Counting Double Tangents of Closed Curves Without Inflection Points

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Contents

Abstract

For a closed, oriented, differentiable curve in an affine plane with a finite number of crossings, a *double tangent* is a tangent line which has exactly two points of tangency on the curve. Double tangents can be *exterior* or *interior* depending on whether the convex arcs of the curve in the neighborhood of the points of tangency are on the *same* or *opposite* sides of the tangent line, respectively. Fabricius-Bjerre proved in 1962 that $t - s = d + \frac{1}{2}i$ for such closed curves in an affine plane, where t is the number of exterior double tangents, s is the number of interior double tangents, d is the number of crossings (double points) and i is the number of inflection points of the curve. We present a technique for counting the number of interior double tangents for a subset of closed curves in an affine plane with no inflection points. Since the number of inflection points, i, is 0, we can use this count together with a count of the number of crossings to determine the number of exterior double tangents of these curves by applying Fabricius-Bjerre's formula.

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Chapter 1

Introduction

This paper explores questions regarding double tangents of closed differentiable curves in the affine plane without inflection points. Smooth closed curves in the plane have been studied in a variety of contexts. This introduction will give a brief overview of some work that has been done on plane curves, and important connections to topology.

There is a parallel between knot theory and the theory of plane curves because while knots are embeddings of a circle in $\mathbb{R}^3$, plane curves are immersions of the circle into the plane. It is useful to manipulate and visualize knots using knot projections in the plane, which are smooth closed curves. Since the crossings of the knot correspond to the crossings (double points) of the closed curve (projection), it is easy to generate the knot from its projection in the plane by distinguishing the under-strand from the over-strand at each crossing. Many people have also studied projections of knots in the 3-sphere to the 2-sphere [1]. I will discuss closed curves in the 2-sphere and in the real projective plane in chapter 3.

V. I. Arnold has done work on invariants of plane curves [3]. He examined three types of equivalences of plane curves. A *self-tangency* is a point at which the oriented plane curve is tangent to itself. A *direct self-tangency* is a self-tangency such that the positive half-tangents at the point of tangency point in the same direction, and an *inverse self-tangency* is a self-tangency such that the positive half-tangents at the point of tangency point in opposite directions. A *triple point* is a crossing of the oriented plane curve in which three arcs of the curve cross. A path $\Phi_t : S^1 \to \mathbb{R}^2$ is *generic* if for all t, Φ_t contains at most one direct self-tangency, inverse self-tangency, or triple point.

Two plane curves $\Phi_0, \Phi_1 : S^1 \to \mathbb{R}^2$ are J^+*-equivalent* if there exists a generic path Φ_t between them such that, for all t, the path does not contain any immersions with direct self-tangencies.

Two plane curves $\Phi_0, \Phi_1 : S^1 \to \mathbb{R}^2$ are J^-*-equivalent* if there exists a generic path Φ_t between them such that, for all t, the path does not contain any immersions with inverse self-tangencies.

Two plane curves $\Phi_0, \Phi_1 : S^1 \to \mathbb{R}^2$ are St*-equivalent* if there exists a generic path Φ_t between them such that, for all t, the path does not contain any immersions with triple points.

The Arnold invariants of plane curves determine when two plane curves are not J^+-/J^--/St-equivalent respectively, using the *discriminant hypersurface* for the three types of equivalence. I will not go into detail about that here. It is discussed in Emi Arima's masters thesis [3], which I read in preparation for this thesis.

At the intersection of knot theory and contact topology is the study of *Legendrian knots* [7]. A *plane field* is like a vector field, except instead of assigning a vector to each point in space, a plane field assigns a plane of directions to each point. A *contact structure* on $\mathbb{R}^3$ is a special type of plane field such that there is no surface in $\mathbb{R}^3$ whose tangent planes are all part of the contact structure. There are curves, called *Legendrian curves*, whose tangent vectors do lie in a contact structure, and a knot in $\mathbb{R}^3$ which is a Legendrian curve is called a *Legendrian knot*.

A contact structure is described by (q, p, z) coordinates in $\mathbb{R}^3$, and a Legendrian knot can be given by $\gamma(t) = (q(t), p(t), z(t))$. The constraints imposed on a knot in order for it to be Legendrian are best visualized by looking at projections of the knot to the $qz-$ and $qp-$ planes, and the p coordinate of a Lengendrian curve is determined by the slope of the $qz-$projection [7]. Any closed curve in the $qz-$plane, smoothly immersed except at finitely many cusps and having no vertical tangent lines, is the $qz-$projection of a Legendrian knot.

Thus, the study of plane curves is important for understanding Legendrian knot theory, which provides tools for the investigation of contact topology and knot theory [7].

In this paper, I am particularly interested in a line of research on smooth plane curves started by Fabricuis-Bjerre. In 1962, he presented a theorem [4] that I will discuss in chapter 2, and generalize in section 4.1. This theorem and generalization play a very important role in the proofs of all the other results in this paper.

In chapters 2 and 3, I will discuss Fabricius-Bjerre's main theorem and its history. In chapter 4, I will present my own generalization of this theorem to component curves overlayed in the plane. Finally, in chapter 5, I will present a method for counting the number of double tangents of a subset of closed curves in the plane without inflection points. The thesis concludes with a chapter on questions to look at further which goes into detail on some observations and conjectures I have made about plane curves.

Chapter 2

Fabricius-Bjerre's Main Theorem

This chapter discusses the main result of Fabricius-Bjerre's paper [4] and its proof. In section 4.1, I will generalize the theorem. Fabricius-Bjerre's entire proof is included in this chapter because it is closely related to the proof of the generalization.

2.1 Statement of the Theorem

Before stating Fabricius-Bjerre's main theorem, we need some definitions.

Let c be a closed, differentiable, oriented, curve in an affine plane, such that

1. c consists of a finite number of convex arcs that do not touch one another other than at crossings (there are no self-tangencies).
2. Crossings are *simple*, meaning that through each crossing passes exactly two branches of the curve (crossings are double points).

Definition 1. [4] *A* ***double tangent*** *is a line which is tangent to c at exactly two points.*

Assume that no double tangents are tangent at an inflection point of c. Then, each point of contact with a double tangent is an interior point of a convex arc belonging to c.

Definition 2. [4] *An* ***exterior double tangent*** *is a double tangent in which the convex arcs in the neighborhood of the points of contact are on the same side of the line.*

Definition 3. [4] *An* ***interior double tangent*** *is a double tangent in which the convex arcs in the neighborhood of the points of contact are on opposite sides of the line.*

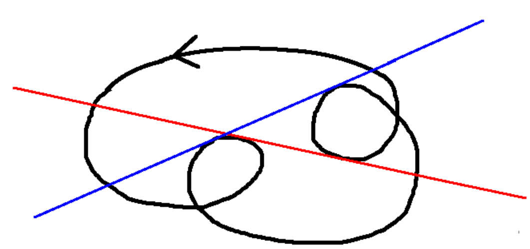


Figure 2.1: For this closed, differentiable, oriented, curve in the plane, there are two crossings and no inflection points. An exterior double tangent is shown in blue and and interior double tangent is shown in red.

The assumptions about c imply that c has a finite number of crossings, inflection points, and double tangents. Let t denote the number of exterior double tangents of c, s denote the number of interior double tangents of c, d denote the number of crossings of c, and i denote the number of inflection points of c. The following is the statement of Fabricius-Bjerre's main theorem:

Theorem 1. [4] *For the closed curve c, the difference between the numbers of exterior and interior double tangents is equal to the sum of the number of crossings and half the number of inflection points:*

$$t - s = d + \frac{1}{2}i \tag{2.1}$$

2.2 Proof of the Theorem

The proof involves categorizing exterior and interior double tangents into three different types based on the orientation of the closed curve.

- An exterior double tangent (respectively, interior double tangent) is type E_1 (resp. I_1) if the positive half-tangents at each point of tangency point in the same direction.
- An exterior double tangent (resp. interior double tangent) is type E_2 (resp. I_2) if the positive half-tangents at each point of tangency point towards each other.
- An exterior double tangent (resp. interior double tangent) is type E_3 (resp. I_3) if the positive half-tangents at each point of tangency point away from each other.

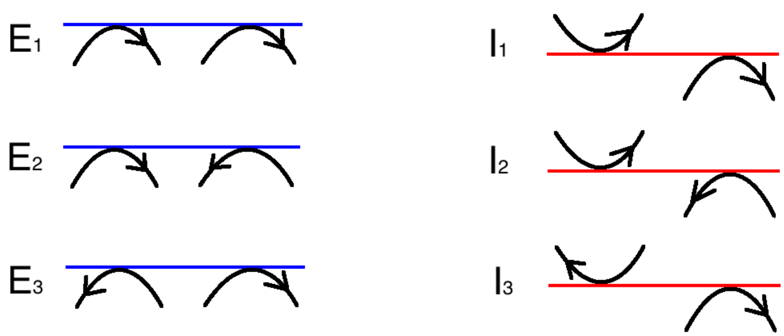


Figure 2.2: The three types of exterior and interior double tangents. The exterior double tangents are in blue, the interior double tangents are in red, and the convex arcs in the neighborhood of the points of contact with the double tangent are in black, with orientation noted.

Now, we are ready to prove Fabricius-Bjerre's main theorem.

Proof. [4] Let P be a point that traverses the curve c. The number of points common to c and the positive half-tangent $p+$ or the negative half-tangent $p-$ will remain unchanged except when P passes a crossing, inflection point, or if the tangent line p passes a double tangent. When P passes a crossing or inflection point, a common point of c and the tangent line p will go from $p+$ to $p-$. That is, one point on $p+$ is lost and one point on $p-$ is gained. By looking at Figure 2.3, we can intuitively see how the number of points of intersection of the green tangent line with the black curve change as P traverses the black curve in the direction of the orientation, passing through a crossing or an inflection point.

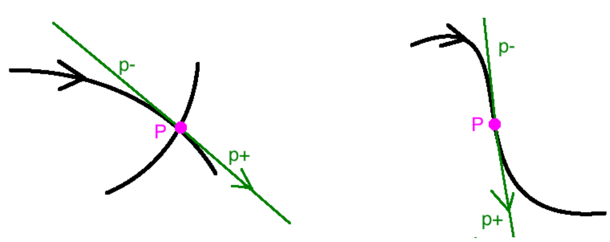


Figure 2.3: P passing through a crossing and an inflection point as it traverses the curve.

When p passes a double tangent of the type E_1 (resectively, I_1), two points are gained (respectively, lost) on $p+$ by the first passing, and two points are lost (respectively, gained) on $p-$ by the second passing. When p passes a double tangent of the type E_2 (resectively, I_2), two points are gained (respectively, lost) on $p+$ by the each passing. For the type E_3 (respectively, I_3), two points are lost (respectively, gained) on $p-$ by each passing.

