A BOUND FOR ORDERINGS OF REIDEMEISTER MOVES

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ABSTRACT. We provide an upper bound on the number of ordered Reidemeister moves required to pass between two diagrams of the same link. This bound is in terms of the number of unordered Reidemeister moves required.

In 1927 Kurt Reidemeister proved that any two link diagrams representing the same link may be joined by a finite sequence of Reidemeister moves. The importance of this theorem to knot theory cannot be understated. Mathematicians like Alexander Coward [1, 2], Marc Lackenby [2], Bruce Trace [4], Joel Hass and Jeffery Lagarias [3] have all explored properties of sequences of Reidemeister moves.

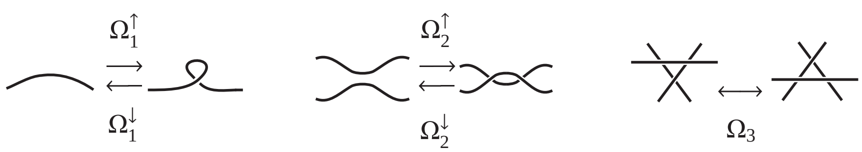

FIGURE 1. Reidemeister Moves.

In 2006, Alexander Coward showed [1] that given any sequence of Reidemeister moves between link diagrams D_1 and D_2, it is possible to construct a new sequence ordered in the following way: first $\Omega_1^{\uparrow}$ moves, then $\Omega_2^{\uparrow}$ moves, then Ω_3 moves, finally $\Omega_2^{\downarrow}$ moves. We present, via the following theorem, an upper bound on the number of moves required for an ordered sequence in terms of the number of moves present in any sequence of Reidemeister moves.

Theorem 1. *Let D_1 and D_2 be diagrams for the same link that are joined by a sequence of M Reidemeister moves. Let $N = 6^{M+1}M$. Then there exists a sequence of no more than $exp^{(N)}(N)$ moves from D_1 to D_2 ordered in the following way: first $\Omega_1^{\uparrow}$, then $\Omega_2^{\uparrow}$, then Ω_3, then $\Omega_2^{\downarrow}$ and finally $\Omega_1^{\downarrow}$.*

Here the function exp is defined as $\exp(x) = 2^x$ and $\exp^{(r)}(x)$ is the function exp iterated r times on input x.

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We define a link diagram to be a 4-valent graph embedded in $\mathbb{R}^2$ with crossing information recorded at each vertex. All diagrams will be oriented, so that they represent oriented links. We regard two diagrams as the same if there is an ambient isotopy of $\mathbb{R}^2$ taking one diagram to the other, preserving crossing information and the orientation of each link component. To prove Theorem 1, we will adapt the methods Alexander Coward uses in [1] and borrow the following terminology.

Definition: Let D be a link diagram and suppose $c : [0,1] \to \mathbb{R}^2$ is an embedded path whose image C intersects D transversely at finitely many points, where $c(0) \in D$ and $c(1) \notin D$. We stipulate that no point of intersection of D and C is a vertex of D. At each such point, apart from $c(0)$, we designate whether C passes over or under D.

Let $C \times [-\epsilon, \epsilon]$ be a small enough neighborhood of C so that $(C \times [-\epsilon, \epsilon]) \cap D = (C \cap D) \times [-\epsilon, \epsilon]$. Then define the diagram D' as the 4-valent graph

$$D \cup \partial(C \times [-\epsilon, \epsilon]) \setminus (c(0) \times (-\epsilon, \epsilon))$$

with crossing information induced by the path c. We write $D \rightsquigarrow D'$ and say that D' is obtained from D by *adding a tail along* C. Additionally, we will call C the *core of this tail*. We require that adding a tail to a diagram D produces a diagram D' where $c(D') > c(D)$. Figure 2 illustrates the construction of a tail.

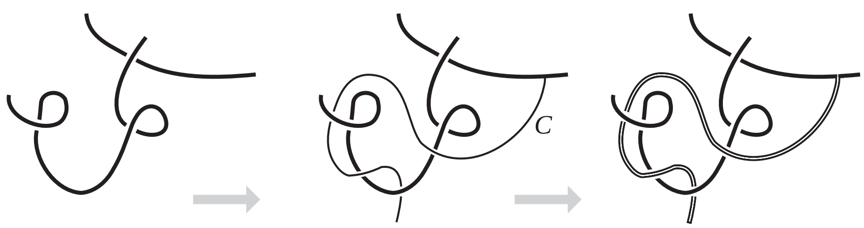


FIGURE 2. Adding a tail.

