Cut-Generating Functions for Integer Linear Programming

By: Masumi Sugiyama

Faculty advisor: Matthias Köppe

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UNIVERSITY OF CALIFORNIA, DAVIS

COLLEGE OF LETTERS AND SCIENCE

DEPARTMENT OF MATHEMATICS

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Abstract

Linear programming (LP) is an optimization method to achieve the best outcome subject to linear constraints. When the unknown variables are required to be integers, one method of obtaining integer solutions involves refining the feasible regions using general-purpose cutting planes. The coefficients of a cutting plane are given by a valid function. In this thesis, the development of extending an existing software for valid functions in Gomory–Johnson's model to a more general model introduced recently by Yildiz and Cornuejols is presented. The generalized model for positive integers case is studied by using quasi-periodic functions. Developing a computational method of constructing quasi-periodic functions allows investigation of the conversion between group and lifting functions. We focus on the continuous piecewise lifting functions (CPL_3 functions) and their group functions. A computational method for finding the exact parameter region of valid functions is also investigated. Different experiments of solving a set of inequalities using Maple software interfaced in Sage software are discussed.

1 Introduction

The development of linear programming is – in my opinion – the most important contribution of the mathematics of the 20th century to the solution of practical problems arising in industry and commerce.
– Martin Grötschel, 2006 [1].

Optimization problems arise in many situations in everyday life. For example, business applications include maximizing the profit and minimizing the total production cost in a factory. In mathematics, linear programming (LP) is an optimization method to achieve the best outcome subject to linear constraints. If, in addition, the unknown variables are required to be integers, then it is called integer linear programming (ILP). LP is a powerful framework for describing and solving optimization problems, and originated from George Dantzig's work on the military's program-planning process. In 1947, the first algorithm for solving LP problem called the simplex method was proposed by Dantzig. This algorithm remains one of the most efficient and most reliable methods for solving LP problems at this date. However, the simplex algorithm cannot directly handle the integrality conditions of ILP problems. One method of obtaining solutions to such problems involves iteratively refining the feasible regions using general-purpose cutting planes. By removing undesirable fractional solutions, cutting planes tighten the formulation but leave the feasible solutions to the original ILP problems. The coefficients of a cutting plane are given by a valid function which is hierarchized on the strength of cutting planes.

1.1 Generalized Infinite Group Relaxation Problem

Let us consider the optimal simplex tableau of a linear integer program

$$\max\{c \cdot x | Ax = b, x \in \mathbb{Z}_+^m\}.$$

It takes the form

$$x_B = A_B^{-1} b + (-A_B^{-1} A_N) x_N, \quad x_B \in \mathbb{Z}_+^B, x_N \in \mathbb{Z}_+^N$$

where the subscripts B and N denote the basic and non-basic parts of the solution x and matrix A, respectively.

We select n rows of the tableau, corresponding to n basic variables x_i for $i \in B$. The tableau corresponding to these n rows is of the following form

$$x = f + \sum_{j \in N} r_j x_j, \quad x \in \mathbb{Z}_+^n, x_j \in \mathbb{Z}_+ \tag{1}$$

where $r_j \in \mathbb{R}^n$ is the vector of coefficients of decision variable x_j and $f \in \mathbb{R}_+^n$ is the vector of constant values of the original linear constraints.

By relaxing $x_B \in \mathbb{Z}_+^B$ to $x_B \in \mathbb{Z}^B$, we obtain Gomory–Johnson's group relaxation:

$$x_B = A_B^{-1} b + (-A_B^{-1} A_N) x_N, \quad x_B \in \mathbb{Z}^B, x_N \in \mathbb{Z}_+^N.$$

Now, n rows of the tableau is of the following form

$$x = f + \sum_{j \in N} r_j x_j, \quad x \in \mathbb{Z}^n, x_j \in \mathbb{Z}_+.$$

We re-write non-basic variables x_j as a function of r which we write as y_r such that $y : \mathbb{R}^n \to \mathbb{Z}_+$. The function y has finite support which means that the infinite-dimensional vector has a finite number of nonzero entries. By introducing infinitely many new variables y_r for every $r \in \mathbb{R}^n$, we obtain Gomory–Johnson's infinite group relaxation:

$$x = f + \sum_{r \in \mathbb{R}^n} r y_r, \quad x \in \mathbb{Z}^n, y_r \in \mathbb{Z}_+. \tag{2}$$

If we assume $f \in \mathbb{R}^n \setminus \mathbb{Z}$, then the basic solution $x = f, y = 0$, is not feasible. Therefore, we would like to generate cutting planes that cut off this infeasible solution. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *valid function* for (2) if a *valid inequality* $\sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ with $\pi \geq 0$ holds for any feasible solutions (x, y) to (2). The valid function gives the coefficients of a cutting plane for (2) to cut off the infeasible solution.
Instead of restricting a valid function to be nonnegative, Yildiz and Cornuéjols considered the generalization of the Gomory–Johnson infinite group relaxation which allows a valid function to take all real values between $-\infty$ and ∞. The generalization of the Gomory–Johnson infinite group relaxation is defined as

$$\begin{gathered} x = f + \sum_{r \in \mathbb{R}^n} r y_r, \\ x \in S, \\ y_r \in \mathbb{Z}_+, \forall r \in \mathbb{R}^n, \\ y \text{ has finite support,} \end{gathered} \tag{3}$$

where S is an non-empty subset of $\mathbb{R}^n$ and $f \in \mathbb{R}^n_+$.

Definition 1 (Valid function for the generalized case)**.** *A function π: $\mathbb{R}^n \to \mathbb{R}$ is a valid function for* (3) *if the valid inequality $\sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ holds for any feasible solutions (x, y) to* (3)*.*

Valid functions are hierarchized on the strength of the corresponding valid inequalities. Minimal valid functions are considered to be *stronger* than non-minimal valid functions. In the next section, we will look at the characterization of minimal valid functions for (3).

2 Minimal Valid Functions

Definition 2 (Minimal valid function)**.** *A valid function π is minimal if there is no valid function π' distinct from π such that $\pi'(r) \leq \pi(r)$ for every $r \in \mathbb{R}^n$.*

2.1 Generalized Case

The main proofs of the following Lemma 3, 5, 7, and Theorem 8 are from the paper ([5]), and more detailed derivations were added to the original proofs.

Let $\mathbb{Z}_{++}$ be the set of strictly positive integers.

Lemma 3 ([5])**.** *If π is a minimal valid function for* (3)*, then* $\pi(0) = 0$.

Proof. Suppose $\pi(0) < 0$. Let $(\bar{x}, \bar{y})$ be a feasible solution of (3). Then there exists some $\bar{k} \in \mathbb{Z}_{++}$ such that $\pi(0)\bar{k} < 1 - \sum_{r\in\mathbb{R}^n\setminus\{0\}} \pi(r)\bar{y}_r$ since the right hand-side of the inequality is a constant.

Define $\tilde{y}$ as $\tilde{y}_0 = \bar{k}$ and $\tilde{y}_r = \bar{y}_r$ for all $r \neq 0$. Then $(\bar{x}, \tilde{y})$ is a feasible solution of (3) since

$$f + \{0 \cdot (\tilde{y}_0) + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{0\}} r\tilde{y}_r\} = f + \{0 \cdot (\bar{y}_0) + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{0\}} r\bar{y}_r\} = x.$$

This contradicts the assumption that π is a valid function since

$$\textstyle\sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r = \pi(0)\bar{k} + \sum_{r\in\mathbb{R}^n\setminus\{0\}} \pi(r)\tilde{y}_r < 1 - \sum_{r\in\mathbb{R}^n\setminus\{0\}} \pi(r)\bar{y}_r + \sum_{r\in\mathbb{R}^n\setminus\{0\}} \pi(r)\tilde{y}_r = 1.$$

Hence, $\pi(0) \geq 0$.

Let $(\bar{x}, \bar{y})$ be a feasible solution of (3). Define $\tilde{y}$ as $\tilde{y}_0 = 0$ and $\tilde{y}_r = \bar{y}_r$ for every $r \neq 0$. Then as before $(\bar{x}, \tilde{y})$ is a feasible solution of (3). Now define the function π' as $\pi'(0) = 0$ and $\pi'(r) = \pi(r)$ for every $r \neq 0$. Since $\pi'(0) = \tilde{y}_0 = 0$,

$$\textstyle\sum_{r\in\mathbb{R}^n} \pi'(r)\bar{y}_r = \pi'(0)\bar{y}_0 + \sum_{r\in\mathbb{R}^n\setminus\{0\}} \pi'(r)\bar{y}_r = \pi(0)\tilde{y}_0 + \sum_{r\in\mathbb{R}^n\setminus\{0\}} \pi(r)\tilde{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r.$$

Since π is a valid function, $\sum_{r\in\mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$. This implies that π' is also a valid function for (3). Since π is minimal and $\pi' \leq \pi$, we have $\pi' = \pi$. Hence, $\pi(0) = 0$. $\square$

Definition 4 (Subadditivity)**.** *A function π is subadditive if $\pi(r^1 + r^2) \leq \pi(r^1) + \pi(r^2)$ for $r^1, r^2 \in \mathbb{R}^n$.*

Lemma 5 ([5])**.** *If π is a minimal valid function for* (3)*, then π is subadditive.*

Proof. Let $r^1, r^2 \in \mathbb{R}^n$. When $r^1 = 0$ or $r^2 = 0$, $\pi(r^1 + r^2) = \pi(r^1) + \pi(r^2)$ by the Lemma 3.