The curve c has d crossings and i inflection points (since c is closed, i is an even integer, so $\frac{1}{2}i$ is an integer). As P traverses the curve once, it goes through each crossing twice, and each inflection point once. Let c have t_1, t_2, and t_3 exterior double tangents of types E_1, E_2, and E_3, respectively, and s_1, s_2, and s_3 interior double tangents of types I_1, I_2, and I_3, respectively. As P traverses c once, since c is closed, the number of points of c gained on $p+$ must be equal to the number of points lost on $p+$. This means

$$2t_1 + 4t_2 = 2s_1 + 4s_2 + 2d + i \tag{2.2}$$

Similarly for the negative half-tangent $p-$, we have

$$2t_1 + 4t_3 = 2s_1 + 4s_3 + 2d + i \tag{2.3}$$

Adding equations (2.2) and (2.3) gives

$$4t_1 + 4t_2 + 4t_3 = 4s_1 + 4s_2 + 4s_3 + 4d + 2i \tag{2.4}$$

Equivalently,

$$4(t_1 + t_2 + t_3) - 4(s_1 + s_2 + s_3) = 4d + 2i \tag{2.5}$$

Dividing everything by 4 gives

$$t - s = d + \frac{1}{2}i \tag{2.6}$$

where t is the total number of exterior double tangents and s is the total number of interior double tangents. This proves the theorem. □

Fabricius-Bjerre makes the following remarks that follow immediately from the theorem [4]:

- The number of exterior double tangents is greater than or equal to the sum of the number of crossings and half the number of inflection points ($t \geq d + \frac{1}{2}i$). Equality holds if and only if c has no interior double tangents.
- The total number of double tangents, $t + s$, has the same parity as the number $d + \frac{1}{2}i$.
- If c has no inflection points, $t - s = d$

I am particularly interested in curves, c, without inflection points. I will present a method for counting the number of interior double tangents of some types of such curves. That, along with a count of the number of crossings, will give us the total number of double tangents, because $t - s = d$.

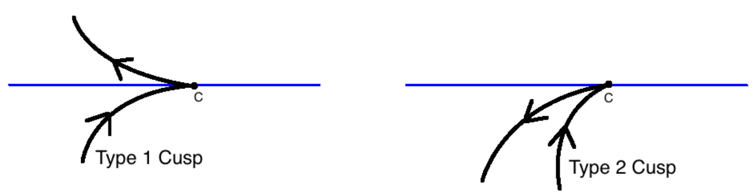
Chapter 3

History of the Theorem

This chapter discusses some results that followed Fabricius-Bjerre's main theorem. I will not go into the details of their proofs.

3.1 Fabricius-Bjerre's Own Extension to Curves with Cusps

Fabricius-Bjerre extended his own theorem to curves with cusps in 1977 [5]. Let γ be a closed curve in the affine plane as before, except now we allow cusps. A neighborhood of a cusp point, C, is composed of two convex arcs of the curve without common points whose endpoint is at C. A *type 1 cusp* is a cusp in which convex arcs lie on opposite sides of the tangent line at C and a *type 2 cusp* is a cusp in which convex arcs lie on the same side of the tangent line at C.



Fabricius-Bjerre's formula (2.1) assumes that the double tangents are *double supporting lines* (lines that touch the curve at exactly two points). A double supporting line can be a double tangent, a tangent line through a cusp, or a line through two cusps. We can define a double supporting line as *exterior* or *interior* as before, depending on whether the neighboring arcs of the points of contact are on the *same* or *opposite* sides of the line, respectively. We assume that through each crossing passes exactly two arcs of the curve, as before.

Theorem 2. [5] *Let γ have t exterior double supporting lines, s interior double supporting lines, d crossings, i inflection points, c_1 type 1 cusps, and c_2 type 2 cusps. Then,*

$$t - s = d + \frac{1}{2}i + c_1 + \frac{1}{2}c_2 \tag{3.1}$$

The proof is very similar to the proof of the main theorem given in chapter 2. All that changes is that we take into consideration how the number of intersection points between the curve and the positive and negative half-tangents change as a point passes through the two types of cusps as it traverses the curve.

3.2 Extension to Curves on S^2

In 1987, Joel Weiner obtained an analogous formula [9] for closed curves on the 2-sphere by adapting Fabricius-Bjerre's original proof so that it works for closed spherical curves (immersions of the circle into the 2-sphere).

Let γ be a closed spherical curve on the 2-sphere of radius 1. That is, γ is an immersion of the circle, C, into the 2-sphere ($\gamma : C \to S^2$). $Q \in S^2$ is a *crossing* of γ if $\gamma^{-1}(Q)$ contains more than one point of C. We assume each crossing has precisely two preimages in C.

$\{P, \overline{P}\}$ is called an *antipodal pair* if P and $\overline{P}$ are antipodes on S^2 and P and $\overline{P}$ are points on the curve γ. An *inflection point* is a point in which the geodesic curvature of γ is 0, and the derivative of the geodesic curvature with respect to arc length is non-zero. Assume points of antipodal pairs are not crossings or inflection points. Assume also that inflection points are not crossings.

A double tangent of γ is a geodesic (great circle) that is tangent to γ at two distinct points. Assume that each point of tangency is not a crossing, point of an antipodal pair, or inflection point. The great circle is an *exterior* double tangent of the curve if the curve near the points of tangency lie on the same side of the geodesic and *interior* if not, as before.

Theorem 3. [9] *Let γ have t exterior double tangents, s interior double tangents, d crossings, i inflection points, and a antipodal pairs. Then,*

$$t - s = d - a + \frac{1}{2}i \tag{3.2}$$

The proof is very similar to Fabricius-Bjerre's original proof, replacing positive and negative half-tangents with half-geodesics (of length π). The main difference is that we observe that when a point traversing the curve passes a point of an antipodal pair, the changes in the number of intersections between the curve and the positive and negative half-geodesics are the opposite of such changes when a point traversing the curve passes a crossing. Thus, we add $-a$ to d in Fabricius-Bjerre's original formula.

This formula can tell us interesting properties about closed curves in Euclidean 3-space (knots). Let $\alpha : C \to \mathbb{R}^3$ be a $\mathscr{C}^4$ immersion, with positive curvature, κ, on C. Let $\gamma : C \to S^2$ be the *tangent indicatrix* of α, meaning that $\gamma(x) = \alpha'(x)$ for all $x \in C$ where α' denotes differentiation of α with respect to arc length. Thus, γ is the curve that is traced on the unit sphere by fixing the tail of each of the unit tangent vectors of α at the origin.

Applying Weiner's formula to the tangent indicatrix provides information about the curve in 3-space. In particular, a crossing of γ corresponds to a pair of parallel tangent vectors of α that point in the same direction (*directly* parallel tangents) and an antipodal pair of γ corresponds to a pair of parallel tangent vectors of α that point in opposite directions (*oppositely* parallel tangents). Let τ be the torsion of α. A point $\alpha(x)$ is a *vertex* of α if $\tau(x) = 0$ and $\tau'(x) \neq 0$. Thus, a vertex of α corresponds to an inflection point of γ.

The *binormal vector* of α at $\alpha(x)$ is the vector, $\beta(x)$, that is orthogonal to the osculating plane, $\mathcal{O}(x)$, of α at $\alpha(x)$ (The tangent vector always lies in this plane). Like tangent vectors, pairs of binormal vectors can be directly parallel or oppositely parallel. Weiner [9] explains that a double tangent of γ with tangencies at $\gamma(x)$ and $\gamma(y)$ corresponds to a pair of parallel osculating planes to α, $\mathcal{O}(x)$ and $\mathcal{O}(y)$. It follows that $\beta(x) = \pm\beta(y)$, and when the double tangent is exterior, α passes through $\mathcal{O}(x)$ and $\mathcal{O}(y)$ in the same direction (*concordant* parallel osculating planes), and when the double tangent is interior, α passes through $\mathcal{O}(x)$ and $\mathcal{O}(y)$ in opposite directions (*discordant* parallel osculating planes).

Theorem 4 follows from Theorem 3 and these parallels. We say that a closed space curve is *generic* when its tangent indicatrix satisfies all the conditions of a closed spherical curve discussed in the beginning of this section.

Theorem 4. [9] *Let α be a generic closed space curve with positive curvature. Then,*

$$t - s = d - a + \frac{1}{2}i \tag{3.3}$$

where

- *t = the number of concordant parallel osculating planes of α*
- *s = the number of discordant parallel osculating planes of α*
- *d = the number of directly parallel tangents of α*

- *a = the number of oppositely parallel tangents of α*
- *i = the number of vertices of α*

Theorem 4 has some interesting consequences that Weiner [9] proved:

1. The number of directly parallel binormals minus the number of oppositely parallel binormals of α is equal to the number of directly parallel tangents minus the number of oppositely parallel tangents of α, $d - a$.
2. α must posess a pair of parallel tangents or a pair of parallel osculating planes.
3. If α has nonvanishing torsion, then α posesses a pair of parallel principal normals.

3.3 First Extension to $\mathbb{R}P^2$

Roberto Pignoni published the first extension of the formula to $\mathbb{R}P^2$ in 1993 [6]. He extended Fabricius-Bjerre's theorem with cusps in section 3.1 to curves in the real projective plane, $\mathbb{R}P^2$.

The real projective plane can be thought of as an extension of the real Euclidean plane $\mathbb{R}^2$. In $\mathbb{R}^2$, a pair of parallel lines never intersect, but $\mathbb{R}P^2$ is equipped with "points at infinity" at which parallel lines intersect. To turn the ordinary Euclidean plane into $\mathbb{R}P^2$,

1. For each class of parallel lines (set of lines with the same slope), assign a single new "point at infinity." It is considered the point of intersection of any two lines in the class of parallel lines.
2. Add a new line that consists of all the points at infinity (and only them). This is the "line at infinity".

Because of this construction, there is a duality between points and lines in $\mathbb{R}P^2$. It is described in the next section. Consider the surface of the unit sphere, S^2, centered at the origin in $\mathbb{R}^3$. Each line in $\mathbb{R}^3$ through the origin intersects S^2 at two antipodal points. Each of these lines is a representative for a class of parallel lines. Thus, a pair of antipodal points on the surface of the 2-sphere is the same as a single point in $\mathbb{R}P^2$. $\mathbb{R}P^2$ has the 2-sphere, S^2, as its double cover and a line in $\mathbb{R}P^2$ is the image of a great circle on the 2-sphere. The intersection of any two great circles on S^2 is a pair of antipodal points, which is a point in $\mathbb{R}P^2$, and any two points in $\mathbb{R}P^2$ have one line (great circle) that is incident to both of them.

