5,\ 7,\ 9,\ \ldots,\ n\}$.

Definition 2.0.23. *The* support *of a permutation g, denoted supp(g), is the set of values that are not fixed by g. For example, if g is the permutation on n elements* $(3\ 6\ 8)$, *supp*$(g) = \{3,\ 6,\ 8\}$. *The support is the complement of the fix, so* $\{1,\ 2,\ \ldots,\ n\}/$*fix*$(g) =$*supp*(g), *and* $\{1,\ 2,\ \ldots,\ n\}/$*supp*$(g) =$*fix*(g).

With these definitions, we can now define Frobenius groups.

Definition 2.0.24. *A* Frobenius group *G is a permutation group on n elements containing a subgroup H such that for all $g \in G\ H$, $H \cap gHg^{-1} = I$. The subgroup H is called the* Frobenius complement*, and the identity element together with all $g \in G$ such that fix$(g) = \{\}$ form a normal subgroup, the* Frobenius kernal. *This second subgroup is denoted by N. Furthermore, G is the direct product of H and N.*

The last definition relating to group theory is a term introduced in [**1**]; it became important as we began to more closely analyze the permutation polytopes of the groups we researched.

DEFINITION 2.0.25. *Let* $g = z_1 \circ z_2 \circ \cdots \circ z_r$ *be the disjoint cycle decomposition of* $g \in G_n$*.* h *is a* subelement *of* g *if there is a subset* $I \subseteq \{1, 2, \ldots, r\}$ *such that* $h = \Pi_{i \in I} z_i$*. If the only subelements of* g *are* I *and* g *itself,* g *is called* indecomposable.

In the process of proving one of our results, we also used the following theorem. The proof relies on the definition of indecomposability, the definition of a group, and the definition of the fix and the support of an element.

THEOREM 2.0.26. *If an element* g *in some permutation group* G *is decomposable into two group elements* h *and* h'*, then* $I + g = h + h'$*, where* $+$ *denotes addition of the associated permutation matrices.*

PROOF. Let $g = z_1 \circ z_2 \circ \cdots \circ z_m$, with z_i disjoint cycles. If g is decomposable into h and h', then without loss of generality $h = z_1 \circ z_2 \circ \cdots \circ z_k$ and $h' = z_{k+1} \circ z_{k+2} \circ \cdots z_m$. Consider the support of g. $\operatorname{supp}(h)$ and $\operatorname{supp}(h')$ must be included in $\operatorname{supp}(g)$, but since h and h' are by definition disjoint, $\operatorname{supp}(h)$ and $\operatorname{supp}(h')$ must be disjoint as well. Thus, the points fixed by h are either fixed by g as well or in $\operatorname{supp}(h')$, and vice versa. So $\operatorname{fix}(h) \cap \operatorname{fix}(h') = \operatorname{fix}(g)$, and similarly $\operatorname{fix}(h) \cup \operatorname{fix}(h') = \operatorname{fix}(I) = G$. Then $h + h'$ includes all of g, but also a copy of I, since h fixes the support of h' and vice versa. $\square$

CHAPTER 3

Graph Automorphisms

In the course of our research, we spent some time collecting data on graph automorphisms, which are also related to our later work on permutation polytopes. These are very specific types of permutations of specific types of graphs, as described in this chapter. For more information on graph theory, see [**2**] and [**11**].

DEFINITION 3.0.27. *A* graph *is a collection of points, called* vertices*, and* edges*, which connect vertices to each other or to themselves. A* simple graph *is a graph with no loops or double edges between any two vertices.*

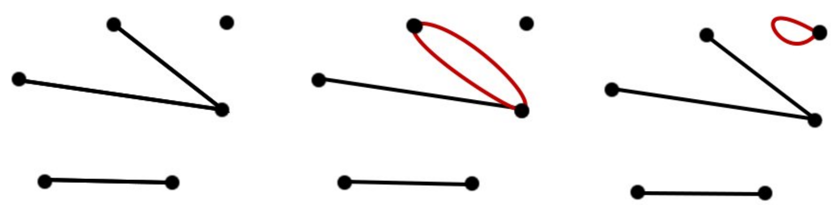


FIGURE 1. A simple graph on the left and two non-simple graphs to the right.

DEFINITION 3.0.28. *A* path *is a set of n vertices connected one to another in a line by $n-1$ edges. The nodes at either end are the* endpoints *of the path.*

DEFINITION 3.0.29. *A* cycle *is a path in which the endpoints are the same.*

We focused our research exclusively on simple graphs, and our work on permutation polytopes is centered on a particular type of simple graph known as a *tree.*

DEFINITION 3.0.30. *A* tree *is a simple graph such that every vertex is connected to every other vertex by a path and there are no closed cycles formed by edges.*

FIGURE 2. A tree on the left and a non-tree on the right, with its closed cycle marked.

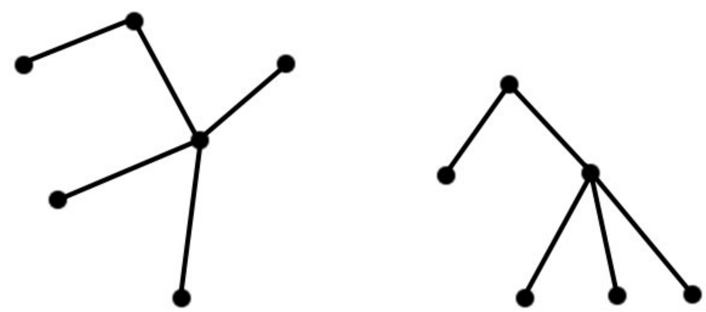


FIGURE 3. A tree on the left, redrawn on the right to highlight the parent-child structure.

Trees can be drawn such that one vertex, or *node*, is at the top, with all edges going down and each subsequent node below the previous one. This allows us to refer to *child* and *parent* nodes, where a child is the node directly below its parent.

This definition of parent and child nodes allows us to add another definition relating to trees.

DEFINITION 3.0.31. *A* binary tree *is a tree where each node has at most two children.*

Graphs are a rather simple concept, but there are all sorts of interesting ideas associated to them, such as the notion of a *graph automorphism*.

DEFINITION 3.0.32. *A* graph automorphism *is a permutation of the vertices of a graph that takes the graph to itself. A* rigid graph *is a graph with no automorphisms beyond the trivial permutation I.*

The set of automorphisms of any graph is a group, since applying multiple automorphisms maps the graph onto itself, if two vertices can be switched, they can be switched back, and the trivial automorphism does not change the graph and so acts as the identity.

The automorphisms of non-trees are often difficult to fully determine, but the automorphisms of binary spanning trees turn out to be quite easy to enumerate. If a tree is rooted at a given vertex, its automorphisms can be fully determined by considering the

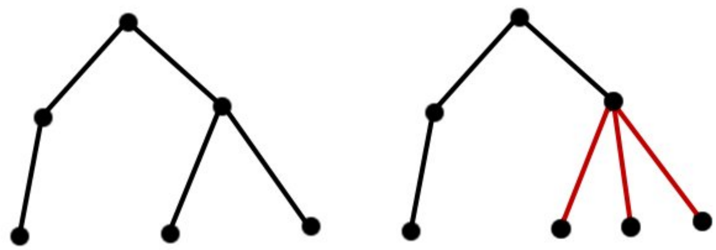


FIGURE 4. A binary tree on the left and a non-binary tree on the right.

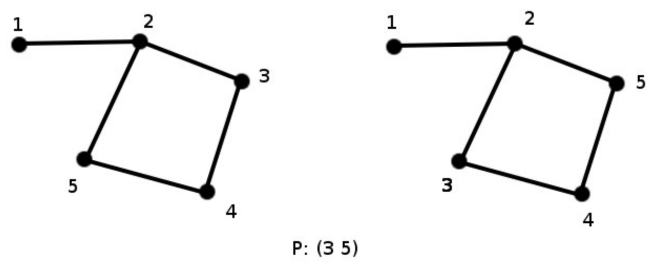


FIGURE 5. A graph pre-automorphism on the left and post-automorphism on the right.

degree of the nodes on a given level and taking direct products and wreath products of permutation groups. For example, consider the tree shown in Figure 6.

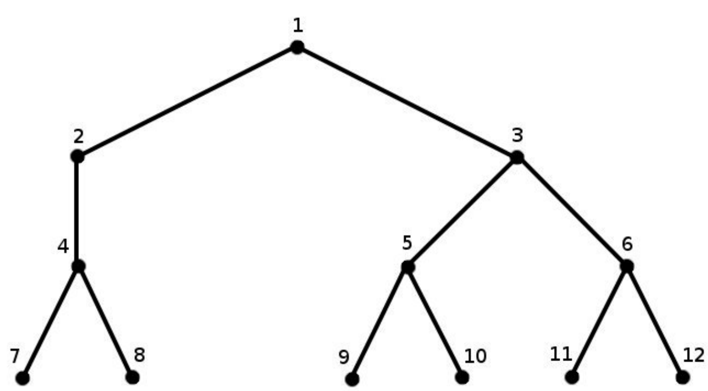


FIGURE 6. A binary tree with nodes labeled.

Start at the lowest level. Nodes 7 and 8 can clearly be mapped to each other, so we can consider the sets of automorphisms switching those two nodes as S_2. Similarly, we can switch 9 and 10 or 11 and 12. So the automorphism groups of the trees rooted at 4, 5, and 6, say $\mathrm{Aut}(T_4)$, $\mathrm{Aut}(T_5)$, and $\mathrm{Aut}(T_6)$, are all S_2. Node 4 cannot be interchanged with any of the other nodes on its level, so the automorphism group of T_2 is just $S_1 \times S_2$, but 5 and 6 can be switched in addition to the permutations on their children. To represent this, we take a wreath product of permutation groups, The set of automorphisms of the tree rooted at 3 is exactly $\mathrm{Aut}(T_5) \wr S_2 = S_2 \wr S_2$. Since T_2 cannot be interchanged with T_3, the final set of automorphisms $\mathrm{Aut}(T_1) = (S_1 \times S_2) \times (S_2 \wr S_2)$. We used this method of describing the automorphisms of a binary tree to determine the Ehrhart polynomial of the polytope associated to the automorphism group of a binary tree.

In order to describe the research done on graph automorphisms, the following two definitions are necessary as well.

Definition 3.0.33. *The* valence *of a vertex is the number of edges that point is a part of. A* regular *graph is a graph where every vertex has the same valence.*

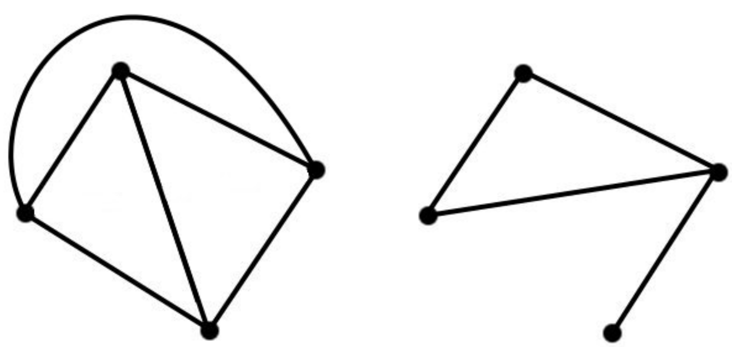


Figure 7. A regular graph on the left and and irregular graph on the right.

We collected a great deal of data on rigid graphs with the help of Courtney Dostie and Mohamed Omar; we also relied on the help of Peter Malkin to work out changes in the programs we used. One of these programs was **nauty** (see [**12**]), which is short for **n**o **aut**omorphisms, **y**es? This program enabled us to check simple graphs for rigidity. **nauty** can also provide the automorphisms of a given graph; however, our focus meant we did not use this functionality much at all.

Once we had a graph that we knew was rigid, we ran it through a pair of programs, **isomorphism** and **nulla**. The algorithm used in both these programs is explained in [**8**]. The first of these, **isomorphism**, produced a series of equations that every non-trivial automorphism of the graph must satisfy.

Any automorphism of a given graph is a permutation, since it permutes the vertices of the graph. The definition of a permutation matrix requires that all entries be either 1 or 0, and that there be exactly one 1 in each row and each column. These requirements can be written as a series of equations using the entries of the automorphism permutation matrix P. $P_{ij}^2 - P_{ij} = 0$ ensures that every entry is 1 or 0, while $\sum_{i=1}^{n} P_{ij} = 1$ requires that the jth column sum to 1, and $\sum_{j=1}^{n} P_{ij} = 1$ requires the ith row to sum to 1. These equations guarantee that the matrix is, in fact, a permutation matrix.

In order for a permutation to be an automorphism of a graph, all edges between vertices must be preserved. But the permutation matrix contains no information about edges - in order to code this information in equations, another matrix must be introduced.

DEFINITION 3.0.34. *The* adjacency matrix A *of a graph with* n *vertices is an* $n \times n$ *matrix, where* $A_{ij} = 1$ *if and only if the vertices* i *and* j *share an edge. If* i *and* j *do share such an edge, then* $A_{ij} = Aji = 1$*, so* A *is by necessity symmetric.*

Th equation $PA = AP$ forces the permutation matrix to maintain edges between vertices. This equation generates n^2 equations, one for each entry in the multiplied matrices. But this one equation can also generate several other equations.

THEOREM 3.0.35. $PA = AP$ *implies* $P^kA^k = A^kP^k$.

PROOF. By assumption, $PA = AP$. Consider P^2A^2. $P^2A^2 = P(PA)A = P(AP)A = (PA)(PA) = (AP)(AP) = A(PA)P = A(AP)P = A^2P^2$. Larger values of k follow inductively. □

Finally, since we were most interested in rigid graphs, we wanted to remove the trivial automorphism from the possible permutation matrices. We checked for this by producing several different sets of equations, with $P_{ii} = 0$ for each possible value of i in each set.