Assume that $r^1 \neq 0$ and $r^2 \neq 0$. Define the function π' as $\pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$ and $\pi'(r) = \pi(r)$ for $r \neq r^1 + r^2$. Let $(\bar{x}, \bar{y})$ be any feasible solution to (3).
Define $\tilde{y}$ as $\tilde{y}_{r^1} = \bar{y}_{r^1} + \bar{y}_{r^1+r^2}, \tilde{y}_{r^2} = \bar{y}_{r^2} + \bar{y}_{r^1+r^2}, \tilde{y}_{r^1+r^2} = 0$, and $\tilde{y}_r = \bar{y}_r$ otherwise.
Then,

$$\begin{aligned}
\textstyle\sum_{r\in\mathbb{R}^n} r\tilde{y}_r &= r^1\tilde{y}_{r^1} + r^2\tilde{y}_{r^2} + (r^1 + r^2)\tilde{y}_{r^1+r^2} + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{r^1,r^2,r^1+r^2\}} r\tilde{y}_r \\
&= r^1(\bar{y}_{r^1} + \bar{y}_{r^1+r^2}) + r^2(\bar{y}_{r^2} + \bar{y}_{r^1+r^2}) + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{r^1,r^2,r^1+r^2\}} r\tilde{y}_r \\
&= r^1\bar{y}_{r^1} + r^2\bar{y}_{r^2} + (r^1 + r^2)\bar{y}_{r^1+r^2} + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{r^1,r^2,r^1+r^2\}} r\bar{y}_r \\
&= \textstyle\sum_{r\in\mathbb{R}^n} r\bar{y}_r.
\end{aligned}$$

Therefore, $(\bar{x}, \tilde{y})$ is a feasible solution to (3). Furthermore, by similar computation, $\sum_{r\in\mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r$. Since π is a valid function, we have $\sum_{r\in R^n} \pi'(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$. This implies that π' is a valid function. Since π is minimal, it follows that $\pi(r^1 + r^2) \leq \pi'(r^1 + r^2) = \pi(r^1) + \pi(r^2)$. □

Definition 6 (Generalized symmetry condition). *A function $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies the generalized symmetry condition if*

$$\pi(r) = \sup_{x,k} \left\{ \frac{1}{k}(1 - \pi(x - f - kr)) : x \in S, k \in \mathbb{Z}_{++} \right\} \quad \forall r \in \mathbb{R}^n. \tag{4}$$

Lemma 7 ([5]). *If π is a minimal valid function for* (3)*, then it satisfies the generalized symmetry condition.*

Proof. Let $\bar{r} \in \mathbb{R}^n$. For any $\bar{x} \in S$ and $\bar{k} \in \mathbb{Z}_{++}$, define $\bar{y}$ as $\bar{y}_{\bar{r}} = \bar{k}, \bar{y}_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise.
Then,

$$\begin{aligned} f + \textstyle\sum_{r\in\mathbb{R}^n} r\bar{y}_r &= f + [\bar{r}\bar{y}_{\bar{r}} + (\bar{x} - f - \bar{k}\bar{r})\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{\bar{r},\bar{x}-f-\bar{k}\bar{r}\}} r\bar{y}_r] \\ &= f + \bar{r}\bar{k} + (\bar{x} - f - \bar{k}\bar{r}) \\ &= \bar{x}. \end{aligned}$$

Therefore, $(\bar{x}, \bar{y})$ is a feasible solution to (3). Since π is a valid function,

$$\begin{aligned} \textstyle\sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r = \pi(\bar{r})\bar{y}_{\bar{r}} + \pi(\bar{x} - f - \bar{k}\bar{r})\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} + 0 \geq 1 \\ \pi(\bar{r}) \geq \tfrac{1}{\bar{k}}(1 - \pi(\bar{x} - f - \bar{k}\bar{r})). \end{aligned}$$

The definition of supremum implies that $\pi(\bar{r}) \geq \sup\{\frac{1}{k}(1-\pi(x-f-k\bar{r})) : x \in S, k \in \mathbb{Z}_{++}\}$. Since π is a real-valued function, the value on the right hand side is bounded from above.

Let the function $\rho : \mathbb{R}^n \to \mathbb{R}$ be defined as $\rho(r) = \sup\{\frac{1}{k}(1-\pi(x-f-kr)) : x \in S, k \in \mathbb{Z}_{++}\}$ for all $r \in \mathbb{R}^n$. Suppose that π does not satisfy the generalized symmetry condition. Then there exists $\tilde{r} \in \mathbb{R}^n$ such that $\pi(\tilde{r}) > \rho(\tilde{r})$. Define the function π' as $\pi'(\tilde{r}) = \rho(\tilde{r})$ and $\pi'(r) = \pi(r)$ for all $r \neq \tilde{r}$. Consider any feasible solution $(\tilde{x}, \tilde{y})$ to (3).

Case 1: If $\tilde{y}_{\tilde{r}} = 0$,

$$\textstyle\sum_{r\in\mathbb{R}^n} \pi'(r)\tilde{y}_r = \pi'(\tilde{r})\tilde{y}_{\tilde{r}} + \sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi'(r)\tilde{y}_r = \pi(\tilde{r})\tilde{y}_{\tilde{r}} + \sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi(r)\tilde{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r.$$

Since π is a valid function, $\sum_{r\in\mathbb{R}^n} \pi'(r)\tilde{y}_r = \sum_{r\in\mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$.

Case 2: If $\tilde{y}_{\tilde{r}} \geq 1$,

$$\pi'(\tilde{r})\tilde{y}_{\tilde{r}} + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi'(r)\tilde{y}_r \geq 1 - \pi(\tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}) + \sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi(r)\tilde{y}_r \geq 1.$$

The first inequality is obtained from $\pi'(\tilde{r}) = \rho(\tilde{r}) \geq \frac{1}{\tilde{y}_{\tilde{r}}}(1 - \pi(\tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}))$ by setting $k = \tilde{y}_{\tilde{r}}$ and $x = \tilde{x}$. Since $\rho(\tilde{r})$ is the supremum of the set, the right-hand side of the equation is less than or equal to $\rho(\tilde{r})$ for the particular choice of $k \in \mathbb{Z}_{++}$ and $x \in S$. The second inequality is obtained from the subadditivity of π and $\sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} r\tilde{y}_r = \tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}$. Then,

$$\pi(\tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}) = \pi(\textstyle\sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} r\tilde{y}_r) \leq \textstyle\sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi(r)\tilde{y}_r.$$

Hence, $1 - \pi(\tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}) + \sum_{r\in\mathbb{R}^n\setminus\{\tilde{r}\}} \pi(r)\tilde{y}_r \geq 1$.

Therefore, π' is a valid function for (3). Since $\pi'(\tilde{r}) \leq \pi(\tilde{r})$ from $\pi'(\tilde{r}) = \rho(\tilde{r}), \pi(\tilde{r}) > \rho(\tilde{r})$ and $\pi'(r) = \pi(r)$ for all $r \neq \tilde{r}$, this is a contradiction to the minimality of π. Thus, π satisfies the generalized symmetry condition. □

The results of Lemma 3, Lemma 5, and Lemma 7 lead to the following Theorem.

Theorem 8 ([5])**.** *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal valid function for* (3) *if and only if $\pi(0) = 0, \pi$ is subadditive and satisfies the generalized symmetry condition.*

Proof. Assume that $\pi(0) = 0, \pi$ is subadditive and satisfies the generalized symmetry condition. Since $\pi(0) = 0$, the generalized symmetry condition implies

$$0 = \sup_{x,k}\{\tfrac{1}{k}(1 - \pi(x - f)) : x \in S, k \in \mathbb{Z}_{++}\}.$$

By considering $x = \bar{x}$ and $k = 1$,

$$0 \geq 1 - \pi(\bar{x} - f).$$

Thus $\pi(\bar{x} - f) \geq 1$ for every $\bar{x} \in S$.
Any feasible solution $(\bar{x}, \bar{y})$ for (3) satisfies $\sum_{r\in\mathbb{R}^n} r\bar{y}_r = \bar{x} - f$, and we have

$$\textstyle\sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r \geq \pi(\sum_{r\in\mathbb{R}^n} r\bar{y}_r) = \pi(\bar{x} - f) \geq 1.$$

Thus, π is a valid function. The first inequality is obtained by using the subadditivity of π.

