FLUCTUATIONS OF MATRIX ENTRIES OF POLYNOMIAL FUNCTIONS OF WIGNER MATRICES

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Abstract. We verify the results of [1] in the special case that the function f is a polynomial. That is, we verify that the fluctuations of the matrix entries of a random matrix Wigner random matrices approach a Gaussian distribution in the limit of large N. We use the technique of Martingale Differences as in the Appendix by J. Baik and J. Silverstein in [2].

1. Introduction

The study of Random Matrices became a subject of great interest to physicists due to the work of Wigner, Dyson, Gaudin, and Mehta [3]. The reasons for this intense interest was due to the desire to model the transitions that represent neutron emission for atoms with large atomic number such as U^{239}. For low atomic numbers, successful models were constructed by analyzing shell-structure of nuclei and individual energy levels. However, for very high atomic numbers and large excited states, the reliability of the shell-structure model decreases [4]. A suggested idea that took hold was to treat the quantum mechanical system itself as a random system – that is, to assume that the effect of the shell structure disappeared.

To clarify this idea it is important to recall that in quantum mechanics all observables are represented by a Hermitian operator. The eigenvalues (which must be real) represents the possible results of a measurement of that observable. The operator that describes how a state evolves in time is known as the Hamiltonian operator $\mathcal{H}$. Its eigenvalues are the energy levels of the system.

What was suggested above as a model was to take the Hamiltonian $\mathcal{H}$ to be a random Hermitian operator – in this case a random Hermitian matrix of large dimension N. This corresponds to a random "system." Note, that this is a radically different approach from traditional statistical physics in which the states are taken as given, and we simply assume that the probability of each is equally likely at equilibrium. In this case, the energy levels themselves are random! A goal then, is to obtain the probability ditribution of the eigenvalues of $\mathcal{H}$ and see its limiting distribution as $N \to \infty$.

The applications of random matrices to fields vastly unrelated to nuclear physics soon revealed themselves in the 1970s due to a meeting between F. Dyson and H. Montgomery [5]. H. Montgomery was studying the distribution of zeros of the Riemann zeta function and discovered that with the rescaling of the zeros γ_j along the critical line $Re(z) = 1/2$, to

$$\tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi},$$

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(note the rescaling means that in the limit that $N \to \infty$ the expression $\#\{j \geq 1 : \tilde{\gamma}_j < N\}/N \to 1$ from the prime number theorem) the resulting two-point correlation function:

$$R(a,b) = \lim_{N\to\infty} \frac{1}{N}\#\{\text{pairs}(j_1, j_2) : 1 \leq j_1, j_2 \leq N, \tilde{\gamma_{j_1}} - \tilde{\gamma_{j_2}} \in (a,b)\}$$

can be expressed as

$$R(a,b) = \int_a^b \left(1 - \left(\frac{\sin 2\pi u}{2\pi u}\right)^2\right) du.$$

Dyson pointed out that this result was identical to the result obtained if one assumed that the zeros of the Riemann zeta function behaved as the eigenvalues of a random matrix of the Gaussian Unitary Ensemble. Similar situations occured in fields as diverse as signal processing and finance.

It is the goal of this paper to verify a special case of a theorem given in [1], which concerns itself with the fluctuations of the entries of functions of what are known as Wigner matrices. Before preceding with this result, it is necessary to first describe what Wigner Matrices are, along with important facts concerning them that are necessary in order to formulate our problem.

2. Wigner Matrices

A Wigner Matrix $X_N = \frac{1}{\sqrt{N}} W_N$ is a random real symmetric (Hermitan) matrix. We consider the real symmetric case. In this situation we have that all of the off-diagonal entries that is $(W_N)_{jk}$ where $j \neq k$ are i.i.d. random variables with probability distribution μ, such that

$$\mathbb{E}(W_N)_{jk} = 0, \quad \mathbb{V}(W_N)_{jk} = \sigma^2, \mathbb{E}(W_N)_{jk}^4 = m_4 < \infty \quad 1 \leq j < k \leq N$$

where $\mathbb{E}$ denotes the expectation and $\mathbb{V}$ denotes the variance. We will also assume that higher moments are finite as necessary – this assumption will be far stronger than the assumption given in [1], which simply assumes only that the fourth moment is finite. The diagonal entries of the matrix $(W_N)_{ii}$ are also i.i.d. random variables independent from the off-diagonal entries such that

$$\mathbb{E}(W_N)_{ii} = 0, \quad \mathbb{V}(W_N)_{ii} = \sigma_1^2, \quad 1 \leq i \leq N.$$

The probability distribution of $\frac{1}{\sqrt{2}}(W_N)_{11}$ by μ_1.

One of the most important results of random matrices is the Wigner Semicircle Law which states [6]:

Theorem 2.1. *Let λ_i^N denote the ordered (smallest to largest) eigenvalues of X_N, and define the empirical distribution of the eigenvalues as*

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N},$$

define also, the semicircle distribution on $\mathbb{R}$ with density

$$\psi(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,\mathbf{1}_{[-2\sigma,2\sigma]}(x),$$

then we have that the empirical distribution converges weakly in probability to the semicircle distribution.

Note that the the moments of the semicircle distribution are:

$$m_{2k} = \int_{-2\sigma}^{2\sigma} \frac{x^{2k}\sqrt{4\sigma^2 - x^2}}{2\pi\sigma^2} dx = \sigma^{2k} C_k, \quad m_{2k+1} = 0,$$

where C_k is the k-th Catalan number:

$$C_k = \binom{2k}{k} \frac{1}{k+1}.$$

The result of [1] that we are concerned with in this paper, is concerning the limiting value of the entries of a function $f(X_N)$ of the wigner matrix (see THeorem 2.3 in [1]). We wish to show that

$$\sqrt{N}\left(f(X_N)_{ii} - \int_{-2\sigma}^{2\sigma} f(x) \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx \right),$$

converges to the sum of two independent random variables $\frac{\alpha(f)}{\sigma} W_{ii}$ and $N(0, \nu_1^2(f))$, where

$$\begin{aligned}
\nu_1^2(f) &= 2\left(\omega^2(f) - \alpha^2(f) + \frac{\kappa_4(\mu)}{2\sigma^4} \beta^2(f) \right), \\
\omega^2(f) &= \frac{1}{2} \int_{-2\sigma}^{2\sigma} \int_{-2\sigma}^{2\sigma} (f(x) - f(y))^2 \frac{1}{4\pi^2\sigma^4} \sqrt{4\sigma^2 - x^2} \sqrt{4\sigma^2 - y^2} dx dy, \\
\alpha(f) &= \frac{1}{\sigma} \int_{-2\sigma}^{2\sigma} x f(x) \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx, \\
\beta(f) &= \frac{1}{\sigma^2} \int_{-2\sigma}^{2\sigma} f(x)(x^2 - \sigma^2) \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx, \\
\kappa_4(\mu) &= m_4 - 3\sigma^4.
\end{aligned}$$

Specifically, we will be analyzing the case when $f(X_N)$ is a polynomial and when $i = j = 1$.