After **isomorphism** produced these sets of equations, we ran the resulting equations through the second of the pair of programs, **nulla**. This program produces a *certificate of infeasibility* - this certificate proves that the system of equations has no solution. Of course, this only works for equations produced from rigid graphs, since otherwise there *is* a non-trivial solution to those equations; any automorphism produces such a solution.

The certificate of infeasibility is known as a *Nullstellensatz certificate.* The idea is to take a system of equations, $f_1 = f_2 = f_n = 0$, and attempt to find a set of polynomials $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ such that $\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n = 1$. If such polynomials exist, there is no solution to the original system. The *degree* of the Nullstellensatz certificate is given by the maximal degree of α_i and corresponds to how difficult the certificate is to obtain.

This work on automorphisms is not directly related to all permutation polytopes, but it is closely linked to the polytopes associated to spanning trees. Rigid trees, which are a subset of the rigid graphs we researched, have a permutation polytope of dimension 0, since they have no automorphisms other than the trivial. Similarly, those of dimension 1 have exactly one automorphism aside from the trivial. With more vertices, the dimension becomes much more difficult to predict, but those trees with few automorphisms are closely related to the rigid graphs we studied with **nauty** and **nulla**.

CHAPTER 4

Results

4.1. Cyclic Permutations

The polytopes of the cyclic groups C_n are exactly the permutation polytopes we are interested in, and they have the advantage of being rather simple objects with a nice structure. Here we show that all polytopes of cyclic groups have the same structure and provide a formula for the volume of said polytopes.

THEOREM 4.1.1. *The polytope of the cyclic group C_n is an $(n-1)$-dimensional simplex for all n. Furthermore, the lattice associated to the vertices of this simplex is exactly the standard lattice.*

PROOF. To prove that $P(C_n)$ is an $(n-1)$-dimensional simplex, it is sufficient to show that C_n produces exactly $n-1$ linearly independent vectors. The group is generated by a single element, so for each $c \in C_n$, the entire term c is determined by the first column of the associated permutation matrix. This means that every term must send 1 to a different value, 1 through n. Now consider each element as a vector, $[R_1,\ R_2,\ \ldots,\ R_n]$, with R_i the ith row of the matrix. Set one of the elements as the origin - I is the easiest choice for this. To adjust for this assignment, subtract I from the remaining $n-1$ vectors - since none of these take 1 to 1, this introduces only -1's. Since each vector takes 1 to a different term, each one takes a different term to 1 as well - this means that R_1 is different for every element, although it always starts with a -1, contains a single 1, and has 0's elsewhere. Since the 1 is in a different place for each element, there is no way to cancel it with a scalar multiple of a different element, so the coefficients for $\lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_{n-1} c_{n-1} = 0$ are all 0. Therefore, there are $n-1$ linearly independent vectors in our polytope, and so it is a simplex.

We now show that the lattice associated to the vertices of the simplex is the standard lattice. Since the vertices are integral, it is sufficient to show that there are no integer points strictly inside the polytope $P(C_n)$. Consider a convex combination of the vertices of $P(C_n)$, $\lambda_1(c_1) + \lambda_2(c_2) + \ldots \lambda_n(c_n)$, and assume that this combination is integral. Consider the first term in this combination. Since each vertex is completely determined by the location of a single 1 in its first row, the first term of this combination is either 1 or 0, as there is only one term with a 1 in the first term. If the term is 1, the coefficient of the identity element must be 1, which forces all the other terms to be 0. If it is 0, the coefficient of the identity element must be 0. Since our combination is convex, at least one of the coefficients must be non-zero, and the argument used for the identity element applies to every other vertex as well. Thus any integral convex combination of the vertices of $P(C_n)$ is exactly one of the vertices itself, so $P(C_n)$ has no strictly interior integer points. $\square$

The indecomposable elements of a group, as defined in 2.0.25, are at the very least an interesting subset of the group. However, they are also very closely related to the process of triangulation mentioned in Chapter 1. The indecomposable elements of a permutation group are exactly those vertices that share an edge with the origin; this is a direct result of Theorem 2.0.26, since any decomposable element can be seen as the product of two elements, and is therefore at least two edges away from the origin. These elements that share an edge with the origin are highly useful for determining a triangulation of the polytope; see [**6**] for more details on this process. As a result of this, identifying the indecomposable elements of a permutation group can be a useful initial step towards determining the volume.

THEOREM 4.1.2. *Every element in C_n is indecomposable.*

PROOF. C_n is by definition a group generated by a single element, so where 1 is mapped to by an element in C_n fully determines that element. Therefore, it is enough to consider the disjoint cycle of each element that contains 1. If an element g was decomposable in C_n, there would have to exist another element in C_n that contained the same disjoint cycle containing 1. But that cycle determines what 1 is mapped to, which in turn determines exactly one element; that is, exactly the g that was originally chosen. Thus, all elements of C_n are indecomposable. $\square$

The theorems in Chapter 1 allow us to directly calculate the volume of the associated polytopes of C_n for all n, as the next theorem demonstrates.

THEOREM 4.1.3. *The volume of $P(C_n)$ is $\frac{1}{(n-1)!}$.*

PROOF. We prove that $P(C_n)$ is unimodular in the standard lattice directly. Since the vertices of $P(C_n)$ span exactly the standard lattice, unimodularity will apply to the standard lattice as well to that associated to $P(C_n)$. Let v_1 be the vertex associated to the identity element of C_n, and then consider $v_i - v_1$ for all i from 2 to $n-1$. Each v_i has a 1 in a different position in its first n entries, so each v_i is linearly independent of the others. Subtracting v_1 does not affect this linear independence, so we end up with $n-1$ linearly independent vectors. Since v_i is a lattice point for i between 1 and $n-1$ and $\dim(P(C_n))= n-1$, these vectors form a basis for the intersection of $\mathrm{aff}(C_n)$ and $\mathbb{Z}^n$. Thus $P(C_n)$ is unimodular, so in the unimodular triangulation T, $f_{n-1} = 1$. If we then apply the formula given in Theorem 1.0.8, we get $i_{P(C_n)}(t) = \sum_{k=0}^{n-1} \binom{t-1}{k} f_k =$

$\binom{t-1}{0}f_0+\binom{t-1}{1}f_1+\ldots+\binom{t-1}{n-1}f_{n-1}=\frac{(t-1)!}{(t-1)!}f_0+\frac{(t-1)!}{(t-2)!}f_1+\ldots+\frac{(t-1)!}{(n-1)!(t-n-2)!}f_{n-1}$. Although the expansion of each of these terms is involved, the dimension of each expansion is easy to see, and the polynomial associated to f_{n-1} clearly has the greatest dimension. This means that the leading term of the Ehrhart polynomial is $\frac{(t-1)!}{(n-1)!(t-n-2)!}$, which has a coefficient of $\frac{1}{n-1!}$; that is, exactly the volume of the simplex. $\square$

The fact that $P(C_n)$ is a simplex can be used to determine the Ehrhart polynomial of $P(C_n)$.

THEOREM 4.1.4. *The Ehrhart polynomial of $P(C_n)$ is $\binom{t+n-1}{n-1}$.*

The proof of this last theorem was found in conjunction with Mohammed Omar and Jesús de Loera, and can be found in [**4**].

4.2. Dihedral Permutations

The polytopes associated to the cyclic permutations, while interesting, are almost too simple to provide a great deal of insight. After charactizing those polytopes, the next step is to move on to the dihedral permutations, which contain the cyclic groups as subsets. This produces a slightly more complex polytope, but one that is still small enough to easily investigate.

THEOREM 4.2.1. *The dimension of the polytope $P(D_n)$ is $(n+1)$ mod $2+4\lfloor\frac{n-1}{2}\rfloor$.*

This was proved in[**13**].

As with the cyclic permutation, a possible entrance into volume is through the indecomposable elements, so the first step is to determine the indecomposable elements of D_n.

THEOREM 4.2.2. *Every element in D_n for $n\neq 4$ is indecomposable.*

PROOF. First consider the case where $n>4$. Take any element g in D_n. If g is decomposable, then $I+g=h+h'$ for $h,h'\in D_n$. Now consider the fixed points of any element in D_n. Other than the identity, every element can be considered as a rotation or a reflection. Rotations do not fix any points, while reflections fix zero, one, or two, depending on the axis of reflection and the parity of n. Note also that $I+g$, in matrix notation, has at least 1 at every diagonal entry. The diagonal 1's in the matrix of h are exactly those elements fixed by h, and the same applies to h'. Therefore, h has at most two 1's along its diagonal. But $h+h'$ must produce at least n 1's along its diagonal, since I contributes n 1's to $I+g$. $h+h'$ can have at most 4 diagonal 1's, so for $n>4$, there is no such decomposable g. Finally, it is easy to see that no element in $D_3=\{(),\ (1\ 2),\ (2\ 3),\ (1\ 3),\ (1\ 2\ 3),\ (1\ 3\ 2)\}$ is decomposable into any other elements in the group. $\square$

D_4 turned out to be a strange case - the odd dihedral groups were always fully indecomposable, as were the even groups of larger size. But the even number of vertices in D_4's associated square, as well as the small number of them, created a single decomposable element in the terms.

THEOREM 4.2.3. *Every element in D_4 except* (1 3)(2 4) *is indecomposable.*

PROOF. As with D_3, this group is small enough that an exhaustive check is easy. $D_4 = \{(), (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2), (1\ 3), (2\ 4), (1\ 4)(2\ 3), (1\ 2)(3\ 4)\}$. Clearly (1 3)(2 4) decomposes into (1 3) and (2 4). Everything else is either one cycle or the product of disjoint cycles not included in D_4. $\square$

We next moved on to examine the Ehrhart polynomial of D_n. This polynomial, as was mentioned in Chapter 1, provides the volume of the polytope and the surface area, as well as the number of lattice points contained within a dilation of the polytope by some integer factor n. We worked on this with Jesús de Loera and Mohamed Omar, and we developed several results, culminating in the complete classification of the Ehrhart polynomial in the standard lattice.

THEOREM 4.2.4. *Let n be an odd positive integer. Then $i_{P(D_n)}(t) =$*

$$\sum_{k=0}^{n-2} \binom{2n}{k+1}\binom{t-1}{k} + \sum_{k=n-1}^{2n-2} \left(\binom{2n}{k+1} - \binom{n}{k-n+1}\right)\binom{t-1}{k}.$$

Let $n = 2m$ be an even positive integer. Then $i_{P(D_n)}(t) =$

$$\sum_{k=0}^{m-2} \binom{2n}{k+1}\binom{t-1}{k} - \sum_{k=m-1}^{2m-2} \left(\binom{2n}{k+1} - 2\binom{2n-m}{k+1-m}\right)\binom{t-1}{k} +$$
$$\sum_{k=2m-1}^{4m-3} \left(\binom{2n}{k+1} - 2\binom{2n-m}{k+1-m} + \binom{2n-2m}{k+1-2m}\right)\binom{t-1}{k}.$$

The proof of this theorem, as well as the propositions and work leading up to it, can be found in [**4**].

This formula for the Ehrhart polynomial also provides the volume of the polytope.

COROLLARY 4.2.5. *The volume of $P(D_n)$ is n if n is odd and $\frac{n^2}{4}$ if n is even.*

4.3. Even Permutations

THEOREM 4.3.1. *For $n \geq 4$, $dim(P(A_n)) = dim(P(S_n)) = (n-1)^2$.*

This result is due to the work of Brualdi and Liu, which can be found in [**3**]. As mentioned earlier, characterizing the indecomposable elements of A_n is an entry point into determining the volume of the $P(A_n)$. This was thus exactly where we started.

THEOREM 4.3.2. *A matrix $A \in A_n$ is indecomposable if and only if $A = [c_1][c_2]$, with c_1 and c_2 disjoint cycles of length l_1 and l_2 with l_1, $l_2 \equiv 0$ mod 2, or $A = [c_1]$, with $[c_1]$ of length $l_1 \equiv 1$ mod 2.*

PROOF. This theorem was first proved in [**3**]; this proof relies on the definition of indecomposability as given in Definition 2.0.25.

Assume $A \in A_n$ is indecomposable and has more than 2 disjoint cycles, so $A = [c_1][c_2]\dots[c_s]$ with $s > 2$ and cycle $[c_i]$ of length l_i. Since $A \in A_n$, $\sum l_i - s \equiv 0$ mod 2, and since A is irreducible, no subset of the cycles can be included in A_n; that is, $\sum l_j - j \equiv 1$ mod 2 for $j < s$. Specifically, this means that $l_1 - 1 \equiv 1$ mod 2, $l_2 - 1 \equiv 1$ mod 2, and $l_1 + l_2 - 2 \equiv 1$ mod 2. But $l_1 - 1 + l_2 - 1 = l_1 + l_2 - 2 \equiv 0$ mod 2, contradicting our assumptions. So $s \leq 2$, so if $A \in A_n$ is indecomposable, then $A = [c_1][c_2]$ with l_1, $l_2 \equiv 0$ mod 2 (otherwise $[c_1]$ A_n and $c_2] \in A_n$ and A would be decomposable), or $A = [c_1]$ with $l_1 \equiv 1$ mod 2 (since $A \in A_n$ itself).

Now assume $A = [c_1][c_2]$, with $[c_1]$ of length l_1, $[c_2]$ of length l_2, and l_1, $l_2 \equiv 0$ mod 2. Since $l_1 \equiv 0$ mod 2, $[c_1] \notin A_n$, and similarly $[c_2] \notin A_n$, so A is indecomposable in A_n. If $A = [c_1]$, then A is trivially indecomposable, since it consists of a single cycle. $\square$

Once the form of the indecomposable elements was determined, it became a combinatorics problem to find out how many of these elements the group contained.

THEOREM 4.3.3. *There are*

$$\sum_{l=1}^{\lceil \frac{n}{2}-1 \rceil} \binom{n}{2l+1}(2l)! + \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor}\sum_{j=1}^{\lfloor \frac{n-2k}{2} \rfloor} \frac{n!}{2k(2j)(n-2j-2k)!)}$$

non-trivial indecomposable elements in A_n.