Assume by contradiction that π is not minimal. Then there exists a valid function π' such that $\pi' \leq \pi$ and $\pi'(\bar{r}) < \pi(\bar{r})$ for some $\bar{r} \in \mathbb{R}^n$. Let $\epsilon = \pi(\bar{r}) - \pi'(\bar{r})$. Because π satisfies the generalized symmetry condition, there exist $\bar{k} \in \mathbb{Z}_{++}$ and $\bar{x} \in S$ such that $\pi(\bar{r}) - \frac{\epsilon}{2} \leq \frac{1}{\bar{k}}(1 - \pi(\bar{x} - f - \bar{k}\bar{r})$. Using $\pi' \leq \pi$ and $\epsilon = \pi(\bar{r}) - \pi'(\bar{r})$, we obtain

$$\begin{aligned} 1 &\geq \bar{k}(\pi(\bar{r}) - \tfrac{\epsilon}{2}) + \pi(\bar{x} - f - \bar{k}\bar{r}) \\ &= \bar{k}(\pi'(\bar{r}) + \epsilon - \tfrac{\epsilon}{2}) + \pi(\bar{x} - f - \bar{k}\bar{r}) \\ &\geq \bar{k}(\pi'(\bar{r}) + \tfrac{\epsilon}{2}) + \pi'(\bar{x} - f - \bar{k}\bar{r}). \end{aligned}$$

This implies that $\bar{k}\pi'(\bar{r}) + \pi'(\bar{x} - f - \bar{k}\bar{r}) < 1$. This contradicts the hypothesis that π' is a valid function. Since $(\bar{x}, \bar{y})$, where $\bar{y}$ is defined as $\bar{y}_{\bar{r}} = \bar{k}$, $\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise, is a feasible to (3),

$$\begin{aligned} \textstyle\sum_{r\in\mathbb{R}^n} \pi'(r)\bar{y}_r &= \pi'(\bar{r})\bar{y}_{\bar{r}} + \pi'(\bar{x} - f - \bar{k}\bar{r})\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} + \textstyle\sum_{r\in\mathbb{R}^n\setminus\{\bar{r},\bar{x}-f-\bar{k}\bar{r}\}} \pi'(r)\bar{y}_r \\ &= \pi'(\bar{r})\bar{k} + \pi'(\bar{x} - f - \bar{k}\bar{r}) \\ &< 1. \end{aligned}$$

Thus, π is minimal. □

2.2 Positive Integer Case

The case $S = \mathbb{Z}_+^n$ is of particular interest since it is closely related to n row of the tableau of an integer linear program (1) described earlier. In this paper, we are interested in one dimensional case where $n = 1$ and $S = \mathbb{Z}_+$.

Assumption 1 *When $S = \mathbb{Z}_+$, assume that $f \in \mathbb{R} \setminus \mathbb{Z}_+$.*

The generalization of Gomory and Johnson model in this setting is of the following form.

$$\begin{aligned} x &= f + \sum_{r \in \mathbb{R}} r y_r, \\ x &\in \mathbb{Z}_+, \\ y_r &\in \mathbb{Z}_+, \forall r \in \mathbb{R}, \\ y &\text{ has finite support}, \end{aligned} \tag{5}$$

where $S = \mathbb{Z}_+$ and $f \in \mathbb{R}_+$.

Theorem 8, the characterization of minimal valid functions, yields that if a function $\pi : \mathbb{R} \to \mathbb{R}$ is a minimal valid function for (5), it satisfies $\pi(0) = 0$, π is subadditive, and π satisfies the generalized symmetry condition (4).

The generalized symmetry condition for the case of $n = 1$ and $S = \mathbb{Z}_+$ is defined as

$$\pi(r) = \sup_{x,k} \left\{ \frac{1}{k}(1 - \pi(x - f - kr)) : x \in \mathbb{Z}_+, k \in \mathbb{Z}_{++} \right\} \quad \forall r \in \mathbb{R}.$$

The generalized symmetry condition can also be satisfied using different conditions.

Definition 9 (Nondecreasing with respect to S)**.** *A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing with respect to S if $\pi(r) \leq \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in S$.*

Proposition 10 ([5])**.** *The function $\pi : \mathbb{R} \to \mathbb{R}$ satisfies the generalized symmetry condition if and only if π is nondecreasing with respect to $\mathbb{Z}_+$ and satisfies the condition*

$$\pi(r) = \sup_{k} \left\{ \frac{1}{k}(1 - \pi(-f - kr)) : k \in \mathbb{Z}_{++} \right\} \quad \forall r \in \mathbb{R}. \tag{6}$$

By satisfying the generalized symmetry condition using Proposition 10, the characterization of minimal valid functions can be described in the following Theorem.

Theorem 11 ([5])**.** *Let $S = \mathbb{Z}_+$. Let $\pi : \mathbb{R} \to \mathbb{R}$. The function π is a minimal valid function for (5) if and only if $\pi(0) = 0, \pi$ is subadditive, nondecreasing with respect to S, and satisfies (6).*

The following proposition gives the condition for a function π to be nondecreasing with respect to $\mathbb{Z}_+$.

Proposition 12 ([5])**.** *Let $\pi : \mathbb{R} \to \mathbb{R}$ be a subadditive function such that $\pi(0) = 0$. The function π is nondecreasing with respect to $\mathbb{Z}_+$ if and only if $\pi(-1) \leq 0$.*

Theorem 11 with Proposition 12 lead to the following simpler result of the characterization of minimal valid function for $S = \mathbb{Z}_+$. The result is that a function π is a minimal valid function (5) if and only if $\pi(0) = 0, \pi(-1) \leq 0, \pi$ is subadditive and satisfies (6).

3 Quasi-Periodic Functions

Definition 13 (Quasi-periodic)**.** *A continuous function $\pi : \mathbb{R} \to \mathbb{R}$ is quasi-periodic with period $d \in \mathbb{R}_+$ if there exists $c \in \mathbb{R}$ such that $\pi(r + d) = \pi(r) + c$ for all $r \in \mathbb{R}$. See Figure 1.*

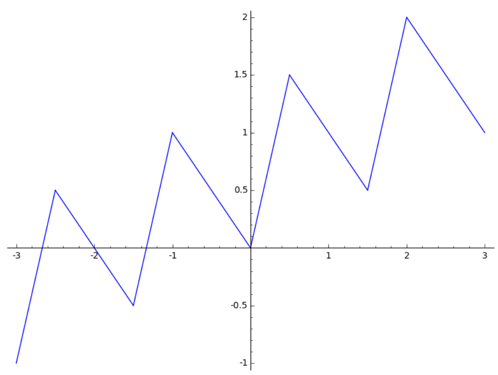


Figure 1: Continuous quasi-periodic function π with period $d = \frac{3}{2}$.[1]

3.1 Minimal and Strongly Minimal Valid Functions

Definition 14 (Symmetric)**.** *A function $\pi : \mathbb{R} \to \mathbb{R}$ is symmetric if $\pi(r)+\pi(-f-r) = \pi(-f)$ for all $r \in \mathbb{R}$.*

Theorem 15 ([5])**.** *Let $S = \mathbb{Z}_+$ and f satisfy Assumption 1. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a quasi-periodic function with period d. The function is a minimal valid function for (3) if and only if $\pi(0) = 0, \pi(-1) \leq 0$, π is subadditive and symmetric. See Figure 2.*

Definition of 'imply' in strongly minimal valid functions
Let $\alpha(x - f) \geq \alpha_0$ be a valid inequality for $S = \mathbb{Z}_+$. A valid function π' for (5) *implies* another valid function π for (5) if there exists a valid inequality $\alpha(x - f) \geq \alpha_0$ for S and $\beta \geq 0$ such that $\pi(r) \geq \alpha r + \beta\pi'(r)$ and $\alpha_0 + \beta \geq 1$.

The above definition makes sense. Let $f + \sum_{r\in\mathbb{R}} ry_r = x$ for any (x, y) feasible to (5). Then, $x - f = \sum_{r\in\mathbb{R}} ry_r$ and $\sum_{r\in\mathbb{R}} \alpha ry_r \geq \alpha_0$. If $\sum_{r\in\mathbb{R}} \pi'(r)y_r \geq 1$ is a valid inequality for (5), then $\sum_{r\in\mathbb{R}} \pi(r)y_r \geq \sum_{r\in\mathbb{R}} \alpha ry_r + \beta \sum_{r\in\mathbb{R}} \pi'(r)y_r \geq \alpha_0 + \beta \geq 1$ is also a valid inequality for (5). The first and second inequalities come from the definition of *imply*.