Pignoni's generalization depended on the selction of a base point for the curve. He encountered difficulties due to problems with distinguishing between two "sides" of a closed geodesic in $\mathbb{R}P^2$, which are overcome by careful attention to the natural metric on the space, which was done in Thompson's paper in 2006 [8]. Thompson resolved the issues raised in Pignoni's paper, so the details of the generalization of Fabricius-Bjerre's theorem to $\mathbb{R}P^2$ is given in the next section.

3.4 General Extension to $\mathbb{R}P^2$

Abigail Thompson generalized Fabricius-Bjerre's formula for curves in $\mathbb{R}P^2$ in 2006 [8]. Consider $\mathbb{R}P^2$ with the spherical metric, inherited from its double cover, the 2-sphere of radius 1. A simple closed geodesic (projective line) in $\mathbb{R}P^2$ has length π.

Let K be a generic (definition 4) oriented closed curve in $\mathbb{R}P^2$, which is smoothly immersed except it has type 1 cusps (type 1 cusps are defined the same way as in section 3.1).

- Let τ_p be the geodesic tangent to K at p, with orientation induced by K.
- The *antipodal point* to p, a_p, is the point on τ_p a distance $\frac{\pi}{2}$ from p.
- An ordered pair (p, q) is an *antipodal pair* if $q = a_p$
- Let c_p be the center of curvature of K at p (the center of the osculating circle of K at p)
- Let v_p be the normal geodesic to K at p.

There is a duality between points and lines in $\mathbb{R}P^2$. A description of this duality is as follows [8]: The two-sphere, S^2, is the double cover of $\mathbb{R}P^2$; a simple closed geodesic in $\mathbb{R}P^2$ lifts to a great circle on S^2. If this is the equator, the dual point in $\mathbb{R}P^2$ is the image of the north (or south) pole.

A generic curve K has a *dual curve* K' under this duality. Let p be a point on K. A *dual point* to p, called p', is the point dual to τ_p, the tangent geodesic at p. The dual curve K' is the curve consisting of the dual points p' for all points p of K.

Let Y_p be the geodesic dual to the point c_p. Let (p, q) be an antipodal pair. Then Y_p and τ_p intersect at q and divide $\mathbb{R}P^2$ into two regions, R_1 and R_2. An antipodal pair, (p, q), is *type 1* if the geodesic tangent at q, τ_q, lies in the same region as c_p, and *type 2* if not [8].

The points p and c_p divide the normal geodesic v_p into two pieces, v_p+ from p to c_p and v_p- from c_p to p. An ordered pair of points (p, q) on K is a *normal-tangent pair* if $\tau_q = v_p$. A normal-tangent pair (p, q) is of *type 1* if q lies on v_p- *type 2* if q lies on v_p+ [8].

T is a *double supporting geodesic* of K if T is either a double tangent geodesic, a tangent geodesic through a cusp or a geodesic through two cusps. The two tangent or cusp points of K divide T into two segments, one of which has length less than $\frac{\pi}{2}$. A double supporting geodesic is *exterior* if the two points of K lie on the same side of the segment, and *interior* if the two points of K lie on the opposite side of the segment [8].

The tangent geodesics at a crossing of K define four angles, two of which, α and β are less than $\frac{\pi}{2}$. In a small neighborhood of a crossing, there are four segments of K. A crossing is *type 1* if one of these segments lies in α and one in β and a crossing is *type 2* if not [8] (see figure 3.1).

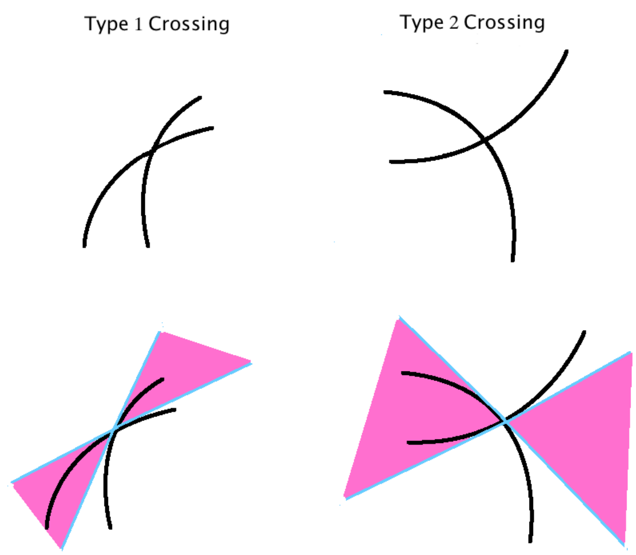


Figure 3.1: Types of crossings. The tangent lines at the crossing are colored blue, and the regions α and β are colored pink.

We can now define what it means for K to be a generic curve.

Definition 4. [8] *K is **generic** if:*

- *K has a finite number of crossings, double tangents, cusps, inflection points, antipodal pairs, and normal-tangent pairs.*
- *Tangent geodesics at self-intersections of K are neither parallel nor perpendicular.*
- *The tangent geodesic through an inflection point or at a cusp is everywhere else transverse to K.*
- *A geodesic goes through at most two tangent points or cusps of K.*
- *No crossings occur at inflection points.*

- *A geodesic normal to K at one point is tangent to K at at most one point and everywhere else transverse to K.*
- *The distance between two points on a double-supporting geodesic is not $\frac{\pi}{2}$.*
- *If (p, q) is an antipodal pair, let Y_p be the geodesic dual to c_p. Then τ_q should be neither τ_p nor Y_p.*
- *If (p, q) on K is a normal-tangent pair, q is not c_p.*

We are now ready to present the general extension of Fabricius-Bjerre's formula to curves in $\mathbb{R}P^2$.

Theorem 5. [8] *Let K be a generic, oriented, closed curve in $\mathbb{R}P^2$ with type 1 cusps. Let*

- *d_1 = the number of type 1 crossings of K,*
- *d_2 = the number of type 2 crossings of K,*
- *a_1 = the number of type 1 andipodal pairs of K,*
- *a_2 = the number of type 2 andipodal pairs of K,*
- *i = the number of inflection points of K,*
- *c = the number of (type 1) cusps of K,*
- *t = the number of exterior double supporting geodesics,*
- *s = the number of interior double supporting geodesics.*

Then,

$$t - s = d_1 + d_2 + \frac{1}{2}i + c - \frac{1}{2}a_1 + \frac{1}{2}a_2 \tag{3.4}$$

Because of the duality between points and lines in $\mathbb{R}P^2$, we have the following proposition:

Proposition 1. [8] *Let K be a generic curve in $\mathbb{R}P^2$ with dual curve K'. Then:*

1. *An exterior double tangent of K is dual to a type 1 crossing of K' and an interior double tangent of K is dual to a type 2 crossing of K'*
2. *A cusp of K is dual to an inflection point of K'*
3. *An antipodal pair of type 1 (respectively, type 2) in K is dual to a normal-tangent pair of type 1 (respectively, type 2) in K'.*

Because the dual of K' is K, these correspondences work in both directions.

Using this proposition, Thompson presented a dual formula for a generic singular curve in $\mathbb{R}P^2$ with type 1 cusps and no inflection points.

Theorem 6. [8] *Let K be a generic, oriented, closed curve in $\mathbb{R}P^2$ with type 1 cusps. Let*

- *d_1 = the number of type 1 crossings of K,*
- *d_2 = the number of type 2 crossings of K,*
- *n_1 = the number of type 1 normal-tangent pairs of K,*
- *n_2 = the number of type 2 normal-tangent pairs of K,*
- *i = the number of inflection points of K,*
- *c = the number of (type 1) cusps of K,*

- *t = the number of exterior double supporting geodesics,*
- *s = the number of interior double supporting geodesics.*

Then,

$$d_1 - d_2 = t + s + \frac{1}{2}c - \frac{1}{2}n_1 + \frac{1}{2}n_2 \tag{3.5}$$

3.5 *Duality Properties of Indicatrices of Knots*

As stated in the introduction, a knot is a simple closed curve in space. Recall from section 3.2 that the tangent indicatrix is a curve in S^2 generated by placing the tail of each unit tangent vector of a knot at the origin. We can do the same thing with each unit normal vector and each unit binormal vector. The normal vector of a knot, α, in $\mathbb{R}^3$ at $\alpha(x)$ is orthogonal to the tangent vector, $\alpha'(x)$, and lies in the osculating plane $\mathcal{O}(x)$, while the binormal vector at $\alpha(x)$ is orthogonal to both the tangent and normal vectors (orthogonal to the osculating plane $\mathcal{O}(x)$).

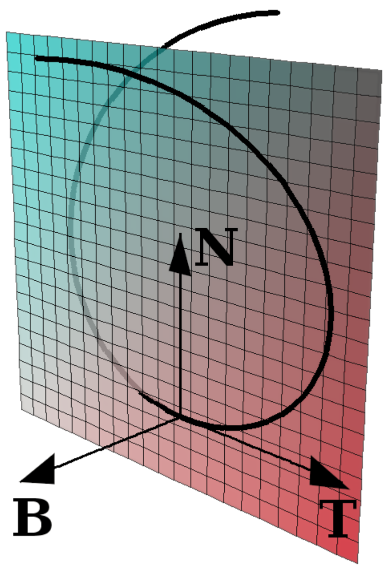


Figure 3.2: [10] An image of an osculating plane at a point on a curve in $\mathbb{R}^3$. The tangent, normal, and binormal vectors are labelled T, N, and B, respectively.

A knot can be generated by these three curves on S^2: the tangent indicatrix, the normal indicatrix, and the binormal indicatrix. This is one example of how curves in 2-dimensional space are related to knot theory. The paper titled *Duality Properties of Indicatrices of Knots* by Adams et al. [2] explains the duaility properties of these indicatricies. These properties are related to the duality discussed in Thompson [8]. As in Weiner [9], these relations can be applied to the curves on S^2 generated by these indicatrices of a knot in S^3. Thus, Fabricius-Bjerre's formula has many interesting applications to knot theory.

Chapter 4

Generalization of Fabricius-Bjerre's Main Theorem

Chapters 4 and 5 present the main results of this thesis. This chapter presents a generalization of Fabricius-Bjerre's main theorem to component curves overlayed in the plane. The proof is similar to his, and the results of this chapter are used in chapter 5 to count the number of interior double tangents of a subset of closed curves in the plane.

4.1 Two Component Curves

Fabricius-Bjerre's formula can be generalized to two curves that are overlayed in the plane, i.e., two curves in general position, such that their intersection is the set of crossings in which exactly one arc of one curve crosses over an arc of the other curve. We can look at a double tangent between the two curves as a line that has exactly one point of tangency on each of the curves (double tangent lines can be *interior* or *exterior*, as before. See Figure 4.1).