PROOF. From Theorem 4.3.2, finding the number of indecomposable elements in A_n is just a matter of counting the possibilities. Consider the indecomposable elements containing a single cycle of length $2l+1$. There are $\lceil \frac{n}{2} \rceil$ possible lengths for these cycles, but one of these is 1, which is just the identity, so we really want to consider $\lceil \frac{n}{2} - 1 \rceil$ lengths. Assume without loss of generality that every cycle starts with the smallest element, so there are $(2l)!$ ways to arrange the remaining terms. Finally, there are $\binom{n}{2l+1}$ different sets of $2l+1$ elements. So there are

$$\sum_{l=1}^{\lceil \frac{n}{2}-1 \rceil} \binom{n}{2l+1}(2l)!$$

non-trivial indecomposable elements in A_n with a single cycle.

Now consider the even permutations that consist of a pair of cycles of even length. Say that the first of these cycles, c_1, has length $2k$. There are n possible choices for the lengths in this cycle, so there are $\binom{n}{2k}$ possible sets of elements and $\binom{n}{2k}(2k-1)!$ unique arrangements of elements. The second cycle, c_2, also has even length - say $2j$. There are $n-2k$ choices for this cycle, and $(2j-1)!$ ways to arange those choices, so there are $\binom{n-2k}{2j}(2j-1)!$ unique cycles. However, any pair of cycles can be reordered, and since the two are disjoint, the rearrangement does not change the permutation. Thus the total number of unique cycles must be divided by 2 to remove this repetition. So we have

$$\frac{1}{2}\binom{n}{2k}\binom{n-2k}{2j}(2j-1)!(2k-1)! =$$

$$\frac{1}{2}(\frac{n!}{(2k!)(n-2k)!})(\frac{(n-2k)!}{(2j!)(n-2k-2j)!})(2j-1)!(2k-1)! =$$

$$\frac{1}{2}(\frac{n!(n-2k)!}{2k(2j)(n-2k)!(n-2k-2j)!}) =$$

$$\frac{1}{2}(\frac{n!}{2k(2j)(n-2k-2j)!})$$

permutations of this type. Since c_1 is a cycle of even length, k must be an integer, so k can vary from 1 to $\lfloor \frac{n}{2} \rfloor$. However, there must be enough elements remaining to produce another cycle of even length, so k actually varies from 1 to $\lfloor \frac{n-2}{2} \rfloor$. Similarly, j varies from 1 to $\lfloor \frac{n-2k}{2} \rfloor$, so all together we have

$$\frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor}\sum_{j=1}^{\lfloor \frac{n-2k}{2} \rfloor}\frac{n!}{2k(2j)(n-2j-2k)!}.$$

Summed with the other type of indecomposable even permutations, we have

$$\sum_{l=1}^{\lceil \frac{n}{2}-1 \rceil}\binom{n}{2l+1}(2l)! + \frac{1}{2}\sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor}\sum_{j=1}^{\lfloor \frac{n-2k}{2} \rfloor}\frac{n!}{2k(2j)(n-2j-2k)!}$$

as the total number of indecomposable elements in A_n. □

The goal is to eventually determine the volume of this polytope; however, the complexity and high dimension of this type made it extremely difficult to gather more information. However, we were able to determine a result relating to the complexity of $P(A_n)$, specifically through determining two-levelness.

THEOREM 4.3.4. *The polytope $P(A_n)$ is two-level if and only if $n \leq 4$. Moreover, for $n \geq 8$, $P(A_n)$ is at least $(\lfloor \frac{n}{4} \rfloor + 1)$-level.*

The proof of this can be found in [**4**].

4.4. Frobenius Groups

Frobenius groups, along with general group theory terms, are defined in Chapter 2. This work was once again done with Mohamed Omar and Jesús de Loera. The following theorem describes the normalized volume of Frobenius groups in terms of the size of the Frobenius kernal and Frobenius complement; the proof of the theorem can be found in [**4**].

THEOREM 4.4.1. *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel N. Let $|N|$ be the size of N and let $|H|$ be the size of H. The normalized volume of $P(G)$ in the sublattice of $\mathbb{Z}^{n\times n}$ spanned by its vertices is*

$$\frac{1}{(|H||N|-|H|)!}\sum_{\ell=0}^{\lfloor\frac{|H|(|N|-1)-1}{|N|}\rfloor}\binom{(|H|-\ell)|N|}{(|H|-\ell)|N|-|H|+1}\binom{|H|-1}{\ell}(-1)^{\ell}.$$

This result is incredibly wide-reaching, as it applies to all Frobenius groups. These groups include A_4 and D_n for n odd, so this theorem provides a description of the normalized volume for D_n with n odd. In addition, it gives us the normalized volume for one of the even permutations, which we do not yet have any formula for.

4.5. Spanning Trees

Our research focused on the polytopes of subgroups of S_n, but this was not limited to the groups mentioned above. Another way to approach these subgroups is to consider the set of automorphisms of a given graph. An automorphism gives a map from a graph onto itself - that is, a permutation of the vertices. The set of all such permutations is the set of automorphisms, as well as a subgroup of the set of all permutations on the number of vertices. Once these permutations are found, the process of converting to a polytope is the same as with any other subgroup, including those above.

Graphs, as defined in Chapter 3, are quite interesting on their own, but for the purposes of our research, we decided to focus on spanning trees. Trees are also defined in Chapter 3; they are graphs without any closed cycles in which every vertex is connected to every other. As with the previous subgroups, the goal was to find some formula for the Ehrhart polynomial of the polytopes of these graphs. While we have not yet found a formula for all spanning trees, the following method allows for the calculation of the Ehrhart polynomials of binary trees. Once again, this research was done with Jesús de Loera and Mohamed Omar.

THEOREM 4.5.1. *Let $G \leq S_n$, and $G \wr S_2$ be the wreath product of G with the symmetric group S_2. Then*

$$i(P(G\wr S_2),t)=\sum_{k=0}^{t} i^2(P(G),k)\cdot i^2(P(G),t-k)$$

for any integer $t \geq 2$.

The proof of this theorem and the work leading up to it can again be found in [**4**].

Using this theorem, we can calculate the Ehrhart polynomial of any binary tree. First, given a rooted binary tree T, we compute the automorphism group $\mathrm{Aut}(T)$ as a sequence of direct products and wreath products. Then we read the group $\mathrm{Aut}(T)$ from left to right. If we encounter a direct product, we compute the Ehrhart polynomials of the corresponding groups and take the product of the polynomials. If we encounter a wreath product, we apply Theorem 4.5.1. This produces the Ehrhart polynomial of the permutation polytope associated to the tree T.

4.6. Graph Automorphisms

The research in this section was done with Mohamed Omar and Courtney Dostie.

While the other research we did focused on polytopes associated to groups or graphs (and hence graphs with non-trivial automorphisms), the work we did using the programs **nauty**, **isomorphism**, and **nulla** focused on *rigid* graphs - that is, graphs with only the trivial permutation as an automorphism. The goal was to determine whether a graph was rigid or not in the shortest amount of time.

As described in Chapter 3, the program **nauty** can determine whether a graph is rigid or not; however, although it has thus far run quite quickly, there is no bound on how long it could take. As a result, our goal was to shorten the runtime of **nulla**, which can not only prove the rigidity of a graph, but also provides a Nullstellensatz certificate as verification.

While **nulla** is an excellent program, producing a certificate of infeasibility is incredibly difficult for certificates of degree greater than one. In order to produce the certificate, the program must run operations on a series of functions, and as the degree of those functions increases, the entire program slows significantly. This means that, while **nulla** produces certificates of degree zero easily, degree one takes a noticable amount of time, and producing a degree two certificate is almost impossible in a reasonable amount of time. The research we did was therefore calculated to lower the degree of certificates with degree higher than one. To achieve this, we added a series of equations to those generated by **isomorphism**. The equations already generated suffice to determine all automorphisms - however, we hoped to lower the runtime and the certificate degree by adding redundant equations.

The first of these equations were $P_{ij}P_{ik} = 0$ and $P_{ji}P_{ki} = 0$. Since all permutation matrices must have row and column sum 0 and entries of 0 or 1, the multiplication of any two entries in the same row or column must equal 0. We hoped this would decrease the runtime for larger graphs - while this information was already provided to **nulla** in a different form, the redundancy might have shortened runtime. Unfortunately, while runtime on regular graphs decreased, runtime actually increased on irregular graphs, as the program had many more equations to deal with. This addition was thus determined to be ineffective.

We considered adding a series of equations of the form $P^kA^k = A^kP^k$, with P our permutation matrix and A the adjacency matrix associated to the graph. As mentioned in Chapter 3, the equation $PA = AP$ included by **isomorphism** ensures that a permutation maintains the edge connections between points. $P^kA^k = A^kP^k$ forces the permutation to maintain the edge connections between neighbor points as well, out to k edges away from the orginal point. In this case, program conflicts prevented us from implementing these equations - while a program was produced that provided the required equations, the format was incompatable with **nulla**, and we were unable to actually test the effectiveness of the equations.

The second approach we used was to run the program **nulla** over fields other than $\mathbb{R}$. **ismorphism** includes a quadratic equation, $P_{ij}^2 - P_{ij} = 0$, to force all entries in P to be 1 or 0. This quadratic equation is a problem, as it is the only non-linear equation given to **nulla** and greatly increases the complexity of the equation set. To avoid this, we ran graphs over the field $\mathbb{F}^2$. Since this field only provides 0 and 1 as possible results, the quadratic equation can be removed entirely. While this greatly simplifies the system of equations for **nulla** to solve, it also weakens the resulting certificate, since the results from $\mathbb{F}^2$ do not necessarily carry over to $\mathbb{R}$.

As mentioned in Chapter 3, in order to remove the trivial automorphism from our permutations, we ran each set of equations once for each vertex, where that vertex was the one chosen by $P_{ii} = 0$ to not map to itself. When the equations associated to a vertex "passed" (that is, received a certificate of infeasibility), we refer to that vertex as passing as well. When we ran graphs over $\mathbb{F}^2$ while looking for a certificate of degree 0, we received a certificate of infeasibility for at most two vertices, regardless of how many passed over $\mathbb{R}$. Every one of the graphs we ran received a degree one certificate of infeasibility for every vertex in $\mathbb{F}^2$. The graphs we ran over $\mathbb{R}$ had a similar degree one certification, but we were unable to run all graphs, as even a degree one certificate over $\mathbb{R}$ takes a prohibitively long time to produce for larger graphs.

To conclude our work on rigid automorphisms, we returned to adding equations to the system given to **nulla**. This time, we looked at the valences of various vertices. Since a vertex cannot map to a vertex with a different number of edges, we determined the valence of each vertex. By comparing these valences via the adjacency matrix, we were able to fill in the permutation matrix with appropriate zeros for vertices that could not be mapped to each other. For two points i and j in our graph with different valences, $P_{ij} = P_{ji} = 0$.

These added equations provided turned out to be quite helpful, as vertex valence was not directly encoded in any of the equations **nulla** dealt with (although it was indirectly included in the edge connections, the $PA = AP$ equations). These additions succeeded in decreasing runtime on all the irregular graphs; since all vertices in regular graphs have the same valence, regular graphs were unaffected by this change. It also greatly increased the number of degree 0 certificates that were produced over both $\mathbb{R}$ and $\mathbb{F}^2$. By adding these equations, nearly all vertices we tested passed at degree 0 over $\mathbb{R}$ and $\mathbb{F}^2$, while previously most vertices failed over $\mathbb{F}^2$ and many failed in $\mathbb{R}$.

CHAPTER 5

Data

In the process of developing the theorems presented in the previous chapter, we collected a great deal of data on a wide variety of groups. That information is presented here, starting in Section 5.1 with the Ehrhart polynomials for the first several cyclic and dihedral groups. These polynomials were calculated using the program lattE (see [**7**]), which can produce a great deal of information related to the triangulations of a given polytope.

The subgroups of S_n are interesting in and of themselves, but for n greater than 5, the number of subgroups and complexity of said groups is too large for an exhaustive study. For n from 3 to 5, however, there are few subgroups, and nearly all are small enough to produce results. These groups, along with the dimension of their associated permutation polytopes and most of their Ehrhart polynomials, are presented in Section 5.2. Beyond the polynomials associated to these, the limitations on group size prevented us from determining any particular results related to these groups. In these tables, $GA(1,5)$ indicates the general affine group of degree one over the field of five elements. This group is generated by taking the semidirect product of the additive and multiplicative groups of the field of five elements. We used the information on subgroups found on the webpages [**14**], [**15**], and [**16**] as the basis for these computations.

We also collected information on spanning trees during our research. Although we only developed a theorem for binary trees, the tables in Section 5.3 collect our research on spanning trees. We found the dimension and Ehrhart polynomials of the polytopes associated to all spanning trees from four to seven vertices, as well as the dimension of the polytope associated to all spanning trees of eight, nine, or ten vertices.

Finally, the rigid graphs shown in Section 5.4 are color-coded regarding which vertices passed or failed over $\mathbb{R}$ or $\mathbb{F}^2$. Blue vertices passed over both $\mathbb{R}$ and $\mathbb{F}^2$ with certificates of degree 0 and the original series of equations. Black vertices passed over $\mathbb{R}$ with degree 0 but failed over $\mathbb{F}^2$ degree 0, again with the original equations. Red vertices failed over both $\mathbb{R}$ and $\mathbb{F}^2$ degree 0 with the original equations, but passed with the addition of the valence equations. Purple vertices failed over $\mathbb{R}$ degree 0 but passed over $\mathbb{F}^2$ degree 0, and passed over both with the addition of the valence equations. Green vertices failed over both fields originally, then passed over $\mathbb{R}$ degree 0 and failed over $\mathbb{F}^2$ degree 0 with the added valence equations. The vertices colored yellow are the rare cases that passed over $\mathbb{R}$ and failed over $\mathbb{F}^2$ both with and without the valence additions. Finally, orange vertices failed over $\mathbb{R}$ degree 0 and passed over $\mathbb{F}^2$ degree 0 with the added valence equations.