[1] q = PiecewiseQuasiPeriodic([[(0,1/2), FastLinearFunction(3,0),[(1/2,3/2),FastLinearFunction(-1,2)]])
[2] q = PiecewiseQuasiPeriodic([[(0,1), FastLinearFunction(3,0)],[(1,4/3), FastLinearFunction(-3,6)]])

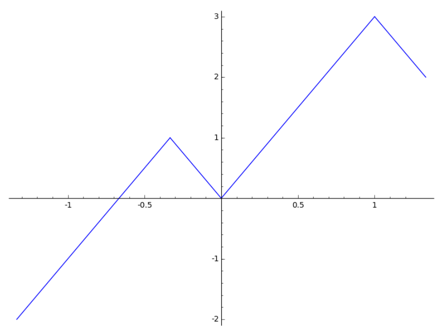


Figure 2: Minimal quasi-periodic function π with period $d = \frac{4}{3}$.[2]

Definition 16 (Strongly minimal valid function)**.** *A valid function π is strongly minimal if there does not exist a valid function π' distinct from π that implies π. See Figure 3.*

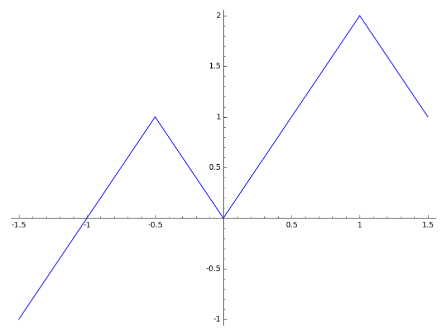


Figure 3: Strongly minimal quasi-periodic function π with period $d = \frac{3}{2}$.[3]

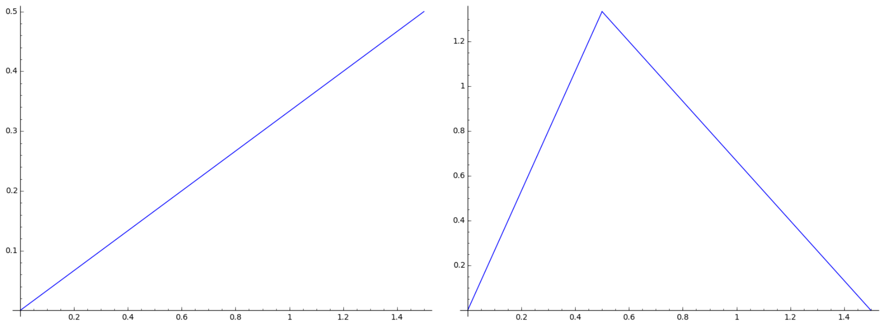
Note that a strongly minimal valid function π is minimal. This follows from the definition above. By taking $\alpha = 0$, $\alpha_0 = 0$, and $\beta = 1$, the definition says that there does not exist a valid function π' distinct from π such that $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}$. Hence, π is minimal. With the fact that strongly minimal valid functions are minimal, the following Theorem gives a simpler way of defining strongly minimal valid functions.

Theorem 17 ([5])**.** *Let $S = \mathbb{Z}_+$ and f satisfy Assumption 1. Let $\pi : \mathbb{R} \to \mathbb{R}$. The function π is a strongly minimal cut-generating function for* (5) *if and only if π is a minimal valid function for* (5)*, $\pi(-f) = 1$ and $\pi(-1) = 0$. See Figure 3.*

3.2 Decomposition of Quasi-Periodic Functions

Let $\tilde{\pi} : \mathbb{R} \to \mathbb{R}$ be a continuous quasi-periodic function with period d. Then, $\tilde{\pi}$ can be written as $\tilde{\pi}(r) = \pi(r) + \phi(r)$ where $\pi : \mathbb{R} \to \mathbb{R}$ is a periodic function with period d and $\phi : \mathbb{R} \to \mathbb{R}$ is a linear function with its slope $\frac{\tilde{\pi}(d)-\tilde{\pi}(0)}{d}$.

[3]q = PiecewiseQuasiPeriodic([[(0,1), FastLinearFunction(2,0)],[(1,3/2),FastLinearFunction(-2,4)]])



(a) Linear function.[4] (b) Periodic function.[5]

Figure 4: Decomposition of Figure 1

Assumption 2: *We assume that the continuous quasi-periodic function takes the value zero at the origin, i.e.,*$\tilde{\pi}(0) = 0$.

From Assumption 2, we have $\phi(r) = \alpha r$ with $\alpha = \frac{\tilde{\pi}(d)}{d}$.
New claim 1: *The decomposition of a continuous quasi-periodic function is unique.*

Proof. Let $\tilde{\pi}(r) = \pi(r) + \phi(r)$ be a quasi-periodic function where $\pi(r)$ is a periodic function and $\phi(r)$ is a linear function. Since $\pi(r)$ is periodic, it must have the same values at $r = 0$ and $r = d$ such that $\pi(0) = \pi(d) = 0$ from Assumption 2. Then, the linear function $\phi(r)$ must have the points such that $(0, \tilde{\pi}(0) - \pi(0)) = (0, 0)$ and $(d, \tilde{\pi}(d) - \pi(d)) = (d, \tilde{\pi}(r))$. Therefore, the function has a slope $\frac{\tilde{\pi}(d) - \pi(d) - \tilde{\pi}(0) - \pi(0)}{d - 0} = \frac{\tilde{\pi}(d)}{d}$. Since the slope depends only on a value of $\tilde{\pi}$, there exists a unique linear function going through the origin associated with a quasi-periodic function. This implies that there exits a unique periodic function. Hence, the decomposition of a quasi-periodic function is unique. $\square$

Definition 18 (Normalized periodic function)**.** *Let* $\pi : \mathbb{R} \to \mathbb{R}$ *be a periodic function with period* d*. A periodic function* $\pi' : \mathbb{R} \to \mathbb{R}$ *is a normalized periodic function of* π *if and only if* $\pi'(r') = \frac{\pi(dr')}{c}$ *where* $r' = \frac{r}{d}$ *and* $c \in \mathbb{R}_+$ *is the value of* $\pi(-f)$*.*

New claim 2.1: *The normalized periodic function* π' *is subadditive if and only if the periodic function* π *is subadditive.*

Proof. Assume that the normalized periodic function π' is subadditive.
Then, $\pi'(r'_1 + r'_2) \leq \pi'(r'_1) + \pi'(r'_2)$ holds for every $r'_1, r'_2 \in \mathbb{R}$.

$$\pi'(r'_1 + r'_2) \leq \pi'(r'_1) + \pi'(r'_2)$$
$$\Longleftrightarrow$$
$$\frac{\pi(dr'_1 + dr'_2)}{c} \leq \frac{\pi(dr'_1)}{c} + \frac{\pi(dr'_2)}{c}$$
$$\Longleftrightarrow$$
$$\pi(r_1 + r_2) \leq \pi(r_1) + \pi(r_2).$$

[4] *q.linear_term*
[5] *q.periodic_term*

Since r_1', r_2' are arbitrary, this implies that r_1, r_2 are also arbitrary.
Hence, $\pi(r_1 + r_2) \leq \pi(r_1) + \pi(r_2)$ holds for all $r_1, r_2 \in \mathbb{R}$, and the periodic function π is subadditive. $\square$

New claim 2.2: *The normalized periodic function π' is not subadditive if and only if the periodic function π is not subadditive.*

Proof. ($\Rightarrow$) Assume by contraposition that the periodic function π is subadditive. From Claim 2, we know that the normalized periodic function π' is subadditive. Hence, if π' is not subadditive, then π is not subadditive.
($\Leftarrow$) Similarly from Claim 2 that if π is not subadditive, then π' is not subadditive. $\square$

New claim 3.1: *The quasi-periodic function $\tilde{\pi}$ is subadditive if and only if the normalized periodic function π' is subadditive.*

Proof. Assume that the normalized periodic function π' is subadditive. From the above Claim 2, we know that the assumption implies that the periodic function π is subadditive. Then, $\pi(r_1 + r_2) \leq \pi(r_1) + \pi(r_2)$ holds for all $r_1, r_2 \in \mathbb{R}$.
Let ϕ be the linear function from the quasi-periodic function $\tilde{\pi}$ such that $\phi = \alpha r$.

$$\pi(r_1 + r_2) \leq \pi(r_1) + \pi(r_2)$$
$$\Longleftrightarrow$$
$$\pi(r_1 + r_2) + \alpha(r_1 + r_2) \leq \pi(r_1) + \alpha r_1 + \pi(r_2) + \alpha r_2$$
$$\Longleftrightarrow$$
$$\tilde{\pi}(r_1 + r_2) \leq \tilde{\pi}(r_1) + \tilde{\pi}(r_2).$$

Since r^1, r^2 are arbitrary, $\tilde{\pi}(r_1 + r_2) \leq \tilde{\pi}(r_1) + \tilde{\pi}(r_2)$ holds for every $r^1, r^2 \in \mathbb{R}$. Hence, the quasi-periodic function $\tilde{\pi}$ is subadditive. $\square$

New Claim 3.2: *The quasi-periodic function $\tilde{\pi}$ is not subadditive if and only if the normalized periodic function π' is not subadditive.*

Proof. ($\Rightarrow$) Assume by contraposition that the normalized periodic function π' is subadditive. From Claim 3, we know that the quasi-periodic function $\tilde{\pi}$ is subadditive. Hence, if $\tilde{\pi}$ is not subadditive, then π' is not subadditive.
($\Leftarrow$) Similarly from Claim 3 that if π' is not subadditive, then $\tilde{\pi}$ is not subadditive. $\square$

As a conclusion of Claim 2.1-3.2, testing subadditivity of a normalized periodic function is sufficient to test subadditivity of a quasi-periodic function.

