Derivation of Finite Sums of Integer t-th Powers: A Historic Perspective

By

BENJAMIN BAIRRINGTON

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ABSTRACT.

In this senior thesis, the sum of integer t-th powers is first presented in a historical context following the writings of Blaise Pascal, John Wallis, and Jacob Bernoulli respectively. Next, a derivation of the sum of integer powers is presented in a novel way starting with nothing more than the fundamental principles of the field axioms along with some simple calculus. After that, the sum of integer t-th powers is lifted to the sum of real powers using an application of the Gamma function. Finally, the results presented are compared to the multidimensional case described by the Ehrhart Polynomials and respective theorem. In closing, a brief discussion is given describing various directions which future study can be engaged.

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CHAPTER 1

Historical Perspective

1.1. Introduction

The sums of integer t-th powers is one of the classical problems in 17^{th} century mathematics that is simple enough to possess a closed expression for a solution, yet is sufficiently complex to be worthy of study. Of course, in the modern era questions being asked of the expression are far different than the original inquiry. As an abbreviated statement, the original problem is as follows:

Find the closed expression for the sum of consecutive integer t-th powers of highest integer n, and power t as a function S, of n and t of the form:

$$S_{n,t} = \sum_{i=0}^{n} i^t.$$

From a broad perspective, there are undoubtedly countless authors who played their role in putting various pieces of this puzzle together. Unfortunately, the volume of work regarding this subject becomes vanishing small moving backwards in time. Moreover, such a task as to accommodate every reputable source would be exhausting in addition to losing sight on the elegance of the solution itself. Hence only three authors, and their contributions, are considered: Blaise Pascal, John Wallis, and Jacob Bernoulli. Each of these authors presented a novel perspective on the problem, illuminating a compelling fundamental relation between the solution and the arithmetical triangle. With this in mind, it is best to begin with a clear image of the arithmetical triangle. No such image could be more clear than to present the arithmetic triangle from the writings of the mathematician of whom the triangle is attributed to: Blaise Pascal.

1.2. Pascal's Arithmetic Relation

One may recall the arithmetical triangle by another name, *Pascal's Triangle*. For reference the double indexed sequence is given below in Figure 1 :

row/column	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0	0	0
4	1	3	3	1	0	0	0	0	0	0
5	1	4	6	4	1	0	0	0	0	0
6	1	5	10	10	5	1	0	0	0	0
7	1	6	15	20	15	6	1	0	0	0
8	1	7	21	35	35	21	7	1	0	0
9	1	8	28	56	70	56	28	8	1	0
10	1	9	36	84	126	126	84	36	9	1

FIGURE 1. A finite partition of the arithmetic triangle

This may appear slightly different than standard convention, although this presentation of the arithmetic triangle holds a lot more information that will be advantageous later on. Any proceeding references to the arithmetic triangle will be to a table of this form.

A complete history of the arithmetic triangle and the corresponding figurate numbers (a synonym for integer entries of the triangle) is a fascinating tale spanning around one-thousand eight hundred years [**5**]. To start from the beginning and lead up to Pascal's contribution would be a tad misleading as to the role that Pascal had with regards to the arithmetic triangle. Provided, his name is attributed to the arithmetic triangle for very good reasons. Although discovery is not one of them. Rather, it is his deep understanding of these numbers, and their application to a wide range of applications that these numbers are attributed to him.

One such application of particular note is the means of which the rows of the arithmetic triangle correspond to the coefficients of a binomial expansion. That is, *The Binomial Theorem.* In *Pascal's Treatise* part II, section four [**5**], states the binomial theorem without any demonstration (proof) regarding its form. Diverging our attention from Pascal, let us consider constructing the binomial theorem, first from intuition, and then as a formal proof.

Let a and b be two numbers, consider the following quantity:

$$(a+b)^1 = (1)a + (1)b.$$

The equivalence of the left and right hand sides should be clear, as a power of one by the definition of multiplication does not change the value of its quantity. Notice furthermore that an additional product by one for both numbers a and b was applied, which again by the standard definition of multiplication does not change the value. The reasoning as to why numbers a and b are multiplied by one explicitly should become clear shortly.

Repeating this process again, but multiplying both sides by $(a+b)$ produces the second power of $(a+b)$.

$$(a+b)^2 = (1)a^2 + (2)ab + (1)b^2.$$

We see here that the constant terms partitioned among the various products of a and b is no longer trivial, as there were two expressions (ab and ba) produced by the product depicted on the right hand side. As standard multiplication is commutative, then $ab = ba$ so that both terms were grouped together into a single term. Repeating this process a third time yields a cubic quantity of the third power on the right hand side:

$$(a+b)^3 = (1)a^3 + (3)a^2b + (3)ab^2 + (1)b^3.$$

Carefully inspecting Figure 1, it should be apparent that the coefficients leading each variant of the product of a and b corresponds to the horizontal rows of the arithmetic triangle. This intuition is not unfounded, as this is exactly the case. Any reasonable attempt to prove the binomial theorem requires a notation to describe where one is inside the arithmetic triangle. Consider the following definition for the choose function, that is the function which by inputting a row (r) and column (c) yields the value in that coordinate.

Definition 1. The *Choose Function* [**4**], is the function which maps a pair of two positive integers n, and k to the positive integers corresponding to the table elements of the arithmetic triangle.

$$\binom{n}{k} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } n = r-1, k = c-1.$$

As a note, the discrepancy in the row and column notation resides in the fact that Figure 1 counts the progression of the arithmetic triangle from one, whereas the choose function counts the arithmetic triangle from zero. The former notation resides in the historical context, while the latter is from convenience. This discrepancy is minimal and does not draw away from the point insomuch as it should be considered any further. With this notation in mind, the binomial theorem lends itself to be stated and proven.

Theorem 1. *(Binomial Theorem) If a and b are real numbers and $n \in \mathbb{N}$, then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. The proof of the binomial theorem is a classical case of standard induction [**9**]. Let us first consider the base case, both $\binom{1}{0}$ and $\binom{1}{1}$ are both one, therefore the binomial theorem is true when $n = 1$. Suppose the equality is true up to some n. This assumption yields the expression:

$$\begin{aligned}
(a+b)^n &= \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}, \text{ multiplying both sides by } (a+b) \\
(a+b)^{n+1} &= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}, \text{let k = k-1} \\
&= a^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}a^k b^{n-k+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k+1} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right)a^k b^{n-k+1} + y^{n+1}.
\end{aligned}$$

An important identity regarding the choose function is that:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

The proof of this identity is placed in the appendix, for now it is sufficient to take this equality as true. Therefore,

$$a^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n+1-k} + b^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n-k+1}.$$

Therefore by induction it has been shown that the binomial theorem holds for all n. □

With the binomial theorem established, let us return to Pascal's work on the sum of integer t-th powers. From the *Potestatum Numericarum Summa* [**5**] Pascal stated the following equality:

$$(n+1)^r - 1 = \binom{r}{1}\sum_{i=1}^{n} i^{r-1} + \binom{r}{2}\sum_{i=1}^{n} i^{r-2} + ... + \binom{r}{r-1}\sum_{i=1}^{n} i.$$

It can be inferred from the above that via an inductive process with respect to r would yield the desired result as a sum of integer t-th powers. For example consider the following steps required to find the sums of powers of three:

$$(n+1)^1 - 1 = n + 1 - 1 = n = \sum_{i=1}^{n} 1.$$

Which follows exactly our intuition of the sums of ones. Taking the next step,

$$2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = (n+1)^2 - 1 = n^2 + 2n + 1 - 1 = (n^2 + n) + n.$$

Although from the previous step we know the sum of ones. Therefore,

$$\begin{aligned} 2\sum_{i=1}^{n} i + n &= n(n+1) + n, \\ \sum_{i=1}^{n} i &= \frac{n(n+1)}{2}. \end{aligned}$$

Finally for the third step, we consider the equality:

$$3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = (n+1)^3 - 1 = n^3 + 3n^2 + 3n.$$

Again, from the previous steps we know the sums of ones, and of the integers so that:

$$\begin{aligned} 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 &= n^3 + 3n^2 + 3n \\ 3\sum_{i=1}^{n} i^2 + \frac{3}{2}n(n+1) + n &= n^3 + 3n^2 + 3n \\ 3\sum_{i=1}^{n} i^2 &= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\ \sum_{i=1}^{n} i^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}. \end{aligned}$$

Pascal did not provide the general solution to the sum of integer powers, but introduced a very powerful relation between the arithmetic triangle and the sum of integer powers that would continue being studied. John Wallis continued the study of the sum of integer powers in his work *The Arithmetic of Infinitesimals* published in 1656 [**10**].

1.3. John Wallis' Observations of Ratios and Arithmetic Numbers

John Wallis' work regarding the sum of integer powers differed from that of Pascal, in that where Pascal focused on securing a concept of the arithmetic triangle first, and found the sum of integer powers as consequence, John Wallis studied the sum of integer powers, and in consequence found various properties of the arithmetic triangle. His inductive process of generating the sum of integer powers is spanned by over forty various propositions in his work *The Arithmetic of Infinitesimals* (for brevity, similar propositions are introduced together in groups).

Propositions 1 2 and 3 [10]:

Wallis introduced the sum of integer powers first by observing the following property regarding strictly the integers:

$$\begin{aligned} \frac{0+1}{1+1} &= \frac{1}{2} \\ \frac{0+1+2}{2+2+2} &= \frac{1}{2} \\ \frac{0+1+2+3}{3+3+3+3} &= \frac{1}{2}. \end{aligned}$$

Of which the sum of the integers up to some fixed value, divided by the number of terms multiplied by the largest integer, yields a constant value of one-half. That is:

$$\frac{\sum_{i=0}^{n} i}{(n+1)n} = \frac{1}{2}.$$

Assuming that n is not zero, then multiplying both sides of the equation by $n(n+1)$ yields:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

John Wallis did not provide a proof for the above relation, but did apply it as a means to prove that the ratio of a triangle and parallelogram of equal height and base have a proportionate ratio of 1/2. Pictorially, consider Figure 2:

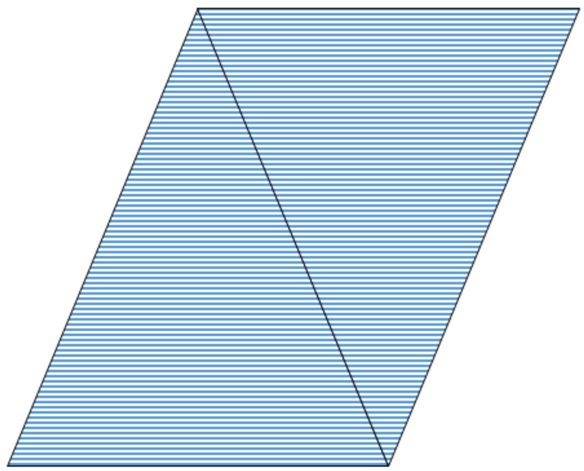

FIGURE 2. A triangle and parallelogram of equal height and base

The lengths of each infinitesimal horizontal bar decreases linearly from bottom to top. That is, in an arithmetic proportion [**10**]. We can consider the parallelogram to be composed of an equal amount of infinitesimal horizontal bars of equal length. Hence, let the sum of all the infinitesimal horizontal bars generate a height of length m, and let the base be of length l. Therefore the ratio of the sum of the arithmetically sized infinitesimal horizontal bars, to the sum of the equally sized horizontal bars is equivalent to the ratio of the areas of the triangle and the parallelogram.

$$\frac{\text{area of triangle}}{\text{area of parallelogram}} = \lim_{l\to\infty} \frac{m\sum_{i=0}^{l}\frac{i}{l}}{m\sum_{i=0}^{l}1} \lim_{l\to\infty} = \lim_{l\to\infty} \frac{\frac{m(l+1)}{2}}{m(l+1)} = \frac{1}{2}.$$

Propositions 19 20 21 and 22 [**10**]:

Wallis continues the investigation observing that the sum of square integers holds a similar proportionality, but with a slight twist. Consider the finite sums below:

$$\begin{aligned}\frac{0+1}{1+1} &= \frac{1}{3}+\frac{1}{6}\\ \frac{0+1+4}{4+4+4} &= \frac{1}{3}+\frac{1}{12}\\ \frac{0+1+4+9}{9+9+9+9} &= \frac{1}{3}+\frac{1}{18}.\end{aligned}$$

It is clear from the above that there remains to be a fixed constant of proportionality, but additionally, there is an extra term which asymptotically approaches zero for large sums. To restate the uniqueness of this sum explicitly:

(1) The ratio of the finite sums is always greater than one-third.
(2) The ratio of the finite sums asymptotically approaches one-third strictly from above.
(3) The secondary term on the right hand side of the equation follows a pattern, that up to a sum of n terms:

$$\frac{\sum_{i=0}^{n} i^2}{(n+1)n^2} = \frac{1}{3} + \frac{1}{6n}.$$

Multiplying both sides of the equation by $(n+1)n^2$ produces the desired result:

$$\sum_{i=0}^{n} i^2 = \frac{n^2(n+1)}{3} + \frac{n^2(n+1)}{6n} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Wallis applied the limiting case to prove that the ratio of volumes between a pyramid of a given base and height is one-third that of a box of equal base and height. Two such cases are described in Figure 3, for both a square and circular based boxes.

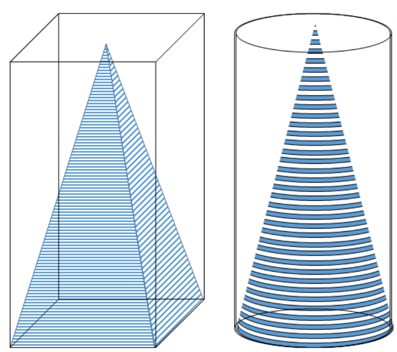

FIGURE 3. A square based, and circular based pyramids enclosed in their respective boxes.

Similar to the previous proof, the pyramid is created by taking the sum of the infinitesimal heights of each planar area that decreases geometrically (on order of squares) from bottom to top. The enclosing box is composed of the sum of infinitesimal heights of equal area. Let the height of the pyramid (and thus box) be l, and let the largest section of the base be area m^2.

$$\frac{\text{volume of pyramid}}{\text{volume of box}} = \lim_{l\to\infty} \frac{m^2 \sum_{i=0}^{l} (\frac{i}{l})^2}{m^2 \sum_{i=0}^{l} 1} = \lim_{l\to\infty} \frac{(\frac{m}{l})^2 (\frac{l^3}{3} + \frac{l^2}{2} + \frac{l}{6})}{m^2(l+1)} = \lim_{l\to\infty} \frac{m^2(\frac{l}{3} + \frac{1}{2} + \frac{1}{6l})}{m^2(l+1)} = \frac{1}{3}.$$

Propositions 39 and 40 [**10**]:

Wallis proceeded with the next inductive step of taking the sum of the cubed integers, observing the relation:

$$\begin{aligned} \frac{0+1}{1+1} &= \frac{1}{4} + \frac{1}{4} \\ \frac{0+1+8}{8+8+8} &= \frac{1}{4} + \frac{1}{8} \\ \frac{0+1+8+27}{27+27+27+27} &= \frac{1}{4} + \frac{1}{12}. \end{aligned}$$

The above quotient of sums can be generalized into the form:

$$\frac{\sum_{i=0}^{n} i^3}{(n+1)n^3} = \frac{1}{4} + \frac{1}{4n}.$$

From here, multiplying both sides of the equation by $(n+1)n^3$ produces the result:

$$\sum_{i=0}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

Of which follows what we already know about such sums. Expressing the sum of integer powers in a quotient like the above possesses a great spectrum of utility. First and foremost it presents an invariant quantity, a constant of proportionality that becomes the limiting value for arbitrarily large sums. One can consider it the reason why the area of a triangle is half of the parallelogram, the volume of a pyramid is one-third that of the enclosing box, and why the region enclosed within infinite-dimensional objects is zero. John Wallis generalizes the first two (thus implying the third) in the following proposition.

Propositions 43 and 44 [**10**]:

THEOREM 2. *(Proportionality of Infinite Geometric Sums) Let S_n denote the sequence of terms depicting the quotient of the sum of the arithmetic series of a power t from zero to a maximal value n^t and the number of terms multiplied by n^t. Then for arbitrarily large n, S_n approaches the limiting value of* $1/(t+1)$.

PROOF. Symbolically we are trying to prove that:

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} i^t}{(n+1)n^t} = \frac{1}{1+t}.$$

Following the spirit of John Wallis, a geometrical approach will be taken which complements his work and greatly reduces the computational difficulty. Observe that such a limit represents the sum of infinitesimals which generate a geometric shape bounded inside a box of equivalent proportions. We recognize immediately that the sum of such infinitesimals represents the integral the the region enclosed. Without loss of generality, let us consider the simplest case of such a geometry with equal dimensions, meaning a t-dimensional pyramid bounded inside of a cube. Let the length of the dimensions of such a cube be b. Therefore:

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} i^t}{(n+1)n^t} = \frac{\int_0^b x^t dx}{b^t \int_0^b 1 dx} = \frac{\frac{b^{t+1}}{t+1}}{b^t b} = \frac{1}{t+1}.$$

□

Without diverging too far from the primary focus regarding the sum of integer powers, consider the limit for the arbitrarily large dimension:

$$\lim_{t\to\infty} \frac{1}{t+1} = 0.$$

This limit briefly mentioned above, touches lightly on a very fundamental point. That is, the region contained in an infinite dimensional object is equivalent to the surface of that object. The contained pyramidal-shape inside the infinite dimensional box does not touch the entire 'surface' of the box, and thus represents none of its total 'volume'. Surprisingly many physics problems regarding multi-particle interactions become much simpler when the amount of dimensions available to the system is infinite.