The technique that we will be using is the central limit theorem of Martingale differences. An example of this technique in use is given in the Appendix of [2] due to J. Baik and J. Silverstein.

The theorem is as follows:

Theorem 2.2. *For each N, let $Z_{N1}, \dots Z_{N_{r_N}}$ be a real martingale difference sequence with respect to the increasing σ-field $\{\mathcal{F}_{N,j}\}$ having second moments. If as $N \to \infty$,*

$$\sum_{j=1}^{r_N} \mathbb{E}(Z_{N_j}^2 | \mathcal{F}_{N,j-1}) \xrightarrow{\mathbb{P}} \nu^2, \tag{2.1}$$

where ν^2 is a positive constant, and for each $\epsilon > 0$,

$$\sum_{j=1}^{r_N} \mathbb{E}(Z_{N_j}^2 \mathbf{1}_{|Z_{N_j}| \geq 0}) \to 0, \tag{2.2}$$

then

$$\sum_{j=1}^{r_N} Z_{N_j} \xrightarrow{\mathcal{L}} N(0, \nu^2). \tag{2.3}$$

3. Cubic Polynomial Case

Let $f(x) = x^3$, in this case then we have that

$$f(X_N)_{11} = \sum_{k=1}^{N}\sum_{j=1}^{N} X_{1j}X_{1k}X_{jk},$$

where we have used the fact that X_{ij} is symmetric. We can rewrite this expression as

$$\begin{aligned} f(X_N)_{11} &= \sum_{k=1}^{N} X_{1k}^2 X_{kk} + \sum_{k=2}^{N}\sum_{j=1}^{k-1} X_{1j}X_{jk}X_{1k} + \sum_{j=2}^{N}\sum_{k=1}^{j-1} X_{1j}X_{jk}X_{1k} \\ &= \sum_{k=1}^{N} X_{1k}^2 X_{kk} + 2\sum_{k=2}^{N}\sum_{j=1}^{k} X_{1j}X_{jk}X_{1k}, \end{aligned}$$

where we have again used the fact that X_{ij} is symmetric.

We will define a sequence of increasing sigma algebras as follows:

$$\mathcal{F}_{N,k-1} = \sigma(X_{st}, \quad 1 \le s,t \le k-1),$$

that is $\mathcal{F}_{N,k-1}$ is the sigma algebra generated by the $k-1 \times k-1$ submatrix of X_N.

We construct a martingale difference by first defining

$$\begin{aligned} \tilde{S}_1 &= \sqrt{N} X_{11}^3 \\ \tilde{S}_k &= \sqrt{N} X_{1k}^2 X_{kk} + 2\sqrt{N}\sum_{j=1}^{k-1} X_{1j}X_{jk}X_{1k} \quad k \ge 2 \end{aligned}$$

then defining

$$S_k = \tilde{S}_k - \mathbb{E}(\tilde{S}_k | \mathcal{F}_{N,k-1}),$$

so that clearly $\mathbb{E}(S_k) = 0$, since

$$\mathbb{E}(S_K) = \mathbb{E}(\tilde{S}_k) - \mathbb{E}(\mathbb{E}(\tilde{S}_k | \mathcal{F}_{N,k-1})) = 0,$$

where we have used the tower property $\mathbb{E}(\mathbb{E}(U|V)) = \mathbb{E}(U)$.

Note that

$$\sum_{k=1}^{N} \tilde{S}_k = \sqrt{N} f(X_N)_{11},$$

so obtaining (2.3) for this expression is exactly what we want.

Now, we proceed to evaluate:

$$\mathbb{E}(\tilde{S}_1 | \mathcal{F}_{N,0}) = \mathbb{E}(\sqrt{N} X_{11}^3) = \sqrt{N}\mathbb{E}(X_{11}^3),$$

and for $k \geq 2$:

$$\begin{aligned}\mathbb{E}(\tilde{S}_k|\mathcal{F}_{N,k-1}) &= \sqrt{N}\mathbb{E}(X_{1k}^2 X_{kk}|X_{ij} \quad 1 \leq i,j \leq k-1) \\ &\quad + 2\sqrt{N}\sum_{j=1}^{k-1}\mathbb{E}(X_{ij}X_{jk}X_{1k}|X_{ij} \quad 1 \leq i,j \leq k-1), \\ &= \sqrt{N}\mathbb{E}(X_{1k}^2)\mathbb{E}(X_{kk}) + 2\sqrt{N}\sum_{j=1}^{k-1}X_{1j}\mathbb{E}(X_{jk}X_{1k}), \\ &= 2\sqrt{N}X_{11}\mathbb{E}(X_{1k}^2),\end{aligned}$$

where we have used the linearity of expectation, along with the fact that for independent variables Y and Z, $\mathbb{E}(YZ) = \mathbb{E}(Y)\mathbb{E}(Z)$.

We now may construct our sequence of martingale differences:

$$\begin{aligned}S_1 &= \sqrt{N}X_{11}^3 - \sqrt{N}\mathbb{E}(X_{11}^3), \\ S_k &= \sqrt{N}X_{1k}^2X_{kk} + 2\sqrt{N}\sum_{j=1}^{k-1}X_{1j}X_{jk}X_{1k} - 2\sqrt{N}X_{11}\mathbb{E}(X_{1k}^2) \quad k \geq 2.\end{aligned}$$

Next, we need to verify (2.1). Since we need to evaluate the convergence of the sum

$$\sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1}),$$

it is reasonable to examine

$$\mathbb{E}\left(\sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1})\right) = \sum_{k=1}^{N}\mathbb{E}(S_k^2),$$

and see what this converges to. We claim that this converges to ν^2, to verify this, we define

$$Z = \sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N_{j-1}}) - \nu^2,$$

and check that

$$\lim_{N\to\infty}\mathbb{E}(Z^2) = 0.$$

This will verify (2.1) since by Chebyshev's inequality:

$$\mathbb{P}(|Z| \geq \epsilon) \leq \epsilon^{-2}\mathbb{E}(Z^2),$$

so that if $\mathbb{E}(Z^2) \to 0$ as $N \to \infty$ then we have that

$$\mathbb{P}(|Z| \geq \epsilon) \to 0,$$

which is the same as (2.1).