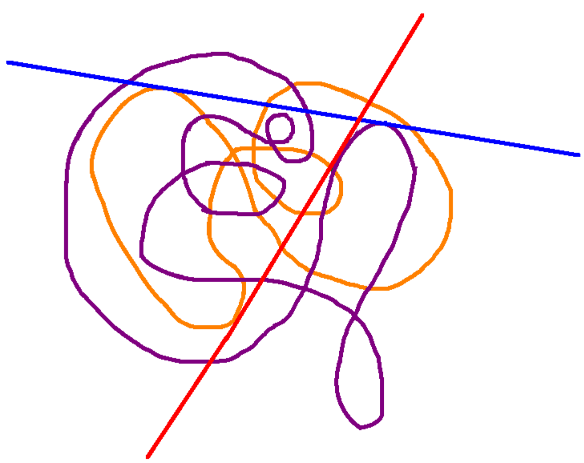


Figure 4.1: Examples of double tangents between two closed differentiable curves overlayed in the plane as described above. An exterior double tangent is shown in blue and an interior double tangent is shown in red.

Let γ and δ be two such curves overlayed in the plane. Let t_β, s_β, d_β, and i_β be the number of exterior double tangents, interior double tangents, crossings, and inflection points of any curve β, respectively. Let

- $t_{\gamma\cup\delta}$ be the number of exterior double tangents in which one point of tangency is on each of the two curves,

- $s_{\gamma\cup\delta}$ be the number of interior double tangents in which one point of tangency is on each of the two curves, and
- $d_{\gamma\cup\delta}$ be the number of crossings in which one arc of one curve crosses over an arc of the other curve.

Proposition 2. *Fabricius-Bjerre's formula holds for two curves overlayed in the plane. That is, for two curves, γ and δ, overlayed in the plane as decribed above,*

$$(t_\gamma + t_\delta + t_{\gamma\cup\delta}) - (s_\gamma + s_\delta + s_{\gamma\cup\delta}) = (d_\gamma + d_\delta + d_{\gamma\cup\delta}) + \frac{1}{2}(i_\gamma + i_\delta) \tag{4.1}$$

The proof is even more straightforward than the proof of Fabricius-Bjerre's theorem because now we only need to traverse one of the curves and observe the changes in the number of intersections between the positive and negative half-tangents and the other curve.

Proof. Let γ and δ be closed curves in the plane overlayed as described above. Since they are closed curves in the plane, Fabricius-Bjerre's formula holds for each of them individually. $t_\gamma - s_\gamma = d_\gamma + \frac{1}{2}i_\gamma$ and $t_\delta - s_\delta = d_\delta + \frac{1}{2}i_\delta$. Thus, it remains to show that $t_{\gamma\cup\delta} - s_{\gamma\cup\delta} = d_{\gamma\cup\delta}$.

Let a point P traverse γ. The number of points common to δ and the positive half-tangent $p+$ or the negative half-tangent $p-$ remains unchanged except when P passes a crossing of γ and δ or when $p+$ or $p-$ is tangent at δ.

When P passes a crossing of γ and δ, a common point of δ and the tangent line p goes from $p+$ to $p-$, that is, $p+$ loses one point on δ and $p-$ gains one point on δ.

We can categorize the exterior double tangents which have tangencies on both γ and δ into two different types depending on whether the positive half-tangent $p+$ points toward the point of tangency on δ (E_1), or away from it (E_2), and we can categorize interior double tangents between γ and δ the same way (I_1 and I_2).

As P traverses γ, when P passes a double tangent of the type E_1, $p+$ gains two points on δ. When P passes a double tangent of the type E_2, $p-$ loses two points on δ. When P passes a double tangent of the type I_1, $p+$ loses two points on δ, and when P passes a double tangent of the type I_2, $p-$ gains two points on δ.

Let γ intersect δ d times. Let t_1 and t_2 be the number of exterior double tangents of type E_1 and E_2, respectively, and let s_1 and s_2 be the number of interior double tangents of type I_1 and I_2, respectively. The number of points of δ that are gained on the positive half-tangent $p+$ are equal to the number of points of δ that are lost on the positive half tangent $p+$ as P traverses γ. Thus, we have the equation,

$$2t_1 = d + 2s_1 \tag{4.2}$$

Similarly, we have the analogous equation considering the negative half tangent $p-$,

$$2t_2 = d + 2s_2 \tag{4.3}$$

Adding those equations together, we have

$$2(t_1 + t_2) = 2d + 2(s_1 + s_2) \tag{4.4}$$

or

$$t = d + s \tag{4.5}$$

where t is the total number of exterior double tangents which have points of tangency on both γ and δ, s is the total number of interior double tangents which have points of tangency on both γ and δ, and d is the number of crossings of one arc of δ and one arc of γ. Thus,

$$t_{\gamma\cup\delta} - s_{\gamma\cup\delta} = d_{\gamma\cup\delta} \tag{4.6}$$

Adding this to the two equations given by Fabricius-Bjerre's formula holding for the two curves individually,

$$(t_\gamma + t_\delta + t_{\gamma\cup\delta}) - (s_\gamma + s_\delta + s_{\gamma\cup\delta}) = (d_\gamma + d_\delta + d_{\gamma\cup\delta}) + \frac{1}{2}(i_\gamma + i_\delta) \tag{4.7}$$

Therefore, Fabricius-Bjerre's formula holds for two curves overlayed in the plane. □

4.2 n Component Curves

Proposition 1 can be generalized to any number of closed curves overlayed in the plane.

Corollary 1. *For n closed differentiable curves overlayed in the plane as described above, consider all the double tangents: lines which have exactly two points of tangency with some curves. Let t be the total number of exterior double tangents and s be the total number of interior double tangents. Let d be the total number of crossings and i be the total number of inflection points. Then,*

$$t - s = d + \frac{1}{2}i \tag{4.8}$$

Proof. (By Induction)

Base Case(s): When n is 1, we have Fabricius-Bjerre's formula, which has already been proven. When n is 2, we have proposition 2, which has already been proven.

Induction Step: Suppose the equation holds for n component curves. Call this collection of n overlayed curves γ. Then t_γ and s_γ are the total number of exterior and interior double tangents, respectively, among all n curves. Also, d_γ and i_γ are the total number of crossings and inflection points, respectively, among all n curves.

Overlay another curve, say δ, so that the intersection of δ and γ is the set of crossings such that one arc of δ crosses over one arc of γ. By the same argument as the proof of proposition 1, $t_{\gamma\cup\delta} - s_{\gamma\cup\delta} = d_{\gamma\cup\delta}$. Since Fabricius-Bjerre's formula holds for δ alone, $t_\delta - s_\delta = d_\delta + \frac{1}{2}i_\delta$. By the induction hypothesis, $t_\gamma - s_\gamma = d_\gamma + \frac{1}{2}i_\gamma$. Adding these three equations together, $(t_\gamma + t_\delta + t_{\gamma\cup\delta}) - (s_\gamma + s_\delta + s_{\gamma\cup\delta}) = (d_\gamma + d_\delta + d_{\gamma\cup\delta}) + \frac{1}{2}(i_\gamma + i_\delta)$. This precisely means that the total number of exterior double tangents minus the total number of interior double tangents is equal to the total number of crossings plus half the total number of inflection points for $n + 1$ overlayed curves, which proves the claim. □

We have now shown that Fabricius-Bjerre's main theorem applies to any number of closed differentiable curves in the plane, and we will use this (corollary 1) in section 5.2.

Chapter 5

Scribbles

I am interested in counting the number of double tangents of any closed, oriented, differentiable curve in an affine plane without inflection points. I will call such curves *scribbles*. This chapter discusses a method for finding the number of interior double tangents of specific types of scribbles. By Fabricius-Bjerre's formula, $t = d + s$ for scribbles, so we can use this count along with a count of the number of crossings to determine the total number of double tangents.

5.1 Definitions

Definition 5. *A **scribble** is a closed, oriented, differentiable curve in an affine plane without inflection points.*

While studying the double tangents of such curves, I decided to divide scribbles into two types. We will first need a few more definitions.

Definition 6. *The **winding number** of a region bounded by arcs of a scribble is an integer, w. For an arbitrary point p in the region, let u be the number of times a point traversing the scribble travels counter-clockwise around p and let v be the number of times a point traversing the scribble travels clockwise around p. The winding number, w, of the region is defined as*

$$w = u - v \tag{5.1}$$

Remark 1. *A region's winding number does not depend on the choice of the point p in the region.*

Definition 7. *An **axis** is a region whose winding number is a local maximum (absolute value). That is, for any region that shares a boundary arc with an axis, the absolute value of its winding number is strictly less than the absolute value of the winding number of the axis.*

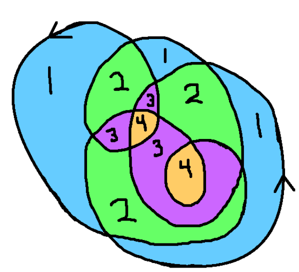


Figure 5.1: An example of a scribble with labelled winding numbers. Each region is colored corresponding to a different winding number. We consider the region outside the boundary of the curve (the exterior of the curve) to have winding number zero. There are two axes, each of which have winding number 4.

Now we are ready to define the two types of scribbles. Consider a path from a point in the exterior of the scribble to a point in a region bounded by arcs of the scribble. This path must cross a finite number (positive integer) of arcs of the scribble. Certain paths cross fewer arcs than others, and there exists a path that crosses the minimum number of arcs. Such a path that crosses the minimum number of arcs is called a ***minimal path***.

Definition 8. *A* ***type I scribble*** *is a scribble such that for all regions, R, bounded by arcs of the scribble, the absolute value of the winding number of the region, w_R, is equal to the minimum number of arcs of the scribble a path must cross from the exterior into R.*

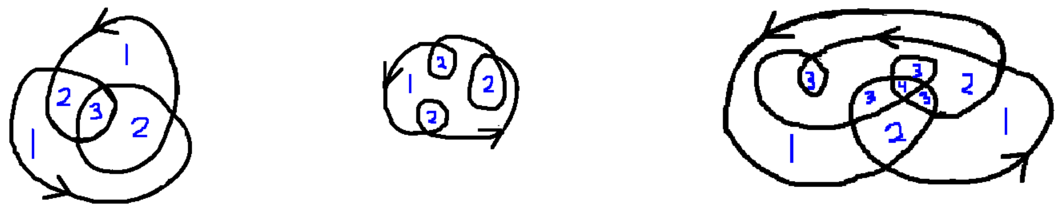


Figure 5.2: Examples of type I scribbles. Figure 5.1 is also a type I scribble.