5.1. Cyclic and Dihedral Permutations

Cyclic Groups		
Group	Dim	Ehrhart Polynomial
C_2	1	$t+1$
C_3	2	$\frac{1}{2}t^2+\frac{3}{2}t+1$
C_4	3	$\frac{1}{6}t^3+t^2+\frac{11}{6}t+1$
C_5	4	$\frac{1}{24}t^4+\frac{5}{12}t^3+\frac{35}{24}t^2+\frac{25}{12}t+1$
C_6	5	$\frac{1}{120}t^5+\frac{1}{8}t^4+\frac{17}{24}t^3+\frac{15}{8}t^2+\frac{137}{60}t+1$
C_7	6	$\frac{1}{720}t^6+\frac{7}{240}t^5+\frac{35}{144}t^4+\frac{49}{48}t^3+\frac{203}{90}t^2+\frac{49}{20}+1$
C_8	7	$\frac{1}{5040}t^7+\frac{1}{180}t^6+\frac{23}{360}t^5+\frac{7}{18}t^4+\frac{967}{720}t^3+\frac{469}{180}t^2+\frac{363}{140}t+1$
C_9	8	$\frac{1}{40320}t^8+\frac{1}{1120}t^7+\frac{13}{960}t^6+\frac{9}{80}t^5+\frac{1069}{1920}t^4+\frac{267}{160}t^3+\frac{29531}{10080}t^2+\frac{761}{280}t+1$
C_{10}	9	$\frac{1}{362880}t^9+\frac{1}{8064}t^8+\frac{29}{12096}t^7+\frac{5}{192}t^6+\frac{3013}{17280}t^5+\frac{95}{128}t^4+\frac{4523}{2268}t^3+\frac{6515}{2016}t^2+\frac{7129}{2520}t+1$
C_{11}	10	$\frac{1}{3628800}t^{10}+\frac{11}{725760}t^9+\frac{11}{30240}t^8+\frac{121}{24192}t^7+\frac{7513}{172800}t^6+\frac{8591}{34560}t^5+\frac{341693}{362880}t^4+\frac{84095}{36288}t^3+\frac{177133}{50400}t^2+\frac{7381}{2520}t+1$
C_{12}	11	$\frac{1}{39916800}t^{11}+\frac{1}{604800}t^{10}+\frac{1}{20736}t^9+\frac{11}{13440}t^8+\frac{10831}{1209600}t^7+\frac{1903}{28800}t^6+\frac{242537}{725760}t^5+\frac{139381}{120960}t^4+\frac{341747}{129600}t^3+\frac{190553}{50400}t^2+\frac{83711}{27720}t+1$
Dihedral Groups		
D_3	4	$\frac{1}{8}t^4+\frac{3}{4}t^3+\frac{15}{8}t^2+\frac{9}{4}t+1$
D_4	5	$\frac{1}{30}t^5+\frac{1}{3}t^4+\frac{4}{3}t^3+\frac{8}{3}t^2+\frac{79}{30}t+1$
D_5	8	$\frac{1}{8064}t^8+\frac{5}{2016}t^7+\frac{5}{192}t^6+\frac{25}{144}t^5+\frac{95}{128}t^4+\frac{575}{288}t^3+\frac{6515}{2016}t^2+\frac{475}{168}t+1$
D_6	9	$\frac{1}{40320}t^9+\frac{3}{4480}t^8+\frac{19}{2240}t^7+\frac{21}{320}t^6+\frac{43}{128}t^5+\frac{741}{640}t^4+\frac{6653}{2520}t^3+\frac{4229}{1120}t^2+\frac{2533}{840}t+1$
D_7	12	$\frac{1}{68428800}t^{12}+\frac{7}{11404800}t^{11}+\frac{91}{6220800}t^{10}+\frac{49}{207360}t^9+\frac{5551}{2073600}t^8+\frac{833}{38400}t^7+\frac{790153}{6220800}t^6+\frac{111083}{207360}t^5+\frac{2486939}{1555200}t^4+\frac{845593}{259200}t^3+\frac{63427}{14850}t^2+\frac{12593}{3960}t+1$
D_8	13	$\frac{1}{389188800}t^{13}+\frac{1}{7484400}t^{12}+\frac{13}{3742200}t^{11}+\frac{1}{17010}t^{10}+\frac{107}{151200}t^9+\frac{713}{113400}t^8+\frac{7117}{170100}t^7+\frac{355}{1701}t^6+\frac{2100947}{2721600}t^5+\frac{1408031}{680400}t^4+\frac{1205899}{311850}t^3+\frac{49018}{10395}t^2+\frac{597941}{180180}t+1$
D_9	16	$\frac{1}{2324754432000}t^{16}+\frac{1}{32288256000}t^{15}+\frac{17}{12915302400}t^{14}+\frac{1}{25625600}t^{13}+\frac{3587}{4257792000}t^{12}+\frac{269}{19712000}t^{11}+\frac{137581}{812851200}t^{10}+\frac{18343}{11289600}t^9+\frac{7292813}{602112000}t^8+\frac{250617}{3584000}t^7+\frac{13195009}{42577920}t^6+\frac{1022521}{985600}t^5+\frac{371446039969}{145297152000}t^4+\frac{8983049989}{2018016000}t^3+\frac{344499373}{67267200}t^2+\frac{275439}{80080}t+1$
D_{10}	17	$\frac{1}{14227497123840}t^{17}+\frac{1}{167382319104}t^{16}+\frac{1}{3804143616}t^{15}+\frac{25}{3218890752}t^{14}+\frac{2017}{11955879936}t^{13}+\frac{235}{83607552}t^{12}+\frac{39409}{1072963584}t^{11}+\frac{111385}{292626432}t^{10}+\frac{3675491}{1170505728}t^9+\frac{24092095}{1170505728}t^8+\frac{12241087}{114960384}t^7+\frac{98775715}{229920768}t^6+\frac{6310439063}{4755179520}t^5+\frac{10646456155}{3487131648}t^4+\frac{242916085}{48432384}t^3+\frac{930775}{169344}t^2+\frac{43467323}{12252240}t+1$

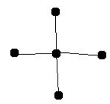
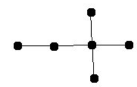
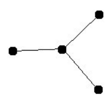
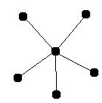
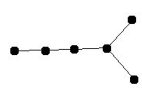
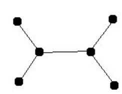
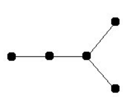
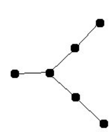
Note that the results found for these groups correspond with the formulas for their Ehrhart polynomials presented in Chapter 4.

5.2. Subgroups of S_3, S_4, and S_5

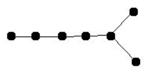
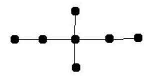
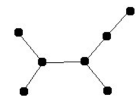
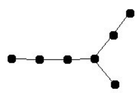
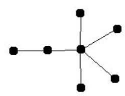
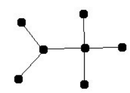
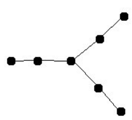
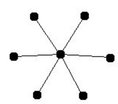
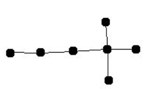
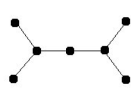
Subgroups of S_3			
Order	Generators	Dim	Ehrhart Polynomial
1	$<() >\cong I$	0	1
2	$<(1\ 2) >\cong C_2$	1	$t+1$
3	$<(1\ 2\ 3) >\cong C_3$	2	$\frac{1}{2}t^2+\frac{3}{2}t+1$
6	$<(1\ 2),(1\ 3),(2\ 3) >\cong S_3$	4	$\frac{1}{8}t^4+\frac{3}{4}t^3+\frac{15}{8}t^2+\frac{9}{4}t+1$
Subgroups of S_4			
Order	Group	Dim	Ehrhart Polynomial
1	$<() >\cong I$	0	1
2	$<(1\ 2) >\cong C_2$	1	$t+1$
2	$<(1\ 2)(3\ 4) >\cong C_2$	1	$t+1$
3	$<(1\ 2\ 3) >\cong C_3$	2	$\frac{1}{2}t^2+\frac{3}{2}t+1$
4	$<(1\ 2),(3\ 4) >\cong C_2\times C_2$	2	t^2+2t+1
4	$<(1\ 2)(3\ 4),(1\ 3)(2\ 4) >\cong C_2\times C_2$	3	$\frac{1}{6}t^3+t^2+\frac{11}{6}t+1$
4	$<(1\ 2\ 3\ 4) >\cong C_4$	3	$\frac{1}{6}t^3+t^2+\frac{11}{6}t+1$
6	$<(1\ 2),(1\ 3),(2\ 3) >\cong S_3$	4	$\frac{1}{8}t^4+\frac{3}{4}t^3+\frac{15}{8}t^2+\frac{9}{4}t+1$
8	$<(1\ 2\ 3\ 4),(1\ 2)(3\ 4) >\cong D_4$	5	$\frac{1}{30}t^5+\frac{1}{3}t^4+\frac{4}{3}t^3+\frac{8}{3}t^2+\frac{79}{30}t+1$
12	$<(1\ 2\ 3),(1\ 2\ 4),(1\ 3\ 4),\ (2\ 3\ 4) >\cong A_4$	9	$\frac{1}{5670}t^9+\frac{1}{504}t^8+\frac{23}{1890}t^7+\frac{1}{15}t^6+\frac{173}{540}t^5+\frac{9}{8}t^4+\frac{29797}{11340}t^3+\frac{1199}{315}t^2+\frac{383}{126}t+1$
24	$<(1\ 2),(1\ 3),(1\ 4),(2\ 3),(2\ 4),(3\ 4) >\cong S_4$	9	$\frac{11}{11340}t^9+\frac{11}{630}t^8+\frac{19}{135}t^7\,\frac{2}{3}t^6+\frac{1109}{540}t^5+\frac{43}{10}t^4+\frac{35117}{5670}t^3+\frac{379}{63}t^2+\frac{65}{18}t+1$
Subgroups of S_5			
Order	Generators	Dim	Ehrhart Polynomial
1	$<() >\cong I$	0	1
2	$<(1\ 2) >\cong C_2$	1	$t+1$
2	$<(1\ 2)(3\ 4) >\cong C_2$	1	$t+1$
3	$<(1\ 2\ 3) >\cong C_3$	2	$\frac{1}{2}t^2+\frac{3}{2}t+1$
4	$<(1\ 2),(3\ 4) >\cong C_2\times C_2$	2	t^2+2t+1
4	$<(1\ 2)(3\ 4),(1\ 3)(2\ 4) >\cong C_2\times C_2$	3	$\frac{1}{6}t^3+t^2+\frac{11}{6}t+1$
4	$<(1\ 2\ 3\ 4) >\cong C_4$	3	$\frac{1}{6}t^3+t^2+\frac{11}{6}t+1$
5	$<(1\ 2\ 3\ 4\ 5) >\cong C_5$	4	$\frac{1}{24}t^4+\frac{5}{12}t^3+\frac{35}{24}t^2+\frac{25}{12}t+1$
6	$<(1\ 2\ 3)(4\ 5) >\cong C_6$	3	$\frac{1}{2}t^3+2t^2+\frac{5}{2}t+1$
6	$<(1\ 2),(2\ 3),(1\ 3) >\cong S_3$	4	$\frac{1}{8}t^4+\frac{3}{4}t^3+\frac{15}{8}t^2+\frac{9}{4}t+1$
6	$<(1\ 2)(4\ 5),(1\ 3)(4\ 5),(2\ 3)(4\ 5) >\cong S_3$	5	$\frac{1}{40}t^5+\frac{1}{8}t^4+\frac{5}{8}t^3+\frac{15}{8}t^2+\frac{47}{20}t+1$
8	$<(1\ 2\ 3\ 4),(1\ 2)(3\ 4) >\cong D_4$	5	$\frac{1}{30}t^5+\frac{1}{3}t^4+\frac{4}{3}t^3+\frac{8}{3}t^2+\frac{79}{30}t+1$
10	$<(1\ 2\ 3\ 4\ 5),(2\ 5)(3\ 4) >\cong D_5$	8	$\frac{1}{8064}t^8+\frac{5}{2016}t^7+\frac{5}{192}t^6+\frac{25}{144}t^5+\frac{95}{128}t^4+\frac{575}{288}t^3+\frac{6515}{2016}t^2+\frac{475}{168}t+1$
12	$<(1\ 2\ 3)(4\ 5),(1\ 2)(4\ 5) >\cong D_6$	5	$\frac{1}{8}t^5+\frac{7}{8}t^4+\frac{21}{8}t^3+\frac{33}{8}t^2+\frac{13}{4}t+1$
12	$<(1\ 2\ 3),(1\ 2\ 4),(1\ 3\ 4),\ (2\ 3\ 4) >\cong A_4$	9	$\frac{1}{5670}t^9+\frac{1}{504}t^8+\frac{23}{1890}t^7+\frac{1}{15}t^6+\frac{173}{540}t^5+\frac{9}{8}t^4+\frac{29797}{11340}t^3+\frac{1199}{315}t^2+\frac{383}{126}t+1$
20	$<(1\ 2\ 3\ 4\ 5),(1\ 2\ 4\ 3) >\cong GA(1,5)$	9	Too large to compute; volume=$\frac{19}{6538371840}$
24	$<(1\ 2),\ (1\ 3),(1\ 4),\ (2\ 3),(2\ 4),\ (3\ 4) >\cong S_4$	9	$\frac{11}{11340}t^9+\frac{11}{630}t^8+\frac{19}{135}t^7+\frac{2}{3}t^6+\frac{1109}{540}t^5+\frac{43}{10}t^4+\frac{35117}{5670}t^3+\frac{379}{63}t^2+\frac{65}{18}t+1$
60	$<(1\ 2\ 3),(1\ 2\ 4),(1\ 2\ 5),(1\ 3\ 4),(1\ 3\ 5),(1\ 4\ 5),$ $(2\ 3\ 4),(2\ 3\ 5),(2\ 4\ 5),(3\ 4\ 5) >\cong A_5$	16	Too large to compute
120	$<(1\ 2),(1\ 3),(1\ 4),(1\ 5),(2\ 3),(2\ 4),(2\ 5),(3\ 4),$ $(3\ 5),(4\ 5) >\cong S_5$	16	$\frac{188723}{836911595520}t^{16}+\frac{188723}{20922789888}t^{15}+\frac{1008757}{5977939968}t^{14}+\frac{112655}{57480192}t^{13}+$ $\frac{72750523}{4598415360}t^{12}+\frac{984101}{10450944}t^{11}+\frac{125188639}{292626432}t^{10}+\frac{55426325}{36578304}t^9+$ $\frac{3541860299}{836075520}t^8+\frac{196563587}{20901888}t^7+\frac{3812839477}{229920768}t^6+\frac{664118435}{28740096}t^5+$ $\frac{438177965089}{17435658240}t^4+\frac{3028287247}{145297152}t^3+\frac{6229735}{494208}t^2+\frac{725}{144}t^1+1$