New claim 4: *The quasi-periodic function $\tilde{\pi}(r)$ is symmetric if and only if the normalized periodic function $\pi'(r')$ is symmetric.*

Proof. Assume that the normalized periodic function $\pi'(r')$ is symmetric.

$$\pi'(r') + \pi'(-f' - r') = \pi'(-f')$$
$$\Longleftrightarrow$$
$$\frac{\pi(dr')}{c} + \frac{\pi(-df' - dr')}{c} = \frac{\pi(-df')}{c}$$
$$\Longleftrightarrow$$
$$\pi(r) + \pi(-f - r) = \pi(-f)$$
$$\Longleftrightarrow$$
$$\pi(r) + \phi(r) + \pi(-f - r) + \phi(-f - r) = \pi(-f) + \phi(-f).$$

Since $\pi(-f) = \tilde{\pi}(-f) - \phi(-f)$, the above equation becomes

$$\tilde{\pi}(r) + \tilde{\pi}(-f - r) = \tilde{\pi}(-f).$$

Since $r \in \mathbb{R}$ is arbitrary, the quasi-periodic function $\tilde{\pi}(r)$ is symmetric. □

New claim 4.1: *The quasi-periodic function $\tilde{\pi}(r)$ is not symmetric if and only if the normalized periodic function $\pi'(r')$ is not symmetric.*

Proof. ($\Rightarrow$) Assume by contraposition that the normalized periodic function π' is symmetric. From Claim 4, we know that the quasi-periodic function $\tilde{\pi}$ is symmetric. Hence, if $\tilde{\pi}$ is not symmetric, then π' is not symmetric.
($\Leftarrow$) Similarly from Claim 4 that if π' is not symmetric, then $\tilde{\pi}$ is not symmetric. □

Therefore, verifying if a normalized periodic function is symmetric is sufficient to test if a quasi-periodic function is symmetric.

3.3 Sage Code

3.3.1 Construction of a Quasi-Periodic Function

The class function called 'PiecewiseQuasiPeriodic' constructs a continuous quasi-periodic function from a list of (interval,function) pairs.

'__*init*__()' assigns an object with its period, its linear and periodic terms decomposed from the quasi-periodic function.

'__*call*__()' evaluates a value of the quasi-periodic function at x.

1. If x is in the interval which is defined from the list of pairs, the function is evaluated at x.

2. If x is outside of the interval which is defined from the list of pairs, x is shifted to an appropriate point in the interval. The linear and periodic terms decomposed from the function are evaluated at the shifted x. A value of the quasi-periodic function at x is the sum of these values with the appropriate factor.

'*normalized_periodic_function*' normalizes a periodic term decomposed from the quasi-periodic function. Break-points are shifted in $[0, 1]$ and function values are normalized in the way that the difference of the maximum and minimum values is 1.

'*plot*()' plots the quasi-periodic function on a given interval.

1. If the minimum value of x is not given or *quasiperiodic_extension* is $False$, the minimum value of x is set to zero. If the maximum value of x is not given or *quasiperiodic_extension* is $False$, the maximum value of x is set to the sum of the minimum value of x and the period of the function.

2. By computing break-points and function values of the first repetition on the given interval from the left, the quasi-periodic function will be plotted with appropriate increases in break-points and function values at each repetition until break-points reach the maximal values of x.

3.3.2 Minimality/Strong Minimality Test

Assume that $S = \mathbb{Z}_+$ and f satisfies Assumption 1. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a continuous quasi-periodic function.

'*quasiperiodic_minimality_test*' tests if the quasi-periodic function π is minimal.

1. $\pi(0) = 0, \pi(-1) \leq 0$ and $\pi(-f) = 1$ are checked using the '*__call__*()' method defined the class 'PiecewiseQuasiPeriodic.' The condition $\pi(-f) = 1$ comes from the symmetric condition with $r = 0$.

2. The conditions for subadditive and symmetric are checked using the normalized periodic function decomposed from the quasi-periodic function.

'*quasiperiodic_strong_minimality_test*' tests if the quasi-periodic function π is strongly minimal.

1. $\pi(-1) = 0$ are checked using the '*__call__*()' method defined the class 'Piecewise-QuasiPeriodic.'

2. Minimality of π is checked using the function '*quasiperiodic_minimality_test*'. The condition $\pi(-f) = 1$ is also checked in the function.

4 Group and Lifting functions

4.1 The conversion between group and lifting functions

Group and lifting functions appear in the two different papers ([4] and [2]). In the paper written by Dey and Richard, they are called *group-space representation* and *lifting-space representation*, respectively. In the paper written by Miller, Li, and Richard, group and lifting functions are called *group representation* and *standard representation*, respectively.

Definition 19 (Lifting function)**.** *Let $r, f \in [0,1)$. Given a valid function $\pi : \mathbb{R} \to \mathbb{R}$, the lifting function of $\phi : \mathbb{R} \to \mathbb{R}$ is defined as $\phi(r) = r - f \cdot \pi(r)$. See Figure 5.*

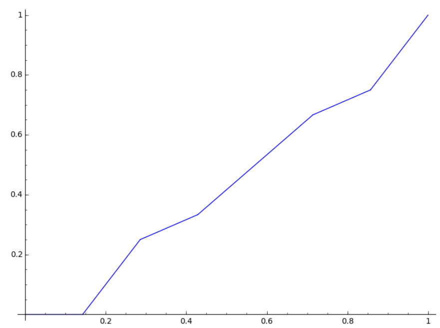


Figure 5: Lifting function ϕ.[6]

Definition 20 (Superadditive)**.** *A function $\pi : \mathbb{R} \to \mathbb{R}$ is superadditive if $\pi(r_1 + r_2) \geq \pi(r_1) + \pi(r_2)$ for all $r_1, r_2 \in \mathbb{R}$.*

Definition 21 (Group function)**.** *Let $r, f \in [0,1)$. Given a superadditive lifting function $\phi : \mathbb{R} \to \mathbb{R}$, the group function $\pi : \mathbb{R} \to \mathbb{R}$ is defined as $\pi(r) = \frac{r-\phi(r)}{f}$. See Figure 6.*

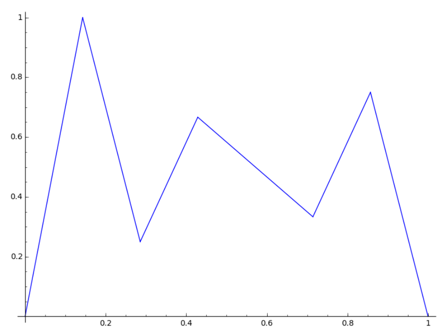


Figure 6: Group function π with $f = \frac{1}{7}$.[7]

Group Functions with Multiple f Values

When there are more than one possible f values in a group function, different super-additive lifting functions are obtained from the group function with a given f value by the conversion. Figure 7 shows a group function with two possible f values. Figure 8a and 8b show two different lifting functions corresponding to the conversion from the group function with the specific f values. When the opposite conversion (from lifting functions to group functions) is performed, however, it may not give back the original group function depending on values

[6] $phi = superadditive\_lifting\_function\_from\_group\_function(mlr\_cpl3\_d\_3\_slope(1/7, 1/7))$

[7] $fn = group\_function\_from\_superadditive\_lifting\_function(cpl3\_function(1/7, 1/7, 1/4, 1/12))$

[8] $h = multiplicative\_homomorphism(gmic(f = 4/5), 2)$

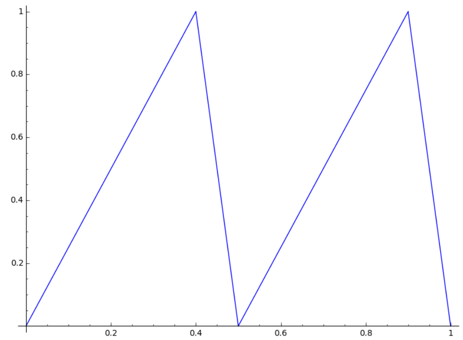
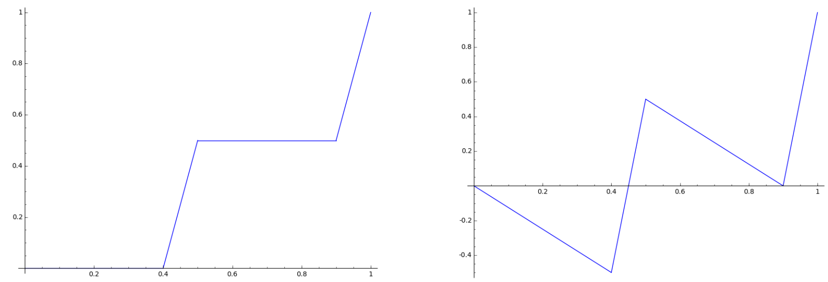


Figure 7: Group function π with $f = \frac{2}{5}$ and $\frac{9}{10}$.[8]



(a) $f = \frac{2}{5}$.[9] (b) $f = \frac{9}{10}$.[10]

Figure 8: Lifting function ϕ from Figure 7 with two different f.

of f chosen. For instance, the conversion from Figure 8a (obtained with $f = \frac{2}{5}$) to a group function with $f = \frac{9}{10}$ gives a group function that takes the function value $\frac{4}{9}$, instead of the function value 1, at $f = \frac{2}{5}$ and $\frac{9}{10}$. On the other hand, the original group function will be obtained by choosing $f = \frac{2}{5}$. This suggests that it is necessary to choose a proper f to obtain a desirable group function.

4.2 CPL-3 function

In Miller, Li, and Richard's paper ([4]), they focus on continuous piecewise linear lifting functions (CPL_n functions). In this section, we will look at the case for $n = 3$, CPL_3 functions.