Definition: Suppose $D_1 \rightsquigarrow D_2$ via some path $c : [0,1] \to \mathbb{R}^2$. Suppose additionally that $c(1)$ lies in a small neighborhood of some crossing χ of D_1. Let D_3 be as in Figure 3, a diagram obtained from D_2 by performing two $\Omega_2^{\uparrow}$ moves followed by one Ω_3 move:

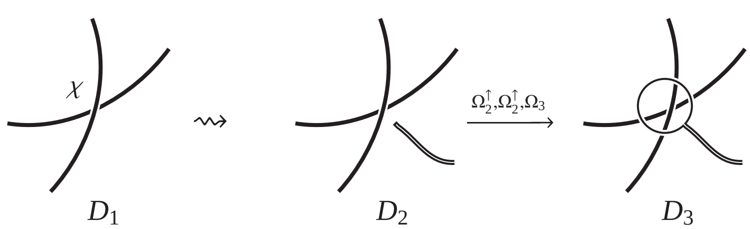


FIGURE 3. Adding a lollipop.

We say D_3 is obtained from D_1 by *adding a lollipop* and write $D_1 \circ\!\!\rightarrow D_3$. The *lollipop* itself is defined as $\overline{D_3 \setminus D_1}$. The *tail part of the lollipop* is $\overline{(D_3 \cap D_2) \setminus D_1}$, and the closure of the rest of the lollipop is the *circle part of the lollipop*. We say that the lollipop is *centered at* χ.

We think of a sequence $\mathcal{S}$ of Reidemeister moves, tails and lollipops between link diagrams L_1 and L_2 in the following way:

$$\mathcal{S} : L_1 = D_0 \xrightarrow{a_1} D_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} D_n = L_2$$

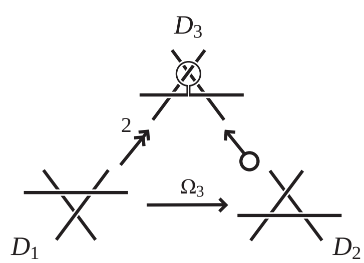
Here each a_i is a Reidemeister move, a tail or a lollipop. A tail or lollipop may be added from D_i to D_{i+1} (eg $D_i \rightsquigarrow D_{i+1}$) or from D_{i+1} to D_i (eg $D_i \leftsquigarrow D_{i+1}$). We say the *length* of $\mathcal{S}$ is n. The intermediate link diagrams D_i are often omitted from the figures in this paper for clarity, but are implicit in any sequence.

If a link diagram D_2 is reached from D_1 by a sequence of $\Omega_2^{\uparrow}$ moves of length n, we write $D_1 \twoheadrightarrow^n D_2$.

The following lemma allows us to take a sequence $\mathcal{S}$ and produce a sequence $\mathcal{S}'$ with one less Ω_3 move.

Lemma 2. *Let D_1 and D_2 be link diagrams such that $D_1 \xrightarrow{\Omega_3} D_2$. Then there exists a diagram D_3 such that $D_1 \twoheadrightarrow^2 D_3$ and $D_2 \circ\!\!\rightarrow D_3$.*

Proof.



□

If an Ω_3 move occurs in a sequence of Reidemeister moves, tails and lollipops $\mathcal{S}$, we can apply Lemma 2 to $\mathcal{S}$ to get a new sequence $\mathcal{S}'$:

$$\mathcal{S} : A \to \cdots \to B \xrightarrow{\Omega_3} C \to \cdots \to D$$

$$\mathcal{S}' : A \to \cdots \to B \xrightarrow{\Omega_2^{\uparrow}} B' \xrightarrow{\Omega_2^{\uparrow}} B'' \leftarrow\!\!\circ\; C \to \cdots \to D$$

When we apply Lemma 2 to construct $\mathcal{S}'$ from $\mathcal{S}$, we call this *capping the* Ω_3 *move* from B to C.

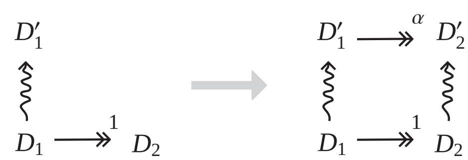
The following proposition and its corollary will also allow us to build new sequences from old ones in a useful way.