So to find ν^2 we find

$$\lim_{N\to\infty}\sum_{k=1}^{N}\mathbb{E}(S_k^2).$$

Now we evaluate:

$$\mathbb{E}(S_1^2) = N(\mathbb{E}(X_{11}^6) - \mathbb{E}(X_{11}^3)^2),$$

and for $k \geq 2$:

$$\begin{aligned}
S_k^2 = N(&X_{1k}^4 X_{kk}^2 + 4\sum_{j=1}^{k-1}\sum_{l=1}^{k-1} X_{1j}X_{1l}X_{jk}X_{lk}X_{1k}^2 + 4X_{11}^2\mathbb{E}(X_{1k}^2)^2,\\
&+ 4X_{1k}^2 X_{kk}\sum_{j=1}^{k-1} X_{1j}X_{jk}X_{1k} - 4X_{1k}^2X_{kk}X_{11}\mathbb{E}(X_{1k}^2),\\
&- 8X_{11}\mathbb{E}(X_{1k}^2)\sum_{j=1}^{k-1} X_{1j}X_{jk}X_{1k}),
\end{aligned}$$

so that

$$\begin{aligned}
\mathbb{E}(S_k^2) = N[&\mathbb{E}(X_{1k}^4)\mathbb{E}(X_{kk}^2) + 4\sum_{j=1}^{k-1}\sum_{l=1}^{k-1}\mathbb{E}(X_{1j}X_{1l}X_{jk}X_{lk}X_{1k}^2) + 4\mathbb{E}(X_{11}^2)\mathbb{E}(X_{1k}^2)^2,\\
&+ 4\mathbb{E}(X_{kk})\sum_{j=1}^{k-1}\mathbb{E}(X_{1j}X_{jk}X_{1k}^3) - 4\mathbb{E}(X_{1k}^2X_{kk}X_{11})\mathbb{E}(X_{1k}^2),\\
&- 8\mathbb{E}(X_{1k}^2)\sum_{j=1}^{k-1}\mathbb{E}(X_{11}X_{1j}X_{jk}X_{1k})],\\
= N[&\mathbb{E}(X_{1k}^4)\mathbb{E}(X_{kk}^2) + 4\sum_{j=1}^{k-1}\mathbb{E}(X_{1j}^2)\mathbb{E}(X_{jk}^2X_{1k}^2) + 4\mathbb{E}(X_{11}^2)\mathbb{E}(X_{1k}^2)^2,\\
&- 8\mathbb{E}(X_{1k}^2)^2\mathbb{E}(X_{11}^2)],
\end{aligned}$$

where we have used independence and the fact that $\mathbb{E}(X_{ij}) = 0$ to obtain the second line.

We main now evaluate the sum

$$\begin{aligned}
\sum_{k=1}^{N}\mathbb{E}(S_k^2) = N[&\mathbb{E}(X_{11}^6) - \mathbb{E}(X_{11}^3)^2 + \sum_{k=2}^{N}\mathbb{E}(X_{1k}^4)\mathbb{E}(X_{kk}^2),\\
&+ 4\sum_{k=2}^{N}\sum_{j=1}^{k-1}\mathbb{E}(X_{1j}^2)\mathbb{E}(X_{jk}^2X_{1k}^2),\\
&- 4\sum_{k=2}^{N}\mathbb{E}(X_{11}^2)\mathbb{E}(X_{1k}^2)^2],\\
= N[&\mathbb{E}(X_{11}^6) - \mathbb{E}(X_{11}^3)^2 + (N-1)\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2),\\
&+ 4\sum_{k=3}^{N}\sum_{j=2}^{k-1}\mathbb{E}(X_{1j}^2)\mathbb{E}(X_{jk}^2X_{1k}^2),\\
&+ 4(N-1)\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^4) - 4(N-1)\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^2)^2],\\
= N[&\mathbb{E}(X_{11}^6) - \mathbb{E}(X_{11}^3)^2 + 5(N-1)\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2)\\
&+ 4\mathbb{E}(X_{12}^2)^3\left(\frac{N^2-3N+2}{2}\right) - 4(N-1)\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^2)^2],
\end{aligned}$$

where we have used the fact that for $k \geq 2$, $\mathbb{E}(X_{1k}^2) = \mathbb{E}(X_{12}^2)$, and also we have evaluated the sum

$$\begin{aligned} 4\sum_{k=3}^{N}\sum_{j=2}^{k-1}\mathbb{E}(X_{1j}^2)\mathbb{E}(X_{jk}^2X_{1k}^2) &= 4\mathbb{E}(X_{12}^2)^3\sum_{k=3}^{N}\sum_{j=2}^{k-1}1, \\ &= 4\mathbb{E}(X_{12}^2)^3\sum_{k=3}^{N}(k-2), \\ &= 4\mathbb{E}(X_{12}^2)^3\left(\frac{N^3-3N+2}{2}\right). \end{aligned}$$

Now since $X_{ij} = W_{ij}/\sqrt{N}$, we know that $X_{ij}^6 = \mathcal{O}(1/N^3)$ so that

$$\mathcal{O}(\mathbb{E}(X_{11}^3))^2 = \mathcal{O}(\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2)) = \mathcal{O}(\mathbb{E}(X_{12}^2)^2\mathbb{E}(X_{11}^2)) = \mathcal{O}(\mathbb{E}(X_{12}^2)^3) = 1/N^3,$$

thus we have that the only non-zero term in the sum above that remains as $N \to \infty$ is

$$2N^3\mathbb{E}(X_{12}^2)^3 = 2\mathbb{E}(W_{12}^2)^3 = \nu^2.$$

Now we define as before

$$Z = \sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1}) - \nu^2$$

so that

$$\begin{aligned} \mathbb{E}Z^2 &= \mathbb{E}\left(\sum_{j=1}^{N}\sum_{l=1}^{N}\mathbb{E}(S_j^2|\mathcal{F}_{N,j-1})\mathbb{E}(S_l^2|\mathcal{F}_{N,l-1}) - 2\nu^2\sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1}) + \nu^4\right) \\ &= \sum_{j=1}^{N}\sum_{l=1}^{N}\mathbb{E}\left(\mathbb{E}(S_j^2|\mathcal{F}_{N,j-1})\mathbb{E}(S_l^2|\mathcal{F}_{N,l-1})\right) - 2\nu^2\sum_{k=1}^{N}\mathbb{E}(S_k^2) + \nu^4, \end{aligned}$$