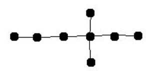
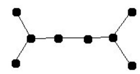
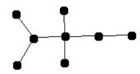
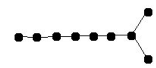
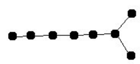
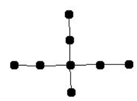
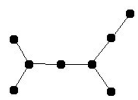
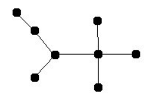
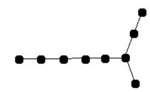
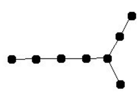
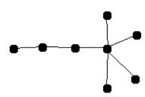
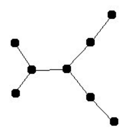
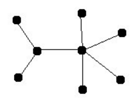
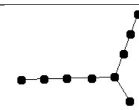
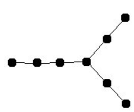
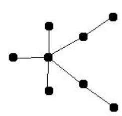
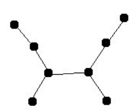
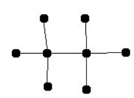
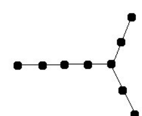
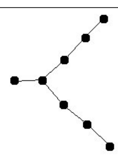
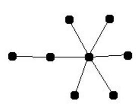
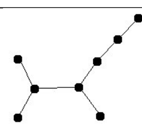
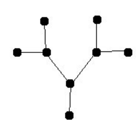
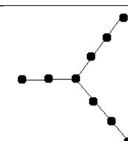
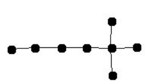
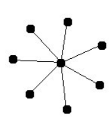
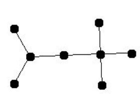
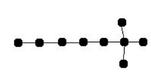
5.3. Spanning Trees

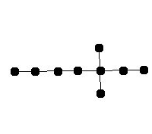
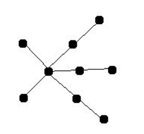
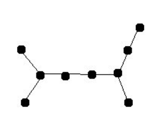
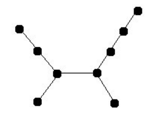
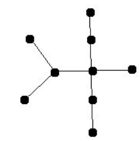
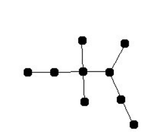
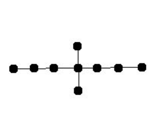
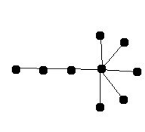
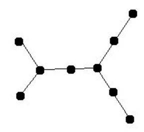
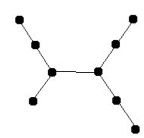
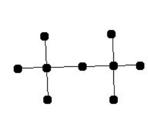
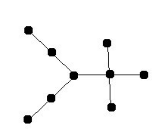
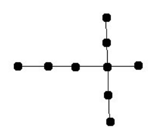
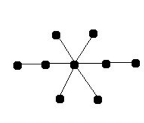
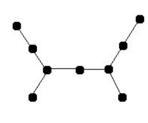
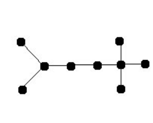
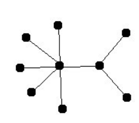
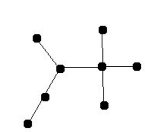
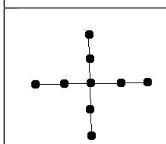
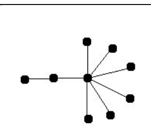
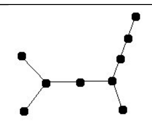
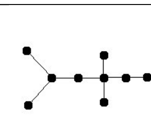
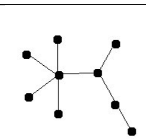
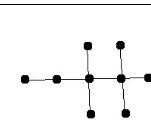
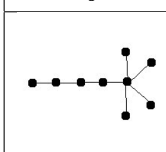
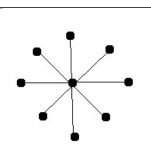
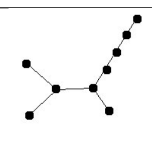
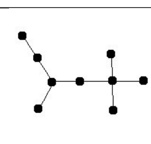
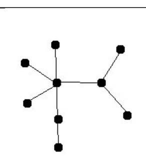
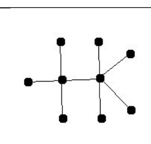
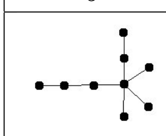
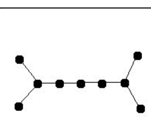
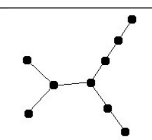
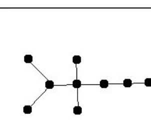
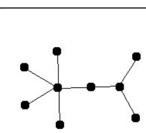
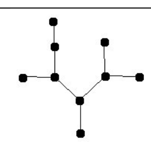
Tree
Dim
Ehrhart Polynomial

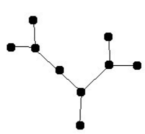
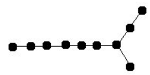
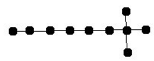
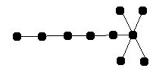
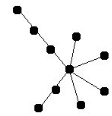
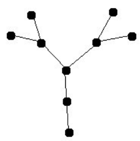
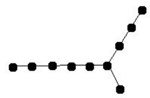
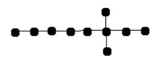
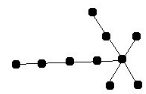
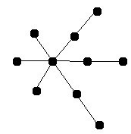
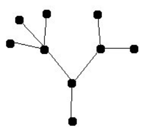
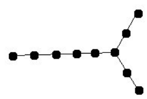
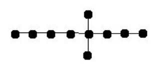
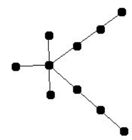
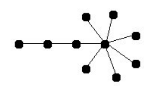
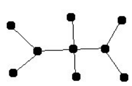
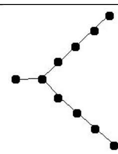
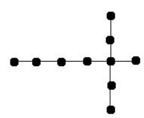
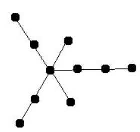
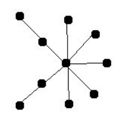
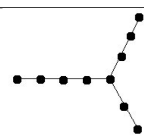
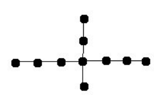
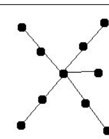
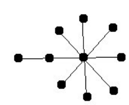
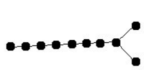
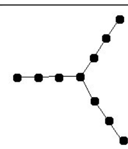
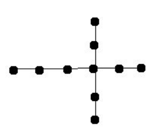
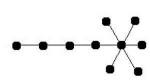
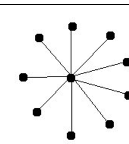
		
ST4.1	ST5.3	ST6.4
1	9	4
$t+1$	$\frac{11}{11340}t^9 + \frac{11}{630}t^8 + \frac{19}{135}t^7 + \frac{2}{3}t^6 + \frac{1}{30}t^5 + \frac{1}{3}t^4 + \frac{4}{3}t^3 + \frac{8}{3}t^2 + \frac{79}{30}t + 1$	$\frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$
		
ST4.2	ST6.1	ST6.5
4	1	16
$\frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$	$t+1$	$\frac{188723}{836911595520}t^{16} + \frac{188723}{20922789888}t^{15} + \frac{1008757}{5977939968}t^{14} + \frac{112655}{57480192}t^{13} + \frac{72750523}{4598415360}t^{12} + \frac{984101}{10450944}t^{11} + \frac{125188639}{292626432}t^{10} + \frac{55426325}{36578304}t^9 + \frac{3541860299}{836075520}t^8 + \frac{196563587}{20901888}t^7 + \frac{3812839477}{229920768}t^6 + \frac{664118435}{28740096}t^5 + \frac{438177965089}{17435658240}t^4 + \frac{3028287247}{145297152}t^3 + \frac{6229735}{494208}t^2 + \frac{725}{144}t + 1$
		
ST5.1	ST6.2	ST6.6
1	1	5
$t+1$	$t+1$	$\frac{1109}{540}t^5 + \frac{43}{10}t^4 + \frac{35117}{5670}t^3 + \frac{379}{63}t^2 + \frac{65}{18}t + 1$
		
ST5.2	ST6.3	ST7.1
1	1	1
$t+1$	$t+1$	$t+1$

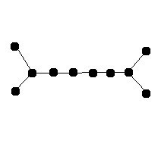
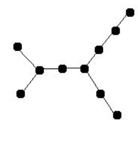
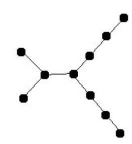
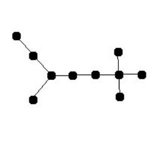
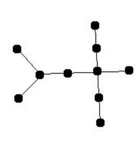
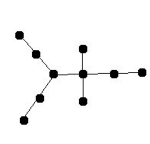
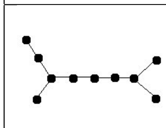
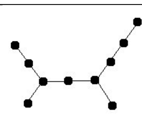
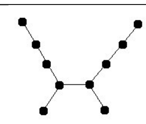
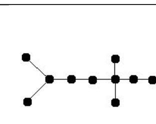
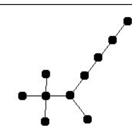
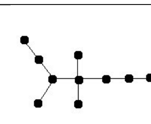
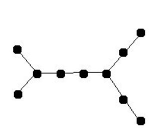
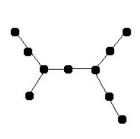
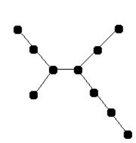
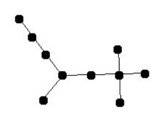
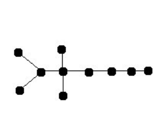
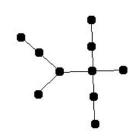
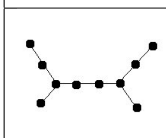
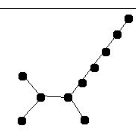
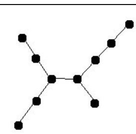
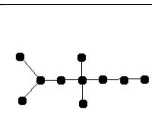
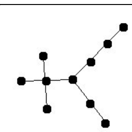
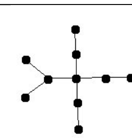
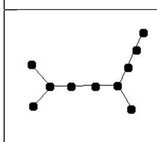
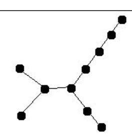
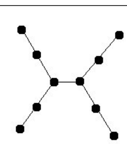
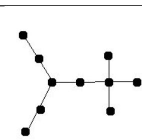
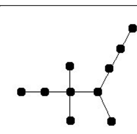
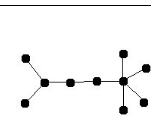
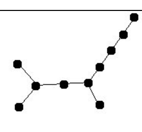
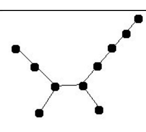
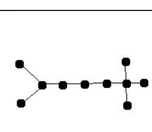
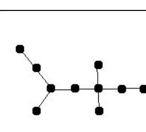
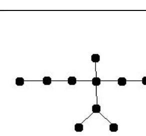
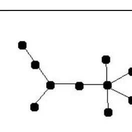
Tree
Dim
Ehrhart Polynomial

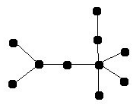
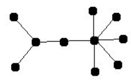
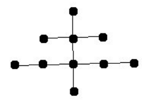
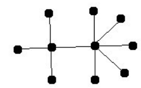
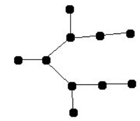
 ST7.2	 ST7.6	 ST7.10
1	2	1
$t+1$	t^2+2t+1	$t+1$
 ST7.3	 ST7.7	 ST7.11
0	9	5
1	$\frac{11}{11340}t^9 + \frac{11}{630}t^8 + \frac{19}{135}t^7 + \frac{2}{3}t^6 + \frac{1109}{540}t^5 + \frac{43}{10}t^4 + \frac{35117}{5670}t^3 + \frac{379}{63}t^2 + \frac{65}{18}t + 1$	$\frac{1}{8}t^5 + \frac{7}{8}t^4 + \frac{21}{8}t^3 + \frac{33}{8}t^2 + \frac{13}{4}t + 1$
 ST7.4	 ST7.8	
4	25	
$\frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$	$\frac{9700106723}{1025873680114483200000}t^{25} + \frac{9700106723}{13678315734859776000}t^{24} + \frac{158824242127}{624444848765337600000}t^{23} + \frac{4394656999}{756902846988288000}t^{22} + \frac{853529939221}{9010748178432000000}t^{21} + \frac{141248912237}{120143309045760000}t^{20} + \frac{2462417656967}{213412456857600000}t^{19} + \frac{18674864899}{203249958912000}t^{18} + \frac{1062348478211833}{1757514350592000000}t^{17} + \frac{77984295979769}{23433524674560000}t^{16} + \frac{1424745952102609}{92060275507200000}t^{15} + \frac{8346012436199}{136385593344000}t^{14} + \frac{570713692223620411}{2761808265216000000}t^{13} + \frac{2000221303490489}{3347646382080000}t^{12} + \frac{1626504818729086 9}{1098446469120000}t^{11} + \frac{736591080322991}{234335246745600}t^{10} + \frac{253578194011961479}{44460928512000000}t^9 + \frac{232132948167689}{26347216896000}t^8 + \frac{243245111626317349}{21118941043200000}t^7 + \frac{14226886368398551}{1126343522304000}t^6 + \frac{155498465793777230567}{13551320502720000000}t^5 + \frac{382955230861099213}{45171068342400000}t^4 + \frac{12246206617138789}{247365374256000}t^3 + \frac{46584105377}{2141691552}t^2 + \frac{3899}{600}t + 1$	
 ST7.5	 ST7.9	
4	5	
$\frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$	$\frac{1}{30}t^5 + \frac{1}{3}t^4 + \frac{4}{3}t^3 + \frac{8}{3}t^2 + \frac{79}{30}t + 1$	