Definition 22 (CPL_3 function)**.** *Let $f \in (0,1)$. Let $z = (z_1, z_2, z_3) \in \mathbb{R}^3_+$ and $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3_+$ be such that $\sum_{j=1}^{3} z_j = \frac{1-f}{2}$ and $\sum_{j=1}^{3} \theta_j = \frac{1}{2}$. Then, a continuous piecewise lifting function, a CPL_3 function $\phi(u)$, is defined as*

$$\phi(r) = \begin{cases} 0, & \text{if } r \in [0, f] \\ \sum_{j=1}^{i-1} \theta_i + \frac{\theta_i}{z_i}(r - f - \sum_{j=1}^{i-1} z_i), & \text{if } r \in (f + \sum_{j=1}^{i-1} z_i, f + \sum_{j=1}^{i} z_i] \\ 1 - \sum_{j=1}^{i} \theta_i + \frac{\theta_i}{z_i}(r - 1 + \sum_{j=1}^{i} z_i), & \text{if } r \in (1 - \sum_{j=1}^{i} z_i, 1 - \sum_{j=1}^{i} z_{i-1}] \end{cases}$$

[9] $phi = superadditive\_lifting\_function\_from\_group\_function(h, f = 2/5)$
[10] $phi = superadditive\_lifting\_function\_from\_group\_function(h, f = 9/10)$

See Figure 5 or 8a.

In the paper, only CPL_3 functions with $z_1 = z_2$ are studied because of the significant reduction in the number of cases under the condition. Note that CPL_3 functions are continuous and nondecreasing. The partition of the interval on $[0, 1]$ is $[0, f, f + z_1, f + 2z_1, 1 - 2z_1, 1 - z_1, 1]$. Values of break-points $(f + 2z_1)$ and $(1 - 2z_1)$ can be the same depending on values of z_1.

4.2.1 Extreme Valid Functions

Group functions that are converted from CPL_3 functions yield strong valid inequalities for the infinite group problem. At the beginning of section 1.1, it is mentioned that minimal valid functions are stronger than non-minimal valid functions. Moreover, extreme valid functions are stronger than minimal valid functions since they are a subset of minimal valid functions. As a result, extreme valid functions are considered to be *strongest* among all of these valid functions.

Definition 23. *A valid function π is extreme if it cannot be written as a convex combination of two other valid functions, i.e., $\pi = \frac{1}{2}(\pi_1 + \pi_2)$ implies $\pi = \pi_1 = \pi_2$. See Figure 6.*

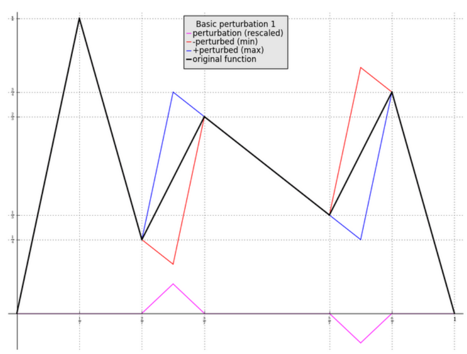


Figure 9: Example of non-extreme function π.[11]

In the paper, there are 14 extreme points for infinite group problem. The cases for extreme points a, h, k, l, o, p, q, and r are two-slope functions. The cases for extreme points b, c and g are three-slope functions. The first two cases have the same extreme conditions with the function called Dey–Richard–Li–Miller's backward 3-slope function. The case for extreme point f is two- or three-slope functions depending on its parameter values. Extreme functions corresponding to each of extreme points can be constructed in two different ways: (1) from break-points and slopes, and (2) from the conversion to group functions from CPL_3 functions. In order to check each of the cases described in the paper, functions constructed using the methods (1) and (2) are checked to be equal functions.

Corrections in the Case l and p

Extreme functions for the case for l and p cannot be constructed from their break-points and slopes given in the paper. The table below summarizes the given slopes in the paper.

[11] *extremality_test*(*group_function_from_superadditive_lifting_function*(*cpl3_function*(1/7, 1/7, 1/4, 1/12)), *show_plots* = *True*)

Interval	$[0, f]$	$[f, f + z_1]$	$[f + z_1, f + 2z_1]$	$[f + 2z_1, 1 - 2z - 1]$	$[1 - 2z_1, 1 - z_1]$	$[1 - z_1, 1]$
Slopes	s_1	s_2	s_3	s_4	s_5	s_6
l	$\frac{1}{f}$	$\frac{2}{2f-1}$	$\frac{-4z_1}{2z_1(1-2f)}$	s_2	s_3	s_2
p	$\frac{1}{f}$	$\frac{2}{2f-1}$	s_1	$\frac{2z_1-10z_1f+f}{f(1-2f)(4z-1-1+f)}$	s_1	s_2

The unavailability of constructing the extreme functions was first realized by comparing figures obtained from the method (1) with the given precise figures in the paper. Since figures constructed by the method (2) are the same as the given figures, it was suspected that some of the given slopes were not correct. By comparing slopes obtained from the method (1) and (2), the incorrect slopes were found. The incorrect slopes are s_3 for the case l, and s_4 for the case p. The correct slopes were found by using the definition of CPL_3 function with $z_1 = z_2$ and the formulas of θ_1, θ_2 for the cases l, p given in the paper, and the conversion from CPL_3 functions to group functions.

For the case l, $\theta_1 = \frac{z_1}{1-2f}$ and $\theta_2 = \frac{2z_1-f}{2-4f}$.
From the definition of CPL_3 function, the function ϕ on $[f + z_1, f + 2z_1]$ is defined as

$$\phi(r) = \theta_1 + \frac{\theta_2}{z_2}(r - f - z_1)$$

From the equation defined in Definition 21, a group function π is defined as

$$\pi(r) = \frac{r-(\theta_1+\frac{\theta_2}{z_2}(r-f-z_1))}{f}$$

By evaluating at the end points, we have $(f + z_1, \frac{f+z_1-\theta_1}{f})$ and $(f + 2z_1, \frac{f-2z_1-\theta_1-\theta_2}{f})$.
Then, the slope of the function π from the two points is given by

$$\frac{z_1-\theta_2}{fz_1} = \frac{z_1-(\frac{2z_1-f}{2-4f})}{fz_1} = \frac{1-4z_1}{2z_1(1-2f)}$$

Therefore, the correct slope for the case l is $\frac{1-4z_1}{2z_1(1-2f)}$.
Similarly, the correct slope for the case p was computed, and it is $\frac{8z_1f-2z_1-f(1-2f)}{f(1-2f)(1-4z_1-f)}$.

Unsolved Problem in the Case k

For the case k, when the break-points (r_0+2z_1) and $(1-2z_1)$ are equal, the break-points and slopes given in the paper do not seem to construct a correct function. This was suspected from the fact that the function constructed from the given slopes does not match with the function constructed from the definition of CPL_3 function with the values of θ_1, θ_2 given in the paper. Unlike the case l and p, there are no typos of the given slopes. It could suggest that the values of θ_1, θ_2 could be incorrect. This problem has not been solved yet.

4.3 Sage Code

4.3.1 Conversions between Group and Lifting Functions

'*superadditive_lifting_function_from_group_function*()' converts a superaditive lifting function ϕ (a superadditive quasiperiodic function) from a group function π (a subadditive periodic function). If f is not given, it is computed using *find_f*() from the group function π.

'$group\_function\_from\_superadditive\_lifting\_function()$' convert a group function π (a subadditive periodic function) from a lifting function ϕ (a superadditive quasiperiodic function). If f is not given, it is the value of f when the difference of $\phi(f) - f$ is minimized over the interval.

4.3.2 Construction of a CPL-3 function

'$cpl3\_function()$' constructs a CPL_3 function from four inputs. Note that $r_0 = f$. The function is a quasi-periodic function which inherits all properties of 'PiecewiseQuasiPeriodic.' If the conditions for parameters and the definition of CPL_3 function are satisfied, it will construct a CPL_3 function. If break-points $(r_0 + 2z_1)$ and $(1 - 2z_1)$ are the same, $(1 - 2z_1)$ will be removed from the set of break-points.

4.3.3 Extreme Functions in the Literature

'$mlr\_cpl3\_\ldots()$' constructs a group function corresponding to an extreme point, $a, b, c, d, f, g, h, k, l, n, o, p, q$, and r, which yields an extreme valid function if the conditions for extremality are satisfied. '$mlr\_cpl3\_\ldots()$' takes values of r_0, z_1 and *conditioncheck* as inputs. Note that $r_0 = f$.

1. For d, f, g, k, l, o and p cases, if z_1 is not given, it is computed using an equality condition for extremality. For the other cases, z_1 must be given as input. If parameter conditions for a CPL_3 function are satisfied, a function will be constructed.

2. If $conditioncheck = True$, the conditions for extremality are checked. If parameter conditions satisfy the extremality condition, an extreme function will be constructed. If $conditioncheck = False$, conditions for extremality are not checked.

3. If break-points $(r_0 + 2z_1)$ and $(1 - 2z_1)$ are the same, $(1 - 2z_1)$ and the corresponding slope will be removed from the set of break-points and slopes.