Propositions 54 58 59 and 64 [10]:

Wallis took the above theorem one more step, allowing the dimension t to be any rational number. This came by expression the integer value of a dimension t as the product of integers p and q such that $t = p/q$. Interpolating values in this manner, Wallis created a table of limiting values for the infinite sums as seen in Figure 4 [**10**]:

p/q	0	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$
2	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	$\frac{2}{9}$	$\frac{2}{10}$	$\frac{2}{11}$	$\frac{2}{12}$
3	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{3}{9}$	$\frac{3}{10}$	$\frac{3}{11}$	$\frac{3}{12}$	$\frac{3}{13}$
4	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	$\frac{4}{9}$	$\frac{4}{10}$	$\frac{4}{11}$	$\frac{4}{12}$	$\frac{4}{13}$	$\frac{4}{14}$
5	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{5}{10}$	$\frac{5}{11}$	$\frac{5}{12}$	$\frac{5}{13}$	$\frac{5}{14}$	$\frac{5}{15}$
6	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{9}$	$\frac{6}{10}$	$\frac{6}{11}$	$\frac{6}{12}$	$\frac{6}{13}$	$\frac{6}{14}$	$\frac{6}{15}$	$\frac{6}{16}$
7	$\frac{7}{7}$	$\frac{7}{8}$	$\frac{7}{9}$	$\frac{7}{10}$	$\frac{7}{11}$	$\frac{7}{12}$	$\frac{7}{13}$	$\frac{7}{14}$	$\frac{7}{15}$	$\frac{7}{16}$	$\frac{7}{17}$
8	$\frac{8}{8}$	$\frac{8}{9}$	$\frac{8}{10}$	$\frac{8}{11}$	$\frac{8}{12}$	$\frac{8}{13}$	$\frac{8}{14}$	$\frac{8}{15}$	$\frac{8}{16}$	$\frac{8}{17}$	$\frac{8}{18}$
9	$\frac{9}{9}$	$\frac{9}{10}$	$\frac{9}{11}$	$\frac{9}{12}$	$\frac{9}{13}$	$\frac{9}{14}$	$\frac{9}{15}$	$\frac{9}{16}$	$\frac{9}{17}$	$\frac{9}{18}$	$\frac{9}{19}$
10	$\frac{10}{10}$	$\frac{10}{11}$	$\frac{10}{12}$	$\frac{10}{13}$	$\frac{10}{14}$	$\frac{10}{15}$	$\frac{10}{16}$	$\frac{10}{17}$	$\frac{10}{18}$	$\frac{10}{19}$	$\frac{10}{20}$

FIGURE 4. Limiting values for the sums of power p, and root q.

The above observation is stated as a lemma regarding theorem 2:

LEMMA 1. *(Limiting Values for Infinite Rational-Powered Sums) The limiting value of the quotient of the sum of integers of power p/q to the product of of the number of terms and the largest term is the quantity $1/(1 + p/q)$.*

PROOF. The proof for this lemma is identical to that of Theorem 2. □

Before continuing further in John Wallis' discussion regarding the sum of integer powers, let us take a brief moment to appreciate what has been accomplished thus far by the means of example. The following is not in Wallis' original text, but rather is a novel application of his formalism that will enlighten the discussion regarding the sums of rational powers later on in chapter three.

Recall the observations which motivated the sums of integers, of square-integers, and cubic integers from above. In all of these examples we observed that for arbitrary sums, the quotient of the sum of a respective power and the product of the number of terms in the sum and the largest term in the sum generated a proportionality that can be broken down into two parts. The first of which we will call ϵ is the limit point of the sequence for arbitrarily large sums. Second is $f(n)$ a function which in addition to ϵ completes the right hand side of the equation for finite sums, as to which asymptotically approaches zero in the infinite sum. That is, we have an expression of the form:

$$\frac{\sum_{i=0}^{n} i^k}{(n+1)n^k} = \epsilon + f(n).$$

Without loss of generality, let us consider the case where $k = 1/2$ such that $\epsilon = 1/(1+1/2) = 2/3$. By intuition observe that for the first few cases $f(n)$ appears in the form:

$$\frac{0+\sqrt{1}}{\sqrt{1}+\sqrt{1}} = \frac{2}{3} - \frac{1}{6},$$
$$\frac{0+\sqrt{1}+\sqrt{2}}{\sqrt{2}+\sqrt{2}+\sqrt{2}} = \frac{2}{3} - \frac{2-\sqrt{2}}{6},$$
$$\frac{0+\sqrt{1}+\sqrt{2}+\sqrt{3}}{\sqrt{3}+\sqrt{3}+\sqrt{3}+\sqrt{3}} = \frac{2}{3} - \frac{5-\sqrt{3}-\sqrt{6}}{12}.$$

Observe that at every step, $f(n)$ is negative. That is, the limit point ϵ is being approached from below. Consider this as opposed to each example of the limit points for the positive integers from before, as to which the limit point was approached from above. Hence we can infer that the form of $f(n)$ is negative. This refines the expression into the form:

$$\frac{\sum_{i=0}^{n} \sqrt{i}}{(n+1)\sqrt{n}} = \frac{2}{3} - f(n).$$

Now taking the difference between the nth and (n+1)th sum produces the expression:

$$\frac{\sum_{i=0}^{n} \sqrt{i}}{(n+1)\sqrt{n}} - \frac{\sum_{i=0}^{n+1} \sqrt{i}}{(n+1)\sqrt{n+1}} = \frac{2}{3} - f(n) - (\frac{2}{3} - f(n+1)) = f(n+1) - f(n).$$

Outputting the (n+1)th term of the second sum allows for both sums to be expressed in the form:

$$\sum_{i=0}^{n} \sqrt{i}\left(\frac{1}{(n+1)\sqrt{n}} - \frac{1}{(n+2)\sqrt{n+1}}\right) - \frac{1}{n+2} = f(n+1) - f(n).$$

Sufficient reordering of the terms on the left hand side yields:

$$\frac{\sum_{i=0}^{n} \sqrt{i}}{(n+1)\sqrt{n}}\left(\frac{n+2-\sqrt{n+1}\sqrt{n}}{(n+1)(n+2)\sqrt{n}}\right) - \frac{1}{n+2} = f(n+1) - f(n).$$

Observe that the far left expression can be substituted by the limiting point and $f(n)$ from above such that:

$$(\frac{2}{3} - f(n))\left(\frac{n+2-\sqrt{n+1}\sqrt{n}}{(n+1)(n+2)\sqrt{n}}\right) - \frac{1}{n+2} = f(n+1) - f(n).$$

With sufficient manipulation, we arrive at two final expressions. The first of which is a recursive relation between $f(n)$ and $f(n+1)$ and the second between the partial sum and $f(n)$.

$$\frac{2}{3} - \frac{2\sqrt{n+1}\sqrt{n}+3}{3(n+2)} + f(n)\frac{\sqrt{n+1}n}{n+2} = f(n+1),$$
$$\sum_{i=0}^{n} \sqrt{i} = (n+1)\sqrt{n}(\frac{2}{3} - f(n)).$$

Of course in this case, $f(1) = 1/6$, so that every consecutive term of f can be solved for recursively. If one were ambitious enough to follow through with this recursion relation, the final result would be of the form:

$$\sum_{i=0}^{n} \sqrt{i} = (n+1)\sqrt{n}\left(\frac{1}{n+1} + \frac{\sqrt{n}}{6}\prod_{j=1}^{n-2}\frac{j+1}{j+2} + \sum_{k=1}^{n}\frac{1}{n+1-k}\prod_{r=0}^{k-1}\frac{\sqrt{n-r-1}\sqrt{n-r}}{n+1-r}\right).$$

The above expression is clearly much more complicated than the original sum. This demonstrates a very fundamental point that irrational numbers do not combine as easily as natural numbers do under standard algebraic operations. That is, each square root of a unique integer can be taken for a unique 'irrational direction'. Consider for example $(3+2)^2 = 5^2 = 25$, while $(\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$. Of which in the latter expression, there is no more fundamental means which the right hand side can be reduced to. Hence each unique integer to a rational power must be written separately, and cannot be re-expressed as a simple polynomial expression like the integer powers. We will return to this point in chapter three. For now, let us return to Wallis as he bridges the gap between the sum of integer powers and the figurate numbers.

Propositions 171 172 176 177 179 and 180 [**10**]:

Recall Pascal's triangle from Figure 1. Consider now the distinct series generated by taking the partial sums of diagonal elements of the form depicted in Figure 5:

1	**1**	1	**1**	1	**1**	1
1	$1 + \mathbf{1}$	2	$1 + \mathbf{2}$	3	$1 + \mathbf{3}$	4
1	$1 + 1 + \mathbf{1}$	3	$1 + 2 + \mathbf{3}$	6	$1 + 3 + \mathbf{6}$	10
1	$1 + 1 + 1 + \mathbf{1}$	4	$1 + 2 + 3 + \mathbf{4}$	10	$1 + 3 + 6 + \mathbf{10}$	20

FIGURE 5. Partial sums of diagonal elements of Pascal's Triangle

Observe that the bold-faced numbers are identical to the partial sums in the previous column. That is, the nth partial sum of a given diagonal in Pascal's triangle is equal to the nth element in the consecutive diagonal. Of course, such a statement would require a satisfactory proof, of which is given in the following section regarding theorem X provided by Bernoulli. For the moment, consider the above to be true on the pretense it will be proven later.

Quantifying the progression of each partial sum, John Wallis observed the partial sums follow a similar pattern to that of the sum of integer powers. For partial sum up to element n of diagonal m:

$$
\begin{aligned}
1 + 1 + 1 + ... + 1 &= \frac{n}{1} \\
1 + 2 + 3 + ... + n &= \frac{n}{1}\frac{n+1}{2} \\
1 + 3 + 6 + ... + \frac{n^2+n}{2} &= \frac{n}{1}\frac{n+1}{2}\frac{n+2}{3} \\
&... \\
\sum_{j=0}^{n} \binom{j+m}{m} &= \prod_{i=1}^{m} \frac{n+i-1}{i}.
\end{aligned}
$$

This allowed for John Wallis to propose a method which the sum of integer powers can be solved for in the general case. Observe that on the left hand side, a given choice of m represents a polynomial depicted by the product of terms similar to the right hand side up to the value $m - 1$. Thus hidden within the left hand side is the sum of integers of $m - 1$ power. All that is required to find the sum of the integers to the $m - 1$ power is to rearrange the terms across the equality. This is best illustrated through an example.

Let us find the nth partial sum of the integers to the fourth power. Looking to the above expression we have:

$$\sum_{j=0}^{n}\binom{j+4}{4}=\prod_{i=1}^{4}\frac{n+i-1}{i}$$

$$\sum_{j=0}^{n}\prod_{k=1}^{3}\frac{j+k-1}{k}=\prod_{i=1}^{4}\frac{n+i-1}{i}$$

$$\sum_{j=0}^{n}\frac{j}{1}\frac{j+1}{2}\frac{j+2}{3}=\frac{n}{1}\frac{n+1}{2}\frac{n+2}{3}\frac{n+3}{4}$$

$$\sum_{j=0}^{n}\frac{j^3}{6}+\frac{j^2}{2}+\frac{j}{3}=\frac{n^4}{24}+\frac{n^3}{4}+\frac{11n^2}{24}+\frac{n}{4}$$

$$\frac{1}{6}\sum_{j=0}^{n}j^3=\frac{n^4}{24}+\frac{n^3}{4}+\frac{11n^2}{24}+\frac{n}{4}-\frac{1}{2}\sum_{j=0}^{n}j^2-\frac{1}{3}\sum_{j=0}^{n}j$$

$$\frac{1}{6}\sum_{j=0}^{n}j^3=\frac{n^4}{24}+\frac{n^3}{4}+\frac{11n^2}{24}+\frac{n}{4}-\frac{1}{2}(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6})-\frac{1}{3}(\frac{n^2}{2}+\frac{n}{2})$$

$$\frac{1}{6}\sum_{j=0}^{n}j^3=\frac{n^4}{24}+\frac{n^3}{12}+\frac{n^2}{24}$$

$$\sum_{j=0}^{n}j^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}.$$

John Wallis provided an intuitive and computationally simple means which the sum of integer powers can be solved for in a recursive manner. The above allows for the partial sum of an mth power to be solved for provided knowledge of the sums of all $m-1$ powers. The above is close, but is not quite the complete solution that we are looking for. Ideally, a solution to the partial sums of integer powers would not require any knowledge of the previous sums, but rather be an independent expression generating a sequence of polynomials which produce the answer. Such a solution was provided by Bernoulli, and leads us to the next section.

1.4. Bernoulli's Conclusion

Jacob Bernoulli's discussion regarding the sum of integer t-th powers lies within chapters two and three of *The Art of Conjecturing* published after his death in 1713 [**2**]. It is these chapters which will be presented below, both establishing his approach to the problem, and his conclusion to the original question.

Bernoulli's approach is quite similar to Pascal's in that he studied the arithmetic triangle first, and used the knowledge he had gained from it to prove the sums of integer powers. Moreover, both presented their findings in the context of probability regarding combinations. Although unlike Pascal, Bernoulli was much more formal in the manner which he presented the arithmetic triangle and the sum of integer powers, beginning first with some definitions:

Definition 2. *Combinations*, are collections of which a given multitude produces a value irregardless of order or position. That such a variant of addition or subtraction has a direct correlation unto the quantity producing again a value [**2**].

What Bernoulli is stating is that when objects (such as numbers) are group together, they can be counted so that their quantity (or multitude) can be assigned a number. For example, I can get a basket of apples. If I unload the basket and count all the apples together in one group, I can assign a number, say twenty, to the apples.

This number defines the quantity of apples that I have. On the other hand, I can reorder the apples into groups of either granny smith, or Fuji and have two numbers, say fourteen and six respectively. I can then state of the twenty apples (of all apples), I have fourteen that are green (of all granny smith) and six that are pink (of all Fuji apples). Irregardless how I order the apples in the basket, there will always be twenty total, with fourteen green and six pink.

Definition 3. *Exponent*, The number of given things conjoined [**2**].

That is, the power of which a quantity is taken such as squares, cubes, and the like. Having established these two definitions, Bernoulli presents the following problem beginning the study of figurate numbers:

If all the things to be combined are diverse and if in no combination should the same things occur twice, to find absolutely all the combinations, that is, of all exponents at once. [**2**]

Fortunately having established the definition of combinations and exponents above, this problem can be tackled merely by the example Bernoulli presents himself. Consider the letters a, b, c, d, e to be the elements to be combined into every possible 'word'. Following the problem statement we iteratively introduce every letter into each word thus creating all possible combinations as in Figure 6.

a
b ab
c ac bc abc
d ad bd cd abd acd bcd abcd
e ae be ce de abe ace bce ade bde cde abce abde acde bcde abcde

FIGURE 6. All possible 'words' created from letters a,b,c,d,e.

Bernoulli points out many observations regarding this example, but the most applicable one for our inquiry is by reordering the above by counting the number of singles, pairs, triplets, and all others into values regarding their quantity per line. That is in this context let 'ac' and 'bc' be identical in regards to a method of combining that only counts the fact that both are a pair of distinct letters. Figure 6 can then be written in the form:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Of which comparing to Figure 1, produces an equivalent result. Bernoulli proceeds by listing some 'Marvelous Properties' [**2**].

(1) Column c begins with c-1 zeros.
(2) The nth diagonal produces the nth column.
(3) The second nonzero column denotes the column number.
(4) Any term equals the sum of all the higher terms in the preceding column.
(5) Any term equals the sum of the two terms above it in the same and preceding column.
(6) Rows increase and decrease symmetrically.
(7) There is symmetry in rows (on bases of equally high columns).
(8) The sum of rows above generate the next row.
(9) The rows are the binomial terms of $(a+b)^n$ for the nth row.
(10) The sums of the rows produce 2^n for the nth row. The sums of these sums produce $2^{n+1}-1$ for the nth row.
(11) The ratio of the terms are arithmetic in nature.
(12) The sum of any number of terms of any column beginning from its zeros has a ratio to the same number of terms all equal to the last term that is equal to the ratio of one to the number of that column. This is also true while starting with one and proceeding one term farther than the maximum.

Our greatest concern lies with property twelve. This property is quite similar in nature to that which John Wallis discussed when describing the quotient of the sum of integers to a power and the product of the number of terms with the greatest term. Although for columns of the arithmetic triangle, there is a critical difference in that there is no extra function $f(n)$ in addition to the limit point ϵ, such that, every ratio of partial sums immediate converges to ϵ. This is best presented through a few examples below in Figure 7:

0	5	1	7
1	5	2	7
2	5	3	7
3	5	4	7
4	5	5	7
5	5	6	7
+		+	
15	30	21	42
1 :	2	1 :	2
0	10	1	28
0	10	3	28
1	10	6	28
3	10	10	28
6	10	15	28
10	10	21	28
+		+	
20	60	56	168
1 :	3	1 :	3

FIGURE 7. A few examples depicting property 12.

As one can see from above, the proportionality is constant irregardless of the number of terms in the partial sum. This intuition is applied for proving the following lemmas:

LEMMA 2. *(Constant of Proportionality of Arithmetic Sequences) In any series if the sum of terms taken from the beginning always has the same ratio to the sum of as many terms equal to the last, no matter how many terms equal to the last, no matter how many terms are taken. Then the number of terms taken minus r will be the same number minus one as the next to last term taken to the last.* [**2**]

PROOF. Unfortunately, it can at times be very difficult to ascertain the full scope of meaning from a literal translation from Latin. Hence, to fully comprehend the lemma in its entirety, let us tread carefully making sure that the statement is clear.