Tree				
Dim				
ST8.1	 ST8.7	 ST8.13	 ST8.19	 ST9.2
1	1	5	2	1
 ST8.2	 ST8.8	 ST8.14	 ST8.20	 ST9.3
1	4	1	4	0
 ST8.3	 ST8.9	 ST8.15	 ST8.21	 ST9.4
0	9	2	10	0
 ST8.4	 ST8.10	 ST8.16	 ST8.22	 ST9.5
1	5	1	17	1
 ST8.5	 ST8.11	 ST8.17	 ST8.23	 ST9.6
1	16	1	5	1
 ST8.6	 ST8.12	 ST8.18	ST9.1	 ST9.7
4	36	5	1	4

Tree					
Dim					
 ST9.8	 ST9.14	 ST9.20	 ST9.26	 ST9.32	 ST9.38
1	5	1	0	2	1
 ST9.9	 ST9.15	 ST9.21	 ST9.27	 ST9.33	 ST9. 39
2	16	2	1	17	5
 ST9.10	 ST9.16	 ST9.22	 ST9.28	 ST9.34	 ST9.40
1	10	1	5	17	4
 ST9.11	 ST9.17	 ST9.23	 ST9.29	 ST9.35	 ST9.41
9	25	1	2	9	5
 ST9.12	 ST9.18	 ST9.24	 ST8.30	 ST9.36	 ST9.42
9	49	1	4	5	13
 ST9.13	 ST9.19	 ST9.25	 ST9.31	 ST9.37	 ST9.43
4	5	1	2	10	1

Tree				
Dim				
 ST9.44	 ST10.3	 ST10.9	 ST10.15	 ST10.21
2	0	4	9	9
 ST9.45	 ST10.4	 ST10.10	 ST10.16	 ST10.22
5	0	1	4	8
 ST9.46	 ST10.5	 ST10.11	 ST10.17	 ST10.23
5	1	1	5	25
 ST9.47	 ST10.6	 ST10.12	 ST10.18	 ST10.24
6	1	1	2	17
ST10.1	 ST10.7	 ST10.13	 ST10.19	 ST10.25
1	0	1	9	36
 ST10.2	 ST10.8	 ST10.14	 ST10.20	 ST10.26
1	4	4	16	64

Tree					
Dim					
 ST10.27	 ST10.33	 ST10.39	 ST10.45	 ST10.51	 ST10.57
5	1	2	4	5	2
 ST10.28	 ST10.34	 ST10.40	 ST10.46	 ST10.52	 ST10. 58
1	0	1	2	4	1
 ST10.29	 ST10.35	 ST10.41	 ST10.47	 ST10.53	 ST10.59
2	1	0	4	2	1
 ST10.30	 ST10.36	 ST10.42	 ST10.48	 ST10.54	 ST10.60
1	1	1	2	4	5
 ST10.31	 ST10.37	 ST10.43	 ST10.49	 ST10.55	 ST10.61
1	1	5	5	1	10
 ST10.32	 ST10.38	 ST10.44	 ST10.50	 ST10.56	 ST10.62
1	0	5	1	1	9

Tree				
Dim				
 ST10.63	 ST10.69	 ST10.75	 ST10.81	 ST10.87
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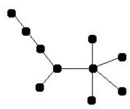
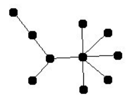
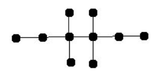
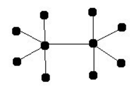
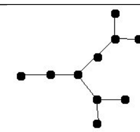
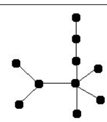
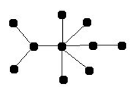
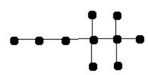
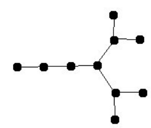
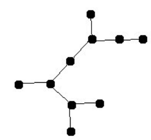
 ST10.64	 ST10.70	 ST10.76	 ST10.82	 ST10.88
9	16	5	37	2
 ST10.65	 ST10.71	 ST10.77	 ST10.83	 ST10.89
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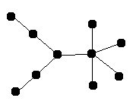
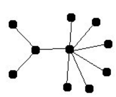
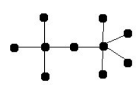
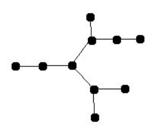
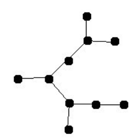
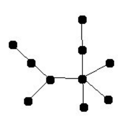
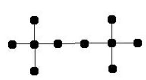
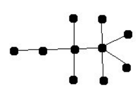
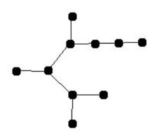
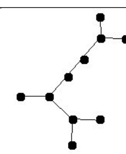
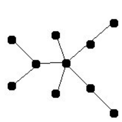
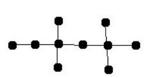
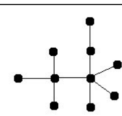
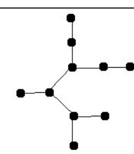
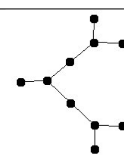
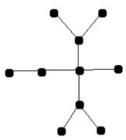
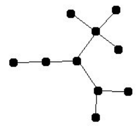
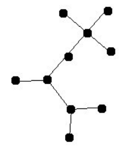
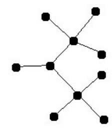
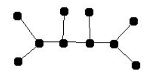
 ST10.66	 ST10.72	 ST10.78	 ST10.84	 ST10.90
10	26	13	1	1
 ST10.67	 ST10.73	 ST10.79	 ST10.85	 ST10.91
4	17	10	1	2
 ST10.68	 ST10.74	 ST10.80	 ST10.86	 ST10.92
3	5	8	2	5

Tree				
Dim				
 ST10.93	 ST10.96	 ST10.99	 ST10.102	 ST10.105
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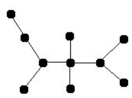
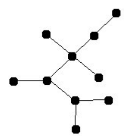
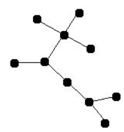
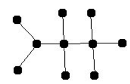
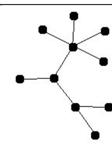
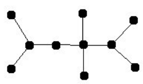
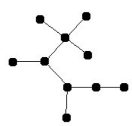
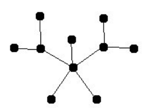
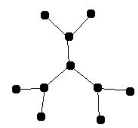
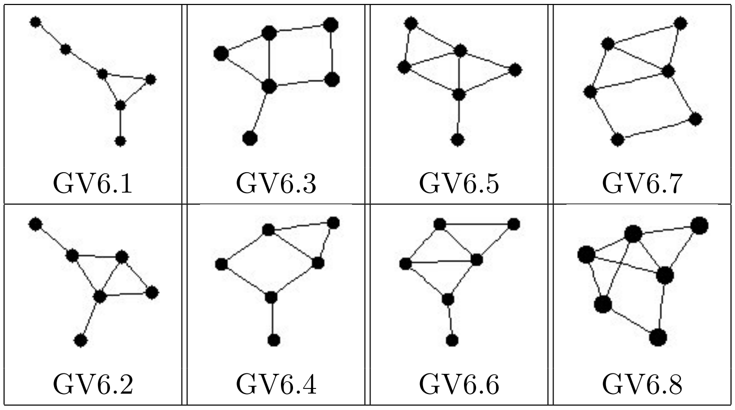
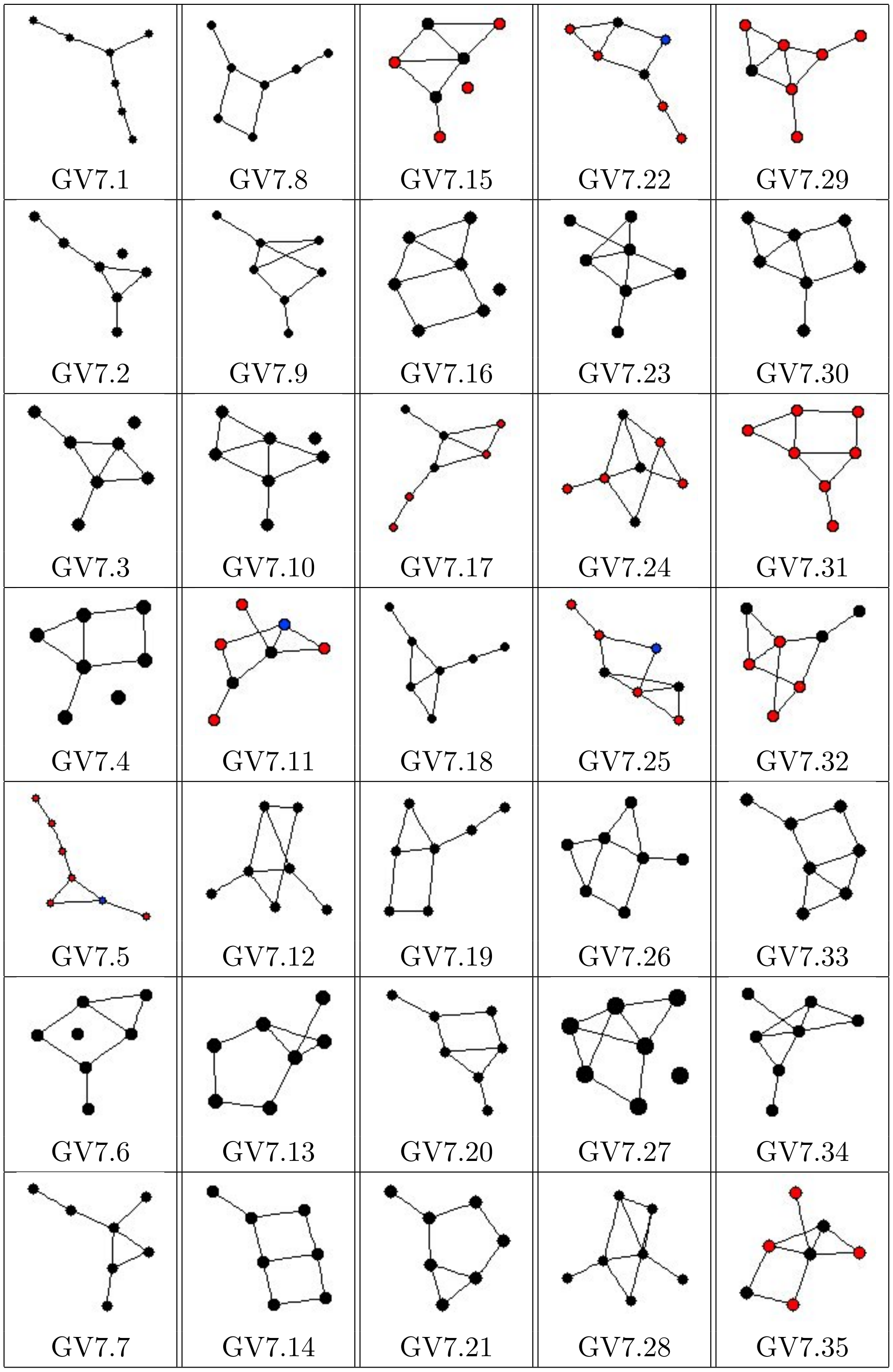
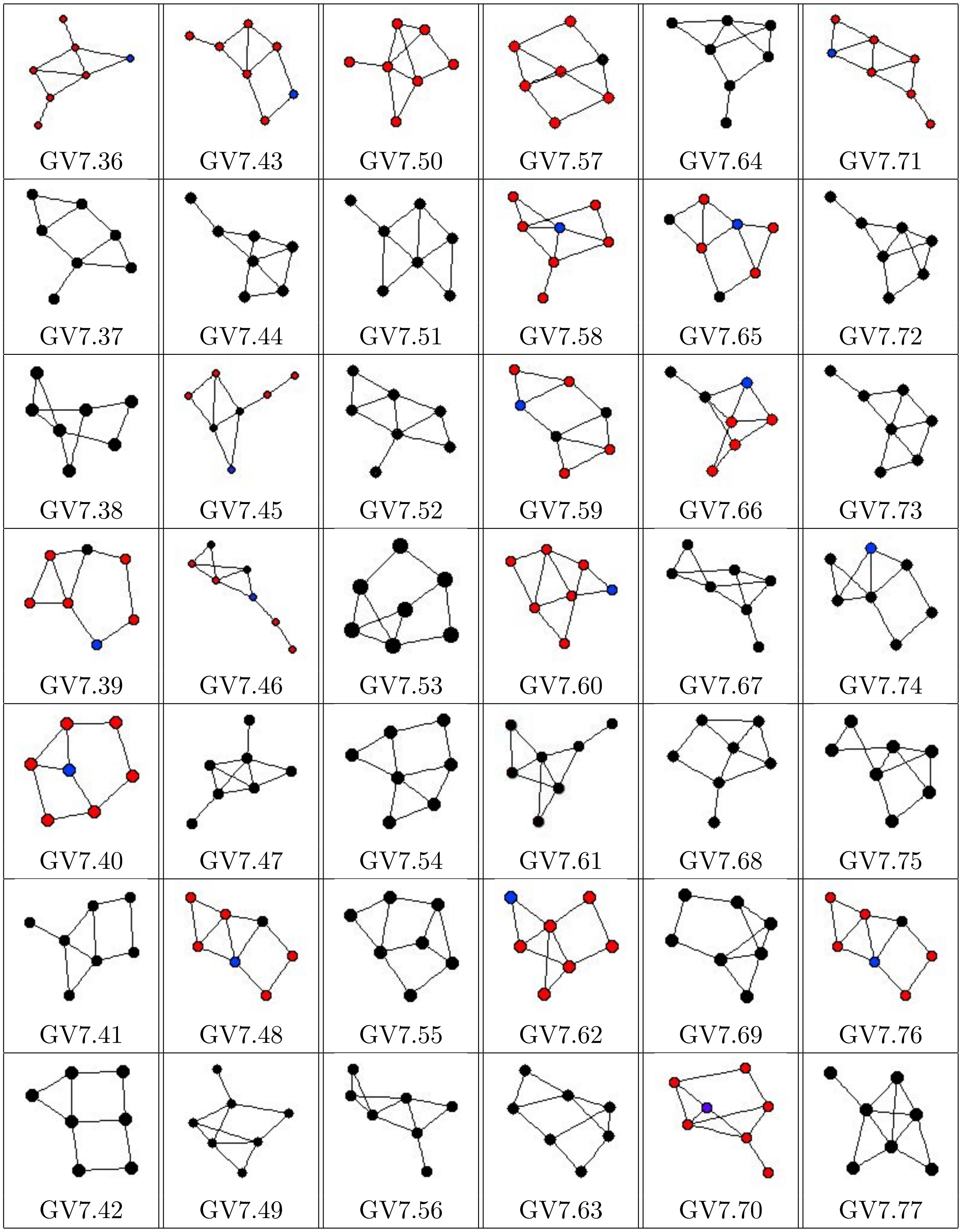
 ST10.94	 ST10.97	 ST10.100	 ST10.103	 ST10.106
2	2	5	6	10
 ST10.95	 ST10.98	 ST10.101	 ST10.104	
3	4	9	13	