Remark 2. *We can always orient a type 1 scribble so that the winding number of each region bounded by the curve is positive.*

Definition 9. *A* ***type II scribble*** *is a scribble that is not type I. That is, there exists a region R bounded by arcs of the scribble such that w_R is not equal to the minimum number of arcs of the scribble a path must cross from the exterior into R.*

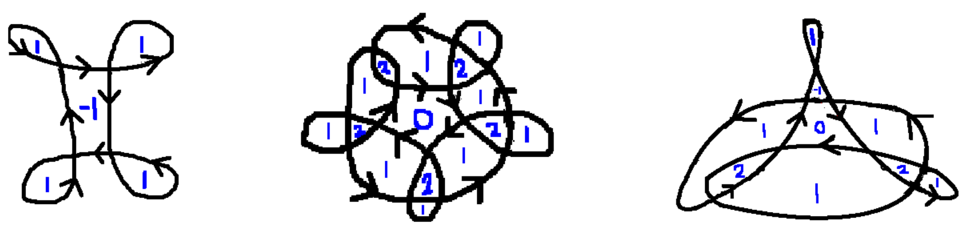


Figure 5.3: Examples of type II scribbles.

For the majority of this paper, I will work with type I scribbles. I will discuss some observations about type II scribbles in section 7.2.

Proposition 3. *Scribbles with one axis are type I scribbles.*

Proof. Let c be a scribble with one axis. Let x be the absolute value of the winding number of the axis. Then each region bounded by arcs of c that is not an axis has winding number strictly less than x.

There are no inflection points, so by the definition of winding number, a point traversing the curve in the counterclockwise direction travels around any point inside the axis x times. Thus, any path from a point in the exterior of c to a point in the axis crosses at least x arcs of c (x is the infemum). Choose a point in the exterior, B, and a point in the axis, A, such that the line segment $\overline{AB}$ does not go through a crossing. This is a path that crosses exactly x arcs because it does not go through any crossings and a point traversing c travels around A x times. Thus, x is the minimum number of arcs of c that a path from the exterior into the axis must cross.

Similarly, by definition of winding number and because there is one axis, for a region with winding number y ($< x$), there are no paths from a point in the exterior to a point in that region that cross fewer than y arcs. We can find a path from the exterior to a point in a region with winding number y that crosses exactly y arcs of c. Consider, WLOG, a region that is not an axis that contains part of the path, line segment $\overline{AB}$. For a point on the path in that region, D, the line segment $\overline{DB}$ is a path that crosses y arcs. Therefore, if a scribble has one axis, then the winding number of each region is equal to the minimum number of arcs a path from the exterior to a point in the region must cross, so it is type I. □

Lemma 1. *If a scribble has one axis, then it has no interior double tangents.*

Proof. Consider a scribble, c, with one axis, oriented counterclockwise such that the winding numbers of every region bounded by arcs of the curve are positive (remark 2). As a point P traverses c in the direction of the orientation, an arc of c or a region bounded by arcs of c is either to the left or to the right of the the line tangent to c at P. We will say these arcs or regions are to the left or to the right of the point P. (Imagine a person walking along the curve in the direction of the orientation. They can either look to the left or to the right). Since c is a scribble with one axis, as a point P traverses c in the direction of the orientation (counterclockwise), the following conditions must be true:

1. The axis is always to the left of P.
2. In a small neigborhood of P, the other points of the curve are to the left of P. That is, $\exists\ \delta > 0$ such that all points except P on the curve that are contained in the open ball $B(P, \delta)$ are to the left of P.

Suppose there exists an interior double tangent line of c. Then, there are arcs of c that are tangent on either side of the line, by definition of interior double tangent.

- ***Case 1:*** *The positive half-tangents at each of the points of tangency point in opposite directions (either towards or away from each other).* Since the axis is always to the left of a point traversing the curve (condition 1), we have that the axis must be on both sides of the tangent line. Contradiction.

Figure 5.4: Case 1 implies the axis is on both sides of the tangent line which is a contradiction.

- ***Case 2:*** *The positive half-tangents at each of the points of tangency point in the same direction.* For one of the points of tangency, Q, for any $\delta > 0$, there are always some points of the scribble in $B(Q, \delta)$ to the right of Q. Contradiction of condition 2.

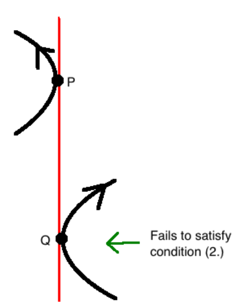


Figure 5.5: In case 2, there is always a point of tangency, Q, for which condition 2 does not hold. We have that in any small neighborhood of Q, there are points of the curve to the right of Q.

Therefore, a scribble with one axis can never have interior double tangents. □

5.2 Counting Double Tangents for One-axis Scribbles

Scribbles with one axis do not have any inflection points or interior double tangents. Thus, by Fabricius-Bjerre's theorem, the number of exterior double tangents is equal to the number of crossings ($t = d$). By Corollary 1 in section 4.2, it follows immediately that for n component scribbles that share a single axis, $t = d$ where t is the total number of exterior double tangents which have exactly two points of tangency with some curves and d is the total number of crossings among all n curves.

Proposition 4. *For two single-axis scribbles in the plane whose intersection is empty and whose axes have winding number x and y, respectively, the number of interior double tangent lines which have one point of tangency on each of the scribbles is $2xy$.*

Proof. Let γ and δ be two scribbles, each with one axis, drawn in the plane so that they do not intersect. Let the winding number of γ's axis be x and the winding number of δ's axis be y. Pick a point P in γ's axis and a point Q in δ's axis, and let ℓ be the line that contains P and Q. Pick P and Q such that ℓ does not pass through any crossings.

Without loss of generality, suppose ℓ is drawn horizontally, so the notions of "to the left" and "to the right" of a point on ℓ, as well as the notions of "above" and "below" ℓ, are well-defined. Since γ is a scribble with one axis, γ intersects ℓ x times to the left of P and x times to the right of P because the segment of ℓ between P and a point on ℓ in the exterior of γ is a path (line segment) from the exterior to a point in the axis.

Let $\{r_1, r_2, ..., r_x\}$ be the the set of points of intersection of γ and ℓ to the left of P, and let $\{s_1, s_2, ..., s_x\}$ be the the set of points of intersection of γ and ℓ to the right of P. By definition of axis and since γ only has one axis, since P is a point in the axis, a point traversing γ in the counterclockwise direction travels around P x times. Thus, each $r_i \in \{r_1, r_2, ..., r_x\}$ is connected to one $s_j \in \{s_1, s_2, ..., s_x\}$ by an arc with no self intersections above ℓ, and one $s_j \in \{s_1, s_2, ..., s_x\}$ by an arc with no self intersections below ℓ. Thus, there are x smooth arcs above ℓ and x smooth arcs below ℓ, with no self-intersections.

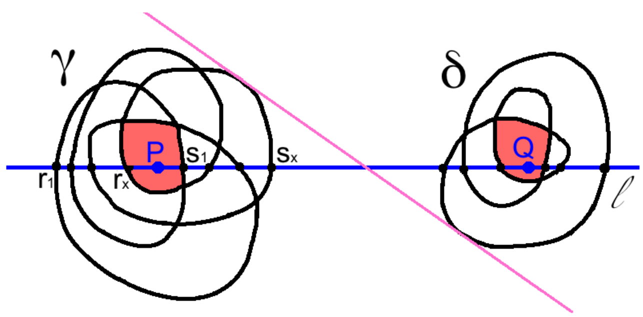


Figure 5.6: This example demonstrates the labelling discussed in the paragraph above. One interior double tangent is drawn in pink.

Similarly, δ intersects ℓ y times to the left of Q and y times to the right of Q, and we have y arcs with no self-intersections above and below the line ℓ surrounding the point Q.

We can draw double tangent lines that are tangent at one of these arcs of γ above ℓ and at one of these arcs of δ below ℓ (like the one drawn in figure 5.6). This is a double tangent in which the convex arcs of the two curves in the neighborhood of the points of contact with the double tangent are on opposite sides of the line, so it is an interior double tangent. Since there are x such arcs of γ and y such arcs of δ, there are xy of these interior double tangents between γ and δ. Similarly, we can draw tangent lines that are tangent at one of the arcs of γ below ℓ and at one of the arcs of δ above ℓ, and there are xy of those. Therefore, there are $2xy$ interior double tangents in which the points of tangency are on each of the two scribbles. □

Remark 3. *If γ and δ were curves that consisted of n component one-axis scribbles that share a single axis (with winding number x and y respectively), by corollary 1 in section 4.2, Fabricius-Bjerre's formula still holds, so there would still be $2xy$ interior double tangents.*

Proposition 4 can be generalized to any number of disjoint single-axis scribbles.

Corollary 2. *If there are n single-axis scribbles which are pairwise disjoint, the number of interior double tangents among all the scribbles is a sum of $\binom{n}{2}$ terms of the form $2xy$. That is, for n disjoint one-axis scribbles whose axes have winding numbers $x_1, ..., x_n$ respectively, the number of interior double tangents is*

$$\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} 2x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} 2x_i x_j - \sum_{i=1}^{n} 2x_i^2 \tag{5.2}$$

Proof. Since each scribble has one axis, by lemma 1, each scribble does not have any interior double tangents. Thus the only interior double tangents among all the scribbles are interior double tangents which have tangencies on two different scribbles. We can count them for each pair in the way that they are counted for the proof of proposition 4. Since there are $\binom{n}{2}$ pairs, the number of interior double tangents is a sum of $\binom{n}{2}$ terms of the form $2xy$ where x and y are the winding numbers of the axes of the scribbles in each pair.

Suppose the winding numbers of the axes of the n scribbles are $x_1, ..., x_n$. Counting the interior double tangents for each pair and taking their sum gives $\sum_{i=1}^{n} \sum_{j=1}^{n} 2x_i x_j - \sum_{i=1}^{n} 2x_i^2$.

$\sum_{i=1}^{n} 2x_i^2$ is subtracted because we cannot count interior double tangents for the same one-axis scribble, and the double sum $\sum_{i=1}^{n} \sum_{j=1}^{n} 2x_i x_j$ counts those. □

Remark 4. *(Generalization of remark 3) By corollary 1, if some of the n single-axis scribbles consist of component scribbles that share a single axis, corollary 2 still holds.*

5.3 Counting Double Tangents for Two-axis Scribbles

In this section, I will discuss a method for counting the number of interior double tangents of a type I scribble with two axes. Recall that a *minimal path* (from a point, a, in the exterior of the scribble to a point, b, in a region bounded by arcs of the scribble) is a path that crosses the minimum number of arcs of the scribble. All minimal paths for type I scribbles cross x arcs where x is the winding number of the region which contains the point b.

Consider a type I scribble with two axes, and let x and y be the winding numbers of the axes. Consider all the minimal paths from a point in the exterior to a point in an axis. Since it is a type 1 scribble, any minimal path from the exterior to a point in the axis with winding number x crosses x arcs of the scribble.