where we have used the tower property in the second line. Now in the limit that $N \to \infty$ we know that

$$\sum_{k=1}^{N}\mathbb{E}(S_k^2) \to \nu^2,$$

so that

$$\lim_{N\to\infty}\mathbb{E}Z^2 = \lim_{N\to\infty}\sum_{j=1}^{N}\sum_{l=1}^{N}\mathbb{E}\left(\mathbb{E}(S_j^2|\mathcal{F}_{N,j-1})\mathbb{E}(S_l^2|\mathcal{F}_{N,l-1})\right) - \nu^4.$$

Thus, all we need to verify is that

$$\lim_{N\to\infty}\sum_{j=1}^{N}\sum_{l=1}^{N}\mathbb{E}\left(\mathbb{E}(S_j^2|\mathcal{F}_{N,j-1})\mathbb{E}(S_l^2|\mathcal{F}_{N,l-1})\right) = \nu^4.$$

We calculate

$$\begin{aligned} \mathbb{E}(S_1^2|\mathcal{F}_{N,0}) &= N\left(X_{11}^6 - 2X_{11}^3\mathbb{E}(X_{11}^3) + \mathbb{E}(X_{11}^3)^2\right) \\ \mathbb{E}(S_k^2|\mathcal{F}_{N,k-1}) &= N\left(\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2) + 4\sum_{j=1}^{k-1}X_{1j}^2\mathbb{E}(X_{jk}^2X_{1k}^2) - 4X_{11}^2\mathbb{E}(X_{12}^2)^2\right), \end{aligned}$$

so that

$$\sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1}) = N(X_{11}^6 - 2X_{11}^3\mathbb{E}(X_{11}^3) + \mathbb{E}(X_{11}^3)^2 + (N-1)\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2)$$
$$+ 4\sum_{k=2}^{N}\sum_{j=1}^{k-1} X_{1j}^2\mathbb{E}(X_{jk}^2X_{1k}^2) - 4(N-1)X_{11}^2\mathbb{E}(X_{12}^2)^2),$$

we need to evaluate the limit of the expectation value of this quantity squared. The only term that is relavant in the square (the rest are of order N^{-k} where k is a positive integer) is

$$\left(4\sum_{k=2}^{N}\sum_{j=1}^{k-1} X_{1j}^2\mathbb{E}(X_{jk}^2X_{1k}^2)\right)^2,$$

but from above we know that the expectation value of this goes to ν^4, thus we have satisfied (2.1).

Next, we will verify (2.2). In order to do so, I will first cite a result, with proof from the appendix of [2]: For random variable Z_1 and Z_2 and positive ϵ, we have that

$$\mathbb{E}(|Z_1+Z_2|^2\mathbf{1}_{|Z_1+Z_2|\geq\epsilon} \leq 4(\mathbb{E}(|Z_1|^2\mathbf{1}_{|Z_1|\geq\epsilon/2} + \mathbb{E}(|Z_2|^2\mathbf{1}_{|Z_2|\geq\epsilon/2}))), \tag{3.1}$$

this follow by analyzing:

$$\begin{aligned}\mathbb{E}(|Z_1|^2\mathbf{1}_{|Z_1+Z_2|\geq\epsilon}) &\leq \mathbb{E}(|Z_1|^2\mathbf{1}_{|Z_1|\geq\epsilon/2}) + \mathbb{E}(|Z_1|^2\mathbf{1}_{(|Z_1|<\epsilon/2),(|Z_2|\geq\epsilon/2)})\\ &\leq \mathbb{E}(|Z_1|^2\mathbf{1}_{|Z_1|\geq\epsilon/2}) + (\epsilon^2/4)\mathbb{P}(|Z_2|\geq\epsilon/2)\\ &\leq \mathbb{E}(|Z_1|^2\mathbf{1}_{|Z_1|\geq\epsilon/2}) + \mathbb{E}(|Z_2|^2\mathbf{1}_{|Z_2|\geq\epsilon/2}).\end{aligned}$$

Exchanging with Z_2 lets us obtain (3.1).

Recall that we need to show that for all $\epsilon > 0$

$$\lim_{N\to\infty}\sum_{k=1}^{N}\mathbb{E}(S_k^2\mathbf{1}_{|S_k|\geq\epsilon}) \to 0.$$

To do this we split for $k \geq 2$ the term S_k^2 as:

$$\begin{aligned}(S_k)^2 &= (\sqrt{N}X_{1k}^2X_{kk} + 2\sqrt{N}\sum_{j=1}^{k-1}X_{1j}X_{jk}X_{1k} - 2\sqrt{N}X_{11}\mathbb{E}(X_{1k}^2))^2,\\ &= (Z_k^{(1)} + Z_k^{(2)})^2,\end{aligned}$$

where

$$Z_k^{(1)} = \sqrt{N}X_{1k}^2X_{kk} - 2\sqrt{N}X_{11}\mathbb{E}(X_{1k}^2), \quad Z_k^{(2)} = 2\sqrt{N}\sum_{j=1}^{k-1}X_{1j}X_{jk}X_{1k},$$

now we notice that the expression:

$$\mathbb{E}(|S_1|^2\mathbf{1}_{|S_1|\geq\epsilon}) + \sum_{k=2}^{N}\mathbb{E}(|Z_k^{(1)}|^2\mathbf{1}_{|Z_k^{(1)}|\geq\epsilon/2}) \to 0$$

as $N \to \infty$, this is because each term of $Z_k^{(1)}$ is of order $\frac{1}{N}$, so that the order of $|Z_k^{(1)}|^2$ is $\frac{1}{N^2}$, which means the sum from $k=2$ to N can be at most order $\frac{1}{N}$ which goes to 0 as $N \to \infty$.