The claim above is that for an arithmetic sequence of n terms, the ratio $(n-r)/(n-1)$ is equal to c/d, for the largest term d in the arithmetic sequence, and second largest term c with r posing for $1/\epsilon$.

This is best illustrated through an example. Consider the partial sequence of the first five terms in the triangular numbers:

$$0 + 0 + 1 + 3 + 6.$$

We know by Figure 7, that their ratio is 1:3, so that $r = 3$. The largest term in this sequence is six, and the second largest is three. Therefore:

$$\frac{n-r}{n-1} = \frac{c}{d}$$
$$\frac{5-3}{5-1} = \frac{2}{4} = \frac{1}{2} = \frac{1}{2} = \frac{3}{6}.$$

For an arbitrary finite arithmetic sequence of length n, let the letters a, b, c, d represent the numbers in the sequence. Therefore:

$$\frac{a+b+c+d}{d+d+d+d} = \frac{1}{r}$$
$$\frac{a+b+c+d}{nd} = \frac{1}{r}$$
$$a+b+c+d = \frac{nd}{r}.$$

Although similarly, because by hypothesis the constant of proportionality r holds for all partial sums, then:

$$\frac{a+b+c}{c+c+c} = \frac{1}{r}$$
$$\frac{a+b+c}{(n-1)c} = \frac{1}{r}$$
$$a+b+c = \frac{c(n-1)}{r}.$$

Therefore we can state that:

$$a+b+c+d = \frac{c(n-1)}{r} + d = \frac{nd}{r}.$$

Subtracting d from both sides and multiplying by r reduces the equality to the form:

$$c(n-1) = d(n-r) \implies \frac{c}{d} = \frac{n-r}{n-1}.$$

Of which is the desired equality. □

Let us take this one step further, by supposing that extending the sequence of a given column in due to extending th partial sum in the previous column preserves the constant of proportionality.

LEMMA 3. *(Relationship Regarding Extended Partial Sums) If there are two adjoining columns in which the prior column any number of terms from the beginning have a constant ratio 1:r to as many terms equal to the last term, and if, indeed, in the following column the sum of any number of terms taken from the beginning has a ratio of 1:r+1 to as many terms equal to the last term, then also, adding the following term, the sum of all the terms together with the added term as there are terms including the added term.* [**2**]

PROOF. Again this is pretty heavy, but is absolutely vital in the pursuit in the sum of integer powers. Taking it slow, let us consider two columns in the arithmetic triangle with elements a, b, c, d in the first column and e, f, g, h, i terms in the second column. Recall property 4 that *any term equals the sum of all the higher terms in the preceding column.* Let us organize the terms in each column in the form:

a	e
b	f
c	g
d	h
	i

Recall by property 4 that *Any term equals the sum of all the higher terms in the preceding column.* Therefore we can state that:

$$h = a + b + c.$$

Multiplying both sides of the equation by the constant of proportionality r allows us to recall the first part of this lemma. That the first column has a constant of proportionality r such that:

$$\frac{a+b+c}{(n-1)c} = \frac{1}{r} \implies r(a+b+c) = (n-1)c.$$

Substituting this back into the above expression yields:

$$rh = (n-1)c = (n-r)d.$$

Whereas in the furthest right hand equality, we utilized the premise of lemma 2. This implies:

$$\frac{n-r}{h} = \frac{r}{d} = \frac{n}{a+b+c+d} = \frac{n}{i}.$$

The third equality is again by the hypothesis of the lemma, and the furthest right equality is from property four.

$$(n-r)i = nh = (r+1)(e+f+g+h).$$

The furthest part of the equality is by the second if statement of the hypothesis of the lemma. Therefore:

$$\frac{n-r}{r+1} = \frac{e+f+g+h}{i}.$$

In this point of the proof Bernoulli introduces a term which is foreign to most modern math textbooks. That is, the mathematical operation called the *composition of a ratio* [**2**]. Let R = a:b, be a ratio. The composition of R is the ratio a+b:b. Let is compose the ratios on both sides of the equality independently such that:

$$\frac{n-r}{r+1} \to \frac{n-r+r+1}{r+1} = \frac{n+1}{r+1}$$
$$\frac{e+f+g+h}{i} \to \frac{e+f+g+h+i}{i}.$$

Subtly, this is simply adding one to both sides of the equation so that the equality still holds. Therefore:

$$\frac{n+1}{r+1} = \frac{e+f+g+h+i}{i} \implies \frac{1}{r+1} = \frac{e+f+g+h+i}{i(n+1)}.$$

This is the desired result. □

This establishes the premise for the following lemma as to where by assuming that a given constant of proportionality of a column is r, we can conclude that the constant of proportionality in the subsequent column is $r+1$.

Lemma 4. *(Relationship Regarding Pairwise Consecutive Columns and Their Constant of Proportionality) In Figure 1, if the sum of terms from the beginning of any column always has a ratio of 1:r to the sum of just as many terms equal to the maximum term, then the sum of terms in the next column will have to the sum of as many terms equal to maximum the ratio of 1:r+1* [**2**].

Proof. Following the same form from the previous lemma, let the values in each column to the nth and (n+1)th term be described with the letters:

a	0
b	g
c	h
d	i
e	l
f	p
	q

Where we observe that the top right element is zero (from property one) and that the bottom left element is empty. By property four and the hypothesis of the lemma:

$$\begin{aligned}
q+p+l+i+h+g+0 &= \frac{nf}{r} + \frac{(n-1)e}{r} + \frac{(n-2)d}{r} + \frac{(n-3)c}{r} + \frac{(n-4)b}{r} + \frac{(n-5)a}{r} \\
&= \frac{n(f+e+d+c+b+a) - e - 2d - 3e - 4b - 5a}{r} \\
&= \frac{nq - (e+d+c+b+a) - (d+c+b+a) - (c+b+a) - (b+a) - (a)}{r} \\
&= \frac{nq - p - l - i - h - g - 0}{r}.
\end{aligned}$$

Multiplying both sides by r establishes the equality in the form:

$$\begin{aligned}
rq + r(p+l+i+h+g+0) &= nq - p - l - i - h - g - 0 \\
nq - rq &= r(p+l+i+h+g+0) + p + l + i + h + g + 0 \\
q(n-r) &= (r+1)(p+l+i+h+g+0) \\
q\frac{n-r}{r+1} &= p+l+i+h+g+0 \\
q\frac{n-r}{r+1} &= p+l+i+h+g+0.
\end{aligned}$$

Adding q to both sides of the equation yields:

$$q\frac{n+1}{r+1} = q+p+l+i+h+g+0 \implies \frac{1}{r+1} = \frac{q+p+l+i+h+g}{q(n+1)}.$$

Of which establishes the desired result. □

This is enough to finally state what our intuition had led us to believe in the very beginning.

LEMMA 5. *(Principle Proposition) For each consecutive column the constant of proportionality r increases linearly such that for each consecutive column: $r = 1, 2, 3, 4, ...$*

PROOF. By lemma3 we know that for a given column with a constant of proportionality r, each subsequent partial sum will retain the same constant of proportionality. By lemma4, we know that for a given column that has a constant of proportionality r, the subsequent column has a constant of proportionality of $r+1$.

Therefore all that is left to be established is the base case. We know already that the sum of the 'ones' is equal to the length of the partial sum itself such that $\sum(1) = n$.

The number of terms in this partial sum is n with a maximal value of one. Therefore the constant of proportionality $r = 1$.

By induction therefore, the lemma holds. $\square$

An important detail regarding the above lemma is that by property one, each column starts with $c - 1$ amount of zeros such that $r = c - 1$. Therefore we can state that for the values at place n going down each consecutive column will be nonzero up to column $n - c$. This allows us to inductively construct the figruate numbers and thus the sums of integer powers in the same manner as presented by John Wallis. Although in the case of Bernoulli, he took the last step in this problem by stating the following regarding the final solution to the sums of integer powers:

$$\begin{aligned}\sum_{i=0}^{n} i^c = {} & \frac{n^{c+1}}{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{2*3*4}Bn^{c-3} \\ & + \frac{c(c-1)(c-2)(c-3)(c-4)}{2*3*4*5*6}Cn^{c-5} \\ & + \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{2*3*4*5*6*7*8}Dn^{c-7} + \ldots\end{aligned}$$

Whereas the coefficients $A = 1/6$, $B = -1/30$, C=1/42, D=$-1/30$,... are known today as the Bernoulli Numbers. In more contemporary notation the above can be rewritten in the form:

$$\sum_{i=0}^{n} i^t = \frac{1}{t+1}\sum_{j=0}^{t}\binom{t+1}{j}B_j n^{t+1-j}.$$

As to which B_j represents the jth Bernoulli number. This is a very powerful result that converts a sum of integers into a sum of decreasing powers for a fixed integer being the maximal in the left hand sum. Before presenting a formal proof of this equality, we could consider what the circle would ask the sphere [**1**] in that 'is there anything more than the sum of integer powers?'. Such an idea of the sum of rational powers John Wallis proposed, and we provided a simple example using his notation to show at the very least, is is a far more complicated query than the integers. Although, it is not a completely lost task, and its approach is enlightening in demonstrating some fundamental concepts regarding these sums as a whole.

Of course from the above, there is a distinctive issue in that between Pascal, Wallis, and Bernoulli all their discussion regarding this problem relied absolutely on the integer relationships between the figurate numbers, and the sum of integer powers. Extending the expression to accommodate the sum of rational powers cannot use this kind of assumption. Therefore everything that was accomplished above must be revised over. This time, using a more geometric picture. This geometric picture will illustrate what the sum of rational powers would look like.

Once the geometric picture is established, then the sum of integer t-th powers will be proven formally with regards to the geometric motivation. Let us begin therefore by establishing the geometric interpretation of the sum of integer t-th powers.

CHAPTER 2

Geometric Approach

2.1. Summary

The complete solution of the sum of integer powers as described in this chapter took many different steps. First, and perhaps the most simple, is to interpret sums of numbers as 'blocks' that can be stacked in various ways to make step-like pyramids. These are easy to draw, and simple to demonstrate fundamental patterns in the sums arbitrary long series of integers. This method worked best with exponent $t = 1, 2$ whereas the results lied in the second and third dimensions respectively.

Second, is what (for lack of a better term) *integration by slicing.* This method represents two basic facts: first, the geometric interpretation has a finite applicability, and that integration can be used for a much wider range of problems than discussed in conventional math courses. This method of integration uses different integrals as 'knives' to cut out the desired pieces of multidimensional shapes solving for the unit volume of each piece individually. This was an extremely powerful tool that worked very well up to $t = 4$ or five-dimensional shapes. At that limit, it was not a failure in the computation, as it had become too exhaustive a calculation to be considered remotely efficient.

Third, used the new data gathered from integration by slice to create much more intricate patters. That is, the integration method produced numeric answers, but provided no means of storing the location of data in space – hence the birth of this method was primarily to organize integration data, though eventually became a method itself. This method presented another step away from the geometric interpretation, but also was a powerful vehicle to assemble relations up to $t = 6$ or seven-dimensional shapes. Beyond eight dimensions it became exhaustive to keep track of all the given shapes.

Fourth, all means of visually cataloging the data were dropped. From here, the general pattern began to rise allowing for a complete divergence away from any type of diagram. That is, enough of the pattern had been thoroughly developed to allow for a recursive relation, which opened up the eight dimension, $t = 7$. At this point it was clear there was enough knowledge to finally put the problem to a close.

Fifth, in the end with eight dimensions to use for an arbitrarily long sum, it became a matter of rigorous algebraic procedure to finally put the entire problem together. The problem came to a close, and the solution was solved for in completion.

2.2. Technique 1: Geometrical Pyramids

Recall from before that the sum of the first n integers is solved via the form:

$$S_{(n,1)} = \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}.$$

The supposed method for solving this was lining up all the integers forwards, and backwards adding up the sum of each, and then dividing by two. This method is simple, compact, and elegant. Although from a geometrical standpoint, it lacks a motivation. Hence (as promised) here is a geometrical means of coming up with the same solution. This geometrical method will become the motor which drives through (though in the end rather abstractly) the final solution of the problem.

2.2.1. Sum of Integers in Two Dimensions. In this case, just adding up n integers, consider each number to be a stack of unit squares (see Figure 1) such that the number one is one unit square, two is two unit squares stacked on top of each other, and so on to an n-height stack of unit squares signifying the number n.

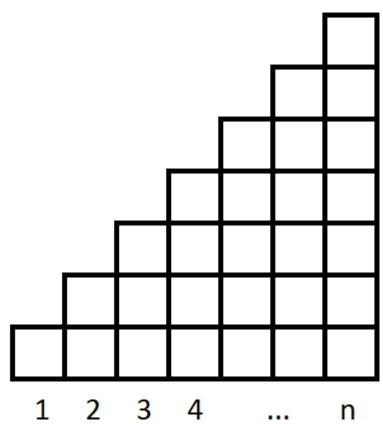

Figure 1. A stepwise pyramid of positive integers.

From here, adding up each consecutive stack yields the desired sum producing a stepwise pyramid. Though of course, this is an awkward shape to be dealing with and needs to be simplified to yield a solution. Hence, the next step is to draw a diagonal line along the jagged edge of the pyramid extending the upper right, and bottom left corners by one unit (see Figure 2).

It is clear to see from Figure 2 that a right pyramid of sides n+1 length has been created. The sum, or area, inside of the pyramid can be expressed as:

$$P_{(n+1)} = \frac{(n+1)(n+1)}{2}.$$

This is not quite the solution specified above, nor should it. The right pyramid *contains* the sum inside of its area, along with a series of blue triangles filling in the far left edge. Simply put, to find the solution those smaller triangles need to be removed. Fortunately there are a few facts that will come in handy that are neatly given by the geometrical picture. First, each of these smaller triangles as an area of one-half. This is given by the fact that each of its edges lies on the faces of the unit squares. Hence, each is one-half the area of a unit square. Moreover, one can follow from a quick venture in induction to show that for a sum of n integers, there will always be n+1 of these blue triangles (see Figure 3). In tabular form the trend becomes intuitive (see Table 1).

squares	1	2	3	...	n
triangles	2	3	4	...	$n+1$

TABLE 1. The number of integer stacks versus the number of triangles.

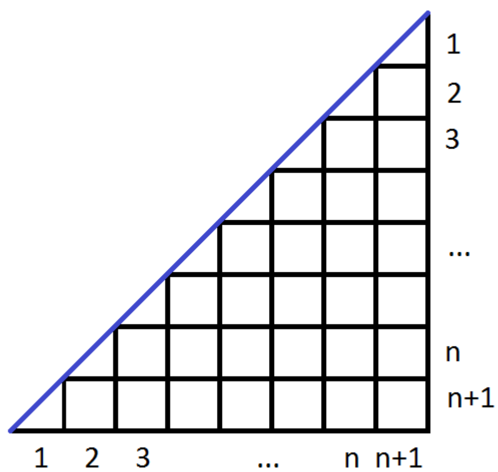

FIGURE 2. A diagonal line meeting each corner on the jagged edge.

Knowing both the area of each of the smaller triangles, and the number of smaller triangles per sequence of integers predetermined – it becomes merely a play of algebra to compute the sum of integers. Taking the difference between the larger triangle, and the smaller triangles will yield the desired result.

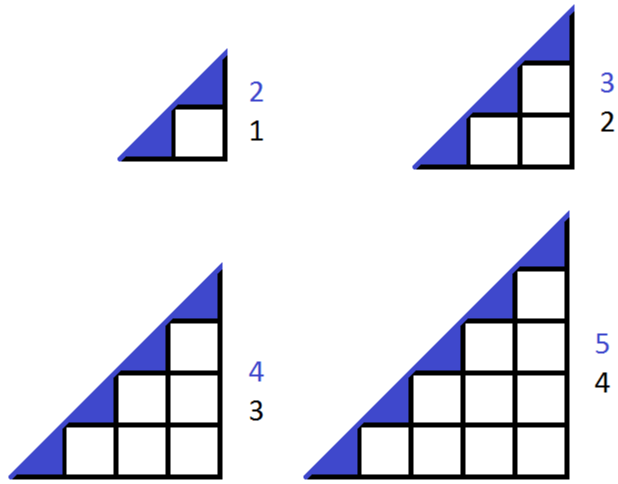

FIGURE 3. Sequential right triangles, blue indicates the number of smaller triangles, and black indicates the number of squares that go across the lowest row (or vertically on the furthest column).

$$\begin{aligned} S_n &= \frac{(n+1)(n+1)}{2} - \frac{n+1}{2} \\ &= \frac{(n+1)}{2}(n+1-1) \\ &= \frac{(n+1)}{2}(n) \\ &= \frac{(n+1)(n)}{2}. \end{aligned}$$

Where the last result is the desired form. This effectively solves the sequence of one dimentional integers *in two dimensions*. Though that is not all that can be said of the direct geometrical means of finding a solution. It is time to find the sum of the square integers.