5.4. Graph Automorphisms

This section includes the results relating to proofs of infeasiblity described in Chapter 4. This includes results for all rigid graphs with 6 and 7 vertices, as well as several rigid graphs with 10, 15, and 17 vertices. In addition, there are results for a handful of famous rigid graphs with a variety of vertex numbers.


GV6.1
GV6.3
GV6.5
GV6.7
GV6.2
GV6.4
GV6.6
GV6.8


GV7.1 GV7.8 GV7.15 GV7.22 GV7.29
GV7.2 GV7.9 GV7.16 GV7.23 GV7.30
GV7.3 GV7.10 GV7.17 GV7.24 GV7.31
GV7.4 GV7.11 GV7.18 GV7.25 GV7.32
GV7.5 GV7.12 GV7.19 GV7.26 GV7.33
GV7.6 GV7.13 GV7.20 GV7.27 GV7.34
GV7.7 GV7.14 GV7.21 GV7.28 GV7.35


GV7.36
GV7.43
GV7.50
GV7.57
GV7.64
GV7.71
GV7.37
GV7.44
GV7.51
GV7.58
GV7.65
GV7.72
GV7.38
GV7.45
GV7.52
GV7.59
GV7.66
GV7.73
GV7.39
GV7.46
GV7.53
GV7.60
GV7.67
GV7.74
GV7.40
GV7.47
GV7.54
GV7.61
GV7.68
GV7.75
GV7.41
GV7.48
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GV7.69
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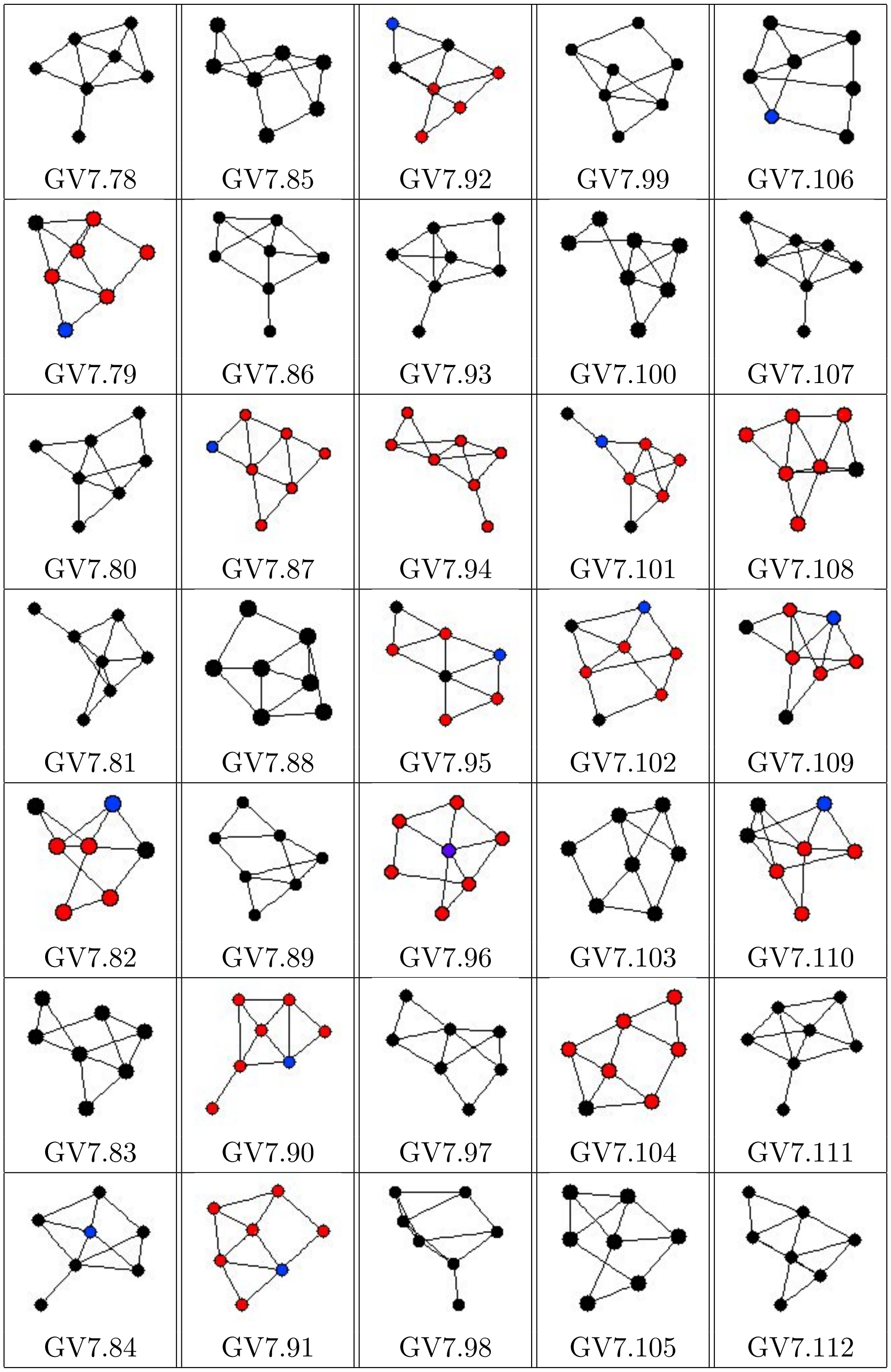
GV7.42
GV7.49
GV7.56
GV7.63
GV7.70
GV7.77


GV7.78
GV7.85
GV7.92
GV7.99
GV7.106
GV7.79
GV7.86
GV7.93
GV7.100
GV7.107
GV7.80
GV7.87
GV7.94
GV7.101
GV7.108
GV7.81
GV7.88
GV7.95
GV7.102
GV7.109
GV7.82
GV7.89
GV7.96
GV7.103
GV7.110
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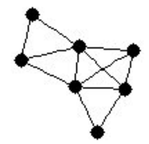
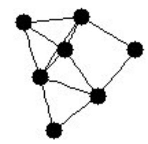
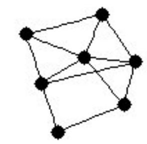
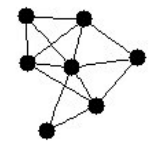
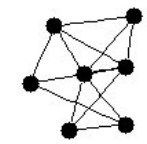
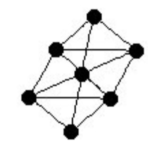
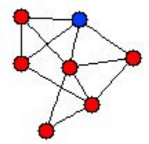
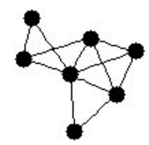
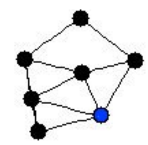
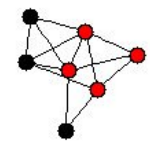
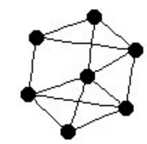
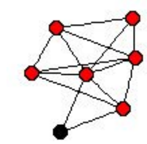
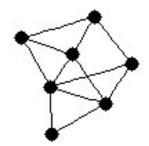
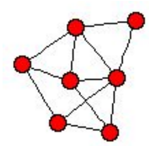
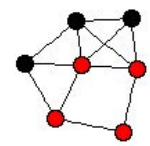
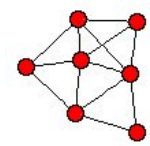
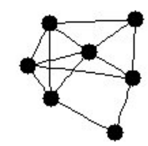
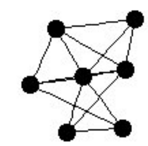
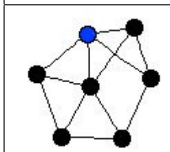
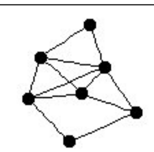
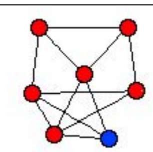
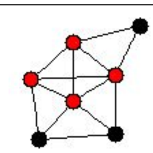
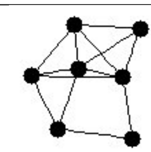
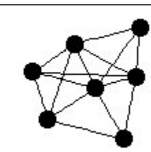
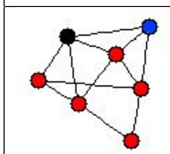
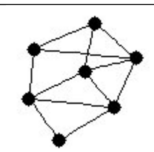
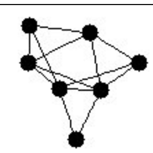
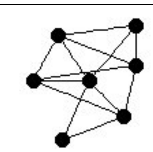
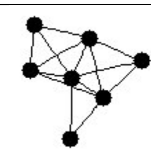
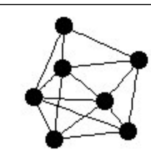
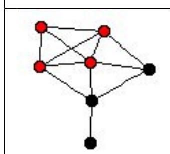
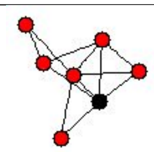
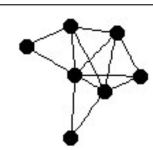
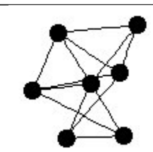
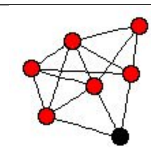
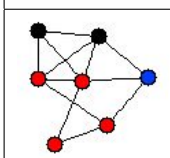
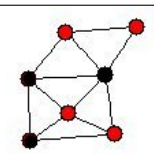
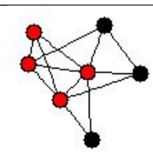
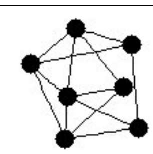
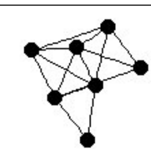
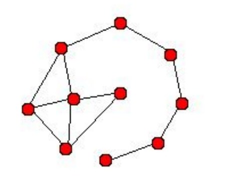
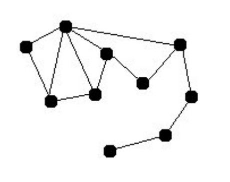
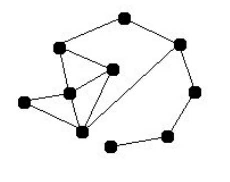
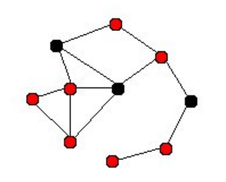
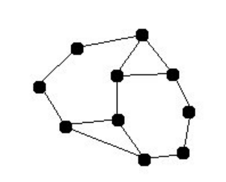
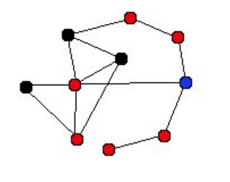
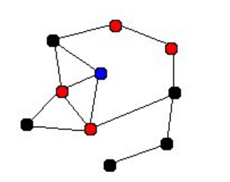
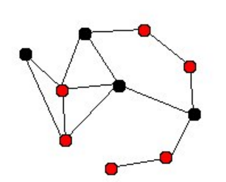
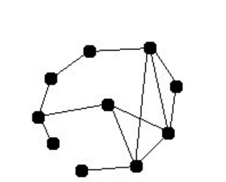
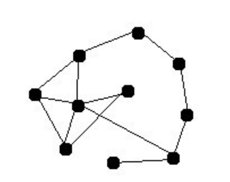
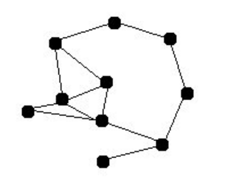
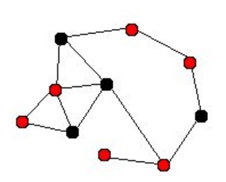
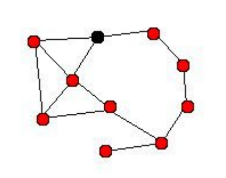
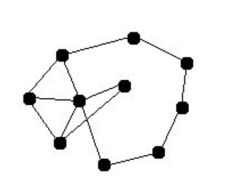
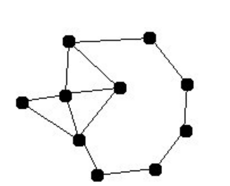
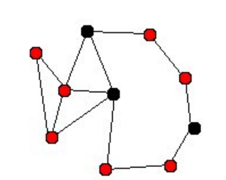
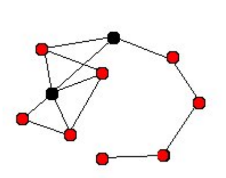
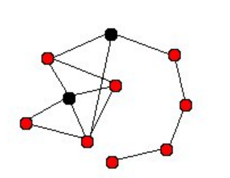
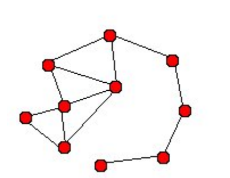
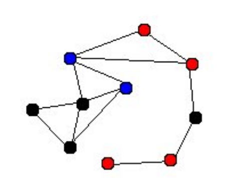
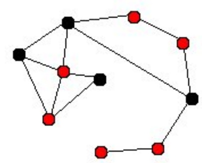
GV7.97
GV7.104
GV7.111
GV7.84
GV7.91
GV7.98
GV7.105
GV7.112