Let S be the set of all arcs of the scribble that are crossed in any minimal path to a point in the axis with winding number x. Let T be the set of all arcs of the scribble that are crossed in any minimal path to a point in the axis with winding number y.

Since it is a scribble, there is at least one arc that any minimal path to a point in either axis will cross, so $S \cap T \neq \emptyset$. I will say that the arcs in this intersection "wind around" both axes. To count interior double tangents for type I scribbles with two axes,

1. Identify the two axes, with winding numbers x and y respectively, and identify the region, c, with highest winding number that is bounded by arcs that wind around both axes. Let c have winding number z.

2. Cut the boundary of the region c at two points on either side of an axis.

3. Reconnect the cut strands so that c is divided into two regions that wind around each axis seperately. Now there is a new region of highest winding number that is bounded by arcs that wind around both axes (with winding number $z - 1$).

4. Repeat steps 2-4 for this new region z times. The result is two disjoint curves (single-axis scribbles or component scribbles that share an axis). There are $2xy$ interior double tangents between these two curves, by proposition 4 (and remark 3).

5. Rejoin the strands that had been cut in each iteration of step 2 as a connect sum z times, to arrive back at the original scribble. We lose the interior double tangents that were tangent on those strands each time we rejoin them, resulting in $2(x-1)(y-1)$ interior double tangents for the first rejoining, $2(x-2)(y-2)$ for the next rejoining...etc. Since we do this z times, the result is $2(x-z)(y-z)$ interior double tangents of the original scribble.

The main theorem of this paper comes from this method and is stated below.

Theorem 7. *For a type 1 scribble with two axes which have winding number x and y, let z be the winding number of the region with highest winding number bounded by arcs that "wind around" both axes. Then, the number of interior double tangents of the scribble is*

$$2(x-z)(y-z) \tag{5.3}$$

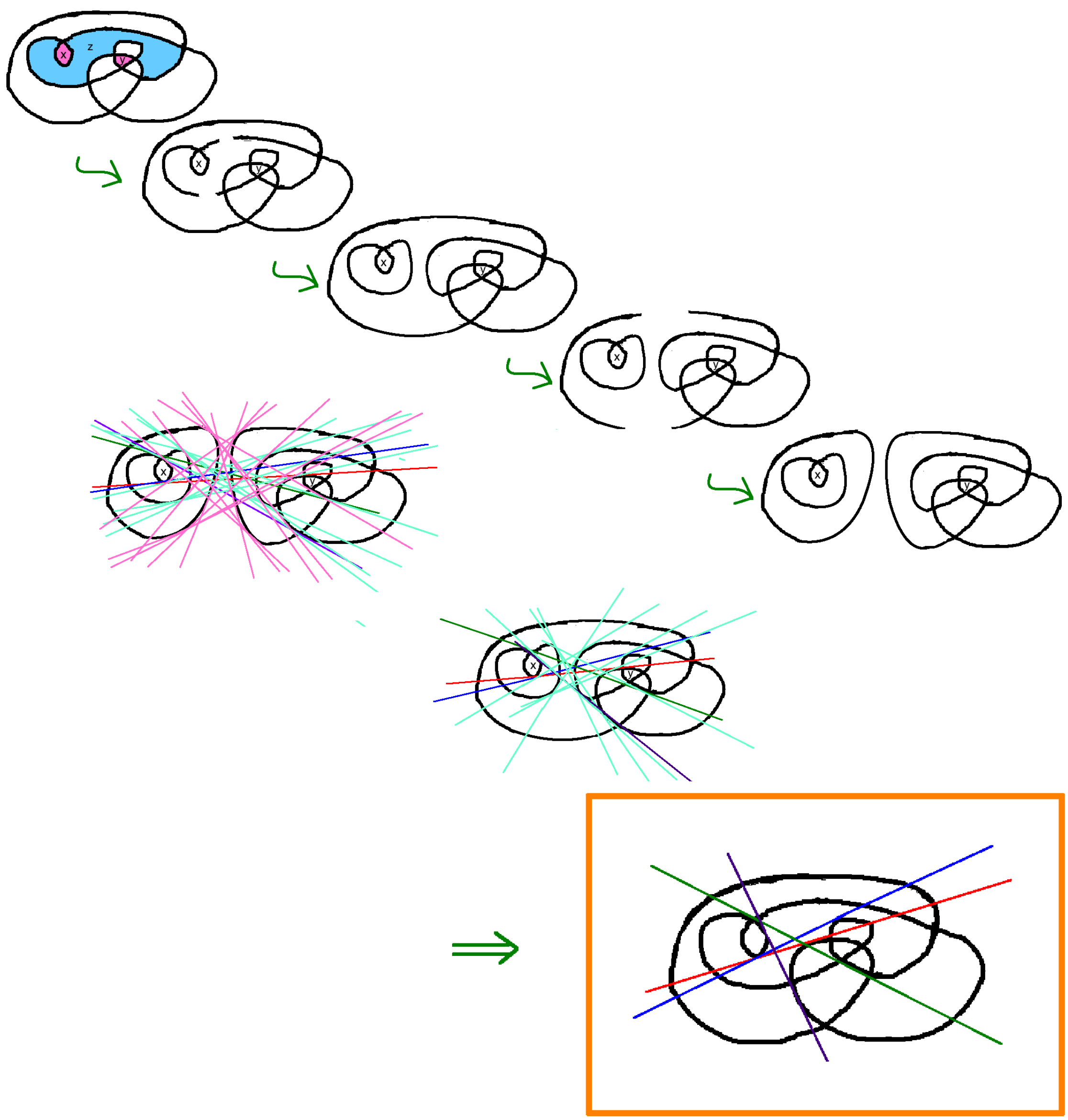


Figure 5.7: This example has $2(3-2)(4-2) = 2(1)(2) = 4$ interior double tangents. It has 8 crossings, so by Fabricius-Bjerre's formula, it has $t = d + \frac{1}{2}i + s = 8 + 0 + 4 = 12$ exterior double tangents. The axes are colored pink and the region c is colored blue.

Chapter 6

Conclusion

There are two main results of this thesis. First, I have generalized Fabricius-Bjerre's formula for any number of closed differentiable curves overlayed in the plane (chapter 4). Second, I have used that to prove that the number of interior double tangents, s, of a type I scribble with two axes is $2(x-z)(y-z)$ where x and y are the winding numbers of the axes and z is the winding number of the region with highest winding number that is bounded by arcs that any path from the exterior of the curve to a point in either axis must cross (chapter 5).

We can now use this count with Fabricius-Bjerre's main formula to conclude that the number of exterior double tangents, t, of a type I scribble with two axes is $d + 2(x-z)(y-z)$ where d is the number of crossings. Therefore, the total number of double tangents of any type I scribble with two axes is $t + s = d + 2(x-z)(y-z) + 2(x-z)(y-z) = d + 4(x-z)(y-z)$.

Other interesting results were found along the way. First, we have proposition 3 and lemma 1, which say that a one-axis scribble is always type I and never has interior double tangents ($t = d$). Second, we have proposition 4 and remark 3, which say that two disjoint, closed, differentiable curves in the plane with no inflection points that each have a single axis have $2xy$ interior double tangents where x and y are the winding numbers of the axes, respectively. Corollary 2 and remark 4 generalized that for n such curves.

In the next chapter, I will look at some open questions that arose through the work on this project including some of my conjectures and ideas for proofs of them.

Chapter 7

Questions to Look At Further

My results have brought up many questions to look at further. I have only found a method for counting double tangents for a specific subset of curves in the plane without inflection points. In sections 7.1 and 7.2, I will explain some ideas I have had about counting double tangents for other scribbles.

Some day, I would like to study how my results can be generalized to counting double supporting lines of curves with cusps, and methods for counting interior double tangents of curves with inflection points. Eventually, I would like to see how the properties I am finding about plane curves can be translated to properties about knots in $\mathbb{R}^3$. Also, I would like to discover what, if anything, changes when we consider scribbles in S^2 or $\mathbb{R}P^2$. Investigating these topics may lead to even more open questions.

For the remainder of this chapter, I will discuss some of my observations regarding double tangents of other scribbles. The "proofs" in these sections are only observations to support my conjectures.

7.1 Counting Double Tangents for Type I Scribbles with n axes

In this section, I will present the method we initially thought of for counting the number of interior double tangents of a scribble. The main conjecture of this section is Conjecture 3 on page 26, which counts the number of interior double tangents of a type I scribble with n axes. It involved using a representation of the scribble that (we think) preserves the number of interior double tangents that I will call the *nested circle representation*. We present definitions and the method for creating the nested circle representation of a type I scribble before discussing conjecture 3.

Definition 10. *The* ***crossing arcs*** *of a scribble are the parts of the curve near an exterior crossing which are on the exterior of the curve but not a part of the boundary of the convex hull of the curve (Fig. 7.1).*

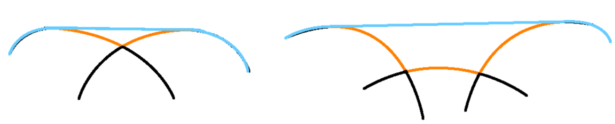


Figure 7.1: Examples of exterior crossings of a scribble. The boundary of the convex hull is shown in blue and the crossing arcs are shown in orange.

The method for constructing a nested circle representation for a scribble is given below in four steps:

1. **Draw the boundary of the convex hull.** This is the exterior circle in the nested circle representation. It is a simple closed curve in the plane.

2. **Change exterior crossings.** At each of the exterior crossings, remove the crossing arcs and "smooth out" the curve at each crossing so that the curve is differentiable everywhere. My conjecture is that this change does not change the number of interior double tangents.

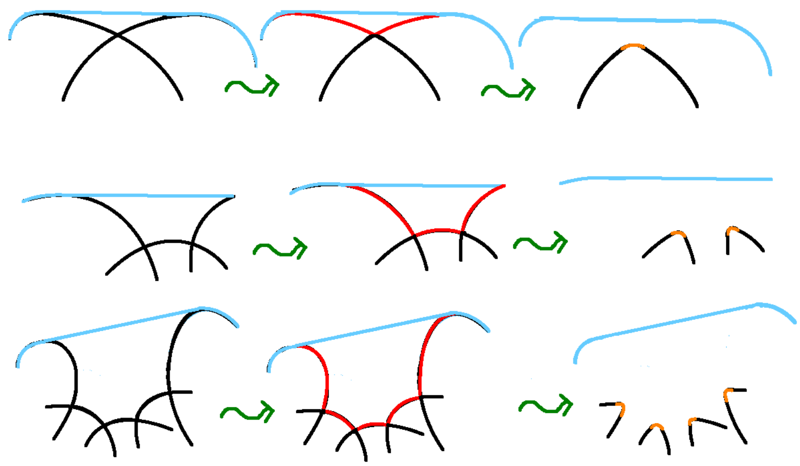


Figure 7.2: Some examples of what step 2 might look like. Exterior crossings are changed by removing the crossing arcs (red). The boundary of the convex hull is shown in blue. Finally, each crossing is smoothed out so that the resulting curve is differentiable everywhere (orange).