All that remains for consideration is the term (we replace $\epsilon/2$ with ϵ for convenience):

$$\sum_{k=2}^{N} \mathbb{E}(|Z_k^{(2)}|^2 \mathbf{1}_{|Z_k^{(2)}|\geq\epsilon})$$

(recall that the in the limit as $N \to \infty$ we had that $\lim_{N\to\infty} \mathbb{E}((Z_k^{(2)})^2) \to \nu^2$). Now we use the following bound, for a random variable Y:

$$\mathbb{E}(|Y|^2 \mathbf{1}_{|Y|\geq\epsilon}) \leq \frac{1}{\epsilon^2}\mathbb{E}(|Y|^4),$$

this follows simply by using the inequality:

$$|Y| \geq \epsilon \implies 1 \leq \frac{|Y|}{\epsilon},$$

in general it follows that

$$\mathbb{E}(|Y|^2 \mathbf{1}_{|Y|\geq\epsilon}) \leq \frac{1}{\epsilon^{2k}}\mathbb{E}(|Y|^{2k}).$$

In our case we may bound:

$$\sum_{k=2}^{N} \mathbb{E}(|Z_k^{(2)}|^2 \mathbf{1}_{|Z_k^{(2)}|\geq\epsilon}) \leq \frac{1}{\epsilon^2}\sum_{k=2}^{N} \mathbb{E}(|Z_k^{(2)}|^4).$$

We proceed to calculate:

$$\begin{aligned}
\mathbb{E}(|Z_k^{(2)}|^4) &= 16N^2 \mathbb{E}\left(\sum_{j_4=1}^{k-1}\sum_{j_3=1}^{k-1}\sum_{j_2=1}^{k-1}\sum_{j_1=1}^{k-1} X_{1j_1}X_{1j_2}X_{1j_3}X_{1j_4}X_{j_1k}X_{j_2k}X_{j_3k}X_{j_4k}X_{1k}^4\right) \\
&= 16N^2 \sum_{j_4=1}^{k-1}\sum_{j_3=1}^{k-1}\sum_{j_2=1}^{k-1}\sum_{j_1=1}^{k-1} \mathbb{E}\left(X_{1j_1}X_{1j_2}X_{1j_3}X_{1j_4}X_{j_1k}X_{j_2k}X_{j_3k}X_{j_4k}X_{1k}^4\right)
\end{aligned}$$

notice that

$$\mathbb{E}\left(X_{1j_1}X_{1j_2}X_{1j_3}X_{1j_4}\right)$$

is non-zero only when there are no single powers of X_{1j}, that is, it is only non-zero in the three cases that two pairs of indices are identical (e.g. $j_1 = j_2$ and $j_3 = j_4$).

The consequence is that

$$\begin{aligned}
\sum_{k=2}^{N}\mathbb{E}(|Z_k^{(2)}|^4) &= 48N^2\sum_{k=2}^{N}\sum_{j_2=1}^{k-1}\sum_{j_1=1}^{k-1}\mathbb{E}\left(X_{1j_1}^2X_{1j_2}^2\right)\mathbb{E}\left(X_{j_1k}^2X_{j_2k}^2X_{1k}^4\right),\\
&= 48N^2\Bigg(\sum_{k=2}^{N}\sum_{j=1}^{k-1}\mathbb{E}(X_{1j}^4)\mathbb{E}(X_{jk}^4X_{1k}^4)\\
&+ 2\sum_{k=3}^{N}\sum_{j_2=2}^{k-1}\sum_{j_1=1}^{j_2-1}\mathbb{E}(X_{1j_1}^2)\mathbb{E}(X_{1j_2}^2)\mathbb{E}(X_{j_1k}^2X_{1k}^4)\mathbb{E}(X_{j_2k}^2)\Bigg),\\
&= 48N^2\Bigg(\mathbb{E}(X_{11}^4)\mathbb{E}(X_{12}^8)\sum_{k=2}^{N}1+\mathbb{E}(X_{12}^4)^3\sum_{k=3}^{N}\sum_{j=2}^{k-1}1\\
&+ 2\mathbb{E}(X_{12}^2)^4\mathbb{E}(X_{12}^4)\sum_{k=4}^{N}\sum_{j_2=3}^{k-1}\sum_{j_1=2}^{j_2-1}1\\
&+ 2\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^2)^2\mathbb{E}(X_{12}^4)\sum_{k=3}^{N}\sum_{j_2=2}^{k-1}1\Bigg),\\
&= 48N^2\Bigg(\mathbb{E}(X_{11}^4)\mathbb{E}(X_{12}^8)(N-1)+\mathbb{E}(X_{12}^4)^3\left(\frac{N^2-3N+2}{2}\right)\\
&+ 2\mathbb{E}(X_{12}^2)^4\mathbb{E}(X_{12}^4)\left(\frac{N^3-12N^2+24N-84}{12}\right)\\
&+ 2\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^2)^2\mathbb{E}(X_{12}^4)\left(\frac{N^2-3N+2}{2}\right)\Bigg),
\end{aligned}$$

note that the order of $\mathcal{O}\left(\mathbb{E}(X_{12}^4)^3\right)=\frac{1}{N^6}$ and also:

$$\mathcal{O}\left(\mathbb{E}(X_{11}^4)\mathbb{E}(X_{12}^8)\right)=\mathcal{O}\left(\mathbb{E}(X_{12}^2)^4\mathbb{E}(X_{12}^4)\right)=\mathcal{O}\left(\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^2)^2\mathbb{E}(X_{12}^4)\right)=\frac{1}{N^6},$$

so that in the limit as $N\to\infty$ we have that

$$\sum_{k=2}^{N}\mathbb{E}(|Z_k^{(2)}|^4)\to 0,$$

thus we have satisfied 2.2.

Now the result of our theorem is that

$$\sum_{k=1}^{N}S_k\to N(0,\nu^2).$$

We check:

$$\sum_{k=1}^{N}S_k=\sqrt{N}f(X_N)_{11}-\sqrt{N}\mathbb{E}(X_{11}^3)+2W_{11}\mathbb{E}(W_{12}^2)-2\sqrt{N}X_{11}\mathbb{E}(X_{12}^2).$$

Now we compare with the result from [1]:

$$\begin{aligned}
\omega^2(f) &= \frac{1}{2}\int_{-2\sigma}^{2\sigma}\int_{-2\sigma}^{2\sigma}(x^3-y^3)^2\frac{1}{4\pi^2\sigma^4}\sqrt{4\sigma^2-x^2}\sqrt{4\sigma^2-y^2}dxdy,\\
&= \sigma^6 C_3 = 5\sigma^6,\\
\alpha(f) &= \frac{1}{\sigma}\int_{-2\sigma}^{2\sigma}x^4\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}dx,\\
&= \sigma^3 C_2 = 2\sigma^3,\\
\beta(f) &= \frac{1}{\sigma^2}\int_{-2\sigma}^{2\sigma}x^3(x^2-\sigma^2)\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}dx,\\
&= 0,\\
\nu_1^2(f) &= 2(5\sigma^6-4\sigma^6) = 2\mathbb{E}(W_{12}^2)^3,
\end{aligned}$$

and we see clearly $\nu_1^2(f)=\nu^2$ as expected. Also, the remaining random variable is $\frac{\alpha(f)}{\sigma}W_{11} = 2\sigma^2 W_{11} = 2\mathbb{E}(W_{12}^2)W_{11}$ which matches with the above expression.