Proposition 3. *Suppose $D_1 \rightsquigarrow D_1'$ (or $D_1 \circ\!\!\rightarrow D_1'$) and also that $D_1 \twoheadrightarrow^1 D_2$. Then there exists a diagram D_2' such that $D_2 \rightsquigarrow D_2'$ ($D_2 \circ\!\!\rightarrow D_2'$ respectively) and $D_1' \twoheadrightarrow^{\alpha} D_2'$, where*

(A) $$c(D_2') - c(D_2) \leq 2(c(D_1') - c(D_1))$$

and

(B) $$\alpha \leq c(D_1') - c(D_1).$$



Proof. The diagram D_2 is obtained from D_1 by a single $\Omega_2^{\uparrow}$ move which takes place over two (possibly non-distinct) edges e_1 and e_2 of D_1. Pick points p_1 and p_2 on e_1 and e_2 respectively, so that p_1 and p_2 are disjoint from a neighborhood of the tail $D_1 \rightsquigarrow D_1'$. We can perform the $\Omega_2^{\uparrow}$ move from D_1 to D_2 by adding a tail along a path P, which starts at p_1 and ends slightly beyond p_2.

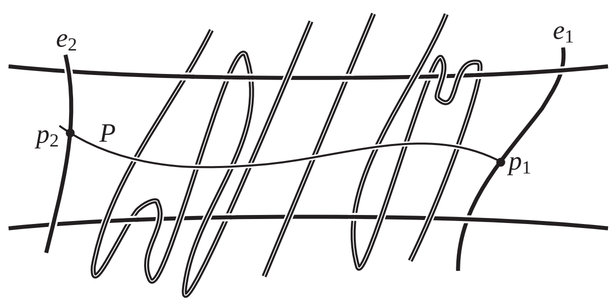


FIGURE 4. Constructing D_2' from D_1' by adding a tail along P.

Diagram D_1' contains the points p_1 and p_2. We may arrange that the intersection of P with the tail $D_1 \rightsquigarrow D_1'$ contains at most $2\lfloor \frac{c(D_1')-c(D_1)}{4} \rfloor$ points. To see this, consider the core C of the tail $D_1 \rightsquigarrow D_1'$. Every instance of the tail corresponding to C passing over or under an edge of D_1 contributes 2 to the quantity $c(D_1') - c(D_1)$. In particular, whenever C spans two edges, the tail corresponding to C contributes 4 to $c(D_1') - c(D_1)$ at the intersection of C and the given edges. Hence there are at most $\lfloor \frac{c(D_1')-c(D_1)}{4} \rfloor$ connected, closed segments $\{\ell_i\}$ of the curve C which lie inside the region between e_1 and e_2. Any path P connecting p_1 and p_2 need only intersect each ℓ_i transversely at most once. Such a path will cross the tail $D_1 \rightsquigarrow D_1'$ in at most $2\lfloor \frac{c(D_1')-c(D_1)}{4} \rfloor$ points. Figure 4 depicts this. Hence, when adding a tail along P, we build a diagram D_2' with

$$c(D_2') - c(D_1') \leq 4\lfloor \frac{c(D_1') - c(D_1)}{4} \rfloor + 2.$$

Hence

$$c(D_2') - c(D_1') \leq c(D_1') - c(D_1) + 2.$$

We note that $c(D_1') - c(D_1) + 2 \leq 2(c(D_1') - c(D_1))$, because adding a tail to a diagram must raise its crossing number by at least two. This implies the desired bound on α. Also

$$c(D_2') - c(D_1') \leq c(D_1') - c(D_1) + 2$$

implies, by adding $c(D_1')$ to both sides and subtracting $c(D_2)$, that

$$c(D_2') - c(D_2) \leq 2c(D_1') - c(D_1) + 2 - c(D_2).$$

Using $c(D_2) = c(D_1) + 2$ we get

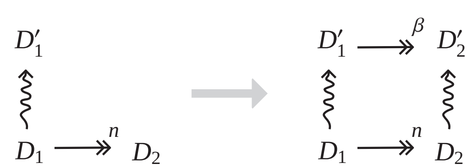
$$c(D_2') - c(D_2) \leq 2c(D_1') - 2c(D_1).$$

In the case that $D_1 \circ\!\!\rightarrow D_1'$, choose p_1 and p_2 to be outside the circle part of the lollipop, and the above considerations go through. □

Corollary 4 is a natural generalization of Proposition 3.

Corollary 4. *Suppose $D_1 \rightsquigarrow D_1'$ (or $D_1 \circ\!\!\rightarrow D_1'$) and also that $D_1 \twoheadrightarrow^n D_2$. Then there exists a diagram D_2' such that $D_2 \rightsquigarrow D_2'$ ($D_2 \circ\!\!\rightarrow D_2'$ respectively) and $D_1' \twoheadrightarrow^\beta D_2'$, where*