2.2.2. Sum of Square Integers in Three Dimensions. To be clear what is meant by finding the sum of square integers – recall the original sum $S_{(n,t)}$ which is expressed in the form:

$$S_{(n,t)} = \sum_{i=0}^{n} i^t.$$

Of which for t = 1 the solution was found to be:

$$S_{(n,1)} = \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}.$$

hence the next step is simply to express t = 2 – the sum of second powers of integers.

$$S_{(n,2)} = \sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + .. + n^2.$$

Unfortunately taking these numbers and arranging them into stacks to produce a stepwise pyramid will not work. This is because the relation between consecutive integers is not linear – it is quadratic. One could attempt to find some two-dimensional relation of the square integers which will produce a generalized answer, but that would be moving away from the simplicity of the triangular shape as described above. Instead consider each unit number not to be a square, but a cube. Therefore each square integers creates a three dimensional rectangle nxnx1 in size (see Figure 4).

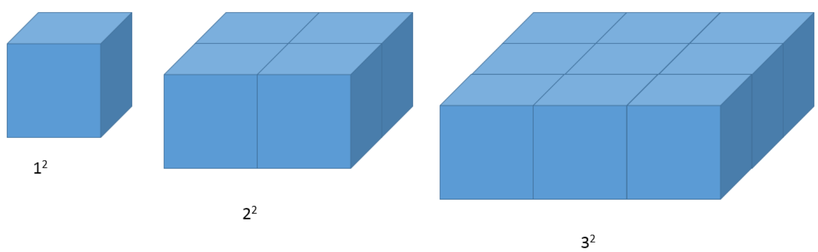

FIGURE 4. Consecutive square integers in three dimensions.

One may observe the advantage this system has in expressing these sums. The simplest means of arriving at the same triangular shape is to choose a 'base corner' and create a right square based stepwise pyramid by stacking consecutive integers on one another (see Figure 5).

As noted in Figure 5, to the far right of the pyramid is the same stacking pattern as seen from the two-dimensional problem. This provides both an indicator and a method of solution. Though unique to the three-dimensional case, three lines need to be drawn – complimenting the three exterior corners on the top cube, and following diagonally down the edges of the pyramid to the bottom (see Figure 6).

Naturally the next step would be to take the difference between the large square based pyramid and its constituent pieces to yield a result. Although in this case, there is not one but two smaller geometries which need to be deleted. First of which is the triangular prism shape – basically a right triangle stretched into the third dimension, and a smaller

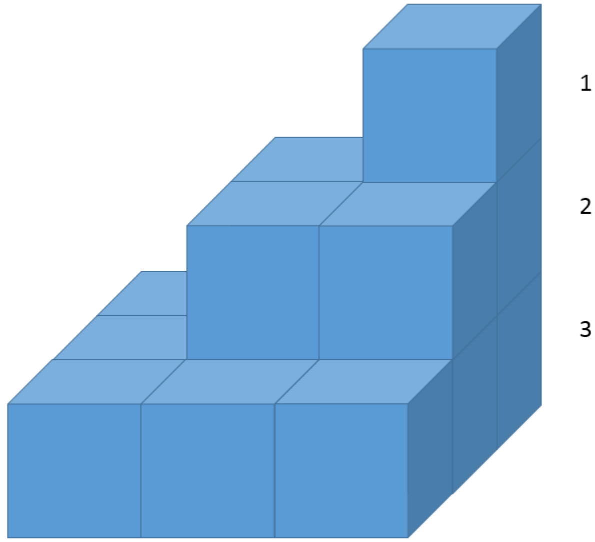

FIGURE 5. A right triangular square based pyramid (note the triangular shape on the far right, as in the last sum).

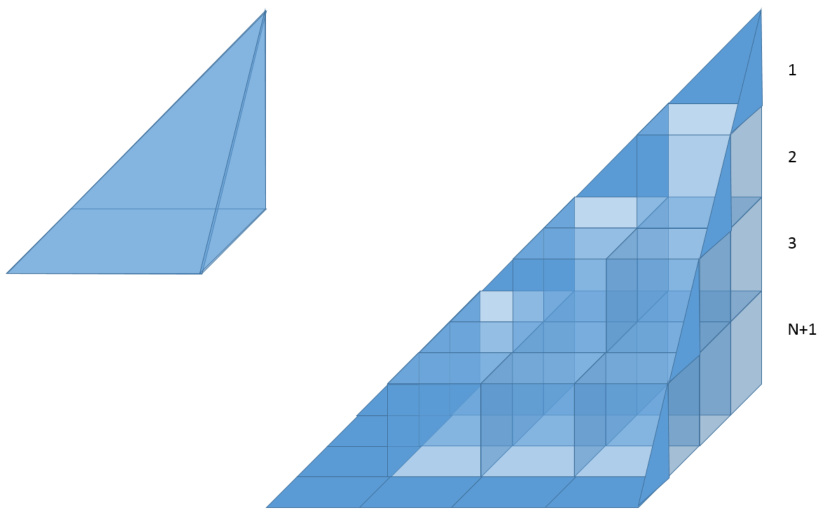

FIGURE 6. The right square based pyramid (left) and broken down into unit cubes (right).

pyramid shape. The latter shape is unique to the three dimensional system, and has no two dimensional analog. Both are detailed in Figure 7.

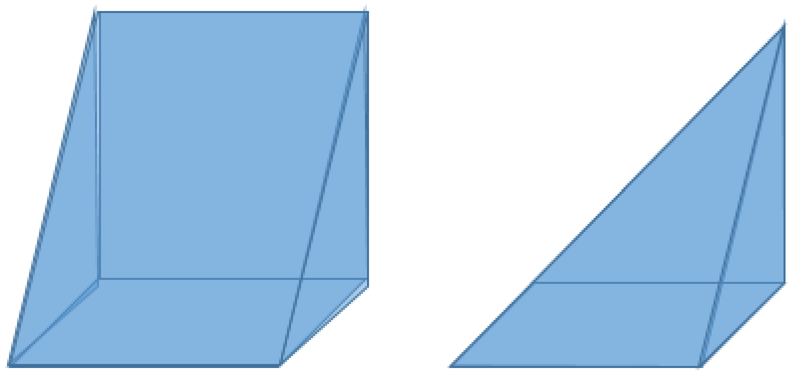

FIGURE 7. The triangular prisim from the two-dimensional case (left) and the square based pyramid from the third-dimensional case (right).

Recalling from before, the unit right triangle had an area of $\frac{1}{2}$. Being stretched into the third dimension, its now volume remains the same (as the extra length is exactly one unit). Recalling basic geometry, the unit pyramid has a volume of $\frac{1}{3}$. This is ony the first two, out of four required pieces of information required to solve this problem. What lies ahead is knowing how many of each shape exists, so a generalized difference can be made. Fortunately the geometry can be used as a guide. Following a basic route of induction, one can see a simple pattern (shown below in Figure 8 and Table 2).

limit of sum	1	2	3	...	n
pyramid	2	3	4	...	$n+1$
prism	2	3	4	...	$(n)(n+1)$.

TABLE 2. The number of prisims and pyramids per endpoint of sum.

Though the pattern may be easy to see – the methodology to arrive at this solution may not. For the sake of the reader, a brief explanation is at hand. As one can see from the inductive iterations following figure eight, that for every unit square added to the height of the pyramid, exactly one more pyramid is added – with a base case of two, covering both the top, and far edge of the first cube. As for the prisms, note that for each iteration of cubes, the prisms behave almost like integer stacks, creating stepwise triangular faces on the surface of the pyramid. Hence, one can use the exact same information from the previous two-dimensional example to deduce that there will be $\frac{(n)(n+1)}{2}$ prisims on one side. As two sides are facing outwards, this value is multipled by two, yielding $(n)(n+1)$. For

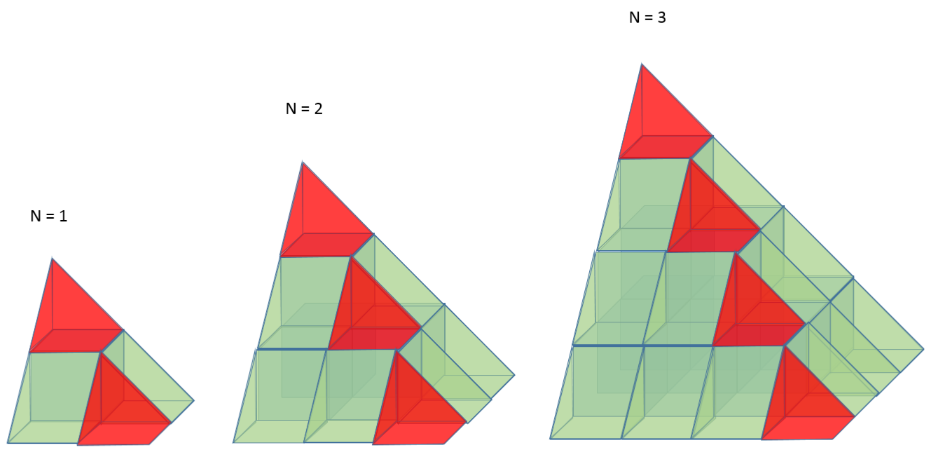

FIGURE 8. A primitive use of induction to find a relation describing the prism and pyramid shapes.

the clever reader, this similarity and convienience is no coincidence. Although a further description of that matter is left for section six.

Hence the volume produced by the stepwise pyramid can be expressed as:

$$\begin{aligned}
S_n &= \frac{(n+1)(n+1)(n+1)}{3} - \frac{n+1}{3} - \frac{(n)(n+1)}{2} \\
&= (n+1)(\frac{(n+1)(n+1)}{3} - \frac{n}{2} - \frac{1}{3}) \\
&= (n+1)(\frac{n^2+2n+1}{3} - \frac{1}{3} - \frac{n}{2}) \\
&= (n+1)(\frac{n^2+2n}{3} - \frac{n}{2}) \\
&= (n+1)(\frac{2(n^2+2n)}{6} - \frac{3n}{6}) \\
&= (n+1)(\frac{2n^2+4n-3n)}{6}) \\
&= (n+1)(\frac{2n^2+n}{6}) \\
&= \frac{(n)(n+1)(2n+1)}{6}.
\end{aligned}$$

The final result may seem counterintuitive, but rest assured this is the correct answer (see Table 3).

n	1	2	3	4	5
$\frac{(n)(n+1)(2n+1)}{6}$	1	5	14	30	55
sum	1	5	14	30	55

TABLE 3. Implied result of the above solution versus the actual sum.

What this solution is simply stating is that the two-dimensional problem is simply solved in *three dimensions*. One may ask whether this would work on cubic integers in the fourth dimension.

The answer to this is both yes and no. In theory, any summation of integers to the power t can be solved using this geometric method. Although in practicality, there is a irrevokable disability that is inherent in a three dimensional space. As of now, the integers, and squared integers could easily be written down on paper – that is both respectfully represent one dimension below and equivilent to the physical universe. Both of these can be checked by visual inspection, and a correct answer can be redeemed with little conceptual trouble. This is not the case for the fourth-dimension.

Without diving too deeply into the details and limitations of three dimensional space (which is discussed thoroughly in section nine), there is a simple joke that expresses the fustration that I as the author had trying to complete this task for an entire summer. It goes as follows:

Question: How do you carve a statue of an elephant?
Answer: Take a block of stone and carve away everything that doesn't look like an elephant.

In this case, the elephant happens to be a four dimensional stepwise pyramid which contains not one, two, but three unique geometries – one of which is completely unique to the fourth dimension. There are not two but three unique faces that need to be considered for the geometries to reside, of which only two can be accurately described on paper. In short, this presses a problem that is not limited by imagination or determination – but by the sheer inadequacy of the second dimension to describe the fourth. If this is not enough proof – look at Figure 9.

On that note: it is time to move on to a new method to find a solution.

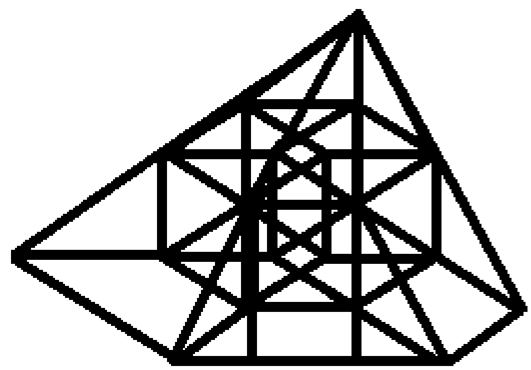

FIGURE 9. A crudly drawn four-dimensional pyramid enclosing a tesseract.

2.3. Technique 2: Integration by Slicing

Before bravely venturing into the fourth-dimension, it is best to cover first why this method works. That is, unlike the geometrical method described above, a level of abstraction is required that prevents confirmation of a solution soley by visual inspection. A method of solution must:

(1) Be logical
(2) Yield the correct answer
(3) Accurately descibe the geometry

All three of these qualities are required to be able to move forward. Missing any one will hinder the completion of the other two. That is, an illogical method may be extremely difficult to repeat, may not work in all cases, and therefore clearly does not yield an adequate interpretation. An incorrect answer will of course fail some sort of logic, and therefore does not reflect the geometry. The inability to effectively describe the geometry leaves the correct answer in doubt and places the logistics in a dubious position. The first two items will be discussed in this section. Though the third requires a much more in depth look at the nature of these sums and is reserved for section nine.

2.3.1. Quality 1: Logistically Sound. The best means to test the logic of this new method is to incorperate it back into a problem with a known answer. If the new method produces the same result *for the same reason* then this is a good indicator that it is safe to move forward and test the method in more abstract systems.

Perhaps the first question that needs to be answered in this case is simply *what is integration by slice?* The answer is rather complicated – hence it is best to have a three-dimensional picture to follow along with the logical process.

The natural form of integration best known in basic calculus classes is that which takes the sum of infantesmal pieces to yield the 'area under the curve'. This works for al functions which a reasonable definition of 'inside' and 'outside' have been established. From the sum of the square integers it is clear that the larger right square based pyramid had a volume of $\frac{(n+1)^3}{3}$ solely from geometrical intuition. Although, can this volume be derived from a different route? The answer to this is yes. Consider the following integral:

$$\int_0^h \int_0^{h-z} \int_0^{h-z} (1) dxdydz.$$

Of which describes the volume of a square based right pyramid of height h. Solving this integral yields:

$$\begin{aligned}
\int_0^h \int_0^{h-z} \int_0^{h-z} (1) dxdydz = & \\
&= \int_0^h \int_0^{h-z} (h-z) dydz \\
&= \int_0^h (h-z)(h-z) dz \\
&= \int_0^h (h^2 - 2hz + z^2) dz \\
&= zh^2 - hz^2 + \frac{z^3}{3}\Big|_0^h \\
&= h^3 - h^3 + \frac{h^3}{3} \\
&= \frac{h^3}{3}.
\end{aligned}$$

Now letting $h = n + 1$ yields:

$$\frac{(n+1)^3}{3}.$$

Which is the desired result. So initially, this is reflects the practical properties of multidimensional integration. Although there is a lot more that can be done with this formula. This is where integration by slice yields is power. Notice that there are three integrals – one for each dimensional direction.

These integrals can represent *planes* through the coordinate axes. That, choosing the correct bounds will define the volume in a given region. For example, if one were unsure of the geometry resting on top of the pyramid of height three, one could arrange the integrals to calculate the volume of that shape alone (see Figure 10).

Computationally this appears in the form:

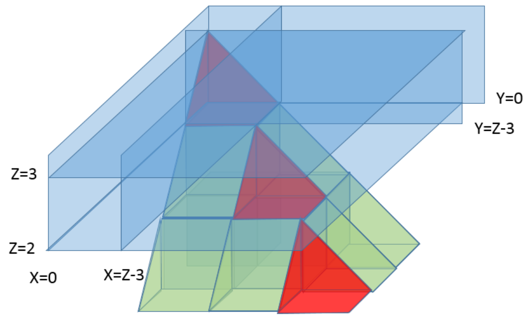

FIGURE 10. The volume associated with the integral: $\int_2^3 \int_0^{3-z} \int_0^{3-z} (1) dxdydz$ relating to the smaller square based right pyramid.

$$
\begin{aligned}
\int_2^3 \int_0^{3-z} \int_0^{3-z} (1) dxdydz = \\
&= \int_2^3 \int_0^{h-z} (3-z) dydz \\
&= \int_2^3 (3-z)(3-z) dz \\
&= \int_0^h (9 - 6z + z^2) dz \\
&= 9z - 3z^2 + \frac{z^3}{3} \Big|_2^3 \\
&= 27 - 27 + \frac{27}{3} - (18 - 12 + \frac{8}{3}) \\
&= \frac{27}{3} - \frac{8}{3} - 6 \\
&= \frac{27 - 8 - 18}{3} \\
&= \frac{27 - 26}{3} = \frac{1}{3}.
\end{aligned}
$$

There are two ways to look at this. First it is clear (and well known) that the volume of a unit pyramid is $\frac{1}{3}$, of which this calculation confirms this fact. On the other hand, this calculation has successfully sliced out of a unit piece exactly $\frac{1}{3}$ of the volume of the larger pyramid – implying the shape *bound by the unit region* is a square based pyramid. It may be difficult to follow this reverse of logic (or to believe in its validity) but rest assured there are some fundamental rules.

As each of the sums (no matter what dimension) is described using integers, and unit blocks – every integral to slice out a shape will always be in-between two integers. Therefore for any integrated bounds both must be at least one unit apart. Moreover the geometries implied here, and beyond will always be less than or equal to one – one being reserved for the unit block, and any value smaller being a piece of the larger pyramid-like structure. The values inputted into the integral will always be discrete, implying that the volumes outputted reflect a definite section of the pyramid. For now, this is as much explanation that seems necessary to move forward. Although, if one is more curious about this reverse in logic this is discussed more thoroughly in section nine.