5 Parametric Search

The parametric search is to compute a parameter region of a given function where functions with given parameter values have the same minimality or extremality conditions. The region is obtained from a set of inequalities and equalities for the parameters that derived based on the valid inequalities. Computing the set of inequalities and equalities to obtain the corresponding parameter region is performed using unpublished version of the software ([3]) developed by Yuan Zhou, a Ph.D student in the Department of Mathematics at UC Davis.

In this section, the following functions were used for one, two, or three parameter cases: GMIC (Gomory mixed integer cut) for one parameter case, Gomory–Johnson's 2-Slope function for two parameter case, and Gomory–Johnson's Forward 3-Slope function for three parameter case ([3]). Each of these functions takes an input called *conditioncheck*. Under '$conditioncheck = False$,' parameters do not need to satisfy parameter conditions for extremality. This produces a set of inequalities and equalities that construct a parameter region where functions can be constructed. On the other hand, under '*conditioncheck* $= True$' parameters must satisfy parameter conditions for extremality. This produces a region where functions are extreme. Under '*conditioncheck* $= False$,' calling '*minimality_test*()' or '*extremality_test*()' to see if functions with specific parameter values are minimal or extreme, can restrict parameter conditions and thus produces a larger set of inequalities and equalities. It can result in a smaller parameter region than the constructible region. This smaller parameter region differs depending on given parameter values and partitions the entire minimal or extreme regions. Since extreme functions must be minimal, parameter conditions of extreme functions can be more strict than that of minimal functions.

5.1 Maple Experiment

A set of inequalities and equalities are solved using Maple software in Sage.

Depending on an order of parameters given, solutions are expressed in different forms. The most left side of parameters in a list has the priority, i.e., in the case of two parameters, $[f, \lambda]$, the parameter f has the priority over λ.

5.1.1 Number of parameters vs. CPU-time for a set of unsimplified inequality

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Number of unsimplified inequalities

Function	Construction	Minimality test	Extremality test
GMIC	3	8	10
Gomory–Johnson's 2-Slope	4	35	44
Gomory–Johnson' Forward 3-Slope	7	86	116

For each of these functions, the number of inequalities increases in the order of constructible, minimal, and extreme conditions. The number of inequalities increases exponentially for minimal and extreme conditions as the number of parameters increases. It can be suspected that a CPU-time for solving a set of inequalities increases with the conditions and number of parameters.

Note: A CPU-time measured is the time obtained for the first run using '*timelimit()*' in Maple. Once an expression is evaluated for the first time, it seems that the CPU-time needed to evaluate the same expression becomes shorter. Therefore, each CPU-times measured was obtained from the first run. Depend on a expression, a CPU-time varies in a wide range in each run.

The amount of CPU-time spent on evaluating the set of inequalities using the Maple function '*solve()*' was set to 600 seconds.

CPU-time

Function	Construction	Minimality test	Extremality test
GMIC	0.136	0.250	0.269
Gomory–Johnson's 2-Slope	0.216	Exceeded	Exceeded
Gomory–Johnson' Forward 3-Slope	0.744	Exceeded	Exceeded

The result above shows the necessity of simplifying a large set of inequalities and equalities so that Maple software can solve within a reasonable time period.

We will look closely at the case of Gomory–Johnson's 2-Slope/Minimality test and /Extremality test to examine the relation between number of inequalities and CPU-time. For the both cases, starting from the original set of inequalities, some number of inequalities were being deleted each time until 6 inequalities remained which are the inequalities to produce the simplified inequalities examined in the next section.

Minimality test

Number of inequalities	35-33	32	27	22	17	12	6
CPU-time	Exceeded	1448.471	854.406	139.364	41.600	6.758	1.172

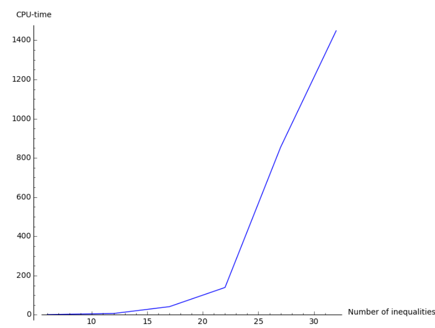


Figure 10: Number of inequalities vs. CPU-time: Gomory–Johnson's 2-Slope/Minimality test.

Extremality test

Number of inequalities	44-29	28	21	14	6
CPU-time	Exceeded	846.895	98.498	14.862	3.850

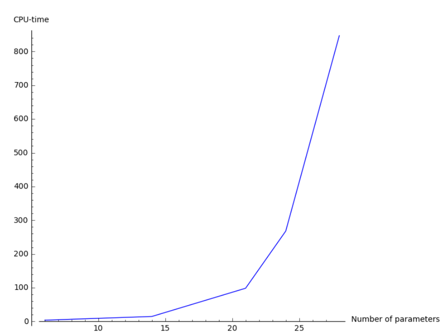


Figure 11: Number of inequalities vs. CPU-time: Gomory–Johnson's 2-Slope/Extremality test.

Figures 10 and 11 show that the CPU-time increases exponentially with the number of inequalities.

5.1.2 Number of parameters vs. CPU-time for a set of simplified inequalities

Simplified inequalities are obtained from a set of unsimplified inequalities by the use of unpublished version of the software ([3]).

Number of simplified inequalities

Function	Construction	Minimality test	Extremality test
GMIC	3	3	3
Gomory–Johnson's 2-Slope	4	6	6
Gomory–Johnson' Forward 3-Slope	7	11	11

Comparing with the table of 'Number of unsimplified inequalities,' the number of inequalities for constructible condition does not change but these for minimal and extreme conditions are significantly reduced.

The table below and figure 12 show CPU-time for solving a set of simplified inequalities with different numbers of parameters.

CPU-time

Function	Construction	Minimality test	Extremality test
GMIC	0.175	0.178	0.182
Gomory–Johnson's 2-Slope	0.267	0.851	1.312
Gomory–Johnson' Forward 3-Slope	0.831	56.32	104.183

As we can see from the table and figure, for one parameter case, there are no significant differences in the CPU-times obtained from sets of unsimplified or simplified inequalities. However, there are significant CPU-time differences for two and three parameters between the case of unsimplified or simplified inequalities. In addition, the CPU-time increases rapidly from two to three parameter cases.

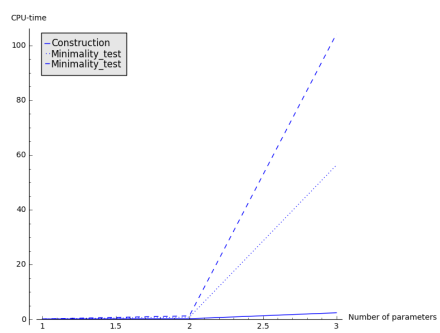


Figure 12: Number of parameters vs. CPU-time.

5.1.3 Priority of parameters vs. CPU-time

Depending on the priority of parameters, CPU-time can vary within the same function. As an example, the table below shows CPU-time for the Gomory–Johnson's 2-Slope function with two different priorities.

Priority	CPU-time		
	Construction	Minimality test	Extremality test
f	0.267	0.597	0.693
λ	0.125	0.416	0.508

When f is the priority over λ, the solution to the set of inequalities is given by one set of simplified inequalities. In the construction case, the solution is

$$[[\lambda < 1, 0 < \lambda, f < 1, \lambda/(\lambda + 1) < f]]$$

When λ is the priority over f, however, the solution to the same set of inequalities is given by two sets of simplified inequalities. In the construction case, the solution is

$$[[f < 1/2, 0 < f, \lambda < -f/(f - 1), 0 < \lambda], [f < 1, 1/2 <= f, \lambda < 1, 0 < \lambda]]$$

This difference in number of sets of simplified inequalities could contribute to CPU-time for solving the set of inequalites within the same function.

5.2 Parameter Regions

As in the above section, GMIC function was used for one parameter case. For two parameter case, Gomory–Johnson's 2-Slope and Dey–Richard–Li–Miller's backward 3-slope functions were examined. As described at the beginning of this section, '$conditioncheck = False$' produces a region where functions are constructible with given parameters. '$Conditioncheck = True$' produces a region where functions are extreme with given parameters. The combination of '$conditioncheck = False$' and '$minimality\_test()$' or '$extremality\_test()$' produces a region where functions with specific parameter values are minimal or extreme.

5.2.1 One Parameter Case

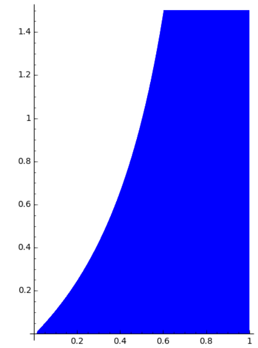
1. GMIC function

Conditioncheck	Parameter Region
False and True	$0 < f < 1$

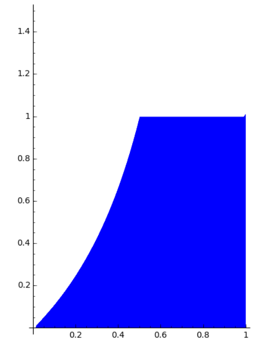
The result shows that every function constructed with a given f will be extreme. Therefore, the function must also be minimal. The parameter region can be further divided based on minimality or extremality conditions with a specific parameter value. The table below summarizes the partitions of minimal and extreme parameter regions.