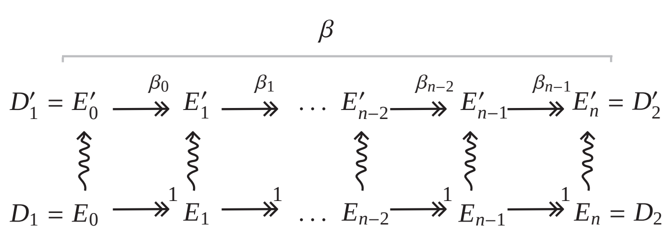
$$\beta \leq 2^n(c(D_1') - c(D_1)).$$



Proof. Let D_1, D_2 and D_1' be as in the statement of the theorem. We work in the case $D_1 \rightsquigarrow D_1'$, but the proof for lollipops is identical. Let $\mathcal{S}$ be the sequence of $\Omega_2^\uparrow$ moves of length n from D_1 to D_2,

$$\mathcal{S} : D_1 = E_0 \twoheadrightarrow^1 E_1 \twoheadrightarrow^1 \cdots \twoheadrightarrow^1 E_n = D_2,$$

and let $E_0' = D_1'$. We use Proposition 3 to construct a diagram E_1' such that $E_1 \rightsquigarrow E_1'$ and $E_0' \twoheadrightarrow^{\beta_0} E_1'$, where $\beta_0 \leq c(E_0') - c(E_0)$. Apply Proposition 3 again to the triple (E_1, E_1', E_2) to build a diagram E_2'. Repeat this application to construct the diagrams E_3' through E_n', as below.



Proposition 3 (**B**) gives us that $\beta_i \leq c(E_i') - c(E_i)$, while Proposition 3 (**A**) tells us $c(E_i') - c(E_i) \leq 2^i(c(E_0') - c(E_0))$. The sequence of Ω_2 moves from E_0' to E_n' has length β, where $\beta = \sum_{i=0}^{n-1} \beta_i$. Hence,

$$\beta \leq (2^n - 1)(c(E_0') - c(E_0)).$$

Take $D_2' = E_n'$ and a larger bound on β to complete the proof. □

Theorem 5 uses the tools we've developed (Lemma 2, Proposition 3 and Corollary 4) to begin building an ordered sequence from an unordered sequence in a special case.

Theorem 5. *Let D_2 be a link diagram obtained from D_1 via a sequence of Ω_2 and Ω_3 moves of length M. Then there exists a diagram D_3 such that $D_1 \twoheadrightarrow^{\gamma} D_3$ and D_3 is obtained from D_2 by adding a sequence no more than M tails and lollipops. Further,*

$$\gamma \leq \exp^{(M)}(6M).$$

Proof. Consider a sequence $\mathcal{S}$ of Ω_2 and Ω_3 moves of length M from D_1 to D_2, α_3 of which are Ω_3:

$$\mathcal{S} : D_1 = A_0 \to A_1 \to \cdots \to A_M = D_2$$

Using Lemma 2, cap every Ω_3 move to build a new sequence $\mathcal{E}_1$ with no Ω_3 moves:

$$\mathcal{E}_1 : D_1 = E_0 \to E_1 \to \cdots \to E_{M+2\alpha_3} = D_2$$

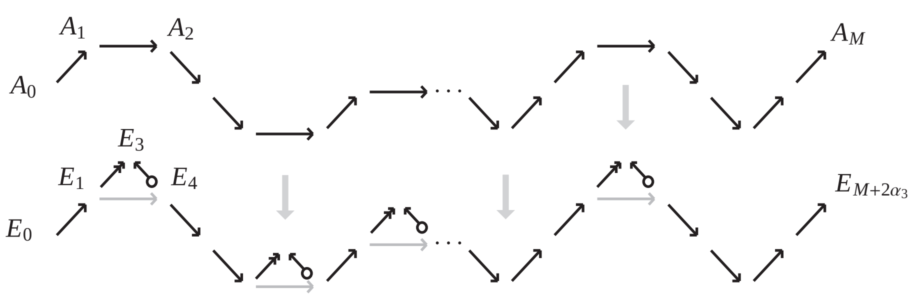


FIGURE 5. Constructing $\mathcal{E}_1$ from $\mathcal{S}$.

If $E_i \xrightarrow{\Omega_2^{\downarrow}} E_{i+1}$, we instead write $E_i \leftsquigarrow E_{i+1}$, because a $\Omega_2^{\uparrow}$ move may be performed by adding a tail. Define a *local minimum* of $\mathcal{E}_1$ to be a diagram E_i such that

$$E_{i-1} \leftsquigarrow E_i \xrightarrow{\Omega_2^{\uparrow}} E_{i+1} \qquad \text{or} \qquad E_{i-1} \leftarrow\!\circ\; E_i \xrightarrow{\Omega_2^{\uparrow}} E_{i+1}.$$

Let $E_x \in \{E_1, \ldots, E_{M+2\alpha_3-1}\}$ be the last local minimum appearing in $\mathcal{E}_1$. Let r_1 be the number of consecutive $\Omega_2^{\uparrow}$ moves in $\mathcal{E}_1$ to the right of E_x. Let ℓ_1 be the number of consecutive $\Omega_2^{\uparrow}$ moves in $\mathcal{E}_1$ to the left of E_{x-1}.