2.3.2. Quality 2: Yield The Correct Answer. From above it is clear that this method works for at least a few unique cases. Although, this can be easily generalized for the pyramidal shape to demonstrate the accuracy, piece by piece. Consider each integral below to represent a unit section of a $3 \times 3 \times 3$ cube (which contains the square based right pyramid of three units tall) in a stack of three 3×3 matrices (a little hint for what is to come in section 3). It is simple to calculate each integral and show the sum of the shape do as a matter of fact add up to the pyramid as a whole.

$$\begin{bmatrix} \int_0^1 \int_0^1 \int_0^1 (1)dxdydz = 1 & \int_0^1 \int_1^2 \int_0^1 (1)dxdydz = 1 & \int_0^1 \int_2^{3-z} \int_0^1 (1)dxdydz = \frac{1}{2} \\ \int_0^1 \int_0^1 \int_1^2 (1)dxdydz = 1 & \int_0^1 \int_1^2 \int_1^2 (1)dxdydz = 1 & \int_0^1 \int_2^{3-z} \int_1^2 (1)dxdydz = \frac{1}{2} \\ \int_0^1 \int_0^1 \int_2^{3-z} (1)dxdydz = \frac{1}{2} & \int_0^1 \int_1^2 \int_2^{3-z} (1)dxdydz = \frac{1}{2} & \int_0^1 \int_2^{3-z} \int_2^{3-z} (1)dxdydz = \frac{1}{3} \end{bmatrix}$$
$$\begin{bmatrix} \int_1^2 \int_0^1 \int_0^1 (1)dxdydz = 1 & \int_1^2 \int_1^{3-z} \int_0^1 (1)dxdydz = \frac{1}{2} & 0 \\ \int_1^2 \int_0^1 \int_1^{3-z} (1)dxdydz = \frac{1}{2} & \int_1^2 \int_1^{3-z} \int_1^{3-z} (1)dxdydz = \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \int_2^3 \int_0^{3-z} \int_0^{3-z} (1)dxdydz = \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Imagine this matrix is a $3 \times 3 \times 3$ cube, and each cell represents the enclosed volume in a unit block. That is, per matrix, the x-axis points downward, the y-axis points across, and the z-axis jumps from one matrix to another (see Figure 11).

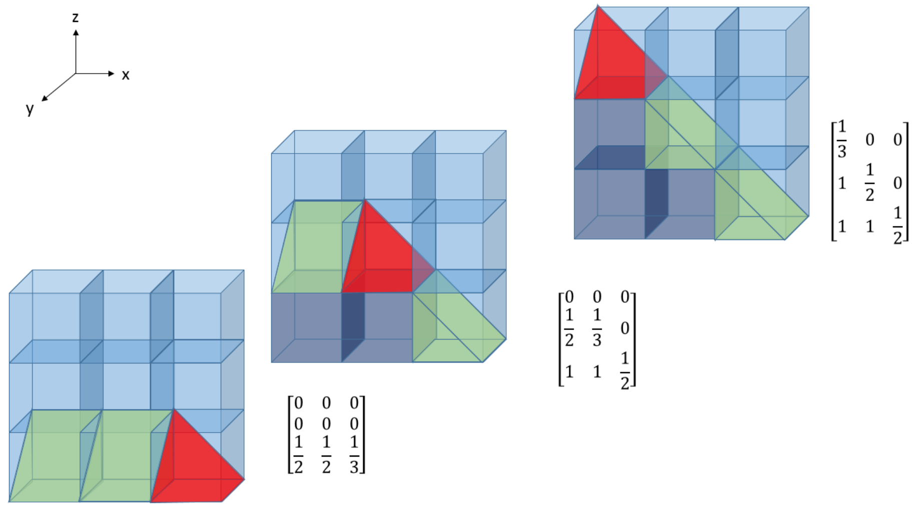

FIGURE 11. The orientation of matrices in space. Note: this matrix representation does not represent the one described above. The coordinate axes have been preserved, but the matrices have been split along the y-axis as opposed to the z-axis. This was done to allow the reader to have a better visual example of how the matrix representation can be used to catalog unit pieces of the pyramid.

Now taking the sum of all the constituent parts:

$$1+1+\frac{1}{2}+1+1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}=9=\frac{27}{3}=\frac{3^3}{3}.$$

It is clear that both methods result in the same numeric value. Although even this is a rather unique case, and a little more can be said abou the general solution before moving on. The clever reader may notice that in the above, there seems to be an axis of symmetry going down the center diagonal of the matrices above. Although, there is more than just the convienient ability to halve the calculations to yield a result, this symmetry reflects the properties of the integrals involved. That is, each of the above integrals can be broken down into three classes:

(1) A unit cube:
in this case each of the integral bounds are integers, resulting in the volume of a cube
$\int_{a-1}^{a}\int_{b-1}^{b}\int_{c-1}^{c}(1)dxdydz=1,$
regardless to where the integral is calculated, if all six boundary numbers are purely integers – the volume is one.

(2) A triangular prism:
in the case only one integral is at a boundary of the pyramid resulting in a one

directional constraint
$\int_{a-1}^{a}\int_{b-1}^{b}\int_{h-z}^{c}(1)dxdydz = \frac{1}{2}$,
This type of integral lies on one edge of the pyramid and yields a volume of $\frac{1}{2}$ irregardless where that edge is.

(3) A pyramid:
in this case, the integral lies on one of the center corners of the pyramid
$\int_{a-1}^{a}\int_{b}^{h-z}\int_{c}^{h-z}(1)dxdydz = \frac{1}{3}$,
this type of integral lies uniquely along the main diagonal, and is bound by two constraints.

It is clear therefore that each geometrical object can either represent a *shape* or a *constraint*. It is this concept of constraints that will move this problem forward into the fourth-dimension and beyond. The three dimensional case allowed for slicing incorporating zero, one, or two unique constraints with respect to the larger pyramid yielding volumes of 1, $\frac{1}{2}$, and $\frac{1}{3}$ respectively. Although what can be said about using more constraints than this – say three? Unfortunately in the three dimensional problem, this is as far as the constraints can go. It is time to move onto the fourth-dimension.

2.3.3. Integration in Four-Dimensions. Now that the background motivation for the integration by slice has been put together. It should be no difficult task to move this method to higher dimensional spaces. For the fourth-dimension, all that is required for computation is to include another integral into the pyramid formula such that the volume-unit is expressed as:

$$\int_0^h \int_0^{h-a} \int_0^{h-a} \int_0^{h-a} (1)dxdydzda.$$

This expression poses no greater challenge than the simple inclusion of one more integral to evaluate. Moreover, it holds the same rules of constraint, with one extra addition. Say for a four-dimensional pyramid of 'height' three from the origin:

$$\int_2^3 \int_0^{3-a} \int_0^{3-a} \int_0^{3-a} (1) dxdydzda =$$

$$\begin{aligned}
&= \int_2^3 \int_0^{3-a} \int_0^{3-a} (3-a) dydzda \\
&= \int_2^3 \int_0^{3-a} (3-a)(3-a) dzda \\
&= \int_2^3 (3-a)(3-a)(3-a) da \\
&= \int_2^3 (27 - 27a + 9a^2 - a^3) da \\
&= 27a - \frac{27a^2}{2} + 3a^3 - \frac{a^4}{4}\Big|_2^3 \\
&= 81 - \frac{243}{2} + 81 - \frac{81}{4} - 54 + \frac{108}{2} - 24 + \frac{16}{4} \\
&= 84 - \frac{135}{2} - \frac{65}{4} \\
&= \frac{336 - 270 - 65}{4} \\
&= \frac{1}{4}.
\end{aligned}$$

Interestingly, this is a volume describing a shape that is neither a block, prism, or pyramid. This is the fourth-dimensional pyramid, enclosing a volume of $\frac{1}{4}$. This represents the shape bound by three constraints in the four-dimensional integral. Integration by slice presents a shape that has no analog to three dimensions, and cannot effectively be drawn on paper. It exists solely by the discrete volume *in a enclosed unit space.*

Using this method the number of shapes within a fourth-dimensional pyramid can be found, and are organized in the following matrices.

$\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

n = 1	
$\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}$	$\begin{bmatrix}\frac{1}{4} & 0\\ 0 & 0\end{bmatrix}$
$\begin{bmatrix}\frac{1}{2} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{4}\end{bmatrix}$	$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

n = 2		
$\begin{bmatrix}1 & 1 & \frac{1}{2}\\ 1 & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$	$\begin{bmatrix}1 & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{3} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 1 & \frac{1}{2}\\ 1 & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$	$\begin{bmatrix}\frac{1}{2} & \frac{1}{3} & 0\\ \frac{1}{3} & \frac{1}{4} & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$
$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

n = 3			
$\begin{bmatrix}1 & 1 & 1 & \frac{1}{2}\\ 1 & 1 & 1 & \frac{1}{2}\\ 1 & 1 & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$	$\begin{bmatrix}1 & 1 & \frac{1}{2} & 0\\ 1 & 1 & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{3} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{1}{4} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 1 & 1 & \frac{1}{2}\\ 1 & 1 & 1 & \frac{1}{2}\\ 1 & 1 & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$	$\begin{bmatrix}1 & 1 & \frac{1}{2} & 0\\ 1 & 1 & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{1}{2} & \frac{1}{3} & 0 & 0\\ \frac{1}{3} & \frac{1}{4} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 1 & 1 & \frac{1}{2}\\ 1 & 1 & 1 & \frac{1}{2}\\ 1 & 1 & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\end{bmatrix}$	$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$
$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4}\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

FIGURE 12. Induction for four-dimensional pyramids.

From here it is simple to use induction to indicate what rate each shape occurs, which should provide a means of producing the generalized solution for the fourth-dimension (see figure 12).

Although at least one more iteration would be preferred, the $5 \times 5 \times 5 \times 5$ matrix is a bit too large to fit on on page. Nevertheless, there is enough useful information provided here to find a general solution. Looking closely at Figure 12, the occurrences of each type of geometry occur at regular patters – some of which have been discovered in previous dimensions (see Table 4).

On a side note relating to table four, the names *triangle*, *prism*, and *pyramid* have been dropped. Although these shapes retain the same volume as in previous dimensions *they*

limit of sum	1	2	3	n
$\frac{1}{4}$	2	3	4	$n+1$
$\frac{1}{3}$	3	9	18	$3(\frac{n(n+1)}{2})$
$\frac{1}{2}$	3	15	42	$3(\frac{n(n+1)(2n+1)}{6})$

TABLE 4. Occurrences of Different Geometries in Four-Dimensions.

are not the same shapes These are all four dimensional objects with the a-axis extend one unit. Consider this like the transition from the two dimensional problem with triangles to the three dimensional problem with prisms and pyramids. That is, the triangular shapes' volume corresponded to a prism – a two dimensional triangle with a length of unit one extending into the third dimension.

This is also seen here, as although the prism retains the volume of $\frac{1}{2}$ and the pyramid retains the volume of $\frac{1}{3}$ both have a unit one axial extension into the fourth dimension. If there were some means of describing this effectively on paper, then there would be a figure to relate to. Unfortunately there is not. In this case, just recall the transition from the triangle to prism with a unit length extension in the latest axis.

One may be curious to know why there is a factor of three for the $\frac{1}{3}$ and $\frac{1}{2}$ shapes. This relates back to the geometrical model for the third-dimension, which multiplied the prisms by a factor of two – that there were two sides for the prisms to stack on. In the fourth-dimensional case there are three.

With the geometries established, it is now possible to find the general formula for the sums of cubic integers. Following the same procedure as before, the expression begins with the difference of the fourth-dimensional pyramid, and the unnecessary geometries that lie under its surface, exposing the desired stepwise pyramid underneath.

$$\begin{aligned}
\frac{(n+1)^4}{4} - \frac{n+1}{4} - \frac{3(\frac{n(n+1)}{2})}{3} - \frac{3(\frac{n(n+1)(2n+1)}{6})}{2} = \\
&= \frac{n^4+4n^3+6n^2+4n+1}{4} - \frac{n+1}{4} - 4\frac{n(n+1)}{4} - \frac{n(n+1)(2n+1)}{4} \\
&= \frac{1}{4}(n^4+4n^3+6n^2+4n+1-n-1-4n^2-4n-2n^3-3n^2+n) \\
&= \frac{1}{4}(n^4+n^3(4-2)+n^2(6-2-3)+n(4-1-2-1)+1-1) \\
&= \frac{n^4+2n^3+n^2}{4}.
\end{aligned}$$

This was clearly a little more technical to calculate, but nevertheless proved itself to produce a very simple quartic polynomial. Although note that there is no n term in the final expression. The reasoning behind this elaborated on in section 9. The values of this polynomial versus the sum is expressed below in Table 5.

limit of sum	1	2	3	4	5	...	n
value of sum	1	9	36	100	125	...	$\frac{n^4+2n^3+n^2}{4}$
$\frac{n^4+2n^3+n^2}{4}$	1	9	36	100	125	...	$\frac{n^4+2n^3+n^2}{4}$

TABLE 5. Generalized Formula Versus Sum of Cubic Integers.

As an observation, consider the relation:

$$\frac{n^4+2n^3+n^2}{4} = (\frac{n(n+1)}{2})^2.$$

Unfortunately this is the only free-bee and does not occur again.

2.3.4. Integration in Five-Dimensions. The power of the integration method provides a very simple means to move up into the next case: the sum of quartic integers in five dimensions via the integral:

$$\int_0^h \int_0^{h-b} \int_0^{h-b} \int_0^{h-b} \int_0^{h-b} (1) dxdydzdadb.$$

There is very little difference in this integral as opposed to the fourth-dimensional integral with exception for the extra constraint, creating another new geometry:

$$\int_{h-1}^{h}\int_{0}^{h-b}\int_{0}^{h-b}\int_{0}^{h-b}\int_{0}^{h-b}(1)dxdydzdadb =$$
$$= \int_{h-1}^{h}\int_{0}^{h-b}\int_{0}^{h-b}\int_{0}^{h-b}(h-b)dydzdadb$$
$$= \int_{h-1}^{h}\int_{0}^{h-b}\int_{0}^{h-b}(h-b)(h-b)dzdadb$$
$$= \int_{h-1}^{h}\int_{0}^{h-b}(h-b)(h-b)(h-b)dadb$$
$$= \int_{h-1}^{h}(h-b)(h-b)(h-b)(h-b)db$$
$$= \int_{h-1}^{h}(h^4-4h^3b+6h^2b^2-4hb^3+b^4)db$$
$$= h^4b-2h^3b^2+2h^2b^3-hb^4+\frac{b^5}{5}\Big|_{h-1}^{h}$$
$$= h^5-2h^5+2h^5-h^5+\frac{h^5}{5}$$
$$-(h^4(h-1)-2h^3(h-1)^2+2h^2(h-1)^3-h(h-1)^4+\frac{(h-1)^5}{5})$$
$$= \frac{h^5}{5}-(h^5-h^4-2h^3(h^2-2h+1)+2h^2(h^3-3h^2+3h-1)$$
$$-h(h^4-4h^3+6h^2-4h+1)+\frac{(h-1)^5}{5})$$
$$= \frac{h^5}{5}-(h^5-h^4-2h^5-2h^4-2h^3+2h^5-6h^4+6h^3-2h^2$$
$$-h^5+4h^4-6h^3+4h^2-h+\frac{(h-1)^5}{5})$$
$$= \frac{h^5}{5}-(h^5-h^4-2h^5+4h^4-2h^3$$
$$+2h^5-6h^4+6h^3-2h^2-h^5+4h^4-6h^3+4h^2-h$$
$$\frac{h^5}{5}-h^4+2h^3-2h^2+h-\frac{1}{5})$$
$$= \frac{h^5}{5}-(h^5(1-2+2-1+\frac{1}{5})+h^4(-1+4-6+4-1)+h^3(-2+6-6+2)$$
$$+h^2(-2+4-2)+h(-1+1)-\frac{1}{5})$$
$$= \frac{h^5}{5}-\frac{h^5}{5}+\frac{1}{5}$$
$$= \frac{1}{5}.$$

Of which is already apparent where this would lie in the matrix diagram. The integration method can therefore be generalized further, with each additional integral produces a new geometry with the reciprocal integer value to the number of integrals present:

$$\int_{h-1}^{h}\int_{0}^{h-k}\dots\int_{0}^{h-k}(1)dx...dk = \frac{1}{k^*}.$$

Where k^* represents the kth integral as an integer. This easy trick above removes the necessity of integrating the more complicated regions. Although now that a pattern is established, why even use integrals at all? The transition poses itself now not like the transition from the geometrical methodology to the integration – where a philosophical deficiency rendered reasonable calculation impossible. In this case, it is a matter of efficiency. Finding the volume of a fifth dimensional shape is an exhaustive task, which if one knows the boundary conditions for – is obsolete. So the transition to the next methodology, matrix formulation, does not come as a closed boundary like the geometrical method, but an open boundary limited by the patience of the user.

2.4. Technique 3: A Matrix Formulation

The rules for the matrix formulation were briefly described in the last section, but it is more than appropriate to review the concept in a more formal matter. Recalling from the integrals above, when one of the integral limits hit and edge of the pyramid shape, its geometry would be reduced from a cubic shape to that which is effectively constrained by the appropriate surfaces. The two-dimensional case only had one constraint – a diagonal line bisecting the x and y axes. This boundary reduced a square into a right triangle. In the three dimensional case, there were two constraints, one of which produced a plane parallel to the xz-axis, and the other the yz-axis. A cube limited by only one of these boundaries would be reduced to a prism, while a cube regarding both constraints would be reduced to a pyramid. This trend continues to higher dimensions, with each constraint adding another boundary, and a further reduction in volume.