 GV7.113	 GV7.120	 GV7.127	 GV7.134	 GV7.141	 GV7.148
 GV7.114	 GV7.121	 GV7.128	 GV7.135	 GV7.142	 GV7.149
 GV7.115	 GV7.122	 GV7.129	 GV7.136	 GV7.143	 GV7.150
 GV7.116	 GV7.123	 GV7.130	 GV7.137	 GV7.144	 GV7.151
 GV7.117	 GV7.124	 GV7.131	 GV7.138	 GV7.145	 GV7.152
 GV7.118	 GV7.125	 GV7.132	 GV7.139	 GV7.146	
 GV7.119	 GV7.126	 GV7.133	 GV7.140	 GV7.147	

 GV10E12.1	 GV10E13.9	 GV10E13.15	 GV10E13.21
 GV10E13.1	 GV10E13.10	 GV10E13.16	 GV10E13.22
 GV10E13.2	 GV10E13.11	 GV10E13.17	 GV10E13.23
 GV10E13.6	 GV10E13.12	 GV10E13.18	 GV10E13.24
 GV10E13.8	 GV10E13.14	 GV10E13.20	 GV10E13.25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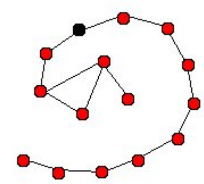
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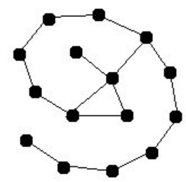
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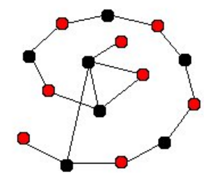
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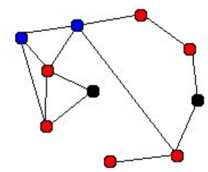
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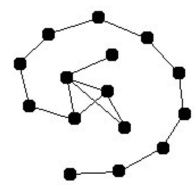
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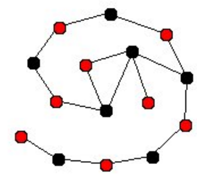
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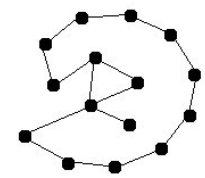
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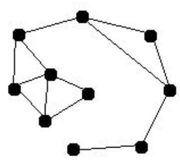
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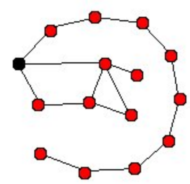
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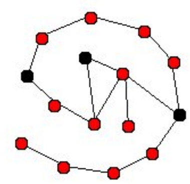
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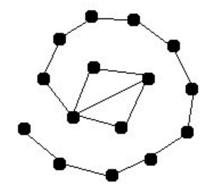
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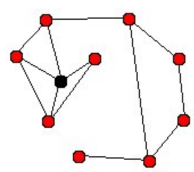
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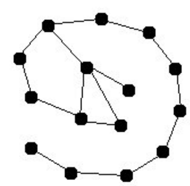
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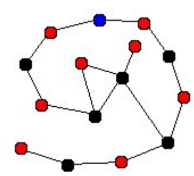
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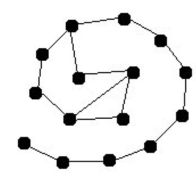
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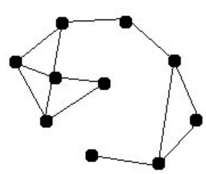
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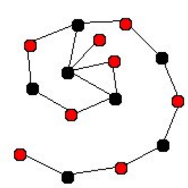
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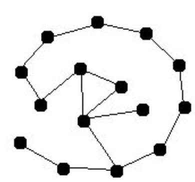
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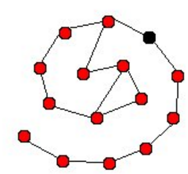
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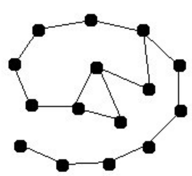
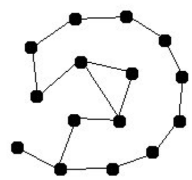
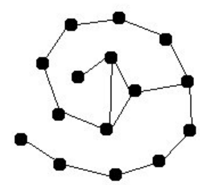
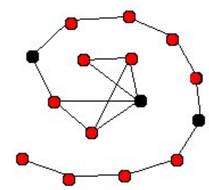
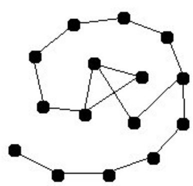
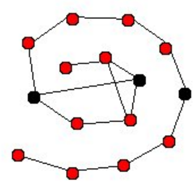
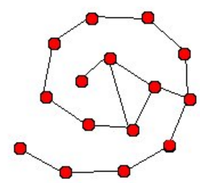
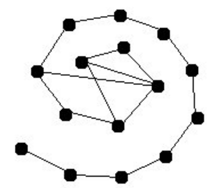
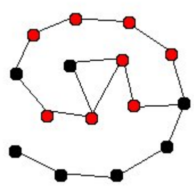
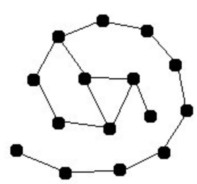
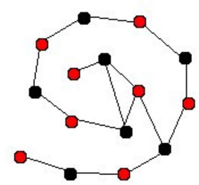
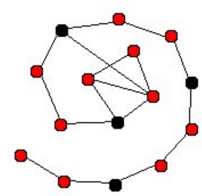
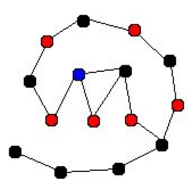
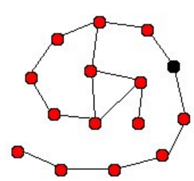
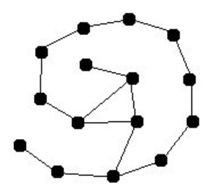
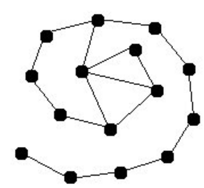
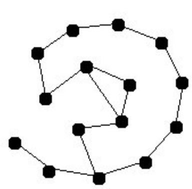
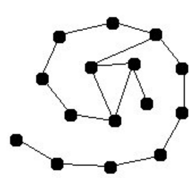
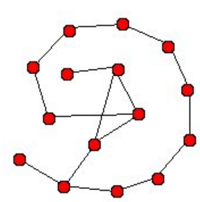
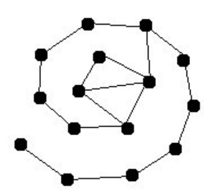
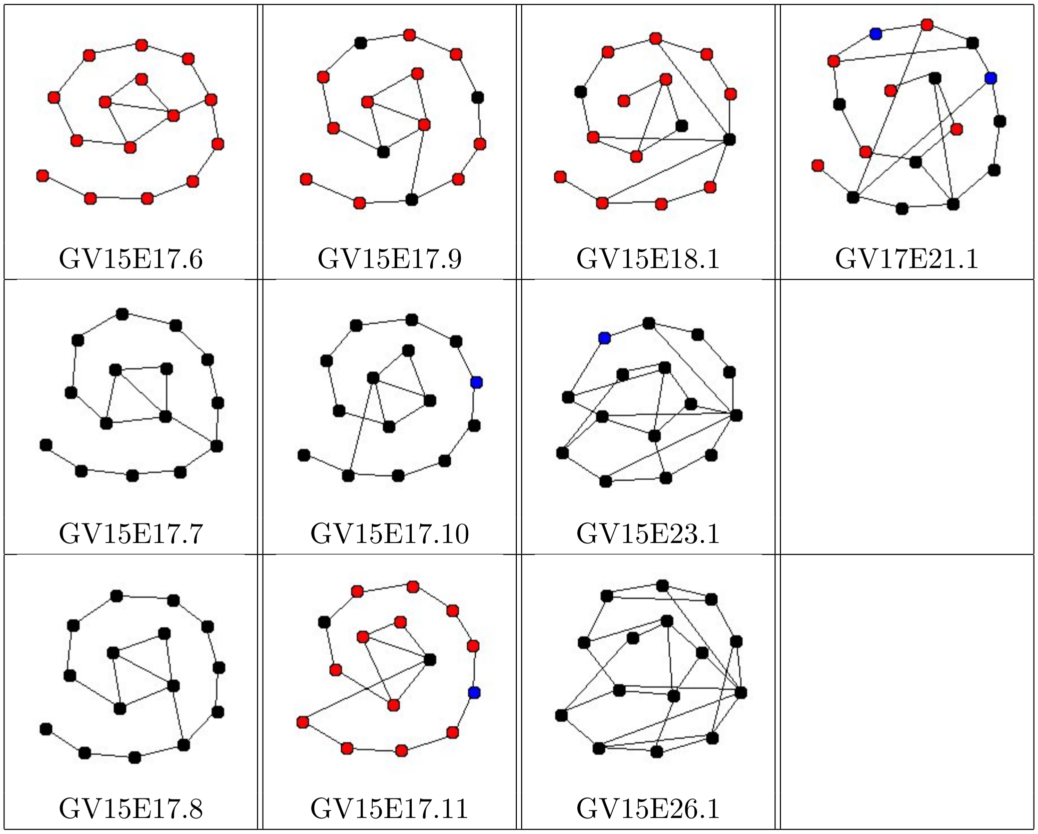
 GV15E16.18	 GV15E16.23	 GV15E16.30	 GV15E17.1
 GV15E16.19	 GV15E16.26	 GV15E16.31	 GV15E17.2
 GV15E16.20	 GV15E16.27	 GV15E16.32	 GV15E17.3
 GV15E16.21	 GV15E16.28	 GV15E16.33	 GV15E17.4
 GV15E16.22	 GV15E16.29	 GV15E16.34	 GV15E17.5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GV15E17.9
GV15E18.1
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GV15E23.1
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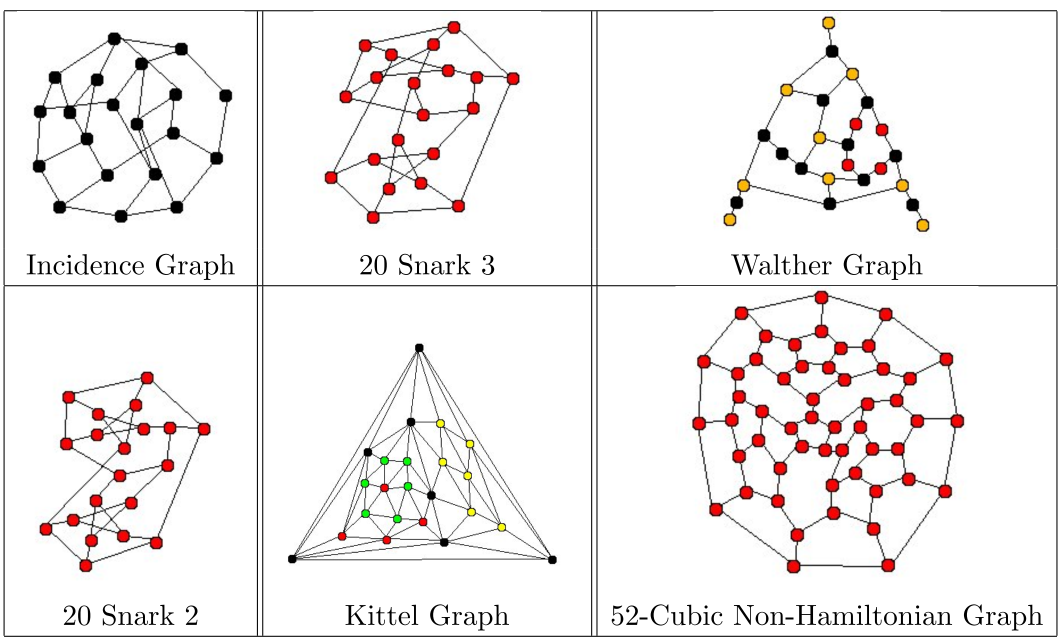
GV15E17.11
GV15E26.1


Incidence Graph
20 Snark 3
Walther Graph
20 Snark 2
Kittel Graph
52-Cubic Non-Hamiltonian Graph

5.5. Conclusions

While we were able to develop a formula for the Ehrhart polynomials of C_n, D_n, and the automorphism groups of binary trees, several questions remain open. We were unable to find a formula for the Ehrhart polynomials of Frobenius groups, and our volume formula is for the normalized volume of Frobenius polytopes rather than the standard volume. We collected a great deal of data on spanning trees, in particular the dimensions of the polytopes associated to automorphism groups of all trees with up to 10 vertices, but we still lack a formula for the dimension of these polytopes, and we do not have results for the Ehrhart polynomials of automorphisms of non-binary trees. We also have a great deal of information regarding the difficulty of proving the rigidity of graphs, but we have not yet been able to implement the $P^k A^k = A^k P^k$ formula into the sets of equations restricting the automorphisms of our graphs.

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