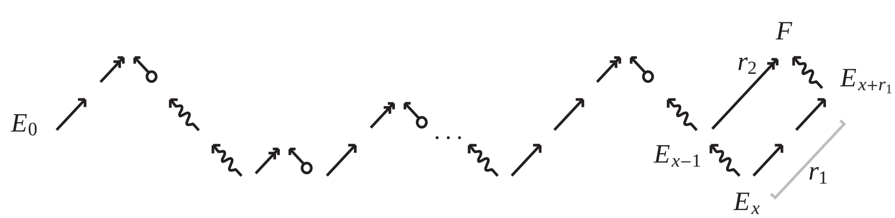


FIGURE 6. Constructing F. In this case, $E_{x+r_1} = E_{M+2\alpha_3}$.

Apply Corollary 4 to the triple $(E_{x-1}, E_x, E_{x+r_1})$ to build a diagram F, where $E_{x-1} \twoheadrightarrow^{r_2'} F$ and where $E_{x+r_1} \circ\!\!\rightarrow F$ if $E_x \circ\!\!\rightarrow E_{x-1}$ or $E_{x+r_1} \rightsquigarrow F$ if $E_x \rightsquigarrow E_{x-1}$. Corollary 4 tells us

$r_2' \leq 4 \cdot 2^{r_1}$, in the worst case that $E_x \circ\!\!\rightarrow E_{x-1}$. This is because any tail in $\mathcal{E}_1$ is an Ω_2 move in disguise, and hence creates two new crossings, while any lollipop in $\mathcal{E}_1$ creates four new crossings. Figure 6 depicts the construction of F.

Define $\mathcal{E}_2$ to be the following sequence:

$$\mathcal{E}_2 : D_1 = E_0 \to E_1 \to \cdots \to E_{x-1} \to \cdots \to F \to E_{x+r_1} \to \cdots \to E_{M+2\alpha_3}$$

Then $\mathcal{E}_2$ is a sequence of diagrams with r_2 consecutive $\Omega_2^{\uparrow}$ moves to the right of its last local minimum, with r_2 bounded by

$$r_2 \leq 4 \cdot 2^{r_1} + \ell_1.$$

Hence

$$r_2 \leq 2^{r_1+2+\ell_1}$$

In general let $\mathcal{E}_k$ be a sequence with r_k the number of $\Omega_2^{\uparrow}$ moves to the right of the last local minimum of $\mathcal{E}_k$. Let ℓ_k be the number of consecutive $\Omega_2^{\uparrow}$ moves preceding the diagram to the immediate left of the last local minimum of $\mathcal{E}_k$. Given the triple $(\mathcal{E}_k, r_k, \ell_k)$, we may apply Corollary 4 as above to produce a triple $(\mathcal{E}_{k+1}, r_{k+1}, \ell_{k+1})$ with r_{k+1} satisfying

$$r_{k+1} \leq 2^{r_k+2+\ell_k}.$$

Inductively,

$$r_{k+1} \leq \exp^{(k)}\left(r_1 + 2k + \sum_{i=1}^{k} \ell_i\right).$$

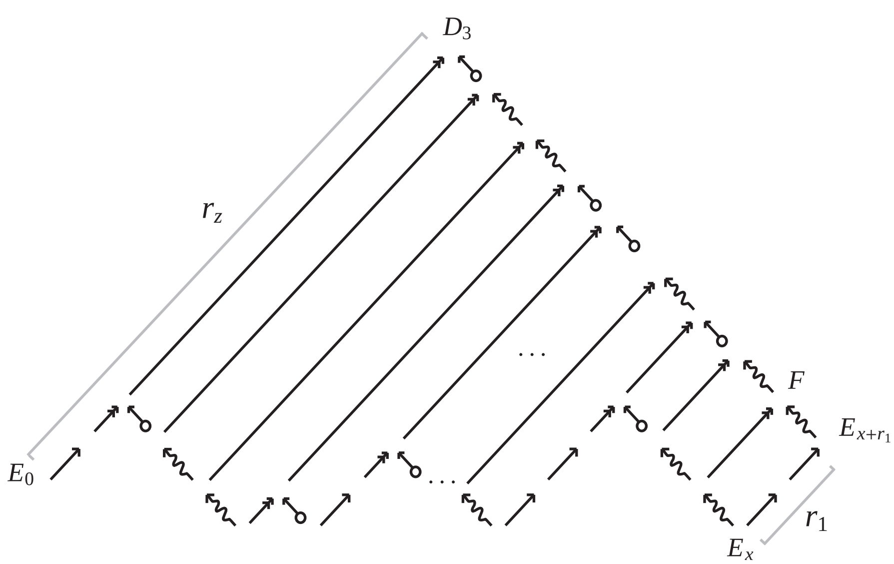


FIGURE 7. Repeatedly applying Corollary 4 to build D_3.