Hence one can look at a matrix not just as a collection of numbers, but rather as a visual representation of a multidimensional function. That the position of an element in the 'function' defines what boundary conditions it must adhere to (see Figure 13).

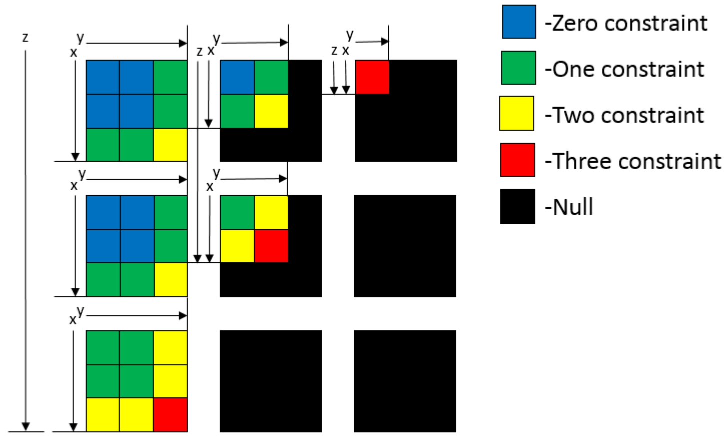

FIGURE 13. How to view a matrix as function of boundaries. Note: although this is for a four-dimensional case, the general procedure is implied.

Now posessing full knowledge of this method, it becomes merely a task of counting the number of shapes within the multidimensional region to find the total volume of each kind of geometry, and then taking the difference versus the volume of the larger pyramid shape. As an example, consider the inductive process associated with the five dimensional pyramid below (see Figures 14 15, and 16).

From these three sums one can find the number of each type of geometry as follows (see Table 6).

limit of sum	1	2	3	...	n
$\frac{1}{5}$	2	3	4	...	$1(n+1)$
$\frac{1}{4}$	4	12	24	...	$4\frac{n(n+1)}{2}$
$\frac{1}{3}$	6	30	84	...	$6\frac{n(n+1)(2n+1)}{6}$
$\frac{1}{2}$	4	36	144	...	$4\frac{n^4+2n^3+n^2}{4}$.

TABLE 6. Number of Geometries Present in Quartic Sum.

Moreover table six also gave a new clue for solving each generalized case. That is, for each new dimension added, each geometry moves up in the polynomial scale – with its respective count becoming that of the sum ahead of it. Although there is one more trick to this that without due consideration will result in a wrong conclusion. That is, it relates back to the number of exposed sides. A clever reader may have gotten the hint with the included scalars in front of each polynomial expression in column six, though solving for the quartic numbers, and comparing them to previous results should produce a dead giveaway.

$$\text{n} = 1$$

$$\begin{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\text{n} = 2$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} & \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} & \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

FIGURE 14. Matrix Representation for the Inductive Process of a Five Dimensional Pyramid I.

$$\text{n} = 3$$

$$\begin{bmatrix}
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\end{bmatrix} \\
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\end{bmatrix} \\
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\1&1&1&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\end{bmatrix} \\
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\end{bmatrix} &
\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{5}\end{bmatrix}
\end{bmatrix}$$

$$\begin{bmatrix}
\begin{bmatrix}1&1&\frac{1}{2}&0\\1&1&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}1&1&\frac{1}{2}&0\\1&1&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \\
\begin{bmatrix}1&1&\frac{1}{2}&0\\1&1&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}1&1&\frac{1}{2}&0\\1&1&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \\
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{3}&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&0\\\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&0\\\frac{1}{4}&\frac{1}{4}&\frac{1}{5}&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \\
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} &
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}
\end{bmatrix}$$

FIGURE 15. Matrix Representation for the Inductive Process of a Five Dimensional Pyramid II.

$$\begin{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

FIGURE 16. Matrix Representation for the Inductive Process of a Five Dimensional Pyramid III.

So solving the quartic numbers as before:

$$\frac{(n+1)^5}{5}-\frac{n+1}{5}-\frac{4(\frac{n(n+1)}{2})}{4}-\frac{6(\frac{n(n+1)(2n+1)}{6})}{3}-\frac{4(\frac{n^4+2n^3+n^2}{4})}{2}=$$
$$=\frac{n^5+5n^4+10n^3+10n^2+5n+1}{5}-\frac{n+1}{5}-\frac{4(\frac{n(n+1)}{2})}{4}-$$
$$\frac{6(\frac{n(n+1)(2n+1)}{6})}{3}-\frac{4(\frac{n^4+2n^3+n^2}{4})}{2}$$
$$=\frac{n^5+5n^4+10n^3+10n^2+5n+1}{5}-\frac{n+1}{5}-\frac{n(n+1)}{2}-$$
$$\frac{n(n+1)(2n+1)}{3}-\frac{n^4+2n^3+n^2}{2}$$
$$=\frac{n^5+5n^4+10n^3+10n^2+5n+1}{5}-\frac{n+1}{5}-\frac{n^2+n}{2}-$$
$$\frac{2n^3+3n^2+n}{3}-\frac{n^4+2n^3+n^2}{2}$$
$$=n^5(\frac{1}{5})+n^4(\frac{5}{5}-\frac{1}{2})+n^3(\frac{10}{5}-\frac{2}{3}-\frac{2}{2})+n^2(\frac{10}{5}-\frac{1}{2}-\frac{3}{3}-\frac{1}{2})+$$
$$n(\frac{5}{5}-\frac{1}{5}-\frac{1}{2}-\frac{1}{3})+(\frac{1}{5}-\frac{1}{5})$$
$$=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}+n^2(0)-\frac{n}{30}+0$$
$$=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}.$$

Which yields a quintic polynomial. Although what is unique in this case is the dropping of the n^2 term as well as the negative $\frac{n}{3}$ term at the end. The literal reason for this is provided in section eight as part of the results – and is revisited again in section nine through an overall interpretation. That is, the fact that the polynomial appears the way it does is very special. There is a very good reason for this, that will inherit a much deeper discussion later.

As for now, this expression opens up another clue to solving this problem, that is – not only is there a 'stepping down' of the summation terms per geometry, there is also a binomial expansion term for each type of geometry, depending on the dimension the sum is bound by (see Table 7).

Now understanding the volume, frequency, and multiple of each type of geometry per dimension, it would seem that the matrix technique has been, like the integration by slice technique – rendered obsolete. Once again this is motivated by the cost of efficiency – as seen from figure fourteen: drawing matrices for five, six, and any dimensions above is highly intuitive (perhaps better so than the integration method) but is also highly exhausting. Diverging from this method only comes as a benefit to the reader, as this reduces computation time down significantly. It is time to throw away all visual example, and use solely recursion and binomial expansions. The problem is nearing the end, but there is still aways to go before the final solution.

t					
0	$(1)n+1[1]$				
1	$(1)\frac{n(n+1)}{2}[1]$	$(1)n+1[\frac{1}{2}]$			
2	$(1)\frac{n(n+1)(2n+1)}{6}[1]$	$(2)\frac{n(n+1)}{2}[\frac{1}{2}]$	$(1)n+1[\frac{1}{3}]$		
3	$(1)\frac{n^4+2n^3+n^2}{4}[1]$	$(3)\frac{n(n+1)(2n+1)}{6}[\frac{1}{2}]$	$(3)\frac{n(n+1)}{2}[\frac{1}{3}]$	$(1)n+1[\frac{1}{4}]$	
4	$(1)\frac{6n^5+15n^4+10n^3-n}{30}[1]$	$(4)\frac{n^4+2n^3+n^2}{4}[\frac{1}{2}]$	$(6)\frac{n(n+1)(2n+1)}{6}[\frac{1}{3}]$	$(4)\frac{n(n+1)}{2}[\frac{1}{4}]$	$(1)n+1[\frac{1}{5}]$

TABLE 7. Pascal's Polynomial Triangle.

2.5. Technique 4: Recursive Relation

This technique is unique from all others, as there is no visual representation to allude to what this mechanism is *physically* doing. The best means of understanding what this technique is, and why it works is to either refer to table seven, or reread from the beginning. From this point on, there is no little back-handed tricks or free-bees given by the system. This may come as a disappointment to many, but perhaps with a little perspective one can appreciate what has been done thus far. The equation below represents a benchmark – a simple, concise expression that in itself is a form of solution. That is, one can use it to solve any kind of sum and come up with the correct answer. It represents a maturing of the problem from something made out of hand-held blocks into an adult form of expression using symbolic language to convey complicated polynomial expressions. The recursive formula is as follows:

$$S_{(n,t)} = \frac{(n+1)^{t+1}}{t+1} - \sum_{i=1}^{t} \frac{\binom{t}{i} S_{(n,t-i)}}{i+1}.$$

This is nothing more than a formal description of Table 7. Although there is no reason that this should not be taken for a 'test drive' to ensure that it yields accurate results. Suppose n is fixed in this case, and that an arbitrary sum is desired. Moving through different choices of t:

Let t = 1:

$$\begin{aligned}
S_{(n,1)} &= \frac{(n+1)^2}{2} - \sum_{i=1}^{1} \frac{\binom{1}{i} S_{(n,1-i)}}{i+1} \\
&= \frac{(n+1)^2}{2} - \frac{\binom{1}{1} S_{(n,0)}}{2} \\
&= \frac{(n+1)^2}{2} - (1)\frac{n+1}{2} \\
&= \frac{n+1}{2}(n+1-1) \\
&= \frac{n(n+1)}{2}.
\end{aligned}$$

Let t = 2:

$$\begin{aligned}
S_{(n,2)} &= \frac{(n+1)^3}{3} - \sum_{i=1}^{2} \frac{\binom{2}{i} S_{(n,2-i)}}{i+1} \\
&= \frac{(n+1)^3}{3} - \frac{\binom{2}{1} S_{(n,1)}}{2} - \frac{\binom{2}{2} S_{(n,0)}}{3} \\
&= \frac{(n+1)^3}{3} - (2)\frac{\frac{n(n+1)}{2}}{2} - (1)\frac{n+1}{3} \\
&= \frac{(n+1)^3}{3} - \frac{n(n+1)}{2} - \frac{n+1}{3} \\
&= \frac{n+1}{3}(n^2+2n+1-\frac{3n}{2}-1) \\
&= \frac{n+1}{3}(n^2+\frac{n}{2}) \\
&= \frac{n+1}{3}(n(n+\frac{1}{2})) \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Let t = 3:

$$
\begin{aligned}
S_{(n,3)} &= \frac{(n+1)^4}{4} - \sum_{i=1}^{3} \frac{\binom{3}{i} S_{(n,3-i)}}{i+1} \\
&= \frac{(n+1)^4}{4} - \frac{\binom{3}{1} S_{(n,2)}}{2} - \frac{\binom{3}{2} S_{(n,1)}}{3} - \frac{\binom{3}{3} S_{(n,0)}}{4} \\
&= \frac{(n+1)^4}{4} - (3)\frac{\frac{n(n+1)(2n+1)}{6}}{2} - (3)\frac{\frac{n(n+1)}{2}}{3} - (1)\frac{n+1}{4} \\
&= \frac{n+1}{4}((n+1)^3 - (12)\frac{\frac{n(n+1)(2n+1)}{6}}{2} - (12)\frac{\frac{n(n+1)}{2}}{3} - 1) \\
&= \frac{n+1}{4}((n+1)^3 - n(2n+1) - 2n - 1) \\
&= \frac{n+1}{4}(n^3 + 3n^2 + 3n + 1 - 2n^2 - n - 2n - 1) \\
&= \frac{n+1}{4}(n^3 + n^2) \\
&= \frac{n^4 + 2n^3 + n^2}{4}.
\end{aligned}
$$

Let t = 4:

$$
\begin{aligned}
S_{(n,4)} &= \frac{(n+1)^5}{5} - \sum_{i=1}^{4} \frac{\binom{4}{i} S_{(n,4-i)}}{i+1} \\
&= \frac{(n+1)^5}{5} - \frac{\binom{4}{1} S_{(n,3)}}{2} - \frac{\binom{4}{2} S_{(n,2)}}{3} - \frac{\binom{4}{3} S_{n,1}}{4} - \frac{\binom{4}{4} S_{(n,0)}}{5} \\
&= \frac{(n+1)^5}{5} - (4)\frac{\frac{n^4+2n^3+n^2}{4}}{2} - (6)\frac{\frac{n(n+1)(2n+1)}{6}}{3} - (4)\frac{\frac{n(n+1)}{2}}{4} - (1)\frac{n+1}{5} \\
&= \frac{n+1}{30}(6(n+1)^4 - 15n^2(n+1) - 10n(2n+1) - 15n - 6) \\
&= \frac{n+1}{30}(6n^4 + 24n^3 + 36n^2 + 24n + 6 - 15n^3 - 15n^2 - 20n^2 - 10n - 15n - 6) \\
&= \frac{n+1}{30}(6n^4 + 9n^3 + n^2 - n) \\
&= \frac{6n^5 + 9n^4 + n^3 - n^2 + 6n^4 + 9n^3 + n^2 - n}{30} \\
&= \frac{6n^5 + 15n^4 + 10n^3 - n}{30}.
\end{aligned}
$$

Of which at this point, the recursion method has yielded every single answer found before. Without droning further with more computation – below (see Table 8) the next three solutions are provided with all the ones before it.

t	
0	$n+1$
1	$\frac{n^2}{2}+\frac{n}{2}$
2	$\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$
3	$\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$
4	$\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$
5	$\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$
6	$\frac{n^7}{7}+\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}$
7	$\frac{n^8}{8}+\frac{n^7}{2}+\frac{7n^6}{12}-\frac{7n^4}{24}+\frac{n^2}{12}$

TABLE 8. Generalized Solutions for t-th Dimensional Sums.

In short, the recursion formula is a success – although as said before, this is not the end. The recursion formula only states the solution *with respect to previous iterations of the solution* that, it is not an independent object, but requires previous knowledge of the expression to produce future computation. Bear in mind, there are some systems like this – called hysteresis loops, and occur in many engineering topics (among other places). Fortunately in this case, this problem is *privileged* to continue further(as there is no reason that it necessarily has to).

At this point, table eight gives enough information to pull everything together for a final showdown. This final step will require extensive use of algebra, and symbolic representation, but in the end it is well worth the trouble. This final exit of one technique for another presents itself as neither a philosophical constraint, or a matter of efficiency, but rather as a completion of one simple question.

2.6. Technique 5: Ruthless Computation

The objective of this section is to use the recursion formula provided above to produce a complete, generalized solution to the problem. That is, assume that each $S_{(n,t)}$ is solved up to some value $t-1$, then the recursion formula states that taking the difference of the pyramid of $t+1$-dimensions with each of the constituent parts will solve for $S_{(n,t)}$. The only trick to this is working from the top down and choosing and arbitrarily large t.

There are a few clues in table eight that will guide the solution. Notice that the first term on the far left (the largest coefficient of the polynomial) always has a fractional denominator equal to its respective exponent. This will come in handy for the computation. Notice also that the next coefficient to the right (one degree lower) always has a fractional denominator of two. Finally, notice that the first three polynomial terms are all integer steps of each other, while the rest step down by two integers, and oscillate positive and negative terms (take the last clue as an ansatz).