4. Quartic Polynomial Case

To illustrate the difficulties of generalizing this method to higher degree polynomials, we will consider the case that $f(x)=x^4$. In this situation:

$$f(X)_{11} = \sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{N}X_{1j}X_{jk}X_{kl}X_{l1},$$

in order to write this in terms of martingale differences, we cannot simply take our martingale differences to be:

$$S_k = \sum_{l=1}^{N}\sum_{j=1}^{N}X_{1j}X_{jk}X_{kl}X_{l1} - \mathbb{E}\left(\sum_{l=1}^{N}\sum_{j=1}^{N}X_{1j}X_{jk}X_{kl}X_{l1}\right),$$

for it can be easily seen that $\mathbb{E}(S_1^2)$ diverges as $N\to\infty$. Rather, we wish to approach this problem in a similar fashion as in the previous section, and write the k-th martingale difference as a sum up to $k-1$. In order to contain every element of the sum in $f(X)_{11}$, we must consider sevaral possibilities. There are a total of 13 cases as follows:

$$\begin{aligned}
l>k>j,&\quad l>j>k,\quad j>k>l,\\
j>l>k,&\quad k>l>j,\quad k>j>l,\\
j=l>k,&\quad k=l>j,\quad j=k>l,\\
k>j=l,&\quad j>k=l,\quad l>j=k,\\
&\quad j=k=l,
\end{aligned}$$

that is, we can break up the sum $f(X)_{11}$ into thirteen sums over defined by the above relations between the indices $j,k,$ and l. For each of these thirteen cases, we can create a sequence of martingale differences which we can put together to create the needed martingale difference sequence.

For example, the case where $j = k = l$ we have

$$\sum_{k=1}^{N} X_{1k}^2 X_{kk}^2,$$

so that if we define:

$$\tilde{S}_k^{(1)} = \sqrt{N} X_{1k}^2 X_{kk}^2,$$

and then subtract out its expectation (note that $\mathcal{F}_{N,k-1}$ will refer to the same sigma algebra as before):

$$\mathbb{E}(\tilde{S}_k^{(1)} | \mathcal{F}_{N,k-1}) = \sqrt{N}\mathbb{E}(X_{1k}^2 X_{kk}^2 | \mathcal{F}_{N,k-1}) = \mathbb{E}(X_{1k}^2 X_{kk}^2),$$

so that

$$S_k^{(1)} = \sqrt{N}(X_{1k}^2 X_{kk}^2 - \mathbb{E}(X_{1k}^2 X_{kk}^2)),$$

is a martingale difference.

We list all 13 martingale differences that represent the cases listed above:

$$\begin{aligned}
S_k^{(1)} &= \sqrt{N}\left(X_{1k}^2 X_{kk}^2 - \mathbb{E}(X_{1k}^2 X_{kk}^2)\right), \quad k \geq 1 \\
S_k^{(2)} &= \sqrt{N}\left(\sum_{j=1}^{k-1} X_{1j}X_{jj}X_{jk}X_{1k} - X_{11}^2\mathbb{E}(X_{12}^2)\right), \quad k \geq 2 \\
S_k^{(3)} &= \sqrt{N}\left(\sum_{j=1}^{k-1} X_{1j}X_{jj}X_{jk}X_{1k} - X_{11}^2\mathbb{E}(X_{12}^2)\right), \quad k \geq 2 \\
S_k^{(4)} &= \sqrt{N}\left(\sum_{j=1}^{k-1} X_{1j}^2 X_{jk}^2 - \mathbb{E}(X_{12}^2)\sum_{j=1}^{k-1} X_{1j}^2\right), \quad k \geq 2 \\
S_k^{(5)} &= \sqrt{N}\left(\sum_{j=1}^{k-1} X_{1k}X_{kk}X_{jk}X_{1j}\right), \quad k \geq 2 \\
S_k^{(6)} &= \sqrt{N}\left(\sum_{j=1}^{k-1} X_{1k}X_{kk}X_{jk}X_{1j}\right), \quad k \geq 2 \\
S_k^{(7)} &= \sqrt{N}\left(\sum_{j=1}^{k-1} X_{1k}^2 X_{jk}^2 - \sum_{j=1}^{k-1}\mathbb{E}(X_{1k}^2 X_{jk}^2)\right), \quad k \geq 2 \\
S_k^{(8)} &= \sqrt{N}\left(\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1j}X_{1l}X_{jk}X_{lk}\right), \quad k \geq 3
\end{aligned}$$

$$S_k^{(9)} = \sqrt{N}\left(\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1j}X_{1l}X_{jk}X_{lk}\right), \quad k \geq 3$$

$$S_k^{(10)} = \sqrt{N}\left(\sum_{l=2}^{k-1}\sum_{j=1}^{l-1} X_{1k}X_{1l}X_{jk}X_{lj} - \mathbb{E}(X_{12}^2)\sum_{l=2}^{k-1} X_{1l}^2\right), \quad k \geq 3$$

$$S_k^{(11)} = \sqrt{N}\left(\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1k}X_{1l}X_{jk}X_{lj}\right), \quad k \geq 3$$

$$S_k^{(12)} = \sqrt{N}\left(\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1j}X_{1k}X_{jl}X_{lk} - \mathbb{E}(X_{12}^2)\sum_{j=2}^{k-1} X_{1j}^2\right), \quad k \geq 3$$

$$S_k^{(13)} = \sqrt{N}\left(\sum_{l=2}^{k-1}\sum_{j=1}^{l-1} X_{1j}X_{jl}X_{lk}X_{1k}\right), \quad k \geq 3$$