Iterate the constructions of the $(\mathcal{E}_k, r_k, \ell_k)$ until we produce a sequence $\mathcal{E}_z$ with no local minima and with r_z consecutive $\Omega_2^{\uparrow}$ moves following E_0. The number of times we apply

Corollary 4 to construct $\mathcal{E}_z$ from $\mathcal{E}_1$ is exactly the number of tails and lollipops in $\mathcal{E}_1$, which is less than or equal to M. So $z \leq M + 1$, and via our above formula,

$$r_z \leq \exp^{(z-1)}\left(r_1 + 2(z-1) + \sum_{i=1}^{z-1} \ell_i\right).$$

We note that $r_1 \leq M$ and $\sum_{i=1}^{z-1} \ell_i \leq M + 2\alpha_3 \leq 3M$. Substituting, we get

$$r_z \leq \exp^{(M)}(6M).$$

There are r_z moves of type $\Omega_2^{\uparrow}$ following $D_1 = E_0$ in $\mathcal{E}_z$, so let D_3 be the diagram obtained by performing these moves on D_1. Because D_3 is obtained from $E_{M+2\alpha_3} = D_2$ by at most M tails and lollipops, Theorem 5 holds. □

The following theorem allows us to construct an ordered sequence of Ω_2 and Ω_3 moves from the tails and lollipops arising in Theorem 5.

Theorem 6. *Suppose D_2 is obtained from D_1 by a sequence $\mathcal{T}$ of tails and lollipops of length S:*

$$\mathcal{T} : D_1 = T_0 \xrightarrow{a_1} T_1 \xrightarrow{a_2} \dots \xrightarrow{a_S} T_S = D_2$$

where either $T_i \rightsquigarrow T_{i+1}$ or $T_i \circ\!\!\rightarrow T_{i+1}$. Then there exists a diagram D_3 obtained from D_2 by a sequence of $\Omega_2^{\uparrow}$ moves of length no more than $\frac{S}{2}(c(D_2) - c(D_1)) + 2S$, followed by a sequence of Ω_3 moves of length no more than S. Additionally D_1 is obtained from D_3 by a sequence of $\Omega_2^{\downarrow}$ moves of length at most $\frac{S+1}{2}(c(D_2) - c(D_1)) + 2S$.

Proof. Consider a crossing χ of the diagram D_2 about which the circle part of a lollipop in $\mathcal{T}$ is centered. There may be multiple lollipops (suppose there are k) centered at χ, so consider a point p_k on the outermost one. Let q be a point in a small enough neighborhood of χ such that a straight line segment from q to χ does not intersect D_2 except at χ.

Consider a path $c : [0, 1] \to \mathbb{R}^2$ such that $c(0) = p_k$ and $c(1) = q$. Choose c in such a way that its image C intersects each concentric lollipop at only one point. The point of intersection of C and the ith concentric lollipop is denoted p_i. Let $\delta_k = 0$ and let $\delta_{k-1} < \delta_{k-2} < \cdots < \delta_1$ be the real numbers in $(0, 1)$ such that $c(\delta_i) = p_i$.

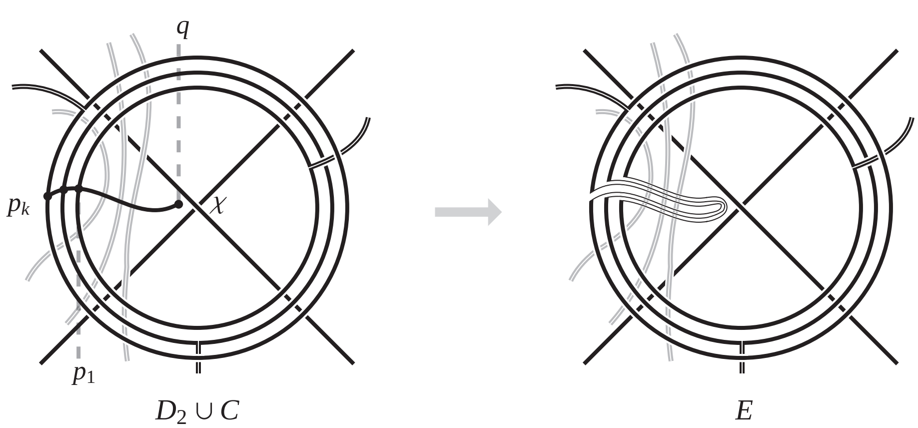


FIGURE 8. Adding concentric tails at the crossing χ.