Hence, two coefficients are already generalized just by using some simple logic! Thus what is now known about the general solution is:

$$S_{(n,t)} = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + ...$$

It is now time to find the third coefficient. Using the recursion relation, one can express all possible polynomials in the form:

$$S_{(n,t)} = \frac{(n+1)^{t+1}}{t+1} - \sum_{i=1}^{t} \frac{\binom{t}{i} S_{(n,t-i)}}{i+1}$$

$$\begin{aligned} = \frac{\binom{t+1}{0} n^{t+1}}{t+1} + \frac{\binom{t+1}{1} n^t}{t+1} + \frac{\binom{t+1}{2} n^{t-1}}{t+1} \\ -\frac{1}{2}\binom{t}{1}\frac{n^t}{t} - \frac{1}{2}\binom{t}{1}\frac{n^{t-1}}{2} \\ -\frac{1}{3}\binom{t}{2}\frac{n^{t-1}}{t-1} \end{aligned}$$

$$\begin{aligned} = \frac{n^{t+1}}{t+1} + \frac{(t+1)n^t}{t+1} + \frac{(t)(t+1)n^{t-1}}{2(t+1)} \\ -\frac{tn^t}{2t} - \frac{tn^{t-1}}{4} \\ -\frac{(t)(t-1)n^{t-1}}{6(t-1)} \end{aligned}$$

$$\begin{aligned} = \frac{n^{t+1}}{t+1} + n^t(1 - \frac{1}{2}) + n^{t-1}(\frac{t}{2} - \frac{t}{4} - \frac{t}{6}) \\ = \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \frac{tn^{t-1}}{12}. \end{aligned}$$

This yields the third coefficient in its generalized form. The information just acquired about the third coefficient can be used to find the fourth. One may infer the solution by looking at table eight, though algebraically the solution is given below:

$$
\begin{aligned}
= \frac{\binom{t+1}{0} n^{t+1}}{t+1} + \frac{\binom{t+1}{1} n^{t}}{t+1} + \frac{\binom{t+1}{2} n^{t-1}}{t+1} + \frac{\binom{t+1}{3} n^{t-2}}{t+1} \\
- \frac{1}{2}\binom{t}{1}\frac{n^t}{t} - \frac{1}{2}\binom{t}{1}\frac{n^{t-1}}{2} - \frac{1}{2}\binom{t}{1}\frac{(t-1)n^{t-2}}{12} \\
- \frac{1}{3}\binom{t}{2}\frac{n^{t-1}}{t-1} - \frac{1}{3}\binom{t}{2}\frac{n^{t-2}}{2} \\
- \frac{1}{4}\binom{t}{3}\frac{n^{t-2}}{t-2}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{n^{t+1}}{t+1} + \frac{(t+1)n^t}{t+1} + \frac{(t+1)(t)n^{t-1}}{2(t+1)} + \frac{(t+1)(t)(t-1)n^{t-2}}{6(t+1)} \\
- \frac{tn^t}{2t} - \frac{tn^{t-1}}{4} - \frac{(t)(t-1)n^{t-2}}{24} \\
- \frac{(t)(t-1)n^{t-1}}{6(t-1)} - \frac{(t)(t-1)n^{t-2}}{12} \\
- \frac{(t)(t-1)(t-2)n^{t-2}}{24(t-2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n^{t+1}}{t+1} + (n^t)(1 - \frac{1}{2}) + n^{t-1}(\frac{t}{2} - \frac{t}{4} - \frac{t}{6}) \\
&+ n^{t-2}(\frac{(t)(t-1)}{6} - \frac{(t)(t-1)}{24} - \frac{(t)(t-1)}{12} - \frac{(t)(t-1)}{24}) \\
&= \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \frac{tn^{t-1}}{12} + n^{t-2}(0) \\
&= \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \frac{tn^{t-1}}{12} + 0 \\
&= \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \frac{tn^{t-1}}{12} + 0\frac{(t)(t-1)n^{t-2}}{1}.
\end{aligned}
$$

Hence it has been shown that the $(n-2)$th term is trivial. Although it would be easy to disregard this term – it is best to leave it for the sake of the larger pattern. Continuing on to the $(n-3)$th expression:

$$
\begin{aligned}
&= \frac{\binom{0}{t+1}n^{t+1}}{t+1} + \frac{\binom{1}{t+1}n^{t}}{t+1} + \frac{\binom{2}{t+1}n^{t-1}}{t+1} + \frac{\binom{3}{t+1}n^{t-2}}{t+1} + \frac{\binom{4}{t+1}n^{t-3}}{t+1} \\
&\quad -\frac{1}{2}\binom{1}{t}\frac{n^t}{t} - \frac{1}{2}\binom{1}{t}\frac{n^{t-1}}{2} - \frac{1}{2}\binom{1}{t}\frac{(t-1)n^{t-2}}{12} - \frac{0}{2}\binom{1}{t}n^{t-3} \\
&\quad -\frac{1}{3}\binom{2}{t}\frac{n^{t-1}}{t-1} - \frac{1}{3}\binom{2}{t}\frac{n^{t-2}}{2} - \frac{1}{3}\binom{2}{t}\frac{(t-2)n^{t-3}}{12} \\
&\quad -\frac{1}{4}\binom{3}{t}\frac{n^{t-2}}{t-2} - \frac{1}{4}\binom{3}{t}\frac{n^{t-3}}{2} \\
&\quad -\frac{1}{5}\binom{4}{t}\frac{n^{t-3}}{t-3}.
\end{aligned}
$$

As the first four coefficients have already been found, it is only a matter of solving the n^{t-3} terms that will yield the next coefficient:

$$
\begin{aligned}
&= \frac{\binom{4}{t+1}n^{t-3}}{t+1} - \frac{0}{2}\binom{1}{t}n^{t-3} - \frac{1}{3}\binom{2}{t}\frac{(t-2)n^{t-3}}{12} - \frac{1}{4}\binom{3}{t}\frac{n^{t-3}}{2} - \frac{1}{5}\binom{4}{t}\frac{n^{t-3}}{t-3} \\
&= n^{t-3}\left(\frac{(t+1)(t)(t-1)(t-2)}{24(t+1)} - \frac{(t)(t-1)(t-2)}{72} - \frac{(t)(t-1)(t-2)}{48} - \frac{(t)(t-1)(t-2)(t-3)}{120(t-3)}\right) \\
&= n^{t-3}\left(\frac{(t)(t-1)(t-2)}{24} - \frac{(t)(t-1)(t-2)}{72} - \frac{(t)(t-1)(t-2)}{48} - \frac{(t)(t-1)(t-2)}{120}\right) \\
&= n^{t-3}(t)(t-1)(t-2)\left(\frac{1}{24} - \frac{1}{72} - \frac{1}{48} - \frac{1}{120}\right) \\
&= -\frac{n^{t-3}(t)(t-1)(t-2)}{720}.
\end{aligned}
$$

Before going further with the coefficients, it is apparent there is a definite pattern to the mechanics of the solution. If the pattern can be logically expressed, then it should prove to also solve the next coefficient (of which now there is a simple method to compute it, as above). Consider the fist five terms thus far:

$$= \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \frac{tn^{t-1}}{12} + 0\frac{(t)(t-1)n^{t-2}}{1} - \frac{n^{t-3}(t)(t-1)(t-2)}{720}...$$

As the polynomial to power $t+1$ steps down, it releases its respective power as part of the product – like a derivative. This is coupled with a sequence of rational numbers (left) that appear very similar to the Bernoulli numbers (right):

$$\{1, \frac{1}{2}, \frac{1}{12}, 0, -\frac{1}{720}\}, \{1, \pm\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}\}.$$

Taking the ratio of each (and ignoring the zero terms, and holding the $\pm\frac{1}{2}$ to be solely positive) the coefficients can be related through:

$$\{\frac{1}{1}, \frac{\frac{1}{2}}{\frac{1}{2}}, \frac{\frac{1}{6}}{\frac{1}{12}}, 0, \frac{\frac{1}{30}}{-\frac{1}{720}}\},$$

$$\{1, 1, 2, 0, 24\},$$

$$\{0!, 1!, 2!, 0, 4!\}.$$

If the relation holds true, then the next nontrivial coefficient should be $\frac{1}{6!*42}$, which are the related term, and the sixth Bernoulli number in the denominator, respectively. In the numerator the expression should be $n^{t-5}(t)(t-1)(t-2)(t-3)(t-4)$. This results in the first seven coefficients:

$$= \frac{n^{t+1}}{t+1} + \frac{n^t}{2} + \frac{tn^{t-1}}{12} + 0 - \frac{n^{t-3}(t)(t-1)(t-2)}{720} + 0 + \frac{n^{t-5}(t)(t-1)(t-2)(t-3)(t-4)}{30240} - ...$$

Testing this for $t = 6$ and $t = 7$ yields (respectively):

$$= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} - ...$$
$$= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12} - ...$$

Of which is *exactly* the correct solution.

2.7. The Result

Using the culmination of everything described above, the solution is expressed in the form:

$$S_{(n,t)} = \sum_{i=0}^{t} \frac{B_i}{i!} (\frac{d}{dn})^i \frac{n^{t+1}}{t+1}.$$

Despite being a sum, it is unlike the sum before, as this generates all the correct coefficients, requiring only knowing the Bernoulli numbers (B_i), and the use of a derivative! A few things to note: it would appear that the Bernoulli numbers appeared out of thin air. In actuality, Bernoulli was one of the first people to generalize this expression (published in 1713), and the set of rational numbers associated with these types of sums is accredited to him. This paper could have either deviated a generating function for the Bernoulli numbers (which would have probably taken another twenty pages or so) or just outright provided their existence to complete the problem at hand. Also to note, there is one peculiarity with this expression, namely that:

$$\begin{aligned} S_{(n,0)} &= \sum_{i=0}^{0} \frac{B_i}{i!}(\frac{d}{dn})^i \frac{n^{t+1}}{t+1} \\ &= \frac{B_0}{0!}n \\ &= n. \end{aligned}$$

This is in conflict with one of the original assumptions that $S_{(n,0)} = n + 1$, and was used as the basis of the recursive formula. Perhaps the simplest explanation for this is simply that $0^0 = 1$, that for every other expression other than the 0^{th} exponent, the sum from $i = 0$ to n incorporates zero to be... zero. Perhaps this is a confusing explanation. Hopefully Table 9 can enlighten the matter.

$t = 0$ starting $i = 0$	$0^0 = 1$	$0^0 + 1^0 = 2$	...	$0^0 + 1^0 + 2^0 + ... + n^0 = n + 1$
$t = 0$ starting with $i = 1$	$1^0 = 1$	$1^0 + 2^0 = 2$	...	$1^0 + 2^0 + 3^0 + ... + n^0 = n$

TABLE 9. The 0^{th} Issue Explained.

It is this simple discrepancy in notation that leaves the summation shifted one to the right, which aside from the trivial case is irrelevant. To prevent further confusion consider the following amendment to the final solution which incorporates the 0-dimensional case.

$$S_{(n,t)} = 0^t + \sum_{i=0}^{t} \frac{B_i}{i!}(\frac{d}{dn})^i \frac{n^{t+1}}{t+1}.$$

This question is left open for further investigation by the reader, but was not the original focus of the paper, and hence will not be discussed any further. Also left for investigation is *Pascal's polynomial triangle* which made its appearance in table seven. This is a fascinating concept that may yield even more results in the future. As for now, the original task is done. The expression is solved.

CHAPTER 3

From Integer to Real Exponents

3.1. Closure Regarding Elements of Chapters One and Two

What had been left hanging from the end of Chapter one and chapter two will finally be addressed in chapter three. Largely, what had been left out were proofs of the above claims regarding the sum of integer powers. This was due to the fact that such proofs placed in the rigorous standards of modern mathematics differ greatly then the original motivations that led to their formulations in the first place. It seemed to be the wiser choice to introduce the motivation, and hold back on the proof until now. Without introducing too much complexity in our approach, let us consider starting with the generator of the Bernoulli numbers. This is stated as a definition below:

Definition 4. The *Generating Function of Bernoulli Numbers*, is the function of which is equivalent to the Bernoulli numbers in the following form:

$$\frac{x}{e^x - 1} \equiv \sum_{l=0}^{\infty} \frac{B_l x^l}{l!}.$$

There is a small discrepancy in the above expression that differs from our previous convention. That is, in the above $B_1 = -\frac{1}{2}$ which does not follow from before. This is due to the underlining fact that there are two types of Bernoulli numbers, the first kind whose set of rational numbers includes negative one-half and the second kind of Bernoulli numbers which includes positive one half. Aside from this distinction, both types of Bernoulli numbers are the same. This definition is slightly counterintuitive with regards to computation, significantly reduces the complexity of the proof itself. Consider below the second kind of Bernoulli numbers, with regards to the following theorem. It exists in a form very different from the Bernoulli numbers of the first kind in a way that cannot be used in the proof. [**11**] Secondly, this choice of definition give the last (but certainly not least) bit of credit to the German mathematician Johann Faulhaber who presented the proof of sums of integer powers, and whose name is given to its respective theorem [**6**].

$$B_n \equiv \frac{1}{n!} \int_0^1 dx \prod_{i=0}^{n-1} (x - i)$$

THEOREM 3. *(Faulhaber's Formula) For $k \in \mathbb{N}, n \in \mathbb{Z}$, and $2 \leq n$, the sums of the integer to a power k is equal to the sums of the first k powers of the below form:*

$$\sum_{m=1}^{n-1} m^k = \frac{1}{1+k} \sum_{i=0}^{k} \binom{k+1}{i} B_i n^{k+1-i}.$$

PROOF. Consider first the sum of the exponential terms of the form: $\sum_{m=0}^{n-1} e^{mx}$. This is the pivotal expression which will be accessed two different ways yielding both the left hand, and right hand expressions. Considering first the left hand side of the formula we can rearrange the sum in the form:

$$\sum_{m=0}^{n-1} e^{mx} = \sum_{m=0}^{n-1} \sum_{k=0}^{\infty} \frac{m^k x^k}{k!} = \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} \frac{m^k x^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{n-1} m^k \right) \frac{x^k}{k!}.$$

Whereas the swapping of the sums is allowed by Tonelli's Theorem regarding summands. Now reconsidering the exponential sum, we can also interpret it as a geometric series such that for finite but arbitrary m:

$$\sum_{m=0}^{n-1} e^{mx} = \frac{1-e^{nx}}{1-e^x} = \frac{e^{nx}-1}{e^x-1} = \frac{e^{nx}-1}{x} \frac{x}{e^x-1} = \left(\frac{e^{nx}-1}{x} \right) \sum_{l=0}^{\infty} \frac{B_l x^l}{l!}.$$

In the last step the second fraction was replaced by the definition of Bernoulli numbers of the first kind. The first partition of the fraction can be expressed in the following form:

$$\frac{e^{nx}-1}{x} = \sum_{j=0}^{\infty} \frac{\frac{n^j x^j}{j!} - 1}{x} = \sum_{j=1}^{\infty} \frac{\frac{n^j x^j}{j!}}{x} \xrightarrow[\text{r=j-1}]{} \sum_{r=0}^{\infty} \frac{n^{r+1} x^r}{(r+1)!}.$$

Plugging this back into the above yields:

$$\sum_{m=0}^{n-1} e^{mx} = \sum_{r=0}^{\infty} \frac{n^{r+1} x^r}{(r+1)!} \sum_{l=0}^{\infty} \frac{B_l x^l}{l!} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{B_i x^i n^{k+1-i} x^{k-i}}{(k+1-i)!} \left(\frac{k+1}{k+1} \right) \left(\frac{k!}{k!} \right),$$

where the third equality is as consequence of interpreting the double sequence in the second part of the equality as a Cauchy product $\sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} b_m = \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j b_{k-j}$. Rearranging the terms above yields:

$$\sum_{m=0}^{n-1} e^{mx} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i n^{k+1-i} \right) \frac{x^k}{k!}.$$

Now equating the two branches of the equality together we find that:

$$\sum_{k=0}^{\infty}\left(\sum_{m=0}^{n-1} m^k\right)\frac{x^k}{k!} = \sum_{m=0}^{n-1} e^{mx} = \sum_{k=0}^{\infty}\left(\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i n^{k+1-i}\right)\frac{x^k}{k!}.$$

Considering the content of each of the terms in parenthesis yields:

$$\sum_{m=0}^{n-1} m^k = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i n^{k+1-i}$$

$$\implies \sum_{m=1}^{n} m^k = \frac{1}{k+1}\sum_{i=0}^{k}(-1)^i\binom{k+1}{i}B_i n^{k+1-i}.$$

If we then choose to use the Bernoulli numbers of the second kind, the additional negative one can be deleted yielding the original result as was presented in chapters one and two. This completes the proof of Faulhaber's Formula. □

The next thing on the intellectual agenda would be to demonstrate the equivalence of the solution presented in Chapter one versus the form of the solution presented in Chapter two. Although coming from very different interpretations, and presented in different forms both are exactly the same. To establish this, a few more definitions must be presented largely on the consideration of analytic extensions of formulas as to which were already presented above. The first of which is the gamma function ($\Gamma(x)$) presented below [**8**]:

Definition 5. *Gamma Function*, For $0 < x < \infty$ the following function converges

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt.$$

There are other texts which go into greater detail regarding various properties of the gamma function. Fortunately, only three specific qualities characterize $\Gamma(x)$ completely [**8**] discussed in the following theorem:

Theorem 4. *Consider the following:*

(1) $\Gamma(x+1) = x\Gamma(x)$, *for* $0 < x < \infty$.
(2) $\Gamma(n+1) = n!$, *for* $n \in \mathbb{N}$.
(3) *log*$(\Gamma(x))$ *is convex on* $(0, \infty)$.

Proof. From the definition, integration by parts proves (1). Moreover, as $\Gamma(1) = 1$, (2) is implied via induction. (3) is not required for this text, but application of Hölder's inequality proves that result. □

Having established that the Γ function is the analytical extension of the the factorial function (!), we can proceed to apply it in a novel manner to describe the derivative. Consider the following definition. [**3**]

Definition 6. *Fractional Derivative Operator*, is the function which takes the derivative of a differential function to a non-integer order. Such that, for a differential function $f^0(t)$:

$$D_t^\alpha(f^0(t)) \equiv f^\alpha(t).$$

For example, consider the monomial polynomial t^β. Taking the derivative of order α yields:

$$D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}.$$

This may appear strange, but as a matter of fact this is no different from the ordinary derivative. As a more specific example, let us set $\alpha = 1/2$, and let $\beta = 1$.

$$D_t^{\frac{1}{2}} t = \frac{\Gamma(2)}{\Gamma(3/2)}\sqrt{t} = \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

Applying the operator one more time:

$$D_t^{\frac{1}{2}} \frac{2\sqrt{t}}{\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(3/2)}{\Gamma(1)}(1) = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}(1) = 1.$$

Which is equivalent to taking the integer derivative ($1/2+1/2 = 1$) once yielding a constant value. This is sufficiently enough to move onto the next step and demonstrate the equivalence between the sums of integer powers presented in chapter one, and that which was presented in chapter two. The equivalence is presented in the following theorem below:

Theorem 5. *(Pyramid and Box Equivalence) The following two expressions for the sums of integer powers:*

(1) $\frac{1}{t+1}\sum_{i=0}^{t}\binom{t+1}{j}B_i n^{t+1-i}$.
(2) $\sum_{i=0}^{t}\frac{B_i}{i!}(\frac{d}{dn})^i\frac{n^{t+1}}{t+1}$.

Are equivalent.