Now we sum over all of these martingale differences which gives us:

$$\begin{aligned}
S_1 &= \sqrt{N}[X_{11}^4 - \mathbb{E}(X_{11}^4)] \\
S_2 &= \sqrt{N}\big[X_{12}^2X_{22}^2 - \mathbb{E}(X_{12}^2)\mathbb{E}(X_{11}^2) + 3X_{11}^2X_{12}^2 - 3\mathbb{E}(X_{12}^2)X_{11}^2 \\
&\qquad + 2X_{12}^2X_{11}X_{22} + X_{12}^4 - \mathbb{E}(X_{12}^4)\big], \\
S_k &= \sqrt{N}\Bigg[X_{1k}^2X_{kk} - \mathbb{E}(X_{1k}^2X_{kk}^2) + 2\sum_{j=1}^{k-1} X_{1j}X_{jj}X_{jk}X_{1k} - 2X_{11}^2\mathbb{E}(X_{12}^2) \\
&\qquad + \sum_{j=1}^{k-1} X_{1j}^2X_{jk}^2 - \mathbb{E}(X_{12}^2)\sum_{j=1}^{k-1} X_{1j}^2 + 2\sum_{j=1}^{k-1} X_{1k}X_{kk}X_{jk}X_{1j} \\
&\qquad + \sum_{j=1}^{k-1} X_{1k}^2X_{jk}^2 - \sum_{j=1}^{k-1}\mathbb{E}(X_{1k}^2X_{jk}^2) + 2\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1j}X_{1l}X_{jk}X_{lk} \\
&\qquad + 2\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1j}X_{1k}X_{jl}X_{lk} - 2\mathbb{E}(X_{12}^2)\sum_{j=2}^{k-1} X_{1j}^2 \\
&\qquad + 2\sum_{j=2}^{k-1}\sum_{l=1}^{j-1} X_{1k}X_{1l}X_{jk}X_{lj}\Bigg], \quad k \geq 3,
\end{aligned}$$

Recall that we wish to find ν^2 such that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1}) \xrightarrow{\mathbb{P}} \nu^2,$$

this means we must evaluate $\mathbb{E}(S_k^2|\mathcal{F}_{N,k-1})$ for all k.

We begin by listing the result for $k = 1$ and $k = 2$:

$$
\begin{aligned}
\mathbb{E}(S_1^2|\mathcal{F}_{N,0}) &= N[\mathbb{E}(X_{11}^8) - \mathbb{E}(X_{11}^4)^2], \\
\mathbb{E}(S_2^2|\mathcal{F}_{N,1}) &= N[\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^4) + \mathbb{E}(X_{12}^2)^2\mathbb{E}(X_{11}^2)^2 \\
&\quad + 9X_{11}^4\mathbb{E}(X_{12}^4) + 9X_{11}^4\mathbb{E}(X_{12}^2)^2 \\
&\quad + 4X_{11}^2\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2) + \mathbb{E}(X_{12}^8) + \mathbb{E}(X_{12}^4)^2 \\
&\quad - 2\mathbb{E}(X_{11}^2)^2\mathbb{E}(X_{12}^2)^2 + 6X_{11}^2\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2) \\
&\quad - 6X_{11}^2\mathbb{E}(X_{12}^2)^2\mathbb{E}(X_{11}^2) + 4X_{11}\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^3) \\
&\quad + 2\mathbb{E}(X_{12}^6)\mathbb{E}(X_{11}^2) - 2\mathbb{E}(X_{12}^2)\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^2) \\
&\quad - 18\mathbb{E}(X_{12}^2)^2X_{11}^4 + 6X_{11}^2\mathbb{E}(X_{12}^6) \\
&\quad - 6X_{11}^2\mathbb{E}(X_{12}^4)\mathbb{E}(X_{12}^2) - 2\mathbb{E}(X_{12}^4)^2],
\end{aligned}
$$

we will not write the complete expression of the conditional expectation value of S_k^2 – the sum is intractable. The proper strategy it seems is to determine what terms are relavant in the limit. However, this will require a detailed analysis of the asymptotics of each term in S_k^2. We begin by writing S_k^2:

$$
\begin{aligned}
S_k^2 = N\Bigg[& X_{1k}^4X_{kk}^4 - \mathbb{E}(X_{1k}^2X_{kk}^2)^2 + 4\sum_{j=1}^{k-1}\sum_{l=1}^{k-1} X_{1j}X_{1l}X_{jj}X_{ll}X_{jk}X_{lk}X_{1k}^2 \\
& + 4X_{11}^4\mathbb{E}(X_{12}^2)^2 + \sum_{j=1}^{k-1}\sum_{l=1}^{k-1} X_{1j}^2X_{1l}^2X_{jk}^2X_{lk}^2 \\
& + \mathbb{E}(X_{12}^2)^2\sum_{j=1}^{k-1}\sum_{l=1}^{k-1} X_{1j}^2X_{1l}^2 + 4\sum_{j=1}^{k-1}\sum_{l=1}^{k-1} X_{1k}^2X_{kk}^2X_{jk}X_{lk}X_{1j}X_{1l} \\
& + \sum_{j=1}^{k-1}\sum_{l=1}^{k-1} X_{1k}^4X_{jk}^2X_{lk}^2 + \sum_{j=1}^{k-1}\sum_{l=1}^{k-1} \mathbb{E}(X_{1k}^2X_{jk}^2)\mathbb{E}(X_{1k}^2\mathbb{E}(X_{lk}^2) \\
& + 4\sum_{j_2=2}^{k-1}\sum_{l_2=1}^{j_2-1}\sum_{j_1=2}^{k-1}\sum_{l_1=1}^{j_1-1} X_{1j_1}X_{1j_2}X_{1l_1}X_{1l_2}X_{j_1k}X_{j_2k}X_{l_1k}X_{l_2k} \\
& + 4\sum_{j_2=2}^{k-1}\sum_{l_2=1}^{j_2-1}\sum_{j_1=2}^{k-1}\sum_{l_1=1}^{j_1-1} X_{1j_1}X_{1j_2}X_{1k}^2X_{j_1l_1}X_{j_2l_2}X_{l_1k}X_{l_2k} \\
& + 4\mathbb{E}(X_{12}^2)^2\sum_{j=2}^{k-1}\sum_{l=2}^{k-1} X_{1j}^2X_{1l}^2 \\
& + 4\sum_{j_2=2}^{k-2}\sum_{l_2=1}^{j_2-1}\sum_{j_1=2}^{k-1}\sum_{l_1=1}^{j_1-1} X_{1j_1}X_{1j_2}X_{1k}^2X_{j_1l_1}X_{j_2l_2}X_{l_1k}X_{l_2k} \\
& + \cdots \Bigg],
\end{aligned}
$$

where the cross terms have been omitted (these are simply the sum of the squares of each terms). We wish to determine from the unommitted terms in this sum, which terms will be relavant in the limit that $N \to \infty$, and then from that determine which cross terms will need to be present.