3. **Repeat.** After changing the exterior crossings, we are left with another scribble or overlayed scribbles nested inside a simple closed curve. Repeat this process again beginning with the first step on this new curve. The process is repeated $n - 2$ times, where n is the highest winding number among all regions of the scribble. This process results in a collection of simple closed curves that do not intersect, which are homeomorphic to nested circles.

Figure 7.3 shows this process of creating the nested circle representation for a particular scribble. The boundary of the convex hull is drawn in blue and the crossing arcs are shown in orange. The process is then repeated (step 3) on the next line with the boundary of the convex hull of the next iteration shown in purple. The steps of the process above are indicated with the green numbers. Since the highest winding number among all the regions in this scribble is 3, the process is repeated one time ($n - 2 = 3 - 2 = 1$).

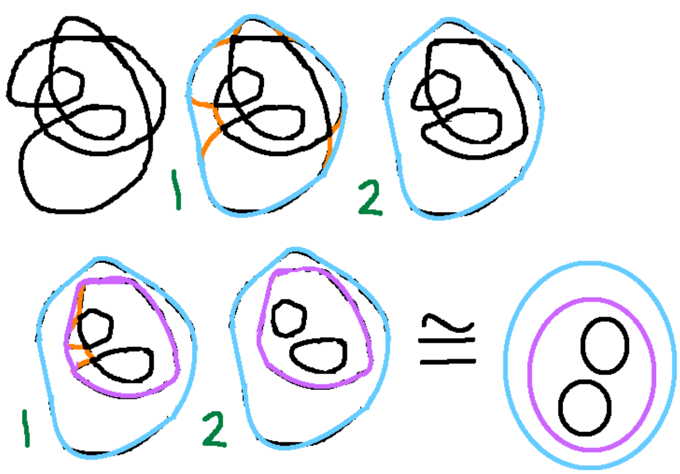


Figure 7.3: The method for creating a nested circle representation for a given scribble

Remark 5. *We can assign winding numbers to the regions bounded by nested circles in the same way we can assign winding numbers to the regions bounded by arcs of the scribble. In section 5.2, "the region with highest winding number which winds around both axes" (with winding number z) would be the region that corresponds to the region in the nested circle representation with highest winding number bounded by a circle that surrounds both of the circles that correspond to the boundaries of the axes.*

To show that the nested circle representation preserves the number of interior double tangents of a scribble, I will need the following definition.

Definition 11. *A tangent line **intersects a crossing specially** if the crossing is also a point on the line, such that the line goes between arcs that curve the same way as they come into or out of the crossing.*

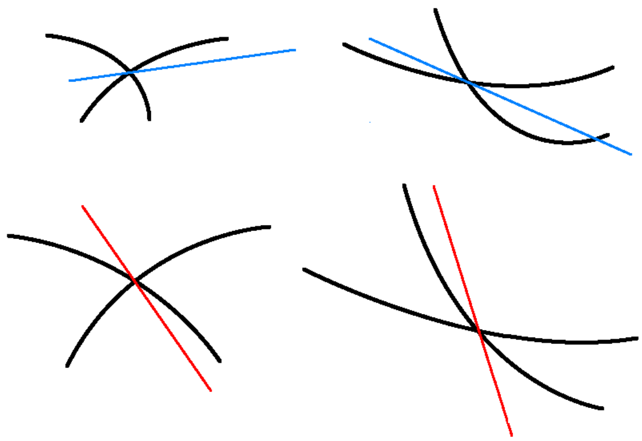


Figure 7.4: Suppose the black arcs are arcs of a scribble. Suppose the colored lines are tangent to the scribble at one point elsewhere. The blue tangent lines intersect a black crossing specially. The red lines, however, while they do pass through a crossing, do not intersect the crossing specially because there is no part of the line that goes between two arcs that curve the same way coming into or out of the crossing.

The following two conjectures are accompanied by "proofs" that are still a work in progress.

Conjecture 1. *For a crossing, c, there exists an interior double tangent which is tangent at one of c's crossing arcs if and only if there exists a tangent line which intersects c specially.*

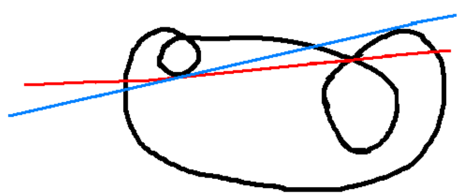


Figure 7.5: Conjecture 1 says the red tangent exists if and only if the blue double tangent exists.

Proof. (Idea) (⇒) Suppose there exists an interior double tangent which is tangent at a crossing arc of a given exterior crossing of a scribble.

- ***Case 1:*** *There is one point of tangency on an arc that is not a crossing arc.* Let p be the point of tangency at the crossing arc, and let q be the other point of tangency. The positive half-tangent at q (that is not at a crossing arc) either points toward or away from p. We can let q traverse the curve (and observe the tangent line as it traverses) in the direction toward p or away from p. Traverse the curve at q in the direction away from p until we have drawn a tangent that goes through the crossing. This will always happen because scribbles do not have inflection points. This is a tangent line which goes through a crossing and part of it lies between two arcs that curve the same way. Thus, it intersects the crossing specially.

- ***Case 2:*** *Both the points of tangency for the double tangent are on crossing arcs.* Choose one point of tangency to traverse away from the other point of tangency as in case 1. Just as in the proof for case 1, there is a tangent line that intersects the crossing specially. Similarly, if we had chosen the other point of tangency, we would get another tangent line that intersects a crossing specially. In this case, there are two tangent lines which intersect crossings specially since there are two crossings which have a double tangent that is tangent at one of their crossing arcs. They intersect specially by the same argument that is given in case 1.

($\Leftarrow$) (Proof by contraposition) Suppose for all interior double tangents, no points of tangency lie on a crossing arc. Then, if there are any tangent lines that go through crossings, they must not intersect the crossing specially (they must look like the red lines in figure 7.4). Thus, there are no tangent lines that intersect crossings specially. □

Conjecture 2. *The nested circle representation preserves the number of interior double tangents of a scribble.*

Proof. (Idea) Let c be a scribble with $n \geq 2$ interior double tangents.

- ***Case 1:*** *For all nested scribbles (or links of scribbles) formed in the repeated process above, no points of tangency of an interior double tangent lie on a crossing arc.* Then, no interior double tangents are lost when the crossing arcs are removed in step 2 (Fig. 7.3). By conjecture 1, there are no tangents that intersect a crossing specially corresponding to an interior double tangent (Fig. 7.5). Then, in step 2 when crossing arcs are removed, no new interior double tangents are formed when the crossings are smoothed out. Therefore, in this case, no interior double tangents are lost or gained in the process of creating the nested circle representation.

- textit**Case 2:** The negation of case 1: There exists an interior double tangent such that at least one of its points of tangency lies on a crossing arc of some scribble in some iteration of the repeated process above. That interior double tangent is lost when the crossing arcs are removed. However, for each of those that are lost, by conjecture 1, there is also a tangent that intersects that crossing specially. This tangent becomes a new interior double tangent that is gained and is tangent at the smoothed-out arc that is formed at the point where the crossing was. Therefore, since conjecture 1 is a biconditional statement, whenever an interior double tangent is lost, another is gained, and vice versa.

Therefore, by exhaustion, the number of interior double tangents of a scribble does not change through the process of creating a nested circle representation. □

Definition 12. *A **unit** in a nested circle representation is a collection of one or more parallel circles (for example, each colored collection of circles in figure 7.6 is a unit).*

Definition 13. *Two units are **disjoint** if neither is nested in the other in a nested circle representation (See figure 7.6).*

Definition 14. *The **winding number** of a unit is equal to the number of circles in the unit plus the sum of the number of circles in all units it is nested in.*

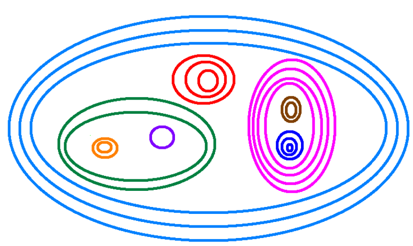


Figure 7.6: In this nested circle representation of a scribble, each colored collection of circles is a unit. For example, the red and green units are disjoint since neither unit is nested inside the other. The pink and brown units are not disjoint since the brown unit is nested inside the pink unit.

Remark 6. *(Conjecture:) In any type I scribble's nested circle representation, each pair of disjoint units is nested in another unit.*

Based on all the conjectures and definitions given in this section, we are now finally ready to make a claim about the number of interior double tangents of a type I scribble with n axes.

Conjecture 3. *For a type I scribble with n axes, ($n \geq 2$) the number of interior double tangents is a sum. The number of terms in the sum is the number of pairs of disjoint units in the nested circle representation. Let m be this number.*

Let T_i be the number of interior double tangents with tangencies at circles in the ith such pair of disjoint units in the nested circle representation, say a_i and b_i. Let x_i be the winding number of a_i and y_i be the winding number of b_i.

Let z_i be the winding number of the unit with highest winding number that surrounds a_i and at least one other unit that is disjoint to a_i (such a unit exists by remark 6). Let w_i be the winding number of the unit with highest winding number that surrounds b_i and at least one other unit that is disjoint to b_i (such a unit exists by remark 6).

Then $T_i = 2(x_i - z_i)(y_i - w_i)$ and

$$s = \sum_{i=1}^{m} T_i = \sum_{i=1}^{m} 2(x_i - z_i)(y_i - w_i) \tag{7.1}$$

where s is the number of interior double tangents of the scribble.

Proof. (By Induction)
Base Case: For a type I scribble with 2 axes, there are $2(x-z)(y-z)$ interior double tangents by Theorem 7. There is one pair of disjoint units because there are only two axes, so there is one term in the sum ($m = 1$). They are both nested in the same unit, so $z = z_i = w_i$ in this case. The winding number of the first axis is $x = x_i$ and the winding number of the second axis is $y = y_i$.

Induction Step: Suppose for some $n \geq 2$, conjecture 3 holds for any type I scribble with n axes. Consider a type I scribble with $n+1$ axes. Now, there exists another unit of circles that surround the $(n+1)^{th}$ axis and only this axis in the nested circle representation. Call this unit a. We are assuming that conjecture 3 holds for all pairs of disjoint units that don't include a.