Via the argument used in the proof of Proposition 3, we also choose c so that $C \cap D_2$ consists of no more than $2\lfloor \frac{c(D_2)-c(D_1)}{4} \rfloor$ points, excluding the points p_1 through p_k.

Add a tail along the path $c|_{[\delta_1,1]}$ to construct a diagram E_1 from D_2, where $c(E_1) - c(D_2) \leq c(D_2) - c(D_1)$. Perturb this tail slightly, so that it is closer to the crossing χ, and now add a second tail disjoint from the first tail along the path $c|_{[\delta_2,1]}$. This second tail introduces no more than $c(D_2) - c(D_1)$ crossings.

Repeating this process of perturbing and adding tails along $c|_{[\delta_i,1]}$ for all $i \in [1, \dots, k]$, we produce a diagram E_k where $c(E_k) - c(D_2) \leq k(c(D_2) - c(D_1))$. To build a diagram E, add nested tails in the same way for every crossing of D_2 that is the center of some lollipop, so that $c(E) - c(D_2) \leq S(c(D_2) - c(D_1))$. Then E may be obtained from D_2 by a sequence of $\Omega_2^{\uparrow}$ moves of length at most $\frac{S}{2}(c(D_2) - c(D_1))$.

Now construct the diagram E' from E by doing the following for each crossing: If there are k concentric circles centered at a crossing χ, perform $2k$ type $\Omega_2^{\uparrow}$ moves, forking the previously constructed tails over the edges of the crossing χ, as Figure 9 illustrates.

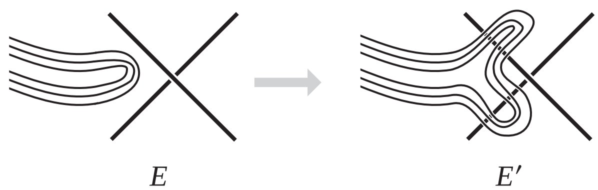


FIGURE 9. Perform $2k$ type $\Omega_2^{\uparrow}$ moves, so that each tail 'forks' over the crossing.

The diagram E' may be reached from D_2 via a sequence of $\Omega_2^{\uparrow}$ moves with length at most $\frac{S}{2}(c(D_2) - c(D_1)) + 2S$. Finally, construct the diagram D_3 by performing at most S moves of type Ω_3, as in Figure 10.

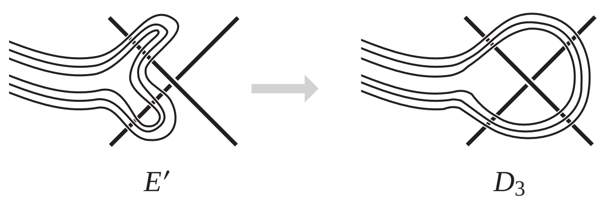


FIGURE 10. Performing Ω_3 moves to pass from E' to D_3.

We may now pass from D_3 to D_1 by performing $\Omega_2^{\downarrow}$ moves as follows. Each tail and lollipop of $\mathcal{T}$ in D_2 is still present in D_3, with the circle parts of each lollipop modified. We remove them one at a time starting with the last tail or lollipop a_S in the sequence. If a_S is a lollipop, it now has the form depicted by Figure 11 in D_3, and may be removed by $\Omega_2^{\downarrow}$ moves. If a_S is a tail, it may likewise be removed by $\Omega_2^{\downarrow}$ moves. We continue to remove tails and lollipops in the reverse order they are added in $\mathcal{T}$ until we obtain D_1.

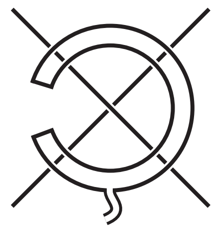


FIGURE 11

Because $c(D_3) = c(E')$, we know $c(D_3) - c(D_1)$ is exactly $c(E') - c(D_2) + c(D_2) - c(D_1)$, which is at most $S(c(D_2) - c(D_1)) + 4S + c(D_2) - c(D_1)$. Simplifying and then halving this bound gives us the number of $\Omega_2^{\downarrow}$ moves from D_3 to D_1. ◻

We consolidate the results of Theorem 5 and Theorem 6 into Theorem 7. This theorem bounds the length of an ordered sequence of Reidemeister moves, given the sequence we started with consists only of Ω_2 and Ω_3 moves, and hence solves the central problem of this paper in a special case.