Parameter Region
$0 < f < \frac{1}{2}$
$f = \frac{1}{2}$
$\frac{1}{2} < f < 1$

Every function constructed with a value $f \in (0, \frac{1}{2})$ have the same minimal and extreme properties. Similarly, any functions constructed with a value $f \in (\frac{1}{2}, 1)$ have the same minimal and extreme properties.

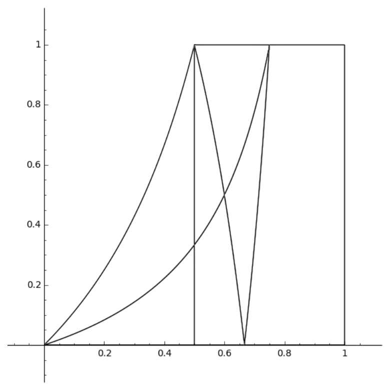


(a) Constructible parameter region.[12]

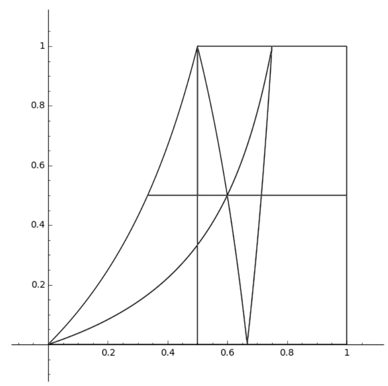


(b) Extreme parameter region.[13]

Figure 13: Gomory–Johnson's 2-Slope function.



(a) Partitions for minimal parameter region.



(b) Partition for extreme parameter region.

Figure 14: Gomory–Johnson's 2-Slope function.

5.2.2 Two Parameters Case

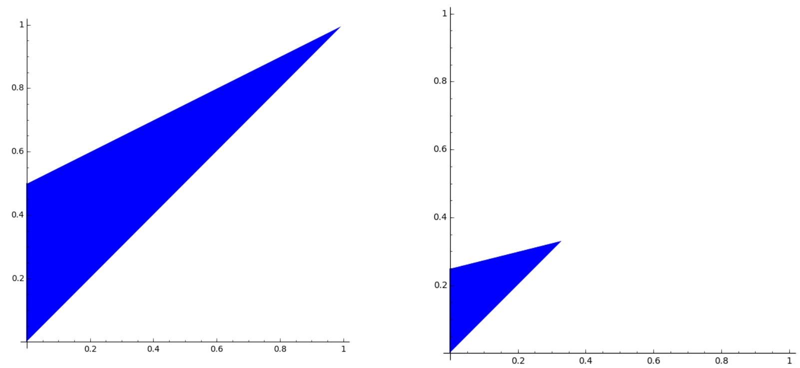
1. Gomory–Johnson's 2-Slope function

$Conditioncheck$	Parameter Region
False	$0 < \lambda$, $\frac{\lambda}{\lambda+1} < f < 1$
True	$0 < \lambda < 1$, $\frac{\lambda}{\lambda+1} < f < 1$

Figure 13a is the constructible parameter region while Figure 13b is the extreme parameter region for the function. As we can see from the figures, the extreme parameter region is smaller than the constructible parameter region. The minimal parameter region is the same as the extreme parameter region in this case since functions with λ greater than 1 take a value greater than 1.

[12] $g = plot\_region([0 < lam, lam/(lam+1) < f, f < 1], (f, 0, 1), (lam, 0, 3/2), plot\_points = 500)$

[13] $g = plot\_region([0 < lam, lam < 1, lam/(lam+1) < f, f < 1], (f, 0, 1), (lam, 0, 3/2), plot\_points = 500)$



(a) Parameter region for construction.[14]

(b) Parameter region for extremality.[15]

Figure 15: Dey–Richard–Li–Miller's backward 3-slope function.

Figure 14b shows the partitions of the extreme region, and Figure 14a shows the partitions of the minimal region. By comparing these two figures, it is seen that the extreme region is more divided than the minimal region. This tells us that the parameter conditions are more strict for extreme functions than minimal functions.

2. Dey–Richard–Li–Miller's backward 3-slope function

$Conditioncheck$	Parameter Region
False	$0 < f < 1,\ f < q < \frac{1}{2}(f+1)$
True	$0 < f < \frac{1}{3},\ f < q < \frac{1}{4}(f+1)$

Figure 15a is the constructible parameter region while Figure 15b is the extreme parameter region for the function. Unlike part(a), the extreme parameter region is much smaller than the constructible parameter region. In this case, the minimal parameter region is the same as the constructible parameter region.

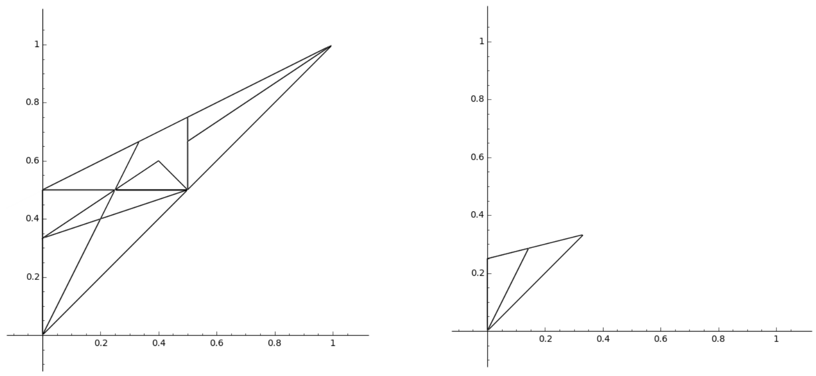
Figure 16b shows the partitions of the extreme region, and Figure 16a shows the partitions of the minimal region. By comparing these figures, it is seen that the extreme parameter region is contained in the minimal parameter region. Again, this shows that the parameter conditions for extreme functions is more strict than the conditions for minimal functions.

5.3 Sage Code

'$solve\_inequalities\_via\_maple()$' solves a set of inequalities and equalities via Maple software in sage. It takes three arguments; a set of inequalities and equalities, a list of parameters, and a time-limit (seconds) of evaluating a set of inequalities and equalities. The Maple function '$solve()$' is used to solve the set.

[14] $g = plot\_region([0 < f, f < 1, f < bkpt, bkpf < f/2 + 1/2], (f, 0, 1), (bkpt, 0, 1), plot\_points = 500)$

[15] $g = plot\_region([0 < f, f < 1, f < bkpt, bkpf < f/4 + 1/4], (f, 0, 1), (bkpt, 0, 1), plot\_points = 500)$



(a) Partitions for minimal parameter region.

(b) Partition for extreme parameter region.

Figure 16: Dey–Richard–Li–Miller's backward 3-slope function.

1. If a time-limit is not given, the set is evaluated without a time restriction.

2. If a time-limit is given, the set is evaluated within the time-limit if possible; otherwise an error message is received.

Difficulties of interfacing to Maple software

1. All inequalities and equalities have to be converted into strings in order to use the Maple function '$solve()$' in sage.

2. The Maple function '*timelimit()*' in sage does not work properly in the form of

   $$maple.timelimit(time, expression)$$

   The upper limit of CPU-time is applied after the expression is evaluated. In order to set the limit of CPU-time of evaluating the expression, we need to call '*timelimit()*' in the form of

   $$maple(\text{“}timelimit(time, expression)\text{”})$$

6 Conclusions

By considering the generalization of the Gomory–Johnson infinite group relaxation model by Yildiz and Cornuéjols with the case $n = 1$ and $S = \mathbb{Z}_+$, we developed a computational method of testing the minimality and strong minimality conditions for quasi-periodic functions. These conditions for quasi-periodic functions can be checked from function values, subadditive, and symmetric conditions. The decomposition of a quasi-periodic function into a linear and a periodic functions allowed to check subadditive and symmetric conditions for the quasi-periodic function by using the existing software for valid functions in Gomory–Johnson model.

There were a couple of interesting findings in the conversion between group and superadditive lifting functions. First of all, converting between a superadditive lifting function and a group function requires with more than one possible f values require a specific f value chosen so that a desired result will be obtained. Next, there were incorrect slopes for group functions from CPL_3 functions given in the paper. Also, when two of break-points are equal, a group functions from CPL_3 function for the case k does not seem to be constructed correctly. This needs a further investigation to be solved. This could suggest a further investigation on the case when extreme parameter conditions are not checked (*,.i.e, conditioncheck=False*) and on the relation between the definition of CPL_3 function and the conversion.

For the parametric search, there were limitations of solving a large set of inequalities using Maple software and therefore needed a computational method of simplifying the set of inequalities and equalities. My investigations of Maple software and its limitations prompted Yuan Zhou to develop a computational simplification method based on polyhedral computations. Using the new software developed by her, simplifying the large set of inequalities significantly reduced number of inequalities and consequently computational time. Since the priority of parameters can also affect computational time, it may be useful to know what order of parameters produce the least number of inequalities. For CPU-time measurements, there needs a further investigation in the interfacing to Maple software in Sage to understand why CPU-time varies in a wide range for some expressions. As a result of simplified inequalities, parameter regions were plotted according to the strength of valid inequalities. These parameter regions were further divided depending on specific parameter values of given functions.

Acknowledgments

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