We will evaluate the expectation of each of these terms and sum from $k = 3$ to N, and verify if the limit is non-zero:

$$\sum_{k=3}^{N} N\mathbb{E}(X_{1k}^4 X_{kk}^4) = N\mathbb{E}(X_{12}^4)\mathbb{E}(X_{11}^4)(N-2) \to 0,$$

since $\mathcal{O}(X_{12}^4) = \mathcal{O}(X_{11}^4) = 1/N^2$ so that the entire term is $\mathcal{O}(1/N^2)$. The same argument holds for the second term. Next,

$$4N\sum_{k=3}^{N}\sum_{j=1}^{k-1}\sum_{l=1}^{k-1}\mathbb{E}(X_{1j}X_{1l}X_{jj}X_{ll}X_{jk}X_{lk}X_{1k}^2) = 4N\sum_{k=3}^{N}\mathbb{E}(X_{1j}^2X_{jj}^2X_{jk}^2X_{1k}^2),$$

so in the limit this goes to zero as well (by the same argument). The fourth term disappears in the same fashion. The fifth term:

$$\begin{aligned}
N\sum_{k=3}^{N}\sum_{j=1}^{k-1}\sum_{l=1}^{k-1}\mathbb{E}(X_{1j}^2X_{1l}^2X_{jk}^2X_{lk}^2) &= N\sum_{k=3}^{N}\sum_{j=1}^{k-1}\mathbb{E}(X_{1j}^4)\mathbb{E}(X_{jk}^4) \\
&\quad +2N\sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{l=1}^{j-1}\mathbb{E}(X_{1j}^2)\mathbb{E}(X_{1l}^2)\mathbb{E}(X_{jk}^2)\mathbb{E}(X_{lk}^2)] \\
&= N\mathbb{E}(X_{11}^4)\mathbb{E}(X_{12}^4) + N\mathbb{E}(X_{12}^4)^2\sum_{k=3}^{N}\sum_{j=2}^{k-1}1 \\
&\quad + 2N\mathbb{E}(X_{11}^2)\mathbb{E}(X_{12}^2)^3\sum_{k=3}^{N}\sum_{j=2}^{k-1}1 \\
&\quad + 2N\mathbb{E}(X_{12}^2)^4\sum_{k=4}^{N}\sum_{j=3}^{k-1}\sum_{l=2}^{j-1}1,
\end{aligned}$$

the only term that does not vanish in the limit is the final term. This last term is:

$$\lim_{N\to\infty} 2N\mathbb{E}(X_{12}^2)^4\left(\frac{N^3+9N^2+21N}{6}\right) = \frac{\mathbb{E}(W_{12}^2)^4}{3},$$

The sixth term:

$$\begin{aligned}
N\mathbb{E}(\mathbb{E}(X_{12}^2)\sum_{j=1}^{k-1}\sum_{l=1}^{k-1}X_{1j}^2X_{1l}^2) &= N\mathbb{E}(X_{12}^2)^2\sum_{j=1}^{k-1}\sum_{l=1}^{k-1}\mathbb{E}(X_{1j}^2X_{1l}^2) \\
&= N\mathbb{E}(X_{12}^2)^2[\sum_{j=1}^{k-1}\mathbb{E}(X_{1j}^4) + 2\sum_{j=2}^{k-1}\sum_{l=1}^{j-1}\mathbb{E}(X_{1j}^2)\mathbb{E}(X_{1l}^2)],
\end{aligned}$$

which, when we sum from $k = 3$ to N and take the limit as $N \to \infty$, the only term to remain:

$$\frac{1}{3}\mathbb{E}(W_{12}^2)^4.$$

Continuing in this fashion, we obtain that the sum of expectations of the expression above is

$$\frac{13}{3}\mathbb{E}(W_{12}^2)^4 + \frac{5}{3}\mathbb{E}(W_{12}^2)^2\mathbb{E}(W_{12}^4),$$

not including cross terms. The only terms that gave a non-negative expectation were the fifth, sixth, and the eigth through thirteenth. Now, the cross terms yield:

$$\frac{22}{3}\mathbb{E}(W_{12}^2)^2\mathbb{E}(W_{12}^4) - \frac{34}{3}\mathbb{E}(W_{12}^2)^4,$$

so that in sum,

$$\nu^2 = 9\mathbb{E}(W_{12}^2)^2\mathbb{E}(W_{12}^4) - 7\mathbb{E}(W_{12}^2)^4.$$

As in the previous section, we can verify 2.1 and 2.2 using the same techniques.

We compare with the result obtained using the methods from [1]:

$$\begin{aligned}
\omega^2(f) &= \frac{1}{2}\int_{-2\sigma}^{2\sigma}\int_{-2\sigma}^{2\sigma}(x^4-y^4)^2\frac{1}{4\pi^2\sigma^4}\sqrt{4\sigma^2-x^2}\sqrt{4\sigma^2-y^2}dxdy\\
&= \sigma^8C_4 - \sigma^8C_2^2 = 10\sigma^8,\\
\alpha(f) &= \frac{1}{\sigma}\int_{-2\sigma}^{2\sigma}x^5\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}dx\\
&= 0,\\
\beta(f) &= \frac{1}{\sigma^2}\int_{-2\sigma}^{2\sigma}x^4(x^2-\sigma^2)\frac{1}{2\pi\sigma^4}\sqrt{4\sigma^2-x^2}dx\\
&= \sigma^4(C_3-C_2) = 3\sigma^4,\\
\kappa_4(\mu) &= m_4 - 3\sigma^4,\\
\nu_1^2(f) &= 2(10\sigma^8 + \frac{(m_4-3\sigma^4)}{2\sigma^4}9\sigma^8)\\
&= 9\sigma^4m_4 - 7\sigma^8,
\end{aligned}$$

thus we see that $\nu_1^2(f) = 9\mathbb{E}(W_{12}^2)^2\mathbb{E}(W_{12}^4) - 7\mathbb{E}(W_{12}^2)^4$ which matches ν^2 exactly.

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