Theorem 7. *Let D_2 be a link diagram obtained from D_1 by a sequence of Ω_2 and Ω_3 moves of length M. Then there is a sequence of at most $\exp^{(2M)}(6M)$ Reidemeister moves from D_1 to D_2 ordered in the following way: first $\Omega_2^{\uparrow}$ moves, then Ω_3 moves and finally $\Omega_2^{\downarrow}$ moves.*

Proof. Given D_1 and D_2, construct a diagram D_3 using Theorem 5, where D_3 is obtained from D_1 by no more than $\exp^{(M)}(6M)$ type $\Omega_2^{\uparrow}$ moves. Additionally, D_3 is obtained from D_2 by no more than M tails and lollipops. Note that $c(D_3) - c(D_2) \leq 2 \cdot \exp^{(M)}(6M) + 2M$.

From D_2 and D_3, apply Theorem 6 to construct a diagram D_4 with the following properties. There is a sequence of $\Omega_2^{\uparrow}$ moves whose length is no more than $M \cdot \exp^{(M)}(6M) + M^2 + 2M$, followed by a sequence of Ω_3 moves of length no more than M from D_3 to D_4. There is also a sequence of $\Omega_2^{\downarrow}$ moves whose length is at most $(M+1) \cdot \exp^{(M)}(6M) + M^2 + 3M$ from D_4 to D_2.

Following the sequences of moves constructed from D_1 to D_3, then to D_4 and finally to D_2, we have a sequence of no more than $(2M+2) \cdot \exp^{(M)}(6M) + (2M+6)M$ Reidemeister moves ordered as desired. For $M \geq 1$, $\exp^{(2M)}(6M) \geq (2M+2) \cdot \exp^{(M)}(6M) + (2M+6)M$. ◻

Before we prove Theorem 1, we need two lemmas relating to Ω_1 moves. These lemmas allow us to take a sequence of Reidemeister moves and build a new sequence in which the Ω_1 moves occur only at the beginning and end.

Lemma 8. *Let A, B and C be link diagrams such that*

$$A \xrightarrow{\Omega} B \xrightarrow{\Omega_1^{\uparrow}} C$$

where Ω is an Ω_2 or Ω_3 move. Then there exists a diagram B' which may be obtained from A by a single $\Omega_1^{\uparrow}$ move, and where C is obtained from B' by no more than six Ω_2 or Ω_3

moves. Additionally, if instead $\Omega = \Omega_1^{\downarrow}$*, there is a diagram* B' *such that*

$$A \xrightarrow{\Omega_1^{\uparrow}} B' \xrightarrow{\Omega_1^{\downarrow}} C.$$

Lemma 9. *Let* A*,* B *and* C *be link diagrams such that* $A \xrightarrow{\Omega_1^{\downarrow}} B \xrightarrow{\Omega} C$*, where* Ω *is an* Ω_2 *or* Ω_3 *move. Then there exists a diagram* B' *such that* B' *is obtained from* A *by no more than six* Ω_2 *or* Ω_3 *moves and where* C *may be obtained from* B' *by a single* $\Omega_1^{\downarrow}$ *move.*

The proofs of Lemma 8 and Lemma 9 are left to be verified by the reader, and Corollary 10 is a rapid consequence of these lemmas.

Corollary 10. *Let* D_2 *be obtained from* D_1 *by an arbitrary sequence of* M *Reidemeister moves,* α *of which are* $\Omega_1^{\uparrow}$ *and* β *of which are* $\Omega_1^{\downarrow}$*. Then there exist diagrams* D_1' *and* D_2' *such that* D_1' *is obtained from* D_1 *by* α *type* $\Omega_1^{\uparrow}$ *moves and* D_2 *is obtained from* D_2' *by* β *type* $\Omega_1^{\downarrow}$ *moves. Additionally,* D_2' *is obtained from* D_1' *by no more than* $6^M M$ *Reidemeister moves of type* Ω_2 *and* Ω_3*.*

Proof of Theorem 1. Begin with an arbitrary sequence of M Reidemeister moves from diagram D_1 to diagram D_2, α of which are $\Omega_1^{\uparrow}$ and β of which are $\Omega_1^{\downarrow}$. Construct D_1' and D_2' as in Corollary 10. Then apply Theorem 7 to the sequence of Ω_2 and Ω_3 moves from D_1' to D_2' to obtain a sorted sequence of Reidemeister moves from D_1 to D_2 of length at most

$$\exp^{(6^M M)}(6 \cdot 6^M M) + \alpha + \beta \leq \exp^{(6^{M+1} M)}(6^{M+1} M).$$

□

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