Proof. Considering expression (1) we can first disassemble the choose function into a composition of factorials such that:

$$\begin{aligned}\frac{1}{t+1}\sum_{i=0}^{t}\binom{t+1}{i}B_i n^{t+1-i} &= \frac{1}{t+1}\sum_{i=0}^{t}\frac{(t+1)!}{i!(t+1-i)!}B_i n^{t+1-i}\\ &= \frac{1}{t+1}\sum_{i=0}^{t}\frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)}B_i n^{t+1-i}.\end{aligned}$$

Where in the last step, the factorials were then converted to Γ functions. Now considering expression (2), we can rewrite the derivative using the fractional-order definition from above, as well as apply the definition of the Γ function. Moreover, applying the example of the monomial allows us to rewrite the expression in the following form:

$$\sum_{i=0}^{t} \frac{B_i}{i!}(\frac{d}{dn})^i \frac{n^{t+1}}{t+1} = \frac{1}{t+1} \sum_{i=0}^{t} \frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)} B_i n^{t+1-i}.$$

Hence, the two expressions are equivalent. $\square$

This resolves all loose ends from the previous chapters. From here, the objective is to extend the sum of integer powers to that of real powers with the final goal of approximating the sums of the square-roots (of power one-half).

3.2. Sums of Real Powers

Unlike the sums of integer powers, the sums of rational, or real powers cannot be stated in a closed form such as a finite polynomial. Surprisingly though, the transition between the integers to rationals, and integers to reals is equivalent. This is largely due to what was discussed in Chapter one, as integers brought up to rational or real powers as to which is not a root, generates an irrational number. These numbers cannot be added the same way as integer or rational numbers can. The solution is presented as the Theorem 6 below, with the proceeding discussion regarding its derivation sufficient 'proof'.

THEOREM 6. *(Sums of Real Powers) For $n \in \mathbb{N}, t \in (-1, \infty)$, the sum of real powers appears in the following form:*

$$\sum_{i=1}^{n} i^t = (n+1)^{t+1-j}(-1)^j \binom{t+1}{j} B_{n,j}.$$

$$B_{n,j} \equiv (n+1)^j \left(\sum_{i=1}^{n} \frac{1}{i} + \frac{j}{n+1} \right) - \sum_{k=0}^{j-1} \sum_{r=0}^{k} (n+1)^{j-r} \frac{\binom{k+1}{r} B_r}{k+1}.$$

A slightly different route must be taken when taking the sums of non-integer powers. This route will be marked out with the following steps. First of which we need to convert this sum of irrational numbers into a sum of integers. Recall from chapter one the binomial theorem which states for two numbers a and b and n$\in \mathbb{N}$:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} b^i a^{n-i}.$$

This can be neatly rearranged to represent the difference in numbers a and b:

$$(a-b)^n = (a+(-b))^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} b^i a^{n-i}.$$

Where the alternating sign is attributed to the original theorem whereas every odd exponent of b is negative. Observe from the above that extending the summation to be from n to infinity does not change the result at all. Meaning that:

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}b^ia^{n-i}=\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}b^ia^{n-i}.$$

This is due to the point that the choose function is zero for all i greater than n, when n is an integer. In examining the right hand side, consider the case as to which n is a positive, real number. As stated above, the choose function can be rewritten in the form of gamma functions such that:

$$\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}b^ia^{n-i}=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(n+1)}{\Gamma(i+1)\Gamma(n+1-i)}b^ia^{n-i}.$$

Of which, follows from the definition of the Γ function up to i=n. In the i=n+1 step, we arrive at an issue as we are exiting the defined domain of Γ. Without creating unnecessary complexity, let it be stated that Γ is still analytic everywhere on the complex plane with exception to zero, and the negative integers.

Therefore we can restate the equality in the following form:

$$(n+1-m)^t=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)}\frac{m^i}{n+1-m}(n+1)^{t+1-i}.$$

Now consider the incremental steps of m such that:

$$(n+1-(n))^t=1^t=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)}\frac{n^i}{1}(n+1)^{t+1-i}$$
$$(n+1-(n-1))^t=2^t=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)}\frac{(n-1)^i}{2}(n+1)^{t+1-i}$$
$$(n+1-(n-2))^t=3^t=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)}\frac{(n-2)^i}{3}(n+1)^{t+1-i}$$
$$...+...$$
$$(n+1-1)^t=n^t=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(t+2)}{\Gamma(i+1)\Gamma(t+2-i)}\frac{1^i}{n}(n+1)^{t+1-i}.$$

It is clear that each term generates the singular integer to a power t. Taking the sum of all terms yields the expression:

$$\sum_{k=1}^{n} k^t = \sum_{i=0}^{\infty}(n+1)^{t+1-i}(-1)^i \binom{t+1}{i} \sum_{j=1}^{n} \frac{j^i}{n+1-j}.$$

In a simple analogy, we have taken a deck of n cards of irrational numbers and "reshuffled" it to an infinite rational coefficient cards with regards to one irrational number n. Let us now focus on the inner sum on the right hand side of the equation. Using a similar imagery from chapter two, let us approach this sum geometrically. Although unlike chapter two, it is far easier to consider this sum to be contained in a box of dimension i. The first three inductive dimensional steps are considered below in Figure 1:

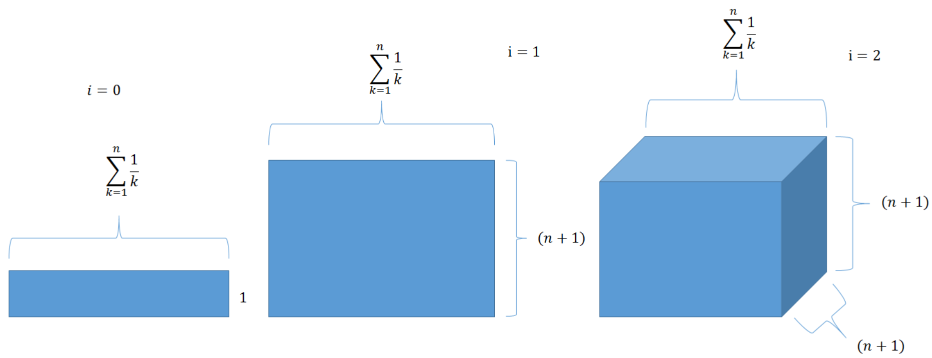

FIGURE 1. The finite box bounding the sums in dimensions i = 0,1,2.

In each case, the sum lies inside the box. Recall from Chapter two the intuition behind bounding the sum inside of a shape. Presenting the problem in this form allows for the answer to be stated as the difference between the volume of the box and some smaller section of the box such that:

$$\sum_{j=1}^{n} \frac{j^i}{n+1-j} = (n+1)^i \left(\sum_{j=1}^{n} \frac{1}{n+1-j} \right) - \sum_{k=1}^{i} A_k n^i.$$

Where the coefficients A_k is what is desired to be solved. Rearranging the above equality yields:

$$\sum_{j=1}^{n} \frac{(n+1)^i - j^i}{n+1-j} = \sum_{k=1}^{i} A_k n^i.$$

There is exactly one root in the numerator on the left hand side of the form j = n+1. Applying polynomial division to the above yields the result:

$$\sum_{k=1}^{i} A_k n^i = \sum_{j=1}^{n}\sum_{r=0}^{i-1} \frac{(n+1-j)(n+1)^{i-1-r} j^r}{n+1-j} = \sum_{j=1}^{n}\sum_{r=0}^{i-1}(n+1)^{i-1-r} j^r.$$

In the above we observe two different summations: a binomial expansion regarding n, and the sum of the integers to a power r. To keep consistent with the other partitions of the equation, we extend the sum of the integers one value more, making sure to subtract it off of the expression:

$$= \sum_{j=1}^{n}\sum_{r=0}^{i-1}(n+1)^{i-1-r} j^r = \sum_{j=1}^{n+1}\sum_{r=0}^{i-1}(n+1)^{i-1-r} j^r - i(n+1)^{i-1}.$$

Now consider on the right hand side there still remains a sum of integers to a power r:

$$\begin{aligned}
&= \sum_{j=1}^{n+1}\sum_{r=0}^{i-1}(n+1)^{i-1-r} j^r - i(n+1)^{i-1} \\
&= \sum_{r=0}^{i-1}(n+1)^{i-1-r}\frac{1}{r+1}\sum_{s=0}^{r}\binom{r+1}{s} B_r (n+1)^{r+1-s} - i(n+1)^{i-1} \\
&= \sum_{r=0}^{i-1}\sum_{s=0}^{r}(n+1)^{i-s}\frac{1}{r+1}\binom{r+1}{s} B_r - i(n+1)^{i-1}.
\end{aligned}$$

Of which determines our coefficients A_k. Placing the above back into the context of the original equation yields:

$$\sum_{k=1}^{n} k^t = \sum_{i=0}^{\infty}(n+1)^{t+1-i}(-1)^i\binom{t+1}{i}\Big((\sum_{k=1}^{n}\frac{1}{k})(n+1)^i - (\sum_{r=0}^{i-1}\sum_{s=0}^{r}(n+1)^{i-s}\frac{1}{r+1}\binom{r+1}{s}B_r - i(n+1)^{i-1})\Big),$$

$$\sum_{k=1}^{n} k^t = \sum_{i=0}^{\infty}(n+1)^{t+1-i}(-1)^i\binom{t+1}{i}\Big((\sum_{k=1}^{n}\frac{1}{k} + \frac{i}{n+1})(n+1)^i - \sum_{r=0}^{i-1}\sum_{s=0}^{r}(n+1)^{i-s}\frac{\binom{r+1}{s}B_r}{r+1}\Big). \tag{3.1}$$

We can restate the above more compactly in the form:

$$\sum_{i=1}^{n} i^t = (n+1)^{t+1-j}(-1)^j\binom{t+1}{j}B_{n,j}.$$

$$B_{n,j} \equiv (n+1)^j\left(\sum_{i=1}^{n}\frac{1}{i} + \frac{j}{n+1}\right) - \sum_{k=0}^{j-1}\sum_{r=0}^{k}(n+1)^{j-r}\frac{\binom{k+1}{r}B_r}{k+1}.$$

Whereas $B_{n,j}$ is to be considered the two dimensional Bernoulli array. Its first few terms are considered below 2.

j	$B_{n,j}$
0	$\sum_{i=1}^{n}(n+1)^0\frac{1}{i}$
1	$\sum_{i=1}^{n}(n+1)^1\frac{1}{i}-n$
2	$\sum_{i=1}^{n}(n+1)^2\frac{1}{i}-\frac{3}{2}n(n+1)$
3	$\sum_{i=1}^{n}(n+1)^3\frac{1}{i}-\frac{n(n+1)(11n+10)}{6}$
4	$\sum_{i=1}^{n}(n+1)^4\frac{1}{i}-\frac{5n(n+1)^2(5n+4)}{12}$
5	$\sum_{i=1}^{n}(n+1)^5\frac{1}{i}-\frac{n(n+1)(137n^3+368n^2+327n+98)}{60}$

FIGURE 2. Components of the Bernoulli Array.

As a simple example, let us take the sums of the 0th powers. Intuitively we know that this should be n (for a sum of the first n '1's). Using this formulation we can state that:

$$\sum_{i=0}^{n} 1 = (n+1)(\sum_{i=0} \frac{1}{i}) - (1)((n+1)(\sum_{i=1} \frac{1}{i} - n)) = n.$$

Of which is the correct result.

3.3. Conclusion

The formula is valid for dimensions $t \in (-1, \infty)$. One may observe that the only reason that the sum of integers is closed (or conversely why the sum of any real, non-integer power t requires an infinite sum) is that the choose function shuts down at j = t+1. It is this distinction that separates the integer powers from the real powers, and that which prevents a closed solution.

Having completed the goal of this paper, it is best to take a step back and reflect on the problem as a whole. The journey that led to our solution was based on a few unspoken premises. Assumptions admittedly, and these assumptions are not set in stone. With only a few simple changes the problem, journey, and solution dramatically changes. It is such changes that entertained the curiosity of many great mathematicians and revealed powerful truths of mathematics. It would seem best at this time to list out these assumptions, and see how the problem changes as the assumptions change.

The first of these assumptions was the set being summed. In this paper there were two sets directly mentioned: the figurate numbers and the positive integers. Both sets were presented in a similar manner, and the transition from the figurate numbers to positive integers through chapter one was subtle. Although, the journey most certainly does not end there. Refining the set being summed from the positive integers to the real numbers is one of the ways which calculus was introduced. This was hinted at with the work of John Wallis back in chapter one with the concept of the summation of infinitesimal slices of various shapes. Hence, varying the set of numbers being summed changes the form of the solution. We observed that the ratio of the sum of figurate numbers to the product of the largest figurate number and the number of terms yielding a constant ratio. This was true primarily as the sum of figurate numbers yielded a number within the figurate numbers. Employing the same ratio over the integers did not produce a constant ratio, but rather a constant with an asymptotically decreasing term dependent on the number of terms being summed. We can observe that in the limit of infinite terms (in the manner which John Wallis presented integration) that the asymptotic term vanishes, leaving a solution identical to the integral over the real numbers on an interval.

The second of these assumptions was the exponent each term was being taken. For most of the duration of chapter one, and all of chapter two, only positive integer exponents were considered. Hinted in chapter one, and tackled in chapter three the set which the chosen exponent was taken to was extended to real numbers within the interval $(-1, \infty)$. As mentioned above, with further work one could extend this even further capturing complex powers leading into finite sums resembling solutions to the Riemann Zeta function (which takes a complex power over the sum of integers to the infinite sum).

Finally, and perhaps most interestingly was the assumption regarding the dimension of the elements being summed. Throughout this entire paper, the elements were zero dimensional numbers as points on a one dimensional line. Bear in mind that these sums were presented as geometric objects in chapters two and three, but this was merely an interpretation to better illuminate the means which the problem to be solved. To truly take the problem to higher dimensions would be to consider taking the sum of integer points in a convex polytope [**7**]. The dimension can be any positive integer, and the polytope any shape that bounds lattice points. It is breaking this assumption regarding dimension that revealed much regarding the nature of the Bernoulli numbers. That is, for any positive integer dimension, the Ehrhart Theorem states that there is a polynomial relationship between the volume of a convex polytope and the number of integer lattice points within its hull. As a closing example let us consider the zero dimensional case discussed in this paper regarding this method of formalism. Consider the polynomial sum of the form:

$$x^1 + x^2 + x^3 + ... + x^n = \frac{x^{n+1} - x}{x - 1}.$$

Whereas the sum is assumed to be finite.

Let us apply the operator $x\frac{d}{dx}$. Observe for the left hand side yields:

$$x\frac{d}{dx}(x^1 + x^2 + x^3 + ... + x^n) = 1x^1 + 2x^2 + ... + nx^n.$$

This conveniently contains the sum of integer powers in the form of the scalar coefficients to each polynomial term. If we were to apply that operator a second, third, or$k$-th time, then we would find the scalar coefficients would imply the sum of the integers to the k-th power. The right hand side yields:

$$x\frac{d}{dx}(\frac{x^{n+1}-x}{x-1}) = \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}.$$

Now evaluating the polynomial for $x = 1$ produces the equality:

$$\begin{aligned} x\frac{d}{dx}\bigg|_{x=1} \frac{x^{n+1}-x}{x-1} = \lim_{x\to 1} \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2} &\to \lim_{x\to 1} \frac{n(n+2)x^{n+1}-(n+1)(n+1)x^n+1}{2(x-1)}, \\ &\to \lim_{x\to 1} \frac{n(n+1)}{2} = \frac{n(n+1)}{2}. \\ x\frac{d}{dx}\bigg|_{x=1} & x^1+x^2+x^3+...+x^n = 1+2+...+n. \\ &\implies \frac{n(n+1)}{2} = 1+2+...+n. \end{aligned}$$

Of which was the desired equality. Again, this was only applied to one dimensional lattice points on a line. This method can be applied to multivariable polynomials in the hull of any integer based convex polytope, of which the theorem states that there will always be a polynomial solution similar to the one dimensional case we have studied thus far.

There is much more to explore with regards to this problem, but that will all be for another time. Thank you for taking the time to read this.

Bibliography

1. E. Abbott, *Flatland a romance of many dimensions*, Penguin Classics, New York, NY, 1998.
2. J. Bernoulli and E. Sylla, *The art of conjecturing*, John Hopkins University Press, Baltimore, MD, 2006.
3. M. Bologna, *Short introduction to fractional calculus*, Lecture Notes, University Tarpeca, Chile, 2005, 41-54.
4. M. Bona, *A walk through combinatorics an introduction to enumeration and graph theory*, World Scientific Publishing, Hackensack, NJ, 2006.
5. A. W. F. Edwards, *Pascal's arithmetical triangle*, Garles Griffin & Co., London, U.K., 1987.
6. R. Lipsett, *Proof of faulhaber's formula*, http://planetmath.org/proofoffaulhabersformula, 2009, Accessed: 2015-03-30.
7. E. Mezzetti and S. Pacha, *Proceedings of the tenth general meeting european women in mathematics*, European women in mathematics, World Scientific Publishingl, River Edge, NJ, 2003.
8. W. Rudin, *Principles of mathematical analysis*, Pure and Applied Mathematics, vol. 1, McGraw-Hill, New York, NY, 1964.
9. J. Taylor, *Foundations of analysis*, Pure and Applied Undergraduate Texts, vol. 18, American Mathematical Society, Providence, RI, 2012.
10. J. Wallis and J. Stedall, *The arithmetic of infinitesimals*, History of Mathematics and Physical Sciences, Springer-Verlag, New York, NY, 2004.
11. P. Young, *A 2-adic formula for bernoulli numbers of the second kind and for the Nörlund numbers*, Journal of Number Theory **128** (2008), no. 11, 2951–2962.