Let p be the number of units which are disjoint with a (the circles in a are not nested inside p units). a introduces new interior double tangents that are tangent at the "top" of a circle in a and the "bottom" of a circle in each of those p units. Similarly, a introduces new interior double tangents that are tangent at the "bottom" of a circle in a and the "top" of a circle in each of those p units as well. This introduces p more terms of the form T_i to the sum of interior double tangents of the scribble, which is the number of pairs of disjoint units that have been added with the addition of a.

We assumed conjecture 3 held before a was added, and the addition of a to the nested circle representation adds exactly p more terms of that form to the number of interior double tangents. Therefore, the number of terms in the sum is the number of pairs of disjoint units in the nested circle representation. Each of these terms is in the form of T_i for each of these pairs of units since the interior double tangents are counted by looking at tangencies on the "top" and "bottom" of the nested circles. Thus, the conjecture holds when there are $n+1$ axes. □

Example 1:
Consider, for example, the scribble in figure 7.7:

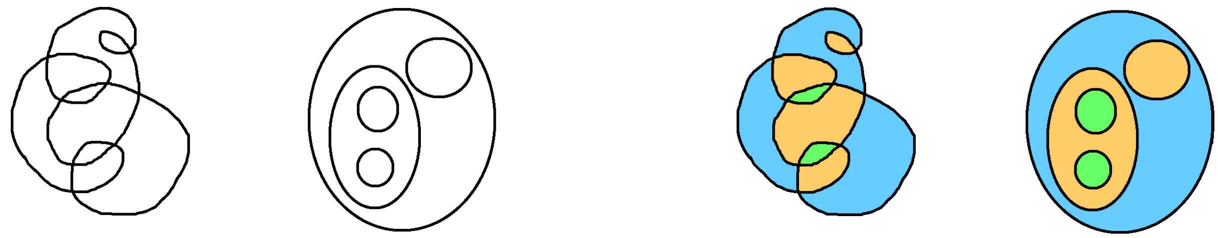


Figure 7.7: An example of a scribble and its corresponding nested circle representation. To the right, the scribble and representation are colored corresponding in regions with different winding number, to help see the correspondence between the scribble and its nested circle representation.

There are 3 axes, so $n = 3$. To count the number of disjoint pairs of units, we first note that there are 4 units that are nested in the outermost unit. Thus, there are $\binom{4}{2} = 6$ pairs of units nested in the outermost unit. Two of those units are nested in another of these four units, so two of those pairs of units are not disjoint. Then, there are $m = 6 - 2 = 4$ pairs of disjoint units. Thus, the number of interior double tangents on that scribble is a sum of 4 terms: $s = 2(3-2)(3-2)+2(3-2)(2-1)+2(3-2)(2-1)+2(2-1)(2-1) = 2+2+2+2 = 8$ interior double tangents.

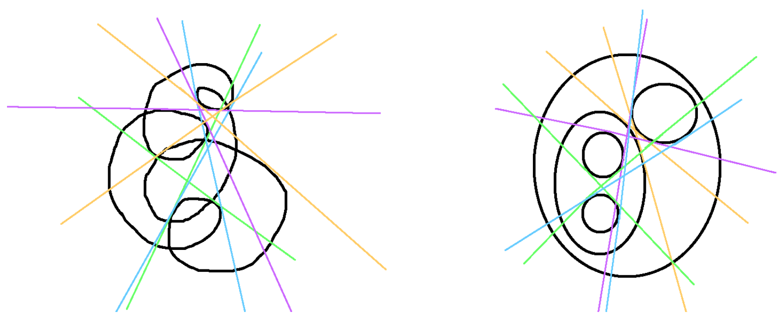


Figure 7.8: There are 8 interior double tangents for this scribble. Corresponding double tangents on the scribble and on its nested circle representation are color-coordinated.

Since there are 8 crossings and no inflection points, by Fabricius-Bjerre's formula, there are $t = d + \frac{1}{2}i + s = 8 + 0 + 8 = 16$ exterior double tangents, meaning that this scribble has a total of 24 double tangents.

Example 2:
Consider, for a more complicated example, the scribble in figure 7.7:

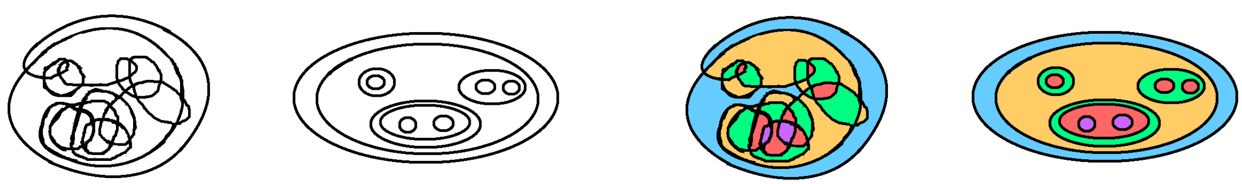


Figure 7.9: An example of a scribble and its corresponding nested circle representation. To the right, the scribble and representation are colored corresponding in regions with different winding number, to help see the correspondence between the scribble and its nested circle representation.

There are 5 axes, so $n = 5$. To count the number of disjoint pairs of units, we first note that there are 7 units that are nested in the outermost unit. Thus, there are $\binom{7}{2} = 21$ pairs of units nested in the outermost unit. Four of those units are nested in another of these seven units, so four of those pairs of units are not disjoint. Then, there are $m = 21 - 4 = 17$ pairs of disjoint units. Thus, the number of interior double tangents on that scribble is a sum of 17 terms: $s = 2(5-4)(5-4)+2(4-3)(4-3)+2(4-2)(4-3)+2(4-2)(4-3)+2(4-2)(5-4)+2(4-2)(5-4)+2(4-2)(4-2)+2(4-2)(3-2)+2(5-4)(4-3)+2(5-4)(4-3)+2(5-4)(4-3)+2(5-4)(4-3)+2(5-4)(3-2)+2(5-4)(3-2)+2(4-3)(4-2)+2(4-3)(4-2)+2(4-2)(3-2) = 2+2+4+4+4+4+8+4+2+2+2+2+2+2+4+4+4 = 56$ interior double tangents. Since there are 26 crossings and no inflection points, by Fabricius-Bjerre's formula, there are $t = d + \frac{1}{2}i + s = 26 + 0 + 56 = 82$ exterior double tangents, meaning that this scribble has a total of 138 double tangents.

7.2 Counting Double Tangents for Type II Scribbles

Counting double tangents of type II scribbles is more complicated. I made one observation for a particular subset of type II scribbles. Consider a type II scribble (as shown in figure 7.10), c, with just one region whose winding number is not equal to the minimum number of arcs a path must cross from the exterior into the region. There are n crossings on the boundary of that region.

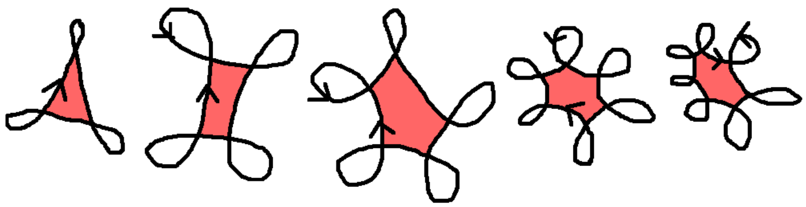


Figure 7.10: Examples on some of these type II scribbles for $n = 3, 4, ..., 7$. The region which defines the scribble as type II is colored.

Conjecture 4. *There are $n^2 - 3n$ interior double tangents of c, where c is a type II scribble with one region whose winding number is not equal to the minimum number of arcs a path must cross from the exterior into the region (as shown in Figure 7.10).*

Proof. (Observations) Let a be the one region that defines c as a type II scribble. Because there are no inflection points, there are n "loops" of the curve at each crossing on the boundary of the region a. The only interior double tangents are tangent on arcs of those loops. In particular, there are two interior double tangents that are tangent on every pair of nonadjacent loops.

We can look at the crossings that bound a as vertices of a graph, and connect two vertices that correspond to a pair nonadjacent loops (which each have two interior double tangents) with an edge. The number of edges is precisely the number of edges of the complete graph K_n, minus n because there are n edges of K_n that would correspond to an edge connecting adjacent vertices. Thus, the total number of edges of this graph is $\binom{n}{2} - n = \frac{n!}{2!(n-2)!} - n = \frac{n(n-1)}{2} - n = \frac{n^2-n-2n}{2} = \frac{n^2-3n}{2}$. Since there are 2 interior double tangents for every edge, the total number of interior double tangents is $2(\frac{n^2-3n}{2}) = n^2 - 3n$. □

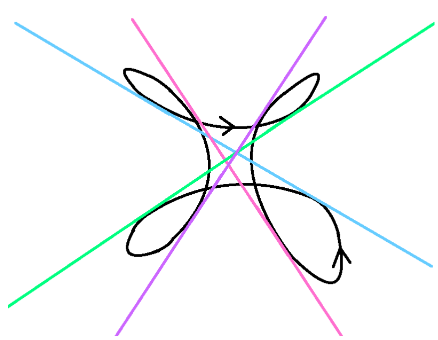


Figure 7.11: For example, this is a type II scribble with $n = 4$ "loops", and there are $n^2 - 3n = 4^2 - 3(4) = 16 - 12 = 4$ interior double tangents.

Bibliography

[1] Adams, Colin C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. W. H. Freeman and Company, New York, 1994.

[2] Adams, Colin; Collins, Dan; Hawkins, Katherine; Sia, Charmaine; Silversmith, Rob; and Tshishiku, Bena. *Duality Properties of Indicatrices of Knots*. 2000 Mathematics Subject Classification. Primary 53A04, Secondary 57M25. June 11, 2010.

[3] Arima, Emi. *The Arnold Invariants of Plane Curves* A.B./M.A. Thesis. Department of Mathematics, University of California, Davis. May 3, 2005

[4] Fabricius-Bjerre, Fr. *On the Double Tangents of Plane Closed Curves*. Math. Scand 11 (1962), 113-116.

[5] Fabricius-Bjerre, Fr. *A Relation Between the Numbers of Singular Points and Singular Lines of a Plane Closed Curve*. Math. Scand 40 (1977), 20-24.

[6] Pignoni, Roberto *Integral Relations for Pointed Curves in A Real Projective Plane*. Geom. Dedicata 45 (1993), 263-287.

[7] Sabloff, Joshua M. *What is...a Legendrian Knot?* Notices of the AMS, Vol. 26, No. 10. November, 2009.

[8] Thompson, Abigail. *Invariants of Curves in* $\mathbb{R}P^2$ *and* $\mathbb{R}^2$. Algebraic and Geometric Topology 6 (2006) 2175-2187.

[9] Weiner, Joel L. *A Spherical Fabricius-Bjerre Formula With Applications To Closed Space Curves*. Math. Scand 61 (1987), 286-291

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